1 Variational form

For purposes of testing, I took the homogeneous Helmholtz equation

$$\begin{aligned} k^2 u + \Delta u = 0 \quad & \text{on } \Omega \\ -iku + \frac{\partial u}{\partial n} = g, \quad & \text{on } \partial\Omega. \end{aligned}$$

Defining the mesh Ω_h and mesh skeleton Γ_h, an integration by parts leads to Jay's variational formulation

$$b(u,v) = k^2 \left(u,v\right)_{L^2(\Omega)} - \left(\nabla u, \nabla v\right)_{L^2(\Omega)} + \left\langle \widehat{\frac{\partial u}{\partial n}}, v \right\rangle_{\Gamma_h}.$$

In order to incorporate impedance boundary conditions, I took a slightly different approach by adding the boundary term $\langle iku, v\rangle_\Gamma$ to the variational form and redefining the flux as the impedance condition, such that the variational form is

$$b(u,v) = k^2 \left(u,v\right)_{L^2(\Omega)} - \left(\nabla u, \nabla v\right)_{L^2(\Omega)} + \langle iku, v\rangle_\Gamma + \left\langle \widehat{-iku + \frac{\partial u}{\partial n}}, v \right\rangle_{\Gamma} + \left\langle \widehat{\frac{\partial u}{\partial n}}, v \right\rangle_{\Gamma_h^0}.$$

Then, the variational problem for the Helmholtz equation becomes

$$k^2 \left(u,v\right)_{L^2(\Omega)} - \left(\nabla u, \nabla v\right)_{L^2(\Omega)} + \langle iku, v\rangle_\Gamma + \left\langle \widehat{\frac{\partial u}{\partial n}}, v \right\rangle_{\Gamma_h^0} = \langle g, v\rangle_\Gamma,$$

which more closely resembles the form of a standard Galerkin problem.

2 Primal DPG for Helmholtz

The test norm is taken to be

$$\|v\|_V^2 \coloneqq \alpha \|v\|_{L^2(\Omega)}^2 + \|\nabla v\|_{L^2(\Omega)}^2$$

In Jay's paper, $\alpha = k^2$. Here, we take $\alpha = 1$. Empirically, this appears to produce lower L^2 errors than $\alpha = k^2$. The resulting spaces are discretized in the same fashion as in Jay's paper.

3 Overlapping additive Schwarz with coarse grid for Helmholtz

4 Overlapping multiplicative Schwarz with coarse grid for Helmholtz