# Discrete Mathematics & Mathematical Reasoning
# Basic Structures: Sets, Functions and Relations


Colin Stirling

Informatics


Some slides based on ones by Myrto Arapinis

# Some important sets

$\mathbb{B} = \{\text{true}, \text{false}\}$ Boolean values

$\mathbb{N} = \{0, 1, 2, 3, \ldots\}$ Natural numbers

$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$ Integers

$\mathbb{Z}^+ = \{1, 2, 3, \ldots\}$ Positive integers

$\mathbb{R}$ Real numbers

$\mathbb{R}^+$ Positive real numbers

$\mathbb{Q}$ Rational numbers

$\mathbb{C}$ Complex numbers

$\emptyset$ Empty set

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- $S = \{x \mid P(x)\}$ where $P(x)$ is a predicate
- Example Subsets of sets upon which an order is defined

$$
\begin{array}{lcll}
[a, b] & = & \{x \mid a \leq x \leq b\} & \text{closed interval} \\
[a, b) & = & \{x \mid a \leq x < b\} & \\
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- $A \times B$ cartesian product (tuple sets)

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- $S \in S$ provided that $S \notin S$; $S \notin S$ provided that $S \in S$
- Modern formulations (such as Zermelo-Fraenkel set theory) restrict comprehension. (However, it is impossible to prove in ZF that ZF is consistent unless ZF is inconsistent.)

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- $f : A \to B$ if $f$ is a function from $A$ to $B$

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# Function composition

**Definition**

Let $f : B \rightarrow C$ and $g : A \rightarrow B$. The composition function $f \circ g : A \rightarrow C$ is $(f \circ g)(a) = f(g(a))$

# Results about function composition

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*The composition of two functions is a function*

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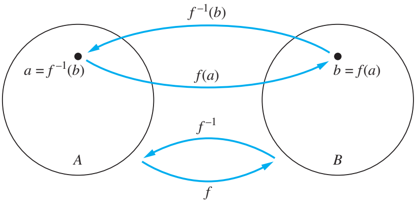
**Corollary**

*The composition of two bijections is a bijection*

# Inverse function

## Definition

If $f : A \to B$ is a bijection, then the inverse of $f$, written $f^{-1} : B \to A$ is $f^{-1}(b) = a$ iff $f(a) = b$



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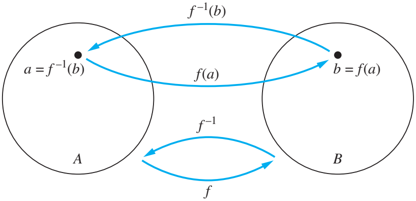


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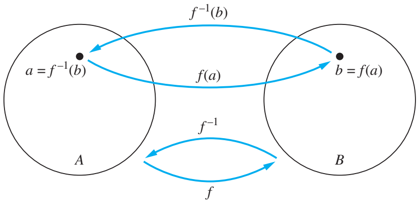


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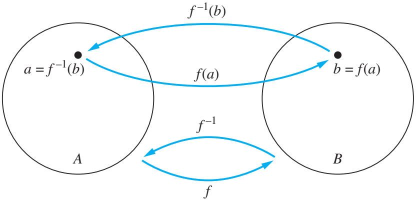


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What is $f^{-1} \circ f$? and $f \circ f^{-1}$?

# The floor and ceiling functions

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The floor function $\lfloor\ \rfloor : \mathbb{R} \rightarrow \mathbb{Z}$ is $\lfloor x \rfloor$ equals the largest integer less than or equal to $x$

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$$\lfloor -6.1 \rfloor = -7 \qquad \lceil 6.1 \rceil = 7$$

# Useful tips about floors and ceilings

- When showing properties of floors is to let $x = n + \epsilon$ if $\lfloor x \rfloor = n$ where $0 \leq \epsilon < 1$
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- Proof in book

# Prove $\lceil x \rceil + \lceil y \rceil = \lceil x + y \rceil$

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False; counterexample $x = 1/2$ and $y = 1/2$

# The factorial function

## Definition

The factorial function $f : \mathbb{N} \to \mathbb{N}$, denoted as $f(n) = n!$ assigns to $n$ the product of the first $n$ positive integers

$$f(0) = 0! = 1$$

and

$$f(n) = n! = 1 \cdot 2 \cdot \cdots \cdot (n-1) \cdot n$$

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**Definition**

Given sets $A_1, \ldots, A_n$, a subset $R \subseteq A_1 \times \cdots \times A_n$ is an $n$-ary relation

# Examples

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- Let $m > 1$ be an integer. $R = \{(a, b) \mid a \bmod m = b \bmod m\}$
- Written as $a = b \pmod{m}$

# Properties of binary relations

A binary relation $R$ on $A$ is called

- reflexive iff $\forall x \in A \; (x, x) \in R$

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- $\leq$, $=$, $<$, and $|$ are antisymmetric
- transitive iff $\forall x, y, z \in A\, (((x, y) \in R \wedge (y, z) \in R) \;\rightarrow\; (x, z) \in R)$
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- $|$ on integers is not an equivalence relation.
- For $m > 1$ be an integer the relation $\equiv \pmod{m}$ is an equivalence relation on integers

# Equivalence classes

## Definition

Let $R$ be an equivalence relation on a set $A$ and $a \in A$. Let

$$[a]_R = \{s \mid (a, s) \in R\}$$

be the equivalence class of $a$ w.r.t. $R$

If $b \in [a]_R$ then $b$ is called a representative of the equivalence class. Every member of the class can be a representative

# Theorem

## Result

Let $R$ be an equivalence on $A$ and $a, b \in A$. The following three statements are equivalent

1. $aRb$
2. $[a]_R = [b]_R$
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Proof in book

# Partitions of a set

## Definition

A partition of a set $A$ is a collection of disjoint, nonempty subsets that have $A$ as their union. In other words, the collection of subsets $A_i \subseteq A$ with $i \in I$ (where $I$ is an index set) forms a partition of $A$ iff

1. $A_i \neq \emptyset$ for all $i \in I$
2. $A_i \cap A_j = \emptyset$ for all $i \neq j \in I$
3. $\bigcup_{i \in I} A_i = A$

# Result

## Theorem

1. If $R$ is an equivalence on $A$, then the equivalence classes of $R$ form a partition of $A$
2. Conversely, given a partition $\{A_i \mid i \in I\}$ of $A$ there exists an equivalence relation $R$ that has exactly the sets $A_i, i \in I$, as its equivalence classes

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