DMMR Tutorial 7 (with solutions)

Graphs

November 4, 2015

1. Consider a (simple, undirected) bipartite graph $G = (V, E)$, with bipartition (V_1, V_2). In other words, $V = V_1 \cup V_2$, $V_1 \cap V_2 = \emptyset$, and for every edge $e \in E$, $e = \{u, v\}$ such that $u \in V_1$ and $v \in V_2$.
   Suppose that every vertex in V has degree exactly 7.
   Prove that $|V_1| = |V_2|$, and that there must exist a perfect matching in such a bipartite graph.

   [Hint: apply the generalized pigeonhole principle in order to show that the key condition for Hall's theorem does hold.]

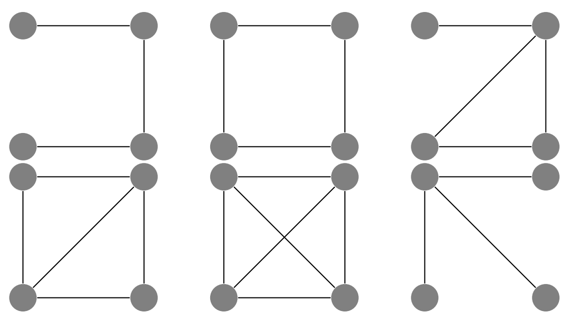
   Solution. For $k \geq 1$, let $G = (V, E)$ be any bipartite graph with bipartition (V_1, V_2), which is "k-regular", meaning that every vertex in V has degree exactly k.

   First, let us show that $|V_1| = |V_2|$. Since G is k-regular, the total number of edges is both $k|V_1|$ and $k|V_2|$. Thus $k|V_1| = k|V_2|$. In other words, $|V_1| = |V_2|$.

   Recall that Hall's theorem says there is a perfect matching in such a bipartite graph if and only if for every non-empty subset $A \subseteq V_1$, $|A| \leq N(A)$, where $N(A)$ denotes the set of neighbors of vertices in A. We prove by contradiction that this condition holds for any k-regular bipartite graph. Suppose not. Then there is a non-empty subset $C \subseteq V_1$, such that $|C| > |N(C)|$. This means that the $k|C|$ edges incident on vertices in C are all incident on vertices in $N(C)$. But then, by the generalized pigeonhole principle, this means that some vertex in $N(C)$ has degree greater than k. But this contradicts the fact that G is k-regular. Thus, Hall's condition holds and any k-regular bipartite graph has a perfect matching.

2. How many non-isomorphic simple undirected graphs are there with exactly 4 vertices? Justify your answer.

   Solution. Let us divide in cases, according to the number of vertices of the largest connected subgraph (LCS). Clearly, there is only one graph where the LCS has 1 vertex (all vertices have degree 0). There are 2 graphs where LCS has 2 vertices. There are 2 graphs where LCS has 3 vertices, and there are 6 graphs where the LCS has 4 vertices. These six graphs are represented below.



   Thus, the total number of non-isomorphic simple undirected graphs with 4 vertices is $1 + 2 + 2 + 6 = 11$.

3. Suppose $G = (V, E)$ is a directed graph, and u and v are vertices of G. Show that either u and v are in the same strongly connected component of G, or they are in disjoint strongly connected components of G.

   Solution. Suppose the strongly connected components C_u of u and C_v of v are not disjoint, i.e., there is a non-empty intersection. We will prove that the subgraph with vertices $C_u \cup C_v$ is strongly connected. It suffices to prove that any point in C_u is strongly connected to any point in C_v. Let $u' \in C_u$ and $v' \in C_v$, and $x \in C_u \cap C_v$. Clearly, there are paths from $u'tox$ and from $xtov'$. Thus, there is a path from u' to v'. Similarly we prove that there is a path from v' to u'. Thus, the subgraph with vertices $C_u \cup C_v$ is strongly connected.

4. Recall that the n-dimensional **hypercube**, or n-cube, is the simple undirected graph whose nodes are bit strings of length n, and such that there is an edge between a pair of nodes if and only if their bit strings differ in exactly one bit position.

   (a) For what values of $n \geq 1$ does the n-cube have an Euler circuit? (1 point)

   (b) Prove by induction that for all $n \geq 2$, the n-cube has a Hamiltonian circuit. (9 points)

   Solution.

   (a) Any vertex in a n-cube has degree n and every n-cube is connected (to give a path, change the bits one by one from one vertex to the other). An Euler circuit exists in a connected graph if and only if every vertex has even degree. Thus, there is an Euler circuit if and only if n is even.

   (b) The base case ($n = 2$) is trivial. Then, let us assume that there is a Hamiltonian circuit for the n-cube and prove that there must also be one for the $(n+1)$-cube. Take the $(n+1)$-cube and consider the subgrah G_0 with the vertices of the form $(b_1, \ldots, b_n, 0)$, and the subgraph G_1 with the vertices of the form $(b_1, \ldots, b_n, 1)$. Clearly, G_0 and G_1 are isomorphic to the n-cube. By the inductive hypothesis we can find a Hamiltonian circuit for each. Let us take the 'same' circuit for both (we can get one of the circuits by only changing the last coordinate of each vertex in the other circuit). Now take two vertices adjacent in the Hamiltonian path for the G_0: $(x_1, \ldots, x_n, 0)$, $(y_1, \ldots, y_n, 0)$, and the corresponding vertices in G_1: $(x_1, \ldots, x_n, 1)$, $(y_1, \ldots, y_n, 1)$. Then, there is a Hamiltonian circuit $C_0 = (x_1, \ldots, x_n, 0), \ldots, (y_1, \ldots, y_n, 0)$ of G_0 and one $C_1 = (y_1, \ldots, y_n, 1) \ldots (x_1, \ldots, x_n, 1)$ of G_1 going in the opposite direction. The sequence that results from composing the paths C_0 and C_1 is clearly a Hamiltonian circuit of the $(n+1)$-cube.

5. Recall that in a (simple undirected) graph, a **cycle** (or circuit) is a **walk** (or path) that begins and ends in the same vertex. A **simple cycle** is a cycle on which no edge occurs more than once.

   Prove that, for each integer k greater than 2, the existence of a simple cycle of length k is an **isomorphism invariant**, meaning that if one graph has a simple cycle of length k, but another graph does not, then the two graphs can not be isomorphic.

   Solution.

   Suppose we have two isomorphic graphs, $G = (V, E)$ and $H = (V', E')$, and that there is a simple cycle of length k in G ($k > 2$). We prove that H must also have a simple cycle of length k.

   Since G and H are isomorphic, there is a bijective function $f : V \to V'$, such that $\{a, b\} \in E$ if and only if $\{f(a), f(b)\} \in E'$. G contains a simple cycle of length k. In other words, in G there is a cycle $x_1, e_1, x_2, e_2, \ldots x_k, e_k, x_1$ where $x_i \in V$, where $e_i = \{x_i, x_{i+1}\} \in E$ for $1 \leq i \leq k - 1$, and $e_k = \{x_k, x_1\} \in E$, and where the edges $e_1, \ldots, e_k$ are all distinct.

Consider $y_i = f(x_i)$, $i \in \{1, \ldots, k\}$. Using the isomorphism given by the function f, we know that there is a circuit in H given by:

$$y_1 e'_i y_2 \ldots y_k e'_k y_1$$

such that $e'_i = \{y_i, y_{i+1}\} \in E'$, for $i \in \{1, \ldots, k-1\}$, and $e'_k = \{y_k, y_1\} \in E'$. Moreoever, this circuit must also be a simple circuit, meaning that the edges $e'_1, \ldots, e'_k$ must all be distinct. For suppose, for contradiction, that $e'_i = e'_j$, for $i < j$. Since f is a bijection between V and V', this implies we must also have $e_i = e_j$. But by assumption this is not the case.

Thus, H must also have a simple cycle of length k.

The distribution of the points for question 5 is: 11 points