A direct consequence of linearity of expectation is the monotonicity (of expectation).

Cor: If (X, Y) is a (discrete or cont) random vector such that X, Y have finite mean and $X \leq Y$, then $E(X) \leq E(Y)$.

Proof: Note that by the theorem (stated in Pg 151) on linearity of expectation, it follows that $X-Y$ has finite mean and, by hypothesis, $X - Y \leq 0$. Therefore

$$0 \geq E(X-Y) = E(X) - E(Y)$$

$$\Rightarrow \quad E(X) \leq E(Y).$$

Note that the above monotonicity result assumes that both X and Y have finite mean. If X and Y are both nonnegative r.v.s, with $X \leq Y$, then the finiteness of $E(Y)$ implies the finiteness of $E(X)$ and the above monotonicity holds. In order to prove this, we shall need the following computational recipe for calculation of mean of a nonnegative r.v.

Thm: If X is a (discrete or continuous) nonnegative r.v. (which is integer valued in the discrete case), then

$$(\text{en}) \ldots \quad E(X) = \begin{cases} \sum_{i=0}^{\infty} P(X > i) & \text{when } X \text{ is discrete,} \\ \int_0^{\infty} P(X > x)\,dx & \text{when } X \text{ is cont.} \end{cases}$$

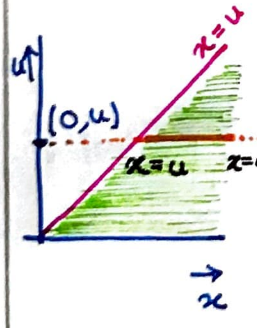
(It may be the case that LHS of (en) = RHS of (en) = $+\infty$)

Proof: We give the proof in the cont case (and the discrete case is left as an exercise).

Since $X \geq 0$ with a pdf f_X (say),

$$E(X) = \int_0^{\infty} x f_X(x)\,dx \qquad [\text{Possibly } +\infty]$$

$$= \int_0^{\infty} \int_0^{x} f_X(x)\,du\,dx$$



$$\underset{\text{Pg } 88}{\overset{\text{Fubini } ①}{=}} \int_0^{\infty} \int_u^{\infty} f_X(x)\,dx\,du = \int_0^{\infty} P(X > u)\,du.$$

(For the discrete case you will need Fubini ① as ~~a~~ stated in Pg (157).)

~~Cor: If X and Y are both jointly distributed nonnegative r.v.s (such that either both are discrete and integer valued, or both are cont)~~

Cor: Suppose (X, Y) is a random vector satisfying $0 \leq X \leq Y$. Assume that either both X, Y are marginally cont r.v.s or both X, Y are integer valued (and hence discrete) r.v.s. Then if Y has finite mean, then so does X and

$$0 \leq E(X) \leq E(Y).$$

Proof: We give the proof when X and Y are both nonneg cont r.v.s. The discrete (and nonneg integer valued) case is left as an exercise.

Since $X \geq 0$, using Thm (of Pg (159)), we get

$$0 \leq E(X) = \int_0^\infty P(X>u)\,du$$

$$\leq \int_0^\infty P(Y>u)\,du \qquad \left[\because \forall\ u \geq 0,\ (X>u) \subseteq (Y>u) \text{ since } Y \geq X\right]$$

$$= E(Y) \qquad [\because Y \geq 0]$$

$$< \infty.$$

Hence ~~Since~~ X, Y both have finite mean, and

$$0 \leq E(X) \leq E(Y).$$

Remark: Note that the above corollary (in Pg 160) does not require X and Y to be jointly cont. This was not the case in the corollary stated (and proved) in Pg 158 because the proof used linearity of expectation. On the other hand, in the discrete case, the above cor (of Pg 160) is more restristive (it requires the discrete r.v.s to be integer valued) than the cor in Pg 158.

Exc: Using the thm stated (and proved) in Pg 159, compute $E(X)$ when

(i) $X \sim Geo(p)$ and (ii) $X \sim Exp(\lambda)$.

Defn: A r.v. X is said to have finite p^{th} moment (here $p \in \mathbb{N}$) if

$E[|X|^p] < \infty$ and in this case, $E(X^p)$ (which exists finitely) is called the (finite) p^{th} moment of X.

Exc: Suppose $|X|$ satisfies the hypothesis of the Cor stated in Pg 160, and take $1 \leqslant p < q$. Then show that if X has finite q^{th} moment, then X has finite p^{th} moment. ~~In this case,~~ $\cancel{E[|X|^p] \leqslant E[|X|^q]}$.

[Hint: You may first show $|X|^p \leq 1 + |X|^q$.]

Remark: The Cor stated in Pg 160 holds as long as (X, Y) is any random vector (not necessarily discrete or cont) satisfying $0 \leq X \leq Y$. In this case, the finiteness of $E(Y)$ implies the finiteness of $E(X)$ and hence $0 \leq E(X) \leq E(Y)$.

However, the proof in this generality is beyond our scope. We shall assume this ~~result~~ general result in this course and use it whenever necessary.

Another Application of Linearity of Expectation

Example: (Top-to-random Shuffle)

Suppose you have a ~~pack~~ deck of N cards (in most cases, $N = 52$) and you perform shuffling