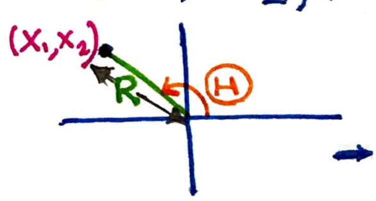
In the previous example and the ~~exercises~~ exercises (based on the change of joint density formula), the choice of I (and ~~hece~~ hence J) was obvious — just take $I = \text{Range}(X_1, X_2)$, the most natural joint range of X_1 and X_2. Now we shall go through an example for which I has to be chosen carefully.

Example: Suppose $X_1, X_2 \overset{iid}{\sim} N(0,1)$.

Define $Y_1 \doteq R = +\sqrt{X_1^2 + X_2^2}$ and

$Y_2 = \Theta =$ Angle (in radian) made by (X_1, X_2) ~~with~~ (in the ~~anti-clock~~ anticlockwise direction) ~~with~~ with the positive side of X_1-axis (horizontal axis).

In other words, $(Y_1, Y_2) = (R, \Theta)$ is the polar coordinate transform of (X_1, X_2). Find the joint dist^{n} of (R, Θ).



Solution: Since $X_1, X_2 \overset{iid}{\sim} N(0,1)$, the most natural choice for I is $\mathbb{R}^2$. However, there are some issues with this ~~chai~~ choice (see below).

Issue #1: Θ is not well-defined for the point $(0,0)$.

One may think that removing origin from $\mathbb{R}^2$ will ~~not~~ resolve the problem but that will still leave us with the following issue.

Issue # 2: The function (H) won't even be continuous at ~~all~~ any points on the positive side of the X_1-axis, i.e., the horizontal axis.

In order to resolve the above issues, we need to take/choose

$$I = \mathbb{R}^2 \setminus \{(x_1, 0) : x_1 \geq 0\}.$$

Exc: Check that I is open (either use the defn or the thm stated in Pg (126)) ~~and~~ and path-connected (verify visually).

We also need to ~~take~~ use the following joint pdf of (X_1, X_2):

$$f_{X_1,X_2}(x_1,x_2) = \begin{cases} \frac{1}{2\pi} \exp\{-\frac{1}{2}(x_1^2+x_2^2)\} & \text{if } (x_1,x_2) \in I, \\ 0 & \text{if } (x_1,x_2) \notin I. \end{cases}$$

With these choices of I and f_{X_1,X_2}, we define $g: I \to \mathbb{R}^2$ by

$$g(x_1, x_2) = (r, \theta), \quad (x_1, x_2) \in I,$$

where

$$r = +\sqrt{x_1^2 + x_2^2} \in (0, \infty), \quad \text{and}$$

θ = angle (in radian) made by (x_1, x_2) with the positive side of the x_1-axis (i.e., the horizontal axis) in the anticlockwise ~~anti clock~~ direction

$\in (0, 2\pi)$. We want to find the joint distn of $(R, \Theta) = g(X_1, X_2)$.

It is clear that $g(I) \subseteq (0, \infty) \times (0, 2\pi)$.

Exc: Show that $g(I) = (0, \infty) \times (0, 2\pi) =: J$ and $g: I \to J$ is a bijection.

Exc: Show that the inverse map of g is given by $g^{-1}: J \to I$

$$g^{-1}(r, \theta) = (r\cos\theta, r\sin\theta), \quad (r, \theta) \in J.$$

In particular, in the notation of Pg 130 - 131, the maps $h_1 : J \to \mathbb{R}$ and $h_2 : J \to \mathbb{R}$ are given by

$$h_1(r, \theta) = r\cos\theta \,, \qquad (r, \theta) \in J$$

$$h_2(r, \theta) = r\sin\theta \,, \qquad (r, \theta) \in J$$

so that:

$$g^{-1}(r, \theta) = \left(h_1(r,\theta), h_2(r,\theta)\right), \qquad (r, \theta) \in J.$$

Hence the Jacobian matrix of g^{-1} is given by

$$J_{g^{-1}}(r, \theta) = \begin{pmatrix} \frac{\partial h_1}{\partial r} & \frac{\partial h_1}{\partial \theta} \\ \frac{\partial h_2}{\partial r} & \frac{\partial h_2}{\partial \theta} \end{pmatrix}, \qquad (r, \theta) \in J$$

$$= \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}, \qquad (r, \theta) \in J$$

$$\Rightarrow \det\left(J_{g^{-1}}(r, \theta)\right) = r\cos^2\theta - (-r\sin^2\theta)$$
$$= r(\cos^2\theta + \sin^2\theta) = r \,, \qquad (r, \theta) \in J.$$

In particular, this means that g is indeed "smooth" in the sense of Pg 133, i.e., the partial derivatives $\frac{\partial h_1}{\partial r}$, $\frac{\partial h_1}{\partial \theta}$, $\frac{\partial h_2}{\partial r}$, $\frac{\partial h_2}{\partial \theta}$ exist and are cont on J, and

$$\frac{d g^{-1}(r,\theta)}{d(r,\theta)} = \det\left(J_{g^{-1}}(r,\theta)\right) = r \neq 0 \quad \text{on } J.$$

(in fact $r > 0$ on J.)

We have verified that all the assumptions of the bivariate change of joint density formula are satisfied. Therefore, it follows that $(R, \Theta) = g(X_1, X_2)$ is also a cont random vector with a joint pdf

$$f_{R,\Theta}(r,\theta) = f_{X_1,X_2}\left(g^{-1}(r,\theta)\right)\left|\frac{d g^{-1}(r,\theta)}{d(r,\theta)}\right| \quad , \quad (r,\theta) \in J$$

$$= f_{X_1,X_2}(r\cos\theta, r\sin\theta) \cdot r \quad , \quad (r,\theta) \in J$$

$$= \frac{1}{2\pi} r e^{-\frac{1}{2}r^2} \quad \text{if } r > 0, \theta \in (0, 2\pi).$$

Note that the above joint pdf of R and Θ

splits into a product of a function of r (say, $\frac{1}{2\pi} r e^{-r^2/2}$, $r \in (0,\infty)$ and a function of θ (namely, the constant function 1 for $\theta \in (0,2\pi)$)

and the ranges of r and θ do not depend on each other. Therefore it follows that $R \perp\!\!\!\perp \Theta$.

Exc: Show that R follows Rayleigh distribution with a pdf

$$f_R(r) = \begin{cases} r e^{-\frac{1}{2}r^2} & \text{if } r > 0, \\ 0 & \text{if } r \leq 0, \end{cases}$$

$\Theta \sim \text{Unif}(0, 2\pi)$ with a pdf

$$f_\Theta(\theta) = \begin{cases} \frac{1}{2\pi} & \text{if } \theta \in (0,2\pi), \\ 0 & \text{if } \theta \notin (0,2\pi), \end{cases}$$

and $R \perp\!\!\!\perp \Theta$.

Remarks: ① Note that $R^2 = X_1^2 + X_2^2 \sim \chi_2^2$

$$\Rightarrow R^2 \sim \chi^2_2 \equiv \text{Gamma}\left(1, \tfrac{1}{2}\right) \equiv \text{Exp}\left(\tfrac{1}{2}\right)$$

$\Rightarrow$ If $T \sim \text{Exp}\left(\frac{1}{2}\right)$, then $+\sqrt{T}$ follows Rayleigh distn. (Exc: Check this directly)

② Heuristically speaking, the independence of R and Θ, and $\Theta \sim \text{Unif}(0, 2\pi)$ are manifestations of the "rotational symmetry" of (X_1, X_2) ~~when X_1, X_2~~ (when $X_1, X_2 \overset{iid}{\sim} N(0,1)$): ~~More precisely,~~

Exc: Suppose $X_1, X_2 \overset{iid}{\sim} N(0,1)$. ~~Define~~ ~~$Y_1, Y_2$~~ Fix $\phi \in (0, 2\pi)$ and define

$$Y_1 = X_1 \cos\phi - X_2 \sin\phi \quad \text{and}$$

$$Y_2 = X_1 \sin\phi + X_2 \cos\phi.$$

In other words, (Y_1, Y_2) is the random vector obtained by rotating (X_1, X_2) by an angle ϕ (in radian) anticlockwise. Show that $Y_1, Y_2 \overset{iid}{\sim} N(0,1)$. (Therefore, the joint distn. of (X_1, X_2) remains unchanged when we apply a rotation.)

③ It can be shown that the converse of the above example (given in Pg 140) also holds. That is, if we take

$$\text{ind} \begin{cases} R \sim \text{Rayleigh dist}^{n} \\ \Theta \sim \text{Unif}(0, 2\pi) \end{cases},$$

then

$$X_1 := R\cos\Theta, \quad X_2 := R\sin\Theta \overset{iid}{\sim} N(0,1).$$

④ ~~Using~~ Thanks to ③, one can simulate a pair of iid standard normal random variables using the following method (known as the Box-Muller method):

Step 1: Generate $U_1, U_2 \overset{iid}{\sim} \text{Unif}(0,1)$.

Step 2: Define $Z_1 := \sqrt[+]{-2\log_e U_1}\,\cos(2\pi U_2)$,

$$Z_2 := \sqrt[+]{-2\log_e U_1}\,\sin(2\pi U_2).$$

Then $Z_1, Z_2 \overset{iid}{\sim} N(0,1)$.

Exc: Assuming 3, show that the Box-Muller method actually works.

Exc: Write a program and simulate a random sample (i.e., iid sample) of size 10000 from the standard normal distn using Box-Muller method. Draw the histogram of ~~these poin~~ this sample and check how close it is ~~looks~~ to the bell-shaped normal curve.

Linearity and Monotonicity of Expectation

Recall that for a (discrete or continuous) r.v. X and a function $h : \text{Range}(X) \longrightarrow \mathbb{R}$, the r.v. $h(X)$ has finite mean provided

$$\sum_{x \in \text{Range}(X)} |h(x)|\, p_X(x) < \infty$$ when X is discrete ~~and~~ (with pmf p_X), and

$$\int_{-\infty}^{\infty} |h(x)|\, f_X(x)\,dx < \infty$$ when X is cont (with a pdf f_X).

In the above situation,

$$E[h(X)] = \begin{cases} \displaystyle\sum_{x \in \text{Range}(X)} h(x)\, p_X(x) & \text{when } X \text{ is discrete, } \cancel{\text{and}} \\ \displaystyle\int_{-\infty}^{\infty} h(x)\, f_X(x)\,dx & \text{when } X \text{ is cont.} \end{cases}$$

This univariate fact has a multivariate generalization, which we shall state without proof.

Now suppose that $\underset{\sim}{X} = (X_1, X_2, \dots, X_k)$ is a ~~ran~~ (discrete or continuous) random vector and $h : \cancel{\text{Range}(\underset{\sim}{X})} \ \text{Range}(\underset{\sim}{X}) \, (\subseteq \mathbb{R}^k) \longrightarrow \mathbb{R}$ is a function. Then the r.v. $h(\underset{\sim}{X}) = h(X_1, X_2, \dots, X_k)$ has finite mean provided

$\sum_{\underset{\sim}{x} \in \text{Range}(\underset{\sim}{X})} |h(\underset{\sim}{x})| \, p_{\underset{\sim}{X}}(\underset{\sim}{x}) < \infty$ when $\underset{\sim}{X}$ is a discrete random vector with joint pmf $p_{\underset{\sim}{X}}$,

and

$\cancel{\int_{\mathbb{R}} \cdots \int_{\mathbb{R}} |h(\underset{\sim}{x})| \, dx_1 \dots dx_k < \infty \text{ when } X}$

$\int_{\mathbb{R}^k} |h(\underset{\sim}{x})| \, f_{\underset{\sim}{X}}(\underset{\sim}{x}) \, d\underset{\sim}{x}$ when $\underset{\sim}{X}$ is a cont random vector with a joint pdf $f_{\underset{\sim}{X}}$.

In the above situation,

$$E[h(\underset{\sim}{X})] = \begin{cases} \sum_{\underset{\sim}{x} \in \text{Range}(\underset{\sim}{X})} h(\underset{\sim}{x}) \, p_{\underset{\sim}{X}}(\underset{\sim}{x}) & \text{when } \underset{\sim}{X} \text{ is discrete,} \\ \int_{\mathbb{R}^k} h(\underset{\sim}{x}) f_{\underset{\sim}{X}}(\underset{\sim}{x}) \, d\underset{\sim}{x} & \text{when } \underset{\sim}{X} \text{ is cont.} \end{cases}$$

(e) . . . .