Probability I: Assignment 9

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Submit solutions to Problems 1, 3 and 7 by January 8th, 11 PM on Moodle.

1. Let X be a continuous random variable with probability density function f and distribution function F. Suppose f is a symmetric function, i.e. $f(x) = f(-x)$ for all $x \in \mathbb{R}$. Then show that
   (a) $\mathbb{P}(X \leq 0) = \mathbb{P}(X \geq 0) = \frac{1}{2}$.
   (b) for $x \geq 0, F(x) = \frac{1}{2} + \int_0^x f(s)\, ds$.
   (c) for $x \leq 0, F(x) = \int_{-x}^{\infty} f(s)\, ds = \frac{1}{2} - \int_x^0 f(s)\, ds$.

2. (*Weibull distribution*) Let $k, \lambda > 0$. Define a function $f : \mathbb{R} \to [0, \infty)$ as
$$f(x) = cx^{k-1}e^{-(\frac{x}{\lambda})^{k-1}}\mathbf{1}[x > 0],$$
for some constant $c > 0$. Find c such that f is a pdf and compute the corresponding CDF.

3. (*Wigner's semicircle distribution*) Let $R > 0$. Define a function $f : \mathbb{R} \to [0, \infty)$ as
$$f(x) = c\sqrt{R^2 - x^2} \times \mathbf{1}[-R < x < R].$$
for some constant $c > 0$. Find c such that f is a pdf and compute the corresponding CDF.

4. (*Beta Distribution.*) Let $\alpha, \beta > 0$. Define a function $f : \mathbb{R} \to [0, \infty)$ as
$$f(x) = cx^{\alpha-1}(1-x)^{\beta-1}\mathbf{1}[x \in (0, 1)],$$
for some constant $c > 0$. Show that there is a c such that f is a pdf.

5. Let X be the $\Gamma(r, \lambda)$ random variable and N be the Poisson (λt) random variable. Show that $\mathbb{P}(X \leq t) = \mathbb{P}(N \geq r)$ for $r \in \mathbb{N}$.

6. Let U be Unif$((0, 1)$ $)$ random variable and $X = U^3$. Find the pdf and CDF of X.

7. Let $a < b \in \mathbb{R}$ and X be a continuous random variable with pdf f such that $f \equiv 0$ outside (a, b). Let g be a differentiable strictly monotonic (either increasing or decreasing) function. Set $Y = g(X)$. Show that Y has a pdf f_Y given by

$$f_Y(y) = f_X(g^{-1}(y)) \times \left|\frac{\mathrm{d}}{\mathrm{d}y}(-g^{-1}(y))\right|, \quad y \in g((a, b)) \ ; \quad f_Y(y) = 0 \text{ else.}$$

8. Let $\alpha > 0$ and X be a random variable with the pdf given by

$$f(x) = \frac{\alpha}{x^{\alpha+1}}\ , 1 \leq x < \infty \ ; \ f(x) = 0, \, x < 1.$$

Show that the above is a pdf and also compute the CDF. Further, find the distribution of the following random variables $X_1 = X^2, X_2 = \log(X)$.