Probability I: Assignment 8

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Submit solutions to Q.1, Q.4 and Q.7 on Moodle by 20th December, 10 PM. Write down the probability space in all questions clearly before writing down the solutions.

1. Suppose that X has a Poisson distribution with parameter λ. Show that $\mathbb{P}(X = k) = \frac{\lambda}{k}\mathbb{P}(X = k-1)$ for $k = 1, 2, \ldots$ Use this to find k so that $\mathbb{P}(X = k)$ is maximal and also show that $\mathbb{E}[Xf(X)] = \lambda\mathbb{E}[f(X+1)]$ for a bounded function $f : \mathbb{N}^* \to \mathbb{R}$.

2. Compute the variance of geometric, negative Binomial and Poisson random variables.

3. Let X be a random variable for which $\mathbb{P}(X = n(n^2+1)) = 2^{-n}$ for all integers $n \geq 1$. Compute $E[X]$ and $E[X^2]$.

4. Let $f_a(n) = cn^{-a}, n \geq 1$ with $a \in (2, \infty)$. Show that for some value of c, f_a is pmf of a random variable X and furthermore show that $\mathbb{E}[X]$ is well-defined.

5. Two players roll a standard dice successively. The first player to get a 6 wins the game. Compute the probability of each player winning the game.

6. Let X be a discrete random variable taking values in $\mathbb{N}^*$. Show that if $x_n \downarrow x \in \mathbb{R}$ as $n \to \infty$ (i.e., x_n is a decreasing sequence converging to x), then $\mathbb{P}(X \leq x_n) \to \mathbb{P}(X \leq x)$ as $n \to \infty$. Is it true that the above claim holds if we assume that $x_n \uparrow x$?

7. Let X be a discrete random variable taking values in $\mathbb{N}^*$. Let $a_j \geq 0, j \geq 1$ be a given sequence. Show the following

$$\sum_{j=1}^{\infty}(a_1 + \ldots + a_j)\mathbb{P}(X = j) = \sum_{i=1}^{\infty} a_i \mathbb{P}(X \geq i),$$

$$\mathbb{E}[X] = \sum_{i=1}^{\infty} \mathbb{P}(X \geq i),$$

$$\mathbb{E}[X(X+1)] = 2\sum_{i=1}^{\infty} i\mathbb{P}(X \geq i).$$