DMMR Tutorial sheet 6

Complexity, Binomial Coefficient

October 28, 2015

Some of the exercises for this tutorial are taken the book: Kenneth Rosen, Discrete Mathematics and its Applications, 7th Edition, McGraw-Hill, 2012.

1. Find the least integer n such that $f(x)$ is $O(x^n)$ for each of these functions.

   (a) $f(x) = 2x^3 + x^2 \log x$

   (b) $f(x) = 3x^3 + (\log x)^4$

   (c) $f(x) = (x^4 + x^2 + 1)/(x^3 + 1)$

   Solution:

   (a) It is clear that n is at least 3. Moreover, 2 is $O(1)$ and $\log x$ is $O(x)$. Thus, the multiplications $2x^3$ and $x^2 \log x$ are $O(1 \cdot x^3) = O(x^2 \cdot x) = O(x^3)$. Then, the sum $2x^3 + x^2 \log x$ is also $O(x^3)$. Therefore, $n = 3$.

   (b) It is clear that n is at least 3 and $3x^3$ is $O(x^3)$. Moreover, $(\log x)^4$ is $O(x)$. Thus, the sum $3x^3 + (\log x)^4$ is $O(x^3)$. Therefore, $n = 3$.

   (c) We rewrite the expression as $x^4/(x^3+1)+(x^2+1)/(x^3+1)$. It is clear that $(x^2+1)/(x^3+1)$ is $O(1)$. Moreover, for $x > 0$ we have that $x^4/(x^3 + 1) < x$ and thus $x^4/(x^3 + 1)$ is $O(x)$. Thus, $x^4/(x^3 + 1) + (x^2 + 1)/(x^3 + 1)$ is $O(x)$. To prove that it is not $O(1)$, we show that for any positive constant c, if $x > c + 1$ then $x^4/(x^3 + 1) > c$. This is true because for any c positive $(c + 1)^4 = c(c + 1)^3 + (c + 1) > c(c + 1)^3 + c = c((c + 1)^3 + 1)$. Thus, $(c + 1)^4/((c + 1)^3 + 1) > c$. Thus, we have shown that $x^4/(x^3 + 1)$ exceeds any constant at some point and thus $x^4/(x^3 + 1)$ is not $O(1)$. From this we can conclude that $n = 1$.

   □

2. Let $f_1(x)$ and $f_2(x)$ be functions from the set of real numbers to the set of positive real numbers. Show that if $f_1(x)$ and $f_2(x)$ are both $\Theta(g(x))$, where $g(x)$ is a function from the set of real numbers to the set of positive real numbers, then $f_1(x) + f_2(x)$ is $\Theta(g(x))$. Is this still true if $f_1(x)$ and $f_2(x)$ can take negative values?

   Solution:
   Let l_1, u_1, k_1 and l_2, u_2, k_2 be witnesses that $f_1(x)$ and $f_2(x)$ are $\Theta(g(x))$, respectively. Let $l = l_1 + l_2$, $u = u_1 + u_2$ and $k = \max(k_1, k_2)$. Then, for $x > k$ we have that $l_1|g(x)| < |f_1(x)| < u_1|g(x)|$ and $l_2|g(x)| < |f_2(x)| < u_2|g(x)|$. If we sum both inequalities we get that for all $x > k$ we have that $l|g(x)| = (l_1 + l_2)|g(x)| = l_1|g(x)| + l_2|g(x)| < |f_1(x)| + |f_2(x)| < u_1|g(x)| + u_2|g(x)| = (u_1 + u_2)|g(x)| = u|g(x)|$. Using the fact that $f_1(x)$ and $f_2(x)$ only take positive values we can see that $|f_1(x)| + |f_2(x)| = |f_1(x) + f_2(x)|$. Thus l, u and k are witnesses that $f_1(x) + f_2(x)$ is $\Theta(g(x))$.

Take $g(x) = f_1(x) = x$ and $f_2(x) = -x$ to show that if the functions can take negative values then the result is not true. It is trivial to check that $f_1(x)$ and $f_2(x)$ are both $\Theta(g(x))$. However, $f_1(x) + f_2(x) = 0$, which is not $\Theta(g(x))$ because for any l, $l|g(x)|$ exceeds 0 for every $x \neq 0$. $\square$

3. Seven women and nine men are on the faculty in the mathematics department at a school.

   (a) How many ways are there to select a committee of five members of the department if at least one woman must be on the committee?

   (b) How many ways are there to select a committee of five members of the department if at least one woman and at least one man must be on the committee?

   Solution:

   (a) The total number of ways to select a committee with 5 members of the department is $\binom{16}{5}$. Of those, $\binom{9}{5}$ have no women in it. Thus, those with at least one woman are $\binom{16}{5} - \binom{9}{5}$. Alternatively, we can divide in four cases and count separately the choices with one woman $\binom{7}{1}\binom{9}{4}$, with two women $\binom{7}{2}\binom{9}{3}$, with three $\binom{7}{3}\binom{9}{2}$, with four $\binom{7}{4}\binom{9}{1}$ and with five $\binom{7}{5}$. Thus, the answer is

   $$\binom{7}{1}\binom{9}{4} + \binom{7}{2}\binom{9}{3} + \binom{7}{3}\binom{9}{2} + \binom{7}{4}\binom{9}{1} + \binom{7}{5}.$$

   (b) Applying a similar reasoning the answer is $\binom{16}{5} - \binom{9}{5} - \binom{7}{5}$, or

   $$\binom{7}{1}\binom{9}{4} + \binom{7}{2}\binom{9}{3} + \binom{7}{3}\binom{9}{2} + \binom{7}{4}\binom{9}{1}.$$

   $\square$

4. What is the coefficient C_8 of x^8y^9 in the expansion of $(3x + 2y)^{17}$ when it is rewritten as a polynomial of the form:

   $$\sum_{i=0}^{17} C_i x^i y^{17-i}$$

   Solution:
   The binomial theorem tells us that the expansion of $(3x + 2y)^{17}$ is

   $$\sum_{i=0}^{17} \binom{17}{i} (3x)^i (2y)^{17-i} = \sum_{i=0}^{17} \binom{17}{i} 3^i x^i 2^{17-i} y^{17-i}.$$

   Thus, $C_8 = \binom{17}{8} 3^8 2^9$. $\square$

5. Prove the following identity holds for all non-negative integers n,r and k, such that $r \leq n$, and $k \leq r$:

   $$\binom{n}{r} \cdot \binom{r}{k} = \binom{n}{k} \cdot \binom{n-k}{r-k}$$

   Solution:
   We show two different proofs:

- Combinatorial: $\binom{n}{r}$ is the number of possibilities to choose r elements from a set of n elements.

  Now consider a set S of n elements. We split S into three disjoint subsets:

  - S_1 contains k elements
  - S_2 contains $r-k$ elements
  - S_3 contains $n-r$ elements

  In total all elements are distributed into one of the sets since $(n-r)+(r-k)+k=n$.

  Now the possibilities to choose S_1,S_2 and S_3 can be calculated in two different ways: We could split S into S_3 and $S_1 \cup S_2$ first. For this we need to choose the r elements for $S_1 \cup S_2$ from S which calculates to $\binom{n}{r}$. Then we split $S_1 \cup S_2$ into S_1 and S_2. This calculates to $\binom{r}{k}$. In total we get $\binom{n}{r} \cdot \binom{r}{k}$ possibilities to choose S_1, S_2, S_3.
  On the other hand we can split S into S_1 and $S_2 \cup S_3$ first. There are $\binom{n}{k}$ possibilities. Then splitting $S_2 \cup S_3$ into S_2 and S_3 is possible in $\binom{n-k}{r-k}$, since $S_2 \cup S_3$ has $n-r+r-k=n-k$ elements and $r-k$ are chosen for S_2. So in total there are $\binom{n}{k} \cdot \binom{n-k}{r-k}$ possibilities to split S in this way.

  Since the two procedures produce the same partitions, the number of possible outcomes have to be equal and we get $\binom{n}{r} \cdot \binom{r}{k} = \binom{n}{k} \cdot \binom{n-k}{r-k}$

- By the formula:

$$\begin{aligned}
\binom{n}{r} \cdot \binom{r}{k} &= \frac{n!}{r! \cdot (n-r)!} \cdot \frac{r!}{k!(r-k)!} \\
&= \frac{n!}{(n-r)! \cdot k! \cdot (r-k)!} \cdot \frac{r!}{r!} \\
&= \frac{n!}{(n-r)! \cdot k! \cdot (r-k)!} \cdot \frac{(n-k)!}{(n-k)!} \\
&= \frac{n!}{k! \cdot (n-k)!} \cdot \frac{(n-k)!}{(r-k)! \cdot (n-r)!} \\
&= \frac{n!}{k! \cdot (n-k)!} \cdot \frac{(n-k)!}{(r-k)! \cdot (n-k-(r-k))!} \\
&= \binom{n}{k} \cdot \binom{n-k}{r-k}
\end{aligned}$$

$\square$

Solutions (to the last question on the sheet) must be handed in on paper at the ITO by Wednesday, 4 November, 4:00pm.