Problem Set 2

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1 Introducing Physical Investment

(a)

$$\pi_t dt = q_{t+\Delta t}k^i_{t+\Delta t} - q_t k^i_t + (a^i(\kappa^i_t)k^i_t - \iota^i_t k^i_t)\Delta t \tag{1}$$

As $\Delta t \to 0$:

$$\boxed{\frac{\pi_t}{q_t k^i_t} = \underbrace{(\frac{\frac{d(q_t k^i_t)}{dt}}{q_t k^i_t})}_{1} + \left[\underbrace{\frac{a^i(\kappa^i_t)}{q_t}}_{2} - \underbrace{\frac{\iota^i_t}{q_t}}_{3}\right]} \tag{2}$$

(b)

So,

$$dr^k_t(\iota_t) = \frac{d(q_t k_t)}{q_t k_t} + \frac{(a^i(\kappa_t) - \iota_t)dt}{q_t} \tag{3}$$

Now, $\frac{dk_t}{k_t} = [\Phi(\iota_t) - \delta]dt + \sigma dZ$
Postulate $\frac{dq_t}{q_t} = \mu^q_t dt + \sigma^q_t dZ$
So, by Ito's:

$$\frac{d(q_t k_t)}{q_t k_t} = [\Phi(\iota_t) - \delta + \mu^q_t + \sigma\sigma^q_t]dt + [\sigma + \sigma^q_t]dZ \tag{4}$$

$$\text{So, } dr^k_t(\iota^i_t) = \left[\frac{a^i(\kappa^i_t) - \iota^i_t}{q_t} + \Phi(\iota_t) - \delta + \mu^q_t + \sigma\sigma^q_t\right]dt + (\sigma + \sigma^q_t)dZ \tag{5}$$

So,

Agents $i \in \{e, h\}$ choose ι_t to maximize:

$$\max_{\iota^i_t} \mathbb{E}dr^k_t(\iota^i_t)/dt \tag{6}$$

$$\Rightarrow \max_{\iota_t} \left[\frac{a^i(k^i_t) - \iota^i_t}{q_t} + \Phi'(\iota^i_t) - \delta + \mu^q_t + \sigma\sigma^q_t\right] \tag{7}$$

FOC:

$$\iota_t^i :\Rightarrow \quad -\frac{1}{q_t} + \Phi'(\iota_t^i) = 0 \tag{8}$$

$$\Leftrightarrow \quad \boxed{\Phi'(\iota_t^i) = \frac{1}{q_t}} \quad \text{(Tobin's } Q \text{ condition)} \tag{9}$$

(c)

For $\Phi(\iota) = \frac{1}{\phi}\log(\phi\iota + 1)$,

$$\Phi'(\iota) = \frac{1}{\phi(\phi\iota + 1)} \cdot \phi = \frac{1}{\phi\iota + 1} \tag{10}$$

So, for q_t optimal investment $\Rightarrow$

$$\frac{1}{\phi\iota_t + 1} = \frac{1}{q_t} \tag{11}$$

$$\Rightarrow \phi\iota_t + 1 = q_t \tag{12}$$

So,

$$\boxed{\iota_t = \frac{1}{\phi}(q_t - 1)} \tag{13}$$

2 Basak-Cuoco

1(a)

Goods market clearing implies:

$$a^e k_t - \iota_t k_t = \rho_t^e \eta_t q_t k_t + \rho_t^h (1 - \eta_t) q_t k_t \tag{14}$$

So,

$$\boxed{q(\eta) = \frac{1 + \phi a^e}{1 + \phi\hat{\rho}(\eta)}, \quad \iota(\eta) = \frac{a^e - \hat{\rho}(\eta)}{1 + \phi\hat{\rho}(\eta)}} \tag{15}$$

(b)

Now, $q(\eta) = \frac{1+a^e\phi}{1+\phi\hat{\rho}(\eta)}$

So, $q' = \frac{1+a^e\phi}{[1+\phi\hat{\rho}(\eta)]^2} \cdot \phi(\rho^h - \rho^e)$

and $q'' = 2\frac{(1+a^e\phi)[\phi(\rho^e-\rho^h)]^2}{[1+\phi\hat{\rho}(\eta)]^3}$

So, by Ito's:

$$\sigma^q(\eta) = -\frac{\phi(\rho^e - \rho^h)}{(1 + \phi\hat{\rho}(\eta))} \cdot \sigma^\eta(\eta)\eta \tag{16}$$

By Capital market clearing:

$$\theta^{k,e} = \frac{1}{\eta} \tag{17}$$

By Ito's using $\eta_t = N_t^e / N_t$:

$$\sigma_t^\eta = -\left[1 - \frac{1}{\eta_t}\right](\sigma + \sigma_t^q) \tag{18}$$

$$= \frac{1-\eta_t}{\eta_t}(\sigma + \sigma_t^q) \tag{19}$$

Combining (1) & (2):

$$\boxed{\sigma_t^\eta = \frac{(1-\eta)(1+\phi\hat{\rho}(\eta))}{\eta(1+\phi\rho^e)}\sigma, \quad \sigma_t^q = -(1-\eta)\frac{\phi(\rho^e - \rho^h)}{1+\phi\rho^e}\sigma} \tag{20}$$

(c)

Putting these values in the drift part of law of motion for η gives:

$$\boxed{\mu^\eta(\eta) = -\rho^e + \hat{\rho} + \left[\frac{(1-\eta)(1+\phi\hat{\rho}(\eta))}{\eta(1+\phi\rho^e)}\right]^2 \sigma^2} \tag{21}$$

2.

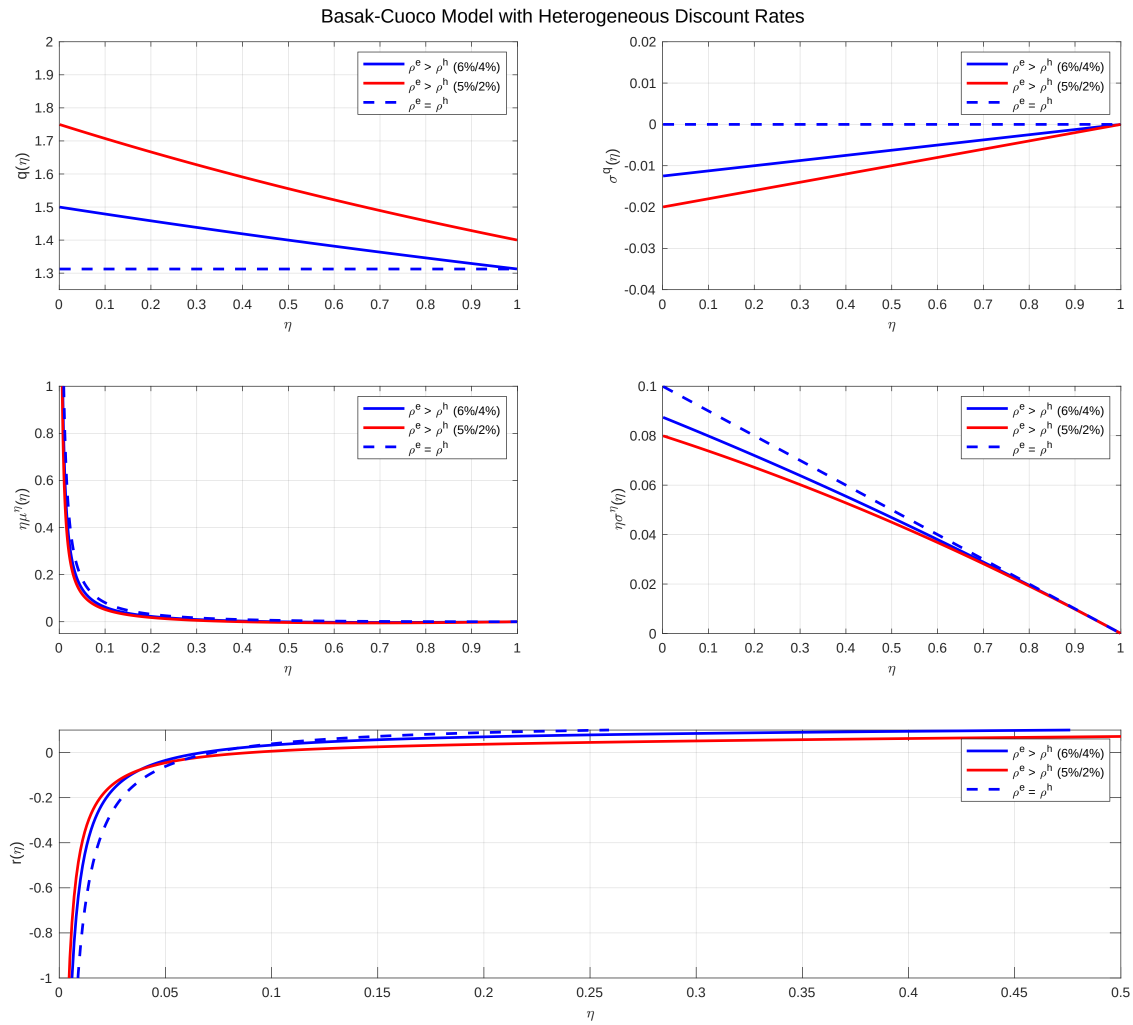

Figure 1: prices and volatility as functions of η

3.

$\sigma + \sigma^q = \frac{1+\phi\hat{\rho}(\eta)}{1+\phi\rho^e}\sigma$

So, $\frac{\sigma+\sigma^q}{\sigma} = \frac{1+\phi\hat{\rho}}{1+\phi\rho^e}$

$\rho^e > \rho^h \Rightarrow \hat{\rho} < \rho^e$

So, $1 + \phi\hat{\rho} < 1 + \phi\rho^e$

So, if $\phi > 0$, then $\frac{1+\phi\hat{\rho}}{1+\phi\rho^e} < 1$

As $\phi \to 0$, $\Phi(\iota) \to \iota$, so no adjustment cost in capital investment. So, this effect is gone.

4.

When $\eta \to 0$, experts become poor, in that situation drift becomes positive. Experts more productive which gives them advantage, though more impatient. So, η pushes back up. When $\eta \to 1$, experts dominate, but households patience make them save more, η pushes back down. Productivity advantage of Experts balances patience advantage of households, creating an interior steady state.