Thesis Title

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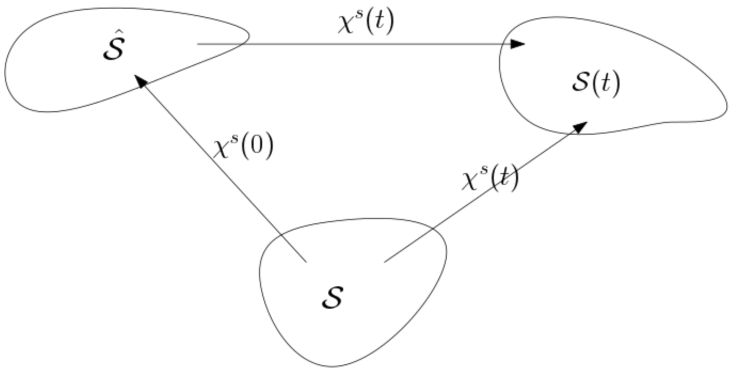
Solid Equations

The solid equations stated here will be in a Lagrangian description. This description fits a solid problem very nicely as the material particles are fixed with grid points. This we will see later is a very nice property when tracking the solid domain. The displacement vector will be the quantity describing the motion of solid.

Reference domain

Mapping and identites

To be able to state the solid equation in a Lagrangian reference configuration, we need to look at some mappings and identities:



We define $\hat{\mathcal{S}}$ as the initial stress free configuration of a given body. $\mathcal{S}$ as the reference and $\mathcal{S}(t)$ as the current configuration. We need to define a smooth mapping that maps from the reference configuration to the current configuration:

$$\chi^s(t) : \hat{\mathcal{S}} \to \mathcal{S}(t)$$

The solid mapping is set as $\chi^s(\mathbf{X}, t) = \mathbf{X} + d^s(\mathbf{X}, t)$ hence giving:

$$d^s(\mathbf{X}, t) = \chi^s(\mathbf{X}, t) - \mathbf{X}$$

$$w(\mathbf{X}, t) = \frac{\partial \chi^s(\mathbf{X}, t)}{\partial t}$$

where $\mathbf{X}$ denote a material point in the reference domain and χ^s denotes the mapping from the reference configuration. $d^s(\mathbf{X}, t)$ denotes the displacement field and w($\mathbf{X}$,t) is the domain velocity.

Deformation gradient

If $d(\mathbf{X}, t)$ is differentiable deformation field in a given body. We define the deformation gradient as:

$$F = \frac{\partial \chi}{\partial \mathbf{X}} = I + \nabla d(\mathbf{X}, t)$$

which denotes relative change of position under deformation in a Lagrangian frame of reference. The similar Eulerian viewpoint is defined as the inverse deformation gradient

$$\hat{\mathbf{F}} = I - \nabla d(\mathbf{X}, t)$$

J is det(F). In continuum mechanics relative change of location of particles is called strain and this is the fundamental quality that causes stress in a material. [godboka]. We say that stress is the internal forces between neighboring particles. E denotes the Green-Lagrangian strain tensor $E = F^T F - I$. This measures the squared length change under deformation.

Solid equation

From the principles of conservation of mass and momentum, we get the solid equation stated in the Lagrangian reference system (Following the notation and theory from Richter, Godboka"):

$$\rho_s J \frac{\partial d^2}{\partial t^2} = \nabla \cdot (F\Sigma) + J\rho_s f \tag{1}$$

where f is the body force and Σ denotes the St. Venant Kirchhoff material law:

$$\Sigma = 2\mu_s E + \lambda_s tr(E) I$$

Multiplying Σ with F we get the 2nd Piola Krichhoff stress tensor. This gives us a non-linear stress tensor.

Some shit about Locking

Fluid Structure Interaction Problem formulation in ALE coordinates

Introduction

Here we will look at the ALE formulation of solving the FSI problem. The ALE approach stands for Arbitrary Lagrangian Eulerian, meaning that we define the fluid problem in an Eulerian framework and the solid problem in an Lagrangian framework. The ALE method can be solved by moving the mesh for each time step, following the structure body movements [Houston paperet]. This approach gives advantages as we can explicitly represent the fluid-structure interface. But problems arise when there are large deformations in solid structure, giving major mesh deformations in the fluid mesh. Another way of approaching the ALE-FSI problem is to used reference or fixed meshes. Instead of updating the mesh for each time step, we instead use a series of mappings to map the solution from a reference mesh and onto our current mesh. First we look at the mapping and identities needed to solve the reference approach to ALE.

Notation

u - Velocity in fluid and structure.
w - Velocity in the domain. It is the velocity of the mesh in the calculations. This will also be the velocity in structure when defined in the lagrangian formulation.
d - Displacement of the solid. The time derivative of the displacement will be the domain velocity.
p - Pressure in the fluid.

Full FSI problem

Find $u \in \mathcal{F}, p \in \mathcal{F}$ and $d \in \mathcal{S}$ such that :

$$\rho_f \frac{\partial u}{\partial t} + (\nabla u)(u - \frac{\partial d}{\partial t}) + \nabla \cdot \sigma_f = 0 \text{ on } \mathcal{F} \tag{2}$$

$$\nabla \cdot u = 0 \text{ on } \mathcal{F} \tag{3}$$

$$\nabla^2 d = 0 \text{ on } \mathcal{F} \tag{4}$$

$$\rho_s \frac{\partial u}{\partial t} + \nabla \cdot \sigma_s = 0 \text{ on } \mathcal{S} \tag{5}$$

$$u - \frac{\partial d}{\partial t} = 0 \text{ on } \mathcal{S} \tag{6}$$

$$\sigma_f n_f = \sigma_s n_s \text{ on } \Gamma \tag{7}$$

Balance laws

We will formulate the equations in the Eulerian, Lagrangian and the ALE description. The Eulerian description suits a fluid problem nicely as the points in the grid are fixed and the fluid particles move through the domain. Whilst the Lagrangian description fits a solid problem as the material particles are fixed with the gridpoints. The fluid velocity vector and the displacement vector are the quantities describing motion of the fluid and solid respectively. Since we here have fluid structure interaction problem we need to formulate the fluid in the fixed mesh description. The fluid velocity will still be the quantity describing motion but it will also have the displacement of the fluid, describing the change in fluid domain. The solid will be described in Lagrangian. We will only look at incompressible fluids where the volume of the fluid domain stays constant.

Solid

We express the solid balance laws in the Lagrangian formulation from the initial configuration

$$\rho_s \frac{\partial^2 d}{\partial t^2} = \nabla \cdot (J\sigma_s F^{-T}) \quad in \quad \hat{\mathbf{S}}$$

Second Piola-Kirchhoff tensor following from the St. Venant-Kirchhoff material with the Green-Lagrange strain tensor: $E = \frac{1}{2}(F^T F - I)$.

$$\begin{aligned} \sigma_s =& \frac{1}{J} F(\lambda_s (trE)I + 2\mu_s E)F^{-T} \\ S_s =& \lambda_s (trE)I + 2\mu_s E \\ \lambda_s =& \frac{2\mu_s nu_s}{1 - 2\nu_s} \end{aligned}$$

Fluid

The fluid equations are denoted from the initial configuration:

$$\rho_f J \big(\frac{\partial u}{\partial t} + ((\nabla u)F^{-1}(u - w)\big) = \nabla \cdot (J\sigma_f F^{-T}) \quad in \hat{\mathbf{F}}$$

$$\nabla \cdot (JuF^{-T}) = 0 \quad in \ \hat{\mathbf{F}}$$

As we see the only difference from the usual way of seeing the N-S equations, is that in the convection term we have $u - w$ which is needed since the not only are the fluid particles moving but the domain, here denoted with w as the domain velocity, is also moving. So $u - w$ will be the actual fluid velocity.

Harmonic extension

To bind together the computation of fluid and structure domain, we need a harmonic extension to the boundary values. The solid deformation d is extended from the interface into the fluid domain and is done to help deal with big deformations in fluid domain. These big deformations can then cause several challenges to the ALE mapping. For this purpose define the following harmonic extension equation in the fluid domain:

$$\nabla^2 d^f = 0 \quad in \ \hat{\mathbf{F}}$$

This This equation is chosen for its good regularity and smoothing properties.
It is also possible to chose an harmonic extension with stiffening, which can give better control of the deformed meshes. This in practice behaves like a transport problem, transporting the deformation into the fluid domain. Another possibility is extension by pseudo-elasticity which defines the extension operator by means of the Navier-Lame equation. And lastly we can chose a biharmonic extension, that is of fourth order character, and thus will have a high computational cost. For now I will stick with the harmonic extension and maybe look at these in the future. [Godboka.]

Boundary conditions

$$u(x, y, t = 0) = u_0$$

In the place where the fluid and structure domains meet, i.e the interface. We set a dynamic condition saying that the normal stresses of the solid and fluid are equal:

$$\sigma_f n_f = \sigma_s n_s \quad on \ \Gamma^0(interface)$$

These will be written in the Lagrangian formulation:

$$J\sigma_f F^{-T} n_f = \sigma_s n_s \quad on \ \Gamma^0(interface)$$

We can introduce a global domain $\Omega \in \mathcal{S} \cup \mathcal{F}$ that is made up of the fluid and the structure and the interface. We define a global velocity function u that is the fluid velocity in the fluid domain and the structure velocity in the structure domain. This can be done due to the interface condition making the velocity field continuous over the entire domain.

0.1 FSI Problem in reference domain

Find $u \in \mathcal{F}, p \in \mathcal{F}$ and $d \in \mathcal{S}$ such that :

$$\rho_f J \frac{\partial u}{\partial t} + (\nabla u) F^{-1}(u - \frac{\partial d}{\partial t}) + \nabla \cdot (J \sigma_f F^{-T}) = 0 \text{ on } \hat{\mathbf{F}} \tag{8}$$

$$\nabla \cdot (J u F^{-T})) = 0 \text{ on } \hat{\mathbf{F}} \tag{9}$$

$$\rho_s \frac{\partial u}{\partial t} + \nabla \cdot F S_s, = 0 \text{ on } \hat{\mathbf{S}} \tag{10}$$

$$\nabla^2 d = 0 \text{ on } \hat{\mathbf{F}} \tag{11}$$

$$u - \frac{\partial d}{\partial t} = 0 \text{ on } \hat{\mathbf{S}} \tag{12}$$

$$J \sigma_f F^{-T} n_f = \sigma_s n_s \text{ on } \Gamma \tag{13}$$

Domain move

Here we will look at the approach involving moving the mesh each time step.
The fluid equation is simply Navier-Stokes with a domain mapping in the transport term giving:

$$\rho_f \big(\frac{\partial u}{\partial t} + (\nabla u)(u - w) \big) = \nabla \cdot \sigma_f \quad in \mathcal{F}$$

$$\nabla \cdot u = 0 \quad in \quad \mathcal{F}$$

After each timestep we update the mesh and compute this equation over the current mesh. The real velocity of the fluid particles is there for the fluid velocity itself minus the velocity of the mesh.

The solid equations will be formulated similar to before but we can write them with the displacement velocity: $\frac{\partial d}{\partial t} = u_f$:

$$\rho_s \frac{\partial u_f}{\partial t} = \nabla \cdot \sigma_s \quad in \quad \hat{\mathbf{S}}$$

Laplace operator:

$$\nabla^2 d_f = 0 \quad in \;\; \hat{\mathbf{F}}$$

Boundary conditions stay the same but without mappings:

The two approaches to the ALE method is equivalent [Godboka]. From a technical point of view, both formulations are equivalent. Wether we use a fixed and reference formulation or a moving mesh and Eulerian formulation.

Finite Element FSI in ALE

Variational formulation

Reference domain

We use 3 testfunctions, ϕ, ψ, γ. As mentioned before we use a global velocity function u for both the solid and fluid.

$$\rho_f J\big(\frac{\partial u}{\partial t} + (\nabla u)F^{-1}(u - \frac{\partial d}{\partial t}), \phi\big)_{\hat{\mathbf{F}}} + (J\sigma_f F^{-T}, \nabla\phi)_{\hat{\mathbf{F}}} = 0 \tag{14}$$

$$\big(\nabla \cdot (JuF^{-T}), \gamma\big)_{\hat{\mathbf{F}}} = 0 \tag{15}$$

$$\big(\rho_s \frac{\partial u}{\partial t}, \phi\big)_{\hat{\mathbf{S}}} + \big(FS_s, \nabla\phi\big)_{\hat{\mathbf{S}}} = 0 \tag{16}$$

$$\big(\nabla d, \nabla\psi\big)_{\hat{\mathbf{F}}} = 0 \tag{17}$$

$$\big(u - \frac{\partial d}{\partial t}, \psi\big)_{\hat{\mathbf{S}}} = 0 \tag{18}$$

Equation (5) has not been addressed and is added since we use a global function for velocity we need to force that the structure velocity is the time derivative of the deformation in the structure domain.

Spaces and Elements

The velocity and pressure copling in the fluid domain must satisfy the inf-sup condition. If not stabilization has to added. We here need to define some spaces that will have these desired properties. We denote $u_h \in V_h$ and $d_h \in W_h$, here the finite element pair og pressure and velocity mush satisfy the inf-sup condition given in ALE coordinates:

$$\inf_{p_h \in L_{h,f}} \sup_{v_h \in V_{h,f}} \frac{(p_h, div(J_f F_f^{-1} u_h))_{\mathcal{F}}}{|||J^{\frac{1}{2}} p_h|||_{\mathcal{F}} |||J_f^{\frac{1}{2}} \nabla u_h F_f^{-T}|||_{\mathcal{F}}} \geq \gamma$$

A good choice of spaces will be P2-P2-P1 for velocity, displacement and fluid pressure respectively.

Locking

The problem og shear locking can happen FEM computations with certain elements. [mek4250 Kent] - Locking occurs if $\lambda >> \nu$ that is, the material is nearly incompressible. The reason is that all the elements discussed in this course are poor at approximating the divergence. Locking refers to the case where the displacement is to small because the divergence term essentially lock the displacement. It is a numerical artifact not a physical feature. [Verbatum]

Domain move

We then look at the variational formulation for moving the mesh, here we employ a global function for u in the fluid and solid. Which is the fluid velocity in the fluid domain and displacement velocity solid domain. And in the last equation we force w to be the solid displacement in the solid domain.

$$\rho_f\big(\frac{\partial u}{\partial t} + (\nabla u)(u - w), \phi\big)_{\mathcal{F}} + (\sigma_f, \nabla\phi)_{\mathcal{F}} = 0$$

$$\big(\nabla \cdot (u), \gamma\big)_{\mathcal{F}} = 0$$

$$\big(\rho_s \frac{\partial u}{\partial t}, \phi\big)_{\mathcal{S}} + \big(\sigma_s(d), \nabla\phi\big)_{\mathcal{S}} = 0$$

$$\big(\nabla d, \nabla\epsilon\big)_{\mathcal{F}} = 0$$

$$\big(w - u, \epsilon\big)_{\mathcal{S}} = 0$$

Introduction

Here we will look at the partitioned approach to solving the FSI problem. This means splitting our scheme into parts where we solve the fluid, structure and extension problem in different steps. This is to greatly reduce the size of the computational cost, and hopefully increase speed. So far the methods for coupling of the fluid and structure, has led to unconditional numerical instabilites and a large added-mass effect. [Explicit coupling thin walled Fernandez]. Here we look at a new approach to explicit coupling, first proposed by Fernadez, which uses a Robin-Neumann explicit treatment of the interface first for thin walled structure but later with an extension to thick walled structures. This combined with a lumped mass approximation of the solid problem ensures added-mass free stability. [Generalized R-N explicit coupling schemes]

Robin-Neumann Interface

The Robin-Neumann treatment of the interface proposed by Fernandez uses a boundary operator $B_h : \Lambda_{\Sigma,h} \rightarrow \Lambda_{\Sigma,h}$ which is used together with the known coupling of stresses:

$$J^n \sigma^f(u^n, p^n)(F^n)^{-T} n^f + \frac{\rho^s}{\tau} B_h u^n = \frac{\rho^s}{\tau} B_h(\dot{d}^{n-1} + \tau \partial_t \dot{d}^*) - \Pi^* n^s$$

- The explicit treatment of the solid ensures uncoupling of the fluid and solid computations. Giving a genuine partitioned system.
- Treating the left hand side solid tensor implicitly ensures added-mass free stability The fluid domain is computed using a generalized Robin condition on the interface, and the solid is computed with the familiar Neumann condition on the interface, equalling the stresses from the fluid and structure.
The general r-order extrapolation x^* is defined:

$$x^* = \begin{cases} 0, & \text{if } r = 0 \\ x^{n-1}, & \text{if } r = 1 \\ 2x^{n-1} - x^{n-2}, & \text{if } r = 2 \end{cases} \tag{19}$$

Boundary interface operator

Using the notation of [Generalized robin-neumann explicit coupling scheme] We denote $(.,.)_{\mathcal{S},h}$ as the lumped mass approximation of the inner product $(.,.)_{\mathcal{S}}$. We will consider a solid and fluid sided discrete lifting operator $\mathcal{L}_h^s : \Lambda_{\Sigma,h} \rightarrow \mathcal{S}$, lifting values from the interface into the solid domain. If $\xi_h, \lambda_h \in \Lambda_{\Sigma,h}$ then $\mathcal{L}_h^s|_\Sigma = \mathcal{L}_h^f|_\Sigma = \xi_h$. We use this to define the boundary operator: $B_h = (\mathcal{L}_h^s)^* \mathcal{L}_h^s$, mapping from interface to interface $B_h : \Lambda_{\Sigma,h} \rightarrow \Lambda_{\Sigma,h}$. Where stars stands for the adjoint operator of $\mathcal{L}_h^s$. We can then write:

$$(B_h \xi_h, \lambda_h)_\Sigma = (\mathcal{L}_h^s \xi_h, \mathcal{L}_h^s \lambda_h)_{\mathcal{S},h}$$

Explicit Robin-Neumann scheme:
Step 1: Fluid domain update

$$
\begin{aligned}
d^{f,n} &= Ext(d^{n-1}) \\
w^n &= \frac{\partial d^{f,n}}{\partial t} \\
with F &= I + \nabla d, J = det(F)
\end{aligned}
$$

Step 2: Fluid step: find u^n, p^n:

$$
\begin{aligned}
\rho^f \big(\frac{\partial u^n}{\partial t} + (u^{n-1} - w^n) \cdot \nabla u^n \big) - \nabla \cdot \sigma(u^n, p^n) &= 0 \in \mathcal{F} \\
\nabla \cdot u &= 0 \in \mathcal{F} \\
\sigma(u^n, p^n) n^f &= f \\
J^n \sigma(u^n, p^n)(F^n)^{-T} n^f + \frac{\rho^s}{\tau} B_h u^n &= \frac{\rho^s}{\tau} B_h(\dot{d}^{n-1} + \tau \partial_t \dot{d}) - \Pi^* n^s
\end{aligned}
$$

Step 3: Solid Step: find d^n

$$
\begin{aligned}
\rho^s \partial_t \dot{d}^n + \alpha \rho^s \dot{d}^n - \nabla \cdot \Pi^n &= 0 \in \mathcal{S} \\
\dot{d} &= \partial_t d^n \\
d^n = 0, \beta \dot{d}^n &= 0 \in \Gamma^d \\
\Pi^n n^s &= 0 \in \Gamma^n \\
\Pi^n n^s &= -J^n \sigma(u^n, p^n)(F^n)^{-T} n^f \in \Sigma
\end{aligned}
$$

The solid stress tensor is given as $\Pi^n = \pi(d^n) + \beta \pi^?(d^{n-1}) \dot{d}^n$