Probability I: Assignment 3

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Submit solutions to Q.2,Q.4, Q.6 and Q.8 by 10 PM on October 28th. Write down the probability space in all questions clearly before writing down the solutions.

1. Let $\Omega = [n] \cup \{0\}$ for $n \geq 1$ and let $m, r \leq n$. Define
$$p(k) = \frac{\binom{m}{r}\binom{n-m}{r-k}}{\binom{n}{r}}, k \leq r, k \leq m.$$
If $k > r$ or $k > m$, set $p(k) = 0$. Show that (Ω, p) is a probability space.

2. Suppose r labelled balls are dropped into n labelled cells uniformly at random. Define the probability space (i.e., sample space and pmf) corresponding to this experiment. Let $(r_1, \ldots, r_n)$ be a vector such that $r_i \geq 0 \, \forall i$. What is the probability that there are exactly r_i balls in cell i for $i = 1, \ldots, n$ (i.e., r_1 balls in the first cell, r_2 balls in the second cell and so on).

3. Consider the sample space $\Omega_{r,n}$ of n unlabelled balls in n labelled cells as considered in the class. Define
$$p((r_1, \ldots, r_n)) = \frac{r!}{r_1! \ldots r_n!} n^{-r}.$$
Show that $(\Omega_{r,n}, p)$ is a probability space.

4. Consider the above probability space $(\Omega_{r,n}, p)$. What is the probability that the first or second cell is empty ?

5. Consider Bose-Einstein and Fermi-Dirac probability distributions. In each of them compute the probability that the first or second cell is empty ?

6. Two gamblers bring their own deck of cards and shuffle their deck of cards i.e., each of them arrange their cards uniformly at random. What is the probability that the first card on their decks match with each other or the second card on their decks match with each other ?

7. Suppose there are r red marbles in a jar containing m marbles in total. A person selects n marbles from the jar uniformly at random with replacement. Compute the probability that the person has selected k red marbles in total for $k \in \{0, \ldots, r\}$.

8. In a party, n women throw their hat into a basket and select a hat uniformly at random from the basket. What is the probability that no woman selects her own hat ? [Hint : Let A_n be the number of ways in which n women do not select their own hat. Show that $A_n = (n-1)[A_{n-1} + A_{n-2}]$ with $A_1 = 0, A_2 = 1$.]

9. A group of individuals containing b boys and g girls are lined up in random order i.e., all $(b+g)!$ orderings are equally likely. Compute the probability that either all the girls lined up next to each other or all the boys are lined up next to each other.