Cor: If $X, Y \overset{iid}{\sim} N(0,1)$, then ~~$X^2+Y^2 \sim Exp(\frac{1}{2})$~~

$X^2+Y^2 \sim Exp(\frac{1}{2})$.

Proof: $X, Y \overset{iid}{\sim} N(0,1)$ + Fact in Pg 71

$\Rightarrow X^2, Y^2 \overset{iid}{\sim} \text{Gamma}(\frac{1}{2}, \frac{1}{2})$

(Here we are also using the following fact
$X \perp\!\!\!\perp Y \Rightarrow X^2 \perp\!\!\!\perp Y^2$)

$\Rightarrow X^2+Y^2 \sim \text{Gamma}(1, \frac{1}{2}) \equiv Exp(\frac{1}{2})$
by the propⁿ in Pg 93.

Exc: If $X, Y \overset{iid}{\sim} N(0,1)$, then find the distⁿ of $R = +\sqrt{X^2+Y^2}$. (The distribution of R is called Rayleigh Distribution.)

Exc: If $X, Y \overset{iid}{\sim} N(0,1)$, then using the convolution formula, find the distⁿ of $X+Y$. (In this case, $X+Y \sim N(0,2)$.)

Exc: If $U, V \overset{iid}{\sim} N(0,1)$, then using the convolution formula (i.e., Method 2), find the distⁿ of $Z = U+V$.

A Quick Digression: k-dimensional random vectors and independence

Suppose, $k \geq 2$ and $X_1, X_2, \ldots, X_k$ are k r.v.s defined on the same sample space Ω.

This means that each $X_i : \Omega \to \mathbb{R}$ is a ~~r.v.~~ function. Combining these k functions, we get a function $\Omega \to \mathbb{R}^k$ defined by

$$\omega \longmapsto (X_1(\omega), X_2(\omega), \ldots, X_k(\omega)).$$

$(X_1, X_2, \ldots, X_k) \doteqdot \underset{\sim}{X}$ is called a k-dimensional random vector or jointly distributed k r.v.s.

Defn: For a (k-dimensional) random vector $\underset{\sim}{X} := (X_1, X_2, \ldots, X_k)$, the joint cdf or joint distn function is defined as

$$F_{\underset{\sim}{X}}(\underset{\sim}{x}) = F_{X_1, X_2, \ldots, X_k}(x_1, x_2, \ldots, x_k)$$

$$\doteqdot P(X_1 \leq x_1, X_2 \leq x_2, \ldots, X_k \leq x_k)$$

for all $\underset{\sim}{x} := (x_1, x_2, \ldots, x_k) \in \mathbb{R}^k$.

As before, marginal cdfs can be computed by taking limit as the other variables tend to infinity. For example (when $k=4$),

$$F_{X_1,X_2,X_3}(x_1,x_2,x_3) = \underset{x_4\to\infty}{Lt}\ F_{X_1,X_2,X_3,X_4}(x_1,x_2,x_3,x_4),$$

$$F_{X_1,X_3}(x_1,x_3) = \underset{\substack{x_2\to\infty\\ x_4\to\infty}}{Lt}\ F_{X_1,X_2,X_3,X_4}(x_1,x_2,x_3,x_4),$$

$$F_{X_3}(x_3) = \underset{\substack{x_1\to\infty\\ x_2\to\infty\\ x_4\to\infty}}{Lt}\ F_{X_1,X_2,X_3,X_4}(x_1,x_2,x_3,x_4),$$

etc.

If each of $X_1, X_2, \ldots, X_k$ is a discrete r.v., then we say that $\underset{\sim}{X} = (X_1, X_2, \ldots, X_k)$ is a discrete random vector. For such a random vector, the joint pmf is defined by

$$p_{\underset{\sim}{X}}(\underset{\sim}{x}) = p_{X_1,X_2,\ldots,X_k}(x_1,x_2,\ldots,x_k)$$
$$= P(X_1=x_1, X_2=x_2, \ldots, X_k=x_k)$$

for all $\underset{\sim}{x} = (x_1, x_2, \dots, x_k) \in \mathbb{R}^k$. Clearly the joint range of $X_1, X_2, \dots, X_k$ is

$$\begin{aligned}\text{Range}(\underset{\sim}{X}) &= \text{Range}(X_1, X_2, \dots, X_k) \\ &= \{\underset{\sim}{x} \in \mathbb{R}^k : p_{\underset{\sim}{X}}(\underset{\sim}{x}) > 0\} \\ &\subseteq \text{Range}(X_1) \times \text{Range}(X_2) \times \cdots \times \text{Range}(X_k).\end{aligned}$$

As in the bivariate case, for any $B \subseteq \mathbb{R}^k$,

$$P(\underset{\sim}{X} \in B) = \sum_{\underset{\sim}{x} \in B \cap \text{Range}(\underset{\sim}{X})} p_{\underset{\sim}{X}}(\underset{\sim}{x}).$$

In particular, marginal pmfs can be computed by "summing out" the other variables. For example (when $k=4$),

$$p_{X_1,X_2,X_3}(x_1, x_2, x_3) = \sum_{x_4 \in \text{Range}(X_4)} p_{X_1,X_2,X_3,X_4}(x_1, x_2, x_3, x_4),$$

$$p_{X_1,X_4}(x_1, x_4) = \sum_{(x_2,x_3) \in \text{Range}(X_2,X_3)} p_{X_1,X_2,X_3,X_4}(x_1, x_2, x_3, x_4),$$

$$p_{X_2}(x_2) = \sum_{(x_1, x_3, x_4) \in \text{Range}(X_1, X_3, X_4)} p_{X_1, X_2, X_3, X_4}(x_1, x_2, x_3, x_4),$$

etc.

For a discussion on k-dimensional continuous random vectors, we shall need multiple integrals, which are higher dimensional versions of double integrals. Again, we shall think of them as repeated integrals – each time integrating w.r.t. one variable treating all other variables as constants. For ~~nonnegative functions~~ functions taking nonnegative values, the order of the integration won't matter.

<u>Defn:</u> A (k-dimensional) random vector $\underset{\sim}{X} = (X_1, X_2, \ldots, X_k)$ is called (absolutely) <u>continuous</u> if $\exists$ a function

$$f_{\underset{\sim}{X}} = f_{X_1, X_2, \ldots, X_k} : \mathbb{R}^k \longrightarrow [0, \infty) \quad \text{such that}$$

$$\forall \ \underset{\sim}{u} = (u_1, u_2, \ldots, u_k) \in \mathbb{R}^k,$$

$$F_{\underset{\sim}{X}}(\underset{\sim}{u}) = F_{X_1, X_2, \ldots, X_k}(u_1, u_2, \ldots, u_k)$$

$$= P(X_1 \leq u_1, X_2 \leq u_2, \ldots, X_k \leq u_k)$$

$$= \int_{-\infty}^{u_k} \int_{-\infty}^{u_{k-1}} \cdots \int_{-\infty}^{u_1} f_{X_1, X_2, \ldots, X_k}(x_1, x_2, \ldots, x_k)\, dx_1\, dx_2 \cdots dx_k$$

$$= \int_{-\infty}^{u_k} \int_{-\infty}^{u_{k-1}} \cdots \int_{-\infty}^{u_1} f_{\underset{\sim}{X}}(\underset{\sim}{x})\, d\underset{\sim}{x} \, .$$

In this case, $f_{\underset{\sim}{X}} = f_{X_1, X_2, \ldots, X_k}$ is called a joint pdf or joint density function of $\underset{\sim}{X} = (X_1, X_2, \ldots, X_k)$. We also say that $X_1, X_2, \ldots, X_k$ are jointly (absolutely) continuous.

It can be shown that for all "nice" $B \subseteq \mathbb{R}^k$,

$$(*) \ldots \; P(\underset{\sim}{X} \in B) = \int\!\!\int_B \cdots \int f_{\underset{\sim}{X}}(\underset{\sim}{x})\, d\underset{\sim}{x} \, , \text{ i.e.,}$$

$$(*) \ldots \; P((X_1, X_2, \ldots, X_k) \in B)$$
$$= \int\!\!\int_B \cdots \int f_{X_1, X_2, \ldots, X_k}(x_1, x_2, \ldots, x_k)\, dx_1\, dx_2 \ldots dx_k .$$

In particular, marginal pdfs can be computed by "integrating out" the other variables. For example (when $k=4$),

$$f_{X_1,X_3,X_4}(x_1,x_3,x_4) = \int_{-\infty}^{\infty} f_{X_1,X_2,X_3,X_4}(x_1,x_2,x_3,x_4)\,dx_2,$$

$$f_{X_1,X_2}(x_1,x_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X_1,X_2,X_3,X_4}(x_1,x_2,x_3,x_4)\,dx_3\,dx_4,$$

$$f_{X_4}(x_4) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X_1,X_2,X_3,X_4}(x_1,x_2,x_3,x_4)\,dx_1\,dx_2\,dx_3,$$

etc.

The interpretation of joint density function is as given in Pg 29 for the bivariate case: whenever $\underset{\sim}{x} \in \mathbb{R}^k$ is a continuity point of $f_{\underset{\sim}{X}}$, we have

$$\lim_{\substack{\Delta \underset{\sim}{x} \to \underset{\sim}{0}^+ \\ \| \\ (\Delta x_1, \dots, \Delta x_k)}} \frac{P[\underset{\sim}{X} \in (\underset{\sim}{x}, \underset{\sim}{x} + \Delta \underset{\sim}{x}]]}{|\Delta \underset{\sim}{x}|} = f_{\underset{\sim}{X}}(\underset{\sim}{x}),$$

where $|\Delta \underset{\sim}{x}| = \Delta x_1 \Delta x_2 \cdots \Delta x_k$ = k-dimensional volume of $(\underset{\sim}{x}, \underset{\sim}{x} + \Delta \underset{\sim}{x}]$.