DMMR Tutorial sheet 1

Propositional Logic, Predicate Logic, Proof techniques

September 25, 2015

Some of the exercises for this tutorial are taken from Chapter 1 of the book: Kenneth Rosen, Discrete Mathematics and its Applications, 7th Edition, McGraw-Hill, 2012.

1. Construct the truth table for the formula $(A \to B) \to [((B \to C) \wedge \neg C) \to \neg A]$.

   Solution:

A	B	C	$\neg A$	$\neg C$	$(A \to B)$	$(B \to C)$	$(B \to C) \wedge \neg C$	$((B \to C) \wedge \neg C) \to \neg A$	X
T	T	T	F	F	T	T	F	T	T
T	T	F	F	T	T	F	F	T	T
T	F	T	F	F	F	T	F	T	T
T	F	F	F	T	F	T	T	F	T
F	T	T	T	F	T	T	F	T	T
F	T	F	T	T	T	F	F	T	T
F	F	T	T	F	T	T	F	T	T
F	F	F	T	T	T	T	T	T	T

   X stands for the formula $(A \to B) \to [((B \to C) \wedge \neg C) \to \neg A]$ □

2. Let $P(m, n)$ be the statement "m divides n", where the domain for both variables consists of all positive integers. (By "m divides n" we mean that $n = km$ for some integer k.) Determine the truth values of each of these statements.

   (a) $P(4, 5)$

   (b) $P(2, 4)$

   (c) $\forall m \, \forall n \, P(m, n)$

   (d) $\exists n \, \forall m \, P(m, n)$

   (e) $\exists m \, \forall n \, P(m, n)$

   (f) $\forall n \, P(1, n)$

   Solution:

   (a) False, since 4 does not divide 5

   (b) True, since $4 = 2 \cdot 2$

   (c) False, see (a)

   (d) False, for all n we get that $P(2n, n)$ is not True, since $\frac{1}{2} \notin \mathbb{Z}$

   (e) True: $m = 1$ see (f)

   (f) True: $n = n \cdot 1$

   □

3. Prove by contraposition, that if m and n are integers and mn is even, then m is even or n is even.

   Solution:
   We have to prove
   $$mn \text{ even} \rightarrow (m \text{ even} \vee n \text{ even}$$
   The contraposition is
   $$\neg(m \text{ even} \vee n \text{ even}) \rightarrow \neg(mn \text{ even})$$
   which can be transformed using DeMorgan and even $\equiv \neg$ odd
   $$(m \text{ odd} \wedge n \text{ odd}) \rightarrow mn \text{ odd}$$
   We assume m is odd and by the definition of odd there exists a $k \in \mathbb{Z}$ with $m = 2k + 1$. Similar there exists a $l \in \mathbb{Z}$ with $n = 2l + 1$. Therefore we get
   $$\begin{aligned} mn &= (2k+1) \cdot (2l+1) \\ &= 4lk + 2k + 2l + 1 \\ &= 2(2lk + k + l) \\ &= 2l' + 1 \end{aligned}$$
   where $l' = 2lk + k + l \in \mathbb{Z}$. By definition mn is therefore odd.

   $\square$

4. Prove that the sum of an irrational number and a rational number is irrational.

   Solution:
   We prove it by contradiction. Assume that the sum of an irrational number i and a rational number $\frac{a}{b}$ is rational. Then, let us take c and d integers such that $i + \frac{a}{b} = \frac{c}{d}$. Therefore $i = \frac{c}{d} - \frac{a}{b} = \frac{bc-da}{db}$. Given that a, b, c and d are integers, $bc - da$ and db are also integers, proving that i is rational and contradicting our initial assumption. Therefore, the sum of a rational and an irrational number must be irrational. $\square$

5. Write the numbers $1, 2, \ldots, 2n$ on a blackboard, where n is an odd integer. Pick any two of the numbers, j and k write $|j - k|$ on the board and erase j and k. Continue this process until only one integer is written on the board. Prove that this integer must be odd.

   Hint consider what happens to the parity of the combined sum of the numbers that are left on the blackboard at each stage.

   Solution:
   If j and k are both even or both odd, then their sum and their difference are both even, and we are replacing the even sum $j + k$ by the even difference $|j - k|$, leaving the parity of the total unchanged. If j and k have different parities, then erasing them changes the parity of the total, but their difference $|j - k|$ is odd, so adding this difference restores the parity of the total. Therefore the integer we end up with at the end of the process must have the same parity as $1+2+\ldots+(2n)$. It is easy to compute this sum. If we add the first and last terms we get $2n + 1$; if we add the second and next-to-last terms we get $2 + (2n - 1) = 2n + 1$; and so on. In all we get n sums of $2n + 1$, so the total sum is $n(2n + 1)$. If n is odd, this is the product of two odd numbers and therefore is odd, as desired.

   $\square$

   Solutions (to the last question on the sheet) must be handed in on paper at the ITO by Wednesday, 30 September, 4:00pm.