## **ANALYSIS-I**

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- Thus the rearranged series may converge to a sum different from that of the given series.
- However, things are not that bad when we deal with absolutely convergent series.



- ▶ Theorem (Rearrangement theorem). If  $\sum_{n=1}^{\infty} a_n$  is absolutely convergent, then any rearrangement  $\sum_{n=1}^{\infty} b_n$  of  $\sum_{n=1}^{\infty} a_n$  converges to the same value.
  - Proof: Let  $\{s_n\}_{n\in\mathbb{N}}$  be the sequence of partial sums of  $\sum_{n=1}^{\infty} a_n$  and let  $\sum_{n=1}^{\infty} a_n = a$ .

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Let  $\epsilon > 0$  be arbitrary.

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Since  $\lim_{n \to \infty} s_n = extit{a}$ , there exists  $extit{K}_1 \in \mathbb{N}$  such that

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Since  $\epsilon>0$  is arbitrary, we conclude that  $\lim_{n\to\infty}t_n=a$ .



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- ➤ This theorem should convince us of the danger of manipulating an infinite series without any attention to rigorous analysis.
- ► To prove this theorem, we need the notions of positive and negative parts of a series.

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▶ We call the series  $\sum_{n=1}^{\infty} a_n^+$  as the series of positive terms of  $\sum_{n=1}^{\infty} a_n$ . Similarly, we call series  $\sum_{n=1}^{\infty} a_n^-$  as the series of negative terms of  $\sum_{n=1}^{\infty} a_n$ .

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- ▶ Note that all the terms of both these series are non-negative.
- ▶ For example, if  $a_n = \frac{(-1)^{n+1}}{n}$ , then

$$\sum_{n=1}^{\infty} a_n^+ = 1 + 0 + \frac{1}{3} + 0 + \frac{1}{5} + \cdots$$

and

$$\sum_{n=1}^{\infty} a_n^- = 0 + \frac{1}{2} + 0 + \frac{1}{4} + 0 + \cdots$$

Proof: Let  $\{s_n\}_{n\in\mathbb{N}}$ ,  $\{t_n\}_{n\in\mathbb{N}}$ ,  $\{u_n^+\}_{n\in\mathbb{N}}$  and  $\{u_n^-\}_{n\in\mathbb{N}}$  be the sequence of partial sums of  $\sum_{n=1}^{\infty}a_n$ ,  $\sum_{n=1}^{\infty}|a_n|$ ,  $\sum_{n=1}^{\infty}a_n^+$  and  $\sum_{n=1}^{\infty}a_n^-$ , respectively.

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Therefore we have

$$t_n = \sum_{k=1}^n |a_k| = u_n^+ + u_n^-$$
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Observe that both  $\{u_n^+\}_{n\in\mathbb{N}}$  and  $\{u_n^-\}_{n\in\mathbb{N}}$  are increasing.

Proposition. If  $\sum_{n=1}^{\infty} a_n$  is conditionally convergent, then  $\sum_{n=1}^{\infty} a_n^+$  and  $\sum_{n=1}^{\infty} a_n^-$  are both divergent.

Proof: Let  $\{s_n\}_{n\in\mathbb{N}}$ ,  $\{t_n\}_{n\in\mathbb{N}}$ ,  $\{u_n^+\}_{n\in\mathbb{N}}$  and  $\{u_n^-\}_{n\in\mathbb{N}}$  be the sequence of partial sums of  $\sum_{n=1}^{\infty}a_n$ ,  $\sum_{n=1}^{\infty}|a_n|$ ,  $\sum_{n=1}^{\infty}a_n^+$  and  $\sum_{n=1}^{\infty}a_n^-$ , respectively.

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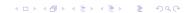
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By hypothesis  $\sum_{n=1}^{\infty} |a_n|$  is divergent, which implies that  $\lim_{n \to \infty} t_n = \infty$ .



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Then 
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, implying that 
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Therefore  $\lim_{n\to\infty}u_n^+=\infty$ . A similar argument shows that  $\lim_{n\to\infty}u_n^-=\infty$ .

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- ► Reference: Theorem 3.54 in [Walter Rudin, Principles of Mathematical Analysis, Third Edition, McGraw Hill Inc., 1976]

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Theorem 8.33 in [Tom M. Apostol, Mathematical Analysis, Addison-Wesley Publishing Company, Inc., 1974]

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$$a_1a_2\cdots a_N\prod_{n=N+1}^{\infty}a_n.$$

(iv)  $\prod_{n=1}^{\infty} a_n$  is called divergent if it does not converge as described in (ii) or (iii).

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- ▶ For this reason, the factors of a product are written as  $1 + a_n$  instead of just  $a_n$ . Thus, if  $\prod_{n=1}^{\infty} (1 + a_n)$  is convergent, then  $\lim_{n \to \infty} a_n = 0$ .

▶ Theorem. Let  $a_n > 0$  for all  $n \in \mathbb{N}$ . Then  $\prod_{n=1}^{\infty} (1 + a_n)$  is convergent if and only if  $\sum_{n=1}^{\infty} a_n$  is convergent.

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- ► Reference: pp. 206-209 of [Tom M. Apostol, Mathematical Analysis, Addison-Wesley Publishing Company, Inc., 1974]