DMMR Tutorial sheet 3

Functions, Sequences

October 7, 2015

Some of the exercises for this tutorial are taken from Chapter 2 and 8 of the book: Kenneth Rosen, Discrete Mathematics and its Applications, 7th Edition, McGraw-Hill, 2012.

1. Determine whether the function $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ is surjective if

   (a) $f(m, n) = m^2 + n^2$

   (b) $f(m, n) = m$

   (c) $f(m, n) = |n|$

   (d) $f(m, n) = m - n$

   Solution:

   (a) The function is not onto because not every integer is the sum of two perfect squares. For example $-|n|$ and 3 are not the sum of two perfect squares (for any n).

   (b) The function is onto because for any $z \in \mathbb{Z}$ we can choose a pair $(z, x) \in \mathbb{Z} \times \mathbb{Z}$ and $f(z, x) = z$.

   (c) The function is not onto because $|n|$ is always positive, so there exists no (x, y) such that $f(x, y) = -|n|$.

   (d) The function is onto because for every z integer $f(z, 0) = z - 0 = z$.

   $\square$

2. (a) Prove that a strictly decreasing function from $\mathbb{R}$ to itself is one-to-one.

   (b) Give an example of a decreasing function from $\mathbb{R}$ to itself that is not one-to-one.

   Solution:

   (a) By contrapositive: Assume f is not one-to-one. Take $x < y$ such that $f(x) = f(y)$. Then, f is not strictly decreasing.

   (b) The constant function $f(x) = 0$ works because for every $x < y$ we have that $f(x) \leq f(y)$.

   $\square$

3. Determine (and prove) whether each of these sets is countably infinite or uncountable. For those that are countably infinite, exhibit a one-to-one correspondence (i.e., bijection) between the set of positive integers and that set.

   (a) the odd negative integers

   (b) the real numbers in the open interval $(0, 2)$

   (c) the irrational numbers in the open interval $(0, 2)$

(d) the set $A \times \mathbb{Z}^+$ where $A = \{2, 3\}$

Solution:
Let X be the set in question.

(a) We prove that X is countably infinite by showing that the function $f : \mathbb{Z}^+ \to X$ defined by $f(x) = -2x + 1$ is bijective. First we have to prove that the codomain of f is actually X. Let $x \in \mathbb{Z}^+$. Then, $x \geq 1$, so $-2x \leq -2$. Therefore $f(x) = -2x + 1 \leq -1$, which means that $f(x)$ is negative and clearly odd. To show that it is one-to-one, let $f(x) = f(y)$. Then, $-2x + 1 = -2y + 1$, from which we get $x = y$. To show that it is onto we take an arbitrary odd negative integer y. Let $x = -(y-1)/2$. Clearly $f(x) = y$ so we just have to show that $x \in \mathbb{Z}^+$. Given that y is negative and odd, $y - 1$ is negative and even. Then, $(y-1)/2$ is a negative integer, and thus $x = -(y-1)/2$ is a positive integer.

(b) From Cantor's diagonalisation argument we know that the set of real numbers between 0 and 1 is uncountable. We call this set A. Then, we can prove that $|X| = |A|$ by showing that the function $f : A \to X$ given by $f(a) = 2a$ is a bijection. It is clearly injective because if $2a = 2b$ then $a = b$, and it is surjective because if $x \in X$ then $x/2 \in A$ and $f(x/2) = x$. Then, if X were countable A would also be countable, which is a contradiction.

(c) The set of real numbers between 0 and 2 is the union of X and the set of rationals between 0 and 2 (we call this set A). The set of rationals between 0 and 2 is countable because it can be injected in $\mathbb{Q}$ (by the inclusion). If X were countable then $X \cup A$ would also be countable, which contradicts the result of (b).

(d) Let $f : A \times \mathbb{Z}^+ \to \mathbb{Z}^+$ be defined by

$$f(a, x) = \begin{cases} 2x & \text{if } a = 2 \\ 2x - 1 & \text{if } a = 3 \end{cases}$$

To prove that this function is injective, let $f(a, x) = f(b, y)$. If this number is even, then $a = b = 2$, which means that $f(a, x) = f(2, x) = 2x$ and $f(b, y) = f(2, y) = 2y$. Then, $2x = 2y$, which implies that $x = y$ and therefore $(a, x) = (b, y)$. If the number is odd then $a = b = 3$, which means that $f(a, x) = f(3, x) = 2x - 1$ and $f(b, y) = f(3, y) = 2y - 1$. Then, $2x - 1 = 2y - 1$, which implies that $x = y$ and therefore $(a, x) = (b, y)$. To prove that f is surjective, let $y \in \mathbb{Z}^+$. If y is even then let $(a, x) = (2, y/2)$, which is clearly an element of $A \times \mathbb{Z}^+$. Then, $f(a, x) = 2(y/2) = y$. If y is odd then let $(a, x) = (3, (y+1)/2)$, which is clearly an element of $A \times \mathbb{Z}^+$. Then, $f(a, x) = 2(y + 1)/2 - 1 = y$.

$\square$

4. A vending machine dispensing books of stamps accepts only \$1 coins, \$1 bills and \$5 bills.

(a) Find a recurrence relation for the number of ways to deposit n dollars in the vending machine, where the order in which the coins and bills are deposited matters.

(b) What are the initial conditions?

(c) How many ways are there to deposit \$10 for a book of stamps?

Solution:

(a) Let a_n be the number of ways to deposit n dollars in the vending machine. We must express a_n in terms of earlier terms in the sequence. If we want to deposit n dollars, we may start with a dollar coin and then deposit $n - 1$ dollars. This gives us a_{n-1} ways to deposit n

dollars. We can also start with a dollar bill and then deposit $n-1$ dollars. This gives us a_{n-1} more ways to deposit n dollars. Finally, we can deposit a five-dollar bill and follow that with $n-5$ dollars; there are a_{n-5} ways to do this. Therefore the recurrence relation is $a_n = 2a_{n-1} + a_{n-5}$. Note that this is valid for $n \geq 5$, since otherwise a_{n-5} makes no sense.

(b) We need initial conditions for all subscripts from 0 to 4. It is clear that $a_0 = 1$ (deposit nothing) and $a_1 = 2$ (deposit either the dollar coin or the dollar bill). It is also not hard to see that $a_2 = 2^2 = 4, a_3 = 2^3 = 8$ and $a_4 = 2^4 = 16$, since each sequence of n coins and Bills corresponds to a way to deposit n dollars.

(c) We will compute a_5 through a_{10} using the recurrence relation:

$$\begin{aligned}
a_5 &= 2a_4 + a_0 = 2 \cdot 16 + 1 = 33 \\
a_6 &= 2a_5 + a_1 = 2 \cdot 33 + 1 = 68 \\
a_7 &= 2a_6 + a_2 = 2 \cdot 68 + 1 = 140 \\
a_8 &= 2a_7 + a_3 = 2 \cdot 140 + 1 = 288 \\
a_9 &= 2a_8 + a_4 = 2 \cdot 288 + 1 = 592 \\
a_{10} &= 2a_9 + a_5 = 2 \cdot 592 + 1 = 1217
\end{aligned}$$

□

5. Show that the set of functions from positive integers to the set $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ is uncountable.

Solution:
Assume a subset F' of these functions F, i.e. $F' \subseteq F$. We will show there is a one-to-one correspondence between $(0, 1)$ to F'. We set $f \in F'$ as $f(n) = d_n$ for each real number $r = 0.d_1d_2 \ldots d_n \ldots$. For each such function f, it corresponds to only one real number. So it is an injection from $(0, 1)$ to F'. It is also onto from $(0, 1)$ to F' since each function in F' is set according to a real number in $(0, 1)$. So it is one-to-one correspondence between $(0, 1)$ to F'. $(0, 1)$ to F' have the same cardinality. Since $|(0, 1)|$ = R is uncountable and $F' \subseteq F$, F is uncountable. □

Solutions (to the last question ONLY on the sheet) must be handed in on paper to the ITO by Wednesday, 14 October, 4:00pm. Please post it into the grey metal box on the wall outside the ITO.