2) Ballot Problem: In election between two candidates A & B, A gets a votes & B gets b votes.

Suppose $a < b$.

Suppose the votes are counted in a random order. What is the prob. that B leads throughout the counting?

Ω_{ab} = Possible arrangements of $\underbrace{AA\ldots A}_{a}\underbrace{BB\ldots B}_{b}$

$P(\omega) = \frac{1}{(a+b)!}$

F - Final vote is for A

F^c - . . . B.

$|F| = a\,(a+b-1)!$

$\downarrow$ a choices for last vote; $(a+b-1)!$ $\longrightarrow$ possible ordering of the remaining votes.

$$\mathbb{P}(F) = \frac{|F|}{|\Omega|} = \frac{a}{a+b}, \quad \mathbb{P}(F^c) = 1 - \frac{a}{a+b} = \frac{b}{a+b}$$

$$\underbrace{\mathbb{P}(\text{B leads always})}_{C} = \mathbb{P}(C|F)\mathbb{P}(F) + \mathbb{P}(C|F^c)\mathbb{P}(F^c)$$

$$= \frac{a}{a+b}\mathbb{P}(C|F) + \mathbb{P}(C|F^c)\frac{b}{a+b}.$$

<u>Ex</u>* $(F, \mathbb{P}_F(\cdot))$ is equivalent to $(\Omega_{a-1,b}, \mathbb{P})$

i.e., $\exists$ a bijn $f : F \longrightarrow \Omega_{a-1,b}$

& $\quad p_F(\omega) = p(f(\omega)) \quad \forall\, \omega \in F.$

$$f(C) = \{\text{B is leading always in } \Omega_{a-1,b}\}$$

Denote $p_{a,b} = \mathbb{P}(C)$ then $\mathbb{P}(f(C)) = p_{a-1,b}$.

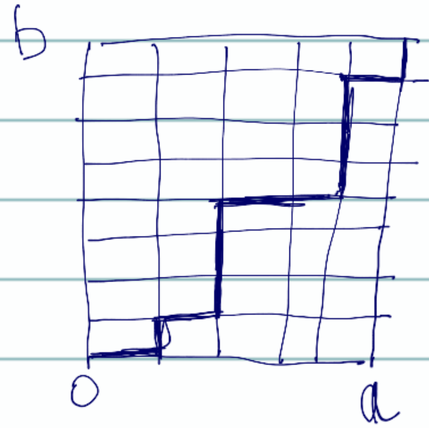
|||^{ly} $\mathbb{P}(C|F^c) = p_{a,b-1}$

$$p_{a,b} = \frac{a}{a+b} p_{a-1,b} + \frac{b}{a+b} p_{a,b-1}.$$

Solve the recursion.

ALTERNATE SOLUTION.

Counting of votes is a path that goes right or up in the rectangular grid from $(0,0)$ to (a,b).



i.e, $x_0 = (0,0)$ $x_{i+1} = \begin{cases} x_i + (1,0) & \text{if vote } i \text{ is } A \\ x_i + (0,1) & \text{is } B \end{cases}$

$\Omega = \{ (x_0, \ldots, x_{a+b}) : x_{i+1} = x_i + (1,0)$ or $x_i + (0,1)$
$x_0 = (0,0)$; $x_{a+b} = (a,b) \}$.

$$|\Omega| = \binom{a+b}{a} = \binom{a+b}{b}$$

In a+b jumps, choose a to the left

In a+b jumps, choose b to up.

$C = \{ B \text{ always leads} \}$
$= \{ x : a_i < b_i \ \forall i \quad x_i = (a_i, b_i) \}$

$D_1 = \{ B \text{ doesn't always lead } \& \ x_1 = (1,0) \}$

$D_2 = \{ B \qquad\qquad . \qquad\qquad \& \ x_1 = (0,1) \}.$

$D_1 = \underset{\substack{\| \\ \{B \text{ doesn't always lead}\}}}{E} \cap F = \{ x_1 = (1,0) \}$

Clearly $F \subseteq E$ & so $D_1 = F = \{ x : x_1 = (1,0) \}$

So D_1 is in bijection with set of R-U paths from
$(1,0)$ to (a,b)
which is in ... R-U paths from
$(0,0)$ to $(a-1, b)$.

Let $\underset{\substack{\| \\ (y_0, y_1, \dots, y_{a+b})}}{y}$ be a R-U path from $(0,0)$ to $(a-1,b)$.

Then $x = ((0,0), y_0 + (1,0), y_1 + (1,0), \dots, y_{a+b} + (1,0))$ is
a R-U path from $(0,0)$ to (a,b).

This gives the bijection (verify)

So

$$|D_1| = \binom{a+b-1}{a-1} = \binom{a+b-1}{b}.$$

What about $|D_2|$? We'll show $|D_2| = |D_1|$.

Then $|C| = |\Omega| - 2|D_1|$

$$= \binom{a+b}{b} - 2\binom{a+b-1}{b}$$

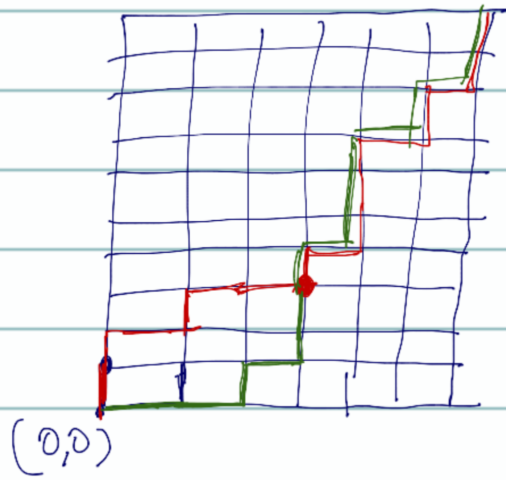
$$\mathbb{P}(C) = 1 - 2\frac{\binom{a+b-1}{b}}{\binom{a+b}{b}}$$

$$= 1 - 2\frac{(a+b-1)!\,a!}{(a+b)!\,(a-1)!} = 1 - \frac{2a}{a+b}$$

$$= \frac{b-a}{a+b}.$$

To show that $|D_1| = |D_2|$.

Let $x \in D_2$; since B doesn't always lead but $x_1 = (0,1)$ & so for some i, $x_{2i} = (i,i)$.
& $b_j > a_j \;\forall\, j < 2i$.



Define the path y in D_1 as follows.

$y_j = (b_j, a_j) \quad \forall\ 0 \le j \le 2i$

$y_j = x_j \quad \forall\ j > 2i.$

Clearly $y \in D_1$.

Ex* S.T. map $x \longmapsto y$ as above is a bijn. from D_2 to D_1. ▨

Proof of Ballot problem complete.