5. Joint pdf is not unique as shown by the following exercise.

Exc: Consider a cont random vector (X, Y) with a joint pdf

$$f(x,y) = \begin{cases} 1 & \text{if } 0<x<1,\ 0<y<1, \\ 0 & \text{o.w.} \end{cases}$$

Compute the joint cdf of (X, Y) and show that

$$g(x,y) = \begin{cases} 1 & \text{if } 0 \leq x \leq 1,\ 0 \leq y \leq 1 \\ 0 & \text{o.w.} \end{cases}$$

is also a joint pdf of X and Y.

Note that in the above exercise, two joint pdfs of the same random vector (X, Y) differ at uncountably many points.

6. We shall compute various joint probabilities for X and Y with the help of $(\ast)$, i.e., $\forall$ "nice" $B \subseteq \mathbb{R}^2$

$$(\ast) \quad P[(X,Y) \in B] = \iint_B f_{X,Y}(x,y)\,dx\,dy.$$

Example: Suppose (X, Y) is a continuous random vector with joint pdf

$$f_{X,Y}(x,y) = \begin{cases} c\,(x+y)^2 & \text{if } x>0,\ y>0,\ x+y<1, \\ 0 & \text{o.w.}, \end{cases}$$

where c is a constant.

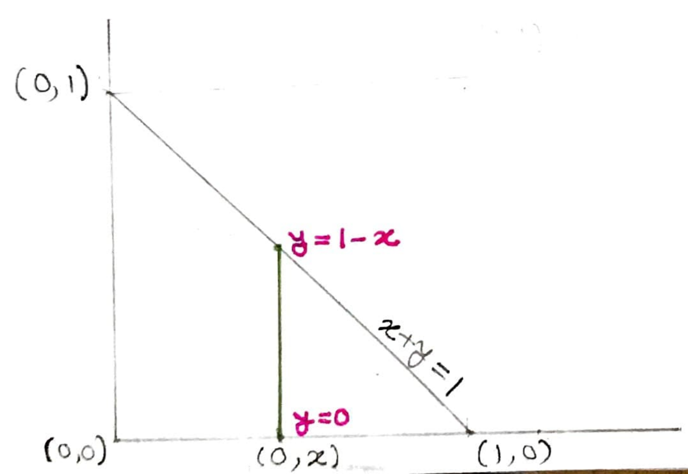
a) Find c.

b) Compute $P(X<Y)$ and $P(Y<X)$.

c) Calculate the marginal pdfs of X and Y.

Solution: Note that

$$\text{Range}(X,Y) = \{(x,y) \in \mathbb{R}^2 : x>0,\ y>0,\ x+y<1\}$$
$$= \{(x,y) : 0<x<1,\ 0<y<1-x\}$$



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a) We know that the joint pdf $f_{X,Y}$ satisfies

(I) $f_{X,Y}(x,y) \geq 0 \quad \forall (x,y) \in \mathbb{R}^2 \Rightarrow \quad c \geq 0$

(II) $\iint_{\mathbb{R}^2} f_{X,Y}(x,y)\,dx\,dy = 1$, from which we shall compute the value of c.

From (II), we get

$$1 = \iint_{\mathbb{R}^2} f_{X,Y}(x,y)\,dx\,dy$$

$$= \iint_{\text{Range}(X,Y)} c\,(x+y)^2\,dx\,dy$$

$$= \iint_{\substack{x>0,\,y>0,\\ x+y<1}} c\,(x+y)^2\,dx\,dy$$

$$= c\int_0^1\int_0^{1-x} (x+y)^2\,dy\,dx$$

$$= c\int_0^1\int_x^1 z^2\,dz\,dx \qquad \left[\begin{matrix}\text{Put } z = x+y\\ dz = dy\end{matrix}\right]$$

$$= c\int_0^1 \left[z^3/3 \right]_{z=x}^{z=1} \, dx$$

$$= c\int_0^1 \frac{1-x^3}{3} \, dx$$

$$= \frac{c}{3} \int_0^1 (1-x^3) \, dx$$

$$= \frac{c}{3} \left(\left[x \right]_{x=0}^{x=1} - \left[\frac{x^4}{4} \right]_{x=0}^{x=1} \right)$$

$$= \frac{c}{3} \left(1 - \frac{1}{4} \right) = \frac{c}{3} \cdot \frac{3}{4} = \frac{c}{4}$$

$$\Rightarrow \frac{c}{4} = 1 \Rightarrow \underline{c = 4.}$$

b) Note that

$$P(X<Y) + \cancel{P(Y>X)}\ P(X>Y) = P(X \neq Y)$$

$$= 1 - P(X=Y)$$

$$= 1$$

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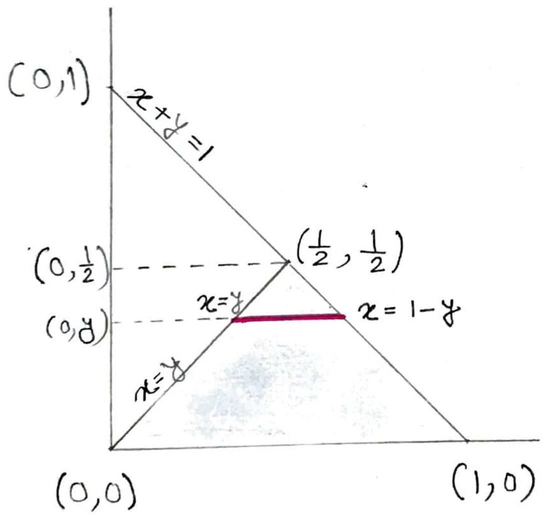
since $P(X=Y) \overset{(*)}{=} \iint\limits_{\substack{x>0, y>0, \\ x+y<1, x=y}} 4(x+y)^2 dx dy = 0.$

Now $P(X>Y) \overset{(*)}{=} \iint\limits_{\substack{x>0, y>0, \\ x+y<1, x>y}} 4(x+y)^2 dx dy$



From the above picture, we get that

$$P(X>Y) = 4\int_0^{1/2}\int_y^{1-y} (x+y)^2 dx\, dy$$

since $\{(x,y) : x>0, y>0, x+y<1, x>y\}$

$= \{(x,y) : 0<y<\frac{1}{2}, y<x<1-y\}.$

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Therefore $P(X>Y)$

$$= 4\int_0^{1/2}\int_y^{1-y} (x+y)^2 \, dx \, dy$$

$$= 4\int_0^{1/2}\int_{2y}^{1} z^2 \, dz \, dy \qquad \left[\begin{array}{l}\text{Put } z = x+y \\ \Rightarrow dz = dx\end{array}\right]$$

$$= 4\int_0^{1/2} \left[z^3/3 \right]_{2y}^{1} dy$$

$$= 4\int_0^{1/2} \frac{1-8y^3}{3} \, dy$$

$$= \frac{4}{3}\int_0^{1/2} (1-8y^3) \, dy$$

$$= \frac{4}{3}\left[y - 2y^4 \right]_{y=0}^{y=\frac{1}{2}}$$

$$= \frac{4}{3}\left[\frac{1}{2} - 2\cdot\frac{1}{2^4} \right]$$

$$= \frac{4}{3}\left(\frac{1}{2} - \frac{1}{8} \right) = \frac{4}{3} \times \frac{3}{8} = \frac{1}{2}.$$

Since $P(X<Y) + P(X>Y) = 1$, it follows that

$P(X<Y) = \frac{1}{2}$.

Remark: Note that ~~the~~ a joint pdf of (X,Y) is

$$f_{X,Y}(u,v) = \begin{cases} 4(u+v)^2 & \text{if } u>0,\ v>0,\ u+v<1, \\ 0 & \text{o.w.} \end{cases}$$

Hence by symmetry, ~~Also,~~ ~~the~~ a joint pdf of (Y,X) is

$$f_{Y,X}(u,v) = \begin{cases} 4(u+v)^2 & \text{if } u>0,\ v>0,\ u+v<1, \\ 0 & \text{o.w.} \end{cases}$$

(Exc: Check this yourself from the def^n of joint pdf.)

Therefore the random vectors (X,Y) and (Y,X) have ~~the~~ same joint pdf. Define

$B = \{(u,v) \in \mathbb{R}^2 : u<v\}$. Then using $(*)$,

we get

$$P(X<Y) = P((X,Y) \in B)$$

$$\overset{(*)}{=} \iint_B f_{X,Y}(u,v)\,du\,dv$$

$$= \iint_B f_{Y,X}(u,v)\,du\,dv$$

$$\overset{(*)}{=} P\left[(Y,X) \in B\right]$$

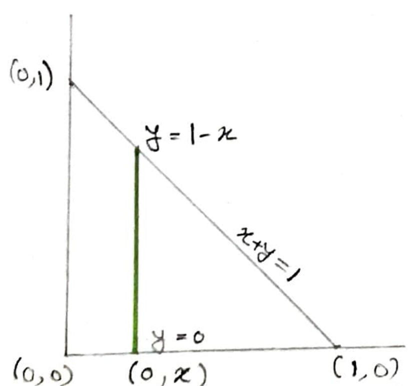
$$= P(Y<X) = P(X>Y).$$

Hence $P(X<Y) = P(X>Y) = \frac{1}{2}$

since $P(X<Y) + P(X>Y) = 1$. Using this trick we can avoid computation of ~~integral~~ integrals in this case.

c) A marginal pdf of X is

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy, \quad x \in \mathbb{R}.$$



Range (X) = Projection of Range (X,Y) on the horizontal axis $= (0,1)$.

This just means

$f_X(x) = 0$ if $x \notin (0,1)$.

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Therefore take $x \in (0,1)$.

Then
$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy$$

$$= \int_0^{1-x} 4(x+y)^2\,dy$$

$$= \int_x^1 4z^2\,dz \qquad \left[\begin{array}{l}\text{Put } z = x+y \\ \Rightarrow dz = dy\end{array}\right]$$

$$= \left[4\frac{z^3}{3}\right]_{z=x}^{z=1}$$

$$= \frac{4}{3}(1-x^3).$$

Therefore, a marginal pdf of X is

$$f_X(x) = \begin{cases} \frac{4}{3}(1-x^3) & \text{if } x \in (0,1), \\ 0 & \text{o.w..} \end{cases}$$

Similarly, a marginal pdf of Y is

$$f_Y(y) = \frac{4}{3}(1-y^3), \qquad 0<y<1.$$

Remark: Note that in the above example, the value taken by X influences the value taken by Y because $P(X+Y<1) = 1$. Hence X and Y are not "independent". In fact, since $X+Y$ is less than 1 with prob 1, it follows that a bigger value of X will ensure a smaller value of Y and ~~vie~~ vice versa.

Independence of Two Random Variables

Suppose (X, Y) is any (not necessarily discrete or continuous) random vector.

Defn: We say that the r.v.s X and Y are independent (and write $X \perp\!\!\!\perp Y$) if $\forall\ (u,v) \in \mathbb{R}^2$,

$$P(X \leq u,\ Y \leq v) = P(X \leq u,\ Y \leq v),$$

i.e., $\forall\ (u,v) \in \mathbb{R}^2$, $F_{X,Y}(u,v) = F_X(u)\, F_Y(v)$.

Roughly speaking, $X \perp\!\!\!\perp Y$ means that

X and Y do not influence each other.

Exc: Show from def^n that for each of the random vectors (X, Y) introduced in Pages 9, 15-16, and 31, $X \not\perp\!\!\!\perp Y$ (i.e., X and Y are not independent).

Exc: Show from def^n that for the random vector (X, Y) introduced in Page 30, $X \perp\!\!\!\perp Y$.

Question: Suppose X, Y are both discrete r.v.s. How to check ~~wth~~ whether X and Y are independent?

~~The answer~~

Answer: Either do it from def^n (very tedious) or use the following theorem.

Thm: Two discrete r.v.s X and Y are independent if and only if

$$p_{X,Y}(x, y) = p_X(x)\, p_Y(y) \quad \cdots \quad (2)$$

$\forall (x, y) \in \mathbb{R}^2$.

(41)

Proof: If part

Suppose (2) holds $\forall\ (x,y) \in \mathbb{R}^2$.

Then
$$\text{Range}(X,Y) = \{(x,y) \in \mathbb{R}^2 : p_{X,Y}(x,y) > 0\}$$
$$= \{(x,y) \in \mathbb{R}^2 : p_X(x) > 0,\ p_Y(y) > 0\}$$
$$= \text{Range}(X) \times \text{Range}(Y).$$

Therefore, $\forall\ (u,v) \in \mathbb{R}^2$, we get

$$P(X \leq u, Y \leq v) = \sum_{\substack{(x,y) \in \text{Range}(X,Y), \\ x \leq u,\ y \leq v}} p_{X,Y}(x,y)$$

$$= \sum_{\substack{x \in \text{Range}(X), \\ x \leq u}} \sum_{\substack{y \in \text{Range}(Y) \\ y \leq v}} p_X(x)\, p_Y(y)$$

$$= \left(\sum_{\substack{x \in \text{Range}(X) \\ x \leq u}} p_X(x) \right) \left(\sum_{\substack{y \in \text{Range}(Y) \\ y \leq v}} p_Y(y) \right)$$

$$= P(X \leq u)\, P(Y \leq v)$$

- establishing $X \perp\!\!\!\perp Y$.

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Only if part

Suppose $X \perp\!\!\!\perp Y$.

To show: (2) holds $\forall$ $(x,y) \in \mathbb{R}^2$.

Fix $(x,y) \in \mathbb{R}^2$.

$\forall$ $n \geq 1$, define events

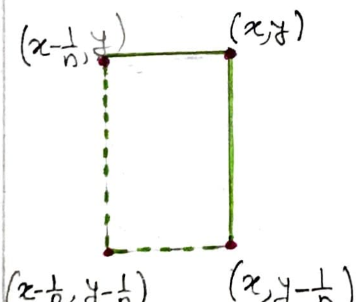
$$A_n := \left[x - \tfrac{1}{n} < X \leq x,\ y - \tfrac{1}{n} < Y \leq y\right].$$

Clearly $A_1 \supseteq A_2 \supseteq \dots$ and

$$\bigcap_{n \geq 1} A_n = [X = x,\ Y = y] =: A.$$

In other words, $A_n \searrow A$

$$\Longrightarrow \quad P(A_n) \searrow P(A).$$



For each $n \geq 1$,

$$P(A_n) = P\left[(X,Y) \in (x-\tfrac{1}{n}, x] \times (y - \tfrac{1}{n}, y]\right]$$
$$= F_{X,Y}(x,y) - F_{X,Y}(x-\tfrac{1}{n}, y) - F_{X,Y}(x, y-\tfrac{1}{n}) + F_{X,Y}(x-\tfrac{1}{n}, y-\tfrac{1}{n})$$

$$\overset{\bullet}{=} F_X(x)F_Y(y) - F_X\left(x-\tfrac{1}{n}\right)F_Y(y) - F_X(x)F_Y\left(y-\tfrac{1}{n}\right) + F_X\left(x-\tfrac{1}{n}\right)F_Y\left(y-\tfrac{1}{n}\right) \qquad \left[\because X \perp\!\!\!\perp Y\right]$$

$$= \left(F_X(x) - F_X\left(x-\tfrac{1}{n}\right)\right)\left(F_Y(y) - F_Y\left(y-\tfrac{1}{n}\right)\right)$$

$$= P\left(x-\tfrac{1}{n} < X \leq x\right) P\left(y-\tfrac{1}{n} < Y \leq y\right)$$

Note that $\left[x-\tfrac{1}{n} < X \leq x\right] \searrow (X = x)$

and $\left[y-\tfrac{1}{n} < Y \leq y\right] \searrow (Y = y)$

and hence

$$P(A_n) = P\left(x-\tfrac{1}{n} < X \leq x\right) P\left(y-\tfrac{1}{n} < Y \leq y\right)$$

$$\searrow P(X = x)\, P(Y = y) \quad \text{as } n \to \infty.$$

$$\Rightarrow P(A) = P(X = x, Y = y) = P(X = x)\,P(Y = y).$$

Since this holds for each $(x, y) \in \mathbb{R}^2$, it follows that (2) ~~hat~~ holds.