DMMR Tutorial sheet 4

Inductions, Modulo, Primes

October 14, 2014

Some of the exercises for this tutorial are taken from the book: Kenneth Rosen, Discrete Mathematics and its Applications, 7th Edition, McGraw-Hill, 2012.

1. Prove the following claim by induction: "Every positive integer $n \in \mathbb{N}, n > 1$ is divisible by a prime number"

2. Use strong induction to show that every positive integer n can be written as a sum of distinct powers of two, that is, as a sum of a subset of the integers $2^0 = 1$, $2^1 = 2$, $2^2 = 4$, and so on. [Hint: For the inductive step, separately consider the case where $k + 1$ is even and where it is odd. When it is even, note that $(k + 1)/2$ is an integer.]

   Before beginning your proof, state the property (the one you are asked to prove for every integer n) in completely formal notation with all quantifiers.

3. What is wrong with this "proof"?

   "Theorem" For every positive integer n, if x and y are positive integers with $\max(x, y) = n$, then $x = y$.

   Base case: Suppose that $n = 1$. If $\max(x, y) = 1$ and x and y are positive integers, we have $x = 1$ and $y = 1$.

   Inductive Hypothesis: Let k be a positive integer. Assume that whenever $\max(x, y) = k$ and x and y are positive integers, then $x = y$. Now let $\max(x, y) = k + 1$, where x and y are positive integers.

   Inductive Step: Then $\max(x - 1, y - 1) = k$, so by the inductive hypothesis, $x - 1 = y - 1$. It follows that $x = y$, completing the inductive step.

4. Let a, b, c, d, m be integers. Find counter examples to each of the following statements about congruences:

   (a) if $ac \equiv bc \pmod{m}$ with $m \geq 2$, then $a \equiv b \pmod{m}$

   (b) if $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$ with c and d positive and $m \geq 2$, then $a^c \equiv b^d \pmod{m}$

5. Let $n \geq 0$ be an integer. Prove by induction:

   (a) 8 divides $3^{2n+2} + 7$

   (b) 64 divides $3^{2n+2} + 56n + 55$

   Solutions (to the last question on the sheet) must be handed in on paper at the ITO by Wednesday, 21 October, 4:00pm. Please post it into the grey metal box on the wall outside the ITO.