Problem set – I

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1. Warmup

① $dS_t = \mu S_t dt + \sigma S_t dZ_t$

$f(S_t) = \ln S_t$

$f'(S_t) = \frac{1}{S_t}$ $\quad f''(S_t) = -\frac{1}{S_t^2}$

So, Using Ito's lemma ⇒

$$d(\ln S_t) = \left[\frac{1}{S_t}\mu S_t - \frac{1}{2S_t^2}\cdot(\sigma S_t)^2\right]dt + \frac{1}{S_t}\sigma S_t dZ_t$$

$$= \left[\mu - \frac{\sigma^2}{2}\right]dt + \sigma dZ_t$$

So, process for $d(\ln S_t)$ ⇒

$$\boxed{\left(\mu - \frac{\sigma^2}{2}\right)dt + \sigma dZ_t}$$

②

$dC_t = \mu_c C_t dt + \sigma_c C_t dZ_t$

$f(C_t) = C_t^{-\gamma}$

so, $f'(C_t) = -\gamma C_t^{-\gamma-1}$

$f''(C_t) = (-\gamma)(-\gamma-1) C_t^{-\gamma-2}$

$= \gamma(\gamma+1)C_t^{-\gamma-2}$

so, By Ito's lemma ⇒

$$dC_t^{-\gamma} = \left[-\gamma C_t^{-\gamma-1}\cdot \mu_c C_t + \frac{1}{2}\gamma(\gamma+1)C_t^{-\gamma-2}\cdot(\sigma_c C_t)^2\right]dt + (-\gamma C_t^{-\gamma-1})\sigma_c C_t dZ_t$$

$$= \left[-\gamma\mu_c C_t^{-\gamma} + \frac{1}{2}\gamma(\gamma+1)\sigma_c^2 C_t^{-\gamma}\right]dt - \gamma\sigma_c C_t^{-\gamma} dZ_t$$

$$= \left[-\gamma\mu_c + \frac{1}{2}\gamma(\gamma+1)\sigma_c^2\right]C_t^{-\gamma}dt - \gamma\sigma_c\cdot C_t^{-\gamma}dZ_t$$

So, process for $dC_t^{-\gamma} \Rightarrow$

$$\boxed{\left[-\gamma\mu_c + \tfrac{1}{2}\gamma(\gamma+1)\sigma_c^2\right] C_t^{-\gamma} dt - \gamma\sigma_c \cdot C_t^{-\gamma} dZ_t}$$

(3) $\xi_t = e^{-\rho t} C_t^{-\gamma} = f(t, C_t)$ so, $f_t' = -\rho e^{-\rho t} C_t^{-\gamma}$

$f_c' = -\gamma e^{-\rho t} C_t^{-\gamma-1}$

$f_{cc}'' = \gamma(\gamma+1) e^{-\rho t} C_t^{-\gamma-2}$

So, By Ito's lemma $\Rightarrow$

$$d\xi_t = \left[-\rho e^{-\rho t} C_t^{-\gamma} - \gamma e^{-\rho t} C_t^{-\gamma-1} \cdot \mu_c C_t + \tfrac{1}{2}\gamma(\gamma+1) e^{-\rho t} C_t^{-\gamma-2} \cdot \sigma_c^2 C_t^2\right] dt - \gamma e^{-\rho t} C_t^{-\gamma-1} \cdot \sigma_c C_t \, dZ_t$$

$$= \left[-\rho e^{-\rho t} C_t^{-\gamma} - \gamma\mu_c e^{-\rho t} C_t^{-\gamma} + \tfrac{1}{2}\gamma(\gamma+1)\sigma_c^2 e^{-\rho t} C_t^{-\gamma}\right] dt - \gamma\sigma_c e^{-\rho t} C_t^{-\gamma} dZ_t$$

$$= \left[-\rho - \gamma\mu_c + \tfrac{1}{2}\gamma(\gamma+1)\sigma_c^2\right] \xi_t \, dt + \gamma\sigma_c \xi_t \, dZ_t$$

So, $\mu_t^{\xi} = -\rho - \gamma\mu_c + \tfrac{1}{2}\gamma(\gamma+1)\sigma_c^2$

So, $\boxed{r^f = -\mu_t^{\xi} = \rho + \gamma\mu_c - \tfrac{1}{2}\gamma(\gamma+1)\sigma_c^2}$ [Analogous to discrete time]

2 Portfolio Choice Problem With Log Utility

1 $U_0 := E\left[\int_0^\infty e^{-\rho t} \log c_t \, dt\right] \qquad \{c_t\}_{t\geq 0}$

s.t $dn_t = -c_t dt + n_t\left[(1-\theta_t^s) r^b dt + \theta_t^s (r^s dt + \sigma dZ_t)\right]$

$$\Leftrightarrow dn_t = \Big[\underbrace{-c_t + n_t\left((1-\theta_t^s) r^b + \theta_t^s r^s\right)}_{\mu_{n,t}}\Big] dt + \underbrace{\theta_t^s \sigma n_t}_{\sigma_{n,t}} dZ_t$$

(a) HJB eq^n:

$$\boxed{\rho V(n_t) dt = \max_{c_t, \theta_t^s}\left[\log(c_t) dt + E_t[dV]\right]} \quad \text{— (1)}$$

Now, By Ito's lemma $\Rightarrow$

$$dV = \left[V'(n)\mu_{n,t} + \frac{1}{2} V''(n)\cdot \sigma_{n,t}^2\right] dt + V'(n)\sigma_{n,t} dZ_t$$

So, $$E_t dV = \left[V'(n)\cdot\left(-c_t + n_t(1-\theta_t^s) r^b + \theta_t^s r^s n_t\right) + \frac{1}{2} V''(n) \theta_t^{s^2} \sigma^2 n_t^2\right] dt$$

Putting this in HJB eq^n (1) $\Rightarrow$

$$\rho V = \max_{c, \theta^s}\left[\log c + V'\left(-c + n(1-\theta^s) r^b + n\theta^s r^s\right) + \frac{1}{2} V'' \theta^{s^2} \sigma^2 n^2\right]$$

$$= \max_c\left[\log c - cV'\right] + \max_{\theta^s}\left[nV'\left((1-\theta^s) r^b + \theta^s r^s\right) + \frac{1}{2}\theta^{s^2} n^2 \sigma^2 V''\right]$$

(b) Taking FOC $\Rightarrow$

c: $\quad \frac{1}{c} - V' = 0 \quad \Rightarrow \quad \boxed{V'(n) = \frac{1}{c}}$ — (2)

θ^s: $\quad nV'(-r^b + r^s) + \theta^s n^2 \sigma^2 V'' = 0$

$$\Rightarrow \theta^s(n) = -\frac{nV'}{n^2 V'' \sigma^2}(r^s - r^b)$$

$$\boxed{\theta^s(n) = -\frac{V'(n)}{nV''(n)\sigma^2}(r^s - r^b)} \quad \text{— (3)}$$

(c) let, $c(n) = an$ for some $a > 0$.

So, using ② $\Rightarrow$

$$V'(n) = \frac{1}{an}$$

$$\Rightarrow \boxed{V(n) = \frac{1}{a}\log n + b \quad \text{for some } b \text{ integ. const.}} \quad -④$$

(d) $$\left.\begin{aligned} \text{So, } V'(n) &= \frac{1}{an} \\ V''(n) &= -\frac{1}{an^2} \end{aligned}\right\} \Rightarrow -\frac{V'(n)}{nV''(n)} = -\frac{1/an}{n\left(-\frac{1}{an^2}\right)}$$

$$= 1$$

So, $$\boxed{\theta^s(n) = \frac{(r^s - r^b)}{\sigma^2}} \quad -⑤$$

(e) Putting these optimal choices in HJB $\Rightarrow$

$$\rho\left[\frac{1}{a}\log n + b\right] = \left[\log(an) - 1\right] + \left[n \cdot \frac{1}{an}\left(\frac{(r^s - r^b)}{\sigma^2} r^s + \left(1 - \frac{r^s - r^b}{\sigma^2}\right) r^b\right) + \frac{1}{2} n^2 \sigma^2 \left(-\frac{1}{an^2}\right) \frac{(r^s - r^b)^2}{\sigma^4}\right]$$

$$\Rightarrow \frac{\rho}{a}\log n + \rho b = (\log a + \log n - 1) + \left[\frac{1}{a\sigma^2}\left((r^s)^2 - r^b r^s + \sigma^2 r^b - r^s r^b + (r^b)^2\right)\right] - \frac{1}{2a\sigma^2}(r^s - r^b)^2$$

$$= (\log a + \log n - 1) + \frac{1}{2a\sigma^2}\left[2(r^s)^2 + 2(r^b)^2 - 4 r^b r^s - (r^s)^2 - (r^b)^2 + 2 r^b r^s + 2\sigma^2 r^b\right]$$

$$= (\log a + \log n - 1) + \frac{1}{2a\sigma^2}\left[(r^s - r^b)^2 + 2\sigma^2 r^b\right]$$

$$= \log a + \log n - 1 + \frac{(r^s - r^b)^2}{2a\sigma^2} + \frac{r^b}{a}$$

(f) This should hold for all $n > 0$,
Comparing co-efficient of $\log n$ & const. So, $\frac{\rho}{a} = 1 \Rightarrow \boxed{a = \rho}$

$$b = \frac{1}{\rho}\left[\log \rho - 1 + \frac{(r^s - r^b)^2}{2\rho\sigma^2} + \frac{r^b}{\rho}\right]$$

So,
$$\boxed{b = \frac{1}{\rho}\left[\log\rho - 1 + \frac{r^b}{\rho} + \frac{(r^s - r^b)^2}{2\rho\sigma^2}\right]} \quad \text{—} (*)$$

(2)

(a) Let, ξ_t costate for networth n_t

$$d\xi_t = \mu_{\xi,t}\,dt + \sigma_{\xi,t}\,dZ_t$$

Hamiltonian ⇒

$$H_t = e^{-\rho t}\log c_t + \xi_t n_t \mu_t^n + \sigma_{\xi,t} n_t \sigma_t^n$$
$$= e^{-\rho t}\log c_t + \xi_t\left(-c_t + n_t\left(r^b + \theta_t^s(r^s - r^b)\right)\right) + \sigma_{\xi,t}\,\theta_t^s \sigma n_t$$

So,
$$\boxed{H_t = e^{-\rho t}\log c_t + \xi_t\left(-c_t + n_t\left(r^b + \theta_t^s(r^s - r^b)\right)\right) + \sigma_{\xi,t}\,\theta_t^s \sigma n_t}$$

(b) choice variables c_t, θ_t^s max. H_t $\forall t$.

<u>FOC</u>

c: $$\frac{e^{-\rho t}}{c_t} - \xi_t = 0 \quad \Rightarrow \quad \boxed{\xi_t = \frac{e^{-\rho t}}{c_t}} \quad \text{—} (6)$$

θ^s: $$\xi_t n_t (r^s - r^b) + \sigma_{\xi,t}\,\sigma n_t = 0$$

$$\Rightarrow \boxed{\frac{\sigma_{\xi t}}{\xi_t} = -\frac{(r^s - r^b)}{\sigma}} \quad \text{—} (7)$$

(c) let, $\varphi = a\eta_t$

So, ⑥ gives $\Rightarrow$ $$\boxed{\xi_{\varphi_t} = e^{-\rho t} \cdot \frac{1}{a\eta_t}} \quad \text{— ⑧}$$

⑦ gives $\Rightarrow$ $$\boxed{\sigma_{\xi,t} = -\frac{e^{-\rho t}}{a\eta_t} \cdot \frac{(r^s - r^b)}{\sigma}} \quad \text{— ⑨}$$

(d) Now, ⑧ gives $\Rightarrow$

Using Ito's lemma —

$$d\xi_{\varphi_t} = \left[-\rho e^{-\rho t}\frac{1}{a\eta_t} - \frac{e^{-\rho t}}{a\eta_t^2} \cdot \left[-a\eta_t + \eta_t\left(r^b + \theta_t^s(r^s - r^b)\right)\right] + \frac{1}{2} \cdot \frac{2e^{-\rho t}}{a\eta_t^3} \theta_t^{s\,2} \sigma^2 \eta_t^2\right] dt - \frac{e^{-\rho t}}{a\eta_t^2} \theta_t^s \sigma \eta_t \, dZ_t$$

$$\Rightarrow d\xi_{\varphi_t} = \left[-\rho + a - r^b - \theta_t^s(r^s - r^b) + (\theta_t^s)^2 \sigma^2\right] \frac{e^{-\rho t}}{a\eta_t} dt - \theta_t^s \sigma \frac{e^{-\rho t}}{a\eta_t} dZ_t$$

— ⑩

Comparing ⑩ & ⑨ $\Rightarrow$

$$\cancel{-} \frac{e^{-\rho t}}{a\eta_t} \cdot \frac{(r^s - r^b)}{\sigma} = \cancel{-} \theta_t^s \sigma \frac{e^{-\rho t}}{a\eta_t}$$

$$\Rightarrow \boxed{\theta_t^s = \frac{(r^s - r^b)}{\sigma^2}}$$

(e) Now co-state equation $\Rightarrow$

$$d\xi_{\varphi_t} = -\partial_\eta H \, dt + \sigma_{\xi_\varphi, t} \, dZ_t$$

$$\partial_\eta H = \xi_{\varphi_t}\left[r^b + \theta_t^s (r^s - r^b)\right] + \sigma_{\xi,t} \theta_t^s \sigma$$

Comparing this with (10) $\Rightarrow$

$$-\xi_t\left[r^b + \theta_t^s(r^s - r^b) - \frac{r^s - r^b}{\sigma}\theta_t^s\sigma\right] = \xi_t(-\rho + a) + \xi_t\left[-r^b - \theta_t^s(r^s - r^b) + (\theta_t^s)^2\sigma^2\right]$$

$$\Rightarrow -\xi_t\left[r^b + \theta_t^s\cancel{(r^s - r^b)} - (\theta_t^s)^2\sigma^2\right] = \xi_t(-\rho + a) - \xi_t\left[r^b + \theta_t^s\cancel{(r^s - r^b)} - (\theta_t^s)^2\sigma^2\right]$$

$$\Rightarrow \boxed{a = \rho}$$

(f) Solution for Stoch. Max. principle exactly matches HJB.

Now, $$\xi_t = e^{-\rho t} \cdot \frac{1}{a n_t} = e^{-\rho t} \cdot \frac{1}{\rho n_t}$$

Now, $$V(n_t) = \frac{1}{\rho}\log n_t + b \qquad [b \text{ given by } (*)]$$

So, $$V'(n_t) = \frac{1}{\rho n_t}$$

So, $$\boxed{\xi_t = e^{-\rho t} V'(n_t)}$$ (Proved)