Lecture 19/03/2022

Virtual abscissa

$$\underbrace{x_0 = x_0 = \dots = x_0}_{n_0} = \dots = \underbrace{x_m = x_m = \dots = x_m}_{n_m}$$

$$[f^{(0)}(x_0), \dots, f^{(n_0-1)}(x_0)] \quad \dots \quad f^{(n_m-1)}(x_m)$$

$$n = \left[\sum_{i=0}^{m} n_i\right] - 1$$

$$(0,0), \dots, (0,k), \dots, (m,0), \dots, (m,n_m-1)$$

$$x_0 \quad f^{(0)}(x_0) \quad x_0 \quad f^{(k)}(x_0) \quad x_m \quad f^{(0)}(x_m)$$

$0 \le i \le m$

$\forall i, 0 \le k \le n_i - 1$

$0 = t_0 \le t_1 \le t_2 \dots \le t_n = x_m$

$$P^{(j-1)}(t_j) = f^{(j-1)}(t_j)$$

$$S_j \triangleq \{t_0 \le t_1 \le \dots \le t_j\} = S \quad [0 \le j \le n]$$

(i) subsequence S_j contains all 'virtual' abscissae $[t_0, t_j]$ begins from t_0

(ii) Concatenate t_j value, x_k on $[0 \le k \le m]$

(iii) S_j = 'multiplicity' of x_k in S.

Interp. Poly. satisfies

$$P_{0 \dots n}^{(s_j-1)} \bullet (x = t_j) = f^{(s_j-1)}(t_j)$$

$$1 \le s_j \le n_k$$

$$\{s_j = 1 \ \forall j\} \qquad \boxed{0 \le j \le n}$$

Matrix - vector formulation

$$\pi^*(t) = \begin{bmatrix} 1 \\ t \\ \vdots \\ t^n/n! \end{bmatrix} \qquad P(t) \in \Pi_n \qquad P(t) = \sum_{j=0}^{n} b_j \frac{t^j}{j!}$$

$$[\pi^*(t)]^T \equiv [\pi(t)]^* = \left[1, t, \frac{t^2}{2!}, \dots, \frac{t^n}{n!}\right]$$

$$\pi(t) b = \left[1, t, t^2/2!, \dots, \frac{t^n}{n!}\right] \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_n \end{bmatrix}$$

$$\Rightarrow \pi^{(s_j-1)}(t = t_j) b$$

$$= f^{(s_j-1)}(t_j) \quad \dots (1)$$

$\therefore P(t)$ is unique, the $P.$ is uniquely solvable

Generally

- Take any non-decreasing sequence $t_0 \le t_1 \le \dots \le t_n$

$$\begin{bmatrix} \pi^{(s_0-1)}(t_0) \\ \pi^{(s_1-1)}(t_1) \\ \vdots \\ \pi^{(s_n-1)}(t_n) \end{bmatrix} \begin{matrix} \to [1 \times (n+1)] \\ \to [1 \times (n+1)] \\ \\ \to [1 \times (n+1)] \end{matrix} \equiv V_n(t_i)$$

$(n+1) \times (n+1)$ matrix

Generalized Vandermonde matrix.

Interpolation

$$V_n b = f$$

$$Ax = b$$

$$\boxed{b = V_n^{-1} f}$$

$\{f_i\}$ satisfies $P(t)$.

Vandermonde matrix :

$s_j = 1 \ \forall j$

$$V_n = \begin{bmatrix} 1 & t_0 & t_0^2 & \dots & t_0^n \\ 1 & t_1 & t_1^2 & \dots & t_1^n \\ \vdots & & & & \\ 1 & t_n & \dots & & t_n^n \end{bmatrix} \begin{bmatrix} 0! & & & \\ & 1! & & 0 \\ & & 2! & \\ 0 & & & \ddots \ n! \end{bmatrix}^{-1}$$

$\cancel{t_0 = t_1 < t}$ $\quad (t_0 = t_1 = t_2) \times (t_0 = t_4)$

$$f^{(0)}(x_0) = f^{(0)}(t_0)$$

$t_0 = t_1 = t_2 = x_0$

$t_3 = t_4 = x_1$

$f^{(0)}(x_0), \ f^{(1)}(x_0), \ f^{(2)}(x_0)$

$f^{(0)}(x_1), \ f^{(1)}(x_1)$

$$\left.\begin{aligned} P(x_0) &= f^{(0)}(x_0) \\ P^{(1)}(x_0) &= f^{(1)}(x_0) \\ P^{(2)}(x_0) &= f^{(2)}(x_0) \\ P^{(0)}(x_1) &= f^{(0)}(x_1) \\ P^{(1)}(x_1) &= f^{(1)}(x_1) \end{aligned}\right\}$$

$$P(t) = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!}$$

$$\pi(t_0) = f^{(0)}(x_0)$$

$$\Rightarrow \pi^{(0)}(t_0) = \left[1, t_0, \frac{t_0^2}{2!}, \frac{t_0^3}{3!}, \frac{t_0^4}{4!}\right]$$

$$P^{(1)}(t) = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!}$$

$$\pi^{(1)}(t_1) = \left[0, 1, t_0, \frac{t_1^2}{2!}, \frac{t_1^3}{3!}\right]$$

$$\begin{bmatrix} P^{t_0} \\ \vdots \\ P^{t_4} \end{bmatrix}$$