3 In order to address the problems of covariance mentioned in Remark 2 © of Pg 181, covariance is divided by the product of the standard deviations of X and Y. This gives rise to the correlation coefficient of X and Y as follows:

$$\mathrm{Corr}(X,Y) := \frac{\mathrm{Cov}(X,Y)}{+\sqrt{\mathrm{Var}(X)\,\mathrm{Var}(Y)}}.$$

This measure of association is unit-free and always lies in $[-1,1]$ making its value (not just the sign) easier to interpret. We shall disuss this in details soon.

4 Covariance (and even correlation coefficient) ~~have~~ has many drawbacks as a measure of ~~association~~ association. In spite of them, it (at least correlation) is a popular measure because covariance (and hence correlation coefficient) is very easy to compute. This will be confirmed by our next result.

Thm: Suppose X and Y are jointly distributed r.v.s with finite means μ_X and μ_Y, respectively. Then X and Y have finite covariance if and only if XY has finite mean. And in this case,

$$\begin{aligned} Cov(X,Y) &= E(XY) - E(X)E(Y) \\ &= E(XY) - \mu_X \mu_Y . \end{aligned}$$

(Computational recipe for covariance.)

Proof: Exc. (Follow the proof of the thm stated at the end of Pg 172.)

Remark: Thanks to Remark 1 of Pg 179, the above thm yields as a special case the thm stated at the end of Pg 172 when $X \equiv Y$. This explains why the proof is along the same line — after all we are simply generalizing the proofs in Pg 173 - 174.

Exc: Suppose $X \sim \mathrm{Unif}\{-1, 0, 1\}$ and $Y = X^2$. Show that (X and Y have finite covariance and) $\mathrm{Cov}(X, Y) = 0$.

Exc: Suppose $X \sim \mathrm{Unif}(-1, 1)$ and $Y = X^2$. Show that $\mathrm{Cov}(X, Y) = 0$.

Exc: Suppose $(X, Y) \sim \mathrm{Unif}(D)$, where $D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$ is the unit disk. Show that $\mathrm{Cov}(X, Y) = 0$.

Remarks: ① Note that in the above ~~examples~~ exercises, X and Y are dependent (and hence they are "associated") even though $\mathrm{Cov}(X, Y) = 0$.

② These examples clearly show the drawback of covariance (and correlation) as a measure of dependence. As we shall see ~~later,~~ soon correlation coefficient measures the amount of linear association between X and Y. In the first two exercises of Pg 184, ~~X an~~ the association between X and Y is "purely quadratic" and hence the amount of linear association is zero. On the other hand, in the third exercise of Pg 184, the inequality $X^2 + Y^2 \leq 1$ is satisfied "in a uniform manner" (i.e., in a symmetric fashion), that leads to no linear association between X and Y.

3 Correlation coefficient is a very good measure of association when X and Y are jointly normal – we shall learn this later in this course. However, if X and Y are not jointly normal, then Corr(X,Y) is not always a good measure of association. ~~This~~

4 Lack of ~~unders~~ understanding of Remark 3 above by the quants was one of the important mathematical reasons behind the subprime mortgage crisis at the USA in 2007-08.

Exc: Compute Cov(X,Y) ~~in~~ for the random vectors (X,Y) discussed/described in the examples given in the following ~~pages~~ pages:

(i) Pg 13 ;

(ii) Pg 18 with $p = \frac{1}{2}$;

(iii) Pg 31 .

Exc: If X and Y are independent r.v.s with finite means, such that either both are discrete or both are cont, then XY has finite mean and $E(XY) = E(X)E(Y)$.

[Hint: Use (e) of Pg 150.]

Cor: If X and Y are ind r.v.s with finite means (such that either both are discrete or both are cont), then $\text{Cov}(X,Y) = 0$.

Proof: Follows directly from the 2^{nd} Exc of Pg 186. and the thm is Pg 183.

Remark: As mentioned clearly in Pg 184 (see Remark 1 and the exercises above it), the converse of the Cor stated in Pg 186.

That is, ~~independence~~ ~~$\Rightarrow$~~

$$\begin{array}{ccc} X,Y \text{ are independent} & \begin{array}{c}\Longrightarrow \\ \not\Longleftarrow\end{array} & \begin{array}{c} X,Y \text{ are uncorrelated} \\ (\text{i.e., } Cov(X,Y)=0) \end{array} \end{array}$$

Cor: If X and Y are jointly distributed random variables with finite 2^{nd} moments, then X, Y have finite covariance and

$$Cov(X,Y) = E(XY) - E(X)E(Y).$$

(In particular, it means that all of the 3 quantities in the RHS exist ~~finitely~~ and are finite.)

Proof: Note that

$$|XY| \leq \frac{X^2+Y^2}{2}$$

$\Rightarrow$ XY has finite mean

Also X, Y have finite mean.

Therefore by the thm on Pg 183 X, Y have finite covariance

and $\text{Cov}(X,Y) = E(XY) - E(X)E(Y)$.

Remarks: ① The corollary stated in Pg 187 is more restrictive but slightly more useful than the theorem stated in Pg 183 - after all you have to verify ~~marginal~~ finiteness of marginal 2^{nd} moments and the finiteness of covariance (which depends on the joint distn) will follow.

② Note that the proofs of the corollaries in Pg 174 and 187 should be viewed through the lens of the Remark stated in Pg 163.