# DMMR Tutorial sheet 2

Sets, Relations

September 30, 2015

Some of the exercises for this tutorial are taken from Chapter 2 and 9 of the book: Kenneth Rosen, Discrete Mathematics and its Applications, 7th Edition, McGraw-Hill, 2012.

1. Show that the subset relation $\subseteq$ is reflexive.

   **Solution:**
   The claim to prove is that for any $A$, $A \subseteq A$ which is clearly true since for any $x$ if $x \in A$ then $x \in A$.

   It is also transitive: given sets $A, B, C$ with $A \subseteq B$ and $B \subseteq C$ it also holds $A \subseteq C$

   By the definition of the relation $\subseteq$ We need to show that every element of $A$ is also an element of $C$. Let $x \in A$. Then since $A \subseteq B$, we can conclude that $x \in B$. Furthermore, since $B \subseteq C$, the fact that $x \in B$ implies that $x \in C$, as we wished to show $\square$

2. The absorption law for sets is $A \cup (A \cap B) = A$. Give a proof that $A \cup (A \cap B) = A$.

   **Solution:**
   This proof is done in two parts: $A \cup (A \cap B) \subseteq A$ and $A \cup (A \cap B) \supseteq A$

   - For the first consider an element $x$ in $A \cup (A \cap B)$. From the definition of $\cup$ we know that either $x \in A$ or $x \in (A \cap B)$. In the first case we are done. In the other case we know that $x$ is both in $A$ and in $B$ from the definition of $\cap$. Therefore we get $x \in A$ in all cases.
   - Consider an element $x \in A$. From the definition of $\cup$ we immediately get $x \in A \cup (A \cap B)$.

   $\square$

3. For each of the following relations on the set of all real numbers, determine whether it is reflexive, symmetric, antisymmetric, and/or transitive, where $(x, y)$ are related if and only if

   (a) $x - y$ is a rational number.
   (b) $x = 2y$.
   (c) $xy \geq 0$.
   (d) $xy = 0$.
   (e) $x = 1$.
   (f) $x = 1$ or $y = 1$.

   **Solution:**

   (a) **Reflexive**: Yes, because $x - x = 0 \in \mathbb{Q}$ **Symmetric**: Yes, because if $x - y$ is rational then $-(x - y) = y - x$ is also rational. **Antisymmetric**: No, because $2 - 1$ is rational and $1 - 2$ is rational, but $1 \neq 2$. **Transitive**: Yes, because if $x - y$ and $y - z$ are both rational, then $x - z = (x - y) + (y - z)$ is also rational.

(b) **Reflexive**: No, because $1 \neq 2 \cdot 1$. **Symmetric**: No. Let $x = 2$ and $y = 1$. Then, $x = 2 \cdot y$, but $y \neq 2 \cdot x$. **Antisymmetric**: Yes, because if $x = 2y$ and $y = 2x$ then $x = 4x$, which means that $x = 0 = y$. **Transitive**: No. Let $x = 4$, $y = 2$ and $z = 1$. Then, $x = 2 \cdot y$ and $y = 2 \cdot z$ but $x \neq 2 \cdot z$.

(c) **Reflexive**: Yes, because $x^2 \geq 0$. **Symmetric**: Yes, because if $xy \geq 0$ then $yx = xy \geq 0$. **Antisymmetric**: No, because $1 \cdot 2 = 2 \cdot 1 \geq 0$, but $2 \neq 1$. **Transitive**: No. Let $x = -1$, $y = 0$ and $z = 1$. Then, $xy = (-1) \cdot 0 \geq 0$ and $yz = 0 \cdot 1 \geq 0$ but $xz = (-1) \cdot 1 = -1 < 0$.

(d) **Reflexive**: No, because $1 \cdot 1 \neq 0$. **Symmetric**: Yes, because if $xy = 0$ then $yx = xy = 0$. **Antisymmetric**: No. Let $x = 1$ and $y = 0$. Then $xy = 0 = yx$, but $0 \neq 1$. **Transitive**: No. Let $x = z = 1$ and $y = 0$. Then, $xy = 0 = yz$ but $xz = 1 \neq 0$.

(e) **Reflexive**: No. Let $x = 2$. Then, $(x, x)$ is not in the relation. **Symmetric**: No. $(1, 2)$ is in the relation, but $(2, 1)$ is not. **Antisymmetric**: Yes, because if $(x, y)$ and $(y, x)$ both are in the relation then $x = 1 = y$. **Transitive**: Yes, because if $(x, y)$ and $(y, z)$ are both in the relation, then $x = 1$, which means that $(x, z) = (1, z)$, which is in the relation.

(f) **Reflexive**: No, because $(2, 2)$ is not in the relation. **Symmetric**: Yes, because if $x = 1 \vee y = 1$ then $y = 1 \vee x = 1$. **Antisymmetric**: No, because $(1, 2)$ and $(2, 1)$ are in the relation, but $2 \neq 1$. **Transitive**: No. Let $x = z = 2$ and $y = 1$. Then, $(x, y) = (2, 1)$ and $(1, 2)$ are in the relation, but $(x, z) = (2, 2)$ is not.

□

4. Let $A, B, C$ be sets. Derive a formula for $|A \cup B \cup C|$, which only uses the cardinality $|\cdot|$, intersection $\cap$ and arithmetic operators. **Solution:**

$$
\begin{aligned}
|A \cup B \cup C| &= |A \cup (B \cup C)| \\
&= |A| + |B \cup C| - |A \cap (B \cup C)| \\
&= |A| + (|B| + |C| - |B \cap C|) - |(A \cap (B \cup C)| \\
&= |A| + |B| + |C| - |B \cap C| - |(A \cap B) \cup (A \cap B)| \\
&= |A| + |B| + |C| - |B \cap C| - (|A \cap B| + |A \cap C| - |(A \cap B) \cap (A \cap C)| \\
&= |A| + |B| + |C| - |B \cap C| - (|A \cap B| + |A \cap C| - |(B \cap A) \cap (A \cap C)| \\
&= |A| + |B| + |C| - |B \cap C| - (|A \cap B| + |A \cap C| - |B \cap (A \cap (A \cap C))| \\
&= |A| + |B| + |C| - |B \cap C| - (|A \cap B| + |A \cap C| - |B \cap ((A \cap A) \cap C)| \\
&= |A| + |B| + |C| - |B \cap C| - |A \cap B| - |A \cap C| + |B \cap A \cap C|
\end{aligned}
$$

□

5. Many program analysis methods rely on call graphs. A call graph is a relation $R_C$ and a pair $(f, g)$ of function names is in $R_C$, iff the body of function $f$ calls the function $g$. For example for the function $f$

```
function f() {
  g();
  h()
}
```

the pairs $(f, g)$ and $(f, h)$ are in the relation $R_C$.

The *transitive closure* of the relation $R_C$ written as $Trans(R_C)$ is a new relation, which contains a pair $(f, f')$, if there is a chain $(f, f_2), (f_2, f_3), ..., (f_{n-1}, f')$ of pairs all contained in $R_C$. Formally this can be defined as

$$Trans(R_C) = \{(f, f') | \exists(f_1, ..., f_n) \text{ with } f = f_1 \wedge f' = f_n \wedge \forall i \in \{1, ..., (n-1)\}(f_i, f_{i+1}) \in R_C\}$$

The *symmetric closure* of the relation $R_C$ is a new relation written as $Sym(R_C)$, which contains all pairs $(f, g)$ from $R_C$ along with their corresponding pairs $(g, f)$.

(a) Prove that $Trans(R_C)$ is transitive

(b) Explain what information the relations $Trans(Sym(R_C))$ and $Sym(Trans(R_C))$ contain about the program.

(c) Decide which of the two relations $Trans(Sym(R_C))$ and $Sym(Trans(R_C))$ subsumes the other, give a formal proof of your claim and show an example relation $R_C$ and a pair which is contained in exactly one of them.

**Solution:**

(a) (3 points) For transitivity it is to show that for two pairs $(f, g)$ and $(g, h)$ in $Trans(R_C)$ also the transitive pair $(f, h)$ is contained in the relation. Since $(f, g) \in Trans(R_C)$ we know that there exists a chain $(f_1, ..., f_n)$ with $(f_i, f_{i+1}) \in R_C$ for all $i \in \{1, ..., n-1\}$. Furthermore due to $(g, h) \in Trans(R_C)$ we know there exists a chain $(f'_1, ..., f'_m)$ with $(f'_i, f'_{i+1}) \in R_C$ for all $i \in \{1, ..., m-1\}$. By renaming $(f'_2, ..., f'_m)$ to $(f_{n+1}, ..., f_{n+m-1})$ we construct a chain $(f_1, ..., f_{n+m-1})$ with $f_1 = f$, $f_{n+m-1} = h$ and $(f_i, f_{i+1}) \in R_C$ for all $i \in \{1, ..., n+m-1\}$, which is exactly the definition of the transitive closure and therefore $Trans(R_C)$ contains the pair $(f, h)$.

(b) (1 point) The relation $Trans(Sym(R_C))$ contains all pairs of functions $(f, g)$ where there is a chain from $f$ to $g$, where each element in the chain or its mirror pair is in $R_C$. The only functions that are not in $Trans(Sym(R_C))$ are those pairs, which are from completely separate parts of the program. Therefore if a pair $(f, g)$ is not contained in this relation, the program can be split into two programs, one of which contains $f$, the other contains $g$.

(1 point) The relation $Sym(Trans(R_C))$ contains all pairs, where there is either a chain from $f$ to $g$ or a chain from $g$ to $f$, where each element of the chain is in $R_C$. Therefore each pair $(f, g)$ is in this relation, iff the two function $f$ and $g$ could be called in the same execution of the program. This does not mean they can be separated, as two functions which are never called in the same execution together might still need the same sub-function.

(c) (2 points) The example given above is an example, that $Trans(Sym(R_C))$ is strictly bigger than $Sym(Trans(R_C))$: the pair $(g, h)$ is only contained in the first: $(f, g), (f, h) \in R_C$, therefore $(g, f), (f, h) \in Sym(R_C)$ and thus $(g, h) \in Trans(Syn(R_C))$. On the other hand $Trans(R_C) = R_C$ and therefore neither $(g, h)$ nor $(h, g)$ are in $Trans(R_C)$ and $(g, h) \notin Sym(Trans(R_C))$.

(1 point) It holds that $Trans(Sym(R_C)) \supseteq Sym(Trans(R_C))$

(3 points) To show that $Trans(Sym(R_C))$ subsumes $Sym(Trans(R_C))$ consider a pair $(f, g) \in Sym(Trans(R_C))$. From this we know that either $(f, g)$ or $(g, f)$ is in $Trans(R_C)$. By the definition of $Trans()$ we know that therefore the needed chain is contained in $R_C$. Since $Sym(R_C)$ has more elements than $R_C$, we can deduce that this chain and the same chain backwards is contained in $Sym(R_C)$. Therefore $(f, g)$ and $(g, f)$ are both contained in $Trans(Sym(R_C))$.

□

**Solutions (to the last question on the sheet) must be handed in on paper at the ITO by Wednesday, 7 October, 4:00pm.**