Online Summer School
Macro, Money, and Finance
Problem Set 4

June 10, 2025

Please submit your solutions to the dropbox link by 6/29/2025 23:59 (EDT).

1 Nominal Bonds as Safe Assets and Bubble Mining Seigniorage

Consider the following model, which is a variant of the money model covered in Lecture 10. The setup is as on Slide 7 of Lecture 10 with the following modifications:

- We call the liabilities issued by the government "bonds" and denote their nominal quantity by $\mathcal{B}_t$. Bonds are of infinitesimal duration and pay a floating nominal interest rate i_t.
- There are no transaction costs, $\mathfrak{T}_t \equiv 0$.
- There are no government expenditures, $\mathscr{G}_t \equiv 0$.
- Assume that the government holds the nominal bond growth rate $\mu^{\mathcal{B}}$ and the nominal interest rate i constant over time. Treat these two variables as parameters of the model.

In this problem, we focus on monetary steady state equilibria. You may therefore assume from the outset that $q^{\mathcal{B}}$ and q^K are positive and constant over time. Also, assume $\tilde{\sigma}^2 > \rho$.

1. Solve the model by performing the following steps:

   (a) Determine the return expressions for capital and bonds and simplify as much as possible using the assumptions provided above.
   Solution:
   $$dr_t^{K,\tilde{i}}(\iota) = \left(\frac{a-\iota}{q^K} + \frac{q^{\mathcal{B}}}{q^K}\check{\mu}^{\mathcal{B}} + \Phi(\iota) - \delta\right)\mathrm{d}t + \tilde{\sigma}d\tilde{Z}_t^{\tilde{i}}$$
   $$dr_t^{\mathcal{B}} = \left(\mu^K - \check{\mu}^{\mathcal{B}}\right)dt$$
   They are as on Slide 18 but without price drifts since we have already imposed a steady state.

(b) Characterize the optimal consumption, investment, and portfolio choice of households.
Solution: Optimal consumption:

$$c_t^{\tilde{i}} = \rho n_t^{\tilde{i}}.$$

Optimal investment: as first bullet point on Slide 19, from Tobin's Q condition,

$$q_t^K = \frac{1}{\Phi'(\iota_t^{\tilde{i}})} = 1 + \phi\iota^{\tilde{i}}.$$

In particular, all agents choose same investment rate:

$$\iota^{\tilde{i}} = \iota$$

Optimal portfolio choice: Start from martingale conditions and plug in the expected returns from (a),

$$\begin{aligned} -\check{\mu}^{\mathcal{B}} + \Phi(\iota) - \delta &= \frac{\mathbb{E}_t[dr_t^{\mathcal{B}}]}{dt} = r \\ \frac{a-\iota}{q^K} + \frac{q^{\mathcal{B}}}{q^K}\check{\mu}^{\mathcal{B}} + \Phi(\iota) - \delta &= \frac{\mathbb{E}_t[dr_t^K]}{dt} = r + \tilde{\varsigma}\tilde{\sigma} \end{aligned}$$

where $\tilde{\varsigma}_t\tilde{\sigma}$ is the premium for idiosyncratic risk in capital.

Due to log utility,

$$\tilde{\varsigma} = \tilde{\sigma}^n = (1-\theta^{\tilde{i}})\tilde{\sigma}.$$

Then,

$$\frac{a-\iota}{q^K} + \frac{q^{\mathcal{B}} + q^K}{q^K}\check{\mu}^{\mathcal{B}} = (1-\theta^{\tilde{i}})\tilde{\sigma}^2.$$

(c) Combine the equations derived in (b) with market clearing to determine the equilibrium values for asset prices (q, q^K, $q^{\mathcal{B}}$), investment (ι), primary surpluses (s), the real risk-free rate, the output growth rate, and the inflation rate.
Solution: From goods market clearing condition and optimal consumption,

$$\rho q K_t + \iota K_t = a K_t$$

To solve for q, use $q^K = (1-\vartheta)q$ and plug into optimal investment condition,

$$(1-\vartheta)\frac{a-\iota}{\rho} = (1-\vartheta)q = 1 + \phi\iota \qquad \Rightarrow \qquad \iota = \frac{(1-\vartheta)a - \rho}{1-\vartheta+\phi\rho}.$$

Plugging ι_t expression back into $q = (a-\iota)/\rho$:

$$q = \frac{1+\phi a}{1-\vartheta+\phi\rho}$$

$$q^K = (1-\vartheta)\frac{1+\phi a}{1-\vartheta+\phi\rho}$$

$$q^{\mathcal{B}} = \vartheta \frac{1 + \phi a}{1 - \vartheta + \phi \rho}$$

To solve for ϑ, there is a unique solution that corresponds to the requirement "monetary steady state". Plugging asset market clearing into portfolio choice from (b),

$$0 = \rho + \breve{\mu}^{\mathcal{B}} - (1 - \vartheta)^2 \tilde{\sigma}^2,$$

which is as on Slide 20 but without a price drift. Then,

$$\vartheta = \frac{\tilde{\sigma} - \sqrt{\rho + \breve{\mu}^{\mathcal{B}}}}{\tilde{\sigma}}.$$

Plug it into previous equations delivering essentially the same expressions as in the second column on Slide 22 (for s, we also need to use the government budget constraint).

$$\begin{aligned}
q^{\mathcal{B}} &= \frac{\left(\tilde{\sigma} - \sqrt{\rho + \breve{\mu}^{\mathcal{B}}}\right)(1 + \phi\, a)}{\sqrt{\rho + \breve{\mu}^{\mathcal{B}}} + \phi\, \rho\, \tilde{\sigma}}, \\
q^{K} &= \frac{\sqrt{\rho + \breve{\mu}^{\mathcal{B}}}\,(1 + \phi\, a)}{\sqrt{\rho + \breve{\mu}^{\mathcal{B}}} + \phi\, \rho\, \tilde{\sigma}}, \\
\iota &= \frac{a \sqrt{\rho + \breve{\mu}^{\mathcal{B}}} - \tilde{\sigma}\, \rho}{\sqrt{\rho + \breve{\mu}^{\mathcal{B}}} + \phi\, \rho\, \tilde{\sigma}}, \\
s &= -\,\breve{\mu}^{\mathcal{B}} \frac{\left(\tilde{\sigma} - \sqrt{\rho + \breve{\mu}^{\mathcal{B}}}\right)(1 + \phi\, a)}{\sqrt{\rho + \breve{\mu}^{\mathcal{B}}} + \phi\, \rho\, \tilde{\sigma}}.
\end{aligned}$$

The remaining variables r^f, g, and π are not stated on the slides but can be recovered easily. Because the the output is proportional to capital,

$$g = \mu^K = \Phi(\iota) - \delta.$$

To solve π, starting from the definition $\pi := \mu^{\mathcal{P}}$ and

$$\mathcal{P}_t = \frac{1}{q^{\mathcal{B}}} \frac{\mathcal{B}_t}{K_t} \implies \pi = \mu^{\mathcal{P}} = \mu^{\mathcal{B}} - g.$$

Lastly, by the Fisher equation,

$$r^f = i - \pi = i - \mu^{\mathcal{B}} + g = g - \breve{\mu}^{\mathcal{B}}.$$

2. Investigate the potential for the government to extract seigniorage revenues from bubble mining due to the presence of idiosyncratic risk. For your answer, restrict attention to the special case $\phi \to \infty$ without physical investment. In this case, the capital stock grows at the exogenous rate $g = -\delta$.

   (a) Derive a relationship between real seigniorage revenue per unit of K_t, $\mathfrak{s} := -s$, and the policy variables i, $\mu^{\mathcal{B}}$. Explain intuitively why there is a Laffer curve.

Solution: Start from the flow budget constraint of the government, $\mathfrak{s} = -s = \breve{\mu}^{\mathcal{B}} q^{\mathcal{B}}$. In the limit $\phi \to \infty$, $q^{\mathcal{B}}$ from 1(c) simplifies to

$$q^{\mathcal{B}} = \frac{\left(\tilde{\sigma} - \sqrt{\rho + \breve{\mu}^{\mathcal{B}}}\right) a}{\rho \tilde{\sigma}}.$$

Hence,

$$\mathfrak{s} = \breve{\mu}^{\mathcal{B}} \frac{\left(\tilde{\sigma} - \sqrt{\rho + \breve{\mu}^{\mathcal{B}}}\right) a}{\rho \tilde{\sigma}}, \qquad \breve{\mu}^{\mathcal{B}} = \mu^{\mathcal{B}} - i$$

expresses equilibrium seigniorage $\mathfrak{s}$ as a function of $\mu^{\mathcal{B}}$ and i.

Why is there a Laffer curve? Raising $\breve{\mu}^{\mathcal{B}} = \mu^{\mathcal{B}} - i$ affects $\mathfrak{s} = \breve{\mu}^{\mathcal{B}} q^{\mathcal{B}}$ through both factors in the product, the first is a tax rate effect that tends to raise seigniorage for given $q^{\mathcal{B}}$, the second is a tax base effect that results from behavioral changes of households in response to larger debt dilution, they substitute away from bonds to capital. With this interpretation, the standard Laffer curve intuition applies.

(b) Show that there is a unique maximum $\mathfrak{s}$, let's call it $\bar{\mathfrak{s}}$, and determine how $\bar{\mathfrak{s}}$ is affected by a change in $\tilde{\sigma}$?

Hint: You can do this without taking any first-order conditions by arguing that $\mathfrak{s}$ is concave and by investigating how it is affected by a change in $\tilde{\sigma}$ for fixed $\mu^{\mathcal{B}}$ and i.

Solution: We can rewrite $\mathfrak{s}$ from the last question in the form

$$\mathfrak{s} = \frac{a}{\rho} \left(\breve{\mu}^{\mathcal{B}} - \frac{\breve{\mu}^{\mathcal{B}} \sqrt{\rho + \breve{\mu}^{\mathcal{B}}}}{\tilde{\sigma}} \right)$$

as the difference of a linear and a strictly convex function of $\breve{\mu}^{\mathcal{B}}$. It is therefore strictly concave in $\breve{\mu}^{\mathcal{B}}$, so there is at most one maximum. There is also at least one because the function is not strictly monotonic as $\mathfrak{s} = 0$ both for $\breve{\mu}^{\mathcal{B}} = 0$ and $\breve{\mu}^{\mathcal{B}} = \tilde{\sigma}^2 - \rho > 0$.

For the dependence of $\bar{\mathfrak{s}}$ on $\tilde{\sigma}$ we follow the hint and first check how $\mathfrak{s}$ depends on $\tilde{\sigma}$ for fixed $\breve{\mu}^{\mathcal{B}}$. Taking the partial derivative yields

$$\frac{\partial \mathfrak{s}}{\partial \tilde{\sigma}} = \frac{a}{\rho} \frac{\breve{\mu}^{\mathcal{B}} \sqrt{\rho + \breve{\mu}^{\mathcal{B}}}}{\tilde{\sigma}^2} > 0.$$

So even if $\breve{\mu}^{\mathcal{B}}$ was not changed to reoptimize, an increase in idiosyncratic risk $\tilde{\sigma}$ would raise seigniorage. Once $\breve{\mu}^{\mathcal{B}}$ is reoptimized, seigniorage can only get larger, so this certainly also implies that

$$\frac{d\bar{\mathfrak{s}}}{d\tilde{\sigma}} > 0.$$

(c) Seigniorage is often depicted as an "inflation tax". Show that there is a policy choice that extracts seigniorage $\mathfrak{s} = \bar{\mathfrak{s}}$ for arbitrarily large inflation rates. Explain intuitively why high inflation does not necessarily erode the government's ability to extract seigniorage revenue.

Solution: Recall that

$$\mu^{\mathcal{B}} = \pi + g$$

and

$$i = \pi + g - \breve{\mu}^{\mathcal{B}*},$$

where $\breve{\mu}^{\mathcal{B}*}$ is the seigniorage-maximizing bubble mining rate from (b). Then we have

$$\pi = \breve{\mu}^{\mathcal{B}*} + i - g.$$

For arbitrarily large inflation rates, the optimal policy is to set $\breve{\mu}^{\mathcal{B}*}$ and keep the nominal interest rate i high, thereby extracting the maximal seigniorage $\mathfrak{s} = \bar{\mathfrak{s}}$.
The idea here is that $\mathfrak{s}$ depends on $\breve{\mu}^{\mathcal{B}}$ but inflation depends on $\mu^{\mathcal{B}}$. Intuitively, we can have high inflation without going to the bad side of the Laffer curve by also keeping the nominal interest rate high. In this model, households do not mind high inflation if bond growth is used to pay them high nominal interest rates.

2 Monetary Policy in Model with Bonds and Money

Consider a variant of the model from Lecture 10 with money (nominal quantity $\mathcal{M}_t$, nominal interest rate $i_t^{\mathcal{M}}$), nominal short-term bonds (nominal quantity $\mathcal{B}_t$, nominal interest rate $i_t^{\mathcal{B}}$), idiosyncratic risk and transaction services, but no fiscal policy ($\tau_t = \mathscr{G}_t \equiv 0$). Transaction services are modeled by a cash-in-advance constraint in production,

$$\mathcal{P}_t a k_t^{\tilde{i}} \leq \bar{\nu} m_t^{\tilde{i}},$$

where $k_t^{\tilde{i}}$ are the capital holdings and $m_t^{\tilde{i}}$ are the nominal money holdings of agent $\tilde{i}$. Importantly, only money relaxes this constraint, bonds do not.

The government starts with initial nominal liabilities $\mathcal{B}_0, \mathcal{M}_0 > 0$, and it sets a constant interest rate on reserves $i^{\mathcal{M}}$ and constant growth rates of nominal liabilities, $\mu^{\mathcal{B}} = \mu^{\mathcal{M}}$, to balance the budget. The nominal rate on bonds, $i_t^{\mathcal{B}}$, is not directly set by the government but left free to clear the bond market.

Throughout, restrict attention to the (unique) monetary steady-state equilibrium. For each of the policy experiments to be analyzed below, you may assume without justification that the economy immediately jumps to a new steady state.

1. Suppose the government unexpectedly announces one of the following policies at $t = 0$. For each of these policies, explain how this affects (i) the equilibrium value of ϑ, (ii) the equilibrium interest rate on bonds, $i^{\mathcal{B}}$, and inflation rate, π, (iii) the initial price level $\mathcal{P}_0$. Assume in all cases that the economy is initially in the "medium of exchange regime" (in which $\Delta i > 0$, analogous to left-hand column on Slide 32) and the policy change is sufficiently small such that the economy remains in this regime.
   Solution: preliminary observations about this model: [These could also be made as needed when discussing the individual policies, but it is more efficient to present them first.]
   (a) The assumption $\mu^{\mathcal{B}} = \mu^{\mathcal{M}}$ implies that the ratio $\frac{\mathcal{M}_t}{\mathcal{B}_t}$ is held constant over time. This is true for both the original equilibrium and all the policies considered here. Therefore, in all cases, $\vartheta_t^{\mathcal{M}} = \frac{\mathcal{M}_t}{\mathcal{M}_t + \mathcal{B}_t}$ is also constant over time.
   (b) Because, by assumption, we are always in the medium of exchange regime, the quantity equation (binding constraint combined with market clearing) can be used to determine the equilibrium value of ϑ. We can determine the relevant quantity equation here by using the isomorphism outlined on slide 39: we need to replace $\bar{\nu}$ in the equation on slide 32 by $\vartheta_t^{\mathcal{M}} \bar{\nu}$:

$$\frac{1}{\bar{\nu}} \frac{1 - \vartheta + \phi\rho}{\vartheta^{\mathcal{M}} \vartheta} \frac{a}{1 + \phi a} = 1 \Rightarrow \vartheta = \frac{(1 + \phi\rho) a}{a + (1 + \phi a) \bar{\nu} \vartheta^{\mathcal{M}}}$$

[Another way to get to this equation is to derive it: start from the binding constraint and plug in $\theta_t = \vartheta_t$, $\theta_t^M = \vartheta_t^M$ and q_t^K as a function of ϑ_t.]
(c) The equilibrium nominal bond yield $i^{\mathcal{B}}$ must adjust to satisfy the government liabilities valuation equation,

$$\rho = (1-\vartheta)^2\tilde{\sigma}^2 + \vartheta^{\mathcal{M}}\Delta i \Rightarrow \Delta i = \frac{\rho - (1-\vartheta)^2\tilde{\sigma}^2}{\vartheta^{\mathcal{M}}}.$$

Here, $\Delta i = i^{\mathcal{B}} - i^{\mathcal{M}}$ because bonds do not provide transaction services, so

$$i^{\mathcal{B}} = i^{\mathcal{M}} + \frac{\rho - (1-\vartheta)^2\tilde{\sigma}^2}{\vartheta^{\mathcal{M}}}.$$

We could now plug in the equation for ϑ from (b) to write $i^{\mathcal{B}}$ as a function of $\vartheta^{\mathcal{M}}$. However, the final expression is quite involved and not used later on. Let us instead just establish that Δi is strictly decreasing in $\vartheta^{\mathcal{M}}$ (for fixed $i^{\mathcal{M}}$): (i) the denominator in the second term is increasing in $\vartheta^{\mathcal{M}}$; (ii) the numerator is increasing in ϑ, which in turn is decreasing in $\vartheta^{\mathcal{M}}$ by (b), so that the numerator is decreasing in $\vartheta^{\mathcal{M}}$.
(d) The government budget constraint is here (recall there are no taxes)

$$(1-\vartheta^{\mathcal{M}})\breve{\mu}^{\mathcal{B}} + \vartheta^{\mathcal{M}}\breve{\mu}^{\mathcal{M}} = 0.$$

Plugging in the definition of $\breve{\mu}^{\mathcal{B}}$ and $\breve{\mu}^{\mathcal{M}}$ and using $\mu^{\mathcal{B}} = \mu^{\mathcal{M}}$, we obtain

$$\mu^{\mathcal{M}} - (1-\vartheta^{\mathcal{M}})i^{\mathcal{B}} - \vartheta^{\mathcal{M}}i^{\mathcal{M}} = 0 \Rightarrow \mu^{\mathcal{M}} = i^{\mathcal{M}} + (1-\vartheta^{\mathcal{M}})\Delta i = i^{\mathcal{M}} + \frac{1-\vartheta^{\mathcal{M}}}{\vartheta^{\mathcal{M}}}(\rho - (1-\vartheta)^2\tilde{\sigma}^2),$$

where the last equality uses (c). In particular, also the money growth rate $\mu^{\mathcal{M}}$ is strictly decreasing in $\vartheta^{\mathcal{M}}$.
(e) The inflation rate is

$$\pi = \mu^{\mathcal{M}} - g = \mu^{\mathcal{M}} - \Phi(\iota(\vartheta)) + \delta,$$

where $\iota(\vartheta)$ is a strictly decreasing function (due to Tobin effect, compare Slides 12 and 38). By (b), ϑ is a strictly decreasing function of $\vartheta^{\mathcal{M}}$. By (d), $\mu^{\mathcal{M}}$ is strictly decreasing in $\vartheta^{\mathcal{M}}$ and strictly increasing in $i^{\mathcal{M}}$. Therefore, π is strictly decreasing in $\vartheta^{\mathcal{M}}$ and strictly increasing in $i^{\mathcal{M}}$.
(f) The initial price level is

$$\mathcal{P}_0 = \frac{\mathcal{M}_0 + \mathcal{B}_0}{q^{\mathcal{MB}}K_0} = \frac{\mathcal{M}_0 + \mathcal{B}_0}{\frac{a}{\bar{\nu}\vartheta^M}K_0} = \frac{\bar{\nu}(\mathcal{M}_0 + \mathcal{B}_0)}{aK_0}\vartheta^M = \frac{\bar{\nu}\mathcal{M}_0}{aK_0},$$

where $q^{\mathcal{MB}} = \frac{a}{\bar{\nu}\vartheta^{\mathcal{M}}}$ follows from the left column on Slide 32 and the isomorphism between the $\mathcal{MB}$-model and the model with separate $\mathcal{M}$ and $\mathcal{B}$ described on Slide 39 [alternatively, this can be rederived using steps analogous to the ones for the $\mathcal{MB}$-model].

- Policy 1: one-time Helicopter drop of money:
  The government prints $\Delta\mathcal{M} > 0$ units of money at $t = 0$ and distributes them lump-sum to households. Thereafter, it follows the same interest on reserve policy as before but possibly adjusts the growth rate of nominal liabilities to satisfy its budget constraint.
  Solution: The main effect is that the initial money supply increases from $\mathcal{M}_0$ to $\mathcal{M}_0' = \mathcal{M}_0 + \Delta\mathcal{M}$, whereas $\mathcal{B}_0' = \mathcal{B}_0$ is unaffected (throughout, $'$-variables refer to variables after the

policy change). There is another effect on the cross-sectional wealth distribution, which is compressed due to the lump-sum distribution of newly created money. However, this second effect is irrelevant for the variables to be analyzed here (because the cross-sectional wealth distribution is). The consequence of the money supply increase is that

$$\vartheta_0^{M\prime} = \frac{\mathcal{M}_0'}{\mathcal{M}_0' + \mathcal{B}_0'} = \frac{\mathcal{M}_0 + \Delta\mathcal{M}_0}{\mathcal{M}_0 + \mathcal{B}_0 + \Delta\mathcal{M}_0} = \frac{\vartheta_0^M + \frac{\Delta\mathcal{M}_0}{\mathcal{M}_0 + \mathcal{B}_0}}{1 + \frac{\Delta\mathcal{M}_0}{\mathcal{M}_0 + \mathcal{B}_0}}$$

increases. Because $\vartheta^{\mathcal{M}\prime}$, $\vartheta^{\mathcal{M}}$ are constant over time (see (a) above), $\vartheta^{\mathcal{M}\prime}$ rises permanently relative to $\vartheta^{\mathcal{M}}$.

The preliminary discussion above relates ϑ, $i^{\mathcal{B}}$, and π directly to the values of $\vartheta^{\mathcal{M}}$ and $i^{\mathcal{M}}$. Specifically, all three variables are decreasing in $\vartheta^{\mathcal{M}}$ by (b), (c), and (e), respectively. In addition, ϑ is unaffected by $i^{\mathcal{M}}$, whereas $i^{\mathcal{B}}$ and π are increasing in $i^{\mathcal{M}}$. Under the policy considered here, $i^{\mathcal{M}\prime} = i^{\mathcal{M}}$ and $\vartheta^{\mathcal{M}\prime} > \vartheta^{\mathcal{M}}$, so we can conclude

$$\vartheta' < \vartheta, \qquad i^{\mathcal{B}\prime} < i^{\mathcal{B}}, \qquad \pi' < \pi.$$

Finally, the initial price level changes to (by (f) above)

$$\mathcal{P}_0' = \frac{\bar{\nu}\mathcal{M}_0'}{aK_0} = \frac{\bar{\nu}\mathcal{M}_0}{aK_0} + \frac{\bar{\nu}\Delta\mathcal{M}}{aK_0} = \mathcal{P}_0 + \frac{\bar{\nu}\Delta\mathcal{M}}{aK_0} > \mathcal{P}_0.$$

In sum, a helicopter drop of money leads to an initial upward jump in the price level followed by a reduction in subsequent inflation. It also reduces the nominal interest rate on bonds and the nominal wealth share ϑ. [Note that the decrease in inflation arises from two channels, the decrease in money growth necessary to fund interest payments and the increase in the growth rate due to the Tobin effect.]

- Policy 2: one-time Helicopter drop of bonds:
  The government prints $\Delta\mathcal{B} > 0$ units of bonds at $t = 0$ and distributes them lump-sum to households. Thereafter, it follows the same interest on reserve policy as before but possibly adjusts the growth rate of nominal liabilities to satisfy its budget constraint.
  Solution: The effects are largely symmetric to Policy 1 but go in the opposite direction (the exception is the effect on the $\mathcal{P}_0$). The relevant effect of the policy is that the initial bond supply increases to $\mathcal{B}_0' = \mathcal{B}_0 + \Delta\mathcal{B}$, while the initial money supply is unaffected, $\mathcal{M}_0' = \mathcal{M}_0$. As a consequence,

$$\vartheta_0^{M\prime} = \frac{\mathcal{M}_0'}{\mathcal{M}_0' + \mathcal{B}_0'} = \frac{\mathcal{M}_0}{\mathcal{M}_0 + \mathcal{B}_0 + \Delta\mathcal{B}_0} = \frac{\vartheta_0^M}{1 + \frac{\Delta\mathcal{B}_0}{\mathcal{M}_0 + \mathcal{B}_0}}$$

  decreases. So $\vartheta^{\mathcal{M}\prime} < \vartheta^{\mathcal{M}}$. Because it remains true also here that $i^{\mathcal{M}\prime} = i^{\mathcal{M}}$, we can conclude in analogy to the discussion of Policy 1 that

$$\vartheta' > \vartheta, \qquad i^{\mathcal{B}\prime} > i^{\mathcal{B}}, \qquad \pi' > \pi.$$

  The initial price level is unaffected, $\mathcal{P}_0 = \mathcal{P}_0'$ because the price level only depends on the money supply, which has not changed (see (f) above).
  In sum, a helicopter drop of bonds does not affect the initial price level but leads to a subsequent rise in inflation. It also raises the nominal interest rate on bonds and the nominal wealth share ϑ. [Again, the increase in inflation is due to both higher money growth to cover the additional interest payments and lower output growth.]

- Policy 3: open market operation/bond auction:
  The government sells $\Delta\mathcal{B} > 0$ bonds and uses the proceeds to take money out of circulation. Thereafter, it follows the same interest on reserve policy as before but possibly adjusts the growth rate of nominal liabilities to satisfy its budget constraint.
  Solution: There is no need to analyze this policy separately, as it combines a "long" position in Policy 2 with a "short" position in Policy 1 such that $\Delta\mathcal{M} = -\Delta\mathcal{B}$. By the previous results, this unambiguously increases ϑ, $i^{\mathcal{B}}$, and π, and it reduces the initial price level $\mathcal{P}_0$.

- Policy 4: increasing interest on reserves:
  The government raises the reserve rate $i^{\mathcal{M}}$ to a permanently higher level $i^{\mathcal{M}\prime}$. It adjusts the growth rate of nominal liabilities as needed to satisfy its budget constraint.
  Solution: For this policy $\vartheta^{\mathcal{M}}$ does not change, so also ϑ remains unaffected (see (b) in preliminary observations),
  $$\vartheta' = \vartheta.$$
  For given $\vartheta^{\mathcal{M}}$ and ϑ, (c)–(e) of the preliminary observations reveal that both $i^{\mathcal{B}}$ and π move one-for-one with $i^{\mathcal{M}}$, so
  $$i^{\mathcal{B}\prime} = i^{\mathcal{B}} + i^{\mathcal{M}\prime} - i^{\mathcal{M}}, \qquad \pi' = \pi + i^{\mathcal{M}\prime} - i^{\mathcal{M}}$$
  both increase.
  Finally, the initial price level $\mathcal{P}_0$ is unaffected by a change in the reserve rate.

2. How do the answers to the previous question change if the economy is in the "store of value regime" (in which $\Delta i = 0$)? Which of the policies, if sufficiently aggressive, may move the economy out of this regime into the "medium of exchange regime"?
   Solution: The predictions for Policy 4 are unchanged. Policy 3 is neutral (no effects on any of the variables). Policies 1 and 2 are equivalent and only affect $\mathcal{P}_0$ (which increases) but none of the other variables.
   How do we see this? Now, the government liabilities valuation equation with $\Delta i = 0$ determines ϑ, but this does not depend on policy variables in the absence of fiscal policy ($\vartheta = \frac{\tilde{\sigma}-\sqrt{\rho}}{\tilde{\sigma}}$), so any conclusion that depends on ϑ moving is no longer true. $\Delta i = 0$ also implies $i^{\mathcal{B}} = i^{\mathcal{M}}$. The government budget constraint then implies that $\mu^{\mathcal{M}} = i^{\mathcal{M}}$. Hence, both $i^{\mathcal{B}}$ and $\mu^{\mathcal{M}}$ are affected only under Policy 4, which adjusts the reserve rate $i^{\mathcal{M}}$. The same then also applies to inflation π (as the Tobin effect is shut down when ϑ remains unaffected, so only the direct effect from money growth remains). Under Policy 4, all these variables move one-for-one with $i^{\mathcal{M}}$, exactly as in Part 1.
   The conclusions about the initial price level follow with a bit of algebra (note that the quantity equation, which has been used previously, is irrelevant here because the cash-in-advance constraint is slack):
   $$\mathcal{P}_0 = \frac{\mathcal{MB}_0}{q^{\mathcal{MB}}K_0} = \frac{1}{\vartheta}\frac{1-\vartheta+\phi\rho}{1+\phi a}\frac{\mathcal{M}_0+\mathcal{B}_0}{K_0}.$$
   We have just established that none of the policies affect ϑ, so the effect on the initial price level is proportional to the effect on total initial nominal liabilities $\mathcal{M}_0 + \mathcal{B}_0$. Policies 1 and 2 increase this quantity, whereas Policies 3 and 4 leave it unaffected.
   Finally, which policies could move us to the other regime? Only those that raise Δi conditional on being in that regime, which are Policies 2 and 3.