General notes on DPG

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1 Least squares

We begin with a general variational formulation

$$b(u, v) = l(v)$$

DPG begins with the idea that you would like to do least squares on the operator equation

$$Bu = \ell, \quad Bu, \ell \in V'$$

where $\langle Bu, v\rangle_{V'\times V} = b(u, v)$ and $\langle \ell, v\rangle_{V'\times V} = l(v)$. Since $Bu - \ell \in V'$, we minimize the norm of this residual in V' over the finite dimensional space U_h, i.e.

$$\min_{u_h \in U_h} \|Bu_h - \ell\|_{V'}^2 .$$

This leads to the normal equations

$$(Bu - \ell, B\delta u)_{V'}, \quad \forall \delta u \in U_h.$$

The Riesz map gives us the equivalent definition

$$\left(R_V^{-1}(Bu - \ell), R_V^{-1}(B\delta u)\right)_V = 0, \quad \forall \delta u \in U_h.$$

Assuming we've specified the Riesz map through a test space inner product

$$\langle R_V v, \delta v\rangle_{V'\times V} = (v, \delta v)_V,$$

this leads to what I call a Dual Petrov-Galerkin method.

2 Algebraic perspective

In the above example, V is infinite dimensional. If we approximate V by V_h such that $\dim(V_h) > \dim(U_h)$, we get matrix representations of our operators

$$\begin{aligned} B_{ij} &= b(u_j, v_i), \quad u_j \in U_h, v_i \in V_h \\ R_V &= (v_i, v_j)_V \quad v_i, v_j \in V_h \\ \ell_i &= l(v_i), \quad v_i \in V_h. \end{aligned}$$

The resulting normal equations

$$\left(R_V^{-1}(Bu - \ell), R_V^{-1}(B\delta u)\right)_V, \quad \forall \delta u \in U_h.$$

can now be written as

$$\left(R_V^{-1}\left(Bu-\ell\right)\right)^T R_V \left(R_V^{-1}B\right)=0,$$

or, after simplifying to $(Bu-\ell)^T R_V^{-1}B=0$, we get the algebraic normal equations

$$B^T R_V^{-1} B u = B^T R_V^{-1}\ell.$$

This is just the solution to the algebraic least squares problem

$$\min_u \|Bu-\ell\|^2_{R_V^{-1}}.$$

Such problems can also be written using the augmented system for the least squares problem

$$\begin{bmatrix} R_V & B \\ B^T & 0 \end{bmatrix}\begin{bmatrix} e \\ u \end{bmatrix} = \begin{bmatrix} l \\ 0 \end{bmatrix}.$$

This can be interpreted as the mixed form of the Dual Petrov-Galerkin method, which is used by Cohen, Welper, and Dahmen in their 2012 paper "Adaptivity and variational stabilization for convection-diffusion equations".

$$\begin{aligned}(e,v)_V + b(u,v) &= l(v)\\ b(\delta u, e) &= 0\end{aligned}$$

Eliminating e from the above system leads to the above algebraic normal equations.

Note: for general R_V, the algebraic normal equations are completely dense. Cohen, Welper, and Dahmen thus solve the augmented system to get solutions in this setting; however, this is a saddle point problem, and over 2x as large as the trial space, which makes preconditioning and solving more difficult.

3 Deriving the Discontinuous Petrov-Galerkin method

We want to avoid solving either a fully dense system or a saddle point problem, so we introduce Lagrange multipliers $\widehat{u}$ to enforce continuity weakly on e, which we will now approximate using discontinuous functions. These $\widehat{u}$ are defined on element edges only, similarly to hybrid variables or mortars in finite elements. This leads to the new system

$$\begin{aligned}\langle \widehat{u}, [\![v]\!]\rangle_{\Gamma_h} + (e,v)_V + b(u,v) &= l(v)\\ b(\delta u, e) &= 0\\ \langle \hat{\mu}, [\![e]\!]\rangle_{\Gamma_h} &= 0\end{aligned}$$

where Γ_h is the mesh skeleton (union of all element edges). The resulting algebraic system here is

$$\begin{bmatrix} R_V & B & \hat{B} \\ B^T & 0 & 0 \\ \hat{B}^T & 0 & 0 \end{bmatrix}\begin{bmatrix} e \\ u \\ \widehat{u} \end{bmatrix} = \begin{bmatrix} f \\ 0 \\ 0 \end{bmatrix}.$$

Because e is now discontinuous, R_V is made block-diagonal; eliminating e returns the (fairly) sparse symmetric positive-definite DPG system

$$A = \begin{bmatrix} B^T R_V^{-1} B & B^T R_V^{-1}\hat{B} \\ \hat{B}^T R_V^{-1} B & \hat{B}^T R_V^{-1}\hat{B} \end{bmatrix}\begin{bmatrix} u \\ \widehat{u} \end{bmatrix} = \begin{bmatrix} B^T R_V^{-1} f \\ \hat{B}^T R_V^{-1} f \end{bmatrix}.$$

3.1 Primal hybrid formulation: why it works for DPG

The primal hybrid formulation for a symmetric, positive-definite variational form is to enforce a weak continuity using mortar-like basis functions on element interfaces.

$$\sum_K \left(a_K(u,v) + \langle \lambda, v\rangle_{\partial K}\right) = (f,v)_{L^2(\Omega)}$$

$$\sum_K \langle \mu, u\rangle_{\partial K} = 0.$$

The weak continuity is enforced by noticing that $\sum_K \langle \mu, u\rangle_{\partial K} = \langle \mu, [\![u]\!]\rangle_{\Gamma_h}$, where Γ_h is the union of mesh interfaces.

The primal hybrid formulation gives Crouziex-Raviart elements for lowest order $N=1$, with flux order $N_f = N-1$. However, this does not work in defining elements for $N>1$; for N even and $N_f = N-1$, the Lagrange multiplier constraint set will not be linearly independent. Proof: on triangles, the jump constraints for N odd will imply that the solution must be an orthogonal polynomial of order N over the edge. Since N is odd, in 2D, the values at both vertices will be equal, and the trace on the edge will be continuous. Taking u to be any polynomial over T with this trace gives that $Bu=0$, but $u\neq 0$. (I believe Jay Gopalakrishnan also noticed this independently.)

If you take N arbitrary and $N_f \leq N-2$, however, things work fine. This is roughly the situation with DPG - consider primal DPG, with N_{trial}, $N_{\text{test}} = N_{\text{trial}}+2$, and $N_f = N_{\text{trial}}-1$.

4 Primal DPG and preconditioning

The DPG system can be reordered

$$A = \left[\begin{array}{cc} B^T R_V^{-1} B & B^T R_V^{-1}\hat{B} \\ \hat{B}^T R_V^{-1} B & \hat{B}^T R_V^{-1}\hat{B}\end{array}\right]\left[\begin{array}{c} u \\ \widehat{u}\end{array}\right] = \left[\begin{array}{cc} A_h & B_h \\ B_h^T & C_h\end{array}\right]\left[\begin{array}{c} u \\ \widehat{u}\end{array}\right] = \left[\begin{array}{c} f_h \\ g_h\end{array}\right]$$

where

$$\left[\begin{array}{c} f_h \\ g_h\end{array}\right] = \left[\begin{array}{c} B^T R_V^{-1} f \\ \hat{B}^T R_V^{-1} f\end{array}\right].$$

where A_h is the matrix corresponding to field (volume) variables, C_h is the submatrix corresponding to flux (surface/mortar) variables, and B_h couples the system together.

For a first pass, we consider the primal DPG method, which is probably the simplest DPG method available, where field variables are approximated with C_0-continuous piecewise polynomials and the flux variables are approximated using discontinuous mortar basis functions. Test functions are approximated with disjoint discontinuous polynomials of higher order than the trial field variables. (We may also wish to consider the DPG method with ultra-weak variational formulation (discontinuous field variables). In this case, A_h is also block diagonal.)

We assume that preconditioning the A_h submatrix can roughly be done using similar techniques to preconditioning standard elliptic equations. Initial numerical evidence appears to support this. Options include

- Two-level additive Schwarz. A coarse solve (P_1 elements with AGMG?) + (overlapping) block solves. Not working yet.
- Direct AGMG preconditioning. Working for CG and DPG.
- P_1 FEM preconditioning of nodal bases Working for CG, not yet implemented for DPG.

4.1 Block Jacobi

We can write the full DPG system in block form

$$\left[\begin{array}{cc} A_h & B_h \\ B_h^T & C_h \end{array}\right]\left[\begin{array}{c} u \\ \widehat{u} \end{array}\right] = \left[\begin{array}{c} f_h \\ g_h \end{array}\right].$$

We could naively break the system up into two blocks and use a fixed point iterative scheme

$$\begin{aligned} u^{n+1} &= A_h^{-1}(f_h - B_h\widehat{u}^n) \\ \widehat{u}^{n+1} &= C_h^{-1}(g_h - B_h^T u^{n+1}). \end{aligned}$$

or equivalently

$$\left[\begin{array}{c} u^{n+1} \\ \widehat{u}^{n+1} \end{array}\right] = \left[\begin{array}{cc} A_h^{-1} & \\ & C_h^{-1} \end{array}\right]\left(\left[\begin{array}{c} f_h \\ g_h \end{array}\right] - \left[\begin{array}{cc} & B_h \\ B_h^T & \end{array}\right]\left[\begin{array}{c} u^n \\ \widehat{u}^n \end{array}\right]\right)$$

which we can recognize as a block Jacobi iteration. This may be preferrable in a matrix-free environment, as both A_h and C_h can be computed in a matrix-free fashion ($C_h\widehat{u}$ can be computed using local inversions of the Riesz product with $\widehat{u}$ as boundary data). As far as I can tell, the Schur complement $S = C_h - B_h^T A_h^{-1} B_h$ for a general DPG system cannot, due to the non-explicit nature of $A_h = B_h^T R_V^{-1} B_h$.

This iteration was tested on pure convection under both the ultra-weak and primal formulations. As there was no observable difference in the primal formulation, we just show ultra-weak results here. Test norm was taken to be $\|v\|_V^2 = \alpha\|v\|_{L^2(\Omega)}^2 + \|\beta\cdot\nabla v\|_{L^2(\Omega)}^2$.

- Convergence depends on ΔN and N_{flux}.
- If $N_{\text{flux}} = 0$, convergence is roughly independent of ΔN. If $N_{\text{flux}} > 0$, the convergence is both slower than with $N_{\text{flux}} = 0$ and more sensitive to N and ΔN.

Convergence of the fixed point solution to the exact solution is shown here.

We note that these results are sensitive also to the regularization parameter α — smaller α results in faster convergence, but obviously worse conditioning of the local problem.

Questions:

- Do the results carry over for the ultra-weak formulation? Maybe only using the graph norm?
- Why do these results fail for Poisson's equation in the primal formulation?
- **This has to be related to the fixed point/Uzawa iteration of Dahmen.** Issues in generalization described below.

4.2 GMRES preconditioner

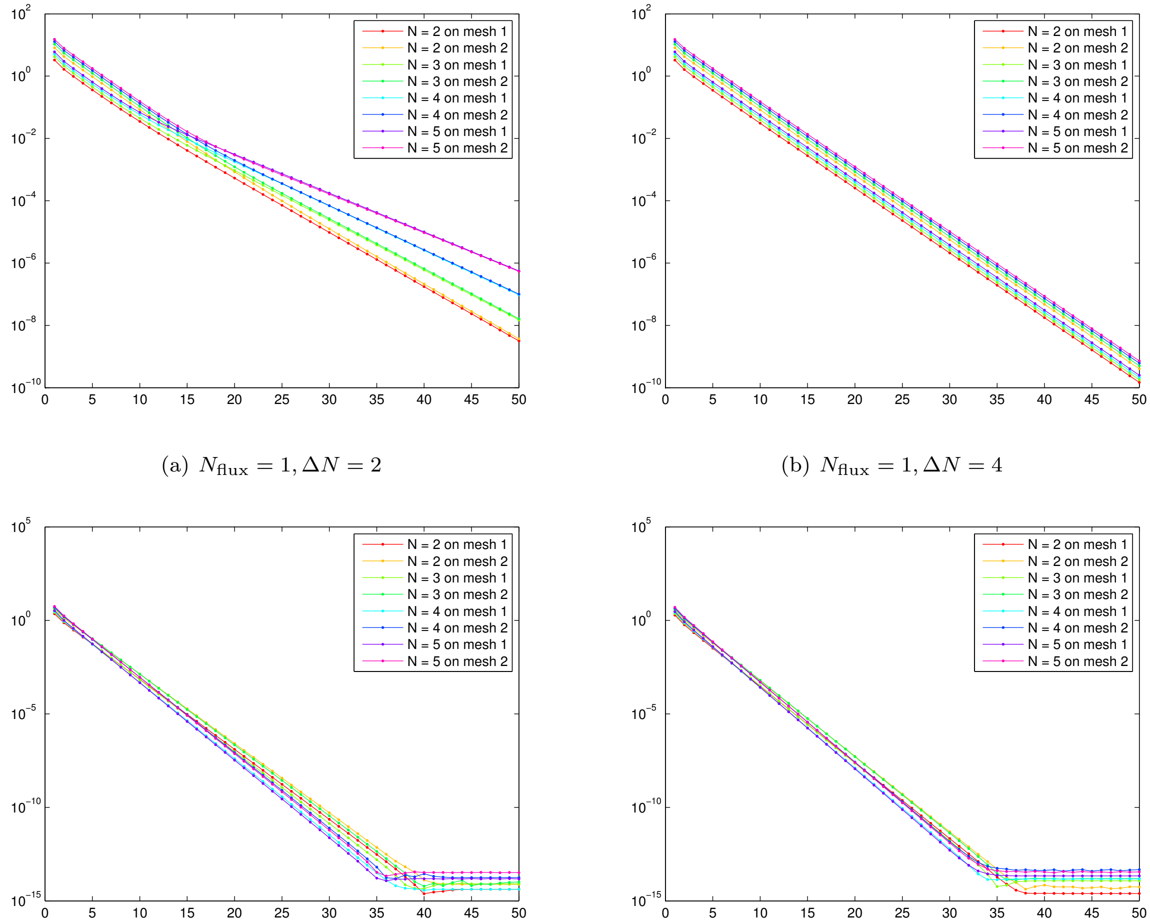
An issue with preconditioning using a fixed point iteration is that the preconditioner is no longer symmetric positive-definite, though we can still take advantage of the positive-definite nature of the blocks in the block Jacobi iteration. We therefore use the block Jacobi iteration as a GMRES preconditioner. We chart the dependence of the method on α and the number of fixed point iterations below.

5 Issues with this idea

- Main problem: for ultra-weak DPG, the matrix

  $$C = \hat{B}R_V^{-1}\hat{B}$$

  has exactly the same sparsity structure as the Schur complement $S = C - B^T A^{-1} B$ if we take discontinuous trial functions. The advantage: C is slightly better conditioned in practice, and there are



(a) $N_{\text{flux}} = 1, \Delta N = 2$

(b) $N_{\text{flux}} = 1, \Delta N = 4$

(c) $N_{\text{flux}} = 0, \Delta N = 4$

(d) $N_{\text{flux}} = 0, \Delta N = 2$

Figure 1: Block Jacobi fixed point iterations for pure convection with $\beta = (1, 0)$, ultra-weak formulation.

better-known ways to precondition mortar systems. However, the preconditioner becomes sensitive to tolerances (AGMG needs a lot of iterations/low tolerance to do well in the inner iteration, though it gets better w/α larger). Do we gain anything else here?

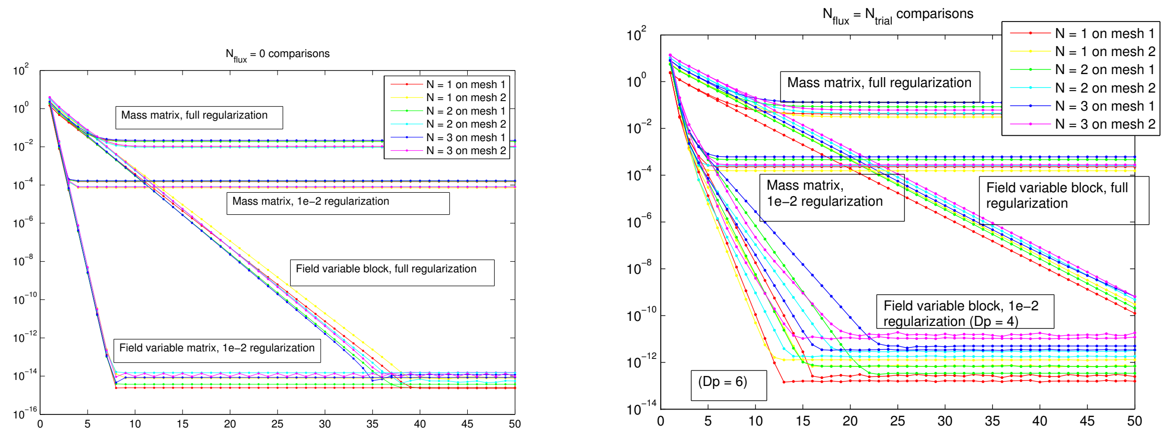
- This approach does not work for second order primal DPG methods - if I take C_0 basis functions and add diffusion, this approach fails. We cannot precondition 2nd order C_0 DPG with this method.

Main contribution of this work so far: connecting the block Jacobi DPG idea to Dahmen's fixed point iteration scheme for the mixed DPG formulation. Noting the dependence of fixed point convergence vs conditioning with α.

5.0.1 Relation to Dahmen fixed point iteration

Dahmen's fixed point iteration considers solving

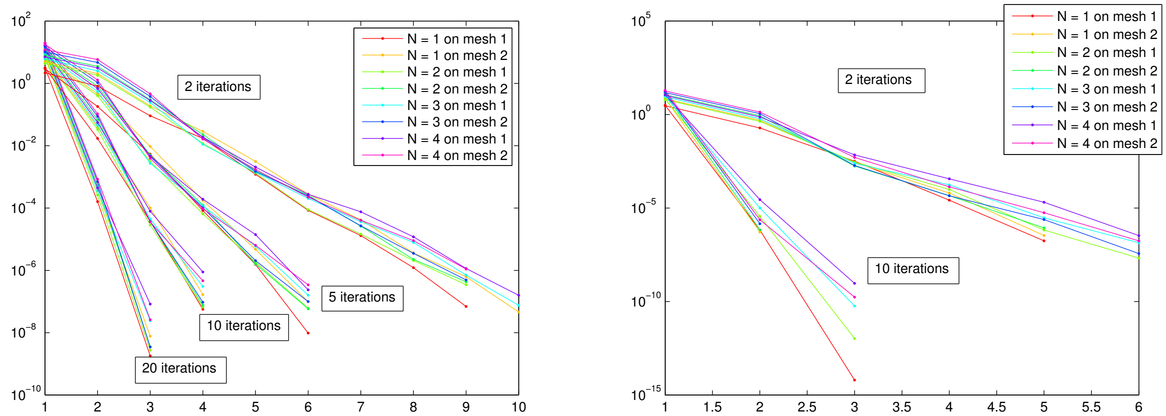
$$\begin{aligned}(e, v)_V + b(u, v) &= l(v)\\ b(\delta u, e) &= 0\end{aligned}$$



(a) $N_{\text{flux}} = 0$

(b) $N_{\text{flux}} = N_{\text{trial}}$

Figure 2: Block Jacobi fixed point iterations for pure convection - dependence on α regularization parameter. “Mass matrix” means that the solution of $B^T R_V^{-1} B$ was replaced by the inversion of an $L^2(\Omega)$ mass matrix instead.



(a) $\alpha = 1$

(b) $\alpha = .1$

Figure 3: Block Jacobi fixed point iterations for pure convection used as a preconditioner for GMRES.

or, in algebraic form

$$\begin{bmatrix} R_V & B \\ B^T & \end{bmatrix} \begin{bmatrix} e \\ u \end{bmatrix} = \begin{bmatrix} f \\ 0 \end{bmatrix}.$$

In the case he examined, he solves the strong equation

$$Au = f$$

such that $v^T B u = (u, A^* v)_{L^2(\Omega)}$ and $(\delta v)^T R_V v = (v, \delta v)_V := (A^* v, A^* \delta v)_{L^2(\Omega)}$. His fixed point iteration is then

$$\begin{aligned} e^{n+1} &= R_V^{-1}(f - Bu^n) \\ Mu^{n+1} &= Mu^n + B^T e^{n+1}. \end{aligned}$$

where M is the standard $L^2(\Omega)$ mass matrix. His key observation is that the Schur complement $B^T R_V^{-1} B$, while completely dense, should converge to M if the test space for v is sufficiently large. He uses C_0 basis functions for the trial space, but could have easily switched to L^2 basis functions instead.

Suppose now that we take the DPG approach to the above iteration. Then, we can solve for the auxiliary variables $\widehat{u}$ instead of solving for e via the system

$$\left[\begin{array}{cc} R_V & \widehat{B} \\ \widehat{B}^T & \end{array}\right]\left[\begin{array}{c} e \\ \widehat{u} \end{array}\right] = \left[\begin{array}{c} f - Bu \\ 0 \end{array}\right]$$

which yields

$$\begin{aligned} \widehat{u} &= (\widehat{B}^T R_V^{-1} \widehat{B})^{-1} \widehat{B}^T R_V^{-1} (f - Bu) \\ e &= R_V^{-1}(f - Bu - \widehat{B}\widehat{u}). \end{aligned}$$

Supposing M can be replaced by the positive definite matrix $B^T R_V^{-1} B$, we can rewrite Dahmen's iteration as the following: given u^n,

$$\begin{aligned} \widehat{u}^n &= (\widehat{B}^T R_V^{-1} \widehat{B})^{-1} \widehat{B}^T R_V^{-1} (f - Bu^n) \\ e^{n+1} &= R_V^{-1}(f - Bu^n - \widehat{B}\widehat{u}^n) \\ u^{n+1} &= u^n + (B^T R_V^{-1} B)^{-1} B^T e^{n+1}. \end{aligned}$$

We can eliminate e^{n+1} to arrive at

$$\begin{aligned} \widehat{u}^n &= (\widehat{B}^T R_V^{-1} \widehat{B})^{-1} \widehat{B}^T R_V^{-1} (f - Bu^n) \\ u^{n+1} &= (B^T R_V^{-1} B)^{-1} B^T R_V^{-1} (f - \widehat{B}\widehat{u}^n). \end{aligned}$$

Substituting in the definitions of A_h, B_h, and C_h, we arrive at

$$\begin{aligned} \widehat{u}^n &= C_h^{-1}(g_h - B_h^T u^n) \\ u^{n+1} &= A_h^{-1}(f_h - B_h \widehat{u}^n) \end{aligned}$$

which is nothing more than a block Jacobi iteration.

5.0.2 Variational crimes

DPG commits two "crimes" compared to Dahmen's work.

- Under the graph norm for DPG, we modify R_V to be

  $$\|v\|_V^2 = \alpha \|v\|_{L^2(\Omega)}^2 + \|A^* v\|_{L^2(\Omega)}^2 .$$

  This makes R_V a positive definite operator. Dahmen relied on the fact that R_V was invertible only under proper assumptions on the test space (i.e. boundary conditions on V). However, this relies on R_V being global.

  We note that this relies on DPG using the graph norm. It's not known whether or not equivalent results hold for robust (but non-graph) test norms.

- We use the primal hybrid formulation as a nonconforming method to approximate e. e is then condensed out to yield Lagrange multipliers $\widehat{u}$, which are appended to the the global system (much like FETI). However, the condensation yields that the total system is now symmetric positive definite.

Convergence analysis of the Jacobi fixed-point iteration will depend on adapting Dahmen's proof to accomodate these two conditions.

Conjecture: convergence of the Jacobi scheme is dependent only on optimal test functions for interface unknowns being approximated sufficiently well. Numerical results indicate that convergence is largely independent of Δp or N in the case where $N_{\text{flux}} = 0$.

6 Block triangular factorization vs fixed point iteration?

We can do a block triangular factorization of this system

$$A = \begin{bmatrix} I & 0 \\ B_h^T A_h^{-1} & I \end{bmatrix} \begin{bmatrix} A_h & 0 \\ 0 & S \end{bmatrix} \begin{bmatrix} I & A_h^{-1} B_h \\ 0 & I \end{bmatrix}.$$

where $S = C_h - B_h^T A_h^{-1} B_h$, the Schur complement. The inverse can be explicitly written

$$A^{-1} = \begin{bmatrix} I & -A_h^{-1} B_h \\ 0 & I \end{bmatrix} \begin{bmatrix} A_h^{-1} & 0 \\ 0 & S^{-1} \end{bmatrix} \begin{bmatrix} I & 0 \\ -B_h^T A_h^{-1} & I \end{bmatrix}$$

Preconditioners based on this factorization often involve an approximation of the Schur complement, which, in this case, is

$$S = \hat{B}^T R_V^{-1} \hat{B} - \hat{B}^T R_V^{-1} B \left(B^T R_V^{-1} B\right)^{-1} B^T R_V^{-1} \hat{B}$$

Suppose we just wish to precondition S with $C = \hat{B}^T R_V^{-1} \hat{B}$. This assumes that $C^{-1} S = I - C^{-1} B^T A^{-1} B$ is close to an identity, or that $C^{-1} B^T A^{-1} B$ is nearly zero. Numerical experiments with Poisson/convection-diffusion indicate this may holds independently of h, though convection-diffusion appears to have dependence on N for $N > 4$ and $\epsilon \ll 1$ or $\epsilon \gg 1$.

Suppose we apply just the rightmost two matrices to the DPG right hand side; we get

$$\begin{bmatrix} u^n \\ \widehat{u}^{n+1} \end{bmatrix} = \begin{bmatrix} A_h^{-1} & 0 \\ 0 & C_h^{-1} \end{bmatrix} \begin{bmatrix} I & 0 \\ -B_h^T A_h^{-1} & I \end{bmatrix} \begin{bmatrix} f_h \\ g_h \end{bmatrix} = \begin{bmatrix} A_h^{-1} f_h \\ C_h^{-1}(g_h - B_h A_h^{-1} f_h) \end{bmatrix} = \begin{bmatrix} A_h^{-1} f_h \\ C_h^{-1}(g_h - B_h u^n) \end{bmatrix}$$

which we can recognize as one iteration of block Jacobi starting with $u^n = A_h^{-1} f_h$ and $\widehat{u}^n = 0$. An application of the leftmost matrix then gives

$$\begin{bmatrix} I & -A_h^{-1} B_h \\ 0 & I \end{bmatrix} \begin{bmatrix} A_h^{-1} f_h \\ C_h^{-1}(g_h - B_h u^{n+1}) \end{bmatrix} = \begin{bmatrix} A_h^{-1}(f_h - B_h \widehat{u}^{n+1}) \\ \widehat{u}^{n+1} \end{bmatrix} = \begin{bmatrix} u^{n+1} \\ \widehat{u}^{n+1} \end{bmatrix}.$$

6.1 Etc notes

- Can we use the idea of reducing to field dofs? i.e. turn the inversion of the flux matrix into just an outer product? Make equiv to testing with null space of $\hat{B}^T$...
- Can we change the coupling to the mortar space for e? i.e. add consistency/penalty/etc?
- Try boundary penalization in test norms.