11/03/2022.

(i) : $T_{i,0} = f_i,\ i = 0(1)n$

(ii) : $\dfrac{\left(T_{i,k-1}(x - x_{i-k}) - (x - x_i)\, T_{i-1,k-1}\right)}{(x_i - x_{i-k})} = T_{i,k}$

$i = 0(1)3$

$T_{0,0} = f_0$ $\quad T_{1,0} = f_1$ $\quad T_{2,0} = f_2$ $\quad T_{3,0} = f_3$

$$\begin{tikzcd}[row sep=small, column sep=small]
 & k=0 & k=1 & k=2 & k=3 \\
i=0 & f_0 \arrow[dr] & & & \\
1 & f_1 \arrow[r] \arrow[dr] & T_{1,1}(P_{01}) \arrow[r] & T_{2,2} \arrow[dr] & \\
2 & f_2 \arrow[r] \arrow[dr] & T_{2,1}(P_{12}) \arrow[ur] \arrow[r] & T_{3,2} \arrow[r] & T_{3,3} \\
3 & f_3 \arrow[r] & T_{3,1}(P_{23}) \arrow[ur] & & P_{0123}(x) \arrow[u, phantom, "\shortparallel"]
\end{tikzcd}$$

$$T_{1,1} = \frac{T_{1,0}(x - x_0) - T_{0,0}(x - x_1)}{x_1 - x_0} = \frac{f_1(x - x_0) - f_0(x - x_1)}{x_1 - x_0} = P_{01}(x)$$

$$T_{2,2}(x) = \frac{(x - x_0)\, T_{2,1}(x) - (x - x_2)\, T_{1,1}(x)}{x_2 - x_0} = \frac{(x - x_0) P_{12}(x) - (x - x_2) P_{01}(x)}{x_2 - x_0} = P_{012}(x)$$

. Evaluate $T_{i,k-1}$ @ $(k-1)^{th}$ level $\forall i$

. Build $T_{i,k}$ using $T_{i,k-1}$ & $T_{i-1,k-1}$

$$P(x) = \sum_{j=0}^{n} a_j x^j \quad \text{s.t. } P(x_k) = f_k \ \forall k = 0(1)n$$

—x—

Newton's 'Divided differences'

$b_i = (x_i, f_i)$ $\quad P \in \Pi_n ; P(x_i) = f_i \quad \forall\, 0(1)n = i$

$$P(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + a_n(x - x_0)(x - x_1)\cdots(x - x_{n-1})$$

Horner

$$P(x = \xi) = a_0 + (\xi - x_0)\left[a_1 + (\xi - x_1)\left[a_2 + \cdots\right]\cdots\right]$$

$P(x = x_0) = a_0 = f_0^k$

$P(x = x_1) = f_0^k + (x_1 - x_0)\, a_1 = f_1^k$

$$\Rightarrow a_1 = \frac{f_1^k - f_0^k}{x_1^k - x_0^k}$$

$\underbrace{P_{i_0 i_1 \ldots i_{k-1}}}_{k\text{-indices}}(x) \in \Pi_{k-1}$ $\quad$ $(k-1)$ zeroes $\{x_{i_0}, x_{i_1}, \ldots, x_{i_{k-2}}\}$

$P_{i_0 \ldots i_k} \in \Pi_k$ $\quad$ $\{x_{i_0}, x_{i_1}, \ldots, x_{i_{k-1}}\}$

$P_{i_0 \ldots i_k} = \alpha(x - x_{i_0}) \cdots (x - x_{i_{k-1}})$,

$P_{i_0 \ldots i_k}(x) - P_{i_0 \ldots i_{k-1}}(x) \in \Pi_k$

$= f_{i_0 i_1 \ldots i_k}(x - x_{i_0}) \cdots (x - x_{i_{k-1}})$

$P_{i_0 \ldots i_{k-1}} = \beta(x - x_{i_0}) \cdots (x - x_{i_{k-2}})$

$$P_{01\ldots n}(x) = a_0 + a_1(x - x_0) + \cdots + a_n(x - x_0)\cdots(x - x_{n-1}) = \sum_{k=0}^{n} a_k \left[\prod_{j=0}^{k-1}(x - x_j)\right]$$

$P_{5\ldots 17} = a_0 + a_1(x - x_5) + \cdots a_{17}(x - x_5) \cdots (x - x_{16})$

Build-up

$P_{i_0}(x) = f_{i_0} \quad \forall i_0 = 0(1)n$

$\boxed{k=1}$ $\Rightarrow P_{i_0 i_1}(x) - P_{i_0}(x) = f_{i_0 i_1}(x - x_{i_0})$

$\Rightarrow P_{i_0 i_1}(x) = f_{i_0} + f_{i_0 i_1}(x - x_{i_0})$

$\boxed{P_{i_1 i_2}(x)} = \boxed{f_{i_1}} + \boxed{f_{i_1 i_2}}(x - x_{i_1})$

$\boxed{k=2}$ $P_{i_0 i_1 i_2}(x) = f_{i_0} + f_{i_0 i_1}(x - x_{i_0}) + \boxed{f_{i_0 i_1 i_2}}(x - x_{i_0})(x - x_{i_1})$

$f_{i_1} + \boxed{f_{i_1 i_2}}(x - x_{i_1})$ $\quad$ ~~$(x - x_{i_2})$~~

$\left[(x - x_{i_0}) P_{i_1 i_2}(x) - (x - x_{i_2}) P_{i_0 i_1}(x)\right] / \boxed{(x_{i_2} - x_{i_0})}$

$f_{i_0 i_1 \ldots i_k}$: k^{th} divided difference

$\boxed{\dfrac{f_{i_1 i_2} - f_{i_0 i_1}}{x_{i_2} - x_{i_0}}}$. $f_{i_0} + \boxed{f_{i_0 i_1}(x)}\, x_{i_0})$

$\{a_0 \ldots a_n\}$

$a_0, a_1, a_2 \ldots$ $\quad i_0 = 5$

Relationship (Neville's P's & DD 'f'-coeffs]

$$P_{i_0 i_1 \ldots i_k}(x) = \frac{\left[(x - x_{i_0}) P_{i_1 i_2 \ldots i_k}(x) - (x - x_{i_k}) P_{i_0 i_1 \ldots i_{k-1}}(x)\right]}{x_{i_k} - x_{i_0}}$$

k=0 $\quad P_{i_0}(x) = f_{i_0} \quad \forall i_0 = 0(1)n$

$[\because P_{i_0}(x_{i_0}) = f_{i_0}$ must be satisfied$]$

$P_{i_0 i_1}(x) = f_{i_0} + f_{i_0 i_1}(x - x_{i_0})$

$$\Rightarrow f_{i_0 i_1}(x - x_{i_0}) = P_{i_0 i_1}(x) - f_{i_0} = \frac{\left[(x - x_{i_0}) f_{i_1} - (x - x_{i_1}) f_{i_0}\right]}{x_{i_1} - x_{i_0}} - f_{i_0}$$

$$\boxed{f_{i_0 i_1 i_2} = \frac{f_{i_1 i_2} - f_{i_0 i_1}}{x_{i_2} - x_{i_0}}}$$

$$f_{i_0 i_1 \ldots i_k} = \frac{f_{i_1 \ldots i_k} - f_{i_0 \ldots i_{k-1}}}{x_{i_k} - x_{i_0}}$$

$\boxed{\text{Newton's representation}}$

$\Phi(x) = P_{i_0 i_1 \ldots i_n}(x) \in \Pi_n$

$= f_{i_0} + f_{i_0 i_1}(x - x_{i_0}) + \cdots f_{i_0 \ldots i_n}(x - x_{i_0}) \cdots (x - x_{i_{n-1}})$

$$\bullet\ f_{i_0 i_1 \ldots i_k} = \frac{\left(-f_{i_0 \ldots i_{k-1}} + f_{i_1 i_2 \ldots i_k}\right)}{x_{i_k} - x_{i_0}}$$

$$\begin{tikzcd}[row sep=small, column sep=small]
 & k=0 & 1 & 2 & 3 \\
x_0 & f_0 \arrow[dr] & & & \\
x_1 & f_1 \arrow[r] \arrow[dr] & f_{01} \arrow[r] & f_{012} \arrow[dr] & \\
x_2 & f_2 \arrow[r] \arrow[dr] & f_{12} \arrow[ur] \arrow[r] & f_{123} \arrow[r] & f_{0123} \\
x_3 & f_3 \arrow[r] & f_{23} \arrow[ur] & &
\end{tikzcd}$$

$$f_{01} = \frac{f_1 - f_0}{x_1 - x_0}\ ; \quad f_{12} = \frac{f_2 - f_1}{x_2 - x_1}\ ; \quad f_{23} = \frac{f_3 - f_2}{x_3 - x_2}\ ;$$

An error expression

$f_i = f(x_i) \quad \forall i = 0(1)n$

$P(x)\ , \quad P(x_i) = f_i \quad \forall i = 0(1)n$

$x \neq \{x_i\},\ [f(x) - P(x)] = \varepsilon(x) \leftarrow$ how big?

(i) $f(x)$ has a $(n+1)^{st}$ derivative in dom$[f]$

(ii) $\forall \bar{x} \in$ dom$[f]$

$I_0[x_0, \ldots, x_n, \bar{x}]$

. $\exists\ \xi \in I_0$ s.t. $\varepsilon(\bar{x}) = f(\bar{x}) - P(\bar{x}) = \omega(\bar{x}) \cdot \left[\dfrac{f^{(n+1)}(\xi)}{(n+1)!}\right]$

$$\omega(\bar{x}) = \prod_{k=0}^{n}(\bar{x} - x_k)$$

$\bar{x} \neq x_i,\ i = 0(1)n$

$F(x) = \varepsilon(x) - K\omega(x)$

$K = \dfrac{\varepsilon(\bar{x})}{\omega(\bar{x})}$ $\quad F(x) = f(x) - P(x) - K\omega(x)$ s.t., $F(\bar{x}) = 0$

$\Rightarrow F(x)$ has $(n+2)$ zeroes (at least) within I_0

WLOG, $x_k < x_{k+1}$ $\quad (x_0 < x_1 < x_2 < \cdots x_n)$

$x_{k^*} < \bar{x} < x_{k^*+1}$

$n-1 \geq k^* \geq 0$

$\Rightarrow F^{(1)}(x)$ has at least $(n+1)$ zeroes (Rolle's theorem)!

$\Rightarrow F^{(n+1)}(x)$ has at least $(n+2) - (n+1) = \boxed{1 \text{ zero}}$ in I_0.

$F^{(n+1)}(x) = f^{(n+1)}(x) - K\omega^{(n+1)}(x)$ $\quad \left[-P^{(n+1)}(x)\right]$

$F^{(n+1)}(\xi) = f^{(n+1)}(\xi) - K\omega^{(n+1)}(\xi)$

$0 = f^{(n+1)}(\xi) - K(n+1)!$

$$\Rightarrow K = \frac{f^{(n+1)}(\xi)}{(n+1)!}$$

$$\varepsilon(\bar{x}) = f(\bar{x}) - P(\bar{x}) = \omega(\bar{x})\, \frac{f^{(n+1)}(\xi)}{(n+1)!}$$