Online Summer School
Macro, Money, and Finance
Problem Set 1

June 2, 2025

Please submit your solutions to the dropbox link by 6/8/2025 23:59 pm (EDT).

1 Warmup

Itô's Lemma is a mathematical tool in stochastic calculus used to calculate the differential of a function involving a stochastic process. Use Itô's lemma to answer the following questions:

1. Assume the process $dS_t = \mu S_t dt + \sigma S_t dZ_t$, what is the process for $d \ln S_t$?
   Solution: $d \ln S_t = (\mu - \frac{1}{2}\sigma^2)dt + \sigma dZ_t$

2. Assume the process $dc_t = \mu_c c_t dt + \sigma_c c_t dZ_t$, what is the process for $d(c_t)^{-\gamma}$?
   Solution: $d(c_t)^{-\gamma} = c_t^{-\gamma}\left[-\gamma(\mu_c dt + \sigma_c dZ_t) + \frac{1}{2}\gamma(\gamma+1)\sigma_c^2 dt\right]$

3. Follow 2., show that the risk-free rate is: $r^f = \rho + \gamma\mu_c - \frac{\gamma(\gamma+1)}{2}\sigma_c^2$.
   Hint: the geometric drift of SDF $\xi_t = e^{-\rho t}(c_t)^{-\gamma}$ is essentially the risk free rate r^f. A discrete time analogy is:
   $$\mathbb{E}[SDF \cdot R^f] = \mathbb{E}[SDF]R^f = 1 \Rightarrow \mathbb{E}[SDF] = \frac{1}{1+r^f}.$$
   Solution: just plug in the expression for $d(c_t)^{-\gamma}/(c_t^{-\gamma})$.

4. (**Not required**) Give a step-by-step derivation for the Itô's quotient rule $\frac{d(X_t/Y_t)}{X_t/Y_t}$ on the slides.
   Hint: consider Itô's lemma with 2 variables.

2 Portfolio Choice Problem with Log Utility

Consider an infinitely-lived household with logarithmic preferences over consumption $\{c_t\}_{t\geq 0}$,

$$U_0 := \mathbb{E}\left[\int_0^\infty e^{-\rho t}\log c_t dt\right].$$

The household has initial wealth $n_0 > 0$ and does not receive any endowment or labor income. Wealth can be invested into two assets. A risk-free bond with (instantaneous) return $r^b dt$ and a risky stock with return $r^s dt + \sigma dZ_t$, where Z_t is a Brownian motion. Here, r^b, r^s, and σ are constant parameters.

The household's net worth evolution is

$$dn_t = -c_t dt + n_t \left(\left(1 - \theta_t^s\right) r^b dt + \theta_t^s \left(r^s dt + \sigma dZ_t\right) \right),$$

where θ_t^s denotes the fraction of wealth invested into the stock. The household chooses consumption $\{c_t\}_{t \geq 0}$ and portfolio shares $\{\theta_t^s\}_{t \geq 0}$ to maximize utility U_0 subject to the net worth evolution (and a solvency constraint $n_t \geq 0$).

1. **Solving the problem using the HJB Equation**:

   In this part, you will solve the consumption-portfolio choice problem using the Hamilton-Jacobi-Bellman (HJB) equation. The state space of this decision problem is one-dimensional with state variable n_t, so you can denote the household's value function by $V(n)$.

   (a) Write down the (deterministic) HJB equation for the value function $V(n)$.
   Solution:

   $$\rho V(n) = \max_{c, \theta^s} \left(\log c + V'(n) \left(-c + \left((1 - \theta^s) r^b + \theta^s r^s \right) n \right) + \frac{1}{2} V''(n) \left(\theta^s n\right)^2 \sigma^2 \right)$$

   (b) Take first-order conditions with respect to all choice variables.
   Solution:

   - for c
   $$V'(n) = \frac{1}{c}$$
   - for θ^s
   $$0 = V'(n) n (r^s - r^b) + V''(n) n^2 \sigma^2 \theta^s$$

   (c) Let's make a guess that optimal consumption is proportional to net worth, $c(n) = an$ with some constant $a > 0$ (to be determined below). Use the first-order condition for consumption derived in part (b) to turn this into a guess for the value function $V(n)$.
   Hint: Don't forget to add an integration constant (call it b) when moving from $V'(n)$ to $V(n)$.
   Solution:
   $c(n) = an$ implies (from consumption first-order condition)

   $$V'(n) = \frac{1}{an}.$$

   Integrating yields

   $$V(n) = \frac{1}{a} \log n + b,$$

   where b is an integration constant.

(d) Use your guess for $V(n)$ to simplify the first-order condition for θ^s and solve the resulting equation for $\theta^s(n)$.

Solution:

Plugging $V'(n) = \frac{1}{an}$, $V''(n) = -\frac{1}{an^2}$ into the first-order condition for θ^s yields

$$0 = \frac{1}{a}(r^s - r^b) - \frac{1}{a}\sigma^2\theta^s.$$

Solving for θ^s and canceling a implies

$$\theta^s = \frac{r^s - r^b}{\sigma^2}.$$

(e) Substitute the optimal choices and the guess for $V(n)$ into the HJB equation to eliminate $V(n)$, $V'(n)$, $V''(n)$, c, θ^s and the max operator.

Solution:

Recall from previous parts

$$\begin{aligned} V(n) &= \frac{1}{a}\log n + b, & V'(n) &= \frac{1}{an}, & V''(n) &= -\frac{1}{an^2}, \\ c(n) &= an. & \theta^s(n) &= \frac{r^s - r^b}{\sigma^2}. \end{aligned}$$

Substituting these expressions into the HJB from part (a) yields

$$\frac{\rho}{a}\log n + \rho b = \log n + \log a - 1 + \frac{1}{a}\left(r^b + \left(\frac{r^s - r^b}{\sigma}\right)^2\right) - \frac{1}{2a}\left(\frac{r^s - r^b}{\sigma}\right)^2.$$

(f) The resulting equation in step (e) has to hold for all $n > 0$ (if it does not, the previous guess was incorrect). Show that this is indeed possible if we choose a and b appropriately. What are the required values for a and b?

Solution:

The equation derived in part (e) can only hold for all n if the $\log n$-terms cancel out. This is the case if and only if $\rho/a = 1 \Leftrightarrow a = \rho$. Making this choice for a, the equation simplifies to

$$\rho b = \log\rho - 1 + \frac{1}{\rho}\left(r^b + \left(\frac{r^s - r^b}{\sigma}\right)^2\right) - \frac{1}{2\rho}\left(\frac{r^s - r^b}{\sigma}\right)^2.$$

This holds (for all n) if and only if

$$b = \frac{\log\rho - 1}{\rho} + \frac{1}{\rho^2}\left(r^b + \frac{1}{2}\left(\frac{r^s - r^b}{\sigma}\right)^2\right).$$

2. **Solving the problem using the Stochastic Maximum Principle**:

Now consider the same decision problem as before but approach it with the stochastic maximum principle instead of the HJB equation.

(a) Denote by ξ_t the costate for net worth n_t and by $\sigma_{\xi,t}$ its (arithmetic) volatility loading (that is $d\xi_t = \mu_{\xi,t}dt + \sigma_{\xi,t}dZ_t$ with some drift $\mu_{\xi,t}$). Write down the Hamiltonian of the problem.
Solution:

$$H_t = e^{-\rho t}\log c_t + \xi_t\left(-c_t + \left((1-\theta_t^s)r^b + \theta_t^s r^s\right)n_t\right) + \sigma_{\xi,t}\theta_t^s\sigma n_t$$

[*Remark*: in the lecture we used $-\varsigma_t\xi_t$ in place of $\sigma_{\xi,t}$. This is just a different way of choosing notation. Important is that what multiplies $\theta_t^s\sigma n_t$ (the arithmetic net worth volatility loading) in the last term is the (arithmetic) volatility loading of the multiplier ξ_t.]

(b) The choice variables have to maximize the Hamiltonian at all times. Take the first-order conditions in this maximization problem.
Solution:

- for c

$$\xi_t = e^{-\rho t}\frac{1}{c_t}$$

- for θ^s (assuming an interior solution)

$$0 = \xi_t\left(r^s - r^b\right) + \sigma_{\xi,t}\sigma$$

(c) Let's again make the guess $c_t = an_t$ with an unknown constant $a > 0$. Use the first-order condition for consumption derived in part (b) to turn this into a guess for the costate ξ_t. Also determine the implied costate volatility $\sigma_{\xi,t}$.
Solution:
We obtain directly from the consumption first-order condition

$$\xi_t = e^{-\rho t}\frac{1}{an_t}.$$

Applying Ito's lemma to this expression yields

$$\sigma_{\xi,t} = -e^{-\rho t}\frac{1}{an_t^2}\theta_t^s\sigma n_t = -\xi_t\theta_t^s\sigma.$$

(d) Determine the optimal solution for θ_t^s.
Solution:
Plugging $\sigma_{\xi,t}$ from part (c) into the first-order condition for θ_t^s yields

$$0 = \xi_t\left(r^s - r^b\right) - \xi_t\theta_t^s\sigma^2.$$

Because $\xi_t > 0$ whenever $n_t > 0$, we can cancel ξ_t. Solving for θ_t^s implies

$$\theta_t^s = \frac{r^s - r^b}{\sigma^2}.$$

(e) Write down the costate equation for ξ_t and substitute in your guess for c_t, the implied guesses for ξ_t and $\sigma_{\xi,t}$, and the implied optimal solution for θ_t^s. Show that the costate equation is indeed satisfied (and hence the guess was correct) if you choose a suitably. Which value(s) for a work?

Solution:

In general, the costate equation is[1]

$$\mathbb{E}_t\left[d\xi_t\right] = -\frac{\partial H_t}{\partial n_t}dt.$$

Here, taking the derivative of the expression for H_t stated in (a) yields

$$\mathbb{E}_t\left[d\xi_t\right] = -\left(\xi_t\left((1-\theta_t^s)r^b + \theta_t^s r^s\right) + \sigma_{\xi,t}\theta_t^s\sigma\right)dt. \tag{1}$$

Using $\xi_t = e^{-\rho t}\frac{1}{an_t}$ and Ito's lemma, we can write the left-hand side of equation (1) as

$$\begin{aligned}\mathbb{E}_t\left[d\xi_t\right] &= -\rho e^{-\rho t}\frac{1}{an_t}dt - e^{-\rho t}\frac{1}{an_t^2}\mathbb{E}_t\left[dn_t\right] + \frac{1}{2}\cdot 2e^{-\rho t}\frac{(n_t\theta_t^s\sigma)^2}{an_t^3}dt \\ &= \xi_t\left(-\rho + \frac{c_t}{n_t} - r^b - \theta_t^s(r^s-r^b) + (\theta_t^s\sigma)^2\right)dt.\end{aligned}$$

Plugging in $c_t = an_t$ (from our guess) and $\theta_t^s = \frac{r^s-r^b}{\sigma^2}$ (from part (d)) allows us to simplify this expression to

$$\mathbb{E}_t\left[d\xi_t\right] = \xi_t\left(a-\rho-r^b\right)dt. \tag{2}$$

For the right-hand side of equation (2), we obtain after plugging in $\sigma_{\xi,t} = -\xi_t\theta_t^s\sigma$ and $\theta_t^s = \frac{r^s-r^b}{\sigma^2}$

$$\begin{aligned}-\left(\xi_t\left((1-\theta_t^s)r^b + \theta_t^s r^s\right) + \sigma_{\xi,t}\theta_t^s\sigma\right)dt &= -\xi_t\left(r^b + \frac{r^s-r^b}{\sigma^2}(r^s-r^b) - (\theta_t^s\sigma)^2\right)dt \\ &= -\xi_t\left(r^b + \left(\frac{r^s-r^b}{\sigma}\right)^2 - \left(\frac{r^s-r^b}{\sigma}\right)^2\right)dt \\ &= -\xi_t r^b dt. \end{aligned}\tag{3}$$

Comparing equations (2) and (3), we see that the costate equation holds if and only if $a=\rho$.

(f) Verify that the optimal solution coincides with the one you obtained from the HJB approach. Also show that $\xi_t = e^{-\rho t}V'(n_t)$, where V is the value function determined previously.

Solution:

This simply involves a comparison between the results:

- Both approaches yield the optimal choices $c_t = \rho n_t$ and $\theta_t^s = \frac{r^s-r^b}{\sigma}$
- The value function determined by the HJB approach is $V(n) = \frac{1}{a}\log n + b$, so that $V'(n_t) = \frac{1}{an_t}$ and $e^{-\rho t}V'(n_t) = e^{-\rho t}\frac{1}{an_t} = \xi_t$.

[1]For an Ito process x_t, $dx_t = \mu_{x,t}dt + \sigma_{x,t}dZ_t$, we denote by $\mathbb{E}_t[dx_t] = \mu_{x,t}dt$ the drift portion (also called the "compensator").