Problem set 5 (1) Bijoy Ratan Ghosh (1)

1. Money Model with Stochastic Volatility

* $\mu^B = i = \mathcal{G} = \tau = 0$ (no FP)
* Log utility

$\tilde{\sigma}_t$ evolves according to stochastic process $\Rightarrow$

$$d\tilde{\sigma}_t = b(\tilde{\sigma}^{ss} - \tilde{\sigma}_t)dt + \nu\sqrt{\tilde{\sigma}_t}\, dZ_t$$

$\tilde{\sigma}_{ss}$, b and ν positive constants.

* Cont. of hh $\tilde{i} \in [0,1]$ choose $c_t^{\tilde{i}}, \theta_t^{\tilde{i}}, \iota_t^{\tilde{i}}$ to maximise $\Rightarrow$

$$E\left[\int_0^\infty e^{-\rho t} \log c_t^{\tilde{i}}\, dt\right]$$

s.t $$\frac{dn_t^{\tilde{i}}}{n_t^{\tilde{i}}} = -\frac{c^{\tilde{i}}}{n^{\tilde{i}}}dt + dr_t^B + (1-\theta_t^{\tilde{i}})(dr_t^{K,\tilde{i}}(\iota_t^{\tilde{i}}) - dr_t^B) \quad -(1)$$

$$n_t^{\tilde{i}} \geq 0 \quad -(2)$$

$$y_t^{\tilde{i}}dt = (ak_t^{\tilde{i}} - \iota_t^{\tilde{i}}k_t^{\tilde{i}})dt$$

$$\frac{dk_t^{\tilde{i}}}{k_t^{\tilde{i}}} = [\Phi(\iota_t^{\tilde{i}}) - \delta]dt + \tilde{\sigma}d\tilde{Z}_t^{\tilde{i}} + d\Delta_t^{K,\tilde{i}}$$

$d\tilde{Z}^{\tilde{i}} \rightarrow$ idiosyncratic Brow. Risk.

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Goods mkt. clearing $\Rightarrow$

$$C_t = \int_0^1 c_t^{\tilde{i}} d\tilde{i}$$

$$= \int_0^1 [ak_t^{\tilde{i}} - \iota_t^{\tilde{i}} k_t^{\tilde{i}}] d\tilde{i}$$

$$= (a - \iota_t)K_t$$

$$C_t = \rho N_t = \rho q_t K_t$$

$q_t^K \rightarrow$ capital price

$q_t^B = \dfrac{B_t}{P_t K_t}$ value of bonds per unit capital

Now, log utility $\Rightarrow$ $\tilde{c}_t^i = \rho \tilde{n}_t^i$

$$q_t^K = \frac{1}{\Phi'(\tilde{\iota}_t^i)} = 1 + \phi \tilde{\iota}^i$$

$$(a - \iota_t) K_t = \rho N_t = \rho q_t K_t$$

So, $q_t = \dfrac{(a - \iota_t)}{\rho}$

$$\Rightarrow (1 - \vartheta_t) q_t = (1 - \vartheta_t) \frac{(a - \iota_t)}{\rho}$$

$$\Rightarrow q_t^K = (1 - \vartheta_t) \frac{(a - \iota_t - \tau_t)}{\rho}$$

$$\Rightarrow 1 + \phi \iota_t = (1 - \vartheta_t) \frac{(a - \iota_t}{\rho}$$

$$\Rightarrow \frac{1 - \vartheta_t + \phi \rho}{\rho} \iota_t = \frac{(1 - \vartheta_t)(a - \iota_t) - \rho}{\rho}$$

$$\Rightarrow \iota_t = \frac{(1 - \vartheta_t) a - \rho}{(1 - \vartheta_t) + \phi \rho}$$

$$q_t^K = 1 + \phi \frac{(1 - \vartheta_t) a - \rho}{(1 - \vartheta_t) + \phi \rho}$$

$$= \frac{(1 - \vartheta_t) + \cancel{\phi \rho} + \phi (1 - \vartheta_t) a - \cancel{\rho \phi}}{(1 - \vartheta_t) + \phi \rho}$$

$$= \frac{(1-\vartheta_t)(1+\phi a)}{(1-\vartheta_t)+\phi\rho}$$

So,
$$q_t^B = \vartheta_t \frac{(1+\phi a)}{(1-\vartheta_t)+\phi\rho}$$

2. "Money Valuation Eq^n":

(a)

Postulate
$$\frac{dq_t^B}{q_t^B} = \mu_t^{q^B} dt + \sigma_t^{q^B} dZ_t$$
$$\frac{dq_t^K}{q_t^K} = \mu_t^{q^K} dt + \sigma_t^{q^K} dZ_t$$

Now,
$$\frac{dk_t^{\tilde{i}}}{k_t^{\tilde{i}}} = [\Phi(\iota_t^{\tilde{i}}) - \delta]dt + \tilde{\sigma} d\tilde{Z}_t^{\tilde{i}} + d\Delta_t^{K,\tilde{i}}$$
$d\tilde{Z}^{\tilde{i}} \to$ idiosyncratic Brow. Risk.

return process:
$$dr_t^{K,\tilde{i}} = \left[\frac{a_t - \iota_t}{q_t} + \Phi(\iota_t) - \delta + \mu_t^{q^K}\right]dt + \sigma_t^{q^K} \cdot dZ_t + \tilde{\sigma}_t d\tilde{Z}_t^{\tilde{i}}$$

GBC $\Rightarrow$ $dB_t = 0$

$$dr_t^B = \frac{d(1/P_t)}{1/P_t}$$
$$= \frac{d(q_t^B K_t/B_t)}{q_t^B K_t/B_t}$$
$$= \frac{d(q_t^B K_t)}{q_t^B K_t} = [\mu_t^{q^B} + \Phi(\iota_t) - \delta]dt + \sigma_t^{q^B} dZ_t$$

So,
$$dr_t^{K,\tilde{i}} - dr_t^B = \left[\frac{a_t - \iota_t}{q_t^K} - \mu_t^{q^B} + \mu_t^{q^K}\right]dt + [\sigma_t^{q^K} - \sigma_t^{q^B}]dZ_t + \tilde{\sigma}_t d\tilde{Z}_t^{\tilde{i}} \quad — (*)$$

$$\frac{dn_t^{\tilde{i}}}{n_t^{\tilde{i}}} = -\frac{c^{\tilde{i}}}{n^{\tilde{i}}}dt + dr_t^B + (1-\theta_t^{\tilde{i}})(dr_t^{k,\tilde{i}}(\iota_t^{\tilde{i}}) - dr_t^B)$$

$$= -\rho dt + [\mu_t^{q^B} + \Phi(\iota_t) - \delta]dt + \sigma_t^{q^B} dZ_t + (1-\vartheta_t)(dr_t^{k,\tilde{i}} - dr_t^B)$$

In eqm $\theta = \vartheta$

$$= \left[-\rho + \mu_t^{q^B} + \Phi(\iota_t) - \delta + (1-\vartheta_t)\left(\frac{a_t - \iota_t}{q_t^k} - \mu_t^{q^B} + \mu_t^{q^K}\right)\right]dt$$

$$+ \underbrace{\left[\sigma_t^{q^B} + (1-\vartheta_t)(\sigma_t^{q^K} - \sigma_t^{q^B})\right]}_{\sigma_t^n} dZ_t + \underbrace{(1-\vartheta_t)\tilde{\sigma}}_{\tilde{\sigma}_t^n} d\tilde{Z}_t^{\tilde{i}}$$

(b) $$\frac{E[dr_t^{K,\tilde{i}}]}{dt} - \frac{E[dr_t^B]}{dt} = \varsigma_t[\sigma_t^{rK,\tilde{i}} - \sigma_t^{rB}] + \tilde{\varsigma}_t[\tilde{\sigma}^{rK,\tilde{i}} - \tilde{\sigma}^{rB}]$$

$$\frac{a_t - \iota_t}{q_t^k} + \mu_t^{q^K} - \mu_t^{q^B} = \sigma_t^n[\sigma_t^{rK,\tilde{i}} - \sigma_t^{rB}] + \tilde{\sigma}_t^n[\tilde{\sigma}^{rK,\tilde{i}} - \underset{0}{\underset{\parallel}{\tilde{\sigma}^{rB}}}]$$

$$\frac{\rho}{1-\vartheta_t} + \mu_t^{q^K} - \mu_t^{q^B} = [\sigma_t^{q^B} + (1-\vartheta_t)(\sigma_t^{q^K} - \sigma_t^{q^B})](\sigma^{q^K} - \sigma^{q^B})$$
$$+ (1-\vartheta_t)\tilde{\sigma}^2 \quad \text{— (1)}$$

Now, $\vartheta_t = \dfrac{q_t^B}{q_t^K + q_t^B}$ $\quad 1-\vartheta_t = q_t^K / (q_t^K + q_t^B)$

$$\frac{\vartheta_t}{1-\vartheta_t} = q_t^B / q_t^K \Leftrightarrow \frac{1-\vartheta_t}{\vartheta_t} = \frac{q_t^K}{q_t^B} \Leftrightarrow \frac{1}{\vartheta_t} - 1 = \frac{q_t^K}{q_t^B}$$

So, $$-\frac{1}{\vartheta_t^2}\vartheta_t \mu_t^{\vartheta} + \frac{1}{2}\cdot\frac{2}{\vartheta_t^3}(\vartheta_t \sigma_t^{\vartheta})^2 = \mu^{q^K} - \mu^{q^B} + \sigma_t^{q^B}(\sigma_t^{q^B} - \sigma_t^{q^K}) \quad \text{— (*)}$$

$$-\frac{1}{\vartheta_t^2}\vartheta_t \sigma_t^{\vartheta} = \sigma^{q^K} - \sigma^{q^B} \Leftrightarrow \sigma_t^{q^B} - \sigma_t^{q^K} = \frac{\sigma_t^{\vartheta}}{\vartheta_t} \quad \text{— (**)}$$

Now, $q_t^B = \vartheta_t \dfrac{1+\phi a}{(1-\vartheta_t)+\phi\rho}$ — ⊛⊛⊛

Let, $d\vartheta_t = \mu_t^{\vartheta} \vartheta_t \, dt + \sigma_t^{\vartheta} \vartheta_t \, dZ_t$ (Using Ito's)

So, $\sigma^{q^B} = \dfrac{1}{q_t^B} \dfrac{((1-\vartheta_t)+\phi\rho)(1+\phi a) + (1+\phi a)\vartheta_t}{(1-\vartheta_t)+\phi\rho}$

$$= \frac{(1-\cancel{\vartheta_t}+\phi\rho+\cancel{\vartheta_t})(1+\cancel{\phi a})}{\cancel{1-\vartheta_t+\phi\rho}} \cdot \frac{\cancel{(1-\vartheta_t)+\phi\rho}}{\vartheta_t(1+\cancel{\phi a})}$$

$$= \frac{1+\phi\rho}{\vartheta_t} = \frac{1}{\vartheta_t}(1+\phi\rho) \quad \text{— ⊛⊛⊛⊛}$$

So, putting ⊛⊛⊛⊛ and ⊛⊛ in ⊛ ⇒

$$-\frac{\mu_t^{\vartheta}}{\vartheta_t} + \frac{(\sigma_t^{\vartheta})^2}{\vartheta_t} = \mu^{q^K} - \mu^{q^B} + \frac{1}{\vartheta_t}(1+\phi\rho)\frac{\sigma_t^{\vartheta}}{\vartheta_t}$$

So, $\mu_t^{q^K} - \mu_t^{q^B} = -\dfrac{\mu_t^{\vartheta}}{\vartheta_t} + \dfrac{(\sigma_t^{\vartheta})^2}{\vartheta_t} - \dfrac{\sigma_t^{\vartheta}(1+\phi\rho)}{\vartheta_t^2}$

Putting this in eqn ① ⇒

$$\frac{\rho}{1-\vartheta_t} - \frac{\mu_t^{\vartheta}}{\vartheta_t} + \frac{(\sigma_t^{\vartheta})^2}{\vartheta_t} - \cancel{\frac{\sigma_t^{\vartheta}(1+\phi\rho)}{\vartheta_t^2}} = (1-\vartheta_t)\tilde{\sigma}_t^2 - \cancel{\frac{1}{\vartheta_t}(1+\phi\rho)\frac{\sigma_t^{\vartheta}}{\vartheta_t}} + (1-\vartheta_t)\left(\frac{\sigma_t^{\vartheta}}{\vartheta_t}\right)^2 \quad \text{— ②}$$

This gives $\underline{\mu_t^{\vartheta}} \Big\} = f(\vartheta_t, \tilde{\sigma}_t)$

$$\Leftrightarrow \frac{\rho}{1-\vartheta_t} - \mu_t^{\vartheta}/\vartheta_t + \frac{(\sigma_t^{\vartheta})^2}{\vartheta_t} = (1-\vartheta_t)\tilde{\sigma}_t^2 + (1-\vartheta_t)\frac{(\sigma_t^{\vartheta})^2}{\vartheta_t}$$

$$\Leftrightarrow \frac{\rho}{1-\vartheta_t} - \mu_t^{\vartheta}/\vartheta_t + (\sigma_t^{\vartheta})^2 = (1-\vartheta_t)\tilde{\sigma}_t^2$$

$$\Rightarrow \rho\vartheta_t^2 - \mu_t^{\vartheta}\vartheta_t(1-\vartheta_t) + (\vartheta_t\sigma_t^{\vartheta})^2(1-\vartheta_t) = (\vartheta_t(1-\vartheta_t)\tilde{\sigma}_t)^2$$

— 2

3. $\nu = 0 \quad \tilde{\sigma}_t = \tilde{\sigma}^{ss} \Rightarrow d\tilde{\sigma}_t = 0$

In steady state $\quad \mu_t^{\vartheta} = \sigma_t^{\vartheta} = 0$

So, from money valn eqn $\Rightarrow$

$$\frac{\rho}{1-\vartheta} = (1-\vartheta)\tilde{\sigma}^{ss^2}$$

$$\Rightarrow (1-\vartheta)^2 = \frac{\rho}{\tilde{\sigma}^{ss^2}}$$

$$1-\vartheta = \pm \frac{\sqrt{\rho}}{\tilde{\sigma}^{ss}}$$

$$\text{so, } \vartheta = 1 \mp \frac{\sqrt{\rho}}{\tilde{\sigma}^{ss}}$$

Now, $\vartheta \in [0,1]$

So, we can reject $\vartheta = 1 + \frac{\sqrt{\rho}}{\tilde{\sigma}^{ss}}$

$$\text{So, } \vartheta = 1 - \frac{\sqrt{\rho}}{\tilde{\sigma}^{ss}}$$

Now, monetary eq exists if $\vartheta > 0$

$$\Rightarrow 1 - \frac{\sqrt{\rho}}{\tilde{\sigma}^{ss}} > 0$$

$$\Rightarrow \frac{\sqrt{\rho}}{\tilde{\sigma}^{ss}} < 1$$

$$\Rightarrow \tilde{\sigma}^{ss} > \sqrt{\rho}$$

So, smallest value of $\tilde{\sigma}^{ss}$ that allows for monetary eq. is $\tilde{\sigma}^{ss}_{min} = \sqrt{\rho}$

If $\tilde{\sigma}^{ss} > \tilde{\sigma}^{ss}_{min}$ then $\frac{\partial \vartheta}{\partial \tilde{\sigma}^{ss}} = \frac{\sqrt{\rho}}{(\sigma^{ss})^2} > 0$

So, $\vartheta \uparrow$

If $0 < \tilde{\sigma}^{ss} < \tilde{\sigma}^{ss}_{min}$ then $\vartheta = 0$

$\frac{\partial \vartheta}{\partial \tilde{\sigma}^{ss}} = 0$

So, no change in ϑ.

4. $\vartheta_t = \vartheta(\tilde{\sigma}_t)$

$$\vartheta_t \mu^{\vartheta}_t = \vartheta' \mu^{\tilde{\sigma}}_t \tilde{\sigma}_t + \frac{1}{2}\vartheta''(\sigma^{\tilde{\sigma}}_t \tilde{\sigma}_t)^2$$

$$= \vartheta' b(\tilde{\sigma}^{ss} - \tilde{\sigma}_t) + \frac{1}{2}\vartheta'' \nu^2 \tilde{\sigma}_t$$

$$\vartheta_t \sigma^{\vartheta}_t = \vartheta' \sigma^{\tilde{\sigma}} \tilde{\sigma}_t = \vartheta' \nu \sqrt{\tilde{\sigma}_t}$$

$$\rho \vartheta_t^2 - \mu^{\vartheta}_t \vartheta_t (1-\vartheta_t) + (\vartheta_t \sigma^{\vartheta}_t)^2 (1-\vartheta_t) = (\vartheta_t(1-\vartheta_t)\tilde{\sigma}_t)^2 \quad (2)$$

$$\rho \vartheta_t^2 - (1-\vartheta_t)\left[\vartheta' b(\tilde{\sigma}^{ss} - \tilde{\sigma}_t) + \frac{1}{2}\vartheta'' \nu^2 \tilde{\sigma}_t\right] + (1-\vartheta_t)(\vartheta'_t)^2 \nu^2 \tilde{\sigma}_t = \vartheta_t^2 (1-\vartheta_t)^2 \tilde{\sigma}_t^2$$

$$\rho \vartheta_t = \underbrace{(1-\vartheta_t)\left[\frac{\vartheta' b(\tilde{\sigma}^{ss} - \tilde{\sigma}_t) + \frac{1}{2}\vartheta''\nu^2\tilde{\sigma}_t}{\vartheta_t} - \frac{(\vartheta'_t)^2 \nu^2 \tilde{\sigma}_t}{\vartheta_t}\right]}_{u(\vartheta_t)} + \underbrace{(1-\vartheta_t)^2 \tilde{\sigma}_t^2}_{M(\vartheta)} \cdot \vartheta_t$$

$\uparrow$ $M(\vartheta) \cdot \vartheta_t$

so, $\rho v_t = u(v_t) + M(v_t) v_r$