Problem Set 5

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1 Problem Setup

Consider the model of Lecture 10 with log utility and without government policy ($\mu^B = i = G = \tau = 0$). Idiosyncratic risk $\tilde{\sigma}_t$ evolves according to:

$$d\tilde{\sigma}_t = b(\tilde{\sigma}^{ss} - \tilde{\sigma}_t)dt + \nu\sqrt{\tilde{\sigma}_t}dZ_t \tag{1}$$

where $\tilde{\sigma}^{ss}$, b and ν are positive constants.

2 Question 1: Market Clearing and Optimal Investment

From goods market clearing:

$$C_t = \int_0^1 C_t^{\tilde{i}} d\tilde{i} \tag{2}$$

$$= \int_0^1 [\alpha K_t^{\tilde{i}} - \iota_t^{\tilde{i}} K_t^{\tilde{i}}] d\tilde{i} \tag{3}$$

$$= (a - \iota_t) K_t \tag{4}$$

$$C_t = \rho N_t = \rho q_t K_t \tag{5}$$

With log utility:

$$C_t^{\tilde{i}} = \rho n_t^{\tilde{i}} \tag{6}$$

$$q_t^K = \frac{1}{\Phi'(\iota_t^{\tilde{i}})} = 1 + \phi \iota_t^{\tilde{i}} \tag{7}$$

where $q_t^K \to$ capital price and $q_t^B = \frac{B_t}{P_t K_t} \to$ value of bonds per unit capital.

2.1 Expressing Variables in Terms of ϑ

Define $\vartheta_t := \frac{q_t^B}{q_t^K + q_t^B}$

From market clearing:

$$(a - \iota_t)K_t = \rho N_t = \rho q_t K_t \tag{8}$$

So: $q_t = \frac{(a-\iota_t)}{\rho}$

Therefore: $(1 - \vartheta_t)q_t = (1 - \vartheta_t)\frac{(a-\iota_t)}{\rho}$

$$q_t^K = (1 - \vartheta_t)\frac{(a - \iota_t)}{\rho} \tag{9}$$

$$\iota_t = \frac{(1 - \vartheta_t)a - \rho}{(1 - \vartheta_t) + \phi\rho} \tag{10}$$

Therefore:

$$q_t^K = 1 + \phi\frac{(1 - \vartheta_t)a - \rho}{(1 - \vartheta_t) + \phi\rho} = \frac{(1 - \vartheta_t)(1 + \phi a)}{(1 - \vartheta_t) + \phi\rho} \tag{11}$$

$$q_t^B = \vartheta_t \frac{(1 + \phi\alpha)}{(1 - \vartheta_t) + \phi\rho} \tag{12}$$

3 Question 2: Money Valuation Equation

3.1 Part (a): Returns and Wealth Dynamics

Returns on capital:

$$dr_t^{K,\tilde{i}} = \left[\frac{\alpha_t - l_t}{q_t^K} + \Phi(\iota_t) - \delta + \mu_t^{q^K}\right]dt + \sigma_t^{q^K} dZ_t + \tilde{\sigma}_t d\tilde{Z}_t^{\tilde{i}} \tag{13}$$

Returns on bonds:

$$dr_t^B = [\mu_t^B + \Phi(\iota_t) - \delta]dt + \sigma_t^{q^B} dZ_t \tag{14}$$

Individual wealth dynamics:

$$\frac{dn_t^{\tilde{i}}}{n_t^{\tilde{i}}} = -\frac{C_t^{\tilde{i}}}{n_t^{\tilde{i}}}dt + dr_t^B + (1 - \theta_t^{\tilde{i}})(dr_t^{K,\tilde{i}} - dr_t^B) \tag{15}$$

where $n_t^{\tilde{i}} > 0$.

Prices of risk: - Aggregate risk price: ς_t - Idiosyncratic risk price: $\tilde{\varsigma}_t$

3.2 Part (b): Martingale Pricing Condition

Using the martingale pricing condition:

$$\frac{E[dr_t^{K,\tilde{i}}]}{dt} - \frac{E[dr_t^B]}{dt} = \varsigma_t(\sigma_t^{r^{K,\tilde{i}}} - \sigma_t^{r^B}) + \tilde{\varsigma}_t(\tilde{\sigma}_t^{r^{K,\tilde{i}}} - \tilde{\sigma}_t^{r^B}) \tag{16}$$

In equilibrium $\theta = \vartheta$:
This gives us:

$$\frac{\rho}{1-\vartheta_t} + \mu_t^K - \mu_t^B = [\sigma_t^{q^B} + (1-\vartheta_t)(\sigma_t^{q^K} - \sigma_t^{q^B})](\sigma_t^{q^K} - \sigma_t^{q^B}) + (1-\vartheta_t)\tilde{\sigma}^2 \tag{17}$$

Using Ito's lemma for $\vartheta_t = \frac{q_t^B}{q_t^K + q_t^B}$:

$$\frac{\vartheta_t}{1-\vartheta_t} = \frac{q_t^B}{q_t^K} \Leftrightarrow \frac{1-\vartheta_t}{\vartheta_t} = \frac{q_t^K}{q_t^B} \tag{18}$$

$$\iff \quad \frac{\rho}{1-\vartheta_t} - \frac{\mu_t^\vartheta}{\vartheta_t} + (\sigma_t^\vartheta)^2 = (1-\vartheta_t)\,\tilde{\sigma}_t^2, \tag{19}$$

$$\implies \quad \rho\,\vartheta_t^2 - \mu_t^\vartheta\,\vartheta_t\,(1-\vartheta_t) + (\vartheta_t\,\sigma_t^\vartheta)^2\,(1-\vartheta_t) = \big[\vartheta_t(1-\vartheta_t)\,\tilde{\sigma}_t\big]^2 \tag{2}$$

4 Question 3: Steady State Analysis ($\nu = 0$)

In steady state with $\tilde{\sigma}_t = \tilde{\sigma}^{ss}$:

$$\gamma = 0, \quad \tilde{\sigma}_t = \tilde{\sigma}^{ss} \Rightarrow d\tilde{\sigma}_t = 0 \tag{20}$$

$$\mu_t^\vartheta = \sigma_t^\vartheta = 0 \tag{21}$$

From the money valuation equation:

$$\frac{\rho}{1-\vartheta} = (1-\vartheta)\tilde{\sigma}^{ss} \tag{22}$$

$$\Rightarrow (1-\vartheta)^2 = \frac{\rho}{\tilde{\sigma}^{ss}} \tag{23}$$

$$1-\vartheta = \pm\frac{\sqrt{\rho}}{\tilde{\sigma}^{ss}} \tag{24}$$

Since $\vartheta \in [0,1]$, we reject $\vartheta = 1 + \frac{\sqrt{\rho}}{\tilde{\sigma}^{ss}}$.
Therefore: $\vartheta = 1 - \frac{\sqrt{\rho}}{\tilde{\sigma}^{ss}}$

4.1 Monetary Equilibrium Condition

For monetary equilibrium to exist, we need $\vartheta > 0$:

$$\Rightarrow 1 - \frac{\sqrt{\rho}}{\tilde{\sigma}^{ss}} > 0 \tag{25}$$

$$\Rightarrow \frac{\sqrt{\rho}}{\tilde{\sigma}^{ss}} < 1 \tag{26}$$

$$\Rightarrow \tilde{\sigma}^{ss} > \sqrt{\rho} \tag{27}$$

Smallest value: $\tilde{\sigma}^{ss}_{min} = \sqrt{\rho}$

4.2 Comparative Statics

If $\tilde{\sigma}^{ss} > \tilde{\sigma}^{ss}_{min}$, then:

$$\frac{\partial \vartheta}{\partial \tilde{\sigma}^{ss}} = \frac{\sqrt{\rho}}{(\tilde{\sigma}^{ss})^2} > 0 \tag{28}$$

So $\vartheta \uparrow$ (increases)
If $0 < \tilde{\sigma}^{ss} < \tilde{\sigma}^{ss}_{min}$, then $\vartheta = 0$ and:

$$\frac{\partial \vartheta}{\partial \tilde{\sigma}^{ss}} = 0 \tag{29}$$

So no change in ϑ.
We already know the relationship of prices and ϑ from part 1.

4. $\vartheta_t = \vartheta(\tilde{\sigma}_t)$

$$\vartheta_t \mu_t^{\vartheta} = \vartheta' \mu_t^{\tilde{\sigma}} \tilde{\sigma}_t + \frac{1}{2}\vartheta''(\sigma_t^{\tilde{\sigma}}\tilde{\sigma}_t)^2$$

$$= \vartheta' b(\tilde{\sigma}^{ss} - \tilde{\sigma}_t) + \frac{1}{2}\vartheta''\nu^2\tilde{\sigma}_t$$

$$-\vartheta_t \sigma_t^{\vartheta}\sigma_t^{\tilde{\sigma}} = \vartheta'\nu\sqrt{\tilde{\sigma}_t}$$

$$\rho\vartheta_t^2 - \mu_t^{\vartheta}\vartheta_t(1-\vartheta_t) + (\vartheta_t\sigma_t^{\vartheta})^2(1-\vartheta_t) = (\vartheta_t')^2(1-\vartheta_t)\tilde{\sigma}_t\nu^2$$

$$\rho\vartheta_t^2 - (1-\vartheta_t)\left[\vartheta' b(\tilde{\sigma}^{ss} - \tilde{\sigma}_t) + \frac{1}{2}\vartheta''\nu^2\tilde{\sigma}_t\right] + (1-\vartheta_t)(\vartheta_t')^2\nu^2\tilde{\sigma}_t = \vartheta_t^2(1-\vartheta_t)^2\tilde{\sigma}_t\nu^2$$

$$\rho\vartheta_t = \underbrace{(1-\vartheta_t)\left[\frac{\vartheta' b(\tilde{\sigma}^{ss} - \tilde{\sigma}_t) + \frac{1}{2}\vartheta''\nu^2\tilde{\sigma}_t - (\vartheta_t')^2\nu^2\tilde{\sigma}_t}{\vartheta_t}\right]}_{U(\vartheta)} + \underbrace{(1-\vartheta_t)^2\tilde{\sigma}_t^2}_{M(\vartheta)}\vartheta_t$$

Thus,

$$\rho\vartheta_t = U(\vartheta_t) + M(\vartheta_t)\vartheta_t$$

MATLAB code:

```
#Main code
% PS5- M matrix and plots

clear all; clc;

a       = 0.2;
phi     = 1;
delta   = 0.05;
rho     = 0.01;
sigma_ss = 0.2;
b       = 0.05;
nu      = 0.02;
sigmaN  = 0.05;       % aggregate vol. of num raire
xi      = 1;
theta   = 0.5;

N          = 200;
sigma_min = 0.01;
sigma_max = 0.5;
sigma_grid= linspace(sigma_min, sigma_max, N)';

mu_sigma  = b*(sigma_ss - sigma_grid);
sig_sigma = nu*sqrt(sigma_grid);

M = build_M(sigma_grid, mu_sigma, sig_sigma);

%  V a l u e Function  Iteration (Implicit)
% Initial guess
vartheta = 0.5*ones(N,1);

% Drift term u(vartheta)
u_fun = @(v,s) -v.*(1-v).*sigmaN.*xi.*(1-theta).*s ...
              + 0.5*v.*(1-v).*(1-2*v).*(xi.*(1-theta).*s).^2;

dt        = 0.01;
max_iter = 1e4;
tol       = 1e-6;

for it=1:max_iter
    u_new = u_fun(vartheta, sigma_grid);
    rhs   = dt*u_new + vartheta;
    LHS   = (eye(N)*(1+rho*dt) - dt*M);
```

```
    vartheta_next = LHS\rhs;
    if max(abs(vartheta_next - vartheta)) < tol
        fprintf('Converged in %d iterations\n', it);
        vartheta = vartheta_next;
        break;
    end
    vartheta = vartheta_next;
end

% Compute Equilibrium Objects
qK            = ones(N,1);
qB            = vartheta ./ (1 - vartheta);
rf            = rho - u_fun(vartheta,sigma_grid)./vartheta;
varsigma_agg = sigmaN * ones(N,1);
varsigma_idio= sigma_grid;

figure;

%
subplot(3,2,1);
plot(sigma_grid, vartheta,'b');
xlabel('$\tilde{\sigma}_t$','Interpreter','latex');
ylabel('$\vartheta$','Interpreter','latex');
title('Portfolio Share $\vartheta$','Interpreter','latex');

% q^B
subplot(3,2,2);
plot(sigma_grid, qB,'r');
xlabel('$\tilde{\sigma}_t$','Interpreter','latex');
ylabel('$q^B$','Interpreter','latex');
title('Bond Price $q^B$','Interpreter','latex');

% q^K
subplot(3,2,3);
plot(sigma_grid, qK,'g');
xlabel('$\tilde{\sigma}_t$','Interpreter','latex');
ylabel('$q^K$','Interpreter','latex');
title('Capital Price $q^K$','Interpreter','latex');

% r^f
subplot(3,2,4);
plot(sigma_grid, rf,'k');
xlabel('$\tilde{\sigma}_t$','Interpreter','latex');
ylabel('$r^f$','Interpreter','latex');
title('Risk-free Rate $r^f$','Interpreter','latex');

% Aggregate price of risk
```

```
subplot(3,2,5);
plot(sigma_grid, varsigma_agg,'m');
xlabel('$\tilde{\sigma}_t$','Interpreter','latex');
ylabel('$\varsigma$','Interpreter','latex');
title('Aggregate Price of Risk $\varsigma$','Interpreter','latex');

% Idiosyncratic price of risk
subplot(3,2,6);
plot(sigma_grid, varsigma_idio,'c');
xlabel('$\tilde{\sigma}_t$','Interpreter','latex');
ylabel('$\tilde{\varsigma}$','Interpreter','latex');
title('Idiosyncratic Price of Risk $\tilde{\varsigma}$','Interpreter','latex');

% Tidy up & save
set(gcf,'PaperPositionMode','auto');
print('PS5_plots','-dpdf','-r300');
```

```
#Function build_M
function M = build_M(x, mu, sig)
    N   = length(x);
    dx  = x(2)-x(1);
    dx2 = dx^2;

    dD = -min(mu,0)/dx       + sig.^2/(2*dx2);
    dM = -max(mu,0)/dx + min(mu,0)/dx - sig.^2/dx2;
    dU =  max(mu,0)/dx       + sig.^2/(2*dx2);

    M = spdiags([dD dM dU],[1 0 -1],N,N)';
end
```

Equilibrium Plots:

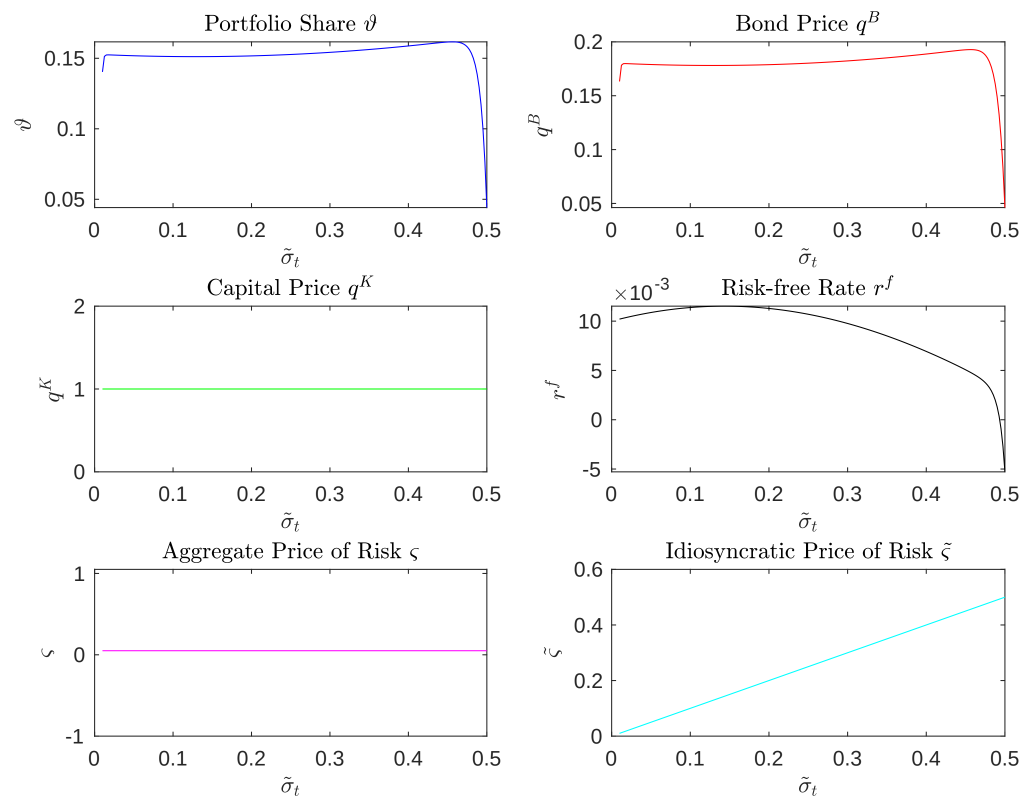

Figure 1: Equilibrium functions of ϑ, q^B, q^K, r^f, ς, and $\tilde{\varsigma}$ against $\tilde{\sigma}_t$.