

Then, with $g_n(x) = \frac{1}{n}$, $x \in [\varepsilon, 2\pi - \varepsilon]$, we conclude by the (full) Dirichlet test, that

$$\sum_{n=1}^{\infty} \frac{1}{n} \cos nx \quad \text{converges uniformly on } [\varepsilon, 2\pi - \varepsilon].$$

$\forall 0 < \varepsilon < 2\pi.$

□

(eg: Let $p > 1$. Then, by M-test,

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \cos nx$$

converges uniformly to some fn $f(x)$ on all of \mathbb{R} .

\therefore the convergence is unif. on \mathbb{R} , ~~in~~ & since

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{n^p} \cos nx \quad (x \in \mathbb{R}),$$

by term-by-term integration:

$$\int_0^{\pi/2} f = \int_0^{\pi/2} \left(\sum_{n=1}^{\infty} \frac{1}{n^p} \cos nx \right) dx$$

$$= \sum_{n=1}^{\infty} \frac{1}{n^p} \int_0^{\pi/2} \cos nx \, dx.$$

$$= \sum_{n=1}^{\infty} \frac{1}{n^{p+1}} \sin \left(\frac{n\pi}{2} \right).$$

$$\Rightarrow \int_0^{\pi/2} \left(\sum_{n=1}^{\infty} \frac{1}{n^p} \cos nx \right) dx = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)^{p+1}}. \quad \forall p > 1.$$

Next, Consider $p \in (0, 1]$.

Then, as in the previous example, use the Dirichlet test to

Conclude that

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \cos nx \text{ converges unif. on } [\varepsilon, 2\pi - \varepsilon]$$

$\forall \varepsilon \in (0, 2\pi)$
 $\nexists p \in (0, 1]$.

□

Eg:

Let $0 < \varepsilon < 1$. We know:

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

uniformly on $[-\varepsilon, \varepsilon]$.

$\therefore \forall x \in [-\varepsilon, \varepsilon]$, we have:

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \text{ on } (-1, 1)$$

$$\int_0^x \left(\sum_{n=0}^{\infty} t^n \right) dt = \int_0^x \frac{1}{1-t} dt$$

$$\Rightarrow \sum_{n=0}^{\infty} \left(\int_0^x t^n dt \right) = -\log(1-x)$$

BTW: What is the defn. of log?

$$\Rightarrow -\log(1-x) = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}$$

$$\Rightarrow \log(1-x) = - \sum_{n=1}^{\infty} \frac{x^n}{n} \quad \forall x \in [-\varepsilon, \varepsilon]$$

Thus: $\log(1-x) = - \sum_{n=1}^{\infty} \frac{x^n}{n} \quad \forall x \in (-1, 1)$

\nexists the convergency is unif. on $[-\varepsilon, \varepsilon]$
 $\forall 0 < \varepsilon < 1$.

eg:

Note that:

$$\sum_{n=0}^{\infty} (-1)^n x^{2n} = \frac{1}{1+x^2} \quad \forall x \in (-1, 1).$$

uniformly on all compact subsets of $(-1, 1)$.

$\therefore \forall x \in (-1, 1)$, we have:

why? \rightarrow

$$\begin{aligned} \int_0^x \frac{1}{1+t^2} dt &= \int_0^x \left(\sum_{n=0}^{\infty} (-1)^n t^{2n} \right) dt \\ &= \sum_{n=0}^{\infty} (-1)^n \int_0^x t^{2n} dt \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \end{aligned}$$

$$\Rightarrow \tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}$$

$\forall x \in (-1, 1)$
 $\&$ unif. on $[-\varepsilon, \varepsilon]$
 $\forall 0 < \varepsilon < 1$.

□

In particular: if $x = \frac{1}{2}$ & $x = \frac{1}{3}$, then

$$\tan^{-1} \frac{1}{2} + \tan^{-1} \frac{1}{3} = \sum_{n=0}^{\infty} \left(\frac{1}{2^{2n+1}} + \frac{1}{3^{2n+1}} \right) x \frac{(-1)^n}{2n+1}$$

$$\tan^{-1} \left(\frac{\frac{1}{2} + \frac{1}{3}}{1 - \frac{1}{2} \cdot \frac{1}{3}} \right) = \tan^{-1} 1 = \frac{\pi}{4}$$

$$\Rightarrow \pi = 4 \times \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \left(\frac{1}{2^{2n+1}} + \frac{1}{3^{2n+1}} \right)$$

□

eg: (Derivatives).

Consider the series:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \cos\left(\frac{x}{n}\right).$$

$$x=0 \Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$$

Altering series.
It converges.

$$\text{Also, } \frac{d}{dx} \left(\frac{(-1)^{n-1}}{n} \cos\left(\frac{x}{n}\right) \right) = \frac{(-1)^{n-1}}{n^2} \sin\left(\frac{x}{n}\right).$$

Now by M-test,

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} \sin\left(\frac{x}{n}\right)$$

Converges unif. on \mathbb{R} .

$$\therefore f(x) := \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \cos\left(\frac{x}{n}\right)$$

defines a diff. f_n on \mathbb{R} . & the above series

converges unif. on \mathbb{R} . Moreover:

$$f'(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} \sin\left(\frac{x}{n}\right) \quad \forall x \in \mathbb{R}.$$

Also, u.c.

Sequences of improper integrals:

Let $\{f_n\} \subseteq \mathbb{R}[a, \infty)$

& let $f_n \rightarrow f$ or $\sum f_n = f$

pointwise or uniformly.

$$\lim_{n \rightarrow \infty} \int_a^{\infty} f_n = \int_a^{\infty} f \quad \text{or} \quad \sum \int_a^{\infty} f_n = \int_a^{\infty} f \quad ??$$

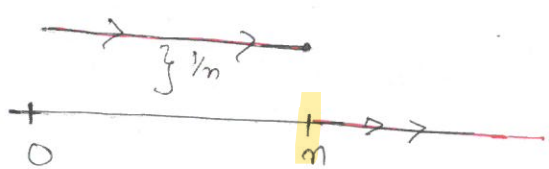
Of course, we need the assurance of $f \in \mathcal{R}[a, \infty)$!!

Note: perhaps with pointwise convergence, there is no/less hope.
What about u.c. ??

↓ Again, no hope.

eg:

$\forall n \in \mathbb{N}$, define $f_n : [0, \infty) \rightarrow \mathbb{R}$ by:



$$\text{i.e. } f_n(x) = \begin{cases} \frac{1}{n} & 0 \leq x \leq n \\ 0 & x > n. \end{cases}$$

Clearly, $f_n \in \mathcal{R}[0, \infty)$ & $\int_0^\infty f_n = 1 \quad \forall n$.

Also, $\lim_{n \rightarrow \infty} f_n(x) = 0 \quad \forall x \in [0, \infty)$ & uniformly.

$\therefore f_n \rightarrow 0$ unif. on $[0, \infty)$.

$$\therefore \lim_{n \rightarrow \infty} \int_0^\infty f_n = 1 \neq 0 = \int_0^\infty \lim_{n \rightarrow \infty} f_n. \quad \left[\because \|f_n\| \leq \frac{1}{n} \right]$$

Observe that $\nexists g \in \mathcal{R}[0, \infty)$ s.t.

$$f_n(x) \leq g(x) \quad \forall x \in [0, \infty) \quad \forall n \in \mathbb{N}.$$

And this lack of dominance is the key!

For instance (a baby version of dominated convergence thm). (DCT)

Thm (DCT). \leftarrow Very useful theorem !!

Let $\{f_n\} \subseteq C[a, \infty)$ s.t. $f_n \rightarrow f$ unif. on $[a, b]$
 $\forall b > a$.

Let $g \in C[a, \infty)$ ~~$\cap R[a, \infty)$~~ s.t. $\int_a^\infty g$ exists &
 $|f_n(x)| \leq g(x) \quad \forall x \in [a, \infty), n \geq 1$.

$$\text{Then } \lim_{n \rightarrow \infty} \int_a^\infty f_n = \int_a^\infty f.$$

$$\boxed{\begin{array}{l} f(x) = x \\ \int_a^\infty x \text{ DNE} \end{array}}$$

Proof: HW.

§ Power Series.

Def: Let $c \in \mathbb{R}$ & $\{a_n\}_{n \geq 0} \subseteq \mathbb{R}$. The formal sum

$$\sum_{n=0}^{\infty} a_n (x-c)^n \quad \text{--- (P)}$$

is called a power series about c (or center c) ^{with}

with coefficients $\{a_n\}$.

eg: Polynomials are power series & admit (P) for any $c \in \mathbb{R}$! Q: Given a polynomial, how you write it as (P)?

Note 8

(51)
Jaydeb Sarkar

- (1) Setting $f_n(x) = a_n(x-c)^n$, we realize that the P.S. (P) is perhaps ^{the} simplest series of f_n 's $\sum_{n=0}^{\infty} f_n(x)$.
- (2) Our interest: Convergence (pointwise/absolute/uniform) of the series (P) & then determine the limit f_n .
- (3) Adopt all ^{the} ideas of series of f_n 's.
- (4) If $S \subseteq \mathbb{R}$ s.t. (P) converges $\forall x \in S$, then $\exists f: S \rightarrow \mathbb{R}$ s.t. $f(x) = \sum_{n=0}^{\infty} a_n(x-c)^n \quad \forall x \in S$.

Clearly, $S \neq \emptyset$ as $c \in S$!

- (5) eg: $\sum_{n=0}^{\infty} x^n$ is a P.S. ^{with} center at 0 & coefficients $\{1\}$.

We know: $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$; $x \in [0, 1)$
 $\quad \quad \quad \downarrow$ $\quad \quad \quad \downarrow$
 $\quad \quad \quad f(x)$ $\quad \quad \quad S$
 $\quad \quad \quad \downarrow$
 $\quad \quad \quad$ Defined only on $(-1, 1)$ $\quad \quad \quad$ Defined on $\mathbb{R} \setminus \{1\}$

Geometric series.

$\therefore \{P.S.\} \longleftrightarrow \{\text{Functions}\}$

We need to understand this link (if any).
& Corresponding domain of definitions!!

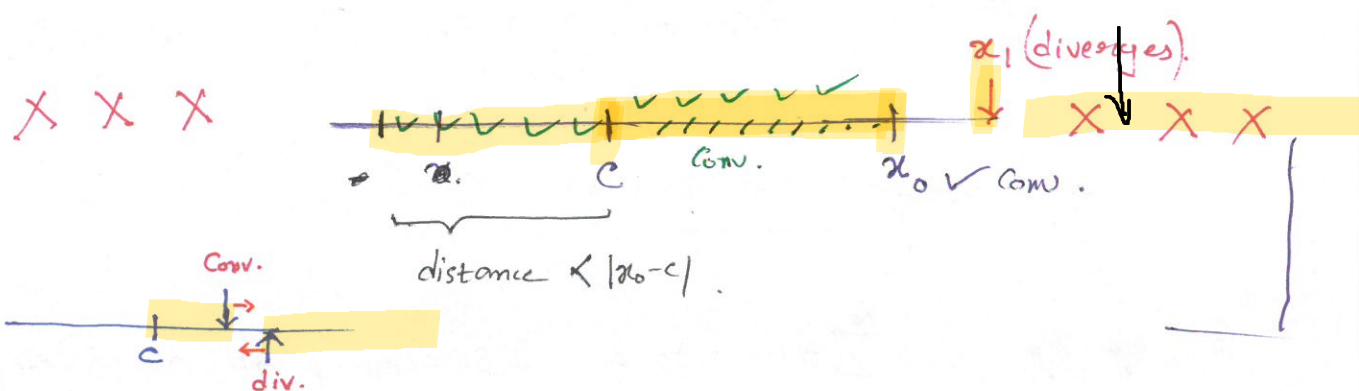
The following is remarkable:

Thm: Consider the P.S. $\sum_{n=0}^{\infty} a_n(x-c)^n$ — (P).

(1) If (P) converges for some $x_0 \in \mathbb{R}, \{x_0\}$, then it converges absolutely for all x s.t. $|x-c| < |x_0-c|$.

(2) If (P) diverges at $x_1 \in \mathbb{R}$, then it diverges $\forall x \in \mathbb{R}$ s.t. $|x-c| > |x_1-c|$.

\therefore We have the following picture:



Proof: Let (P) converges at $x = x_0$. (Also, $x_0 \neq c$).

$$\Rightarrow \sum_{n=0}^{\infty} a_n(x_0 - c)^n \text{ Converges.}$$

← A series of real no's.

$$\Rightarrow a_n(x_0 - c)^n \rightarrow 0 \text{ as } n \rightarrow \infty.$$

\therefore For $\varepsilon = \frac{1}{2}$, $\exists N \in \mathbb{N}$ s.t.

$$|a_n(x_0 - c)^n| < \frac{1}{2} \quad \forall n \geq N.$$

Let $x \in \mathbb{R}$ & suppose $|x-c| < |x_0-c|$.

$$\therefore |a_n(x-c)^n| = |a_n(x_0-c)^n| \times \left| \frac{x-c}{x_0-c} \right|^n$$

$$\forall n \geq N \quad \frac{1}{2} \times \left| \frac{x-c}{x_0-c} \right|^n$$

But $|x-c| < |x_0-c| \Rightarrow \left| \frac{x-c}{x_0-c} \right| =: r$ for some $r \in (0,1)$.

$$\therefore |a_n(x-c)^n| < \frac{1}{2} r^n \quad \forall n \geq N$$

\therefore By Comparison test, $\sum_{n=0}^{\infty} |a_n(x-c)^n|$ converges.

i.e. (P) is A.C. $\forall x \in \mathbb{R}$ s.t. $|x-c| < |x-x_0|$. ✓

For (2): Let $x_1 \in \mathbb{R}$ s.t. $|x_1-c| > |x_0-c|$.

If (P) converges at x , then by (1), (P) will

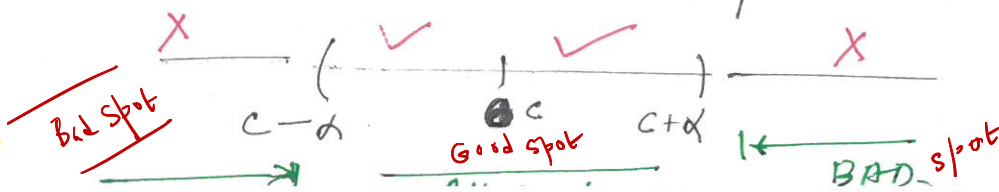
converge at x_1 $\rightarrow \leftarrow$.

□

The above result is curious: We need to find the

"maximum" of x_0 s.t. $\sum a_n(x_0-c)^n$ converges &
"min. of x_1 s.t. $\sum a_n(x_1-c)^n$ diverges. ✓

If there is such a min-max, say α , then



Q: How to find/determine that "min-max"?

Recall: (Root test)

Let $\sum_{n=0}^{\infty} a_n$ be a series of the nos.

Set $\rho = \limsup \sqrt[n]{|a_n|}$

possibly $\rho = +\infty$

- (1) If $\rho < 1$, then the series converges.
- (2) If $\rho > 1$, then the series diverges.
- (3) If $\rho = 1$, the test is inconclusive.

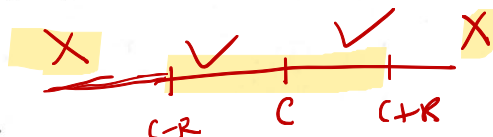
CONVENTION: $\frac{1}{\infty} = 0$; $\frac{1}{0} = \infty$.

HW

Thm: (Cauchy - Hadamard Thm).

Consider the P.S. $\sum_{n=0}^{\infty} a_n(x-c)^n$. & Set

$$\frac{1}{R} = \limsup \sqrt[n]{|a_n|}$$



Then the ~~the~~ (P.S.) converges $\forall x \in \mathbb{R}$ s.t. $|x-c| < R$.

& diverges $\forall x \in \mathbb{R}$ s.t. $|x-c| > R$.

Proof:

Observe that $\forall x \in \mathbb{R}$,

$$\limsup |a_n(x-c)^n|^{1/n} = \frac{|x-c|}{R}$$

\therefore The result follows from the Root test.

□