Probability I: Assignment 4

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Submit solutions to Q.1,Q.6, Q.7 and Q.9(b),(e) by 10.00 PM on November 9th. Solutions are to be submitted on Moodle.

Write down the probability space in all questions clearly before writing down the solutions.

1. Complete the proof of Bonferroni inequalities for case $m \geq 3$.

2. Let $A_1, \ldots, A_n$ be events in a probability space $(\Omega, \mathbb{P})$. Define the event B_m that "at least m events of $A_1, \ldots, A_n$ occur" as
$$B_m := \cup_{1 \leq i_1 < i_2 \ldots i_m \leq n} (A_{i_1} \cap \ldots A_{i_m}).$$
Show that
$$\mathbb{P}(B_m) = \sum_{k=m}^{n} (-1)^{k-m} \binom{k-1}{k-m} S_k,$$
where $S_k = \sum_{1 \leq i_1 < \ldots i_k \leq n} \mathbb{P}(A_{i_1} \cap \ldots \cap A_{i_k})$.

3. In the above question, show that
$$\mathbb{P}(B_2) = S_2 - 2S_3 + 3S_4 + \ldots + (-1)^n (n-1) S_n.$$

4. As in the previous questions, define the event C_m that "exactly m events of $A_1, \ldots, A_n$ occur" as $C_m = B_m \setminus B_{m+1}$. Show that
$$\mathbb{P}(C_m) = \sum_{k=m}^{n} (-1)^{k-m} \binom{k}{m} S_k.$$

5. Compute the above probabilities $(\mathbb{P}(B_m), \mathbb{P}(C_m))$ for Maxwell-Boltzmann distribution where A_i is the event that the ith urn is empty.

6. Compute the above probabilities $(\mathbb{P}(B_m), \mathbb{P}(C_m))$ for Fermi-Dirac distribution where A_i is the event that the ith urn is empty.

7. Two permutations are chosen randomly (i.e., an element is chosen uniformly at random from $S_n \times S_n$ where S_n is the set of permutations on $[n]$). Let A_m be the event that there are exactly m matches between

the permutations. In other words, if π, σ are the two permutations, we say that they have exactly m matches if $\pi(i) = \sigma(i)$ for exactly m many indices i in $1, \ldots, n$. Compute the probability of A_m. [1]

8. Ten pair of shoes are in a closet. Four shoes are selected at random. Find the probability that there will be at least one pair among the four shoes selected.

9. In each of the following cases, try to guess whether A and B are independent and if not, whether $\mathbb{P}(B|A)$ is smaller or larger than $P(B)$. Then calculate the probabilities and verify the validity of your guesses.

   (a) A box contains n coupons labelled $1, 2, \ldots, n$. Coupons are drawn one after another randomly with replacement (a coupon is drawn, the number noted, the coupon is returned to the box, the next coupon is drawn, etc). Let A be the event that the first coupons drawn is an even number. Let B be the event that the second coupon drawn is an even number.

   (b) Same as first situation (draw with replacement). Let A be the event that the first two coupons drawn are different. Let B be the event that the second and third coupons drawn are different.

   (c) 10 balls are labelled $1, 2, \ldots, 10$ are thrown at random into 4 labelled bins. Let A be the event that the first bin contains a ball with label 4 or less. Let B be the event that the second bin contains a ball labelled 7 or more.

   (d) Same situation as the previous one. Let A be the event that the first bin contains a ball with label 4 or less. Let B be the event that the first bin contains a ball labelled 7 or more.

   (e) Same situation as the previous one. Let A be the event that the first bin contains a ball with label 4 or less. Let B be the event that the second bin contains a ball labelled 4 or less.

[1] EXTRA QUESTION : For every fixed m, can you find out the value of $\mathbb{P}(A_m)$ as $n \to \infty$?