29/09/2020, Section 1.

error analysis

round-off error (1)

- truncation error
- discretization error

$A1 \longrightarrow$ input/data error

$A2 \longrightarrow$ algo/ accuracy

$$x = \frac{1}{2}(a_0 10^0 + a_{-1} 10^{-1} + a_{-2} 10^{-2} + \dots)$$

$$a_{-k} \in \{0,1\}$$

$x = 1.25$ $= 1 \times 2^0 + 0 \times 2^{-1} + 0 \times 2^{-2} + 1 \times 2^{-3} + 0 \times 2^{-4}$
$+ 1 \times 2^{-5} + \dots + 1 \times 2^{-11}$

$\boxed{x = 1.0 \cdots 1}$

- Non-unique: $0.999 \cdots = 1$

finite rep.!

round length 'n'

$n = a_1 + a_{-1} \longrightarrow$ fixed point representation

Reading - point representation

$$x = a \times 10^b \qquad x = \tilde{a} \times 10^{\tilde{b}}$$
$$|a| < 1 \qquad \tilde{a} < 1$$
$$b \in \mathbb{Z} \qquad \tilde{b} \in \mathbb{Z}$$

$0.3014 = 0.3014 \times 10^0$ $\quad a = 0.3014$
$\quad b = 0$

$n = 4$

$n = \tilde{b} + 3$ $\quad x = 5.6200$
$\quad \tilde{b} = 1$ $\quad a = 0.56200 \times 10^1$ $\quad b = 1$
$\quad \tilde{a} = 5$

$\boxed{5.6200}$ $\boxed{0.5620}$

Normalize :-

$0.0\underline{3}0140 \times 10^{-1}$ $\quad |a| < 1 \quad b \in \mathbb{Z}$

$0.1 \le |a| < 1$, $\quad 0.560 \times 10^{-1}$

$0.5 \le |a| < 1$, $\quad a \times 10^b$

$|a| < 1 \quad b \in \mathbb{Z}$

$\boxed{n^{-1} \le |a| < 1}$

$x = a \times 10^b \quad \boxed{b \in \mathbb{Z}}$

$\boxed{100}$ $\boxed{3b = 5}$ $\quad \boxed{a \le 1}$

Normalized $a \in \mathbb{R}$.

$x = a \times 10^b$

$n^{-1} \le |a| < 1, \quad b \in \mathbb{Z}$

$x = 10$

$a = 0.a_{-1} a_{-2} a_{-3} a_{-4} \cdots$
$\times 10^1$
$a_{-k} \in \{0,1\}, k \ge 2$
$a_{-1} = \{1\}$

Min. $\quad 0.a_{-1} a_{-2} a_{-3} a_{-4} \cdots$

$x = a \times 10^b$

$a = 0.a_{-1} a_{-2} \cdots a_{-k} \cdots$

$0 \le |a| < 1 \qquad a_{-k} = \{0,1\}$
$k \ge 2$
$a_{-1} = 1$

$x = 6 \qquad x = a \times 10^b$

$0.5 \le |a| < 1, \quad b \in \mathbb{Z}$

$a = 0.a_{-1} a_{-2} \cdots a_{-k} \cdots$

$a_{-1} \in \{1, 1.5, 2, 3\}$

$a_{-k} \in \{0, 1.5, 2, 3\}$

$100 \rightarrow 0.5 \times 10^3 \quad b \in \mathbb{Z}$

$\Leftrightarrow x \times 10^3$

0.1000×10^3

$(3,1,5)$

Rounding off : $(3,1,5)$. $\quad A \subset \mathbb{R}$.

Number system

Approximate $\quad x \in \mathbb{R} - A$

$\exists \tilde{x} \in A$

$|x - x_{rd}| \le |x - y| \ \forall y \in A$

$A^{(3,1,5)}$

$x = a \times 10^b \quad 0.5 \le |a| < 1, b \in \mathbb{Z}$

$|a| = 0.a_{-1} a_{-2} \cdots a_{-k} \cdots \quad a_{-k} \in$

round off

$\begin{cases} \text{if } 0 \le a_{-k+1} < 5, & a_{-k} = 0. \\ \quad \tilde{a} = 0.a_{-1} a_{-2} \\ \text{if } 5 \le a_{-k+1} \le 9, & \tilde{a} = 0.a_{-1} a_{-2} + 10^{-k} \end{cases}$

chopping

$x_{rd} = \text{sgn}(a) \cdot \tilde{a} \times 10^b$

$\text{sgn}(a) = +, \text{if } x \ge 0$
$\quad -, \text{if } x < 0$

$\left| \frac{x_{rd} - x}{x} \right| \le \frac{5 \times 10^{-(k+1)}}{|a|} \le \frac{5 \times 10^{-k}}{|a|} \ge 0.1$

$\epsilon_{mr} = 5 \times 10^{-k}$

$\Rightarrow |x_{rd} - x| \le |x| \, \epsilon_{mr}$

$\Rightarrow x_{rd} = x(1+\epsilon), \quad |\epsilon| \le \epsilon_{mr}$

$\boxed{x_{rd} = x(1+\epsilon)}$, $\quad |\epsilon| \le \epsilon_{mr} \ (= 5 \times 10^{-k})$

$\underbrace{\text{round off}}_{\text{error}}$ $\quad (3,1,5) \rightarrow$ algorithm $\rightarrow$ error analysis

$x_1 (3,1,5)$ $\longrightarrow A$ $\quad x \in A$

Aim: $x_{rd} \in A$

s.t $|x - x_{rd}| \le |x - y| \ \forall y \in A$

$x = a \times 10^b$

$|a| = 0.a_{-1} \cdots a_{-k} a_{-k+1} \cdots$

$a_{-k+1} < 5 \quad \tilde{a} = 0.a_{-1} \cdots a_{-k}$

$a_{-k+1} \ge 5 \quad \tilde{a} = 0.a_{-1} \cdots a_{-k} + 10^{-k}$

$x_{rd} = \text{sgn}(x) \cdot \tilde{x} \cdot 10^b$

$\tilde{x}_{rd} = 10^{-k}$

$x_{rd} = x(1+\epsilon), \quad |\epsilon| \le \tilde{\epsilon}_{mr}$

✓ If $x_{rd} \in A$, $x \xrightarrow{\text{round off}} x_{rd}$ then

$|x - x_{rd}| \le |x - y| \ \forall y \in A$

- $\boxed{x_{rd} \notin A}$

- algorithms $\rightarrow$ error