DMMR Tutorial 7

Graphs

November 4, 2015

1. Consider a (simple, undirected) bipartite graph $G = (V, E)$, with bipartition (V_1, V_2). In other words, $V = V_1 \cup V_2$, $V_1 \cap V_2 = \emptyset$, and for every edge $e \in E$, $e = \{u, v\}$ such that $u \in V_1$ and $v \in V_2$.

   Suppose that every vertex in V has degree exactly 7.

   Prove that $|V_1| = |V_2|$, and that there must exist a perfect matching in such a bipartite graph.

   [Hint: apply the generalized pigeonhole principle in order to show that the key condition for Hall's theorem does hold.]

2. How many non-isomorphic simple undirected graphs are there with exactly 4 vertices? Justify your answer.

3. Suppose $G = (V, E)$ is a directed graph, and u and v are vertices of G. Show that either u and v are in the same strongly connected component of G, or they are in disjoint strongly connected components of G.

4. Recall that the n-dimensional **hypercube**, or n-cube, is the simple undirected graph whose nodes are bit strings of length n, and such that there is an edge between a pair of nodes if and only if their bit strings differ in exactly one bit position.

   (a) For what values of $n \geq 1$ does the n-cube have an Euler circuit? (1 point)

   (b) Prove by induction that for all $n \geq 2$, the n-cube has a Hamiltonian circuit. (9 points)

5. Recall that in a (simple undirected) graph, a **cycle** (or circuit) is a **walk** (or path) that begins and ends in the same vertex. A **simple cycle** is a cycle on which no edge occurs more than once.

   Prove that, for each integer k greater than 2, the existence of a simple cycle of length k is a **isomorphism invariant**, meaning that if one graph has a simple cycle of length k, but another graph does not, then the two graphs can not be isomorphic.

Solution to question 5 (and only for question 5) to be handed in on paper at the ITO by Wednesday, November 11 at 4:00pm. In the solution you hand in, please write down your <u>name</u>, <u>matriculation number</u> and the <u>tutorial group</u> you belong to.