Chapter 1

Introduction

The Cerebrospinal Fluid (CSF) surrounds the brain and acts as a protection to the brain inside the skull. As a result of the cardiac cycle the CSF will flow up and down the subarachnoid space (SAS) surrounding the spinal cord. The Chiari malformation is a displacement of the cerebellar tonsils that partially blocks CSF flow entering the SAS around the spinal cord. This causes abnormal CSF flows which sometimes results in in a syringomyelia inside the spinal cord filled with fluid. Treatment may include surgery to remove parts of the bones of the skull to relieve pressure. Studies have shown that the syrinx gradually vanishes after surgery. The mechanisms behind this are not yet fully understood. Many researchers have suggested Computational Fluid Dynamics (CFD) to give useful insight, as experiments are very difficult and expensive.

Chapter 2

Problem definition

The model consists of a spinal cord surrounded by space filled with CSF. The cord has a diameter of 10 mm and the fluid space has a diameter of 18 mm. The model also has a inner central spinal canal, 2 mm in diameter. The central spinal canal has height 4 cm and is placed in the center of the mesh which has heigth 6 cm.

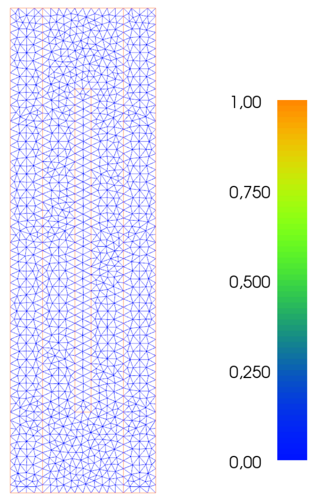

Figure 2.1: Coarse version of the mesh

2.1 Governing equations

2.1.1 Fluid Space

The fluid in the SAS as well as the central spinal canal is governed by the Navier-Stokes equations

$$\frac{\partial u_f}{\partial t} + u \cdot \nabla u = -\frac{1}{\rho_f}\nabla p_f + \frac{1}{\rho}\nabla \cdot \tau + F \qquad \text{in } \Omega_f \tag{2.1}$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho u_f) = 0 \qquad \text{in } \Omega_f \tag{2.2}$$

where u_f is the fluid velocity, ρ_f is the fluid denisty, τ is the stress tensor and F are all external volume forces. In addition to the equations (2.1-2.2) comes

boundary conditions at the domain boundary. When solving the euqations numerically, manipulation of the coupled equations into simpler equations may lead to problems involving the boundary conditions. Therefore a proper choice of boundary conditions is crucial for stability and existence.
In this case we will consider incompressible flows, where ρ is constant and $\nabla \cdot u_f = 0$. The stress tensor is given by

$$\tau_f = -pI + 2\mu_f \epsilon(u_f)$$

where $\epsilon(u_f) = \frac{1}{2}(\nabla u_f + (\nabla u_f)^T)$ When taking the divergence of the stress tensor the Navier-Stokes equations for incompressible flows take the form

$$\frac{\partial u_f}{\partial t} + u \cdot \nabla u_f = -\frac{1}{\rho}\nabla p_f + \nu \nabla^2 u_f + F \qquad \text{in } \Omega_f \tag{2.3}$$

$$\nabla \cdot u_f = 0 \qquad \text{in } \Omega_f \tag{2.4}$$

2.1.2 Spinal Cord

The spinal cord is modeled as a poroelastic medium, which allows for some fluid to flow from the SAS to the central spinal canal. In this case the Biot-equations have been used to include both elasticity and flow in the spinal cord.

2.1.3 Weak form

Equations (3) and (5) will be multiplied by a vector test function, v, and equations (4) and (6) with a scalar test function, q. The equations are then integrated over the entire domain. By using two different test functions the coupled equations can be written as one equation combining the two.

2.1.4 Stationary problem, Stokes

In stokes flow, a steady state pattern have developed for the flow, and convection is neglected. We start with coupling the Stokes flow with a stationary Biot problem. In the CSF the governing equations are

$$\begin{aligned} -\mu_f \nabla^2 u_f + \nabla p_f &= 0 \\ \nabla \cdot u_f &= 0 \end{aligned} \tag{2.5}$$

And in the cord

$$\begin{aligned} -\mu_s \nabla^2 u_s - (\mu_s + \lambda_s)\nabla\nabla \cdot u_s + \nabla p_s &= 0 \\ -\nabla \cdot \frac{\kappa}{\mu_f} \nabla p_s &= 0 \end{aligned} \tag{2.6}$$

Since this is a stationary problem, the first lines of the two equations can be written $-\nabla \cdot \sigma_f = 0$ in Ω_f and $-\nabla \cdot \sigma_s = 0$ in Ω_s

2.1.5 Boundary conditions

We denote the outer wall, $\partial\Omega_f$,, interface Γ, the top of the cord, $\partial\Omega_{s_t}$, the bottom of the cord $\partial\Omega_{s_b}$, the top of the fluid space $\partial\Omega_{f_t}$ and the bottom of the fluid space $\partial\Omega_{f_b}$. The boundary conditions are as follows:
No-slip on the outer walls

$$u = 0 \text{ on } \partial\Omega_{f_o} \tag{2.7}$$

Conservation of mumentum at the interface:

$$\sigma_f \cdot n = \sigma_s \cdot n \text{ on } \Gamma \tag{2.8}$$

Normal velocity equal at the interface:

$$u_f \cdot n = -\frac{\kappa}{\mu_f} \nabla p_s \cdot n \text{ on } \Gamma \tag{2.9}$$

In the stationary case, the filtration velocity, $-k\nabla p$, corresponds to the total fluid velocity in Ω_s
Prescribed stress on the top and bottom

$$\begin{aligned} -\sigma_s \cdot n &= p0 \text{ on } \partial\Omega_{f_t} \\ -\sigma_f \cdot n &= p0 \text{ on } \partial\Omega_{s_t} \\ -\sigma_s \cdot n &= -p0 \text{ on } \partial\Omega_{f_b} \\ -\sigma_f \cdot n &= -p0 \text{ on } \partial\Omega_{s_b} \end{aligned} \tag{2.10}$$

2.1.6 Weak form

Equations (7) and (8) will be multiplied with testFunctions and integrated over their respective domains. The vector equations are multiplied with a vector testFunction, v, and the scalar equations with a scalar testFunction, q. By using two different test functions the coupled equations can be written as one equation combining the two.
We now integrate the simplified versions of 7-8. Both stress terms and the continuity equation in the Biot problem are integrated by parts.

$$\begin{aligned} (\sigma_f, \nabla v)_{\Omega_f} - (v, \sigma_f \cdot n)_{\partial\Omega_f} &= 0 \\ (\nabla \cdot u_f, q)_{\Omega_f} &= 0 \end{aligned} \tag{2.11}$$

$$\begin{aligned} (\sigma_s, \nabla v)_{\Omega_s} - (v, \sigma_s \cdot n)_{\partial\Omega_s} &= 0 \\ (\frac{\kappa}{\mu_f} \nabla p_s, \nabla q)_{\Omega_s} - (q, \frac{\kappa}{\mu_f} \nabla p \cdot n)_{\partial\Omega_s} &= 0 \end{aligned} \tag{2.12}$$

First of all, using condition (10) we can eliminate the stress terms on the interface, when combining the two equations. Condition (12) can be used to handle the stress term on the top and bottom of the domain. Since we have a no-slip condition on the outer wall, the testFunction v is set to be zero on this boundary so the boundary stress term has now been eliminated from the equations by employing the conditions (10) and (12).

If we use continuous elements, $u_f = u_s$ on Γ by the method itself. For now we settle with this approximation, which might be realistic for low x-velocities and small discplacements. Therefore we can use the condition (11) by rewriting the boundary term in eqn (14). The final form F is:

$$(\sigma_f, \nabla v)_{\Omega_f} + (\sigma_s, \nabla v)_{\Omega_s} + (\nabla \cdot u_f, q)_{\Omega_f} + (\frac{\kappa}{\mu_f} \nabla p_s, \nabla q)_{\Omega_s} + (q, u_s \cdot n)_{\partial \Omega_s} \tag{2.13}$$

And we seek to solve F = 0. In this case, we assume $u_f = u_s$ on the interface, then (11) is satisfied. At the top and bottom of the cord we also assume $u_s = -\frac{\kappa}{\mu_f} \nabla p_s$(???)

Chapter 3

Mathematical background

The flow of CSF around the spinal cord requires equations for fluid flow to be coupled with equations for poroelasticity. The underlying concepts of these kinds of problems were originally developed somewhat independently within petroleum engineering, geomechanics and hydrogeology. The equations will first be presented separately. Later in the chapter, coupling conditions will be discussed.

3.1 Fluid flow

The most fundamental equations in fluid flow are conservation laws. These equations are based on classical mechanics and states conservation of mass, momentum and energy.

3.1.1 Reynolds Transport Theorem

The famous engineer and scientist Osbourne Reynold stated the general conservation law the following way [1]:

Any change whatsoever in the quantity of any entity within a closed surface can only be effected in one or other of two distinct ways:

1. it may be effected by the production or destruction of the entity within the surface, or

2. by the passage of the entity across the surface.

Now, consider a control volume, V_0 and some fluid property $Q(\mathbf{x}, t)$. The rate of change of Q within the control volume can be written

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{V_0} Q(\mathbf{x}, t) \, \mathrm{d}V \tag{3.1}$$

The net change of Q must be equal the rate of change in Q within the control volume plus the net rate of mass flow out of the volume. In other words

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{V_0} Q(\mathbf{x},t)\,\mathrm{d}V = \int_{V_0} \frac{\partial Q(\mathbf{x},t)}{\partial t}\mathrm{d}V + \int_{S_0} Q(\mathbf{x},t)\mathbf{u}\cdot\mathbf{n}\mathrm{d}S \tag{3.2}$$

Here, $\mathbf{u}$ denotes the fluid velocity. This equation is known as the Reynold's transport theorem. The right hand side could be rewritten by using Gauss' theorem on the last term.

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{V_0} Q(\mathbf{x},t)\,\mathrm{d}V = \int_{V_0} \big[\frac{\partial Q(\mathbf{x},t)}{\partial t} + \nabla\cdot(Q(\mathbf{x},t)\mathbf{u})\big]\mathrm{d}V \tag{3.3}$$

where

$$\nabla = \mathbf{i}_j\frac{\partial}{\partial x_j}$$

3.1.2 Conservation of mass and momentum

Choose $Q(\mathbf{x},t) = \rho$. Conservation of mass means that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{V_0} \rho\,\mathrm{d}V = 0$$

Which in turn implies

$$\int_{V_0} \big[\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{u})\big]\mathrm{d}V = 0 \tag{3.4}$$

This should hold for any volume V_0, hence the integrand has to be zero:

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{u}) = 0 \tag{3.5}$$

(3.5) is known as the continuity equation and states conservation of mass.

To derive a simliar property for the momentum, Newtons second law of motion is used. The net change of momentum must be equal to the applied forces to the system. The forces can be divided into volume forces, acting on the entire control volume, and forces acting only on the control surface. The forces acting on the surface can be written $\sigma\cdot n$, where $\sigma = \sigma(\mathbf{u},p)$ is the tensor denoting the total stress. By inserting $Q(\mathbf{x},t) = \rho\mathbf{u}$, and using Gauss' theorem again we end up with

$$\frac{\partial \rho\mathbf{u}}{\partial t} + \nabla\cdot(\rho\mathbf{u}\mathbf{u}) = \nabla\cdot\sigma + \mathbf{F}_v \tag{3.6}$$

The stress tensor, σ, depends on fluid properties and will be defined in the next subsection. Equation 3.6 is known as the momentum equation as it states conservation of momentum.

3.1.3 Incompressible Newtonian fluids

In this text we will only consider incompressible fluid flow for a Newtonian fluid. The assumption of a Newtonian fluid requires the viscous stresses to be linear functions of the components of the strain-rate tensor, denoted by $\epsilon_{ij} = \frac{\partial u_i}{\partial x_j}$. These assumptions were first made by Stokes in 1845. These assumptions have later proven to be quite accurate for all gases and most common fluids. Stokes' three postulates regarding the deformation laws are: [2]

1. The fluid is continuous, and its stress tensor, σ_{ij} is at most a linear function of the strain rates, ϵ_{ij}
2. The fluid is isotropic, i.e., its properties are independent of direction, and therefore the deformation law is independent of the coordinate axes in which it is expressed.
3. When the strain rates are zero, the deformation law must reduce to the hydrostatic pressure condition, $\sigma_{ij} = -p\delta_{ij}$, where δ_{ij} is the Kroenecker delta function.

From the first and third conition the following assumption can be made

$$\sigma_{ij} = -p\delta_{ij} + K_{ijkl}\epsilon_{kl} \tag{3.7}$$

It can be shown that symmetry of σ and ϵ also requires symmetry of K. This assumption reduces the number of coefficients in 3.7 from 36 to 21. If Stokes' second condition is also taken into account and the fluid properties are identical in each direction the number of coefficients are further reduced to 2. These simplifications allow us to denote the stress tensor the following way:

$$\sigma_{ij} = -p\delta_{ij} + 2\mu\epsilon_{ij} + \lambda\nabla\cdot\mathbf{u} \tag{3.8}$$

where $\epsilon = \frac{1}{2}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i})$, p is the fluid pressure and μ and λ are known as Lame's constants. In the present study we only consider incompressible flow where ρ is constant. From 3.5, this implies $\nabla\cdot\mathbf{u} = 0$ and the last term in 3.8 vanishes. Furthermore,

$$\nabla\cdot\epsilon = \frac{\partial}{\partial x_j}(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j})\mathbf{i}_i = (\frac{\partial}{\partial x_i}\frac{\partial u_j}{\partial x_j} + \frac{\partial u_i}{\partial x_j\partial x_j})\mathbf{i}_i = \frac{\partial u_i}{\partial x_j\partial x_j}\mathbf{i}_i$$

Which simplifies the representation of $\nabla\cdot\sigma$ in 3.6 for an incompressible fluid

3.1.4 Navier-Stokes equations for incompressible flow

Stating both conservation of mass and momentum of a fluid together with suitable boundary conditions gives us all the information we need to be able to define the flow field and the corresponding pressure. This requires a solution to the system (3.5-3.6), equations which are commonly referred to as the Navier-Stokes equations. With the simplifications described in the previous section the system of equations can be written

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\frac{1}{\rho}\nabla p + \nu \nabla^2 \mathbf{u} + \mathbf{F}_v \tag{3.9}$$

$$\nabla \cdot \mathbf{u} = 0 \tag{3.10}$$

These equations are coupled and non-linear and can generally not be solved analytically. Hence, numerical solutions are a necessity to obtain useful solutions to real-life problems. Such metods will be discussed in chapter xxx.

3.2 Linear Poroelasticity

In this section, the equations describing fluid flowing trough a elastic medium is presented. For a more detailed discussion, derivation and history within the field we refer to [3] on Linear Poroelasticity. To keep the mathematics as similar to the fluid case as possible, we use μ and λ instead of the possion ratio. Many of the same considerations as for the free fluid flow still applies. The most noteably difference is that $\mathbf{u}$ now represents the displacement in the solid rather than the velocity. The reason for such a notation will be more clear as different numerical solution strategies are presented in chapter xxx.

3.2.1 Biot's Equations

The stress tensor for the Biot problem is

$$\sigma = -\alpha p I + 2\mu\epsilon(\mathbf{u}) + (\mu + \lambda)\mathrm{tr}(\epsilon(\mathbf{u}))I$$

Here μ and λ are Lame's parameters for the solid. The parameter $\alpha = \frac{K}{H}$ is known as the Biot-Willis coefficient. K is known as the drained bulk modulus, and $\frac{1}{K}$ denotes compressibility. H is a poroelastic parameter describing how much the bulk volume changes due to a change in pore pressure while holding the applied stress constant. Again, conservation of momentum and mass, respectively, yields

$$-\mu\nabla^2\mathbf{u} - \lambda\nabla\nabla \cdot \mathbf{u} + \nabla p = 0 \tag{3.11}$$

$$(\nabla \cdot \mathbf{u})_t - \nabla \cdot (\frac{\kappa}{\mu_f}\nabla p) = 0 \tag{3.12}$$

Where μ_f is the dynamic viscosity of the fluid. Here, $-\frac{\kappa}{\mu_f}\nabla p$ represents the fluid velocity in the porous medium relative to the solid movement. In other words, the total fluid movement in the poroelastic medium is $\mathbf{u}_t - \frac{\kappa}{\mu_f}\nabla p$. κ is known as the permeability, with units m^2.

Chapter 4

Numerical Methods

4.1 The finite element Method

Consider the Poisson-equation

$$\Delta u = f \quad \text{in } \Omega \tag{4.1}$$

$$u = u_0 \quad \text{on } \partial\Omega_D \tag{4.2}$$

$$\frac{\partial u}{\partial n} = g \quad \text{on } \partial\Omega_N \tag{4.3}$$

where $\Omega \in \mathbb{R}^d$ is a domain, $u = u(x)$ is an unknown field and f is a source function. The boundary, $\partial\Omega$ is divided into two parts. $\partial\Omega_D$ for the Dirichlet boundary condition, and $\partial\Omega_N$ for the Neumann condition.

4.1.1 Weak formulation

(4.1) is known as the strong form of the equation. To reformulate the problem and state a weak formulation we multiply (4.1) with a test function $v \in V$, where V is a function space, and integrate over the domain. Weak formulations are important in the sense that differential equations can be transformed into a system of linear equations. In the rest of this text the following notation is used for the inner product of two functions

$$(u, v)_\Omega = \int_\Omega u\, v \mathrm{d}x \tag{4.4}$$

By multiplying 4.1 with a test function, v and integrating over the domain, the weak form is obtained

$$(\Delta u - f, v)_\Omega = 0 \quad \forall v \in V \tag{4.5}$$

or, after integrating by parts

$$-(\nabla u, v)_\Omega + (g, v)_{\partial\Omega_N} = (f, v)_\Omega \quad \forall v \in V \tag{4.6}$$

The formulation (4.5) is known as the projection of a function $u - f$ onto the function space V. In other words, the error is orthogonal to all functions in V.

4.1.2 Finite elements

The next step is to approximate u with a sum of basisfunctions in a finite-dimensional function space, $V = \text{span}\{\psi_0, \psi_1, ..., \psi_N\}$. Here, ψ_i represents the basis functions and we search for a solution $u_h \in V$ such that u_h can be written as a linear combination of the basis functions. The first step in the finite element method consists of dividing the domain into smaller parts

$$\Omega = \Omega_0 \cup \Omega_1 \cup ... \cup \Omega_{N_e}$$

where N_e is the number of elements. Each element have a number of nodes within them depending on what type of basis functions to be used. Let's first consider the continuous Galerkin basis functions, ϕ_i, in a one-dimensional domain. There is exactly one basisfunction for each node located at x_i. These basis functions have the property that

$$\phi_i(x_j) = \begin{cases} 1 & \text{for } i = j \\ 0 & \text{for } i \neq j \end{cases}$$

That is, the basis functions ϕ_i are zero on all nodes except at node i. Each basis function is constructed by taking the Lagrange-polynomial which is 1 at the given node and 0 on all other nodes within the same element. Note that the basis functions for a node on the boundary of an element will have two Lagrange-polynomials "pieced together" depending on at which element the basis function is considered. For the rest of the domain the basis functions are defined to be 0.

Now, let's return to the original problem (4.1-4.3). We start by approximating u as a linear combination of all the basis functions.

$$u_h = \sum_{i=0}^{N} c_i \phi_i \tag{4.7}$$

The definitions of u_h and V now give rise to a linear system. Using the summation convention $x_i\, y_i = \sum_0^N x_i y_i$ the system (4.6)

$$-c_i(\nabla \phi_i, \phi_j)_\Omega = (f, \phi_j) - (g, \phi_j)_{\partial\Omega_N} \tag{4.8}$$

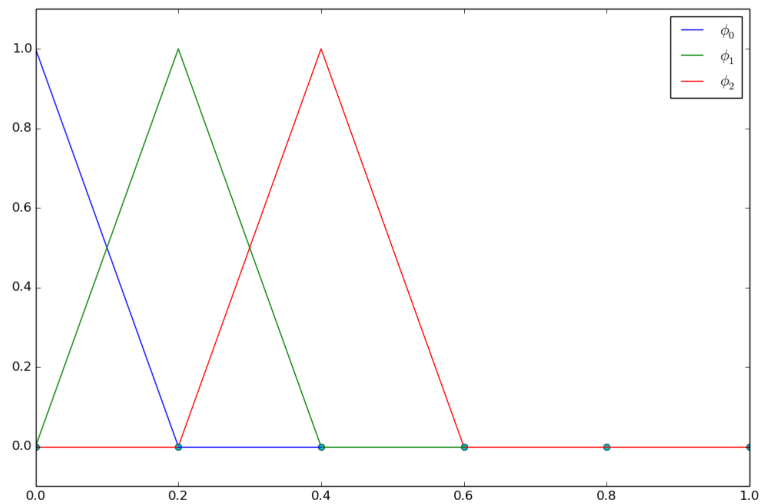

Figure 4.1: The three first linear basis functions on the unit interval divided uniformly into 5 elements

The right hand side of (4.8) is known as the bilinear form while the left hand side is the linear form, assuming the normal derivative is known on the boundary. In the case of Dirichlet boundary conditions all test functions ϕ_j will take the value 0 on $\partial\Omega_D$ and the linear system will be adjusted to take these boundary conditions into account.
The system can be written in matrix form, and in the end the problem consists of solving the linear system

$$A_{i,j}c_i = b_j \tag{4.9}$$

Chapter 5

Material parameters

CSF is modeled as water at 37^oC, i.e

$$\mu_f = 0.653 \cdot 10^{-3}\ Ns/m^2$$

$$\nu_f = 0.658 \cdot 10^{-6}\ m^2/s$$

$$\rho_f = 1000\ kg/m^3$$

For the spinal cord, studies have shown a huge variety in material parameters. One of the most measured properties in the mammalian central nervous system is probably the Young's modulus, E. [Smith, Humphrey 2006]. In addition to this, values for the Poisson ratio, ν_P needs to be found. Smith and Huphrey used the following values for these parameters.

$$E = 5 * 10^4 \text{dyn}/cm^2 = 5000 \text{ Pa}$$

$$\nu_P = 0.479$$

From this, Lame's parameters for the spinal cord were determined as

$$\mu_s = \frac{E}{2(1+\nu_P)} = 1.7 \cdot 10^3 \text{ Pa}$$

$$\lambda_s = \frac{\nu_P E}{(1+\nu_P)(1-2\nu_P)} = 3.9 \cdot 10^4 \text{ Pa}$$

The permeability, κ is used as a measurement for the how fluid flows in a porous medium. A large permeability indicates a pervious medium. We use the value from [Karen, Ida]

$$\kappa = 1.4 \cdot 10^{-15} m^2$$

Bibliography

[1] O. Reynolds, *Papers on Mechanical and Physical Subjects, The Sub-Mechanics of the Universe*, vol. 3. Cambridge University Press, Cambridge, 1903.

[2] F. M. White, *Viscous Fluid Flow.* McGraw-Hill, 3 ed., 2006.

[3] H. F. Wang, *Theory of Linear Poroelasticity with Applications to Geomechanics and Hydrogeology.* Princeton University Press, 2000.