Discrete Dirichlet Problem

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Discrete Laplace Equation:

$$4f(u,v) - f(u+1,v) - f(u-1,v) - f(u,v+1) - f(u,v-1) = 0$$

Consider 'n' points in the interior (set of all interior points). Number them arbitrarily and name the functional values as $x_1, \dots, x_n$. Write the Discrete Dirichlet equation by ~~\$~~ transposing the values at boundary points to RHS. Clearly, the RHS is constant for a given set-up. So, we have 'n' distinct equations in 'n' variables. Construct an $n \times n$ matrix A as follows:

$A = (a_{ij})$

$a_{ii} = 4$,

$a_{ij} = -1 \iff x_i$ & x_j are values at "Neighbouring points"

(Neighbour means the points differ in exactly one coordinate by 1)

if i & j ~~are ne~~ represent neighbours, so do j & i

By construction, A is an $n \times n$ symmetric matrix. The 'n' equations can be represented as:

$AX = B$, where B is the column vector containing the constant terms of the n equations and $X = (x_1 \; x_2 \; \cdots \; x_n)^T$.

(a) Consider the case of $n = 5$.

$$4\beta_{00} - \beta_{10} - \beta_{01} - \beta_{\bar{1}0} - \beta_{0\bar{1}} = 0$$

~~$4\beta_{10} - \beta_{00}$~~

$$-\beta_{00} + 4\beta_{10} + 0 + 0 + 0 = \beta_{11} + \beta_{1\bar{1}} + \beta_{20} = 1$$

$$-\beta_{00} + 0 + 4\beta_{01} + 0 + 0 = \beta_{11} + \beta_{\bar{1}1} + \beta_{02} = 3$$

$$-\beta_{00} + 0 + 0 + 4\beta_{\bar{1}0} + 0 = \beta_{\bar{1}1} + \beta_{\bar{1}\bar{1}} + \beta_{\bar{2}0} = 1$$

$$-\beta_{00} + 0 + 0 + 0 + 4\beta_{0\bar{1}} = \beta_{0\bar{2}} + \beta_{1\bar{1}} + \beta_{\bar{1}\bar{1}} = 0$$

$\therefore$ The augmented matrix is:

$$\left[\begin{array}{ccccc|c} 4 & -1 & -1 & -1 & -1 & 0 \\ -1 & 4 & 0 & 0 & 0 & 1 \\ -1 & 0 & 4 & 0 & 0 & 3 \\ -1 & 0 & 0 & 4 & 0 & 1 \\ -1 & 0 & 0 & 0 & 4 & 0 \end{array}\right]$$

$R_2' \to R_2 + R_1/4$
$R_3' \to R_3 + R_1/4$
$R_4' \to R_4 + R_1/4$
$R_5' \to R_5 + R_1/4$

$$\left[\begin{array}{ccccc|c} 4 & -1 & -1 & -1 & -1 & 0 \\ 0 & \frac{15}{4} & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & 1 \\ 0 & -\frac{1}{4} & \frac{15}{4} & -\frac{1}{4} & -\frac{1}{4} & 3 \\ 0 & -\frac{1}{4} & -\frac{1}{4} & \frac{15}{4} & -\frac{1}{4} & 1 \\ 0 & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & \frac{15}{4} & 0 \end{array}\right]$$

$R_3' \rightarrow R_3 + \frac{1}{15} R_2$

$R_4' \rightarrow R_4 + \frac{1}{15} R_2$

$R_5' \rightarrow R_5 + \frac{1}{15} R_2$

$$\left(\begin{array}{ccccc|c} 4 & -1 & -1 & -1 & -1 & 0 \\ 0 & \frac{15}{4} & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & 1 \\ 0 & 0 & \frac{56}{15} & -\frac{4}{15} & -\frac{4}{15} & \frac{46}{15} \\ 0 & 0 & -\frac{4}{15} & \frac{56}{15} & -\frac{4}{15} & \frac{16}{15} \\ 0 & 0 & -\frac{4}{15} & -\frac{4}{15} & \frac{56}{15} & \frac{1}{15} \end{array}\right)$$

$\cancel{R_2' \rightarrow 4R_2}$

$\cancel{R_3' \rightarrow 15R_3}$

$R_4' \rightarrow R_4 + \frac{R_3}{14}$

$R_5' \rightarrow R_4 + \frac{R_3}{14}$

$$\left(\begin{array}{ccccc|c} 4 & -1 & -1 & -1 & -1 & 0 \\ 0 & \frac{15}{4} & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & 1 \\ 0 & 0 & \frac{56}{15} & -\frac{4}{15} & -\frac{4}{15} & \frac{46}{15} \\ 0 & 0 & 0 & \frac{26}{7} & -\frac{2}{7} & \frac{9}{7} \\ 0 & 0 & 0 & -\frac{2}{7} & \frac{26}{7} & \frac{2}{7} \end{array}\right)$$

$R_5' \rightarrow R_5 + \frac{R_4}{13}$

$$\begin{pmatrix} 4 & -1 & -1 & -1 & -1 & 0 \\ 0 & \frac{15}{4} & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & 1 \\ 0 & 0 & \frac{56}{15} & -\frac{4}{15} & -\frac{4}{15} & \frac{46}{15} \\ 0 & 0 & 0 & \frac{26}{7} & -\frac{2}{7} & \frac{9}{7} \\ 0 & 0 & 0 & 0 & \frac{48}{13} & \frac{5}{13} \end{pmatrix}$$

$$B_{0T} = \frac{5}{48}$$

$$B_{T0} = \frac{7}{26}\left(\frac{2}{7} + \frac{5}{48} + \frac{9}{7}\right) = \cancel{\frac{127}{338}} = \frac{17}{48}$$

$$\boxed{\begin{aligned} B_{01} &= \frac{15}{56}\left(\frac{46}{15} + \cancel{\frac{15}{4}}\,\frac{4}{15}\left(\frac{5}{13} + \frac{127}{338}\right)\right) \\ &= \frac{148}{169} \\ B_{10} &= \frac{4}{15}\left(1 + \frac{1}{4}\left(\frac{148}{169} + \frac{7}{26} + \frac{5}{48}\right)\right)\end{aligned}}$$

$$B_{01} = \frac{15}{56}\left(\frac{46}{15} + \frac{4}{15}\left(\frac{17}{48} + \frac{5}{48}\right)\right) = \frac{41}{48}$$

$$B_{10} = \frac{4}{15}\left(1 + \frac{1}{4} + \frac{41+17+5}{48}\right) = \frac{17}{48}$$

$$B_{00} = \frac{1}{4}\,\frac{5+17+41+17}{48} = \frac{20}{48}$$

(c) The discrete Dirichlet problem has a unique solution iff 'A' has an inverse.

- A is symetric
- All diagonal entries are +4
- # of (-1)'s per row or per column is 4
- Rest elements are 0.

B is the coefficient matrix, which is a constant.

For creating $x_1, x_2, \ldots x_n$, Sort the points $P_1, \ldots P_n$ according to the x-coordinate. Whenever a match is found, sort according to the y-coordinate.

Now, in A, a '4' entry can have atmost two '-1's above it.

Now apply the following algorithm :

Begin at $i=1$.

At a point let $A^* = (a_{ij})$
Consider $a_{ii} > 0$ (we assume inductively that an element is (+)ve iff it is a diagonal entry)

Then

Then if $a_{ji} < 0$

apply $R_j \to R_j - \frac{R_i}{a_{ii}} \cdot a_{ji}$

The least value of a diagonal entry a_{kk} becomes:-

$$4 - \left| \frac{a_{kk'} \, a_{k'k}}{a_{k'k'}} \right| - \left| \frac{a_{kk''} \, a_{k''k}}{a_{k''k''}} \right|$$

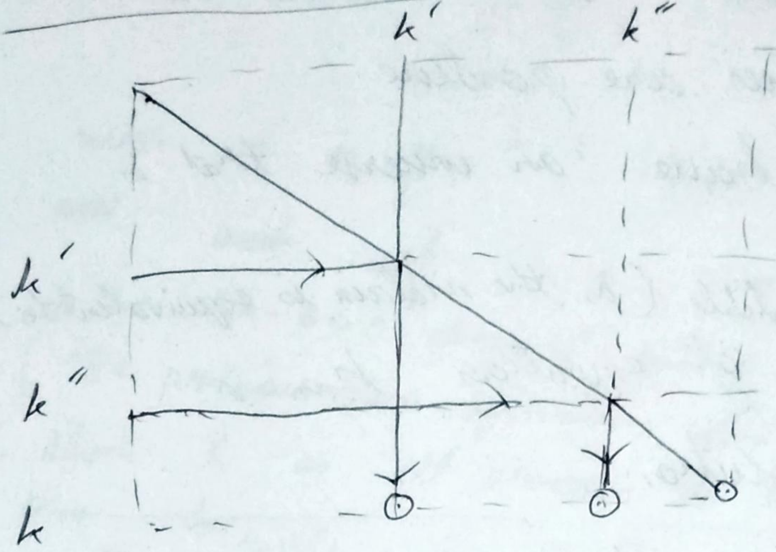
The off-diagonal nonzero entries below diagonal do not affect the diagonal entries.
For the rest, the max. value of absolute value of non-diagonal entry

$$= 1 + (1) \cdot \left| \frac{a_{ji}}{a_{ii}} \right|$$

Let, $k' < k'' < k$

then, $|a_{k'k}| = 1$

(as all entries above it are 0) ~~in the~~ in every step.



By symmetry of matrix, there can be atmost two off-diagonal nonzero elements to the left of a_{kk}.

Thus, there was NO row operation on k^{th} row other than those indicated by k' and k''. (as the rest of the row is zero).

Inductively, assume, $|a_{ii}| > 2$

$\therefore$ the max. value of the 2nd non-diagonal entry in the row 'k' & column 'k' to left of $a_{k,k}$ has absolute value $< 1 + \frac{1}{2} = \frac{3}{2}$.

$\therefore$ after operation,

$$a_{kk} \geq 4 - \frac{1}{2} \cdot 1 \cdot 1 - \frac{1}{2} \cdot \frac{3}{2} \cdot \frac{3}{2} > 2$$

$\therefore$ ~~At~~ At the end, we get an upper triangular matrix where ~~non~~ diagonal entries are positive

$\Rightarrow$ It must have an inverse that is unique.

$\Rightarrow$ A is invertible (as the matrix is equivalent to A)

$\Rightarrow$ The system of equations has a unique solution.

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* Proof of the fact that upper Δr matrix with nonzero ~~el~~ diagonal entries is invertible:

Consider the statement for an $n \times n$ matrix.

This is true for $n = 1$.

Let it be true for $n = k$.

Let $B_{(k+1) \times (k+1)}$ be such a matrix.

$b_{(k+1),(k+1)} \neq 0$.

$\therefore$ By Type 2 elem. operation, we can reduce it to B' where

$b'_{i,(k+1)} = 0 \quad \forall \; i \neq (k+1)$

$B' \Longrightarrow$ Set $b'_{(k+1),(k+1)}$ to 1 by type 3 operation.

Let $B'' = \left(\begin{array}{c|c} C & 0 \\ \hline 0 & 1 \end{array} \right)$

where C is of $k \times k$ size.

~~and~~ and C is upper triangular with non-zero diagonal entries.

(The previous row operations did not affect the block C as all elements of last row other than $b_{(k+1),(k+1)}$ are 0.)

Now, row reduce C and apply the very same operation on B''.

By induction, C can be reduced to I_k

$\therefore$, We can reduce B to I_{k+1} by a series of operation.

QED.

(b) Assume, for contradiction, maximum occurs at an interior point. Let max. value be M, and let it occur at i^{th} position.

$\Rightarrow$ The four neighbours of i must have same value. Let ~~to~~ the neighbours be

j_1, j_2, j_3, j_4.

$M = f(i) = f(j_1)$, where j_1 lies to the right of i. If j_1 is interior point Apply the same argument on j_1 ~~to~~ and so on, to get a series of points

$j_1, j_2, \ldots$ each lying to the right of the other. Since number of interior points 'n' is finite, the sequence must end somewhere, which must be a boundary point.

[N.B. $a, b, c, d \leq M$

& $\frac{a+b+c+d}{4} = M$

$\Rightarrow a = b = c = d = M$]