### LINEAR ALGEBRA -II

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▶ We note that  $\langle x, x \rangle \ge 0$  for every x in  $\mathbb{R}^n$  and  $\langle x, x \rangle = 0$  if and only if x = 0.

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such that

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- Some authors take inner product as linear in the first variable. It is a matter of convention. A vector space with a specified inner product is called an inner product space.



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- ▶ On the other hand if the field  $\mathbb{F} = \mathbb{C}$ , from linearity and symmetry of the inner-product we get anti-linearity in the first variable.
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$$\mathbb{R}^2$$
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This suggests the following definitions.

## The norm on an inner product space

▶ Definition 7.5: Let  $(V, \langle \cdot, \cdot \rangle)$  be an inner product space. Then the norm of a vector  $x \in V$  is defined as

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► The 'distance function'  $d: V \to V \to \mathbb{R}$  is also known as metric.



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- $(iii) ||x + y|| \le ||x|| + ||y||, \forall x, y \in V.$

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(iii) This is a consequence of Cauchy-Schwarz inequality and will be proved in the next class.



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- ► Hence z is a scalar multiple of x. In particular, x, y are linearly dependent.
- Exercise: We proved the Cauchy-Schwarz inequality with the assumption of definiteness, that is  $\langle x, x \rangle = 0$  implies x = 0. Prove the inequality without this assumption. (Hint: Consider the function  $p(t) = \|y t\langle x, y \rangle x\|^2$  for  $t \in \mathbb{R}$ .)

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- ► All these are immediate from respective properties of the norm.

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- Exercise: Find out as to why this is called parallelogram law.

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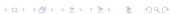
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Now the result is clear by taking the difference of two equations and dividing by four.



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- END OF LECTURE 8.