# Real Analysis II Notes

B. Math(Hons.) 1st years

# Contents

# Lecture 1, January 24

[Note: $\blacksquare$ marks the end of a proof. If used immediately after a statement to be proved, indicates that the proof is trivial and left as an exercise to the reader]

### Assumptions:

- $\mathbb{N}$, the set of all natural numbers, is defined by $\mathbb{N} = \{0, 1, \dots\}$.
- $\mathbb{Z}$, the set of all integers, is defined the usual way.
- $\mathbb{Z}_+$, the set of all non negative integers, is defined by $\mathbb{Z}_+ = \{0, \pm 1, \pm 2, \dots\}$.
- Any function $f\colon [a, b] \subset \mathbb{R} \to \mathbb{R}$ shall always be bounded.

## 1.1 Partitions

**Definition 1.1.1.** A partition $P$ of $I = [a, b] \subset \mathbb{R}$ is a set of reals $\{x_0, x_1, \dots, x_n\}$ for some $n \in \mathbb{N}$ such that

$$x_0 < x_1 < \cdots < x_n$$

We shall denote the interval $[x_{j-1}, x_j]$ by the expression $I_j$.

**Definition 1.1.2.** If $I = (a, b)$ or $(a, b]$ or $[a, b)$ or $[a, b]$, we define

$$|I| = b - a$$

We shall informally refer to $|I|$ as the *length* of $I$.

**Claim 1.1.1.** If $P = \{a = x_0, x_1, \dots, x_n = b\}$ is a partion of $I = [a, b] \subset \mathbb{R}$,

$$|I| = \sum_{i=1}^{n} |I_i|$$

$\blacksquare$

**Claim 1.1.2.** If $P$ and $\tilde{P}$ are both partitions of an interval $[a, b] \subset \mathbb{R}$, so is $P \cup \tilde{P}$. $\blacksquare$

**Definition 1.1.3.**

1) We define $\mathbb{P}[a, b]$ to be the set of all partitions (not just those of a fixed cardinality) of $[a, b]$. If the interval is clear from the context, we shall suppress it, writing $\mathbb{P}[a, b]$ as $\mathbb{P}$.

2) Let $f\colon [a, b] \subset \mathbb{R} \to \mathbb{R}$ be a (bounded) function. Given a partition $P = \{a = x_0, x_1, \ldots, x_n = b\}$ of an interval $I = [a, b] \subset \mathbb{R}$, we define

$$M_j = \sup_{x \in I_j} f(x) \qquad \text{and} \qquad m_j = \inf_{x \in I_j} f(x)$$

for all $1 \leq j \leq n$. We also define

$$M = \sup_{x \in I} f(x) \qquad \text{and} \qquad m = \inf_{x \in I} f(x)$$

**Claim 1.1.3.** If $S_1 \subset S_2 \subset \mathbb{R}$,

$$\sup S_1 \leq \sup S_2 \qquad \text{and} \qquad \inf S_1 \geq \inf S_2$$

■

**Corollary 1.1.3.1.** Using the notation of item 2 of definition 1.1.3,

$$m \leq m_j \leq M_j \leq M$$

for all $1 \leq j \leq n$.

*Proof.* After choosing $S_1$ and $S_2$ to be the relevant images of $f$ (see definition 1.1.3), the statement follows trivially. ■

**Definition 1.1.4.** Given an interval $[a, b] \subset \mathbb{R}$, we define $\mathbb{B}[a, b]$ to be the set of all bounded functions from $[a, b]$ to $\mathbb{R}$.

# Lecture 2, January 26

## 2.2 Properties of Lower and Upper Riemann Sums

**Proposition 2.2.1.** Let $f \in \mathscr{B}[a,b]$ and let $P, \tilde{P} \in \mathscr{P}[a,b]$, if $\tilde{P} \supset P$ then

$$L(f,P) \leq L(f,\tilde{P}) \leq U(f,\tilde{P}) \leq U(f,P)$$

*Proof.* We prove it for the case $\tilde{P} = P \cup \{c\}$. Suppose $c \in [x_i, x_{i-1}]$ where $P = \{x_1, \ldots, x_n\}$. Then we can write

$$U(f,\tilde{P}) = \sum_{\substack{k=1\\k\neq i}}^{n} M_k \Delta x_k + \tilde{M}_i(c - x_i) + \tilde{M}_{i+1}(x_{i+1} - c) \tag{2.1}$$

where $\tilde{M}_i = \sup\{f(x) : x \in [x_i, c]\}$ and $\tilde{M}_{i+1} = \sup\{f(x) : x \in [c, x_{i+1}]\}$. Now since

$$[x_i, c], [c, x_{i+1}] \subset [x_i, x_{i+1}]$$

its obvious that $\tilde{M}_i \leq M_i$ and $\tilde{M}_{i+1} \leq M_i$. But then from equation (2.1) we get that

$$\begin{aligned} U(f,\tilde{P}) &\leq \sum_{\substack{k=1\\k\neq i}}^{n} M_k \Delta x_k + M_i(c - x_i) + M_i(x_{i+1} - c) \\ &= \sum_{k=1}^{n} M_k \Delta x_k \\ &= U(f,P) \end{aligned}$$

Now by induction it easily follows that for any $\tilde{P} \supset P$, we have $U(f,\tilde{P}) \leq U(f,P)$. The proof of the other part is similar, just that in place of $\tilde{M}_i$ and $\tilde{M}_{i+1}$ we will be working with $\tilde{m}_i$ and $\tilde{m}_{i+1}$, where $\tilde{m}_i = \inf\{f(x) : x \in [x_i, c]\}$ and $\tilde{m}_{i+1} = \inf\{f(x) : x \in [c, x_{i+1}]\}$, and we will use that fact that $\tilde{m}_i, \tilde{m}_{i+1} \geq m_i$.

Now since for any $P \in \mathscr{P}[a,b]$, we have $L(f,P) \leq U(f,P)$, we get that

$$L(f,P) \leq L(f,\tilde{P}) \leq U(f,\tilde{P}) \leq U(f,P)$$

which completes the proof. ■

**Corollary 2.2.1.1.** Let $f \in \mathscr{B}[a,b]$ and $P, Q \in \mathscr{P}[a,b]$, then

$$L(f,P) \leq U(f,Q)$$

*Proof.* We take $\tilde{P} = P \cup Q$, then we have $\tilde{P} \supset P$ and $\tilde{P} \supset Q$ then using **Proposition** 2.2.1, we get that

$$L(f,P) \le L(f,\tilde{P}) \le U(f,\tilde{P}) \le U(f,Q)$$

which completes the proof. ■

**Corollary 2.2.1.2.** Let $f \in \mathscr{B}[a,b]$, then

$$\underline{\int_a^b} f \le \overline{\int_a^b} f$$

*Proof.* From **Corollary** 2.2.1.1, we know that for any $P, Q \in \mathscr{P}[a,b]$, we have $L(f,P) \le U(f,Q)$. Now fix $Q$ thus we get that $U(f,Q)$ is an upper bound for $L(f,P)$ for all $P \in \mathscr{P}[a,b]$, hence

$$\underline{\int_a^b} f = \sup\{L(f,P) : P \in \mathscr{P}[a,b]\} \le U(f,Q)$$

But then we get that $\underline{\int_a^b} f$ is an lower bound for $U(f,Q)$ for all $Q \in \mathscr{P}[a,b]$, thus we get that

$$\underline{\int_a^b} f \le \inf\{U(f,Q) : Q \in \mathscr{P}[a,b]\} = \overline{\int_a^b} f$$

■

Now the question that arises is whether $\mathscr{B}[a,b] = \mathscr{R}[a,b]$, i.e., are all bounded functions Riemann integrable? And as it turns out this is not true, consider the following counter example.

**Counter Example 2.2.2.** Consider the **Dirichlet function** $f : [0,1] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1 & \text{if } x \in [0,1] \cap \mathbb{Q} \\ 0 & \text{if } x \in [0,1] \cap \mathbb{Q}^c \end{cases}$$

Clearly $f \in \mathscr{B}[0,1]$. But note that for any partition $P = \{x_1, \dots, x_n\} \in \mathscr{P}[0,1]$, we have

$$I_j \cap \mathbb{Q} \ne \emptyset \text{ and } I_j \cap \mathbb{Q}^c \ne \emptyset,\ \forall\, j = 1, \dots, n-1$$

where $I_j = [x_j, x_{j+1}]$. And hence we trivially get that

$$L(f,P) = 0 \text{ and } U(f,P) = 1,\ \forall\, P \in \mathscr{P}[0,1]$$

and hence we get that

$$\underline{\int_0^1} f = 0 \ne 1 = \overline{\int_0^1} f$$

and thus we get that $f \notin \mathscr{R}[0,1]$.

We conclude this section with two examples.

**Example 2.2.3.** The set of Riemann integrable functions on $[a,b]$ is non-empty. Consider $f : [a,b] \to \mathbb{R}$ defined by $f(x) = c$ for all $x \in [a,b]$, where $c$ is any real number. Then its trivial to show that

$$L(f,P) = U(f,P) = c(b-a),\ \forall\, P \in \mathscr{P}[a,b]$$

Thus, we obvious have $f \in \mathscr{R}[a,b]$, in particulat we get that $\int_a^b f = c(b-a)$.

**Example 2.2.4.** Can we find a function $f \in \mathscr{B}[a,b]$ such that $f \notin \mathscr{R}[a,b]$ but $|f| \in \mathscr{R}[a,b]$ ? Consider $f : [0,1] \to \mathbb{R}$ as follow

$$f(x) = \begin{cases} 1 & \text{if } x \in [0,1] \cap \mathbb{Q} \\ -1 & \text{if } x \in [0,1] \cap \mathbb{Q}^c \end{cases}$$

Then its trivial to show that $f \notin \mathscr{R}[0,1]$ (the proof is exactly same as the arguments given in **Counter Example** 2.2.2), whereas $|f|$ is simply a constant function, and from **Example** 2.2.3, it follow that $|f| \in \mathscr{R}[0,1]$.