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Chapter 1

Introduction

The Cerebrospinal Fluid (CSF) surrounds the brain and acts as a protection to the brain inside the skull. As a result of the cardiac cycle the CSF will flow up and down the subarachnoid space (SAS) surrounding the spinal cord. The Chiari malformation is a displacement of the cerebellar tonsils that partially blocks CSF flow entering the SAS around the spinal cord. This causes abnormal CSF flows which sometimes results in in a syringomyelia inside the spinal cord filled with fluid. Treatment may include surgery to remove parts of the bones of the skull to relieve pressure. Studies have shown that the syrinx gradually vanishes after surgery. The mechanisms behind this are not yet fully understood. Many researchers have suggested Computational Fluid Dynamics (CFD) to give useful insight, as experiments are very difficult and expensive.

Chapter 2

Medical Background

In this chapter, relevant background information from medicine is presented.

2.1 Anatomy of the central nervous system

The central nervous system (CNS) constits of two parts: the brain and the spinal cord. The CNS receives and process information from all parts of the body. Consequently, studies on the CNS are crucial for our understanding of the human anatomy.

2.2 The Spinal Cord

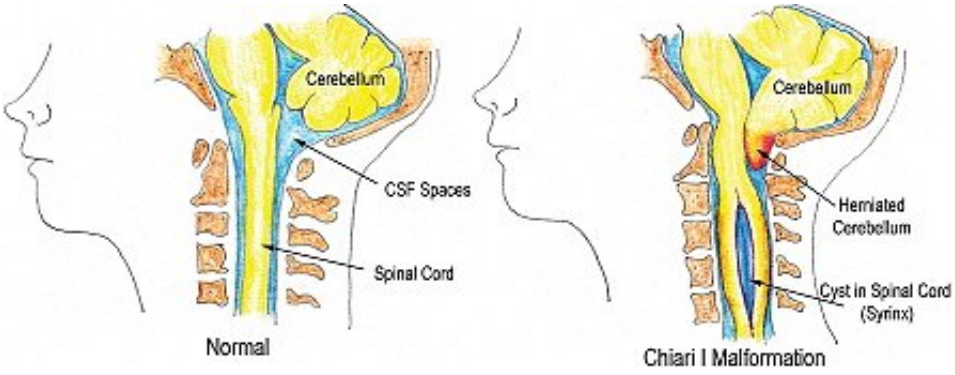

Figure 2.1: hei

Chapter 3

Mathematical background

The flow of CSF around the spinal cord requires equations for fluid flow to be coupled with equations for poroelasticity. The underlying concepts of these kinds of problems were originally developed somewhat independently within petroleum engineering, geomechanics and hydrogeology. The equations will first be presented separately. Later in the chapter, coupling conditions will be discussed.

3.1 Fluid flow

The most fundamental equations in fluid flow are conservation laws. These equations are based on classical mechanics and states conservation of mass, momentum and energy.

3.1.1 Reynolds Transport Theorem

The famous engineer and scientist Osbourne Reynold stated the general conservation law the following way [1]:

Any change whatsoever in the quantity of any entity within a closed surface can only be effected in one or other of two distinct ways:

1. it may be effected by the production or destruction of the entity within the surface, or
2. by the passage of the entity across the surface.

Now, consider a control volume, V_0 and some fluid property $Q(\mathbf{x}, t)$. The rate of change of Q within the control volume can be written

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{V_0} Q(\mathbf{x}, t)\,\mathrm{d}V$$

The net change of Q must be equal the rate of change in Q within the control volume plus the net rate of mass flow out of the volume. In other words

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{V_0} Q(\mathbf{x}, t)\,\mathrm{d}V = \int_{V_0} \frac{\partial Q(\mathbf{x}, t)}{\partial t}\mathrm{d}V + \int_{S_0} Q(\mathbf{x}, t)\mathbf{u}\cdot\mathbf{n}\mathrm{d}S$$

Here, $\mathbf{u}$ denotes the fluid velocity. This equation is known as the Reynold's transport theorem. The right hand side could be rewritten by using Gauss' theorem on the last term.

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{V_0} Q(\mathbf{x},t)\,\mathrm{d}V = \int_{V_0}\big[\frac{\partial Q(\mathbf{x},t)}{\partial t} + \nabla\cdot(Q(\mathbf{x},t)\mathbf{u})\big]\mathrm{d}V \tag{3.1}$$

where

$$\nabla = \mathbf{i}_j\frac{\partial}{\partial x_j}$$

3.1.2 Conservation of mass and momentum

Choose $Q(\mathbf{x},t) = \rho$. Conservation of mass means that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{V_0}\rho\,\mathrm{d}V = 0$$

And by using the transport theorem (3.1)

$$\int_{V_0}\big[\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{u})\big]\mathrm{d}V = 0$$

This should hold for any volume V_0, hence the integrand has to be zero:

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{u}) = 0 \tag{3.2}$$

(3.2) is known as the continuity equation and states conservation of mass.

To derive a simliar property for the momentum, Newtons second law of motion is used. The net change of momentum must be equal to the applied forces to the system. The forces can be divided into volume forces, acting on the entire control volume, and forces acting only on the control surface. The forces acting on the surface can be written $\sigma\cdot n$, where $\sigma = \sigma(\mathbf{u},p)$ is the tensor denoting the total stress. By inserting $Q(\mathbf{x},t) = \rho\mathbf{u}$, and using Gauss' theorem again we end up with

$$\frac{\partial\rho\mathbf{u}}{\partial t} + \nabla\cdot(\rho\mathbf{u}\mathbf{u}) = \nabla\cdot\sigma + \mathbf{F}_v \tag{3.3}$$

The stress tensor, σ, depends on fluid properties and will be defined in the next subsection. Equation (3.3) is known as the momentum equation as it states conservation of momentum.

3.1.3 Incompressible Newtonian fluids

In this text we will only consider incompressible fluid flow for a Newtonian fluid. The assumption of a Newtonian fluid requires the viscous stresses to be linear functions of the components of the strain-rate tensor, denoted by $\epsilon_{ij} = \frac{\partial u_i}{\partial x_j}$. These assumptions were first made by Stokes in 1845. These assumptions have later proven to be quite accurate for all gases and most common fluids. Stokes' three postulates regarding the deformation laws are: [2]

1. The fluid is continuous, and its stress tensor, σ_{ij} is at most a linear function of the strain rates, ϵ_{ij}

2. The fluid is isotropic, i.e., its properties are independent of direction, and therefore the deformation law is independent of the coordinate axes in which it is expressed.

3. When the strain rates are zero, the deformation law must reduce to the hydrostatic pressure condition, $\sigma_{ij} = -p\delta_{ij}$, where δ_{ij} is the Kroenecker delta function.

From the first and third conition the following assumption can be made

$$\sigma_{ij} = -p\delta_{ij} + M_{ijkl}\epsilon_{kl} \tag{3.4}$$

It can be shown that symmetry of σ and ϵ also requires symmetry of M. This assumption reduces the number of coefficients in (3.4) from 36 to 21. If Stokes' second condition is also taken into account and the fluid properties are identical in each direction the number of coefficients are further reduced to 2. These simplifications allow us to denote the stress tensor the following way:

$$\sigma_{ij} = -p\delta_{ij} + 2\mu\epsilon_{ij} + \lambda\nabla \cdot \mathbf{u} \tag{3.5}$$

where $\epsilon = \frac{1}{2}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i})$, p is the fluid pressure and μ and λ are known as Lame's constants. In the present study we only consider incompressible flow where ρ is constant. From (3.2), this implies $\nabla \cdot \mathbf{u} = 0$ and the last term in (3.5) vanishes. Furthermore,

$$\nabla \cdot \epsilon = \frac{\partial}{\partial x_j}(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j})\mathbf{i}_i = (\frac{\partial}{\partial x_i}\frac{\partial u_j}{\partial x_j} + \frac{\partial u_i}{\partial x_j \partial x_j})\mathbf{i}_i = \frac{\partial u_i}{\partial x_j \partial x_j}\mathbf{i}_i$$

Which simplifies the representation of $\nabla \cdot \sigma$ in (3.3) for an incompressible fluid

3.1.4 Navier-Stokes equations for incompressible flow

Stating both conservation of mass and momentum of a fluid together with suitable boundary conditions gives us all the information we need to be able to define the flow field and the corresponding pressure. This requires a solution to the system (3.2)-(3.3), equations which are commonly referred to as the Navier-Stokes equations. With the simplifications described in the previous section the system of equations can be written

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\mathbf{u} + \mathbf{F}_v$$

$$\nabla \cdot \mathbf{u} = 0$$

These equations are coupled and non-linear and can generally not be solved analytically. Hence, numerical solutions are a necessity to obtain useful solutions to real-life problems. Such metods will be discussed in chapter xxx.

3.2 Linear Elasticity

From the Eulerian point of view, an elastic medium is described by the balance equation for the momentum

$$\rho\frac{\partial \mathbf{u}}{\partial t} + \rho(\nabla(\mathbf{u})) = \nabla \cdot \sigma^s + \rho\mathbf{f}$$

3.3 Linear Poroelasticity

In this section, the equations describing fluid flowing trough a elastic medium is presented. For a more detailed discussion, derivation and history within the field we refer to [3] on Linear Poroelasticity. To keep the mathematics as similar to the fluid case as possible, we use μ and λ instead of the possion ratio. The most noteably difference between the solid case and the fluid case is that $\mathbf{u}$ now represents the displacement in the solid rather than the velocity. The reason for such a notation will be more clear as different numerical solution strategies are presented in chapter xxx.

3.3.1 Biot's Equations

The stress tensor for the Biot problem is

$$\sigma = -\alpha p I + 2\mu\epsilon(\mathbf{u}) + (\mu + \lambda)\mathrm{tr}(\epsilon(\mathbf{u}))I$$

Here μ and λ are Lame's parameters for the solid. The parameter $\alpha = \frac{K}{H}$ is known as the Biot-Willis coefficient. K is known as the drained bulk modulus, and $\frac{1}{K}$ denotes compressibility. H is a poroelastic parameter describing how much the bulk volume changes due to a change in pore pressure while holding the applied stress constant. Again, conservation of momentum and mass, respectively, yields

$$-\mu\nabla^2\mathbf{u} - \lambda\nabla\nabla\cdot\mathbf{u} + \nabla p = \mathbf{F}_v \tag{3.6}$$

$$(\nabla\cdot\mathbf{u})_t - \nabla\cdot(\mu_f^{-1}\mathbf{K}\nabla p) = 0 \tag{3.7}$$

Where μ_f is the dynamic viscosity of the fluid and the t-subscript denotes time derivative. As described in [4], $-\mu^{-1}\mathbf{K}\nabla p$ represents the fluid velocity in the porous medium relative to the solid movement. In other words, the total fluid movement in the poroelastic medium is $\mathbf{u}_t - \mu_f^{-1}\mathbf{K}\nabla p$. $\mathbf{K}$ is known as the permeability matrix. In an isotropic medium we can assume that $\mathbf{K}$ is a scalar constant, K.
Equations (3.6)-(3.7) are nothing more than superpositioning of the tho phases. The material derivatives in (3.6) has been dropped under the assumption that these terms are small. This assumption is known as a quasistatic approximation.

3.4 Coupling Fluid flow with Poroelasticy

Coupling viscous fluid flow with porous flow in an elastic medium is [5]

Chapter 4

Numerical Methods

4.1 The Finite Element Method

The theory presented in this section is inspired by the great work of Langtangen in "Finite Element Method - INF5620 lecture notes" [**?**]
Consider the Poisson-equation

$$\nabla^2 \mathbf{u} = \mathbf{f} \quad \text{in } \Omega \tag{4.1}$$

$$\mathbf{u} = \mathbf{u_0} \quad \text{on } \partial\Omega_D$$

$$\frac{\partial \mathbf{u}}{\partial n} = \mathbf{g} \quad \text{on } \partial\Omega_N \tag{4.2}$$

where $\Omega \in \mathbb{R}^d$ is a domain, $\mathbf{u} = \mathbf{u}(\mathbf{x})$ is an unknown field and $\mathbf{f}$ is a source function. The boundary, $\partial\Omega$ is divided into two parts. $\partial\Omega_D$ for the Dirichlet boundary condition, and $\partial\Omega_N$ for the Neumann condition.

4.1.1 Weak formulation

(4.1) is known as the strong form of the equation. To reformulate the problem and state a weak formulation we multiply (4.1) with a test function $\mathbf{v} \in V$, where V is a function space, and integrate over the domain. Weak formulations are important in the sense that differential equations can be transformed into a system of linear equations. In the rest of this text the following notation is used for the inner product of two functions

$$(\mathbf{u}, \mathbf{v})_\Omega = \int_\Omega \mathbf{u} : \mathbf{v} \mathrm{d}V$$

By multiplying (4.1) with a test function, $\mathbf{v}$ and integrating over the domain, the weak form is obtained

$$(\nabla^2 \mathbf{u} - \mathbf{f}, \mathbf{v})_\Omega = 0 \quad \forall \mathbf{v} \in V \tag{4.3}$$

or, after integrating by parts

$$-(\nabla \mathbf{u}, \mathbf{v})_\Omega + (\mathbf{g}, \mathbf{v})_{\partial\Omega_N} = (\mathbf{f}, \mathbf{v})_\Omega \quad \forall \mathbf{v} \in V \tag{4.4}$$

The formulation (4.3) is known as the projection of a function $\mathbf{u} - \mathbf{f}$ onto the function space V. In other words, the error is orthogonal to all functions in V.

4.1.2 Finite elements

The next step is to approximate $\mathbf{u}$ with a sum of basisfunctions in a finite-dimensional function space, $V = \text{span}\{\psi_0, \psi_1, ..., \psi_N\}$. Here, ψ_i represents the basis functions and we search for a solution $\mathbf{u}_h \in V$ such that $\mathbf{u}_h$ can be written as a linear combination of the basis functions. The first step in the finite element method consists of dividing the domain into smaller parts

$$\Omega = \Omega_0 \cup \Omega_1 \cup ... \cup \Omega_{N_e}$$

where N_e is the number of elements. Each element have a number of nodes within them depending on what type of basis functions to be used. Let's first consider the continuous Galerkin basis functions, ϕ_i, in a one-dimensional domain. There is exactly one basisfunction for each node located at x_i. These basis functions have the property that

$$\phi_i(x_j) = \begin{cases} 1 & \text{for } i = j \\ 0 & \text{for } i \neq j \end{cases}$$

That is, the basis functions ϕ_i are zero on all nodes except at node i. Each basis function is constructed by taking the Lagrange-polynomial which is 1 at the given node and 0 on all other nodes within the same element. Note that the basis functions for a node on the boundary of an element will have two Lagrange-polynomials "pieced together" depending on at which element the basis function is considered. For the rest of the domain the basis functions are defined to be 0.

Now, let's return to the original problem (4.1)-(4.2). We start by approximating u as a linear combination of all the basis functions.

$$u_h = \sum_{i=0}^{N} c_i \phi_i$$

The definitions of u_h and V now give rise to a linear system. Using the summation convention $x_i\, y_i = \sum_0^N x_i y_i$, (4.4) is now written

$$-c_i(\nabla \phi_i, \phi_j)_\Omega = (f, \phi_j) - (g, \phi_j)_{\partial\Omega_N} \tag{4.5}$$

The right hand side of (4.5) is known as the bilinear form while the left hand side is the linear form, assuming the normal derivative is known on the

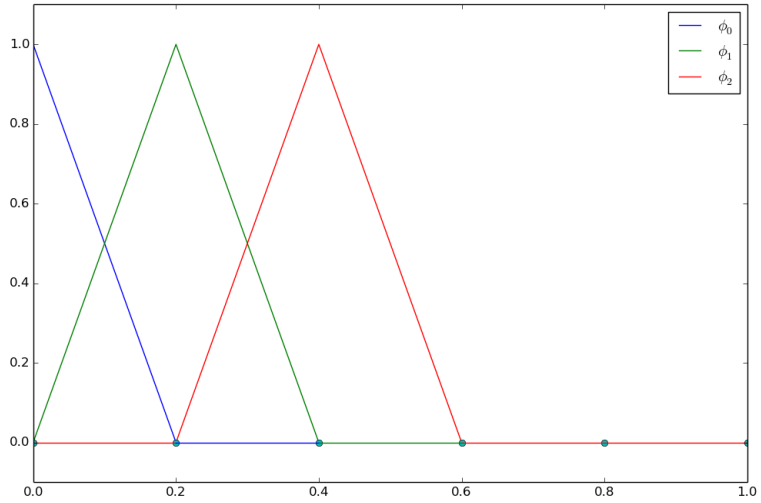

Figure 4.1: The three first linear basis functions on the unit interval divided uniformly into 5 elements

boundary. In the case of Dirichlet boundary conditions all test functions ϕ_j will take the value 0 on $\partial\Omega_D$ and the linear system will be adjusted to take these boundary conditions into account.
The system can be written in matrix form, and in the end the problem consists of solving the linear system

$$A_{i,j}c_i = b_j$$

4.2 Implementation in FEniCS

When the variational form has been carried out, implementation in FEniCS is relatively simple. We need a computational mesh, appropriate function spaces, a variational form and boundary conditions to have a well-posed problem.

4.3 A Discontinuous Galerkin method

Discontinuous Galerkin (DG) methods is a relatively new tool for CFD simulations. The method itself was developed during the 1970s and has been used increasingly the last few decades. Unlike with the Continuous Galerkin elements, we now allow the solution to be discontiuous, i.e cells do not share nodes

anymore. Instead of solving over the whole domain, we now seek approximate continuous solutions on each cell independently of the others. To this end it is convenient to define the average and jump of a discontinuous variable

$$\{\mathbf{u}\} = \frac{1}{2}(\mathbf{u}^+ + \mathbf{u}^-) \qquad\qquad [\mathbf{u}] = \mathbf{u}^+ - \mathbf{u}^-$$

where $\mathbf{u}^+$ and $\mathbf{u}^-$ is the solution at two neighboring cells at cell E^+ and E^-. The normal vectors are denoted $\mathbf{n}^+$ and $\mathbf{n}^-$. If consistent the choice of $\mathbf{n}$ is arbitrary.

4.3.1 Stokes flow

By integrating Stokes equation by parts, adding symmetry and penalty terms as in [5] we end up with a DG-method ready for use. Since the method involves many terms we derive a weak formulation for each term. All sets of facets are denoted as e, interior facets as Γ and exterior facets as $\partial\Omega$. Starting with the diffusion term $-\mu\nabla^2 u$, we multiply with a test function and integrate by parts:

$$\mu \sum_{E\in e}(\nabla u, \nabla v)_E - \mu \sum_{E\in\Gamma}([v], \{\nabla u\}\cdot n_e)_E - \mu \sum_{E\in\Gamma}([u], \{\nabla v\}\cdot n_e)_E + \frac{\alpha}{h}\sum_{E\in\Gamma}([u],[v])_E$$
$$-\mu \sum_{E\in\partial\Omega}(v, \nabla u\cdot n_e)_E - \mu \sum_{E\in\partial\Omega}(u, \nabla v\cdot n_e)_E + \frac{\beta}{h}\sum_{E\in\partial\Omega}(u,v)_E$$

Similiarly, the contributions from the term ∇p will be

$$-\sum_{E\in e}(p, \nabla\cdot v)_E - \sum_{E\in\Gamma}$$

Chapter 5

Material parameters

CSF is modeled as water at 37^oC, i.e

$$\mu_f = 0.653 \cdot 10^{-3}\ Ns/m^2$$

$$\nu_f = 0.658 \cdot 10^{-6}\ m^2/s$$

$$\rho_f = 1000\ kg/m^3$$

For the spinal cord, studies have shown a huge variety in material parameters. One of the most measured properties in the mammalian central nervous system is probably the Young's modulus, E. [Smith, Humphrey 2006]. In addition to this, values for the Poisson ratio, ν_P needs to be found. Smith and Huphrey used the following values for these parameters.

$$E = 5 * 10^4 \text{dyn}/cm^2 = 5000 \text{ Pa}$$

$$\nu_P = 0.479$$

From this, Lame's parameters for the spinal cord were determined as

$$\mu_s = \frac{E}{2(1+\nu_P)} = 1.7 \cdot 10^3 \text{ Pa}$$

$$\lambda_s = \frac{\nu_P E}{(1+\nu_P)(1-2\nu_P)} = 3.9 \cdot 10^4 \text{ Pa}$$

The permeability, κ is used as a measurement for the how fluid flows in a porous medium. A large permeability indicates a pervious medium. We use the value from [Karen, Ida]

$$\kappa = 1.4 \cdot 10^{-15} m^2$$

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