Online Summer School
Macro, Money, and Finance
Problem Set 3

June 23, 2025

Please submit your solutions to the dropbox link by 6/22/2025 23:59 pm (EDT).

1 Fire Sales

In this exercise you will solve the model from Lecture 04 numerically, under the assumption of log utility and with agents' deaths.

1. Our goal is to construct functions $q(\eta), \iota(\eta), \kappa(\eta)$ and $\sigma^q(\eta)$ on the $[0,1]$ grid. Slides 46-47 provide the parameter values, and slide 43-45 provides the set of equations and the algorithm.

   (a) Solve the model at the boundaries: for $\eta = 0$ and $\eta = 1$.

   Solution:

   Both $\eta = 0$ and $\eta = 1$ are absorbing states meaning $\sigma^q(0) = \sigma^q(1) = 0$. If there are no experts, then $\kappa(0) = 0$, otherwise $\kappa(1) = 1$. The capital price can be derived from the goods market clearing condition: $q(0) = \frac{1+a^h\phi}{1+\rho^h\phi}$, $q(1) = \frac{1+a^e\phi}{1+\rho^e\phi}$.

   (b) Create a uniform grid for $\eta \in [0.0001, 0.9999] = \{\eta_1 = 0.0001, \eta_2, \dots, \eta_N = 0.9999\}$.

   Solution: See code.

   (c) Using the implicit method with the one-step Newton's algorithm, solve the system of equations on slide 43 for $\eta_1, \eta_2, \dots$ and so on.

   Solution:

   At each grid point we are looking for $q_i \equiv q(\eta_i)$, $\kappa_i \equiv \kappa(\eta_i)$ and $ssq_i \equiv \sigma + \sigma^q(\eta_i)$,[1] that satisfy the following system of equations:

$$F(q_i, \kappa_i, ssq_i) = \begin{bmatrix} \kappa_i(a^e - a^h) + a^h - \frac{q_i - 1}{\phi} - q_i(\eta_i\rho^e + (1-\eta_i)\rho^h) \\ ssq_i\left(q_i - \frac{q_i - q_{i-1}}{\Delta\eta_i}(\kappa_i - \eta_i)\right) - \sigma q_i \\ a^e - a^h - q_i\frac{\kappa_i - \eta_i}{\eta_i(1-\eta_i)}ssq_i^2 \end{bmatrix} = 0$$

[1] We can of course use directly $\sigma^q(\eta)$ as the third variable, but since σ and $\sigma^q(\eta)$ often enter the equations as a sum, using $\sigma + \sigma^q(\eta)$ is algebraically slightly more convenient.

To apply Newton's method, we guess that $q_i = q_{i-1}$, $\kappa_i = \kappa_{i-1}$, $ssq_i = ssq_{i-1}$, derive the Jacobian J_i of F at that point (see code) and update once using:

$$\begin{bmatrix} q_i \\ \kappa_i \\ ssq_i \end{bmatrix} = \begin{bmatrix} q_{i-1} \\ \kappa_{i-1} \\ ssq_{i-1} \end{bmatrix} - J_i^{-1} F(q_{i-1}, \kappa_{i-1}, ssq_{i-1})$$

Note: at the first grid point $\eta_1 = 0.0001$, the grid point $(i-1)$ corresponds to our solution for $\eta = 0$. That means we have to make sure that $\Delta\eta_1 = \eta_1$, whereas for all other grid points $\Delta\eta_i$ is determined by the grid step of the uniform grid.

(d) Stop once you reach $\kappa \geq 1$. From here on, set $\kappa = 1$, solve for q and σ^q.

Solution:

Once we reach $\kappa_i \geq 1$ at some grid point i, we know that $\kappa_j = 1$ for all $j \geq i$. We can then directly compute q_j and ssq_j from the goods market clearing condition and the expression for ssq_j (see code).

(e) Verify your solution by plotting $q(\eta)$ and $\sigma^q(\eta)$ and comparing it with the graph on slide 46. Do your functions converge to the boundary solution for $\eta = 1$ that you obtained in (a) as $\eta \to 1$?

Solution: See Figure 1.

(f) Plot the remaining variables: $\iota(\eta), \kappa(\eta)$.

Solution: See Figure 1.

(g) We can also look at the experts' balance sheet: derive expression for the scaled version of issued debt: $\frac{D_t^e}{q_t K_t}$ and plot it against η.

Solution: See Figure 2. We use here $D_t^e \geq 0$ as value of issued debt, while experts' portfolio share of debt is of course non-positive: $\theta_t^{e,D} \leq 0$.

$$\frac{D_t^e}{q_t K_t} = \frac{D_t^e N_t^e}{N_t^e q_t K_t} = -\theta_t^{e,D}\eta_t = -(1-\theta_t^{e,K})\eta_t = -(1-\frac{\kappa_t}{\eta_t})\eta_t = \kappa_t - \eta_t$$

2. Recall from the lecture that drift and volatility of η in the general case are given by:

$$\begin{aligned}
\mu_t^\eta = (1-\eta_t)\Big[&(\varsigma_t^e - \sigma - \sigma_t^q)(\sigma_t^\eta + \sigma + \sigma_t^q) - (\varsigma_t^h - \sigma - \sigma_t^q)\Big(-\tfrac{\eta_t}{1-\eta_t}\sigma_t^\eta + \sigma + \sigma_t^q\Big) \\
&- \Big(\tfrac{C_t^e}{N_t^e} - \tfrac{C_t^h}{N_t^h}\Big) + \frac{\rho_d^h \zeta(1-\eta_t) - \rho_d^e(1-\zeta)\eta_t}{\eta_t(1-\eta_t)}\Big], \\
\sigma_t^\eta = \frac{\kappa_t - \eta_t}{\eta_t}&(\sigma + \sigma_t^q).
\end{aligned}$$

(a) Which terms in the above equations can we simplify/substitute because of log utility and why? Perform these substitutions and derive the drift and volatility of η under log utility.

Solution:

Because of log utility, agents consume a fixed fraction of their net worth, equal to their discount factor: $\frac{C_t^e}{N_t^e} = \rho^e$, $\frac{C_t^h}{N_t^h} = \rho^h$. In addition, the price of risk is given by agent's net worth volatility: $\varsigma_t^e = \sigma_t^{n^e}$, $\varsigma_t^h = \sigma_t^{n^h}$. As a result, the expression for volatility of η remains as above, and the drift becomes:

$$\mu_t^\eta = (1-\eta_t)\left[(\kappa_t - \eta_t)\left(\frac{\kappa_t}{\eta_t^2} + \frac{1-\kappa_t}{(1-\eta_t)^2}\right)(\sigma + \sigma_t^q)^2 - (\rho^e - \rho^h) + \frac{\rho_d^h \zeta(1-\eta_t) - \rho_d^e(1-\zeta)\eta_t}{\eta_t(1-\eta_t)}\right]$$

(b) Verify your solution by plotting $\eta\mu^\eta(\eta)$ and $\eta\sigma^\eta(\eta)$ and comparing them with the graph on slide 47.

Solution: See Figure 3.

Figure 1: Equilibrium variables

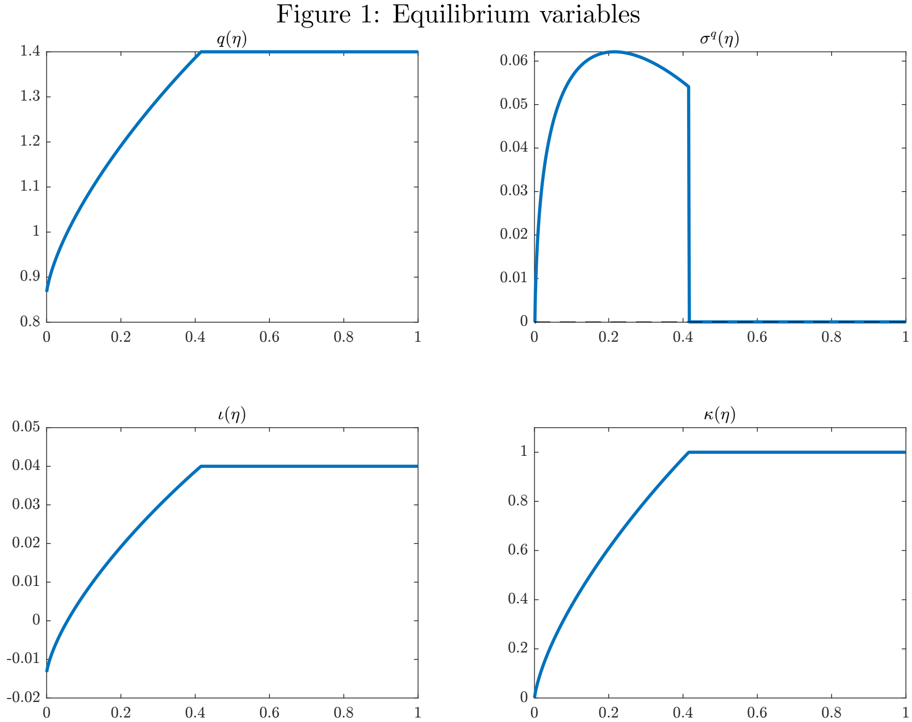

Figure 2: Balance Sheet of Experts

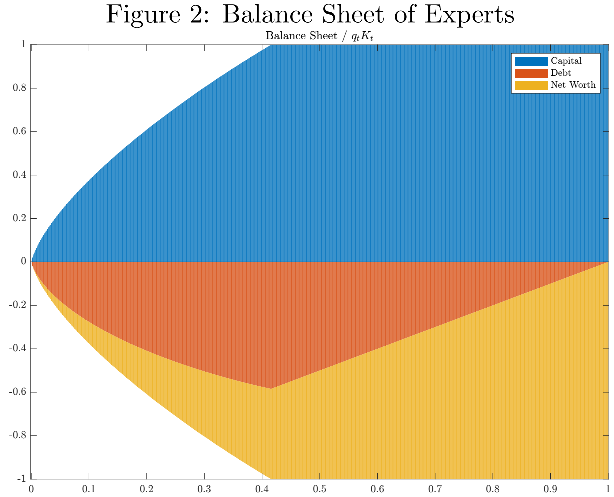

Figure 3: Drift and Volatility of η