Probability I: Assignment 6

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Submit solutions to Problems 2, 6, 7 and 9 on Moodle by 30th November, 10 PM. Write down the probability space in all questions clearly before writing down the solutions.

1. A string of n bits needs to be sent across a noisy communication channel. Any bit sent across the channel is corrupted with probability p independently. To minimize the error, the sender replicates each bit k times (say k is an odd number) and sends it across the channel. The receiver decodes the message by choosing the majority in each block of k bits. For any $1 \leq m \leq n$, compute the probability that there are exactly m wrongly decoded bits.

2. Suppose that $A_1, \ldots, A_n$ are independent events. Let $I_1, \ldots, I_k$ be a partition of $[n]$ i.e., $[n] = \sqcup_{j=1}^{k} I_j$. For each $1 \leq j \leq k$, let $B_j = \cup_{i \in I_j} A_i$. Show that $B_1, \ldots, B_k$ are independent events.

3. In the above problem, define $B_j = \cap_{i \in I_j} A_i$, $j = 1, \ldots, k$. Show again that $B_1, \ldots, B_k$ are independent events.

4. There is a parallel system with n components. Each component works independently with probability p. The system is said to work if at least one of the components work. Find the conditional probability that there are at least k components working given that the system is working.

5. Suppose that there are m different types of coupons. A coupon of type i ($i = 1, \ldots, m$) is chosen with probability p_i. You select n coupons independently. What is the probability that the nth coupon is new i.e., not selected before ? [1]

6. There is a class of n students. Each pair of students become friends with probability p and independently of other pairs of students. A group of k students are said to form a *a clique* if all of them are friends with each other. Compute the probabilities that (i) a group of k students form a clique and (ii) a group of k students form a clique and are not part of any larger clique.

7. Population genetics: Suppose there are two types of genes, say A, a. Genes appear in (unordered) pairs in each individual i.e., AA, Aa, aa. This

[1] **Extra:** Can you find n to make this probability very small.

is called as *genotype* of an individual. [2]. In the first generation, the proportion of the three gene types are $u, 2v, w$ respectively where $u, v, w \geq 0$ and $u + 2v + w = 1$. A child is born to two randomly chosen individuals from the first generation. The child choses one gene at random from each individual. Let $u_1, 2v_1, w_1$ be the probabilities that the child has genotype AA, Aa, aa respectively. Compute $u_1, 2v_1$ and w_1. [3]

8. In the Urn model with $c = -1, d = 0$ compute the probabilities of selection of red and black balls i.e., compute $\mathbb{P}(w)$ for $w \in \{0, 1\}^n$ where $n = r + b$ and $0, 1$ denoting red and black balls respectively.

9. In Polya's Urn model compute $\mathbb{P}(B_m \cap B_n)$ for any $m < n$.

[2] For ex., A could cause blue-eyes and a could cause black-eyes. Then AA is white, Aa is brown and aa is black.

[3] **Extra:** $u_1, 2v_1, w_1$ are interpreted as proportion of gene types in the second generation. Now compute the proportion of gene types in the third generation i.e., $u_2, 2v_2, w_2$.