1 Introduction

The Cerebrospinal Fluid (CSF) surrounds the brain and acts as a protection to the brain inside the skull. As a result of the cardiac cycle the CSF will flow up and down the subarachnoid space (SAS) surrounding the spinal cord. The Chiari malformation is a displacement of the cerebellar tonsils that partially blocks CSF flow entering the SAS around the spinal cord. This causes abnormal CSF flows which sometimes results in in a syringomyelia inside the spinal cord filled with fluid. Treatment may include surgery to remove parts of the bones of the skull to relieve pressure. Studies have shown that the syrinx gradually vanishes after surgery. The mechanisms behind this are not yet fully understood. Many researchers have suggested Computational Fluid Dynamics (CFD) to give useful insight, as experiments are very difficult and expensive.

2 Problem definition

The model consists of a spinal cord surrounded by space filled with CSF. The cord has a diameter of 10 mm and the fluid space has a diameter of 18 mm. The model also has a inner central spinal canal, 2 mm in diameter. The central spinal canal has height 4 cm and is placed in the center of the mesh which has heigth 6 cm.

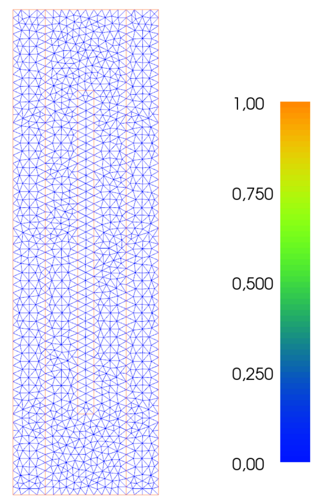

Figure 1: Coarse version of the mesh

2.1 Governing equations

Fluid Space

The fluid in the SAS as well as the central spinal canal is governed by the Navier-Stokes equations

$$\frac{\partial u_f}{\partial t} + u \cdot \nabla u = -\frac{1}{\rho_f}\nabla p_f + \frac{1}{\rho}\nabla \cdot \tau + F \qquad \text{in } \Omega_f \tag{1}$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho u_f) = 0 \qquad \text{in } \Omega_f \tag{2}$$

where u_f is the fluid velocity, ρ_f is the fluid denisty, τ is the stress tensor and F are all external volume forces. In addition to the equations (1)-(2) comes

boundary conditions at the domain boundary. When solving the euqations numerically, manipulation of the coupled equations into simpler equations may lead to problems involving the boundary conditions. Therefore a proper choice of boundary conditions is crucial for stability and existence.
In this case we will consider incompressible flows, where ρ is constant and $\nabla \cdot u_f = 0$. The stress tensor is given by

$$\tau_f = -pI + 2\mu_f \epsilon(u_f)$$

where $\epsilon(u_f) = \frac{1}{2}(\nabla u_f + (\nabla u_f)^T)$ When taking the divergence of the stress tensor the Navier-Stokes equations for incompressible flows take the form

$$\frac{\partial u_f}{\partial t} + u \cdot \nabla u_f = -\frac{1}{\rho}\nabla p_f + \nu \nabla^2 u_f + F \qquad \text{in } \Omega_f \tag{3}$$

$$\nabla \cdot u_f = 0 \qquad \text{in } \Omega_f \tag{4}$$

Spinal Cord

The spinal cord is modeled as a poroelastic medium, which allows for some fluid to flow from the SAS to the central spinal canal. In this case the Biot-equations have been used to include both elasticity and flow in the spinal cord. The stress tensor for the Biot problem is

$$\tau_s = -pI + 2\mu_s \epsilon(u_s) + (\mu + \lambda)\mathrm{tr}(\epsilon(u_s))I$$

Here μ_s and λ_s are Lame's parameters

$$-\mu \nabla^2 u_s - (\lambda + \mu)\nabla \nabla \cdot u_s + \nabla p_s = 0 \qquad \text{in } \Omega_s \tag{5}$$

$$(\nabla \cdot u_s)_t - \nabla \cdot (\frac{\kappa}{\mu_f}\nabla p_s) = 0 \qquad \text{in } \Omega_s \tag{6}$$

Here, $\frac{\kappa}{\mu_f}\nabla p_s$ represents the fluid velocity in the porous medium. p_s is the pressure in the fluid occupying the pores. κ is known as the permeability, with units m^2.

2.2 Weak form

Equations (3) and (5) will be multiplied by a vector test function, v, and equations (4) and (6) with a scalar test function, q. The equations are then integrated over the entire domain. By using two different test functions the coupled equations can be written as one equation combining the two.

2.3 Stationary problem, Stokes

In stokes flow, a steady state pattern have developed for the flow, and convection is neglected. We start with coupling the Stokes flow with a stationary Biot problem. In the CSF the governing equations are

$$\begin{aligned} -\mu_f \nabla^2 u_f + \nabla p_f &= 0 \\ \nabla \cdot u_f &= 0 \end{aligned} \tag{7}$$

And in the cord

$$\begin{aligned} -\mu_s \nabla^2 u_s - (\mu_s + \lambda_s)\nabla\nabla \cdot u_s + \nabla p_s &= 0 \\ -\nabla \cdot \frac{\kappa}{\mu_f} \nabla p_s &= 0 \end{aligned} \tag{8}$$

Since this is a stationary problem, the first lines of the two equations can be written $-\nabla \cdot \sigma_f = 0$ in Ω_f and $-\nabla \cdot \sigma_s = 0$ in Ω_s

Boundary conditions

We denote the outer wall, $\partial\Omega_{f,}$, interface Γ, the top of the cord, $\partial\Omega_{s_t}$, the bottom of the cord $\partial\Omega_{s_b}$, the top of the fluid space $\partial\Omega_{f_t}$ and the bottom of the fluid space $\partial\Omega_{f_b}$. The boundary conditions are as follows:
No-slip on the outer walls

$$u = 0 \text{ on } \partial\Omega_{f_o} \tag{9}$$

Conservation of mumentum at the interface:

$$\sigma_f \cdot n = \sigma_s \cdot n \text{ on } \Gamma \tag{10}$$

Normal velocity equal at the interface:

$$u_f \cdot n = -\frac{\kappa}{\mu_f} \nabla p_s \cdot n \text{ on } \Gamma \tag{11}$$

In the stationary case, the filtration velocity, $-k\nabla p$, corresponds to the total fluid velocity in Ω_s
Prescribed stress on the top and bottom

$$\begin{aligned} -\sigma_s \cdot n &= p0 \text{ on } \partial\Omega_{f_t} \\ -\sigma_f \cdot n &= p0 \text{ on } \partial\Omega_{s_t} \\ -\sigma_s \cdot n &= -p0 \text{ on } \partial\Omega_{f_b} \\ -\sigma_f \cdot n &= -p0 \text{ on } \partial\Omega_{s_b} \end{aligned} \tag{12}$$

2.4 Weak form

Equations (7) and (8) will be multiplied with testFunctions and integrated over their respective domains. The vector equations are multiplied with a vector testFunction, v, and the scalar equations with a scalar testFunction, q. By using two different test functions the coupled equations can be written as one

equation combining the two.
now, we denote the inner product

$$(u, v)_{\Omega_f} = \int_{\Omega_f} u\, v \mathrm{d}x$$

And integrate the simplified versions of 7-8. Both stress terms and the continuity equation in the Biot problem are integrated by parts.

$$\begin{aligned}(\sigma_f, \nabla v)_{\Omega_f} - (v, \sigma_f \cdot n)_{\partial\Omega_f} &= 0 \\ (\nabla \cdot u_f, q)_{\Omega_f} &= 0\end{aligned} \tag{13}$$

$$\begin{aligned}(\sigma_s, \nabla v)_{\Omega_s} - (v, \sigma_s \cdot n)_{\partial\Omega_s} &= 0 \\ (\frac{\kappa}{\mu_f}\nabla p_s, \nabla q)_{\Omega_s} - (q, \frac{\kappa}{\mu_f}\nabla p \cdot n)_{\partial\Omega_s} &= 0\end{aligned} \tag{14}$$

First of all, using condition (10) we can eliminate the stress terms on the interface, when combining the two equations. Condition (12) can be used to handle the stress term on the top and bottom of the domain. Since we have a no-slip condition on the outer wall, the testFunction v is set to be zero on this boundary so the boundary stress term has now been eliminated from the equations by employing the conditions (10) and (12).

If we use continuous elements, $u_f = u_s$ on Γ by the method itself. For now we settle with this approximation, which might be realistic for low x-velocities and small discplacements. Therefore we can use the condition (11) by rewriting the boundary term in eqn (14). The final form F is:

$$(\sigma_f, \nabla v)_{\Omega_f} + (\sigma_s, \nabla v)_{\Omega_s} + (\nabla \cdot u_f, q)_{\Omega_f} + (\frac{\kappa}{\mu_f}\nabla p_s, \nabla q)_{\Omega_s} + (q, u_s \cdot n)_{\partial\Omega_s} \tag{15}$$

And we seek to solve F = 0. In this case, we assume $u_f = u_s$ on the interface, then (11) is satisfied. At the top and bottom of the cord we also assume $u_s = -\frac{\kappa}{\mu_f}\nabla p_s(???)$

3 Material parameters

CSF is modeled as water at $37^o C$, i.e

$$\mu_f = 0.653 \cdot 10^{-3}\ Ns/m^2$$

$$\nu_f = 0.658 \cdot 10^{-6}\ m^2/s$$

$$\rho_f = 1000\ kg/m^3$$

For the spinal cord, studies have shown a huge variety in material parameters. One of the most measured properties in the mammalian central nervous system is probably the Young's modulus, E. [Smith, Humphrey 2006]. In addition to this, values for the Poisson ratio, ν_P needs to be found. Smith and Huphrey used the following values for these parameters.

$$E = 5 * 10^4 \text{dyn}/cm^2 = 5000 \text{ Pa}$$

$$\nu_P = 0.479$$

From this, Lame's parameters for the spinal cord were determined as

$$\mu_s = \frac{E}{2(1+\nu_P)} = 1.7 \cdot 10^3 \text{ Pa}$$

$$\lambda_s = \frac{\nu_P E}{(1+\nu_P)(1-2\nu_P)} = 3.9 \cdot 10^4 \text{ Pa}$$

The permeability, κ is used as a measurement for the how fluid flows in a porous medium. A large permeability indicates a pervious medium. We use the value from [Karen, Ida]

$$\kappa = 1.4 \cdot 10^{-15} m^2$$