Online Summer School
Macro, Money, and Finance
Problem Set 2

June 15, 2025

Please submit your solutions to the dropbox link by 6/15/2025 23:59 pm (EDT).

1 Introducing Physical Investment

In class, we consider a deterministic two sector model with fixed aggregate capital $\bar{K}$. Now we add investment to it. We introduce a concave capital conversion function $\Phi(\iota)$ with assumption $\Phi'(\cdot) > 0, \Phi''(\cdot) < 0$ for investment rate ι and capital depreciation rate δ. For example, consider an agent with capital k_t at time t with investment rate ι_t. His real investment is $\iota_t k_t$, and the capital accumulation is $(\Phi(\iota_t) - \delta)k_t$. This is equivalent to a convex adjustment cost assumption.

1. **Optimal Investment Decision**. Consider the following time line of operating the capitals:
   - At t, agent i purchases capital k_t^i at price q_t.
   - He/she makes the (optimal) investment decision ι_t^i within period $[t, t+dt)$.
   - The capital generates output $a^i(k_t^i)$ at time $t+dt$
   - Finally agent i sells the capital k_{t+dt}^i at market price q_{t+dt}.

   The total gains are:

   $$\pi_t dt = \underbrace{q_{t+dt}k_{t+dt}^i - q_t k_t^i}_{①} + (\underbrace{a^i(k_t^i)}_{②} - \underbrace{\iota_t^i k_t^i}_{③})dt$$

   where ① represents the gain from holding and reselling, ② represents the dividend (output) flow, and ③ represents the investment cost.

   (a) Derive the expression for the return rate, $\frac{\pi_t}{q_t k_t}$. Do the decomposition for ①②③ for π_t.

Solution:

$$
\begin{aligned}
\frac{\pi_t}{q_t k_t^i} &= \frac{1}{q_t k_t^i}\frac{q_{t+dt}k_{t+dt}^i - q_t k_t^i}{dt} + \frac{a^i(k_t^i)}{q_t k_t^i} - \frac{\iota_t^i k_t^i}{q_t k_t^i} \\
&= \frac{1}{q_t k_t^i}\frac{d(q_t k_t^i)}{dt} + \frac{a^i(k_t^i)}{q_t k_t^i} - \frac{\iota_t^i}{q_t} \\
&= \frac{1}{q_t k_t^i}\left(\frac{dq_t}{dt}k_t^i + \frac{dk_t^i}{dt}q_t\right) + \frac{a^i(k_t^i)}{q_t k_t^i} - \frac{\iota_t^i}{q_t} \\
&= \frac{1}{q_t}\frac{dq_t}{dt} + \frac{1}{k_t^i}\frac{\left(\Phi(\iota_t^i) - \delta\right)k_t^i dt}{dt} + \frac{a^i(k_t^i)}{q_t k_t^i} - \frac{\iota_t^i}{q_t} \\
&= \underbrace{\frac{1}{q_t}\frac{dq_t}{dt} + \Phi(\iota_t^i) - \delta}_{①} + \underbrace{\frac{a^i(k_t^i)}{q_t k_t^i}}_{②} - \underbrace{\frac{\iota_t^i}{q_t}}_{③}
\end{aligned}
$$

(b) Show that optimal investment ι_t^i is the same for $i \in \{e, h\}$, and is determined by:

$$\frac{1}{q_t} = \Phi'(\iota_t^i).$$

This is the Tobin's Q condition.

Solution: Maximizing the return on capital ($\frac{\pi_t}{q_t k_t}$) with respect to ι_t^i yields:

$$\Phi'(\iota_t^i) - \frac{1}{q_t} = 0$$

(c) For conversion function $\Phi(\iota) = \frac{1}{\phi}\log(\phi\iota + 1)$, what is the optimal investment ι_t given price q_t?

Solution:

$$\Phi'(\iota_t^i) = \frac{1}{\phi\iota_t^i + 1} \rightarrow \iota_t^i = \frac{q_t - 1}{\phi}$$

2 The Basak-Cuoco Model with Heterogeneous Discount Rates

Consider the Basak-Cuoco Model of the slides. Now we introduce the investment conversion function $\Phi(\iota) = \frac{1}{\phi}\log(\phi\iota + 1)$. Assume that households are more patient than experts, i.e. they have a discount rate $\rho^h < \rho^e$. This is the simplest way to generate both a nondegenerate stationary distribution and some endogenous capital price dynamics.

1. Derive closed-form expressions for ι, q, σ^q, μ^η and σ^η as functions of η and model parameters:

   (a) Start with goods market clearing condition and use $\hat{\rho}(\eta) = \rho^e\eta + \rho^h(1-\eta)$ to ease notation. Derive $q(\eta)$ and $\iota(\eta)$.
   Solution:
   Goods market clearing:

$$C = C^e + C^h = \underbrace{(\rho^e\eta + \rho^h(1-\eta))}_{\hat{\rho}(\eta)} qK = (a - \iota)K$$

Optimal investment choice: $q = 1 + \phi\iota$. Combine with market clearing:

$$q(\eta) = \frac{1+\phi a}{1+\phi\hat{\rho}(\eta)}$$
$$\iota(\eta) = \frac{a-\hat{\rho}(\eta)}{1+\phi\hat{\rho}(\eta)}$$

(b) Use $q(\eta)$ and the law of motion for η to find $\sigma^q(\eta)$ and $\sigma^\eta(\eta)$.
Solution:
Law of Motion for η from LOMs of N^e and qK:

$$\frac{d\eta}{\eta} = \left(\frac{a-\iota^e}{q} - \rho^e + \theta^e(\sigma+\sigma^q-\varsigma^e)(\sigma+\sigma^q)\right)dt - \theta^e(\sigma+\sigma^q)dZ$$

Capital market clearing: $\theta^e = -\frac{1-\eta}{\eta}$, log-utility: $\varsigma^e = \sigma^{n^e} = (1-\theta^e)(\sigma+\sigma^q)$, and $(a-\iota^e) = \hat{\rho}(\eta)q$:

$$\frac{d\eta}{\eta} = \underbrace{\left((1-\eta)(\rho^h-\rho^e) + \left(\frac{1-\eta}{\eta}\right)^2(\sigma+\sigma^q(\eta))^2\right)}_{\mu^\eta(\eta)}dt \; + \; \underbrace{\frac{1-\eta}{\eta}(\sigma+\sigma^q(\eta))}_{\sigma^\eta(\eta)}dZ$$

Apply Ito's formula to $q(\eta)$:

$$dq(\eta) = \underbrace{\left(q'(\eta)\mu^\eta(\eta)\eta + \frac{(\sigma^\eta(\eta)\eta)^2}{2}q''(\eta)\right)}_{\mu^q(\eta)q(\eta)}dt + \underbrace{q'(\eta)\sigma^\eta(\eta)\eta}_{\sigma^q(\eta)q(\eta)}dZ$$

Combine three equations:

$$q(\eta) = \frac{1+\phi a}{1+\phi\hat{\rho}(\eta)}, \qquad \sigma^q(\eta)q(\eta) = q'(\eta)\sigma^\eta(\eta)\eta, \qquad \sigma^\eta(\eta) = \frac{1-\eta}{\eta}(\sigma+\sigma^q(\eta))$$

Obtain:

$$\sigma^q(\eta) = -\frac{\phi(\rho^e-\rho^h)(1-\eta)}{1+\phi\rho^e}\sigma, \qquad \sigma^\eta(\eta) = \frac{1-\eta}{\eta}\frac{1+\phi\hat{\rho}(\eta)}{1+\phi\rho^e}\sigma$$

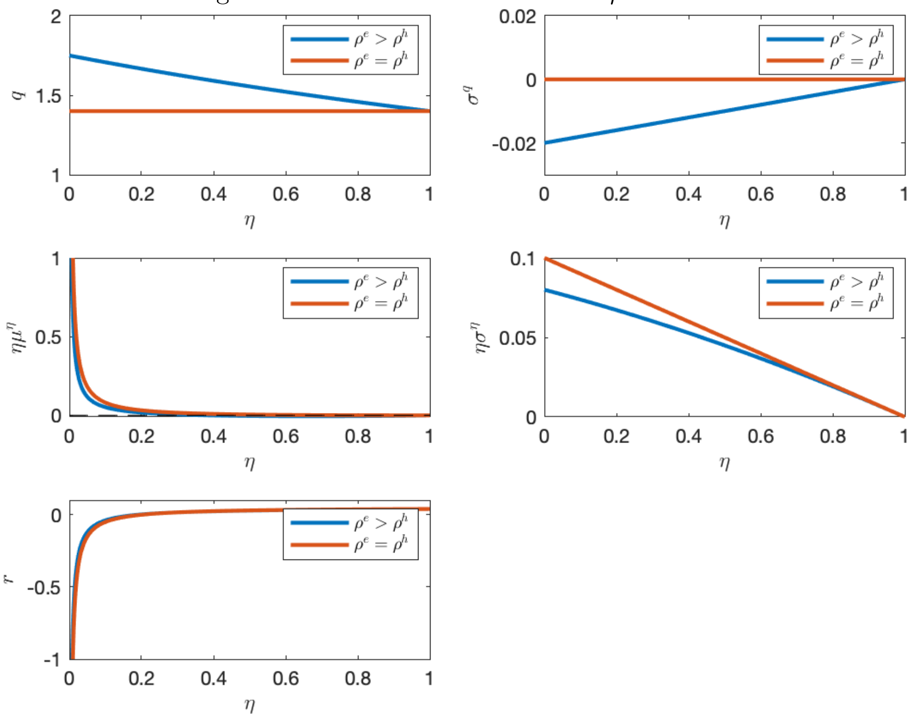
(c) Derive $\mu^\eta(\eta)$.
Solution: already obtained in (b).

2. Replicate the figures from slide 26, setting $\phi = 10$ and $\delta = 0.035$, then add to each plot the corresponding line for the model with $\rho^e = 5\%$ and $\rho^h = 2\%$ (and all other parameters as before).
Solution:

See Figure 1. Risk-free rate can be obtained from experts portfolio choice:

$$r(\eta) = \frac{a-\iota(\eta)}{q(\eta)} + \Phi(\iota(\eta)) - \delta + \underbrace{\mu^q(\eta)}_{\neq 0} + \sigma\underbrace{\sigma^q(\eta)}_{\neq 0} - \varsigma^e(\eta)(\sigma + \underbrace{\sigma^q(\eta)}_{\neq 0})$$

Figure 1: Basak-Cuoco Model with $\rho^e = 0.05$



3. Assume $\phi > 0$. Show that in this model asset price movements mitigate exogenous risk (i.e. $\sigma^q + \sigma < \sigma$). Explain economically why this happens and why the effect disappears if $\phi = 0$.
Solution:

Total risk:

$$\sigma + \sigma^q(\eta) = \frac{\sigma}{1 - \frac{1-\eta}{\eta}\frac{q'(\eta)}{q/\eta}} = \left(1 - \frac{\phi(\rho^e - \rho^h)(1-\eta)}{1 + \phi\rho^e}\right)\sigma \ < \ \sigma \text{ if } \phi > 0$$

Goods market clearing condition:

$$\underbrace{\hat{\rho}(\eta)}_{C/N}\underbrace{q(\eta)K}_{N} + \iota(\eta)K = aK$$

Suppose $K \downarrow \implies \eta \downarrow \implies C/N \downarrow \implies q(\eta)$ and/or $\iota(\eta)$ must go up. Since $\phi > 0$, investment adjustment is costly and the higher demand for capital can not be fully satisfied, which results in an increase of capital price. As a result, a drop in K is compensated by an increase in q, which stabilizes qK and reduces its volatility. If $\phi = 0$, then $\iota(\eta)$ adjusts alone and there is no price effect.

4. Argue that the model must have a nondegenerate stationary distribution (just give some intuition, not a formal proof).
Solution:

Unlike in the model with homogeneous discount rates, the drift of η crosses zero at an interior point:

$$\mu^{\eta}(\eta) = (1-\eta)\left[\underbrace{(\rho^h - \rho^e)}_{<0} + \underbrace{\frac{1-\eta}{\eta^2}(\sigma + \sigma^q(\eta))^2}_{>0}\right] \xrightarrow[\eta \to 1]{} (1-\eta)(\rho^h - \rho^e) + o(1-\eta) < 0$$

Together with vanishing volatility ($\sigma^{\eta}(\eta) \xrightarrow[\eta \to 1]{} 0$) this ensures that η is pushed back as it approaches $\eta = 1$ and there exists a non-degenerate distribution.