Probability II

Recall: In Probability I, you have seen many random variables, which are real-valued functions defined on ~~some~~ sample spaces.

Even though these are functions, we rarely think of them as functions. Rather, we think of them as variables whose values depend on chance. Keeping this point of view in mind, we focus on computations of probabilities of a random variable taking various values.

Defn: For any r.v. X, the cumulative distribution function (cdf) is defined as

$$F_X(x) = P(X \leq x), \qquad x \in \mathbb{R}.$$

The random variables considered in Probability I and II will be either discrete or continuous even though ~~they~~ these two classes do not exhaust all random variables.

Defn: A r.v. X is called discrete if it takes countably many values, i.e., $\exists$ a ctble set $\{x_i : i \in I\}$ of real numbers such that $\sum_{i \in I} P(X = x_i) = 1$. In this situation, $\text{Range}(X) = \{x_i : i \in I\}$ as long as $P(X = x_i) > 0 \ \forall \ i \in I$.

Defn: For a discrete r.v. X, the probability mass function (pmf) is defined as

$$p_X(x) = P(X = x), \quad x \in \mathbb{R}.$$

Clearly, ① $p_X(x) = 0$ whenever $x \notin \text{Range}(X)$.

② $\forall \ u \in \mathbb{R}$, $F_X(u) = P(X \leq u) = \sum_{\substack{x \leq u \\ x \in \text{Range}(X)}} p_X(x)$.

$$= \sum_{x \leq u} p_X(x) \quad \text{(shall write)}$$

③ $\forall \ A \subseteq \mathbb{R}$,

$$P(X \in A) = \sum_{x \in A} p_X(x).$$

Defn: A r.v. X is called (absolutely) continuous if $\exists$ a function $f_X : \mathbb{R} \to [0, \infty)$ such that $\forall\, u \in \mathbb{R}$,

$$F_X(u) = P(X \leq u) = \int_{-\infty}^{u} f_X(x)\, dx.$$

In this case, f_X is called a probability density function (pdf) of X.

In most examples, f_X will have at most finitely many discontinuities even though this is not at all necessary.

Note: (1) The integral in the above def^n is an improper integral. For example,

$$\int_{-\infty}^{7} f_X(x)\, dx = \lim_{A \to -\infty} \int_{A}^{7} f_X(x)\, dx \quad (\text{the limit exists}).$$

(2) If u is a continuity point of f_X, then by Fundamental Theorem of Calculus, F_X is diff.ble at u and $F_X'(u) = f_X(u)$. This means that

$$\lim_{\Delta u \to 0+} \frac{F_X(u+\Delta u) - F_X(u)}{\Delta u} = f_X(u)$$

$$\Rightarrow \lim_{\Delta u \to 0+} \frac{P(u < X \leq u+\Delta u)}{\Delta u} = f_X(u)$$

(this explains why f_X is called a probability "density" function)

$$\Rightarrow \text{“} \; P(u < X \leq u+du) = f_X(u)\,du \; \text{”}$$

3. f_X satisfies $f_X(x) \geq 0 \;\; \forall \; x \in \mathbb{R}$, and

$$\int_{-\infty}^{\infty} f_X(x)\,dx = 1.$$

Exc: Suppose $Y \sim \text{Unif}(-1,1)$, i.e., Y is a cont r.v. with pdf $f_Y(y) = \begin{cases} \frac{1}{2} & \text{if } y \in (-1,1), \\ 0 & \text{o.w.} \end{cases}$

Define $X = \max\{Y, 0\}$.

Find the cdf of X and show that X is neither discrete nor continuous.

Random Vectors or Jointly Distributed Random Variables

We shall start with bivariate random vectors or jointly distributed two random variables.

Suppose X, Y are two r.v.s defined on the same sample space Ω. This means that $X : \Omega \to \mathbb{R}$ and $Y : \Omega \to \mathbb{R}$ are functions.

Therefore combining these two functions, we get a function $\Omega \longrightarrow \mathbb{R}^2$ defined by

$$\omega \longmapsto (X(\omega), Y(\omega)).$$

(X, Y) is called a bivariate random vector or jointly distributed two r.v.s.

Remarks: ① Since X, Y are defined on the same sample space, we can talk about their joint distribution function, we can add them, ~~multip~~ multiply them, etc.

② As in the case of r.v.s, we shall forget that (X,Y) defines a map from Ω to $\mathbb{R}^2$ and think of it as a vector whose value depends on chance. We shall focus on computing ~~various~~ probabilities of X and Y jointly taking various values.

Keeping the above remark in mind, we define the following notion.

Defn: For a bivariate random vector (X,Y), the joint cdf or joint distribution function is defined as

$$F_{X,Y}(x,y) = P(X \leq x, Y \leq y), \quad (x,y) \in \mathbb{R}^2.$$

Note that the cdf of X can be obtained from the joint cdf of X and Y as follows. For any $x \in \mathbb{R}$,

$$F_X(x) = P(X \leq x)$$

$$= P(X \leq x, Y < \infty)$$

$$= P\left(\lim_{y \to \infty} \{X \leq x, Y \leq y\}\right)$$

$$= P\left(\lim_{n\to\infty} \{X \leq x, Y \leq y_n\}\right)$$ For any seq $y_n \uparrow \infty$

$$= \lim_{n\to\infty} P(X \leq x, Y \leq y_n)$$

$$= \lim_{y\to\infty} P(X \leq x, Y \leq y)$$

$$= \lim_{y\to\infty} F_{X,Y}(x, y).$$

We ~~sh~~ have therefore shown:

Fact: 1. $F_X(x) = \lim_{y\to\infty} F_{X,Y}(x, y)$

2. $F_Y(y) = \lim_{x\to\infty} F_{X,Y}(x, y)$

Exc: Show the following:

1. $\forall\ a, b \in \mathbb{R}$,
$$P(X > a, Y > b) = 1 - F_X(a) - F_Y(b) + F_{X,Y}(a, b).$$

2. $\forall\ a_1, b_1, a_2, b_2 \in \mathbb{R}$ with $a_1 < a_2$, $b_1 < b_2$,
$$P(a_1 < X \leq a_2,\ b_1 < Y \leq b_2)$$
$$= F_{X,Y}(a_2, b_2) - F_{X,Y}(a_1, b_2) - F_{X,Y}(a_2, b_1) + F_{X,Y}(a_1, b_1).$$