11/08/2022

$$\begin{bmatrix} x_1^{(0)} \\ \vdots \\ x_n^{(0)} \end{bmatrix} = x^{(0)} \in D_{n_0} \subseteq \mathbb{R}^n$$

$$\varphi(x^{(0)}) = \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix} = y \in D_m \subseteq \mathbb{R}^m$$

$$\varphi = \begin{bmatrix} \varphi_1 \\ \vdots \\ \varphi_m \end{bmatrix}$$

$$= \begin{bmatrix} \varphi_1(x_1^{(0)} \dots x_n^{(0)}) \\ \vdots \\ \varphi_m(x_1^{(0)} \dots x_n^{(0)}) \end{bmatrix}$$

$$= \varphi_j(x^{(0)}), \; j = 1(1)m;$$

$$x = x^{(0)} \to \varphi^{(0)}(x^{(0)}) \, [= x^{(1)}]$$

$$\downarrow$$

$$\dots [= x^{(p)}] \, \varphi^{(p)}(x^{(p)}) \leftarrow \varphi^{(1)}(x^{(1)}) \, [= x^{(2)}]$$

$$\to \varphi^{(p)}[x^{(p)}] = x^{(p+1)} \, [\dim(x^{(p+1)}) = m, \; x^{(p+1)} = y]$$

Remainder funcⁿ

$\psi_k^{(m)}$; Suppose $p = 17$

i.e., $\{\varphi^{(0)} \dots \varphi^{(17)}\}$

↳ 18 elem. maps

$i = 13$; $\psi^{(13)} = \varphi^{(17)} \circ \varphi^{(16)} \circ \varphi^{(15)} \circ \varphi^{(14)} \circ \varphi^{(13)}$

$\psi^{(13)}(x^{(13)})$: domain of $\psi^{(13)} = D_{13} \subseteq \mathbb{R}^{n_{13}}$

range: $D_m \subseteq \mathbb{R}^m$.

$\psi^{(0)} = \varphi$; $\psi^{(i)} : D_i \, (\subseteq \mathbb{R}^{n_i}) \to D_m \subseteq \mathbb{R}^m$

$\psi^{(p)} = \varphi^{(p)}$

$$\Delta y = \tilde{y} - y = D(\varphi(x)) \cdot \Delta x$$

• ∵ $h = f \circ g$; $h(x) = f[g(x)]$

$$D(h(x)) = D[f(g(x))] \, D[g(x)]$$

$$D[\varphi(x)] = D[\varphi^{(p)}(x^{(p)})] \dots \dots D[\varphi^{(1)}(x^{(1)})] \, D[\varphi^{(0)}(x^{(0)})]$$

$$D[\psi^{(i)}(x^{(i)})] = D[\varphi^{(p)}(x^{(p)})] \dots D[\varphi^{(i)}(x^{(i)})]$$

—×—×—

$$\Delta x^{(i)} \equiv \tilde{x}^{(i)} - x^{(i)}$$

$$\Delta x^{(i+1)} \equiv \tilde{x}^{(i+1)} - x^{(i+1)}$$

$$x^{(i+1)} = fl \, \varphi^{(i)}(x^{(i)})$$

$$\cdot \, \varphi^{(i)}(\tilde{x}^{(i)})$$

$$\cdot \, \varphi^{(i)}(\tilde{x}^{(i)})$$

$$\Delta x^{(i+1)} \equiv fl[\varphi^{(i)}(\tilde{x}^{(i)})] - \varphi^{(i)}(x^{(i)})$$

$$= \{fl[\varphi^{(i)}(\tilde{x}^{(i)})] - \varphi^{(i)}(\tilde{x}^{(i)})\} + \{\varphi^{(i)}(\tilde{x}^{(i)}) - \varphi^{(i)}(x^{(i)})\}$$

$$\varphi^{(i)}(\tilde{x}^{(i)}) - \varphi^{(i)}(x^{(i)}) = D\varphi^{(i)}(x^{(i)}) \cdot \Delta x^{(i)} \quad \dots (1)$$

$$\Delta x^{(i+1)} = fl[\varphi^{(i)}(\tilde{x}^{(i)})] - \varphi^{(i)}(\tilde{x}^{(i)}) + D\varphi^{(i)}(x^{(i)}) \cdot \Delta x^{(i)} \quad \dots (2)$$

—×—×—

$\dots$ (2)

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<u>General</u> :- $\varphi^{(i)} : D_i \mapsto D_{i+1}$

$D_i \subseteq \mathbb{R}^{n_i}$

$D_{i+1} \subseteq \mathbb{R}^{n_{i+1}}$

$$\varphi^{(i)}(u) = \begin{bmatrix} \varphi_1^{(i)}(u) \\ \vdots \\ \varphi^{(i)}_{n_{i+1}}(u) \end{bmatrix}$$

$$fl(\varphi_j^{(i)}(u)) = (1+\varepsilon_j)\,\varphi_j^{(i)}(u)$$

i.e., $|\varepsilon_j| \leq \varepsilon_M$, $j = 1 \dots n_{i+1}$

$$fl(\varphi_j^{(i)}(u)) - \varphi_j^{(i)}(u) = \varepsilon_j\, \varphi_j^{(i)}(u)$$

$$fl[\varphi^{(i)}(u)] = \begin{pmatrix} 1+\varepsilon_1 & & O \\ & 1+\varepsilon_2 & \\ O & & \ddots \quad 1+\varepsilon_{n_{i+1}} \end{pmatrix} \left(\varphi^{(i)}(u)\right)$$

$$\Rightarrow fl[\varphi^{(i)}(u)] - \varphi^{(i)}(u) = E_{i+1}\,\varphi^{(i)}(u) \quad \dots (\alpha)$$

$$E_{i+1} = \begin{bmatrix} \varepsilon_1 & & \\ & \ddots & \\ & & \varepsilon_{n_{i+1}} \end{bmatrix} \quad |\varepsilon_j| \leq \varepsilon_M,$$

— × —

$$\Delta x^{(i+1)} = E_{i+1}\,\varphi^{(i)}(\tilde{x}^{(i)}) + D\varphi^{(i)}(\tilde{x}^{(i)}) \cdot \Delta x^{(i)} \quad \dots (6)$$

$$E_{i+1}\,\varphi^{(i)}(\tilde{x}^{(i)}) = E_{i+1}\,\varphi^{(i)}(x^{(i)}(1+\tilde{\delta}^{(i)})) = E_{i+1}\,\varphi^{(i)}(x^{(i)}) + O(E_{i+1}\tilde{\delta}^{(i)})$$

$$\therefore E_{i+1}\,\varphi^{(i)}(\tilde{x}^{(i)}) \approx E_{i+1}\,\varphi^{(i)}(x^{(i)}) + O(\varepsilon_M^2)$$

$\dots$ (7)

$$\Delta x^{(i+1)} = \underbrace{E_{i+1}\, x^{(i+1)}}_{\alpha_{i+1}} + D\varphi^{(i)}(x^{(i)}) \cdot \Delta x^{(i)}$$

$$\Delta x^{(0)} = fl(x^{(0)}) - x^{(0)}$$

$$\Delta x^{(1)} = \alpha_1 + D\varphi^{(0)}(x^{(0)}) \cdot \Delta x^{(0)} = E_1 x^1 + D\varphi^{(0)}(x^{(0)})\, \Delta x^{(0)}$$

$i=1$

$$\Delta x^{(2)} = \alpha_2 + D\varphi^{(1)}(x^{(1)}) \cdot \Delta x^{(1)} = E_2 x^2 + D\varphi^{(1)}(x^{(1)}) \cdot \left[E_1 x^1 + D\varphi^{(0)}(x^{(0)}) \cdot \Delta x^{(0)}\right]$$

$$\Delta x^{(2)} = \underbrace{E_2 x^2}_{(n_2\times 1)} + \underbrace{D\varphi^{(1)}(x^{(1)})}_{(n_2 \times n_1)} \cdot \underbrace{E_1 x^1}_{(n_1\times 1)} + \underbrace{D\varphi^{(1)}(x^{(1)})}_{n_2\times n_1} \cdot \underbrace{D\varphi^{(0)}(x^{(0)})}_{n_1 \times n} \cdot \underbrace{\Delta x^{(0)}}_{n \times 1}$$

$$\Delta x^{(p+1)} = \Delta y = E_{p+1}\, y + D\varphi(x) \cdot \Delta x + \sum_{k=1}^{p} D\psi^{(k)}(x^{(k)}) \cdot E_k x^{(k)}$$

$\dots$ (8)

<u>Example</u> :-

$$x = x^{(0)} = \begin{bmatrix} a \\ b \end{bmatrix} \in \mathbb{R}^2$$

$$y = \varphi(x) \stackrel{\text{def}}{=} a^2 - b^2$$

$$\therefore \varphi : \mathbb{R}^2 \longmapsto \mathbb{R}$$

$$\varphi^{(0)} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a^2 \\ b \end{bmatrix}$$

$$\varphi^{(0)} : \mathbb{R}^2 \longmapsto (\mathbb{R}_+ \cup \{0\}) \otimes \mathbb{R}$$

$$\varphi^{(1)} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} u \\ v^2 \end{bmatrix}$$

$$\varphi^{(1)} : (\mathbb{R}_+ \cup \{0\}) \otimes \mathbb{R} \longmapsto (\mathbb{R}_+ \cup \{0\})^2$$

$$\varphi^{(2)} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \alpha - \beta$$

$$\varphi^{(2)} : (\mathbb{R}_+ \cup \{0\})^2 \longmapsto \mathbb{R}$$

$$\varphi = \varphi^{(2)} \circ \varphi^{(1)} \circ \varphi^{(0)}$$

$D[\varphi(x)] :- (m \times n)$ – matrix

$$\dim[D(\varphi(x))] = 1 \times 2 \qquad m = 1, \; n = 2$$

$$D(\varphi(x)) = \left[\frac{\partial \varphi}{\partial a}, \frac{\partial \varphi}{\partial b} \right]$$

$$\varphi(a, b) = a^2 - b^2$$

$$\Rightarrow D[\varphi(a,b)] = (2a, -2b)$$

$$D[\varphi(a,b)] \begin{pmatrix} \Delta a \\ \Delta b \end{pmatrix} = \underbrace{(2a, -2b)}_{1 \times 2} \underbrace{\begin{pmatrix} \Delta a \\ \Delta b \end{pmatrix}}_{2 \times 1}$$

$$= 2a\,\Delta a - 2b\,\Delta b$$

$$E_{p+1} \cdot y$$

$$p = 2;$$

$$\dim[E_3] = n_3 = m \, (= 1); \quad |\epsilon_3| \leq \epsilon_M$$

$$\therefore E_{p+1} \cdot y = \epsilon_3 (a^2 - b^2)$$

$$\Delta y = 2a\,\Delta a - 2b\,\Delta b + \epsilon_3 (a^2 - b^2) + D\psi^{(1)}(x^{(1)}) E_1 x^{(1)} + D\psi^{(2)}(x^{(2)}) . E_2 x^{(2)}$$

$$\psi^{(1)}(x^{(1)}) : \quad x^{(1)} \in (\mathbb{R}_+ \cup \{0\}) \otimes \mathbb{R}$$

$$\psi^{(1)} : D_1 \longmapsto \mathbb{R} \qquad m \times n_1, \; m = 1, \; n_1 = 2$$

$$\psi^{(1)} \begin{bmatrix} u \\ v \end{bmatrix} = u - v^2$$

$$\therefore D\psi^{(1)}(x^{(1)}) = (1, -2v)$$

$$fl[\varphi^{(0)}(x^{(0)})] - \varphi^{(0)}(x^{(0)})$$

$$= \begin{bmatrix} fl(a \times a) \\ b \end{bmatrix} - \begin{bmatrix} a^2 \\ b \end{bmatrix}$$

$$= \begin{pmatrix} \epsilon_1 a^2 \\ 0 \end{pmatrix}$$

$$E_1 = E_{i+1} \; (i = 0)$$

$$= \begin{bmatrix} \epsilon_1 & 0 \\ 0 & 0 \end{bmatrix} \quad \therefore E_1 x^{(1)} = \begin{bmatrix} \epsilon_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a^2 \\ b \end{bmatrix}$$

$$\therefore D\psi^{(1)}(x^{(1)}) . E_1 x^{(1)}$$

$$= [1, -2b] \begin{bmatrix} \epsilon_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a^2 \\ b \end{bmatrix}$$

$$= (1, -2b) \begin{pmatrix} \epsilon_1 a^2 \\ 0 \end{pmatrix} \rightarrow \alpha_1$$

$$= \epsilon_1 a^2$$

$$\Delta y = 2a\,\Delta a - 2b\,\Delta b + \epsilon_3 (a^2 - b^2) + \epsilon_1 a^2 + D\psi^{(2)}(x^{(2)}) . E_2 x^{(2)}$$

$$\psi^{(2)} = \varphi^{(2)} \qquad \psi^{(2)} \begin{bmatrix} u \\ v \end{bmatrix} = u - v$$

$$\therefore D\psi^{(2)}(x^{(2)}) = (1, -1)$$

$$E_2 x^{(2)} = fl[\varphi^{(2)}(x^{(2)})] - \varphi^{(2)}(x^{(2)})$$

$$= \begin{bmatrix} 0 \\ \epsilon_2 b^2 \end{bmatrix}$$

$$\therefore D\psi^{(2)}(x^{(2)}) E_2 x^{(2)}$$

$$= (1, -1) \begin{pmatrix} 0 & 0 \\ 0 & \epsilon_2 \end{pmatrix} \begin{pmatrix} a^2 \\ b^2 \end{pmatrix}$$

$$= -\epsilon_2 b^2$$

$$\therefore \Delta y_{\text{final}} = 2a\,\Delta a - 2b\,\Delta b + \epsilon_1 a^2 - \epsilon_2 b^2 + \epsilon_3 (a^2 - b^2)$$

$$|\epsilon_i| \leq \epsilon_M, \quad i = 1, 2, 3.$$

HW :- $\begin{bmatrix} a \\ b \end{bmatrix} \rightarrow \begin{bmatrix} a+b \\ a-b \end{bmatrix} \longrightarrow (a+b)(a-b)$