# Real Analysis II Notes

Indian Statistical Institute, Bangalore

Bachelors of Mathematics
First Year(Batch 2021-2024)

# Contents

# Lecture 1, January 24

[Note: ■ marks the end of a proof, i.e., QED. If used immediately after a statement to be proved, indicates that the proof is trivial and left as an exercise to the reader]

### Assumptions:

- $\mathbb{N}$, the set of all natural numbers, is defined by $\mathbb{N} = \{0, 1, \dots\}$.
- $\mathbb{Z}$, the set of all integers, is defined the usual way.
- $\mathbb{Z}_+$, the set of all non-negative integers, is defined by $\mathbb{Z}_+ = \{0, \pm 1, \pm 2, \dots\}$.
- Any function $f\colon [a, b] \subset \mathbb{R} \to \mathbb{R}$ shall always be bounded.

## 1.1 Partitions

**Definition 1.1.1.** A partition $P$ of $I = [a, b] \subset \mathbb{R}$ is a set of reals $\{x_0, x_1, \dots, x_n\}$ for some $n \in \mathbb{N}$ such that

$$x_0 < x_1 < \cdots < x_n$$

We shall denote the interval $[x_{j-1}, x_j]$ by the expression $I_j$.

**Definition 1.1.2.** If $I = (a, b)$ or $(a, b]$ or $[a, b)$ or $[a, b]$, we define

$$|I| = b - a$$

We shall informally refer to $|I|$ as the *length* of $I$.

**Claim 1.1.1.** If $P = \{a = x_0, x_1, \dots, x_n = b\}$ is a partition of $I = [a, b] \subset \mathbb{R}$,

$$|I| = \sum_{i=1}^{n} |I_i|$$

■

**Claim 1.1.2.** If $P$ and $\tilde{P}$ are both partitions of an interval $[a, b] \subset \mathbb{R}$, so is $P \cup \tilde{P}$. ■

**Definition 1.1.3.**

1) We define $\mathscr{P}[a,b]$ to be the set of all partitions (not just those of a fixed cardinality) of $[a,b]$. If the interval is clear from the context, we shall suppress it, writing $\mathscr{P}[a,b]$ as $\mathscr{P}$.

2) Let $f\colon [a,b] \subset \mathbb{R} \to \mathbb{R}$ be a (bounded) function. Given a partition $P = \{a = x_0, x_1, \dots, x_n = b\}$ of an interval $I = [a,b] \subset \mathbb{R}$, we define

$$M_j = \sup_{x \in I_j} f(x) \qquad \text{and} \qquad m_j = \inf_{x \in I_j} f(x)$$

for all $1 \leq j \leq n$. We also define

$$M = \sup_{x \in I} f(x) \qquad \text{and} \qquad m = \inf_{x \in I} f(x)$$

**Claim 1.1.3.** If $S_1 \subset S_2 \subset \mathbb{R}$,

$$\sup S_1 \leq \sup S_2 \qquad \text{and} \qquad \inf S_1 \geq \inf S_2$$

■

**Corollary 1.1.3.1.** Using the notation of item 2 of definition 1.1.3,

$$m \leq m_j \leq M_j \leq M$$

for all $1 \leq j \leq n$.

*Proof.* After choosing $S_1$ and $S_2$ to be the relevant images of $f$ (see definition 1.1.3), the statement follows trivially. ■

**Definition 1.1.4.** Given an interval $[a,b] \subset \mathbb{R}$, we define $\mathscr{B}[a,b]$ to be the set of all bounded functions from $[a,b]$ to $\mathbb{R}$.

**Definition 1.1.5.** Let $f \in \mathscr{B}[a,b]$ and let $P = \{a = x_0, x_1, \dots, x_n = b\} \in \mathscr{P}[a,b]$. Then, using the notation of item 2 of definition 1.1.3, the **upper Riemann sum** of $f$ with respect to $P$ is defined as

$$U(f;P) = \sum_{i=1}^{n} M_i |I_i|$$

and the **lower Riemann sum** of $f$ with respect to $P$ is defined as

$$L(f;P) = \sum_{i=1}^{n} m_i |I_i|$$

**Claim 1.1.4.** Both $U(f;P)$ and $L(f;P)$ must exist.

*Proof.* The maxima and minima exist because $f \in \mathscr{B}[a,b]$, and the sums exist because $P$ has finitely many elements (also called nodes). ■

**Theorem 1.1.5.** Using the notation of definitions 1.1.5 and 1.1.3, given a function $f \in \mathscr{B}[a,b]$,

$$m(b-a) \le L(f;P) \le U(f;P) \le M(b-a)$$

for all $P \in \mathscr{P}[a,b]$.

*Proof.* Let $P = \{a = x_0, x_1, \dots, x_n = b\}$. For all $1 \le i \le n$, using corollary 1.1.3.1

$$m|I_i| \le m_i|I_i| \le M_i|I_i| \le M|I_i|$$

Summing up over all $1 \le i \le n$, we obtain the desired inequality. ■

**Corollary 1.1.5.1.** For all $P \in \mathscr{P}$, $L(f;P)$ and $U(f;P)$ are bounded by $m(b-a)$ and $M(b-a)$. ■

**Definition 1.1.6.** Suppose that $f \in \mathscr{B}[a,b]$. We define the **lower Riemann integral** of $f$ on $[a,b]$ to be

$$\underline{\int_a^b} f = \sup\{L(f,P) \colon P \in \mathscr{P}[a,b]\}$$

and **upper Riemann integral** of $f$ on $[a,b]$ to be

$$\overline{\int_a^b} f = \inf\{U(f,P) \colon P \in \mathscr{P}[a,b]\}$$

Note that both integrals must exist, since they are defined as the supremums/infimums of sets that by corollary 1.1.5.1 are bounded, and because the reals are complete.

**Definition 1.1.7.** Suppose that $f \in \mathscr{B}[a,b]$. We say that $f$ is **Riemann integrable** on $[a,b]$ if

$$\underline{\int_a^b} f = \overline{\int_a^b} f$$

We also define the **Riemann integral** of $f$ on $[a,b]$ to be

$$\int_a^b f := \underline{\int_a^b} f = \overline{\int_a^b} f$$

The remainder of the notes covering the material done in Lecture 1 overlaps with the notes covering the material done in Lecture 2, so there is no point in reproducing it here.

# Lecture 2, January 26

## 2.1 Properties of Lower and Upper Riemann Sums

**Proposition 2.1.1.** Let $f \in \mathscr{B}[a,b]$ and let $P, \tilde{P} \in \mathscr{P}[a,b]$, if $\tilde{P} \supset P$ then

$$L(f,P) \leq L(f,\tilde{P}) \leq U(f,\tilde{P}) \leq U(f,P)$$

*Proof.* We prove it for the case $\tilde{P} = P \cup \{c\}$. Suppose $c \in [x_i, x_{i-1}]$ where $P = \{x_1, \ldots, x_n\}$. Then we can write

$$U(f,\tilde{P}) = \sum_{\substack{k=1 \\ k \neq i}}^{n} M_k \Delta x_k + \tilde{M}_i(c - x_i) + \tilde{M}_{i+1}(x_{i+1} - c) \tag{2.1.1}$$

where $\tilde{M}_i = \sup\{f(x) : x \in [x_i, c]\}$ and $\tilde{M}_{i+1} = \sup\{f(x) : x \in [c, x_{i+1}]\}$. Now since

$$[x_i, c], [c, x_{i+1}] \subset [x_i, x_{i+1}]$$

its obvious that $\tilde{M}_i \leq M_i$ and $\tilde{M}_{i+1} \leq M_i$. But then from equation (2.1.1) we get that

$$\begin{aligned} U(f,\tilde{P}) &\leq \sum_{\substack{k=1 \\ k \neq i}}^{n} M_k \Delta x_k + M_i(c - x_i) + M_i(x_{i+1} - c) \\ &= \sum_{k=1}^{n} M_k \Delta x_k \\ &= U(f,P) \end{aligned}$$

Now by induction it easily follows that for any $\tilde{P} \supset P$, we have $U(f,\tilde{P}) \leq U(f,P)$. The proof of the other part is similar, just that in place of $\tilde{M}_i$ and $\tilde{M}_{i+1}$ we will be working with $\tilde{m}_i$ and $\tilde{m}_{i+1}$, where $\tilde{m}_i = \inf\{f(x) : x \in [x_i, c]\}$ and $\tilde{m}_{i+1} = \inf\{f(x) : x \in [c, x_{i+1}]\}$, and we will use that fact that $\tilde{m}_i, \tilde{m}_{i+1} \geq m_i$.

Now since for any $P \in \mathscr{P}[a,b]$, we have $L(f,P) \leq U(f,P)$, we get that

$$L(f,P) \leq L(f,\tilde{P}) \leq U(f,\tilde{P}) \leq U(f,P)$$

which completes the proof. ■

**Corollary 2.1.1.1.** Let $f \in \mathscr{B}[a,b]$ and $P, Q \in \mathscr{P}[a,b]$, then

$$L(f,P) \leq U(f,Q)$$

*Proof.* We take $\tilde{P} = P \cup Q$, then we have $\tilde{P} \supset P$ and $\tilde{P} \supset Q$ then using **Proposition** 2.1.1, we get that

$$L(f,P) \le L(f,\tilde{P}) \le U(f,\tilde{P}) \le U(f,Q)$$

which completes the proof. ■

**Corollary 2.1.1.2.** Let $f \in \mathscr{B}[a,b]$, then

$$\underline{\int_a^b} f \le \overline{\int_a^b} f$$

*Proof.* From **Corollary** 2.1.1.1, we know that for any $P, Q \in \mathscr{P}[a,b]$, we have $L(f,P) \le U(f,Q)$. Now fix $Q$ thus we get that $U(f,Q)$ is an upper bound for $L(f,P)$ for all $P \in \mathscr{P}[a,b]$, hence

$$\underline{\int_a^b} f = \sup\{L(f,P) : P \in \mathscr{P}[a,b]\} \le U(f,Q)$$

But then we get that $\underline{\int_a^b} f$ is a lower bound for $U(f,Q)$ for all $Q \in \mathscr{P}[a,b]$, thus we get that

$$\underline{\int_a^b} f \le \inf\{U(f,Q) : Q \in \mathscr{P}[a,b]\} = \overline{\int_a^b} f$$

■

Now the question that arises is whether $\mathscr{B}[a,b] = \mathscr{R}[a,b]$, i.e., are all bounded functions Riemann integrable? And as it turns out this is not true, consider the following counter example.

**Counter Example 2.1.2.** Consider the **Dirichlet function** $f : [0,1] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1 & \text{if } x \in [0,1] \cap \mathbb{Q} \\ 0 & \text{if } x \in [0,1] \cap \mathbb{Q}^c \end{cases}$$

Clearly $f \in \mathscr{B}[0,1]$. But note that for any partition $P = \{x_1, \ldots, x_n\} \in \mathscr{P}[0,1]$, we have

$$I_j \cap \mathbb{Q} \ne \emptyset \text{ and } I_j \cap \mathbb{Q}^c \ne \emptyset,\ \forall j = 1, \ldots, n-1$$

where $I_j = [x_j, x_{j+1}]$. And hence we trivially get that

$$L(f,P) = 0 \text{ and } U(f,P) = 1,\ \forall P \in \mathscr{P}[0,1]$$

and hence we get that

$$\underline{\int_0^1} f = 0 \ne 1 = \overline{\int_0^1} f$$

and thus we get that $f \notin \mathscr{R}[0,1]$.

We conclude this section with two examples.

**Example 2.1.3.** The set of Riemann integrable functions on $[a,b]$ is non-empty. Consider $f : [a,b] \to \mathbb{R}$ defined by $f(x) = c$ for all $x \in [a,b]$, where $c$ is any real number. Then it's trivial to show that

$$L(f,P) = U(f,P) = c(b-a),\ \forall P \in \mathscr{P}[a,b]$$

Thus, we obviously have $f \in \mathscr{R}[a,b]$, in particular we get that $\int_a^b f = c(b-a)$.

**Example 2.1.4.** Can we find a function $f \in \mathscr{B}[a,b]$ such that $f \notin \mathscr{R}[a,b]$ but $|f| \in \mathscr{R}[a,b]$ ? Consider $f : [0,1] \to \mathbb{R}$ as follows

$$f(x) = \begin{cases} 1 & \text{if } x \in [0,1] \cap \mathbb{Q} \\ -1 & \text{if } x \in [0,1] \cap \mathbb{Q}^c \end{cases}$$

Then it's trivial to show that $f \notin \mathscr{R}[0,1]$ (the proof is exactly same as the arguments given in **Counter Example** 2.1.2), whereas $|f|$ is simply a constant function, and from **Example** 2.1.3, it follows that $|f| \in \mathscr{R}[0,1]$.

# Lecture 5, February 4

Let $f \in \mathscr{B}[a,b]$. We have already shown that for any partition $P \in \mathscr{P}[a,b]$, we have

$$L(f,P) \le S(f,P) \le U(f,P) \tag{5.0.2}$$

## 5.1 Riemann Sums as a tool for computing integrals

**Definition 5.1.1.** Given $f \in \mathscr{B}[a,b]$, we say that

$$\lim_{\|P\|\to 0} S(f,P) = \lambda \tag{5.1.1}$$

for some $\lambda \in \mathbb{R}$, if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$|S(f,P) - \lambda| < \varepsilon$$

for all $P \in \mathscr{P}[a,b]$, and tag set $T_P$, such that $\|P\| < \delta$.

Observe that whenever the limit in equation (5.1.1) exists, the limit is unique, and hence our definition does not have any ambiguity.

**Theorem 5.1.1.** Let $f \in \mathscr{B}[a,b]$. Then $f \in \mathscr{R}[a,b]$, if and only if there exists a $\lambda \in \mathbb{R}$ such that

$$\lim_{\|P\|\to 0} S(f,P) = \lambda$$

in this case we have $\int_a^b f = \lambda$.

*Proof.* Let $f \in \mathbb{R}[a,b]$, and suppose $\lambda = \int_a^b f$. Fix $\varepsilon > 0$. Then from **Darboux Criterion** we get that there exists a $\delta > 0$ such that

$$U(f,P) - L(f,P) < \varepsilon$$

for all $P \in \mathscr{P}[a,b]$ such that $\|P\| < \delta$. Then note that

$$L(f,P) > U(f,P) - \varepsilon \ge \overline{\int_a^b} f - \varepsilon = \lambda - \varepsilon$$
$$U(f,P) < L(f,P) + \varepsilon \ge \underline{\int_a^b} f + \varepsilon = \lambda + \varepsilon$$

Then from equation (5.0.2) we get that

$$\lambda - \varepsilon < L(f,P) \le S(f,P) \le U(f,P) < \lambda + \varepsilon$$

Hence, we get that $|S(f,P) - \lambda| < \varepsilon$ for all $P \in \mathscr{P}[a,b]$ and tag set $T_P$, such that $\|P\| < \delta$.

Conversely, suppose

$$\lim_{\|P\|\to 0} S(f,P) = \lambda$$

Then for $\varepsilon > 0$, we get there exists a $\delta > 0$ such that $|S(f,P) - \lambda| < \dfrac{\varepsilon}{2}$ for all $P \in \mathscr{P}[a,b]$ and tag set $T_P$, such that $\|P\| < \delta$. Suppose $P = \{x_0, \ldots, x_n\}$, and let $M_j = \sup_{x\in I_j} f(x)$ where $I_j = [x_{j-1}, x_j]$ for $j = 1, \ldots, n$. Then note that from properties of supremum of a set we get that $\forall\ \varepsilon > 0$, there exists a $\zeta_j \in I_j$ such that

$$M_j - \frac{\varepsilon}{2(b-a)} < f(\zeta_j) \leq M_j$$

So if we choose our tag set to be the set of all these $\zeta_j$'s, we get that

$$\begin{aligned} S(f,P) &= \sum_{j=1}^{n} f(\zeta_j)\Delta x_j \\ &> \sum_{j=1}^{n} \left(M_j - \frac{\varepsilon}{2(b-a)}\right)\Delta x_j \\ &= \sum_{j=1}^{n} M_j \Delta x_j - \frac{\varepsilon}{2(b-a)} \sum_{j=1}^{n} \Delta x_j \\ &= U(f,P) - \frac{\varepsilon}{2} \end{aligned}$$

But now since

$$\lambda - \frac{\varepsilon}{2} < S(f,P) < \lambda + \frac{\varepsilon}{2}$$

for all $P \in \mathscr{P}[a,b]$ and tag set $T_P$, such that $\|P\| < \delta$, we get that

$$U(f,P) - \frac{\varepsilon}{2} < S(f,P) < \lambda + \frac{\varepsilon}{2} \Rightarrow U(f,P) < \lambda + \varepsilon$$

and

$$\lambda - \frac{\varepsilon}{2} < S(f,P) \leq U(f,P) \Rightarrow U(f,P) > \lambda - \varepsilon$$

hence, combining everything we get that

$$|U(f,P) - \lambda| < \varepsilon,\ \forall\, P \in \mathscr{P}[a,b] \text{ such that } \|P\| < \delta \tag{5.1.2}$$

Now using similar arguments its easy to show that

$$|L(f,P) - \lambda| < \varepsilon,\ \forall\, P \in \mathscr{P}[a,b] \text{ such that } \|P\| < \delta \tag{5.1.3}$$

Thus from equation (5.1.2) and (5.1.3) we get that

$$U(f,P) - L(f,P) \leq |U(f,P) - \lambda| + |L(f,P) - \lambda| < 2\varepsilon$$

for all $P \in \mathscr{P}[a,b]$ such that $\|P\| < \delta$, hence from **Darboux's criterion** it follows that $f \in \mathscr{R}[a,b]$, and finally since

$$\lambda - \varepsilon < L(f,P) \leq \int_a^b f \leq U(f,P) < \lambda + \varepsilon$$

It follows that $\int_a^b f = \lambda$, which completes the proof. ■

**Theorem** 5.1.1 is a nice tool for computing integrals, if we already know that $f \in \mathscr{R}[a,b]$.

**Theorem 5.1.2.** Suppose $f \in \mathscr{R}[a,b]$ and $\{P_n\}_{n\in\mathbb{N}} \subset \mathscr{P}[a,b]$ such that $\|P_n\| \to 0$ as $n \to \infty$. Then

$$\lim_{n\to\infty} S(f, P_n) = \int_a^b f$$

*Proof.* Since $f \in \mathscr{R}[a,b]$, using **Darboux criterion** we get that for all $\varepsilon > 0$, there exists a $\delta > 0$, such that

$$U(f,P) - L(f,P) < \varepsilon, \ \forall\, P \in \mathscr{P}[a,b] \text{ such that } \|P\| < \delta$$

Now since $\|P_n\| \to 0$, thus for all $\delta > 0$, there exists an $N \in \mathbb{N}$ such that $\|P_n\| < \delta, \ \forall n > N$. Thus, we get

$$U(f,P_n) - L(f,P_n) < \varepsilon, \ \forall n > N$$

But then we get that

$$\left(U(f,P_n) - \int_a^b f\right) + \left(\int_a^b f - L(f,P_n)\right) < \varepsilon, \ \forall n > N$$

Note that $U(f,P_n) - \int_a^b f \geq 0$ and $\int_a^b f - L(f,P_n) \geq 0$, hence we have

$$0 \leq U(f,P_n) - \int_a^b f < \varepsilon \qquad \text{and} \qquad 0 \leq \int_a^b f - L(f,P_n) < \varepsilon \qquad \forall n > N \tag{5.1.4}$$

Thus from equation (5.1.4), we conclude that

$$\lim_{n\to\infty} L(f,P_n) = \lim_{n\to\infty} U(f,P_n) = \int_a^b f$$

Finally from **Squeeze theorem** we get that

$$\int_a^b f = \lim_{n\to\infty} L(f,P_n) \leq \lim_{n\to\infty} S(f,P_n) \leq \lim_{n\to\infty} U(f,P_n) = \int_a^b f$$

and hence $\lim_{n\to\infty} S(f,P_n) = \int_a^b f$, which completes the proof. ∎

**Note:** *The limit* $\lim_{n\to\infty} S(f,P_n) = \int_a^b f$ *does not depend on the tag set, so we can choose our favourite tag set for computing the integral, given that we know that the function is Riemann integrable.*

Suppose we know that $f \in \mathscr{R}[a,b]$, then consider the following partition

$$a = x_0 < x_1 = x_0 + \frac{1}{n}(b-a) < x_2 = x_0 + \frac{2}{n}(b-a) < \cdots < x_n = x_0 + (b-a) = b$$

Then note that if $P = \{x_0, \ldots, x_n\}$, then $\|P\| = \frac{b-a}{n}$, thus we get that $\|P\| \to 0$ as $n \to \infty$. And since $f \in \mathscr{R}[a,b]$ we know that

$$\lim_{n\to\infty} S(f,P_n) = \int_a^b f$$

Now we choose the tag set to be the points $\{x_0, x_1, \ldots, x_{n-1}\}$, then we get that

$$\lim_{n\to\infty} \left(\frac{b-a}{n}\sum_{j=1}^{n} f\left(a + \frac{j-1}{n}(b-a)\right)\right) = \int_a^b f \tag{5.1.5}$$

Thus, if we know that the function is Riemann integrable then the integral can be written as a limit of **Newton sums**.

## 5.2 What else can we say about the set $\mathscr{R}[a,b]$ ?

Observe that the set of bounded functions on $[a,b]$, i.e., $\mathscr{B}[a,b]$ forms a vector space over the field of real numbers, with addition on $\mathscr{B}[a,b]$ as sum of functions, and scalar multiplication as product of a function with a real number.

- Associativity and commutativity follows trivially.
- Additive identity is the zero function, and obviously we have $\mathbf{0} \in \mathscr{B}[a,b]$, and multiplicative identity is $1$.
- And if $f \in \mathscr{B}[a,b]$, then since $|f| = |-f|$, we get that $-f \in \mathscr{B}[a,b]$, and since $f + (-f) = \mathbf{0}$, we get that additive inverse exists for all $f \in \mathscr{B}[a,b]$.
- Distributive properties hold trivially.

Also, it's not difficult to show that if $f : [a,b] \to [c,d]$ and $g : [c,d] \to \mathbb{R}$ such that both $f, g \in \mathscr{B}[a,b]$, then $g \circ f : [a,b] \to \mathbb{R} \in \mathscr{B}[a,b]$.

The next question that arises immediately is can we say $\mathscr{R}[a,b]$ is a vector space over the field of real numbers? Well, with similar operations as in case of $\mathscr{B}[a,b]$, again it's not very difficult to show that indeed $\mathscr{R}[a,b]$ is a vector space over the field of real numbers. Now consider the function $\mathscr{I} : \mathscr{R}[a,b] \to \mathbb{R}$ given by

$$\mathscr{I}(f) = \int_a^b f, \quad \forall\, f \in \mathscr{R}[a,b] \tag{5.2.1}$$

**Theorem 5.2.1.** Suppose $\mathscr{I} : \mathscr{R}[a,b] \to \mathbb{R}$ be defined as in equation (5.2.1), then following conditions are true

(i) For all $r, s \in \mathbb{R}$ and $f, g \in \mathscr{R}[a,b]$ we have

$$\mathscr{I}(rf + sg) = r\mathscr{I}(f) + s\mathscr{I}(g)$$

i.e., $\mathscr{I}$ is a **linear functional**.

(ii) $\mathscr{I}$ preserves the order, i.e., if $f, g \in \mathscr{R}[a,b]$ and we have $f(x) \le g(x),\ \forall\, x \in [a,b]$, then

$$\mathscr{I}(f) \le \mathscr{I}(g)$$

(iii) Let $c \in (a,b)$ then we have

$$\mathscr{I}(f) = \int_a^c f + \int_c^b f$$

*Proof.*

(i) Let $f, g \in \mathscr{R}[a,b]$, we will first show that $f + g \in \mathscr{R}$. Note the since $f, g \in \mathscr{R}$, then using **theorem** 5.1.1 we get for $\varepsilon > 0$, there exists $\delta_1 > 0$ and $\delta_2 > 0$, such that

$$\left| S(f,P) - \int_a^b f \right| < \frac{\varepsilon}{2},\ \forall\, P \in \mathscr{P}[a,b] \text{ such that } \|P\| < \delta_1$$

$$\left| S(g,P) - \int_a^b g \right| < \frac{\varepsilon}{2},\ \forall\, P \in \mathscr{P}[a,b] \text{ such that } \|P\| < \delta_2$$

choose $\delta = \min\{\delta_1, \delta_2\}$, then we get that

$$
\begin{aligned}
\left|S(f+g,P)-\int_a^b f-\int_a^b g\right| &= \left|\left(S(f,P)-\int_a^b f\right)+\left(S(g,P)-\int_a^b g\right)\right| \\
&\leq \left|S(f,P)-\int_a^b f\right|+\left|S(g,P)-\int_a^b g\right| \\
&\overset{(1)}{<} \varepsilon
\end{aligned}
$$

where the $(1)$ holds $\forall\, P \in \mathscr{P}[a,b]$ such that $\|P\| < \delta$. Hence, we have shown that $f+g \in \mathscr{R}[a,b]$, and in particular we have

$$
\int_a^b (f+g) = \int_a^b f + \int_a^b g \tag{5.2.2}
$$

Finally its enough to show that if $f \in \mathscr{R}[a,b]$, then $rf \in \mathscr{R}[a,b]$ for all $r \in \mathbb{R}$. But this is not difficult to prove as it can be easily seen that

$$
S(rf,P) = rS(f,P)
$$

and hence

$$
\left|S(rf,P) - r\int_a^b f\right| = |r|\left|S(f,P)-\int_a^b f\right| \overset{(1)}{<} |r|\varepsilon
$$

where $(1)$ holds for all $P \in \mathscr{P}[a,b]$, such that $\|P\| < \delta$, from here we can conclude that $rf \in \mathscr{R}[a,b]$, and in particular we have shown that

$$
\int_a^b rf = r\int_a^b f \tag{5.2.3}
$$

Hence, using equation (5.2.2) and (5.2.3) we get that if $f, g \in \mathscr{R}[a,b]$, then $rf, sg \in \mathscr{R}[a,b]$, and hence $rf + sg \in \mathscr{R}[a,b]$, and in particular we have

$$
\mathscr{I}(rf+sg) = \int_a^b (rf+sg) = r\int_a^b f + s\int_a^b g = r\mathscr{I}(f) + s\mathscr{I}(s)
$$

Thus we have proved that $I : \mathscr{R}[a,b] \to \mathbb{R}$, is a **linear functional**.

(ii) Let $f, g \in \mathscr{R}[a,b]$ such that $f(x) \leq g(x)$, $\forall\, x \in [a,b]$, then consider a sequence of partitions $\{P_n\} \subset \mathscr{P}[a,b]$ such that $\|P_n\| \to 0$ as $n \to \infty$, then using **theorem** 5.1.2, we get that

$$
\lim_{n\to\infty} S(f,P_n) = \mathscr{I}(f) \qquad \text{and} \qquad \lim_{n\to\infty} S(g,P_n) = \mathscr{I}(g)
$$

then for any $P_n \in \mathscr{P}[a,b]$ and for any tag set $T_{P_n} = \{\zeta_1, \ldots, \zeta_n\}$ we get that

$$
S(f,P_n) = \sum_{j=1}^n f(\zeta_j)\Delta x_j \leq \sum_{j=1}^n g(\zeta_j)\Delta x_j = S(g,P_n)
$$

and hence we get that

$$
\mathscr{I}(f) = \lim_{n\to\infty} S(f,P_n) \leq \lim_{n\to\infty} S(g,P_n) = \mathscr{I}(g)
$$

(iii) Let $c \in (a,b)$. We first need to show that if $f \in \mathscr{R}[a,b]$, then $f \in \mathscr{R}[a,c]$ and $f \in \mathscr{R}[c,b]$ for all $c \in [a,b]$. Now since $f \in \mathscr{R}[a,b]$ **Cauchy criterion** tells us that for all $\varepsilon > 0$, there exists a partition $P_\varepsilon \in \mathscr{P}[a,b]$ such that for all $P \supset P_\varepsilon$, we have

$$
U(f,P) - L(f,P) < \varepsilon
$$

Now without loss of generality we can assume that $c \in P$ (otherwise we can simply work with $P \cup \{c\}$). Let

$$P = \{a = x_0, \ldots, x_{i-1}, x_i = c, x_{i+1}, \ldots, x_n = b$$

and consider $P_1 = \{x_0, \ldots, x_i\}$ and $P_2 = \{x_i, \ldots, x_n\}$, then we have $P_1 \in \mathscr{P}[a, c]$ and $P_2 \in \mathscr{P}[c, b]$ and further $P = P_1 \cup P_2$. Now observe that

$$L(f, P) = L(f, P_1) + L(f, P_2) \qquad \text{and} \qquad U(f, P) = U(f, P_1) + U(f, P_2)$$

Then we get that

$$\underbrace{(U(f, P_1) - L(f, P_1))}_{\geq 0} + \underbrace{(U(f, P_2) - L(f, P_2))}_{\geq 0} = U(f, P) - L(f, P) < \varepsilon \tag{5.2.4}$$

Thus from equation (5.2.4), we deduce that

$$U(f, P_1) - L(f, P_1) < \varepsilon \qquad \text{and} \qquad U(f, P_2) - L(f, P_2) < \varepsilon \tag{5.2.5}$$

Now since equation (5.2.5), will hold for any partition $P', P^{''}$ finer that $P_1, P_2$ respectively we can conclude that $f \in \mathscr{R}[a, c]$ and $f \in \mathscr{R}[c, b]$, so $\int_a^c f$ and $\int_c^b f$ are well-defined. Finally, we have

$$\begin{aligned}
\int_a^b f &\geq L(f, P) \\
&= L(f, P_1) + L(f, P_2) \\
&> (U(f, P_1) - \varepsilon) + (U(f, P_2) - \varepsilon) \\
&= (U(f, P_1) + U(f, P_2)) - 2\varepsilon \\
&\geq \left( \int_a^c f + \int_c^b f \right) - 2\varepsilon
\end{aligned}$$

and on the other hand we get

$$\begin{aligned}
\int_a^b f &\leq U(f, P) \\
&= U(f, P_1) + U(f, P_2) \\
&< (L(f, P_1) + L(f, P_2)) + 2\varepsilon \\
&\leq \left( \int_a^c f + \int_c^b f \right) + 2\varepsilon
\end{aligned}$$

Hence, we have shown that for all $\varepsilon > 0$, we have

$$\left| \int_a^b f - \left( \int_a^c f + \int_c^b f \right) \right| < 2\varepsilon$$

and hence, we can conclude that

$$\mathscr{I}(f) = \int_a^b f = \int_a^c f + \int_c^b f$$

■

# Lecture 6, February 7

Note that the first theorem described in JDS's version of the notes for the lecture on February 7th is included in Theorem 5.2.1, Lecture 5 of this document, and is thus not reproduced here.

**Theorem 6.0.2.** If $B \subset \mathbb{R}$ is a bounded set, $\widehat{B} := \{|x-y| : x, y \in B\}$ is also bounded, and

$$\sup \widehat{B} = \sup B - \inf B$$

*Proof.* There exists ∎

# Lecture 8, Febuary 14

## (Valentine's Special)

### 8.1 Fundamental Theorem of Calculus

**Definition 8.1.1.** Let $S \subseteq \mathbb{R}$, and let $f : S \to \mathbb{R}$ be a function. A differentiable function $F$ is called **antiderivative** or **primitive** of $f$ on $S$, if

$$f(x) = F'(x), \ \forall\, x \in S$$

**Example 8.1.1.** Let $f : [0,1] \to \mathbb{R}$ be defined by $f(x) = x, \ \forall\, x \in [0,1]$ then $F(x) = \frac{1}{2}x^2 + c$ is a antiderivative of $f$ on $[0,1]$, where $c$ is a fixed real number. Thus clearly antiderivatives are not unique, since if $F$ is an antiderivative of $f$ on some set $S$, then so is $F' := F + c$, where $c$ is any real number.

The natural question that arises now is do all functions have antiderivatives?

**Example 8.1.2.** Consider $f : [-1.1] \to \mathbb{R}$ as follows

$$f(x) = \begin{cases} 1 & \text{if } x \in [0,1] \\ 0 & \text{otherwise} \end{cases}$$

Then $f$ doesn't have any antiderivative. To see this observe if $f$ indeed had an antiderivative $F$, then we would have

$$F'(x) = f(x) = \begin{cases} 1 & \text{if } x \in [0,1] \\ 0 & \text{otherwise} \end{cases}$$

But then by **Darboux's theorem**[1], we would get that $\forall\, c \in (0,1)$, there exists a $\eta \in [-1,1]$ such that $f(\eta) = F'(\eta) = c$ (Contradiction! since range of $f$ is simply $\{0,1\}$).

Thus, its evident from **example** 8.1.2, that there are functions which do not have any antiderivatives. Later we will see that for any continuous function there always exists an antiderivative, this will follow as a consequence of **Second Fundamental Theorem of Calculus**. But first we look at the **First Fundamental Theorem of Calculus**.

[1]**Darboux's theorem:** Let $f : [a,b] \to \mathbb{R}$ be a differentiable function such that $f'(a) \neq f'(b)$, then for any $c$ in between $f'(a)$ and $f'(b)$, there exists $\eta \in (a,b)$ such that $f'(\eta) = c$.

### 8.1.1 First Fundamental Theorem of Calculus

**Theorem 8.1.3.** Let $f \in \mathscr{R}[a,b]$, and suppose $F$ is an antiderivative of $f$ on $(a,b)$, then

$$\int_a^b f = F(b) - F(a)$$

*Proof.* Let $P \in \mathscr{P}[a,b]$, suppose $P = \{a = x_0, x_1, \dots, x_{n-1}, x_n = b\}$ where $x_0 < x_1 < \cdots < x_n$. Then observe that we can write

$$F(b) - F(a) = \sum_{j=1}^{n} (F(x_j) - F(x_{j-1}))$$

But then since $F$ is antiderivative of $f$ on $(a,b)$, we have $F$ is differentiable on $(a,b)$, and hence using **Mean Value Theorem** we get that

$$\begin{aligned} F(x_j) - F(x_{j-1}) &= F'(\zeta_j)(x_j - x_{j-1}) \\ &= f'(\zeta_j)\Delta x_j \end{aligned}$$

for some $\zeta_j \in (x_{j-1}, x_j)$. Hence, we get that

$$L(f,P) \le \sum_{j=1}^{n} f(\zeta_j)\Delta x_j = \sum_{j=1}^{n} (F(x_j) - F(x_{j-1})) = \sum_{j=1}^{n} f(\zeta_j)\Delta x_j \le U(f,P)$$

and thus, we have shown that

$$L(f,P) \le F(b) - F(a) \le U(f,P) \qquad \forall\, P \in \mathscr{P}[a,b] \tag{8.1.1}$$

but then from properties of supremum and infimum of a set we deduce that

$$\underline{\int_a^b} f \le F(b) - F(a) \le \overline{\int_a^b} f$$

But since $f \in \mathscr{R}[a,b]$, we get that $\underline{\int_a^b} f = \int_a^b f = \overline{\int_a^b} f$, and thus we have proved that

$$\int_a^b f = F(b) - F(a)$$

■

Thus, **theorem** 8.1.3 tells us that, if we already know the antiderivative of the function $f$, we can easily compute its integral.

### 8.1.2 Second Fundamental Theorem of Calculus

**Theorem 8.1.4.** Let $f \in \mathscr{R}[a,b]$, and define the function $F : [a,b] \to \mathbb{R}$, as

$$F(x) := \int_a^x f(t)\,\mathrm{d}t\,,\ \forall\, x \in [a,b] \tag{8.1.2}$$

then the following statements are true:

(i) $F \in \mathscr{C}[a,b]$.

(ii) If $f$ is continuous at $x_0 \in (a,b)$, then $F$ is differentiable at $x_0$, and in particular we have $F'(x_0) = f(x_0)$.

(iii) If $f$, is continuous from right at $a$, then $F$ has right-hand derivative at $a$, and we have $F'_+(a) = f(a)$. Similarly, if $f$ is continuous from left at $b$, then $F$ has left-hand derivative at $b$, and we have $F'_-(b) = f(b)$.

*Proof.*

(i) Let $M = \sup_{x\in[a,b]} |f(x)|$, let $x, y \in [a,b]$, then we have

$$|F(x) - F(y)| = \left|\int_y^x f(t)\,\mathrm{d}t\right|$$

Now note that from $-M \le f(t) \le M$ we have

$$\begin{aligned}
&\Rightarrow -\int_y^x M\,\mathrm{d}t \le \int_y^x f(t)\,\mathrm{d}t \le \int_x^y M\,\mathrm{d}t \\
&\Rightarrow -M(x-y) \le \int_y^x f(t)\,\mathrm{d}t \le M(x-y) \\
&\Rightarrow \left|\int_y^x f(t)\,\mathrm{d}t\right| \le M|x-y|
\end{aligned}$$

and hence we get that

$$|F(x) - F(y)| \le M|x-y|$$

and hence, $F$ from here we can easily see that $F \in \mathscr{C}[a,b]$, and we further conclude that $F$ is in fact **Lipschitz continuous**.

(ii) Observe that we can write

$$\frac{F(x) - F(x_0)}{x - x_0} - f(x_0) = \frac{1}{x - x_0}\int_{x_0}^x (f(t) - f(x_0))\,\mathrm{d}t$$

But then since $f$ is continuous at $x_0$, we have for all $\varepsilon > 0$, there exists a $\delta > 0$, such that

$$|f(t) - f(x_0)| < \varepsilon,\ \forall\, t \in (x_0 - \delta, x_0 + \delta)$$

and hence for $x \in (x_0 - \delta, x_0 - \delta)$ we have

$$\begin{aligned}
\left|\frac{F(x) - F(x_0)}{x - x_0} - f(x_0)\right| &= \frac{1}{|x - x_0|}\left|\int_{x_0}^x (f(t) - f(x_0))\,\mathrm{d}t\right| \\
&\le \frac{1}{|x - x_0|}\int_{x_0}^x |f(t) - f(x_0)|\,\mathrm{d}t \\
&\overset{(1)}{<} \frac{1}{|x - x_0|}\int_{x_0}^x \varepsilon\,\mathrm{d}t \\
&\le \frac{1}{|x - x_0|}\cdot |x - x_0| \cdot \varepsilon = \varepsilon
\end{aligned}$$

where $(1)$ is true because for $x \in (x_0 - \delta, x_0 + \delta)$, we have $t \in (x_0 - \delta, x_0 + \delta)$. Hence, if $x_0$ is a point continuity of $f$, we have $F$ is differentiable at $x_0$, and further we have $F'(x_0) = f(x_0)$.

(iii) Now if $f$ is continuous from right at $a$, then similar arguments as in part (ii), will work, just that we in this case we have to work on an interval $(a, a + \delta)$, for some $\delta > 0$. The proof for $f$ is continuous from left at $b$, is also similar (in this case we will have to work on an interval $(b - \delta, b)$ for some $\delta > 0$).

■

**Corollary 8.1.4.1.** Let $f \in \mathscr{C}[a,b]$, then

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\int_a^x f(t)dt\right) = f(x),\ \forall\, x \in [a,b]$$

*Proof.* Directly follows from **theorem** 8.1.4. ■

**Example 8.1.5.** Consider $f : [0, 2] \to \mathbb{R}$, defined by

$$f(x) = \begin{cases} 1 & \text{if } x \in [0, 1] \\ 0 & \text{otherwise} \end{cases}$$

Then it is obvious that $f \in \mathscr{R}[0, 2]$. Now the function $F(x) := \int_0^x f(t)\,\mathrm{d}t$, where $x \in [0, 2]$ can be easily computed, and we get

$$F(x) = \begin{cases} x & \text{if } 0 \le x \le 1 \\ 1 & \text{if } 1 < x \le 2 \end{cases}$$

But then we easily observe that $F \in \mathscr{C}[0, 2]$, but evidently $F$ is not differentiable at $1$, which is in fact the point of discontinuity of $f$.

## 8.2 Integration By Parts

**Theorem 8.2.1.** Let $f, g \in \mathscr{D}[a, b]$, and further assume that $f', g' \in \mathscr{R}[a, b]$, then

$$\int_a^b f(x)g'(x)\,\mathrm{d}x + \int_a^b f'(x)g(x)\,\mathrm{d}x = f(b)g(b) - f(a)g(a) \tag{8.2.1}$$

*Proof.* Set $u = fg$, then since $f, g \in \mathscr{D}[a, b]$, we have $g \in \mathscr{D}[a, b]$, and in particular we have $u' = fg' + f'g$, then from **first fundamental theorem of calculus** 8.1.3, we get that

$$\int_a^b u'(x)\,\mathrm{d}x = u(b) - u(a)$$

but then observe that from **linearity of integral** (**theorem** 5.2.1 (i)) we get that

$$\int_a^b f(x)g'(x)\,\mathrm{d}x + \int_a^b f'(x)g(x)\,\mathrm{d}x = f(b)g(b) - f(a)g(a)$$

which completes the proof. ■

## 8.3 Change of Variable in a Riemann Integral

**Theorem 8.3.1.** Let $u \in \mathscr{D}[a, b]$, and $u' \in \mathscr{R}[a, b]$ and suppose $f \in \mathscr{C}[u([a, b])]$ then

$$\int_a^b f\left(u(x)\right) u'(x)\,\mathrm{d}x = \int_{u(a)}^{u(b)} f(x)\,\mathrm{d}x \tag{8.3.1}$$

*Proof.* First of all note that if $u$ is a constant function, then the theorem holds trivially. So we assume that $u$ is not a constant function.

Now since $f \in \mathscr{C}[u([a, b])]$ (note that $u([a, b])$ is a closed interval, since $u$ is a continuous function, hence image of $[a, b]$ under $u$ is also a closed interval), so we have $f \circ u \in \mathscr{R}[a, b]$. Also since $u' \in \mathscr{R}[a, b]$, we get that $(f \circ u) \cdot u' \in \mathscr{R}[a, b]$, hence $\int_a^b f(u(x))u'(x)\,\mathrm{d}x$ is well-defined.

Now $\forall\, x \in u([a, b])$ we define

$$F(x) := \int_{u(a)}^{x} f(t)\,\mathrm{d}t$$

then by **second fundamental theorem of calculus** 8.1.4, we get $F'(x) = f(x),\ \forall\, x \in [a, b]$.

Also observe that from **chain rule** we get that

$$(F \circ u)'(t) = F'(u(t))u'(t) = f(u(t))u'(t)$$

and finally we get that

$$\begin{aligned}\int_a^b f(u(x))u'(x)\,\mathrm{d}x &= \int_a^b (F \circ u)'(x)\,\mathrm{d}x \\ &\stackrel{(1)}{=} F(u(b)) - F(u(a)) \\ &\stackrel{(2)}{=} \int_{u(a)}^{u(b)} f(x)\,\mathrm{d}x\end{aligned}$$

where (1) follows from **first fundamental theorem of calculus** 8.1.3, and (2) follows from the fact that $F(u(a)) = 0$. ■