Problem Set 4

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1 Nominal Bonds as Safe Assets and Bubble Mining Seigniorage

1.1 Part 1: Solving the Model

1.1.1 (a) Return Expressions

Starting with the standard setup where agents choose $c_t^i, \theta_t^i, \iota_t^i$ to maximize:

$$\mathbb{E}\left[\int_0^\infty e^{-\rho t}\log c_t^i dt\right]$$

Subject to:

$$\frac{dn_t^i}{n_t^i} = \frac{c_t^i}{n_t^i}dt + dr_t^B + (1-\theta_t^i)\left(dr_t^{K,i} - (\iota_t^i)dr_t^B\right)$$

Since there are no transaction costs:

$$y_t^i dt = (ak_t^i - \iota_t^i k_t^i)dt$$

For capital returns:

$$\frac{dk_t^i}{k_t^i} = (\phi(\iota_t^i) - \delta)dt + \tilde{\sigma}d\tilde{Z}_t^i + d\Delta_t$$

Using Ito's lemma on price processes and noting that i, q^B, q^K are constant:

$$dr_t^B = idt + \frac{d(\mathcal{P}_t B_t)}{\mathcal{P}_t B_t}$$

Since there's no aggregate risk and $d\tilde{Z}_t$ represents idiosyncratic risk:

$$(\mu^B - i)B_t + \mathcal{P}_t k_t T_t a = 0$$

1.1.2 (b) Optimal Choice

With log utility, optimal consumption is:

$$c_t = \rho n_t = \rho q_t k_t$$

From first-order conditions:

$$\phi'(\iota_t) = \frac{1}{q^K}$$

Portfolio choice with log utility yields:

$$(1-\theta_t) = \frac{a - \iota_t - \rho}{\tilde{\sigma}^2} = \frac{a(1-\rho/r) - \iota}{\tilde{\sigma}^2}$$

1.1.3 (c) Equilibrium Values

Using market clearing and imposing steady state ($\dot{\mu} = 0$):

Asset prices:

$$q^K = 1 + \phi\iota, \quad q = 1 + \phi\iota \Rightarrow \iota_t = \frac{q^K - 1}{\phi}$$

From capital market clearing:

$$C_t = \mathcal{P}N_t = \mathcal{P}q_t K_t$$

Therefore:

$$q_t = \frac{a - \iota_t}{\rho}$$

Combining equations:

$$(1 - \theta_t)q_t = (1 - \theta_t)\frac{a - \iota_t}{\rho}$$

This gives us:

$$\iota_t = \frac{(1 - \theta_t)a - \rho}{1 - \theta_t + \phi\rho}$$

Inflation rate (growth rate of nominal assets minus growth rate of real assets):

$$\pi = \mu^B - g$$

In steady state:

$$\dot{\mu}^B = 0 \Rightarrow \mu^B = -\rho$$

Therefore:

$$q_t^B = \frac{s}{\rho} \quad \text{(by GBC)}$$

1.2 Part 2: Seigniorage Revenue from Bubble Mining

Consider the special case where $\phi \to \infty$ (no physical investment) and capital grows at exogenous rate $g = -\delta$.

1.2.1 (a) Seigniorage-Policy Relationship

From the money valuation equation:

$$\mathbb{E}_t[d\vartheta_t] = (\rho + \hat{\mu}_t^B - (1 - \vartheta_t)^2\tilde{\sigma}^2)\frac{dt}{q_t}$$

For $\mu_t^\vartheta = 0$:

$$s = \frac{(\mu^B - i)(\tilde{\sigma} - \sqrt{\rho + \mu^B - i})a}{\rho\tilde{\sigma}^2}$$

This exhibits a Laffer curve because:

- As $(\mu^B - i)$ increases, the inflation tax base $(1 - \theta)$ increases
- But the tax rate θ decreases
- These opposing effects create an inverted-U shape

1.2.2 (b) Maximum Seigniorage

To find the maximum seigniorage $\bar{s}$, note that s is concave in $(\mu^B - i)$.

Let $M^* = \arg\max_{\mu^B - i} s(\mu^B - i)$

Then:

$$\bar{s} = \frac{M^*(\tilde{\sigma} - \sqrt{\rho + M^*})a}{\rho\tilde{\sigma}^2}$$

And:

$$s^{eq} = \frac{\mu^* a}{\rho}\left[1 - \frac{\sqrt{\rho + M^*}}{\tilde{\sigma}}\right]$$

Taking the derivative with respect to $\tilde{\sigma}$:

$$\frac{\partial \bar{s}}{\partial \tilde{\sigma}} = \frac{\mu^* a}{\rho} \cdot \frac{\sqrt{\rho + M^*}}{\tilde{\sigma}^2}$$

Since $\mu^B > i$ implies $\frac{\partial \bar{s}}{\partial \tilde{\sigma}} > 0$, and $\mu^B < i$ implies $\frac{\partial \bar{s}}{\partial \tilde{\sigma}} < 0$.

1.2.3 (c) Seigniorage with Arbitrary Inflation

The key insight is that seigniorage depends on $(\mu^B - i)$, not on inflation alone.

For any desired seigniorage level $s = \bar{s}$, if inflation is arbitrarily large, we can choose the μ^B and i combination such that:

$$\mu^B - g = \pi \quad \text{(large)}$$

while maintaining:

$$\mu^B - i = M^* \quad \text{(optimal for seigniorage)}$$

This is possible because we can set both μ^B and i high while keeping their difference at the optimal level. High inflation doesn't erode seigniorage because:

- The government controls both the growth rate of bonds AND the interest rate
- What matters for seigniorage is the spread $(\mu^B - i)$, not the absolute levels
- Agents care about real returns, which depend on this spread

2 Monetary Policy in Model with Bonds and Money

2.1 Policy 1: One-time Helicopter Drop of Money

At $t = 0$: $\Delta M > 0$ distributed lump-sum.

(i) Effect on ϑ (money share):

- Initial wealth increases: $W_0^+ = W_0^- + \Delta M$
- Agents initially hold too much money relative to optimal portfolio
- From the discrete-time analog: money demand satisfies $\theta = \frac{M}{W}$
- Since agents now hold excess money, ϑ increases initially

(ii) Effects on i^B and π:

- To restore portfolio balance, i^B must fall (reducing opportunity cost of money)
- From equilibrium condition: lower i^B means lower $\Delta i = i^B - i^M$
- Inflation π increases because:

$$\pi = \mu^B - g = \mu^M - g \tag{1}$$

- Government must increase μ^M to satisfy budget constraint with higher money stock

(iii) Initial price level P_0:

- From the discrete-time equation: $\frac{B_t+M_t}{\mathcal{P}_t} = \mathbb{E}_t \sum_s \frac{\xi_s}{\xi_t} K_s$
- With sudden increase in M_0, and sluggish adjustment in real variables, $\mathcal{P}_0$ must jump up
- This prevents real wealth from increasing proportionally to nominal money increase

2.2 Policy 2: One-time Helicopter Drop of Bonds

At $t = 0$: $\Delta B > 0$ distributed lump-sum.

(i) Effect on ϑ:

- Agents now hold too many bonds relative to money
- Money share $\vartheta = \frac{M}{W}$ decreases
- Creates a shortage of transaction services

(ii) Effects on i^B and π:

- Bond rate i^B must rise to clear the bond market
- Higher i^B increases Δi, making money more attractive at margin
- Government must adjust $\mu^B = \mu^M$ upward to service higher debt
- Inflation increases: $\pi = \mu^B - g$ rises

(iii) Initial price level P_0:

- Unlike money drop, bonds don't relax cash-in-advance constraint
- P_0 rises but less than proportionally to wealth increase
- Real effects are contractionary due to tighter transaction constraints

2.3 Policy 3: Open Market Operation

Government sells $\Delta B > 0$ bonds, using proceeds to reduce money supply.

(i) Effect on ϑ:

- Total nominal wealth unchanged: $\Delta W = 0$
- But composition shifts: $M \downarrow$, $B \uparrow$
- Money share ϑ falls significantly

(ii) Effects on i^B and π:

- Tighter money supply requires higher i^B to equilibrate
- From discrete-time FOC: $(1-\theta)\frac{a}{q^K} = \tilde{\sigma}^2 - \Delta i \frac{q^B}{q^K}$
- Higher Δi reduces capital investment
- Long-run inflation rises due to higher debt service burden

(iii) Initial price level P_0:

- Money contraction causes immediate deflation: P_0 falls
- This is the "normal" monetary policy effect
- But sets stage for higher future inflation (Unpleasant Monetarist Arithmetic)

2.4 Policy 4: Increasing Interest on Reserves

Permanent increase: $i^M \to i^{M\prime} > i^M$.

(i) Effect on ϑ:

- Higher i^M reduces opportunity cost of holding money
- Agents want to hold more money: ϑ increases
- Money becomes better store of value

(ii) Effects on i^B and π:

- Spread $\Delta i = i^B - i^M$ narrows
- If economy approaches "store of value regime" where $\Delta i \to 0$
- Government pays more interest on reserves, requiring higher $\mu^B = \mu^M$
- Inflation increases to finance higher interest payments

(iii) Initial price level P_0:

- Increased money demand at higher i^M creates deflationary pressure
- P_0 falls as agents try to accumulate real balances

2.5 Part 2: Store of Value Regime ($\Delta i = 0$)

When $i^B = i^M$, money and bonds are perfect substitutes.

Key differences:

- Portfolio composition ϑ becomes indeterminate
- Quantity of money irrelevant for price level (liquidity trap)
- Only total government liabilities $B + M$ matter

Policies that can exit this regime:

- Policy 3 (OMO) or Policy 4 (lower i^M) can create $\Delta i > 0$
- Need to make bonds sufficiently more attractive than money
- Requires either reducing money supply or lowering return on money