LINEAR ALGEBRA -II

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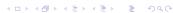
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A linearly independent collection of vectors $\{v_1, v_2, \dots, v_n\}$ is a basis for V if

$$V = span\{v_1, v_2, \dots, v_n\},\$$

▶ that is, given any vector $x \in V$, there exist, c_1, c_2, \ldots, c_n in \mathbb{F} such that $x = c_1v_1 + c_2v_2 + \cdots + c_nv_n$. Note that given x, these coefficients are uniquely determined due to linear independence of v_i 's.



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Proof. We have

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▶ as cross-terms are equal to zero due to orthogonality. ♦ ३०००

Proposition 9.3: Suppose $\{v_1, v_2, \ldots, v_m\}$ is an orthogonal collection of non-zero vectors in an inner product space $(V, \langle \cdot, \cdot \rangle)$. Then the collection $\{v_1, v_2, \ldots, v_n\}$ is linearly independent.

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- For any j, $1 \le j \le m$, taking inner product with v_j , as $\langle v_j, v_i \rangle = \delta_{ij} \langle v_j, v_j \rangle$, we get

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▶ Proof. This is clear, as the dimension of V is same as the maximum possible size of linearly independent sets. ■



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$$\langle v_i, v_j \rangle = \left\{ \begin{array}{ll} 1 & \text{if } i = j; \\ 0 & \text{if } i \neq j. \end{array} \right.$$

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- In other words, an orthonormal basis is a basis consisting of mutually orthogonal unit vectors.
- **Example** 9.6: For \mathbb{R}^n (or \mathbb{C}^n) the standard basis $\{e_1, e_2, \dots, e_n\}$, where e_i is the vector whose j-th coordinate is one and all other coordinates are equal to zero, is an orthonormal basis with respect to the standard inner product.

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- ▶ Theorem 9.7: Let $\{v_1, v_2, \dots, v_n\}$ be an orthonormal basis of an inner product space $(V, \langle \cdot, \cdot \rangle)$. Then for any vector $w \in V$,

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$$w=\sum_{j=1}^n\langle v_j,w\rangle v_j.$$

▶ Proof. As $\{v_1, v_2, \dots, v_n\}$ is a basis for V, $w = \sum_{i=1}^n c_i v_i$ for some c_1, c_2, \dots, c_n in \mathbb{F} .

Now for any *j*, using linearity of the inner product in second variable,

$$\langle v_j, w \rangle = \langle v_j, \sum_{i=1}^n c_i v_i \rangle$$

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▶ As this is true for every j, $w = \sum_{i=1}^{n} \langle v_j, w \rangle v_j$. ■

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- ▶ In particular, $||u_1|| \neq 0$.
- ▶ Take $y_1 = u_1$ and $v_1 = \frac{u_1}{\|u_1\|}$.

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- ▶ Now take, $v_2 = \frac{y_2}{\|y_2\|}$.
- ▶ This way, $\{v_1, v_2\}$ are orthonormal (that is, they have norm one and are mutually orthogonal.)

▶ Inductively, after we construct $\{v_1, v_2, \dots, v_k\}$ such that they are orthonormal and

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Now, for any $1 \le i \le k$,

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 $y_{k+1} \neq 0$ follows as $u_{k+1} \notin \text{span} \{v_1, \dots, v_k\} = \text{span}\{u_1, \dots, u_k\}.$

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- ► Then take $v_{k+1} = \frac{y_{k+1}}{\|y_{k+1}\|}$.
- We see that $\{v_1, \ldots, v_{k+1}\}$ are orthonormal and span $\{v_1, \ldots, v_{k+1}\} = \{u_1, \ldots, u_{k+1}\}$ so that the induction can be continued.



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$$v_{k+1} = \frac{u_{k+1} - \sum_{j=1}^{k} \langle v_j, u_{k+1} \rangle v_j}{\|u_{k+1} - \sum_{j=1}^{k} \langle v_j, u_{k+1} \rangle v_j\|}$$

is worth remembering.

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Exercise: Obtain an orthonormal basis for \mathbb{R}^3 by Gram-Schmidt orthogonalization applied to the basis:

$$\left(\begin{array}{c}1\\1\\0\end{array}\right),\quad \left(\begin{array}{c}0\\1\\1\end{array}\right),\quad \left(\begin{array}{c}1\\0\\1\end{array}\right)$$

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END OF LECTURE 9.

