Probability I: Assignment 2

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Submit solutions to Q.3,Q.5, Q.8 and Q.10 by 10 PM on October 18th. Write down the probability space in all questions clearly before writing down the solutions.

1. Old Indian dice have only four faces labelled 1,2,4,6. Two old Indian dice are thrown and the numbers are noted. Write down the probability space assuming equally likely outcomes. For $k \geq 1$, let A_k denote the event that the sum of dice is k. Calculate $P(A_k)$ for all $k \geq 1$.

2. Suppose that each of n women at a party throws her hat into the center of the room. Then each woman selects an hat from the center of the room uniformly at random. Describe the sample space for the possible selections of hats. There are k women who live in the same street. What is the probability that each of these k women select a hat of another woman in the same street.

3. Consider a deck of 50 cards. Each card has one of 5 colors (black, blue, green, red, and yellow), and is printed with a number (1,2,3,4,5,6,7,8,9, or 10) so that each of the 50 color/number combinations is represented exactly once. A hand is produced by dealing out five different cards uniformly at random from the deck. The order in which the cards were dealt does not matter.

   (a) What is the probability of being dealt a hand consisting of identical colours ?

   (b) What is the probability of being dealt a hand that contains exactly three cards with one number, and two cards with a different number (i.e., For ex., a sequence like $2, 3, 2, 3, 2$ is ok but not $2, 3, 2, 4, 2$ is not)?

4. If r_1 indistinguishable red balls and r_2 indistinguishable blue balls are placed into n urns, find the number of distinguishable arrangements. Assuming that all distinguishable arrangements are equally likely, find the probabilty that the first urn contains balls of both colours.

5. Suppose that 8 rooks (castles) are placed uniformly at random on a chessboard. Compute the probability that no row or column contains more than one rook (i.e., no rooks can capture any other).

6. Suppose a box contains n labelled balls and the balls are of various (say m) colours. Suppose that r balls are *sampled from the box without replacement* (i.e., the balls are chosen one by one from the box in order and the order is noted). Let $k_1, \ldots, k_m \in [n]$. What is the probability of observing k_1 balls of colour 1, k_2 balls of colour 2 and so on upto k_m balls of colour m ?

7. Suppose that $(\Omega_i, p_i), i = 1, \ldots, k$ be finite probability spaces. Show that the cartesian product $\Omega = \prod_{i=1}^{k} \Omega_i := \Omega_1 \times \ldots \times \Omega_k$ with $p((w_1, \ldots, w_k)) = \prod_{i=1}^{k} p_i(w_i)$ for $(w_1, \ldots, w_k) \in \Omega$ is a finite probability space as well. Further, show that $\mathbb{P}(A_1 \times \ldots \times A_k) = \prod_{i=1}^{k} \mathbb{P}_i(A_i)$ for all $A_i \subset \Omega_i$, $i = 1, \ldots, k$.

8. Suppose that $(\Omega_1 \times \Omega_2, \mathbb{P})$ is a finite prob. space. Show that (Ω_1, p_1) is a probability space as well where p_1 is defined as

$$p_1(w_1) := \mathbb{P}(\{w_1\} \times \Omega_2), w_1 \in \Omega_1.$$

   Similarly, we can define a finite probability space (Ω_2, p_2). Is it necessary that $p((w_1, w_2)) = p_1(w_1)p_2(w_2)$ for all $(w_1, w_2) \in \Omega$? Either prove the claim or give a counter-example.

9. A probabilist needs to select a seven letter word (i.e., a string of seven English alphabets and the word need not have a meaning) for a problem in his question paper. Suppose that he choses the seven letter word uniformly at random.

   (a) What is the probability that the randomly chosen word has three vowels and four consonants [1] ?

   (b) What is the probability that there are only five distinct letters ?

   Please write down the probability space before computing the probabilities.

10. There are 50 students in a mathematics program consisting of 3 courses - analysis, number theory and probability. 14 students like Analysis, 13 like Number theory and 8 like Probability. 6 Students like both analysis and number theory, 2 who like both analysis and probability and 3 who like both number theory and probability. 1 student likes all the three courses.

    (a) Suppose a student is chosen randomly, what is the probability that the student does not like any of the three courses ?

    (b) If two students are chosen uniformly at random, what is the probability that both of them do not like any of the three courses ?

[1] The alphabets a,e,i,o,u are the vowels and the remaining 21 alphabets are all consonants.