Real Analysis II Notes

B. Math(Hons.) 1st years

Contents

Lecture 1, January 24

[Note: ■ marks the end of a proof. If used immediately after a statement to be proved, indicates that the proof is trivial and left as an exercise to the reader]

Assumptions:

- $\mathbb{N}$, the set of all natural numbers, is defined by $\mathbb{N} = \{0, 1, \dots\}$.
- $\mathbb{Z}$, the set of all integers, is defined the usual way.
- $\mathbb{Z}_+$, the set of all non-negative integers, is defined by $\mathbb{Z}_+ = \{0, \pm 1, \pm 2, \dots\}$.
- Any function $f\colon [a, b] \subset \mathbb{R} \to \mathbb{R}$ shall always be bounded.

1.1 Partitions

Definition 1.1.1. A partition P of $I = [a, b] \subset \mathbb{R}$ is a set of reals $\{x_0, x_1, \dots, x_n\}$ for some $n \in \mathbb{N}$ such that

$$x_0 < x_1 < \cdots < x_n$$

We shall denote the interval $[x_{j-1}, x_j]$ by the expression I_j.

Definition 1.1.2. If $I = (a, b)$ or $(a, b]$ or $[a, b)$ or $[a, b]$, we define

$$|I| = b - a$$

We shall informally refer to $|I|$ as the *length* of I.

Claim 1.1.1. If $P = \{a = x_0, x_1, \dots, x_n = b\}$ is a partition of $I = [a, b] \subset \mathbb{R}$,

$$|I| = \sum_{i=1}^{n} |I_i|$$

■

Claim 1.1.2. If P and $\tilde{P}$ are both partitions of an interval $[a, b] \subset \mathbb{R}$, so is $P \cup \tilde{P}$. ■

Definition 1.1.3.

1) We define $\mathscr{P}[a,b]$ to be the set of all partitions (not just those of a fixed cardinality) of $[a,b]$. If the interval is clear from the context, we shall suppress it, writing $\mathbb{P}[a,b]$ as $\mathbb{P}$.

2) Let $f\colon [a,b] \subset \mathbb{R} \to \mathbb{R}$ be a (bounded) function. Given a partition $P = \{a = x_0, x_1, \dots, x_n = b\}$ of an interval $I = [a,b] \subset \mathbb{R}$, we define

$$M_j = \sup_{x \in I_j} f(x) \qquad \text{and} \qquad m_j = \inf_{x \in I_j} f(x)$$

for all $1 \leq j \leq n$. We also define

$$M = \sup_{x \in I} f(x) \qquad \text{and} \qquad m = \inf_{x \in I} f(x)$$

Claim 1.1.3. If $S_1 \subset S_2 \subset \mathbb{R}$,

$$\sup S_1 \leq \sup S_2 \qquad \text{and} \qquad \inf S_1 \geq \inf S_2$$

■

Corollary 1.1.3.1. Using the notation of item 2 of definition 1.1.3,

$$m \leq m_j \leq M_j \leq M$$

for all $1 \leq j \leq n$.

Proof. After choosing S_1 and S_2 to be the relevant images of f (see definition 1.1.3), the statement follows trivially. ■

Definition 1.1.4. Given an interval $[a,b] \subset \mathbb{R}$, we define $\mathscr{B}[a,b]$ to be the set of all bounded functions from $[a,b]$ to $\mathbb{R}$.

Lecture 2, January 26

2.1 Properties of Lower and Upper Riemann Sums

Proposition 2.1.1. Let $f \in \mathscr{B}[a,b]$ and let $P, \tilde{P} \in \mathscr{P}[a,b]$, if $\tilde{P} \supset P$ then

$$L(f,P) \leq L(f,\tilde{P}) \leq U(f,\tilde{P}) \leq U(f,P)$$

Proof. We prove it for the case $\tilde{P} = P \cup \{c\}$. Suppose $c \in [x_i, x_{i-1}]$ where $P = \{x_1, \ldots, x_n\}$. Then we can write

$$U(f,\tilde{P}) = \sum_{\substack{k=1 \\ k \neq i}}^{n} M_k \Delta x_k + \tilde{M}_i(c - x_i) + \tilde{M}_{i+1}(x_{i+1} - c) \tag{2.1}$$

where $\tilde{M}_i = \sup\{f(x) : x \in [x_i, c]\}$ and $\tilde{M}_{i+1} = \sup\{f(x) : x \in [c, x_{i+1}]\}$. Now since

$$[x_i, c], [c, x_{i+1}] \subset [x_i, x_{i+1}]$$

its obvious that $\tilde{M}_i \leq M_i$ and $\tilde{M}_{i+1} \leq M_i$. But then from equation (2.1) we get that

$$\begin{aligned} U(f,\tilde{P}) &\leq \sum_{\substack{k=1 \\ k \neq i}}^{n} M_k \Delta x_k + M_i(c - x_i) + M_i(x_{i+1} - c) \\ &= \sum_{k=1}^{n} M_k \Delta x_k \\ &= U(f,P) \end{aligned}$$

Now by induction it easily follows that for any $\tilde{P} \supset P$, we have $U(f,\tilde{P}) \leq U(f,P)$. The proof of the other part is similar, just that in place of $\tilde{M}_i$ and $\tilde{M}_{i+1}$ we will be working with $\tilde{m}_i$ and $\tilde{m}_{i+1}$, where $\tilde{m}_i = \inf\{f(x) : x \in [x_i, c]\}$ and $\tilde{m}_{i+1} = \inf\{f(x) : x \in [c, x_{i+1}]\}$, and we will use that fact that $\tilde{m}_i, \tilde{m}_{i+1} \geq m_i$.

Now since for any $P \in \mathscr{P}[a,b]$, we have $L(f,P) \leq U(f,P)$, we get that

$$L(f,P) \leq L(f,\tilde{P}) \leq U(f,\tilde{P}) \leq U(f,P)$$

which completes the proof. ∎

Corollary 2.1.1.1. Let $f \in \mathscr{B}[a,b]$ and $P, Q \in \mathscr{P}[a,b]$, then

$$L(f,P) \leq U(f,Q)$$

Proof. We take $\tilde{P} = P \cup Q$, then we have $\tilde{P} \supset P$ and $\tilde{P} \supset Q$ then using **Proposition** 2.1.1, we get that

$$L(f,P) \le L(f,\tilde{P}) \le U(f,\tilde{P}) \le U(f,Q)$$

which completes the proof. ∎

Corollary 2.1.1.2. Let $f \in \mathscr{B}[a,b]$, then

$$\underline{\int_a^b} f \le \overline{\int_a^b} f$$

Proof. From **Corollary** 2.1.1.1, we know that for any $P, Q \in \mathscr{P}[a,b]$, we have $L(f,P) \le U(f,Q)$. Now fix Q thus we get that $U(f,Q)$ is an upper bound for $L(f,P)$ for all $P \in \mathscr{P}[a,b]$, hence

$$\underline{\int_a^b} f = \sup\{L(f,P) : P \in \mathscr{P}[a,b]\} \le U(f,Q)$$

But then we get that $\underline{\int_a^b} f$ is an lower bound for $U(f,Q)$ for all $Q \in \mathscr{P}[a,b]$, thus we get that

$$\underline{\int_a^b} f \le \inf\{U(f,Q) : Q \in \mathscr{P}[a,b]\} = \overline{\int_a^b} f$$

∎

Now the question that arises is whether $\mathscr{B}[a,b] = \mathscr{R}[a,b]$, i.e., are all bounded functions Riemann integrable? And as it turns out this is not true, consider the following counter example.

Counter Example 2.1.2. Consider the **Dirichlet function** $f : [0,1] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1 & \text{if } x \in [0,1] \cap \mathbb{Q} \\ 0 & \text{if } x \in [0,1] \cap \mathbb{Q}^c \end{cases}$$

Clearly $f \in \mathscr{B}[0,1]$. But note that for any partition $P = \{x_1, \dots, x_n\} \in \mathscr{P}[0,1]$, we have

$$I_j \cap \mathbb{Q} \ne \emptyset \text{ and } I_j \cap \mathbb{Q}^c \ne \emptyset,\ \forall\, j = 1, \dots, n-1$$

where $I_j = [x_j, x_{j+1}]$. And hence we trivially get that

$$L(f,P) = 0 \text{ and } U(f,P) = 1,\ \forall\, P \in \mathscr{P}[0,1]$$

and hence we get that

$$\underline{\int_0^1} f = 0 \ne 1 = \overline{\int_0^1} f$$

and thus we get that $f \notin \mathscr{R}[0,1]$.

We conclude this section with two examples.

Example 2.1.3. The set of Riemann integrable functions on $[a,b]$ is non-empty. Consider $f : [a,b] \to \mathbb{R}$ defined by $f(x) = c$ for all $x \in [a,b]$, where c is any real number. Then its trivial to show that

$$L(f,P) = U(f,P) = c(b-a),\ \forall\, P \in \mathscr{P}[a,b]$$

Thus, we obviously have $f \in \mathscr{R}[a,b]$, in particular we get that $\int_a^b f = c(b-a)$.

Example 2.1.4. Can we find a function $f \in \mathscr{B}[a,b]$ such that $f \notin \mathscr{R}[a,b]$ but $|f| \in \mathscr{R}[a,b]$? Consider $f : [0,1] \to \mathbb{R}$ as follows

$$f(x) = \begin{cases} 1 & \text{if } x \in [0,1] \cap \mathbb{Q} \\ -1 & \text{if } x \in [0,1] \cap \mathbb{Q}^c \end{cases}$$

Then its trivial to show that $f \notin \mathscr{R}[0,1]$ (the proof is exactly same as the arguments given in **Counter Example** 2.1.2), whereas $|f|$ is simply a constant function, and from **Example** 2.1.3, it follow that $|f| \in \mathscr{R}[0,1]$.

Lecture 5, Febuary 4

Let $f \in \mathscr{B}[a,b]$. We have already shown that for any partition $P \in \mathscr{P}[a,b]$, we have

$$L(f,P) \leq S(f,P) \leq U(f,P) \tag{5.2}$$

5.1 Riemann Sums as a tool for computing integrals

Definition 5.1.1. Given $f \in \mathscr{B}[a,b]$, we say that

$$\lim_{\|P\| \to 0} S(f,P) = \lambda \tag{5.3}$$

for some $\lambda \in \mathbb{R}$, if for every $\epsilon > 0$, there exists a $\delta > 0$ such that

$$|S(f,P) - \lambda| < \epsilon$$

for all $P \in \mathscr{P}[a,b]$, and tag set T_P, such that $\|P\| < \delta$.

Observe that whenever the limit in equation (5.3) exists, the limit is unique, and hence our definition does not have any ambiguity.

Theorem 5.1.1. Let $f \in \mathscr{B}[a,b]$. Then $f \in \mathscr{R}[a,b]$, if and only if there exists a $\lambda \in \mathbb{R}$ such that

$$\lim_{\|P\| \to 0} S(f,P) = \lambda$$

in this case we have $\int_a^b f = \lambda$.

Proof. Let $f \in \mathbb{R}[a,b]$, and suppose $\lambda = \int_a^b f$. Fix $\epsilon > 0$. Then from **Darboux Criterion** we get that there exists a $\delta > 0$ such that

$$U(f,P) - L(f,P) < \epsilon$$

for all $P \in \mathscr{P}[a,b]$ such that $\|P\| < \delta$. Then note that

$$L(f,P) > U(f,P) - \epsilon \geq \overline{\int_a^b} f - \epsilon = \lambda - \epsilon$$
$$U(f,P) < L(f,P) + \epsilon \geq \underline{\int_a^b} f + \epsilon = \lambda + \epsilon$$

Then from equation (5.2) we get that

$$\lambda - \epsilon < L(f,P) \leq S(f,P) \leq U(f,P) < \lambda + \epsilon$$

Hence, we get that $|S(f,P) - \lambda| < \epsilon$ for all $P \in \mathscr{P}[a,b]$ and tag set T_P, such that $\|P\| < \delta$.

Conversely, suppose

$$\lim_{\|P\|\to 0} S(f,P) = \lambda$$

Then for $\epsilon > 0$, we get there there exists a $\delta > 0$ such that $|S(f,P) - \lambda| < \frac{\epsilon}{2}$ for all $P \in \mathscr{P}[a,b]$ and tag set T_P, such that $\|P\| < \delta$. Suppose $P = \{x_0, \ldots, x_n\}$, and let $M_j = \sup_{x\in I_j} f(x)$ where $I_j = [x_{j-1}, x_j]$ for $j = 1, \ldots, n$. Then note that from properties of supremum of a set we get that $\forall\ \epsilon > 0$, there exists a $\zeta_j \in I_j$ such that

$$M_j - \frac{\epsilon}{2(b-a)} < f(\zeta_j) \leq M_j$$

So if we choose our tag set to be the set of all these ζ_j's, we get that

$$\begin{aligned}
S(f,P) &= \sum_{j=1}^{n} f(\zeta_j)\Delta x_j \\
&> \sum_{j=1}^{n} (M_j - \frac{\epsilon}{2(b-a)})\Delta x_j \\
&= \sum_{j=1}^{n} M_j \Delta x_j - \frac{\epsilon}{2(b-a)} \sum_{j=1}^{n} \Delta x_j \\
&= U(f,P) - \frac{\epsilon}{2}
\end{aligned}$$

But now since

$$\lambda - \frac{\epsilon}{2} < S(f,P) < \lambda + \frac{\epsilon}{2}$$

for all $P \in \mathscr{P}[a,b]$ and tag set T_P, such that $\|P\| < \delta$, we get that

$$U(f,P) - \frac{\epsilon}{2} < S(f,P) < \lambda + \frac{\epsilon}{2} \Rightarrow U(f,P) < \lambda + \epsilon$$

and

$$\lambda - \frac{\epsilon}{2} < S(f,P) \leq U(f,P) \Rightarrow U(f,P) > \lambda - \epsilon$$

hence, combining everything we get that

$$|U(f,P) - \lambda| < \epsilon,\ \forall\, P \in \mathscr{P}[a,b] \text{ such that } \|P\| < \delta \tag{5.4}$$

Now using similar arguments its easy to show that

$$|L(f,P) - \lambda| < \epsilon,\ \forall\, P \in \mathscr{P}[a,b] \text{ such that } \|P\| < \delta \tag{5.5}$$

Thus from equation (5.4) and (5.5) we get that

$$U(f,P) - L(f,P) \leq |U(f,P) - \lambda| + |L(f,P) - \lambda| < 2\epsilon$$

for all $P \in \mathscr{P}[a,b]$ such that $\|P\| < \delta$, hence from **Darboux's criterion** it follows that $f \in \mathscr{R}[a,b]$, and finally since

$$\lambda - \epsilon < L(f,P) \leq \int_a^b f \leq U(f,P) < \lambda + \epsilon$$

It follows that $\int_a^b f = \lambda$, which completes the proof. ■

Theorem 5.1.1 is a nice tool for computing integrals, if we already know that $f \in \mathscr{R}[a,b]$.

Theorem 5.1.2. Suppose $f \in \mathscr{R}[a,b]$ and $\{P_n\}_{n\in\mathbb{N}} \subset \mathscr{P}[a,b]$ such that $\|P_n\| \to 0$ as $n \to \infty$. Then

$$\lim_{n\to\infty} S(f,P_n) = \int_a^b f$$

Proof. Since $f \in \mathscr{R}[a,b]$, using **Darboux criterion** we get that for all $\epsilon > 0$, there exists a $\delta > 0$, such that

$$U(f,P) - L(f,P) < \epsilon,\ \forall\, P \in \mathscr{P}[a,b] \text{ such that } \|P\| < \delta$$

Now since $\|P_n\| \to 0$, thus for all $\delta > 0$, there exists a $N \in \mathbb{N}$ such that $\|P_n\| < \delta,\ \forall n > N$. Thus we get

$$U(f,P_n) - L(f,P_n) < \epsilon,\ \forall n > N$$

But then we get that

$$\left(U(f,P_n) - \int_a^b f\right) + \left(\int_a^b f - L(f,P_n)\right) < \epsilon,\ \forall n > N$$

Note that $U(f,P_n) - \int_a^b f \geq 0$ and $\int_a^b f - L(f,P_n) \geq 0$, hence we have

$$0 \leq U(f,P_n) - \int_a^b f < \epsilon \qquad \text{and} \qquad 0 \leq \int_a^b f - L(f,P_n) < \epsilon \qquad \forall n > N \tag{5.6}$$

Thus from equation (5.6), we conclude that

$$\lim_{n\to\infty} L(f,P_n) = \lim_{n\to\infty} U(f,P_n) = \int_a^b f$$

Finally from **Squeeze theorem** we get that

$$\int_a^b f = \lim_{n\to\infty} L(f,P_n) \leq \lim_{n\to\infty} S(f,P_n) \leq \lim_{n\to\infty} U(f,P_n) = \int_a^b f$$

and hence $\lim_{n\to\infty} S(f,P_n) = \int_a^b f$, which completes the proof. ■

Note: *The limit* $\lim_{n\to\infty} S(f,P_n) = \int_a^b f$ *does not depend on the tag set, so we can choose our favourite tag set for computing the integral, given that we know that the function is Riemann integrable.*

Suppose we know that $f \in \mathscr{R}[a,b]$, then consider the following partition

$$a = x_0 < x_1 = x_0 + \frac{1}{n}(b-a) < x_2 = x_0 + \frac{2}{n}(b-a) < \cdots < x_n = x_0 + (b-a) = b$$

Then note that if $P = \{x_0, \ldots, x_n\}$, then $\|P\| = \frac{b-a}{n}$, thus we get that $\|P\| \to 0$ as $n \to \infty$. And since $f \in \mathscr{R}[a,b]$ we know that

$$\lim_{n\to\infty} S(f,P_n) = \int_a^b f$$

Now we choose the tag set to be the points $\{x_0, x_1, \ldots, x_{n-1}\}$, then we get that

$$\lim_{n\to\infty}\left(\frac{b-a}{n}\sum_{j=1}^{n} f\left(a + \frac{j-1}{n}(b-a)\right)\right) = \int_a^b f \tag{5.7}$$

Thus, if we know that the function is Riemann integrable then the integral can be written as a limit of **Newton sums**.

5.2 What else can we say about the set $\mathscr{R}[a,b]$?

Observe that the set of bounded functions on $[a,b]$, i.e., $\mathscr{B}[a,b]$ forms a vector space over the field of real numbers, with addition on $\mathscr{B}[a,b]$ as sum of functions, and scalar multiplication as product of a function with a real number.

- Associativity and commutativity follows trivially.
- Additive identiy is the zero function and obviously we have $\mathbf{0} \in \mathscr{B}[a,b]$, and multiplicative identiy is 1.
- And if $f \in \mathscr{B}[a,b]$, then since $|f| = |-f|$, we get that $-f \in \mathscr{B}[a,b]$, and since $f + (-f) = \mathbf{0}$, we get that additive inverse exists for all $f \in \mathscr{B}[a,b]$.
- Distributive properties hold trivially.

Also its not difficult to show that if $f : [a.b] \to [c,d]$ and $g : [c,d] \to \mathbb{R}$ such that both $f, g \in \mathscr{B}[a,b]$, then $g \circ f : [a,b] \to \mathbb{R} \in \mathscr{B}[a,b]$.

The next question that arises immediately is can we say $\mathscr{R}[a,b]$ is a vector space over the field of real numbers? Well, with similar operations as in case of $\mathscr{B}[a,b]$, its again not very difficult to show that indeed $\mathscr{R}[a,b]$ is a vector space over the field of real numbers. Now consider the function $\mathscr{I} : \mathscr{R}[a,b] \to \mathbb{R}$ given by

$$\mathscr{I}(f) = \int_a^b f, \quad \forall\, f \in \mathscr{R}[a,b] \tag{5.8}$$

Theorem 5.2.1. Suppose $\mathscr{I} : \mathscr{R}[a,b] \to \mathbb{R}$ be defined as in equation (5.8), then following conditions are true

(i) For all $r, s \in \mathbb{R}$ and $f, g \in \mathscr{R}[a,b]$ we have

$$\mathscr{I}(rf + sg) = r\mathscr{I}(f) + s\mathscr{I}(g)$$

i.e., $\mathscr{I}$ is a **linear functional**.

(ii) $\mathscr{I}$ preserves the order, i.e., if $f, g \in \mathscr{R}[a,b]$ and we have $f(x) \le g(x), \ \forall\, x \in [a,b]$, then

$$\mathscr{I}(f) \le \mathscr{I}(g)$$

(iii) Let $c \in (a,b)$ then we have

$$\mathscr{I}(f) = \int_a^c f + \int_c^b f$$