Problem Set 2

Bijoy Ratan Ghosh

1. Introducing Physical Investment

(a) $$\pi_t\, dt = \underbrace{q_{t+\Delta t}\, k^i_{t+\Delta t} - q_t k^i_t}_{①} + [a^i(\kappa^i_t) k^i_t - \iota^i_t k^i_t]\Delta t$$

return rate $\dfrac{\pi_t}{q_t k_t}$

$$dr^k_t(\iota_t) = \underbrace{\frac{a^i(\kappa^i_t) - \iota^i_t}{q_t}}_{\text{Dividend yield}} + \frac{d(q_t k^i_t)}{q_t k^i_t}$$

As, $\Delta t \to 0$

$$\pi_t\, dt = d(q_t k^i_t) + [a^i(\kappa^i_t) k^i_t - \iota^i_t k^i_t]\, dt$$

$$\frac{\pi_t}{q_t k^i_t} dt = \frac{d(q_t k^i_t)}{q_t k^i_t} + \frac{a^i(\kappa^i_t) k^i_t - \iota^i_t k^i_t}{q_t k^i_t} dt$$

$$= \underbrace{\frac{d(q_t k^i_t)}{q_t k^i_t}}_{①} + \left[\underbrace{\frac{a^i(\kappa^i_t) k^i_t}{q_t k^i_t}}_{②} - \underbrace{\frac{\iota^i_t k^i_t}{q_t k^i_t}}_{③}\right] dt$$

So, $$\frac{\pi_t}{q_t k^i_t} = \underbrace{\frac{\frac{d}{dt}(q_t k^i_t)}{q_t k^i_t}}_{①} + \left[\underbrace{\frac{a^i(\kappa^i_t)}{q_t}}_{②} - \underbrace{\frac{\iota^i_t}{q_t}}_{③}\right]$$

(b) So, $$dr^k_t(\iota_t) = \frac{d(q_t k_t)}{q_t k_t} + \frac{(a^i(\kappa^i) - \iota^i_t)}{q_t} dt$$

Now, $$\frac{dk_t}{k_t} = [\Phi(\iota^i_t) - \delta]\, dt + \sigma dZ$$

Postulate $$\frac{dq_t}{q_t} = \mu^q_t\, dt + \sigma^q_t dZ$$

So, By Ito's $\rightarrow$
$$\frac{d(q_t k_t)}{q_t k_t} = [\Phi(\iota_t) - \delta + \mu^q_t + \sigma\sigma^q_t]\, dt + [\sigma + \sigma^q_t]\, dZ$$

So, $$dr^k(\iota_t) = \left[\frac{a^i(\kappa^i_t) - \iota^i_t}{q_t} + \Phi(\iota^i_t) - \delta + \mu^q_t + \sigma\sigma^q_t\right] dt + (\sigma + \sigma^q_t)\, dZ$$

So,
Agents $i \in \{e, h\}$ choose ι_t to maximize $\Rightarrow$

$$\max_{\iota_t} \mathbb{E}\, dr_t^k(\iota_t)/dt$$

$$\Rightarrow \max_{\iota_t} \left[\frac{a^i(k_t^i) - \iota_t^i}{q_t} + \Phi(\iota_t^i) - \delta + \mu_t^q + \sigma\sigma_t^q\right]$$

FOC: $\iota_t^i : \Rightarrow \quad -\frac{1}{q_t} + \Phi'(\iota_t^i) = 0$

$$\Leftrightarrow \boxed{\Phi'(\iota_t^i) = \frac{1}{q_t}} \quad \text{(Tobin's Q condition)}$$

(c) For $\Phi(\iota) = \frac{1}{\phi}\log(\phi\iota + 1)$,

$$\Phi'(\iota) = \frac{1}{\phi(\phi\iota + 1)} \cdot \phi = \frac{1}{\phi\iota + 1}$$

So, for q_t optimal investment $\Rightarrow$

$$\frac{1}{\phi\iota_t + 1} = \frac{1}{q_t}$$

$$\Rightarrow \phi\iota_t + 1 = q_t$$

So, $\boxed{\iota_t = \frac{1}{\phi}(q_t - 1)}$

2. Basak-Cuoco

1>(a) Goods market clearing implies $\Rightarrow$

$$a^e K_t - \iota_t K_t = \rho^e \eta_t q_t K_t + \rho^h (1-\eta_t) q_t K_t$$

So, $\boxed{q(\eta) = \frac{1+\phi a^e}{1+\phi\hat{\rho}(\eta)}}$ $\boxed{\iota(\eta) = \frac{a^e - \hat{\rho}(\eta)}{1+\phi\hat{\rho}(\eta)}}$ (Inserting iota value from previous part)

(b)

Now, $q(\eta) = \dfrac{1+a^e\phi}{1+\phi\hat{\rho}(\eta)}$

so, $q' = -\dfrac{1+a^e\phi}{(1+\phi\hat{\rho}(\eta))^2}\,\phi(\rho^e-\rho^h)$

and $q'' = 2\,\dfrac{(1+a^e\phi)(\phi(\rho^e-\rho^h))^2}{(1+\phi\hat{\rho}(\eta))^3}$

so, By Ito's

$$\sigma^q(\eta) = -\frac{\phi(\rho^e-\rho^h)}{(1+\phi\hat{\rho}(\eta))}\cdot\sigma^\eta(\eta)\,\eta \qquad -①$$

By Capital mkt. clearing $\Rightarrow$

$$\theta^{k,e}_t = \frac{1}{\eta_t}$$

By Ito's using $\eta_t = N^e_t/N_t = N^e_t/q_tK_t$,

$$\frac{d\eta^e_t}{\eta^e_t} = \left[-\rho^e + r_t + \theta^{k,e}_t\varsigma^e_t(\sigma+\sigma^q_t) - r_t + \frac{a^e-\iota_t}{q_t} - \varsigma^e_t(\sigma+\sigma^q_t) + (\sigma+\sigma^q_t)^2(1-\theta^{k,e}_t)\right]dt - (1-\theta^{k,e}_t)(\sigma+\sigma^q_t)\,dZ$$

so, $\sigma^\eta_t = -\left[1-\dfrac{1}{\eta_t}\right](\sigma+\sigma^q_t)$

$$= \frac{1-\eta_t}{\eta_t}(\sigma+\sigma^q_t) \qquad -②$$

Combining ① & ② $\Rightarrow$

$$\boxed{\begin{aligned}\sigma^\eta_t &= \frac{(1-\eta)}{\eta}\cdot\frac{(1+\phi\hat{\rho}(\eta))}{1+\phi\rho^e}\,\sigma \\ \sigma^q_t &= -(1-\eta)\cdot\frac{\phi(\rho^e-\rho^h)}{1+\phi\rho^e}\,\sigma\end{aligned}}$$

(c)

So, $\mu_t^\eta = -\rho^e + \cancel{\iota_t} + \theta_t^{k,e} \varsigma_t^e(\sigma+\sigma_t^q) - \cancel{\iota_t} + \frac{a^e - \iota_t}{q_t} - \varsigma_t^e(\sigma+\sigma_t^q) + (\sigma+\sigma_t^q)^2(1-\theta_t^{k,e})$

$= -\rho^e + \hat{\rho} - (1-\theta_t^{k,e})\varsigma_t^e(\sigma+\sigma_t^q) + (1-\theta_t^{k,e})(\sigma+\sigma_t^q)^2$

$= -\rho^e + \hat{\rho} - (1-\theta_t^{k,e})(\sigma+\sigma_t^q)(\varsigma_t^e - (\sigma+\sigma_t^q))$

$= -\rho^e + \hat{\rho} + (1-\theta_t^{k,e})^2(\sigma+\sigma_t^q)^2$

So, $\boxed{\mu^\eta(\eta) = -\rho^e + \hat{\rho} + \left[\frac{(1-\eta)(1+\phi\hat{\rho}(\eta))}{\eta(1+\phi\rho^e)}\right]^2 \sigma^2}$

2. MATLAB plot.

3. $\sigma + \sigma^q = \frac{1+\phi\hat{\rho}(\eta)}{1+\phi\rho^e}\sigma$

So, $\frac{\sigma+\sigma^q}{\sigma} = \frac{1+\phi\hat{\rho}}{1+\phi\rho^e}$

$\rho^e > \rho^h$

$\Rightarrow \hat{\rho} < \rho^e$

So, $1+\phi\hat{\rho} < 1+\phi\rho^e$

$\Rightarrow \frac{1+\phi\hat{\rho}}{1+\phi\rho^e} < 1$

So, if $\phi > 0$

$\Rightarrow \frac{\sigma+\sigma^q}{\sigma} < 1$

As, $\phi \to 0$ $\quad \Phi(\iota) \to \iota$

so no adjustment cost in capital investment.

So, this effect is gone

4.
- When $\eta \to 0$ experts become poor, in that situation drift becomes positive. Experts more productive which gives them advantage, though more impatient. So, η pushes back up.
- When $\eta \to 1$ experts dominate, but households' patience make them save more, η pushes back down.

- productivity advantage of Experts balances patience advantage of households, creating a stable interior steady state.