Macrofinance

Lecture 03: A Simple Real Macro Model with Heterogeneous Agents

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Summer, 2025

Course Overview

1. Intro
2. Portfolio & Consumption Choice

Real Macrofinance Models with Heterogeneous Agents

3. Simple Real Macrofinance Models
4. Endogenous (Price of) Risk Dynamics
5. Contrasting Financial Frictions

Immersion Chapters

Money Models

International Macrofinance Models

Class Overview

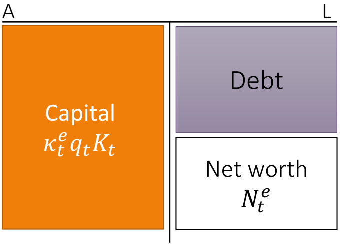
Real Macrofinance Models with Heterogeneous Agents

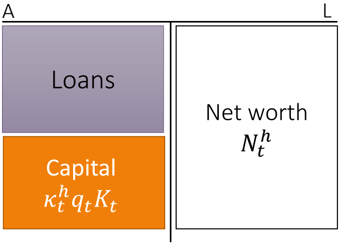
A Two-Sector Macrofinance Model

- Complete Markets Benchmark
  - No risk and no frictions
- Basak-Cuoco Model
  - Risk, one unproductive sector and no frictions
- Kiyotaki-Moore Model
  - No risk, two production sectors and leverage constraints

Two Sector Macrofinance Model Setup

- Expert sector (Farmers) — Household sector (Gatherers)





- Capital shares: κ_t^e (experts), κ_t^h (households), $\kappa_t^e + \kappa_t^h = 1, \kappa_t^e, \kappa_t^h \geqslant 0$
- Experts produce with capital with linear production function $a^e k_t^e (= a^e \kappa_t^e K_t)$.
- Households' production function $a^h(\kappa_t^h) k_t^h$ is fcn. of (aggregate) κ_t^h
  - Productivity $a^h(\kappa^h) \leqslant a^e$ with equality for $\kappa^h = 0$ and strictly decreasing in κ^h
- Experts can only issue debt with collateral constraint: $D_t^e \leqslant \ell \kappa_t^e q_t K_t$
  - $\frac{D_t^e}{N_t^e} \leqslant \ell \frac{\kappa_t^e q_t K_t}{N_t^e} \Leftrightarrow -(1-\theta_t^{K,e}) \leqslant \ell \theta_t^{K,e} \Leftrightarrow (1-\ell)\theta_t^{K,e} \leqslant 1$
- All experts' net worth $N_t^e = \int_0^1 n_t^{e,i} di = n_t^e$; all households' net worth $N_t^h = n_t^h$
- Assumption: aggregate physical capital evolves exogenously

$$\frac{\mathrm{d}K_t}{K_t} = g\mathrm{d}t + \sigma \mathrm{d}Z_t$$

Two Sector Model Setup

Expert sector

- Output: $y_t^e = a^e k_t^e$
- Consumption rate: c_t^e
- Portfolio choice $\theta_t^{K,e} \equiv \kappa_t^e \frac{q_t K_t}{N_t^e}$

- Objective: $\mathbb{E}_0 \left[\int_0^\infty e^{-\rho^e t} \log\left(c_t^e\right) \mathrm{d}t\right]$

Household Sector

- Output: $y_t^h = a^h(\kappa_t^h) k_t^h$
- Consumption rate: c_t^h
- Portfolio choice $\theta_t^{K,h} \equiv \kappa_t^h \frac{q_t K_t}{N_t^h}$

- Objective: $\mathbb{E}_0 \left[\int_0^\infty e^{-\rho^h t} \log\left(c_t^h\right) \mathrm{d}t\right]$

Individual capital evolution: $\frac{\mathrm{d}\check{k}_t^{\tilde{i}}}{\check{k}_t^{\tilde{i}}} = g\mathrm{d}t + \sigma \mathrm{d}Z_t + \mathrm{d}\Delta_t^{k,\tilde{i},i}$

Friction: Can only issue risk-free debt with collateral constraint $(1-\ell)\theta_t^{K,e} \leqslant 1$

Sector Optimization

- Expert's problem

$$
\begin{aligned}
& \max_{\{c_t^e, \theta_t^{K,e}\}_{t=0}^{\infty}} \mathbb{E}_t \left[\int_0^{\infty} e^{-\rho^e t} \log(c_t^e) \mathrm{d}t \right] \\
s.t. \quad & \mathrm{d}n_t^e = \left[-c_t^e \mathrm{d}t + n_t^e \left(\mathrm{d}r_t + \theta_t^{K,e} (\mathrm{d}r_t^{K,e} - r_t) \right) \right] \\
& (1-\ell)\theta_t^{K,e} \leqslant 1, \\
& \theta_t^{K,e} \geqslant 0, \\
& n_0^e \text{ given}
\end{aligned}
$$

- Household's problem

$$
\begin{aligned}
& \max_{\{c_t^h, \theta_t^{K,h}\}_{t=0}^{\infty}} \mathbb{E}_t \left[\int_0^{\infty} e^{-\rho^h t} \log(c_t^h) \mathrm{d}t \right] \\
s.t. \quad & \mathrm{d}n_t^h = \left[-c_t^h \mathrm{d}t + n_t^h \left(\mathrm{d}r_t + \theta_t^{K,h} (\mathrm{d}r_t^{K,h} - r_t) \right) \right] \\
& \theta_t^{K,h} \geqslant 0, \\
& n_0^h \text{ given}
\end{aligned}
$$

Market Clearing

- Goods market

$$C_t^e + C_t^h = a^e K_t^e + a^h(\kappa_t^h) K_t^h$$

- Capital market

$$K_t^e + K_t^h = K_t$$

- Debt market
  clears by Walras law

Solving Macro Models Step-by-Step

0. Postulate aggregates, price processes and obtain return processes
1. For given $\check{\rho} := c^i/n^i$-ratio and SDF^i processes for each i **finance block**

   Toolbox 1: ~~HJB~~, Stochastic Maximum Principle, ~~Martingale approach~~

   a. Goods market clearing *(static)*
   b. Fisher separation theorem

      Portfolio choice θ + asset market clearing
2. Evolution of state variable η (and K) **forward equation**
3. Value functions **backward equation**
   a. Value fcn. as fcn. of individual investment opportunities ω

      Special case: log-utility $\check{\rho} := c/n = \rho$
4. Numerical model solution
5. KFE: Stationary distribution, Fan charts

0. Postulate Aggregates and Processes

- Capital aggregation:
  - Within sector i: $K_t^i \equiv \int k_t^{\tilde{i},i} d\tilde{i}$
  - Across sectors: $K_t = \Sigma_i K_t^i$ $\quad \mathrm{d}K_t/K_t = g\mathrm{d}t + \sigma \mathrm{d}Z_t$
  - Capital share: $\kappa_t^i = K_t^i/K_t$
- Net worth aggregation:
  - Within sector i: $N_t^i \equiv \int n_t^{\tilde{i},i} d\tilde{i}$
  - Across sectors: $N_t = \Sigma_i N_t^i$
  - Net worth share: $\eta_t^i = N_t^i/N_t$,
- Value of capital stock: $q_t K_t$, $\quad dq_t/q_t = \mu_t^q dt + \sigma_t^q dZ_t$
- Postulated SDF-process: $\quad \frac{d\xi_t^i}{\xi_t^i} = \underbrace{\mu_t^{\xi^i}}_{-r_t^i} dt + \underbrace{\sigma_t^{\xi^i}}_{-\varsigma_t^i} dZ_t$

0. Postulate Aggregates and Processes

- Capital aggregation:
  - Within sector i: $K_t^i \equiv \int k_t^{\tilde{i},i} d\tilde{i}$
  - Across sectors: $K_t = \Sigma_i K_t^i$ $\quad \mathrm{d}K_t / K_t = g\mathrm{d}t + \sigma \mathrm{d}Z_t$
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- Value of capital stock: $q_t K_t,$ $\quad dq_t/q_t = \mu_t^q dt + \sigma_t^q dZ_t$
- Postulated SDF-process: $\quad \frac{d\xi_t^i}{\xi_t^i} = \underbrace{\mu_t^{\xi^i}}_{-r_t^i} dt + \underbrace{\sigma_t^{\xi^i}}_{-\varsigma_t^i} dZ_t$

Poll: Why drift of SDF equal is risk-free rate

1. no idio risk
2. $e^{-r_f} = \mathbb{E}[SDF]$
3. no jump in consumption

0. Return Processes

- Using Itô's product rule, the returns on capital follow

$$
\begin{aligned}
\mathrm{d}r_t^{K,e} &= \overbrace{\frac{a^e}{q_t}\mathrm{d}t}^{\text{Dividend Yield}} + \overbrace{\frac{\mathrm{d}(q_t k_t^e)}{q_t k_t^e}}^{\text{Capital Gain}} \\
&= \left(\frac{a^e}{q_t} + g + \mu_t^q + \sigma\sigma_t^q\right)\mathrm{d}t + (\sigma + \sigma_t^q)\mathrm{d}Z_t \\
\mathrm{d}r_t^{K,h} &= \left(\frac{a^h(\kappa_t^h)}{q_t} + g + \mu_t^q + \sigma\sigma_t^q\right)\mathrm{d}t + (\sigma + \sigma_t^q)\mathrm{d}Z_t
\end{aligned}
$$

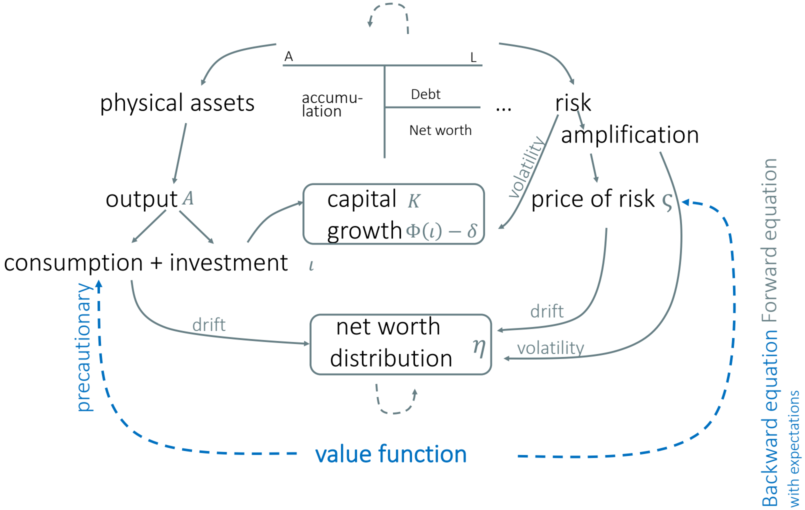
2. GE: Markov States and Equilibria

- Equilibrium is a map

$$\begin{array}{ccc} \text{Histories of shocks} & \longrightarrow & \text{prices } q_t, \varsigma_t^i, \iota_t^i, \theta_t^i \\ \{\mathbf{Z}_{s\in[0,t]}\} \searrow & & \nearrow \\ & \text{net worth distribution} & \\ & \eta_t^e = \frac{N^e}{q_t K_t} \in (0,1) & \end{array}$$

- All agents maximize utility
  - Choose: consumption, portfolio, ...
- All markets clear
  - Consumption, capital, debt,

The Big Picture



Overview: This Lecture

- $\boxed{\textbf{Complete Markets Benchmark}}$
  - $\sigma = 0$ and $\ell \to \infty$
- Basak Cuoco Model
  - $a^h \to -\infty$ and $\ell \to \infty$
- Kiyotaki-Moore Model
  - $\sigma = 0$ and $\ell < \infty$

1. Individual Agent Choice (Complete Markets Benchmark)

- $a^e \geqslant a^h(1-\kappa_t) \Rightarrow \theta_t^{K,h} \geqslant 0$ must bind $\Rightarrow \kappa_t = 1$
- By log-utility $\check{\rho}^i := c_t^i/n_t^i = \rho^i \implies C_t^i = \rho^i N_t^i$ for $i \in \{e, h\}$.
  Using $N_t^e = \eta_t q_t K_t$ and $N_t^h = (1-\eta_t) q_t K_t$, market clearing implies

$$q_t = \frac{a^e \kappa_t + a^h(1-\kappa_t)(1-\kappa_t)}{\rho^e \eta_t + \rho^h(1-\eta_t)}$$

- Using $\kappa_t = 1$ in equilibrium

$$q_t = \frac{a^e}{\rho^e \eta_t + \rho^h(1-\eta_t)}.$$

- Since there is no leverage constraint, the experts instantly take out loans to finance the purchase of all available capital in the economy.
  Capital market clearing condition

$$\theta_t^{K,e} \eta_t q_t K_t + \theta_t^{K,h}(1-\eta_t) q_t K_t = q_t K_t,$$

  implies that $\theta_t^{K,e} = 1/\eta_t$ instantaneously.

2. Evolution of state variable η_t

- Recall $\eta_t = N_t^e / N_t$ so that by Itô's lemma

$$\mu_t^\eta = \mu_t^{N^e} - \mu_t^N = (1-\eta_t)(\mu_t^{N^e} - \mu_t^{N^h})$$

since $\mu_t^N = \frac{1}{N_t}\frac{\mathrm{d}N_t}{\mathrm{d}t} = \eta_t \mu_t^{N^e} + (1-\eta_t)\mu_t^{N^h}$.

- The evolution of experts' net worth

$$\mu_t^{N^e} = -\rho^e + r_t + \theta^{K,e}(r_t^{K,e} - r_t) = -\rho^e + r_t$$

since $r_t^{K,e} = r_t$.

- The evolution of households' net worth

$$\mu_t^{N^h} = -\rho^h + r_t + \theta^{K,h}(r_t^{K,h} - r_t) = -\rho^h + r_t$$

since $\theta^{K,h} = 0$.

- Hence, net worth share follows

$$\mu_t^\eta = -(1-\eta_t)(\rho^e - \rho^h)$$

which yields the following closed-form expression for η_t

$$\eta_t = \frac{\eta_0 e^{-(\rho^e-\rho^h)t}}{1-\eta_0+\eta_0 e^{-(\rho^e-\rho^h)t}}$$

for initial condition η_0.

Frictionless Case: Main Takeaways

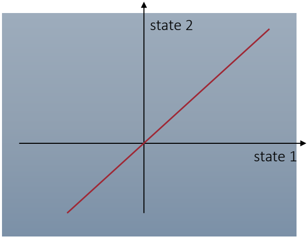
1. **Capital Allocation:** always held by the most efficient sector
2. **Consumption Allocation:** determined by the initial wealth distribution and wealth only moves due to differences in preferences for the timing of consumption
   - $\rho^e = \rho^h$: every initial condition leads to a steady state
   - $\rho^e > \rho^h$: converges in the long run to a boundary $\eta = 0$
   - $\rho^e < \rho^h$: converges in the long run to a boundary $\eta = 1$
3. **Price of Capital:** $q_t = \frac{a^e}{\rho^e \eta_t + \rho^h (1 - \eta_t)}$
   - $\rho^e = \rho^h$: $q_t = a^e / \rho^e$ is constant
   - otherwise, it rises over time because the agents with the lower marginal propensity to consume become richer as time passes.

Overview: This Lecture

- Complete Markets Benchmark
  - $\sigma = 0$ and $\ell \to \infty$
- $\boxed{\textbf{Basak Cuoco Model}}$
  - $a^h \to -\infty$ and $\ell \to \infty$
- Kiyotaki-Moore Model
  - $\sigma = 0$

Financial Frictions and Distortions

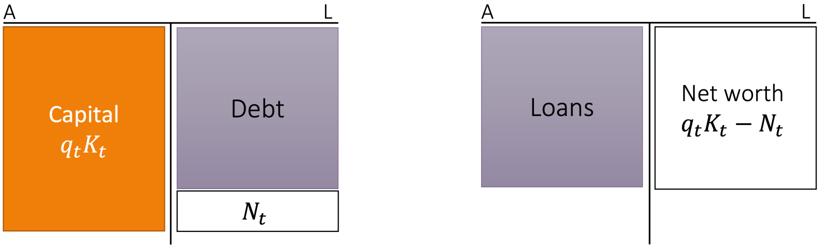
- **Incomplete markets:**
  - **"natural" leverage constraint** (*BruSan*)
  - Costly state verification (*BGG*)
- Leverage constraints
  (no "liquidity creation")
  - Exogenous limit (*Bewley/Ayagari*)
  - Collateral constraint
    - Current price $Rb_t \leqslant q_t k_t$
    - Next period's price $Rb_t \leqslant q_{t+dt} k_t$ (*KM*)
    - Next period's VaR $Rb_t \leqslant VaR_t(q_{t+dt}) k_t$ (*BruPed*)



Simple Two Sector Model: Basak Cuoco (1998)

- Special case with
  - risk $\sigma > 0$
  - no leverage constraint $\ell \to \infty$
  - unproductive household sector $a^h \to -\infty$
- Expert sector

Household sector



See Lecture Notes, Chapter 3 or Handbook of Macroeconomics 2017, Chapter 18

1. Given SDF processes, derive individual FOC

- Hamiltonian for the experts is given by

$$
\begin{aligned}
\mathcal{H}_t^e &= e^{-\rho^e t} \log c_t^e + \xi_t^e n_t^e \mu_t^{n^e} - \varsigma_t^e \xi_t^e n_t^e \sigma_t^{n^e} \\
&= e^{-\rho^e t} \log c_t^e + \xi_t^e \left[-c_t^e + n_t^e r_t + n_t^e \theta_t^{K,e} \left(\frac{a^e}{q_t} + g + \mu_t^q + \sigma \sigma_t^q - r_t \right) \right] \\
&\quad - \varsigma_t^e \xi_t^e n_t^e \theta_t^{K,e} (\sigma + \sigma_t^q)
\end{aligned}
$$

- FOCs wrt c_t^e and $\theta_t^{K,e}$ are given by

$$
e^{-\rho^e t} \left(c_t^e \right)^{-1} = \xi_t^e
$$

$$
\frac{a^e}{q_t} + g + \mu_t^q + \sigma \sigma_t^q - r_t = \varsigma_t^e (\sigma + \sigma_t^q)
$$

- The costate equation reads by virtue of the

$$
\begin{aligned}
d\xi_t^e &= -\frac{\partial H_t^e}{\partial n_t^e} dt - \varsigma_t^e \xi_t^e dZ_t \\
&= -r_t \xi_t^e - \varsigma_t^e \xi_t^e dZ_t
\end{aligned}
$$

where the equality follows from the first-order condition for $\theta_t^{K,e}$.

1. Given SDF processes, derive individual FOC

- Using the first-order condition for consumption, we also find by Itô's lemma that

$$d\xi_t^e = \left[-\rho^e - \mu_t^{c^e} + (\sigma_t^{c^e})^2\right] dt - \sigma_t^{c^e} dZ_t$$

- Since for log-utility $c_t^e / n_t^e = \rho^e$, $\mu_t^{c^e} = \mu_t^{n^e}$ and $\sigma_t^{c^e} = \sigma_t^{n^e}$.
- Then the price of risk is given by $\varsigma_t^e = \sigma_t^{c^e} = \sigma_t^{n^e} = \theta_t^{K,e}(\sigma + \sigma_t^q)$.
- Similarly, the household sector will consume according to $c_t^h / n_t^h = \rho^h$.

2. GE: Markov States and Equilibria

- Equilibrium is a map

$$\begin{array}{ccc} \text{Histories of shocks} & \longrightarrow & \text{prices } q_t, \varsigma_t^i, \iota_t^i, \theta_t^i \\ \{\mathbf{Z}_{s\in[0,t]}\} \quad \searrow & & \nearrow \\ & \text{net worth distribution} & \\ & \eta_t^e = \frac{N^e}{q_t K_t} \in (0,1) & \end{array}$$

- All agents maximize utility
  - Choose consumption, portfolio, ...
- All markets clear
  - Consumption, capital, debt,

2. Evolution of state variable η_t

- Recall that $\eta_t = \frac{N_t^e}{q_t K_t}$
- The total wealth of experts N_t^e follows

$$\frac{dN_t^e}{N_t^e} = \frac{dn_t^e}{n_t^e} = -\frac{c_t^e}{n_t^e}dt + r_t dt + \theta_t^{K,e}\left[dr_t^K - r_t dt\right]$$
$$= -\frac{c_t^e}{n_t^e}dt + r_t dt + \theta_t^{K,e}\left\{\left[\frac{a^e}{q_t} + g + \mu_t^q + \sigma\sigma_t^q - r_t\right]dt + (\sigma + \sigma_t^q)dZ_t\right\}$$
$$= -\frac{c_t^e}{n_t^e}dt + r_t dt + \theta_t^{K,e}\left\{\varsigma_t^e(\sigma + \sigma_t^q)dt + (\sigma + \sigma_t^q)dZ_t\right\}.$$

- Also

$$\frac{d(q_t K_t)}{q_t K_t} = \left[\mu_t^q + g + \sigma\sigma_t^q\right]dt + (\sigma + \sigma_t^q)dZ_t$$
$$= \left[r_t - \frac{a^e}{q_t} + \varsigma_t^e(\sigma + \sigma_t^q)\right]dt + (\sigma + \sigma_t^q)dZ_t.$$

- Apply Itô's quotient rule to $\eta_t = N_t^e / q_t K_t$:

$$\frac{d\eta_t}{\eta_t} = \left[-\frac{c_t^e}{n_t^e} + \frac{a^e}{q_t} - (1 - \theta_t^{K,e})(\sigma + \sigma_t^q)\left(\varsigma_t^e - (\sigma + \sigma_t^q)\right)\right]dt - (1 - \theta_t^{K,e})(\sigma + \sigma_t^q)dZ_t$$

2. Evolution of state variable η_t

- Using $c_t^e = \rho^e n_t^e$ and $\varsigma_t^e = \theta_t^{K,e}(\sigma + \sigma_t^q)$, we have

$$\frac{d\eta_t}{\eta_t} = \left[-\rho^e + \frac{a^e}{q_t} + (1-\theta_t^{K,e})^2(\sigma + \sigma_t^q)^2\right]dt - (1-\theta_t^{K,e})(\sigma + \sigma_t^q)dZ_t$$

- Goods market clearing

$$C_t = \rho^e \eta_t q_t K_t + \rho^h(1-\eta_t)q_t K_t = a^e K_t \implies q_t = \frac{a^e}{\rho^e \eta_t + \rho^h(1-\eta_t)}$$

- Itô's lemma implies

$$\sigma_t^q = -\frac{\rho^e - \rho^h}{\rho^e \eta_t + \rho^h(1-\eta_t)}\sigma_t^\eta \eta_t$$

- Capital market clearing

$$\theta_t^{K,e} = \frac{q_t K_t}{N_t^e} = \frac{1}{\eta_t}$$

- Hence η_t follows

$$\frac{d\eta_t}{\eta_t} = (1-\eta_t)\left[-(\rho^e - \rho^h) + \frac{1-\eta_t}{\eta_t^2}(\sigma + \sigma_t^q)^2\right]dt + \frac{1-\eta_t}{\eta_t}(\sigma + \sigma_t^q)dZ_t$$

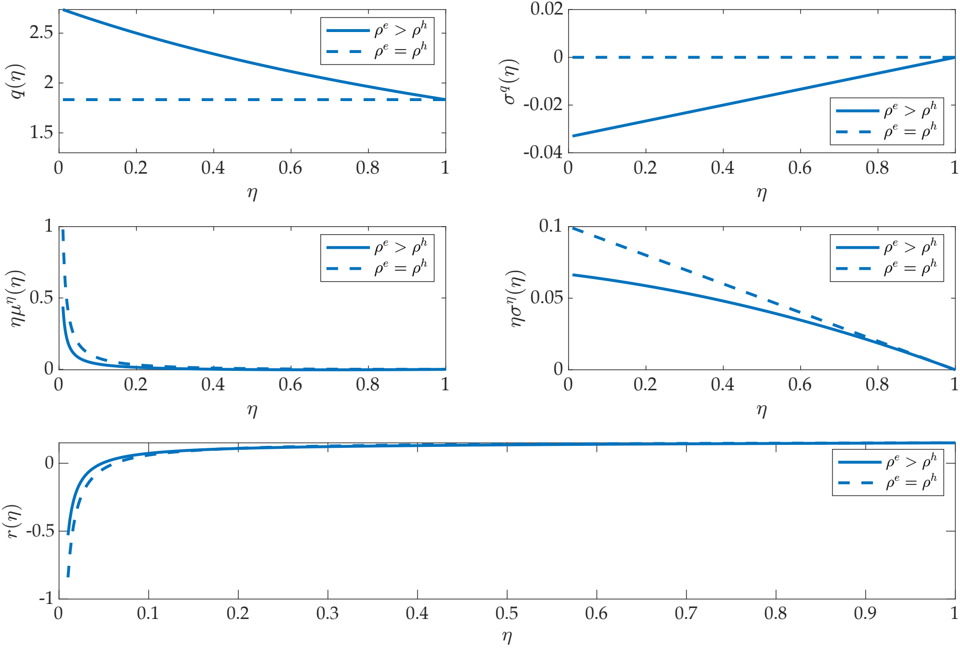
2. Evolution of state variable η_t

- Combining the above we get

$$\begin{aligned}\frac{\mathrm{d}\eta_t}{\eta_t} =&(1-\eta_t)\left[-(\rho^e-\rho^h)+\frac{1-\eta_t}{\eta_t^2}\frac{\left[\rho^e\eta_t+\rho^h(1-\eta_t)\right]^2}{(\rho^e)^2}\sigma^2\right]\mathrm{d}t\\ &+\frac{1-\eta_t}{\eta_t}\frac{\rho^e\eta_t+\rho^h(1-\eta_t)}{\rho^e}\sigma\mathrm{d}Z_t\end{aligned}$$

a simple one-dimensional stochastic differential equation (SDE).

4. Numerical Example of Basak-Cuoco



$$\rho^e = 0.06,\ \rho^h = 0.04,\ a^e = 0.11,\ \sigma = 0.10,\ g{=}0.1$$

Observation of Basak-Cuoco Model

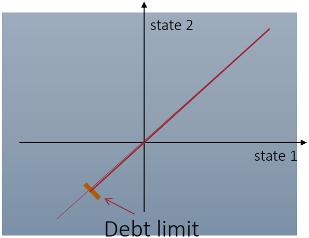
1. **Changes in Risk-free Rate drives Risk Premium**
   Price of risk ς^e, i.e., Sharpe ratio is:
   $$\frac{1}{\eta_t}(\sigma+\sigma_t^q) = \frac{\eta_t\rho^e + (1-\eta_t)\rho^h + g + \mu_t^q + \sigma\sigma_t^q - r_t}{\sigma+\sigma_t^q}$$
   - Goes to ∞ as η_t goes to zero
   - RHS can only go to ∞ if risk-free rate $r_t \to \infty$
     $r_t = \eta_t\rho^e + (1-\eta_t)\rho^h + g + \mu_t^q + \sigma\sigma_t^q - (\sigma+\sigma_t^q)^2/\eta_t \to -\infty$
   - **Special case** $\rho^e = \rho^h$: since $q = \frac{a^e}{\rho^e}$ is constant ($\mu^q = \sigma^q = 0$), easy to see.
2. q_t **Movements Mitigates rather than Amplifies Shocks,** $\sigma^q \leqslant 0$
   Stationary distribution requires $\rho^e > \rho^h \Rightarrow q_t$ appreciates after $dZ_t < 0$-shock
   - Otherwise for $\rho^e \leqslant \rho^h$, $\mu_t^\eta > 0\ \forall\eta, \Rightarrow \eta^{SS} = 1$
     in the long run HH-net worth share vanishes (degenerated stationary distribution)
   - Alternative ways out to obtain a stationary distribution
     without $q(\eta)$ decreasing in η (instead of $\rho^e > \rho^h$):
     - Switching types (jumps) (BGG)
     - 2 types of experts (BruSan AEJ:Macro international paper)

Overview: This Lecture

- Complete Markets Benchmark
  - $\sigma = 0$ and ℓ unconstrained
- Basak Cuoco Model
  - $a^h \to -\infty$ and ℓ unconstrained
- $\boxed{\textbf{Kiyotaki-Moore Model}}$
  - $\sigma = 0$ and $\ell < \infty$, and $a^h(\cdot)$ is a function of κ^h

Financial Frictions and Distortions

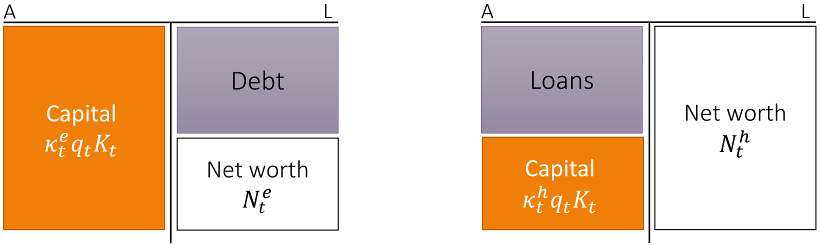
- Incomplete markets:
  - "natural" leverage constraint (*BruSan*)
  - Costly state verification (*BGG*)
- $\boxed{\textbf{Leverage constraints:}}$
  (no "liquidity creation")
  - Exogenous limit (*Bewley/Ayagari*)
  - Collateral constraint
    - **Current price** $Rb_t \leqslant q_t k_t$
    - **Next period's price** $Rb_t \leqslant q_{t+dt} k_t$ **(KM)**
    - Next period's VaR $Rb_t \leqslant VaR_t(q_{t+dt})k_t$ (*BruPed*)



Debt limit
can depend on
prices/volatility

Kiyotaki Moore (1997) in Cts. Time

- No risk: $\sigma = 0$
- Leverage constraint: $\ell < \infty$
- Households can produce: $a^h(\kappa^h)$ is a decreasing function of κ^h
- Expert sector (Farmers) Household sector (Gatherers)



Kiyotaki Moore (1997) in Cts. Time

Expert Sector (Farmers)

- Output: $y_t^e = a^e k_t^e = a^e \kappa_t^e K_t$
- Consumption rate: c_t^e
- Objective: $\int_0^\infty e^{-\rho^e t} \log(c_t^e) \mathrm{d}t$

Household Sector (Gatherers)

- Output: $y_t^h = a^h(\kappa_t^h) k_t^h = a^h(\cdot) \kappa_t^h K_t$
- Consumption rate: c_t^e
- Objective: $\int_0^\infty e^{-\rho^h t} \log(c_t^h) \mathrm{d}t$

Assumptions:

- Experts are more impatient $\rho^e > \rho^h$
- Productivity $a^h(\kappa^h) \leqslant a^e$ with equality for $\kappa^h = 0$ and constant return to scale individually, but $a^h(\cdot)$ decreasing in (aggregate) κ^h
- No equity issuance
- Debt issuance with collateral constraint: $D_t^e \leqslant \ell \kappa_t^e q_t K_t$

  $$\Leftrightarrow \frac{D_t^e}{N_t^e} \leqslant \ell \frac{\kappa_t^e q_t K_t}{N_t^e} \Leftrightarrow -(1 - \theta_t^{K,e}) \leqslant \ell \theta_t^{K,e}$$

  Collateral constraint in KM97: $D_t^e (1 + r_{t+dt} dt) \leqslant \ell \kappa_t^e q_{t+dt} K_t$

1. Portfolio choices: Hamiltonian Approach

- Experts' problem: $\max_{c_t^e,\theta_t^{K,e}} \int_s^\infty e^{-\rho^e t} u(c_t^e) \mathrm{d}t$ s.t. $(1-\ell)\theta_t^{K,e} \leqslant 1$, and

$$\frac{\mathrm{d}n_t^e}{\mathrm{d}t} = \left[-c_t^e + n_t^e \left(r_t + \theta_t^{K,e}(r_t^{K,e} - r_t)\right)\right]$$

- Households' problem: $\max_{c_t^h,\theta_t^h} \int_s^\infty e^{-\rho^h t} u(c_t^h) \mathrm{d}t$, s.t.

$$\frac{\mathrm{d}n_t^h}{\mathrm{d}t} = \left[-c_t^h + n_t^h \left(r_t + \theta_t^{K,h}(r_t^{K,h} - r_t)\right)\right],$$

- The Hamiltonians can be constructed as

$$\begin{aligned}
\mathcal{H}_t^e =& e^{-\rho^e t} u(c_t^e) + \xi_t^e \overbrace{\left[-c_t^e + n_t^e \left(r_t + \theta_t^{K,e}(r_t^{K,e} - r_t)\right)\right]}^{\mu_t^{n^e} n_t^e} + \xi_t^e n_t^e \lambda_t^\ell \left(1 - (1-\ell)\theta_t^{K,e}\right) \\
\mathcal{H}_t^h =& e^{-\rho^h t} u(c_t^h) + \xi_t^h \left[-c_t^h + n_t^h \left(r_t + \theta_t^{K,h}(r_t^{K,h} - r_t)\right)\right]
\end{aligned}$$

- ξ_t^i multiplier on the budget constraint, $\xi_t^e n_t^e \lambda_t^\ell$ multiplier on leverage constraint
  - Later we show that co-state variable ξ_t^i equals SDF, which for log-utility $= e^{-\rho^i t} \frac{1}{\rho^i n_t^i}$
- Fisher Separation Theorem btw. consumption and portfolio choice

1. Hamiltonian Approach: First order conditions

- FOC w.r.t c_t^i:

$$\begin{cases} e^{-\rho^e t} u'(c_t^e) = & \xi_t^e \\ e^{-\rho^h t} u'(c_t^h) = & \xi_t^h \end{cases} \Rightarrow c_t^i = \rho^i n_t^i, \text{ log utility}$$

1. Hamiltonian Approach: First order conditions

- FOC w.r.t c_t^i:

$$\begin{cases} e^{-\rho^e t} u'(c_t^e) = \xi_t^e \\ e^{-\rho^h t} u'(c_t^h) = \xi_t^h \end{cases} \Rightarrow c_t^i = \rho^i n_t^i, \text{ log utility}$$

- FOC w.r.t $\theta_t^{K,i}$:

$$\begin{cases} r_t^{K,e} - r_t = (1-\ell)\lambda_t^\ell \\ r_t^{K,h} - r_t = 0 \end{cases}$$

- Where capital returns are: (dividend + price drift)

$$\begin{cases} r_t^{K,e} = \dfrac{a^e}{q_t} + \dfrac{1}{q_t}\dfrac{\mathrm{d}q_t}{\mathrm{d}t} \\ r_t^{K,h} = \dfrac{a^h(\kappa_t^h)}{q_t} + \dfrac{1}{q_t}\dfrac{\mathrm{d}q_t}{\mathrm{d}t} \end{cases}$$

2. Net Worth Evolution

- Equilibrium objects are functions of state, net worth share, $\eta_t = \frac{N_t^e}{N_t} = \frac{N_t^e}{q_t \bar{K}}$
- Price dynamics: (No arbitrage for households)

$$\frac{a^h(\kappa_t^h)}{q_t} + \frac{1}{q_t}\frac{\mathrm{d}q_t}{\mathrm{d}t} = r_t,$$

- State dynamics:

$$\mu_t^N dt = \frac{dN_t}{N_t} = \underbrace{\frac{N_t^e}{N_t}}_{\eta_t} \mu_t^{N^e} dt + \underbrace{\frac{N_t^h}{N_t}}_{(1-\eta_t)} \mu_t^{N^h} dt$$

$$\mu_t^\eta = \mu_t^{N^e} - \mu_t^N = (1-\eta_t)(\mu_t^{N^e} - \mu_t^{N^h})$$

$$= (1-\eta_t)[-(\rho^e - \rho^h) + \theta_t^{K,e}(\frac{a^e}{q_t} + \frac{1}{q_t}\frac{\mathrm{d}q_t}{\mathrm{d}t} - r_t) - \theta_t^{K,h}\overbrace{(\frac{a^h(\kappa_t^h)}{q_t} + \frac{1}{q_t}\frac{\mathrm{d}q_t}{\mathrm{d}t} - r_t)}^{=0 \text{ from above}}]$$

$$= (1-\eta_t)[-(\rho^e - \rho^h) + \theta_t^{K,e}\underbrace{(\frac{a^e}{q_t} - \frac{a^h(\kappa_t^h)}{q_t})}_{=r_t^{K,e} - r_t^{K,h}}]$$

2. Net Worth Evolution

- Equilibrium objects $(\kappa^e, \kappa^h, q, r)$ are functions of state, net worth share, $\eta_t = \frac{N_t^e}{N_t} = \frac{N_t^e}{q_t \bar{K}}$
- pinned down by:

$$q_t K_t[\rho^e \eta_t + \rho^h(1-\eta_t)] = [a^e \kappa_t^e + a^h(\kappa_t^h)\kappa_t^h]K_t \qquad \text{(Goods market)}$$

$$\underbrace{\theta_t^{Ke}\eta_t}_{=\kappa_t^e} q_t K_t + \underbrace{\theta_t^{Kh}(1-\eta_t)}_{=\kappa_t^h} q_t K_t = q_t K_t \qquad \text{(Capital market)}$$

$$\kappa_t^e \leqslant \frac{\eta_t}{1-\ell} \qquad \text{(Collateral Constraint)}$$

$$\mu_t^\eta = (1-\eta_t)\left[-(\rho^e - \rho^h) + \theta_t^{K,e}\frac{a^e - a^h(\kappa_t^h)}{q_t}\right]$$

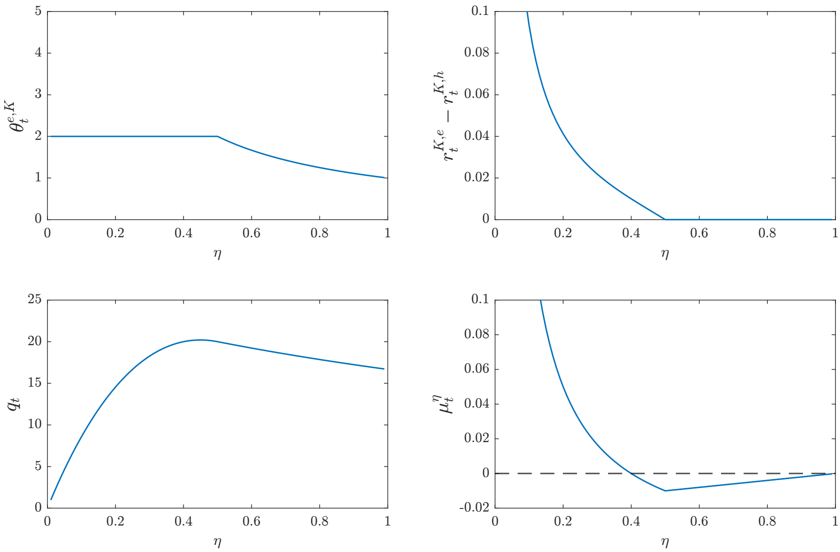
- simplified to (and define $\kappa_t := \kappa_t^e = 1 - \kappa_t^h$)

$$q_t[(\rho^e - \rho^h)\eta_t + \rho^h] = \kappa_t a^e + (1-\kappa_t)a^h(1-\kappa_t)$$

$$\kappa_t \leqslant \frac{\eta_t}{1-\ell}$$

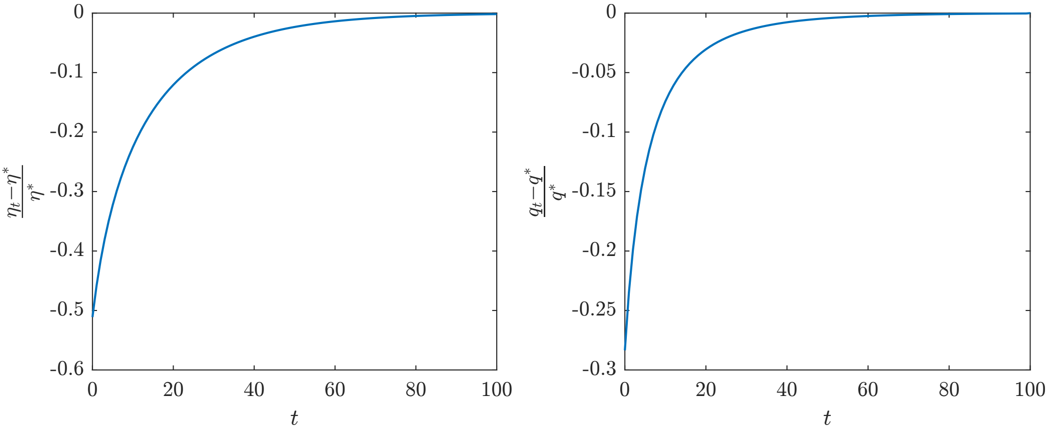
$$\mu_t^\eta = (1-\eta_t)\left[-(\rho^e - \rho^h) + \frac{\kappa_t}{\eta_t}\frac{a^e - a^h(1-\kappa_t)}{q_t}\right]$$

Global Non-linear Solution



Parameters: $\rho^e = 0.06, \rho^h = 0.04, \ell = 0.05, a^e = 1.0, a^h(1-\kappa) = a^e\kappa$

Impulse Responses



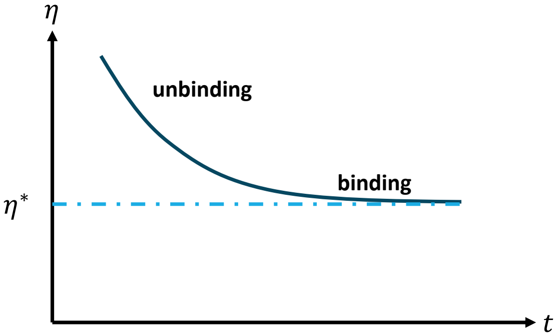
Impulse response function with 30% (of η) negative redistribution shock.
Parameters: $\rho^e = 0.06, \rho^h = 0.04, \ell = 0.5, a^e = 1.0, a^h(1-\kappa) = a^e\kappa$

Log-linearization around Steady State

1. Derive steady state with $\mu^\eta = 0$
   with its properties
2. Log-linearize around steady state
   characterize dynamical system locally around the steady state

The Steady State: Binding Collateral Constraint

- The collateral constraint always binds in the steady state
  - If collateral constraint does not bind $\lambda_t^\ell = 0$ and hence $r^{K,e} = r^{K,h}$, i.e. $a^e = a^h(\cdot)$
- Note, the constraint does not need to bind only if $\kappa_t = 1$.
  - Then $\mu_t^\eta = (1-\eta_t)(\rho^h - \rho^e)$
  - as $\rho^e > \rho^h \Rightarrow \mu_t^\eta < 0$, i.e. η declines
- Characterization of Steady State (Next Page)



Steady State

- Since Collateral constrained binds, steady state capital share
$$\kappa^{SS} = \frac{\eta^{SS}}{1-\ell}$$
- Expert sector's net worth share is $\eta_t := \frac{N_t^e}{q_t \bar{K}}$, is constant, i.e. $\mu_t^\eta := \frac{\mathrm{d}\eta_t}{\mathrm{d}t} = 0$
$$q^{SS}[(\rho^e - \rho^h)\eta^{SS} + \rho^h] = \kappa^{SS} a^e + (1-\kappa^{SS}) a^h (1-\kappa^{SS})$$
$$(\rho^e - \rho^h) = \frac{\kappa^{SS}}{\eta^{SS}} \frac{a^e - a^h(1-\kappa^{SS})}{q^{SS}} \quad \text{for } \mu^\eta = 0$$
- Combine
$$\kappa^{SS} a^e - \kappa^{SS} a^h (1-\kappa^{SS}) + q^{SS}\rho^h = \kappa^{SS} a^e + (1-\kappa^{SS}) a^h (1-\kappa^{SS})$$
$$\Rightarrow \quad q^{SS} = a^h(1-\kappa^{SS})/\rho^h,$$
where the steady state κ^{SS} is implicitly given by:
$$\frac{\rho^e - \rho^h}{\rho^h} = \frac{1}{1-\ell} \frac{a^e - a^h(1-\kappa^{SS})}{a^h(1-\kappa^{SS})}.$$
- For specific functional form $a^h(\cdot) = a^e \kappa_t$:
$$\kappa^{SS} = \frac{1}{(1-\ell)(\rho^e - \rho^h)/\rho^h + 1} \quad \Rightarrow \eta^{SS} = \frac{1-\ell}{(1-\ell)(\rho^e - \rho^h)/\rho^h + 1}$$

Steady State: Comparative Static

- For the specific example $a^h(\cdot) = a^e \kappa$:

- For higher leverage, ℓ, (i.e. less tight collateral constraint)
  - κ^{SS}, SS-capital share, is higher.
  - η^{SS}, SS-net worth share, is lower.
  - $q^{SS} = \frac{a^h}{\rho^h}$, price of capital, is higher.
    $q^{SS}\bar{K}$, total wealth in the economy, is higher too.
  - $N^{e,SS}$ experts' net worth in steady state, is higher (Check?)
  - Comparative Static = permanent (long-run) shift to new steady state
  - Next: Dynamics of how to return to the old steady state
    (after an unanticipated shock)

Log-linearized Dynamics Around Steady State

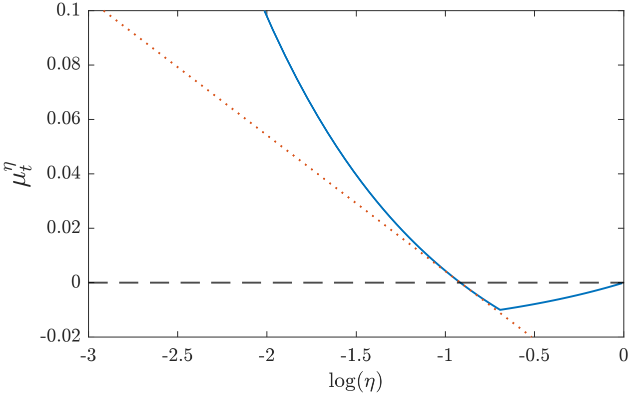
- Analytical solutions to η_t, q_t dynamics are hard to obtain. Expansion around the steady state:

$$\log(\eta_t/\eta^{SS}) = \hat{\eta}_t$$
$$\log(q_t/q^{SS}) = \hat{q}_t$$
$$\log(r_t/r^{SS}) = \hat{r}_t$$
$$\log(a_t^h/a^{h,SS}) = \hat{a}_t^h$$

- Expression for $\hat{a}_t^h, \hat{q}_t^h$ as a function of $\hat{\eta}_t$
- State dynamics and price dynamics become:

$$\frac{\mathrm{d}\hat{\eta}_t}{\mathrm{d}t} = \frac{1-\eta^{SS}}{1-\ell}\left(-\frac{a^{h,SS}}{q^{SS}}\hat{a}_t^h - \frac{a^e - a^{h,SS}}{q^{SS}}\hat{q}_t\right)$$
$$\frac{\mathrm{d}\hat{q}_t}{\mathrm{d}t} = r^{SS}(\hat{r}_t + \hat{q}_t - \hat{a}_t^h)$$

Global vs. Log-linearized Solution for η-drift

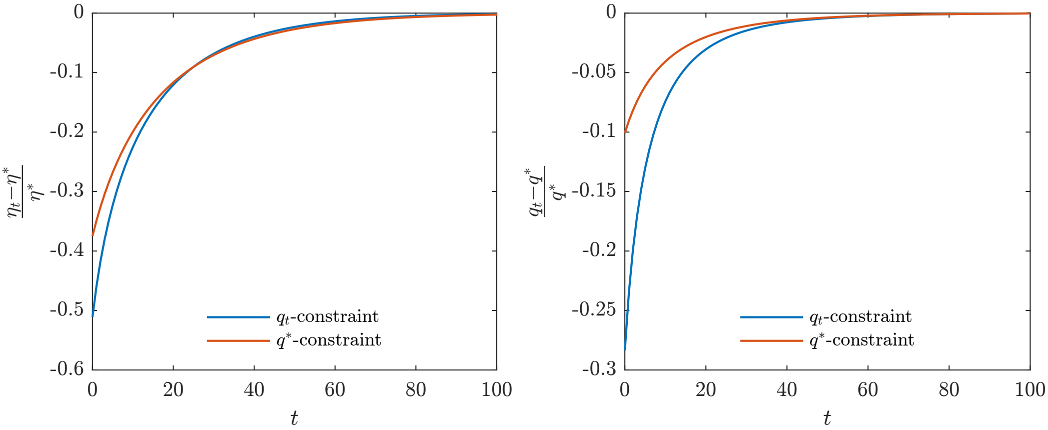


- Note: x-axis is $\log(\eta)$, since log-linearization

Decomposing Amplification Effects

- Start at steady state $\{q^{SS}, \eta^{SS}, \kappa^{SS}\}$
- Shock: redistribution of a fraction of experts' net worth share to households
  - In KM productivity shock lasts for one period (not for an instant), causes initial redistribution
- **Impulse response function** (with deterministic recovery)
- Immediate impact at $t = 0$
  - direct redistributive effect/shock
  - **price-net worth effect**
    decline in q_t reduces experts' net worth share as they are levered $\Rightarrow$ feedback
  - **price-collateral effect**
    decline in q_t tightens collateral constraints $\Rightarrow$ feeds back on price-net worth effect
- Subsequent impact $t > 0$ (which feeds back to immediate impact)
- **Decomposition:**
  Switch off price-collateral effect by assuming that
  collateral constraint is determined by SS-price q^{SS} instead of equilibrium price q_t.
  (Formally, collateral constraint, $\kappa_t \leqslant \frac{\eta_t}{1-\ell}$, becomes $\kappa_t \leqslant \frac{\eta_t}{1-\ell q^{SS}/q_t}$.)

Decomposition of Amplification: Impulse Response Fcn



Impulse response function with 30% (of η) negative redistribution shock.
Parameters: $\rho^e = 0.06, \rho^h = 0.04, \ell = 0.5, a^e = 1.0, a^h(1-\kappa) = \kappa$

Decomposing Amplification at $t = 0$

- At time t, the economy is at steady state $\{q^{SS}, \eta^{SS}, \kappa^{SS}\}$.
- Negative initial/direct redistributive shock $\eta' = (1-\epsilon)\eta^{SS}$, new price q', and capital holding κ' solves:

$$q' = \frac{\kappa' a^e + (1-\kappa')a^h(1-\kappa')}{(\rho^e - \rho^h)\eta' + \rho^h} \qquad \text{(Goods market)}$$

$$\kappa' = \frac{\eta^{SS}(1-\epsilon)}{1-\ell} \qquad (q_t\text{-constraint})$$

$$\kappa' = \frac{\eta^{SS}(1-\epsilon)}{1-\ell q^{SS}/q'} \qquad (q^{SS}\text{-constraint})$$

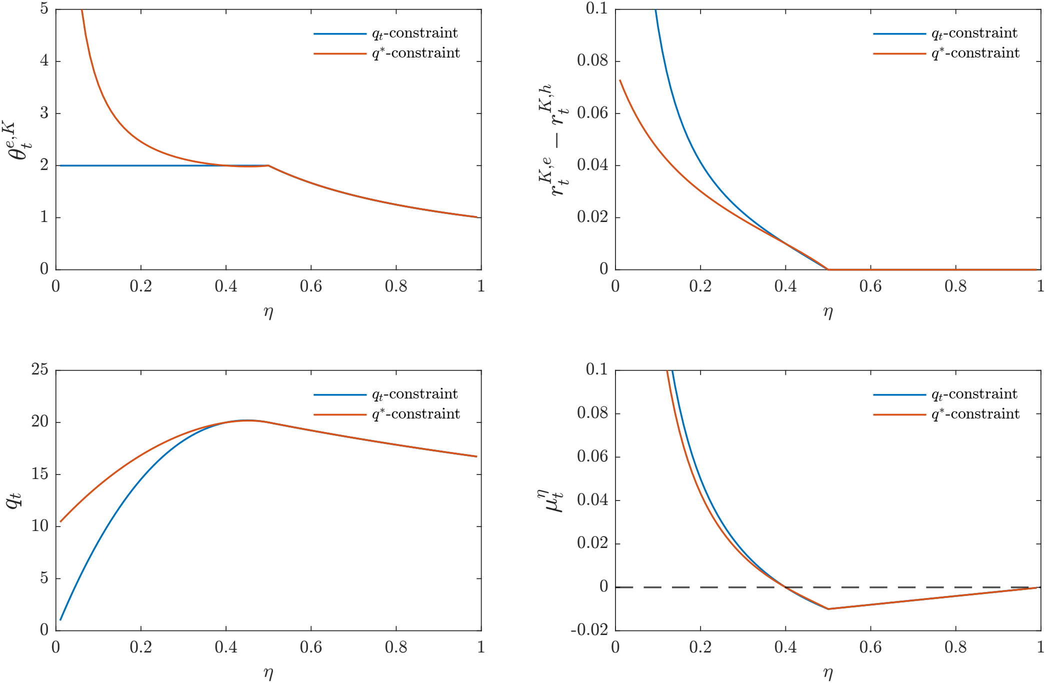
- However, debt contract was signed by old price $q^{SS} \Rightarrow \eta$ drops further
- Consider the balance sheet (first round effect):

$$\frac{\eta'}{1-\ell} q' = \frac{\ell}{1-\ell}\eta' q^{SS} + \eta'' q'$$

To get the convergence result, we need to do this procedure iteratively.

Decomposing Amplification for $t > 0$ (global solution)

$\rho^e = 0.06, \rho^h = 0.04, \ell = 0.05, a^e = 1.0, a^h(1-\kappa) = a^e\kappa$



Decomposing Amplification for $t > 0$ (log-linearized sol.)

- Price dynamics:

$$\frac{\mathrm{d}\hat{q}_t}{\mathrm{d}t} = r^{SS}\hat{r}_t - r^{SS}\hat{a}^h_t + r^{SS}\hat{q}_t$$

- State dynamics with q_t-collateral constraint:

$$\frac{\mathrm{d}\hat{\eta}_t}{\mathrm{d}t} = \frac{1-\eta^{SS}}{1-\ell}\left(-\frac{a^{h,SS}}{q^{SS}}\hat{a}^h_t - \frac{a^e - a^{h,SS}}{q^{SS}}\hat{q}_t\right)$$

- State dynamics with q^{SS}-collateral constraint:

$$\frac{\mathrm{d}\hat{\eta}_t}{\mathrm{d}t} = \frac{1-\eta^{SS}}{1-\ell}\left(-\frac{a^{h,SS}}{q^{SS}}\hat{a}^h_t - \frac{1}{1-\ell}\frac{a^e - a^{h,SS}}{q^{SS}}\hat{q}_t\right)$$

$\hat{q}_t, \hat{a}^h_t, \hat{r}_t$ are different with different constraints.

“Single Shock Critique”

- Critique: After the shock all agents in the economy know that the economy will deterministically return to the steady state.
  - Length of slump is deterministic (and commonly known)
    - No safety cushion needed
- In reality an adverse shock may be followed by additional adverse shocks
  - Build-up extra safety cushion for an additional shock in a crisis
- Impulse response vs. volatility dynamics

Overview: This Lecture

Real Macrofinance Models with Heterogeneous Agents

A Two-Sector Macrofinance Model

- Complete Markets Benchmark
  - No risk and no frictions
- Basak-Cuoco Model
  - Risk, one unproductive sector and no frictions
- Kiyotaki-Moore Model
  - No risk, two production sectors and leverage constraints

- Physical Investment

Adding Investments/Physical Capital Formation

- Instead of exogenous capital stock,
  convert goods into physical capital (not buying it from others)
- Capital conversion function $\Phi(\iota)$ (increasing and concave)

$$dk_t^i = (\Phi(\iota_t^i)k_t^i - \delta k_t^i)dt + \Delta_t^{k,i} k_t^i$$

  - ι_t^i is investment rate of **new** physical capital (real investment is $\iota_t^i k_t^i$)
    - occurs within the period (no "time-to-build") $\Rightarrow$ static problem
  - $\Delta_t^{k,i} k_t^i$ is the purchase/sale of physical capital at q_t from/to others
    - Hint: $\Delta_t^{k,i} k_t^i$ doesn't impact return on capital $r_t^{K,i}$
  - δ is the depreciation rate of capital
- Optimal investment rate ι_t^i depends on price of physical capital q_t.
  - Tobin's Q:

$$q_t = 1/\Phi'(\iota_t)$$

  - attractive functional form with adjustment cost ϕ:
    $\Phi(\iota) = \frac{1}{\phi}\log(\phi\iota + 1)$
- K_t is a second state variable, which can be solved ex-post

- Homework: Redo continuous time KM analysis with ι-investment.

Bernanke, Gertler, Gilchrist 1999

- Fully fledged DSGE Model with price stickiness, idiosyncratic firm risk, ...
- Aggregate shocks are unanticipated zero-probability shocks (MIT shocks)
- No fire-sale to less productive household sector (unlike in KM97)
- Divestment: Convert physical capital back to consumption good at a cost (captured by $\Phi(\cdot)$-adjustment cost function)
- Financial Frictions:
  - No equity issuance
  - Debt issues with costly state verification (instead of collateral constraint)
    - If firm defaults (after negative idiosyncratic shock), creditor has to pay cost to verify true (remaining) cash flow
    - Optimal contract is a debt contract (debt payoff is hockey stick function of cash flow)
    - De-facto borrowing firms pay verification costs in expectations (in form of higher interest rate/funding costs)
  - A negative aggregate shock, lowers firms' net worth $\Rightarrow$ firm's default prob. rises $\Rightarrow$ expected verification cost rise $\Rightarrow$ Firms funding costs rise

Conclusion & Takeaways

- **Equity Issuance Friction**

1. Incomplete Markets (Basak-Cuoco)
   - No fire sales (since $a^h \to -\infty$)
   - Asset Price constant (or appreciates after adverse shock $\sigma^q \leqslant 0$) $\Rightarrow$ Fluctuating risk-free rate
   - No net worth trap, fat tails, spillovers

- **Debt Issuance Friction** (explicit)

2. Collateral Constraint (Kiyotaki-Moore)
   - Fire sales (since $a^h(\cdot) > \infty$)
   - Dynamic amplification
   - Single zero-probability shock
   - No risk dynamics, no net worth trap
   - No Volatility Paradox, Minsky Hypothesis

- **Physical Investment**/ Tobin's Q is within-period/static Decision

Extra Slides for KM97: Understanding Asset Prices

- Price dynamics (with some proper initial conditions):

$$\frac{1}{q_t}\frac{\mathrm{d}q_t}{\mathrm{d}t} + \frac{a^h(\kappa_t^h)}{q_t} = r_t,$$
$$q_t = \int_t^\infty e^{-\int_t^s r_u du} a^h(\kappa_s^h) ds$$

- Discrete time analogy:

$$\frac{q_{t+1} - q_t}{q_t} + \frac{a^h(\kappa_t^h)}{q_t} = r_t$$
$$q_t = \sum_{s=0}^{\infty} \left[\prod_{u=0}^{s} \frac{1}{(1 + r_{t+u})} \right] a^h(\kappa_{t+s}^h)$$

- Asset price = sum of discounted dividend flows.
- Asset prices are **solved backward**