General notes on DPG

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1 Least squares

We begin with a general variational formulation

$$b(u,v) = l(v)$$

DPG begins with the idea that you would like to do least squares on the operator equation

$$Bu = \ell, \quad Bu, \ell \in V'$$

where $\langle Bu, v\rangle_{V' \times V} = b(u,v)$ and $\langle \ell, v\rangle_{V' \times V} = l(v)$. Since $Bu - \ell \in V'$, we minimize the norm of this residual in V' over the finite dimensional space U_h, i.e.

$$\min_{u_h \in U_h} \|Bu_h - \ell\|^2_{V'}.$$

This leads to the normal equations

$$(Bu - \ell, B\delta u)_{V'}, \quad \forall \delta u \in U_h.$$

The Riesz map gives us the equivalent definition

$$\left(R_V^{-1}(Bu - \ell), R_V^{-1}(B\delta u)\right)_V = 0, \quad \forall \delta u \in U_h.$$

Assuming we've specified the Riesz map through a test space inner product

$$\langle R_V v, \delta v\rangle_{V' \times V} = (v, \delta v)_V,$$

this leads to what I call a Dual Petrov-Galerkin method.

2 Algebraic perspective

In the above example, V is infinite dimensional. If we approximate V by V_h such that $\dim(V_h) > \dim(U_h)$, we get matrix representations of our operators

$$\begin{aligned} B_{ij} &= b(u_j, v_i), \quad u_j \in U_h, v_i \in V_h \\ R_V &= (v_i, v_j)_V \quad v_i, v_j \in V_h \\ \ell_i &= l(v_i), \quad v_i \in V_h. \end{aligned}$$

The resulting normal equations

$$\left(R_V^{-1}(Bu - \ell), R_V^{-1}(B\delta u)\right)_V, \quad \forall \delta u \in U_h.$$

can now be written as

$$\left(R_V^{-1}\left(Bu-\ell\right)\right)^T R_V\left(R_V^{-1}B\right)=0,$$

or, after simplifying to $\left(Bu-\ell\right)^T R_V^{-1}B=0$, we get the algebraic normal equations

$$B^T R_V^{-1}Bu = B^T R_V^{-1}\ell.$$

This is just the solution to the algebraic least squares problem

$$\min_u \left\|Bu-\ell\right\|^2_{R_V^{-1}}.$$

Such problems can also be written using the augmented system for the least squares problem

$$\left[\begin{array}{cc} R_V & B \\ B^T & 0 \end{array}\right]\left[\begin{array}{c} e \\ u \end{array}\right]=\left[\begin{array}{c} l \\ 0 \end{array}\right].$$

This can be interpreted as the mixed form of the Dual Petrov-Galerkin method, which is used by Cohen, Welper, and Dahmen in their 2012 paper “Adaptivity and variational stabilization for convection-diffusion equations”.

$$\begin{aligned}(e,v)_V + b(u,v) &= l(v)\\ b(\delta u, e) &= 0\end{aligned}$$

Eliminating e from the above system leads to the above algebraic normal equations.

Note: for general R_V, the algebraic normal equations are completely dense. Cohen, Welper, and Dahmen thus solve the augmented system to get solutions in this setting; however, this is a saddle point problem, and over 2x as large as the trial space, which makes preconditioning and solving more difficult.

3 Deriving the Discontinuous Petrov-Galerkin method

We want to avoid solving either a fully dense system or a saddle point problem, so we introduce Lagrange multipliers $\widehat{u}$ to enforce continuity weakly on e, which we will now approximate using discontinuous functions. These $\widehat{u}$ are defined on element edges only, similarly to hybrid variables or mortars in finite elements. This leads to the new system

$$\begin{aligned}\langle \widehat{u}, [\![v]\!]\rangle_{\Gamma_h} + (e,v)_V + b(u,v) &= l(v)\\ b(\delta u, e) &= 0\\ \langle \hat{\mu}, [\![e]\!]\rangle_{\Gamma_h} &= 0\end{aligned}$$

where Γ_h is the mesh skeleton (union of all element edges). The resulting algebraic system here is

$$\left[\begin{array}{ccc} R_V & B & \hat{B} \\ B^T & 0 & 0 \\ \hat{B}^T & 0 & 0 \end{array}\right]\left[\begin{array}{c} e \\ u \\ \widehat{u} \end{array}\right]=\left[\begin{array}{c} f \\ 0 \\ 0 \end{array}\right].$$

Because e is now discontinuous, R_V is made block-diagonal; eliminating e returns the (fairly) sparse symmetric positive-definite DPG system

$$A=\left[\begin{array}{cc} B^T R_V^{-1}B & B^T R_V^{-1}\hat{B} \\ \hat{B}^T R_V^{-1}B & \hat{B}^T R_V^{-1}\hat{B} \end{array}\right]\left[\begin{array}{c} u \\ \widehat{u} \end{array}\right]=\left[\begin{array}{c} B^T R_V^{-1}f \\ \hat{B}^T R_V^{-1}f \end{array}\right].$$

4 Fast evaluation of the Riesz inverse

Given that the test norm is typically defined

$$\|v\|_V \coloneqq \alpha \|v\|^2_{L^2(\Omega)} + \|v\|^2_{V_0},$$

where $\|v\|_{V_0}$ can be a seminorm. This leads the definition of the Riesz map R_V as

$$R_V = \alpha M + A.$$

Taking eigenvectors and eigenvalues of the generalized eigenvalue problem $AQ = MQ\Lambda$ and normalizing such that $Q^T MQ = I$, we can rewrite R_V as

$$R_V = Q^{-T}(\alpha I + \Lambda)Q^{-1}$$

which is then easy to invert. This technique is especially efficient when tensor-product techniques are applicable, reducing the generalized eigenvalue problem to one for a one-dimensional discretization.

5 OAS of Barker et. al

Only the patch OAS appears to deliver both h and N independent results.

6 Tensor product approaches

Standard FEM/SEM on tensor product elements can take advantage of fast diagonalization using 1D operators. In other words, if M and K are mass and stiffness matrices on a quadrilateral grid, then

$$(M + K)^{-1} = (Q^T \otimes Q^T)(I \otimes I + \Lambda \oplus \Lambda)^{-1}(Q \otimes Q)$$

where Q and Λ are eigenvalues and eigenvectors resulting from the generalized eigenvalue problem $K_1 Q = M_1 Q\Lambda$ (K_1, M_1 are 1D operators), such that $Q^T MQ = I$. This inverts the operator exactly.

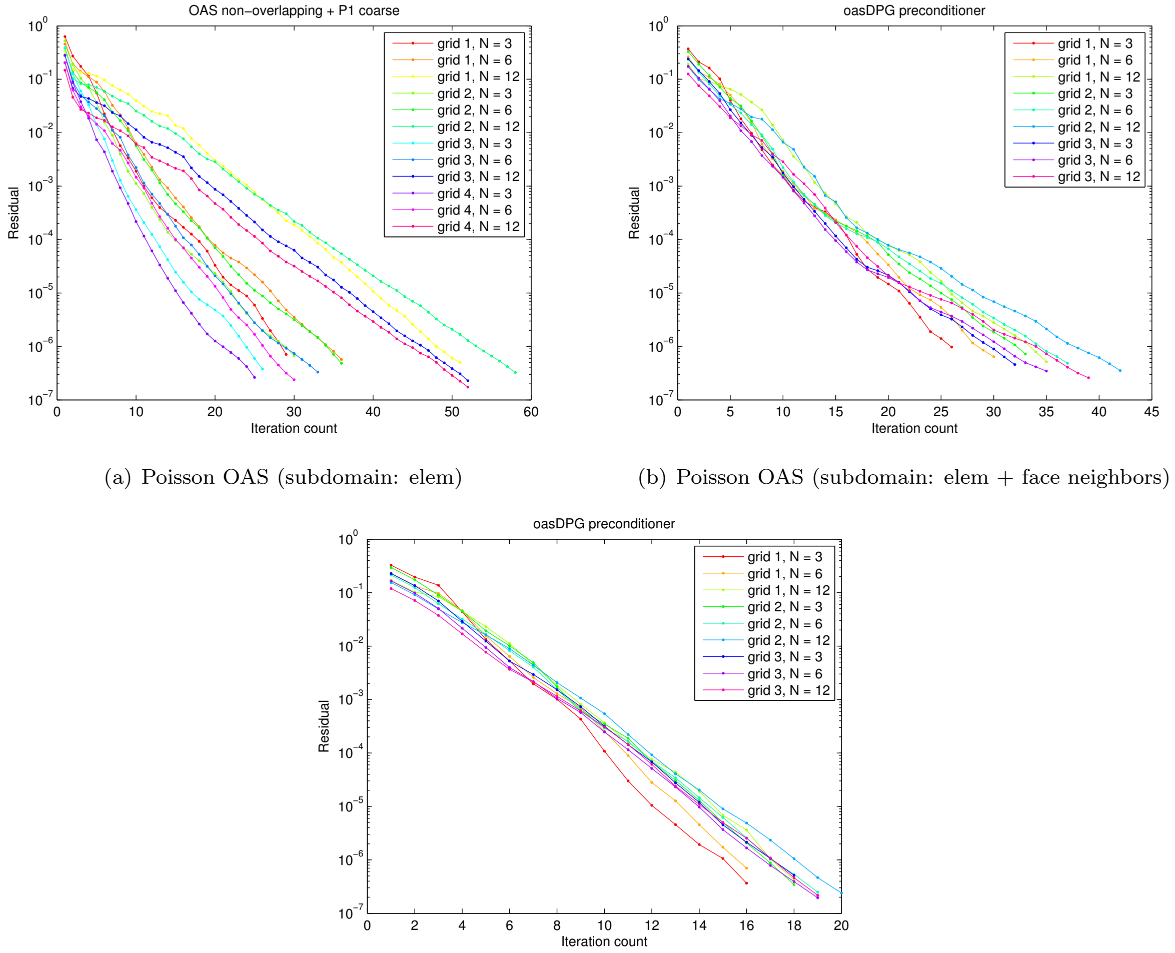
With DPG for Poisson, the stiffness matrix over one element is given by $B^T R_V^{-1} B$, where $B = KI_N^{N+\Delta N}$, and $I_N^{N+\Delta N}$ is an interpolation operator from order N polynomials to order $N + \Delta N$ polynomials. Under tensor product geometries, this reduces the system down to

$$B^T R_V^{-1} B = (I_N^{N+\Delta N} \otimes I_N^{N+\Delta N})^T (Q^{-1} \otimes Q^{-1})(\tilde{\Lambda})(Q^{-T} \otimes Q^{-T})(I_N^{N+\Delta N} \otimes I_N^{N+\Delta N})$$

where $\tilde{\Lambda} = (\Lambda \oplus \Lambda)(I \otimes I + \Lambda \oplus \Lambda)^{-1}(\Lambda \oplus \Lambda)$, or

$$\tilde{\Lambda}_{ij} = 1 + \lambda_i + \lambda_j.$$

Unfortunately, due to the presence of the $(I_N^{N+\Delta N} \otimes I_N^{N+\Delta N})$ matrices, it is difficult to invert this directly. Can we break up $\tilde{\Lambda}$ into a tensor product structure as well? If so, then we could break down a 2D DPG method into 1D lines.



(a) Poisson OAS (subdomain: elem)

(b) Poisson OAS (subdomain: elem + face neighbors)

(c) Poisson OAS (subdomain: vertex patches)

Figure 1: Behavior of OAS for different subdomains (element, neighbors, and one-ring patches). All cases include a P_1/P_0 coarse solver.