Real Analysis II Notes

B. Math(Hons.) 1st years

[Note: ■ marks the end of a proof. If used immediately after a statement to be proved, indicates that the proof is trivial and left as an exercise to the reader]

January 24

Assumptions:

- $\mathbb{N}$, the set of all natural numbers, is defined by $\mathbb{N} = \{0, 1, \dots\}$.
- $\mathbb{Z}$, the set of all integers, is defined the usual way.
- $\mathbb{Z}_+$, the set of all non negative integers, is defined by $\mathbb{Z}_+ = \{0, \pm 1, \pm 2, \dots\}$.
- Any function $f\colon [a,b] \subset \mathbb{R} \to \mathbb{R}$ shall always be bounded.

1 Partitions

Definition 1.1. A partition P of $I = [a,b] \subset \mathbb{R}$ is a set of reals $\{x_0, x_1, \dots, x_n\}$ for some $n \in \mathbb{N}$ such that

$$x_0 < x_1 < \dots < x_n$$

We shall denote the interval $[x_{j-1}, x_j]$ by the expression I_j.

Definition 1.2. If $I = (a,b)$ or $(a,b]$ or $[a,b)$ or $[a,b]$, we define

$$|I| = b - a$$

We shall informally refer to $|I|$ as the *length* of I.

Claim 1.1. If $P = \{a = x_0, x_1, \dots, x_n = b\}$ is a partion of $I = [a,b] \subset \mathbb{R}$,

$$|I| = \sum_{i=1}^{n} |I_i|$$

■

Claim 1.2. If P and $\tilde{P}$ are both partitions of an interval $[a,b] \subset \mathbb{R}$, so is $P \cup \tilde{P}$. ■

Definition 1.3.

1) We define $\mathbb{P}[a, b]$ to be the set of all partitions (not just those of a fixed cardinality) of $[a, b]$. If the interval is clear from the context, we shall suppress it, writing $\mathbb{P}[a, b]$ as $\mathbb{P}$.

2) Let $f\colon [a, b] \subset \mathbb{R} \to \mathbb{R}$ be a (bounded) function. Given a partition $P = \{a = x_0, x_1, \ldots, x_n = b\}$ of an interval $I = [a, b] \subset \mathbb{R}$, we define

$$M_j = \sup_{x \in I_j} f(x) \qquad \text{and} \qquad m_j = \inf_{x \in I_j} f(x)$$

for all $1 \leq j \leq n$. We also define

$$M = \sup_{x \in I} f(x) \qquad \text{and} \qquad m = \inf_{x \in I} f(x)$$

Claim 1.3. If $S_1 \subset S_2 \subset \mathbb{R}$,

$$\sup S_1 \leq \sup S_2 \qquad \text{and} \qquad \inf S_1 \geq \inf S_2$$

■

Corollary 1.3.1. Using the notation of item 2 of definition 1.3,

$$m \leq m_j \leq M_j \leq M$$

for all $1 \leq j \leq n$.

Proof. After choosing S_1 and S_2 to be the relevant images of f (see definition 1.3), the statement follows trivially. ■

Definition 1.4. Given an interval $[a, b] \subset \mathbb{R}$, we define $\mathbb{B}[a, b]$ to be the set of all bounded functions from $[a, b]$ to $\mathbb{R}$.