### **ANALYSIS -I**

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- ▶ Definition 22.7: Let  $A \subseteq \mathbb{R}$ . Then a function  $f : A \to \mathbb{R}$  is said to be continuous if f is continuous at every  $c \in A$ .



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- ▶ Therefore f + g is continuous at c.
- It is easy to see that if f is continuous at c, af is continuous at c. Similarly bg is continuous at c. Combining with the previous result, af + bg is continuous at c.

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- ▶ Hence fg and  $\frac{f}{g}$  are continuous. This completes the proof.

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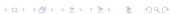
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Proof: This is clear from the previous theorem and the definition of continuous functions.



#### Restrictions of continuous functions

▶ Theorem 23.3: Let  $A \subseteq \mathbb{R}$  and let B be a subset of A and let  $c \in B$ . Suppose  $f : A \to \mathbb{R}$  is a function continuous at c. Then  $g : B \to \mathbb{R}$  defined by

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- Notation: The function g of this theorem is called the restriction of f to B and is denoted by  $f|_B$ .

# Continuity of polynomials

▶ Theorem 23.4: Let  $p : \mathbb{R} \to \mathbb{R}$  be a polynomial defined by  $p(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n, \ \forall x \in \mathbb{R},$ 

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- Proof: It is easy to see that the constant function

$$p_0(x)=a_0, x\in\mathbb{R}$$

and the identity function,

$$p_1(x) = x, x \in \mathbb{R}$$

are continuous. Now by (ii) of Theorem 23.2, and mathematical induction, the polynomials

$$p_k(x) = x^k, \ \forall x \in \mathbb{R}$$

 $k \in \mathbb{N}$ , are continuous. The proof is complete by a simple application of (i) of Theorem 23.2.



#### Rational functions

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- Such functions are known as rational functions.
- ▶ Example 23.6: The function  $g : \mathbb{R} \setminus \{0\} \to \mathbb{R}$  defined by  $g(x) = \frac{1}{x}, \ \forall x \in \mathbb{R} \setminus \{0\}$  is continuous.

▶ Theorem 23.7: Let A, B be subsets of  $\mathbb{R}$  and  $c \in A$ . Suppose f, g are real valued functions on A, B respectively and  $f(A) \subseteq B$ . Suppose f is continuous at c and g is continuous at f(c). Then f(c) is continuous at f(c).

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- ▶ In other words  $\{h(x_n)\}$  converges to h(c). This proves that h is continuous at c.

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- ► Exercise 23.8: Prove the previous theorem directly using the definition of continuity.

▶ Theorem 23.9: Let A, B be subsets of  $\mathbb{R}$ . Suppose f, g are continuous real valued functions on A, B respectively and  $f(A) \subseteq B$ . Then  $h = g \circ f$  is a continuous function.

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- **Example 23.10 (Dirichlet function): Define**  $d : \mathbb{R} \to \mathbb{R}$  by

$$d(x) = \begin{cases} 1 & \text{if } x \text{ is rational;} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

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- **Example 23.11**: Define  $g:[1,2] \to \mathbb{R}$  by

$$g(x) = \begin{cases} 0 & \text{if } x \text{ is irrational;} \\ \frac{1}{q} & \text{if } x = \frac{p}{q}, \ p, q \in \mathbb{N} \\ p, q & \text{relatively prime.} \end{cases}$$

Then g is continuous at irrational points in [1,2], but is discontinuous at rational points in [1,2].



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