NOTES ON PRECONDITIONING

NATHAN V. ROBERTS

1. Introduction

We seek a general iterative solve strategy for DPG systems. Because the DPG system matrices are SPD, we will use preconditioned conjugate gradient. Suppose that the system we seek to solve on the fine grid is of the form

$$A\boldsymbol{x} = \boldsymbol{b}.$$

For an approximate solution $\boldsymbol{x}_k$, the residual will be

$$\boldsymbol{r}_k = A\boldsymbol{x}_k - \boldsymbol{b}.$$

The problem solved on each CG iteration is for the preconditioned residual $\boldsymbol{z}_k = M^{-1}\boldsymbol{r}_k$. We define the preconditioner M^{-1} by

$$M^{-1}\boldsymbol{r}_k = PM_c^{-1}P^T\boldsymbol{r}_k + S^{-1}\boldsymbol{r}_k,$$

where M_c^{-1} is an operator (in what follows, $M_c^{-1} = (P^TAP)^{-1}$) on the coarse grid, P is a prolongation operator converting coefficient data on the coarse grid into coefficient data on the fine grid, and S^{-1} is a *smoother* for A. In what follows, we use additive Schwarz with zero overlap or level-1 overlap[1] for S^{-1}.

1.1. Convergence Criterion.

We measure the residual of the system $\boldsymbol{r}_k$ in a scaled ℓ_2 norm:

$$\|\boldsymbol{r}_k\| = \frac{\|\boldsymbol{r}_k\|_2}{\|\boldsymbol{b}\|_2}.$$

Because we are less concerned with precise convergence on coarse meshes, our usual approach is as follows:

(1) Set initial mesh convergence threshold, $\epsilon = \epsilon_0$.
(2) Measure the energy error e_0 of a zero solution on the coarse mesh.
(3) Perform conjugate gradient iterations until $\|\boldsymbol{r}\| < \epsilon$.
(4) Measure the energy error e—a proxy for the L^2 error.
(5) Refine the mesh according to the energy error on each element.
(6) Set $\epsilon = \max\left(\epsilon_0 \frac{e}{e_0}, \epsilon_{\min}\right)$.
(7) Repeat steps 3-6 as required.

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[1]It is worth noting that we have also experimented with Jacobi smoothers (i.e. $S = D$, the diagonal of A), and that these did not perform well; here, the condition number of AM^{-1} appeared to grow with h^{-2}. It is also worth noting that the zero-overlap Schwarz smoother we use is communication-avoiding, and locally greedy—that is, only local coefficients are used for the block matrices that are inverted, and *all* local coefficients are used for the block. Level-1 overlap means that columns with nonzero entries for the local rank's rows are imported as rows to the local rank, and included in the additive Schwarz block. To maintain symmetry, we use the "Add" combine mode within the IfPack Schwarz preconditioner.

ϵ_0 and $\epsilon_{\min}$ are user-definable tolerances indicating the convergence criterion on the coarse mesh, and a minimal tolerance. As the mesh is refined and the energy error goes to zero, the tolerance becomes more restrictive until it reaches the minimal tolerance.

2. Example: Stokes System

In our research, we often apply DPG to incompressible flow problems. Therefore, a natural place to start experimenting is the Stokes system. We have generally used an ultraweak velocity-gradient-pressure (VGP) formulation, which is as follows:

$$\begin{aligned} b((u,\widehat{u}),v) = (\boldsymbol{\sigma} - p\boldsymbol{I}, \nabla \boldsymbol{v})_{\Omega_h} - \left\langle \widehat{\boldsymbol{t}}_n, \boldsymbol{v} \right\rangle_{\Gamma_h} \\ + (\boldsymbol{u}, \nabla q)_{\Omega_h} - \langle \widehat{\boldsymbol{u}} \cdot \boldsymbol{n}, q \rangle_{\Gamma_h} \\ + (\boldsymbol{\sigma}, \boldsymbol{\tau})_{\Omega_h} + (\boldsymbol{u}, \nabla \cdot \boldsymbol{\tau})_{\Omega_h} - \langle \widehat{\boldsymbol{u}}, \boldsymbol{\tau}\boldsymbol{n} \rangle_{\Gamma_h} = (\boldsymbol{f}, \boldsymbol{v})_{\Omega_h} = l(v). \end{aligned}$$

Here, the group variables u and $\widehat{u}$ refer to our field and trace variables[2], respectively, while the group variable v refers to our test variables. The field variables $\boldsymbol{u}$, p, and $\boldsymbol{\sigma}$ are the velocity, pressure, and gradient variables, respectively—the tensor $\boldsymbol{\sigma}$ is defined as the gradient of vector $\boldsymbol{u}$. The trace variables . . .

The *graph norm* on the test space for this formulation is given by

$$\left\|(\boldsymbol{v}, q, \boldsymbol{\tau})\right\|_{\text{graph}}^2 = \left\|\nabla \cdot \boldsymbol{v}\right\|^2 + \left\|\boldsymbol{\tau} - \nabla \boldsymbol{v}\right\|^2 + \left\|\nabla \cdot \boldsymbol{\tau} + \nabla q\right\|^2 + \left\|\boldsymbol{v}\right\|^2 + \left\|q\right\|^2 + \left\|\boldsymbol{\tau}\right\|^2 .$$

2.1. Condition Numbers and Convergence on Uniform Meshes.

To begin, we use $k_{\text{fine}} = 4$, $k_{\text{coarse}} = 0$ and a tolerance of $\epsilon = 10^{-8}$ on some uniform meshes for 2D lid-driven cavity flow. The number of iterations required for convergence and the estimated condition number are of interest. (The estimated condition number is a feature of Aztec's conjugate gradient solver.) Note that, because of the additive Schwarz smoother we use, the performance may depend on the number of MPI ranks used—for 1 MPI rank, the smoother will be exactly A^{-1}; the more MPI ranks, the worse we expect the smoother to perform. The results for 4 MPI ranks are shown in Table 1. The results for 16 MPI ranks are shown in Table 2. As expected, condition number estimates and iterations counts are somewhat higher for the larger number of MPI ranks, but the difference is not extreme. Importantly, the iteration counts and condition numbers do not appear to increase as the mesh is refined.

We repeat the experiments for a level-1 Schwarz overlap; the results are shown in Tables 3 and 4. In both cases, the number of iterations required for convergence is substantially less than that required for the zero overlap case.

2.2. Condition Numbers and Convergence on h-refined Meshes.

Our second test begins with precisely the 2×2 mesh for the problem in Section 2.1, with initial and minimum tolerances set $\epsilon_0 = \epsilon_{\min} = 10^{-8}$, for the sake of full comparability to the uniform case. We use an adaptive tolerance of $\theta = 0.20$. The results for the 4-rank case are shown in Table 5; those for the 16-rank case are shown in Table 6.

Although the increase in estimated condition number is not uniform, both the iteration count and the condition number do generally appear to be increasing somewhat as we refine. Moreover, the results are substantially worse for the 16-rank case. The conjugate gradient convergence history—specifically, the fact that the decrease of the residual was not monotonic in some instances—suggests

[2] By field variables, we mean those trial space unknowns defined on element interiors; by trace variables, we mean those defined on the mesh skeleton. In ultraweak formulations, the field variables are discontinuous across inter-element boundaries.

elements	Num. Global dofs	Energy Error	Num. Iterations	Condest.
2×2	940	1.18e+00	75	1.34e+04
4×4	3600	6.33e-01	79	1.01e+04
8×8	14080	4.35e-01	77	9.12e+03
16×16	55680	2.20e-01	74	8.84e+03
32×32	221440	7.93e-02	73	8.77e+03
64×64	883200	5.24e-02	70	8.75e+03

TABLE 1. Iteration counts and estimated condition numbers for the cavity flow problem on uniform meshes on 4 MPI ranks with Schwarz overlap level 0.

elements	Num. Iterations	Condest.
4×4	95	1.63e+04
8×8	88	1.06e+04
16×16	84	9.23e+03
32×32	81	8.81e+03

TABLE 2. Iteration counts and estimated condition numbers for the cavity flow problem on uniform meshes on 16 MPI ranks with Schwarz overlap level 0.

elements	Num. Global dofs	Energy Error	Num. Iterations	Condest.
2×2	940	1.18e+00	29	5.61e+03
4×4	3600	6.33e-01	36	1.08e+04
8×8	14080	4.35e-01	41	1.10e+04
16×16	55680	2.20e-01	47	1.10e+04
32×32	221440	7.93e-02	55	1.09e+04
64×64	883200	5.24e-02	61	1.09e+04

TABLE 3. Iteration counts and estimated condition numbers for the cavity flow problem on uniform meshes on 4 MPI ranks with Schwarz overlap level 1.

elements	Num. Iterations	Condest.
4×4	56	2.36e+04
8×8	58	1.37e+04
16×16	63	1.12e+04
32×32	70	1.10e+04
64×64	75	1.09e+04

TABLE 4. Iteration counts and estimated condition numbers for the cavity flow problem on uniform meshes on 16 MPI ranks with Schwarz overlap level 1.

that numerically, the operator AM^{-1} is no longer SPD. Therefore, we repeat the experiment, but this time using an overlap level of 1 for the Schwarz smoother.[3] The results for 4 MPI ranks are shown in Table 7; those for 16 MPI ranks are shown in Table 8.

[3]Level 1 overlap means that columns with nonzero entries for the local rank's rows are imported as rows to the local rank, and included in the additive Schwarz block. Thus far, we have not succeeded in combining non-zero overlap with conjugate gradient—this also behaves as though the preconditioned operator is no longer SPD.

Ref. Number	Num. Elements	Num. Global dofs	Energy Error	Num. Iterations	Condest.
0	4	940	1.18e+00	75	1.34e+04
1	10	2250	6.33e-01	115	6.57e+03
2	16	3540	4.35e-01	173	7.19e+03
3	22	4830	2.21e-01	170	6.74e+03
4	28	6120	8.09e-02	106	5.77e+03
5	34	7410	5.58e-02	156	2.08e+04
6	70	15170	2.97e-02	239	8.84e+04
7	118	25550	1.36e-02	220	1.03e+05
8	136	29380	7.09e-03	374	5.68e+06
9	154	33210	4.07e-03	52	1.09e+02
10	190	40910	2.43e-03	761	9.07e+07

TABLE 5. Iteration counts and estimated condition numbers for the cavity flow problem on adaptively refined meshes on 4 MPI ranks with Schwarz overlap level 0.

Ref. Number	Num. Iterations	Condest.
0	75	1.34e+04
1	134	1.68e+04
2	227	1.71e+04
3	277	1.59e+04
4	393	2.45e+04
5	612	8.42e+04
6	749	1.05e+05
7	1720	1.47e+06
8	1216	5.67e+06
9	911	2.27e+07
10	2375	5.96e+07

TABLE 6. Iteration counts and estimated condition numbers for the cavity flow problem on adaptively refined meshes on 16 MPI ranks with Schwarz overlap level 0.

For comparison, we repeat the 1-overlap experiment using GMRES, again using a level-1 overlap for the Schwarz smoother. The 4-rank results are shown in Table 9; 16-rank results are shown in Table 10. In both cases, the number of iterations and the condition numbers appear to be fairly stable under refinement.

2.3. **Stokes 3D with h-refinements.** For our next experiment, we again examine Stokes driven cavity flow, but now in 3 dimensions, with $k_{\text{fine}} = 2$ and $k_{\text{coarse}} = 0$. (We ran this on Argonne's Cetus.) Results with an adaptive tolerance and zero overlap are in Table 11. Results with fixed tolerance of 10^{-8} and level-1 overlap are in Table 12.

Ref. Number	Num. Elements	Num. Global dofs	Energy Error	Num. Iterations	Condest.
0	4	940	1.18E+000	29	5.62e+03
1	10	2250	6.33E-001	32	4.43e+03
2	16	3540	4.35E-001	29	3.52e+03
3	22	4830	2.21E-001	30	3.44e+03
4	28	6120	8.09E-002	30	3.38e+03
5	34	7410	5.58E-002	29	3.36e+03
6	70	15170	2.97E-002	40	4.33e+03
7	118	25550	1.36E-002	42	3.67e+03
8	136	29380	7.09E-003	43	4.07e+03
9	154	33210	4.07E-003	15	4.07e+03
10	190	40910	2.43E-003	41	4.05e+03

TABLE 7. Iteration counts and estimated condition numbers for the cavity flow problem on adaptively refined meshes on 4 MPI ranks, with Schwarz overlap level 1.

Ref. Number	Num. Iterations	Condest.
0	29	5.62e+03
1	46	1.81e+04
2	55	1.90e+04
3	40	4.76e+01
4	53	6.06e+03
5	49	1.21e+04
6	64	1.22e+04
7	60	6.42e+03
8	60	1.29e+04
9	54	6.41e+03
10	44	4.47e+01

TABLE 8. Iteration counts and estimated condition numbers for the cavity flow problem on adaptively refined meshes on 16 MPI ranks, with Schwarz overlap level 1.

Ref. Number	Num. Iterations	Condest.
0	44	3.74e+05
1	56	8.96e+04
2	51	1.78e+05
3	48	2.83e+05
4	45	2.85e+05
5	44	4.89e+05
6	58	1.86e+05
7	53	7.04e+04
8	60	4.50e+05
9	27*	7.93e+04*
10	61*	1.87e+04*

TABLE 9. Iteration counts and estimated condition numbers using GMRES for the cavity flow problem on adaptively refined meshes on 4 MPI ranks. Here the Schwarz smoother has level 1 overlap. For the entries marked with a *, Aztec warned that the recursive residual indicated convergence even though the true residual was too small. For these, we restarted with the solution thus far as initial guess, and we report the total number of iterations and the estimated condition number for the initial GMRES run (which in each case was the one with the greatest number of iterations).

Ref. Number	Num. Iterations	Condest.
0	44	3.74e+05
1	70	2.82e+05
2	94	5.21e+05
3	94	7.36e+05
4	98	3.09e+06
5	97	7.60e+05
6	114	9.64e+04
7	99*	1.63e+04*
8	97*	3.22e+04*
9	97*	2.23e+04*
10	103*	9.27e+03*

TABLE 10. Iteration counts and estimated condition numbers using GMRES for the cavity flow problem on adaptively refined meshes on 16 MPI ranks. Here the Schwarz smoother has level 1 overlap. For the entries marked with a *, Aztec warned that the recursive residual indicated convergence even though the true residual was too small. For these, we restarted with the solution thus far as initial guess, and we report the total number of iterations and the estimated condition number for the initial GMRES run (which in each case was the one with the greatest number of iterations).

Ref. Number	Num. Elements	Num. Global dofs	Energy Error	Num. Iterations	Condest.
0	8	4752	1.41e+00	92	1.34e+04
1	36	19764	1.43e+00	102	1.13e+04
2	120	63504	8.82e-01	104	5.12e+03
3	316	164700	6.70e-01	110	4.18e+03
4	540	279936	5.04e-01	134	4.08e+03
5	1268	655020	3.13e-01	211	1.16e+04
6	4880	2512188	1.54e-01	342	1.34e+04

TABLE 11. Iteration counts and estimated condition numbers for the 3D cavity flow problem on adaptively refined meshes on 2048 MPI ranks, with Schwarz overlap level 0.

Ref. Number	Num. Elements	Num. Global dofs	Energy Error	Num. Iterations	Condest.
0	8	4752	1.40e+00	47	1.60e+04
1	36	19764	1.43e+00		
2	120	63504	8.82e-01		
3	316	164700	6.70e-01		
4	540	279936	5.04e-01		
5	1268	655020	3.13e-01		
6	4880	2512188	1.54e-01		

TABLE 12. Iteration counts and estimated condition numbers for the 3D cavity flow problem on adaptively refined meshes on 2048 MPI ranks, with Schwarz overlap level 1.