A Small Digression

Thm: Suppose $-\infty < a < b < \infty$ and $f : [a,b] \longrightarrow \mathbb{R}$ is a Riemann integrable function and

$$F(x) := \int_a^x f(t)\,dt \quad , \quad x \in [a,b].$$

Then the following properties hold.

(a) F is a continous function on $[a,b]$.

(b) If f is continuous at $x_0 \in (a,b)$, then F is differentiable at x_0 and

$$F'(x_0) = f(x_0).$$

Remarks: ① Part (b) is known as "First Fundamental Theorem of Calculus". See Page 202 of Calculus, Volume I by Apostol (2nd Edition).

② The above theorem says that integration makes a function smoother — if f is just Riemann integrable, then F is continous, if f is continuous, then F is differentiable, if f is differentiable, then F is twice differentiable and so on.

<u>Exc</u>: Using First Fundamental Theorem of Calculus, show the following: (a) If X is a cont r.v. with cdf F_X and pdf f_X, then for any continuity point x_0 of f_X, ~~we~~ the cdf F_X is differentiable at x_0 and $F_X'(x_0) = f_X(x_0)$.

In particular, if f_X has finitely many ~~discont~~ discontinuities, then

$$f(x) = \begin{cases} F_X'(x) & \text{whenever } F_X \text{ is differentiable at } x, \\ 0 & \text{otherwise} \end{cases}$$

is also a pdf of X.

(b) For any cont r.v. X, the cdf F_X is a continous ~~fun~~ function.

<u>Remark</u>: The converse of Exc (b) does not hold, i.e., there are r.v.s for which the cdf is cont but they do not have pdfs.

Questions: Suppose you have calculated the cdf F_X of a r.v. X. How to verify whether X is a cont r.v. and / or how to compute a pdf of X (if it exists)?

Answer: As seen in the Exc (b) + Remark (in Pg 56), continuity of F_X is a necessary condition (but not a sufficient condition) for continuity of the r.v. X.

Therefore, the problem is two-fold:

(i) we ~~do~~ cannot infer directly from the function F_X whether X is a cont r.v., and

(ii) even if X has a pdf f_X, we do not know a priori ~~that~~ whether f_X only has ~~finitely~~ finitely many discontinuities.

In particular, this means that f defined in Exc (a) of Pg 56 is not guaranteed to be a pdf of X even if X is known to be a cont r.v. In light of this, we shall use the following function as a recipe for guessing a possible pdf of X:

$$(\#)\ldots\ f(x) = \begin{cases} F_X'(x) & \text{if } F_X \text{ is diffble at } x, \\ 0 & \text{otherwise.} \end{cases}$$

After making this educated guess, we shall verify from definition whether f is indeed a pdf of X. That is, we shall check if

$$(\text{V})\ldots F_X(u) = \int_{-\infty}^{u} f(x)\,dx \qquad \text{for each } u \in \mathbb{R}.$$

<u>Question:</u> Will this method always work?

<u>Answer:</u> There is a very deep result in real analysis that guarantees that the recipe (#) will work as long as X is a cont r.v. Of couse, if X is not a cont r.v., then (#) will give a function f that will not satisfy (V).

Back to the ~~problem~~ example

Recall: X ; Y are independent and identically distributed r.v.s with Exp (λ) distribution.

i.i.d.

Notation: $X, Y \overset{iid}{\sim} \text{Exp}(\lambda)$.

Define $Z = \dfrac{X}{Y}$.

Question: What is the dist^n of Z ?

Answer: We have computed the cdf of Z,

$$F_Z(a) = P(Z \leq a) = \begin{cases} \dfrac{a}{1+a} & \text{if } a \geq 0, \\ 0 & \text{if } a < 0. \end{cases}$$

It is easy to check that $F_Z(a)$ is diffble at each $a \neq 0$ and $F_Z'(0)$ does not exist.

~~The~~ Clearly $F_Z'(a) = 0 \quad \forall \; a < 0$

and $\forall \; a > 0$,

$$F_Z'(a) = \frac{d}{da}\left(\frac{a}{1+a}\right) = \frac{d}{da}\left(1 - \frac{1}{1+a}\right) = \frac{1}{(1+a)^2}.$$

Guess of a pdf: Using the recipe (#), we guess that Z has a pdf

$$h(z) = \begin{cases} \frac{1}{(1+z)^2} & \text{if } z > 0, \\ 0 & \text{if } z \leq 0. \end{cases}$$

Claim: Z is a cont r.v. with a pdf

$$h(z) = \frac{1}{(1+z)^2}, \quad z > 0.$$

Proof: The claim will be verified once we establish that $\forall$ $a \in \mathbb{R}$,

$$\int_{-\infty}^{a} h(z)\, dz = F_Z(a) = \begin{cases} \frac{a}{1+a} & \text{if } a \geq 0, \\ 0 & \text{if } a < 0. \end{cases} \quad \cdots (v)$$

Clearly (v) holds if $a < 0$. On the other hand, if $a \geq 0$, then also (v) holds because

$$\int_{-\infty}^{a} h(z)\, dz = \int_{0}^{a} \frac{dz}{(1+z)^2} = \left[-\frac{1}{1+z}\right]_0^a = 1 - \frac{1}{1+a} = \frac{a}{1+a}.$$

This proves the claim.

Remark: In parallel to the univariate case, one can make a guess of a joint pdf from the joint cdf and then verify whether the guess is correct. More precisely, from the joint cdf $F_{X,Y}(u,v)$ of a bivariate random vector (X,Y), we can use the following recipe for guessing a joint pdf (if exists) of (X,Y) as follows:

$$(\#\#) \qquad h(x,y) = \begin{cases} \frac{\partial}{\partial x}\frac{\partial}{\partial y} F_{X,Y}(x,y) & \text{if the partial derivatives exist,} \\ 0 & \text{otherwise.} \end{cases}$$

The order of the partial derivatives does not matter. We can verify if this guess is correct by checking if $\forall\ (u,v) \in \mathbb{R}^2$,

$$F_{X,Y}(u,v) = \int_{-\infty}^{v}\int_{-\infty}^{u} h(x,y)\,dx\,dy \ldots \qquad (vv).$$

__Example:__ Suppose (X,Y) is uniformly distributed ~~in~~ on the unit disk

$$D = \{(x,y) \in \mathbb{R}^2 : \quad x^2+y^2 < 1\}.$$

(a) Write down a joint pdf of (X,Y).

(b) Find ~~the~~ marginal pdfs of X and Y.

(c) Suppose Δ denotes the distance from origin of the random point (X,Y) as above. Find the cdf of Δ.

(d) Find ~~E(D)~~ $E(\Delta)$.

(e) Are X, Y independent?

__Solution:__ The phrase "(X,Y) is uniformly distributed on the unit disk D" means that (X,Y) has a joint pdf

$$f_{X,Y}(x,y) = \begin{cases} c & \text{if } (x,y) \in D, \\ 0 & \text{if } (x,y) \notin D, \end{cases}$$

where c is a constant. Since $f_{X,Y}(x,y)$ takes only nonnegative values, we get $c \geq 0$. On the other hand,

$$\iint_{\mathbb{R}^2} f_{X,Y}(x,y)\,dx\,dy = 1$$

$$\Rightarrow \quad \iint_{D} c \, dx\,dy = 1$$

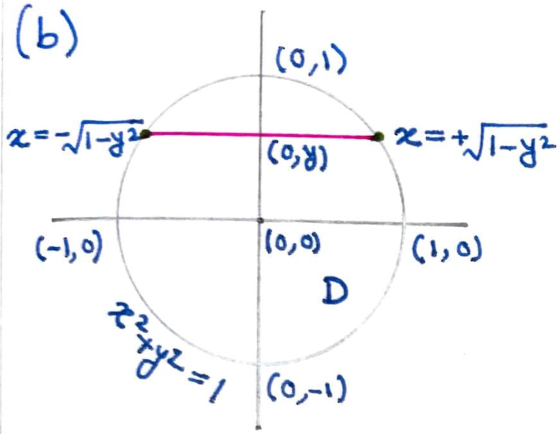
$$\Rightarrow \quad c \; \cancel{\text{Vol}} \; \text{Area}(D) = 1$$

$$\Rightarrow \quad c = \frac{1}{\text{Area}(D)} = \frac{1}{\pi}.$$

Therefore (X, Y) has a joint pdf

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{\pi} & \text{if } x^2+y^2 < 1, \\ 0 & \text{o.w.} \end{cases}$$

(b)



We shall first find a marginal pdf f_Y of Y.

Firstly, note that

$$\text{Range}(Y) = \text{Projection of Range}(X,Y)\ (=D) \text{ on the vertical axis}$$

$$= (-1, 1).$$

In particular, $f_Y(y) = 0$ if $y \notin (-1, 1)$.

If $y \in (-1,1)$, then

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dx$$

$$= \int_{-\sqrt{1-y^2}}^{+\sqrt{1-y^2}} \frac{1}{\pi}\, dx = \frac{2}{\pi}\sqrt{1-y^2}.$$

Hence, Y has a marginal pdf $f_Y(y) = \frac{2}{\pi}\sqrt{1-y^2}$, $-1<y<1$.

Similarly (or using the symmetry), a marginal pdf of X is $f_X(x) = \frac{2}{\pi}\sqrt{1-x^2}$, $-1<x<1$.

(c) Clearly, $\Delta^2 = (X-0)^2 + (Y-0)^2$

$$\Rightarrow \Delta = +\sqrt{X^2+Y^2}.$$

$$\Rightarrow \text{Range}(\Delta) = (0,1).$$

In particular, $F_\Delta(a) = P(\Delta \leq a)$

$$= \begin{cases} 0 & \text{if } a \leq 0, \\ 1 & \text{if } a \geq 1. \end{cases}$$

Take $a \in (0,1)$. Then $F_\Delta(a)$

$$= P(\Delta \leq a) = P\left(+\sqrt{X^2+Y^2} \leq a\right)$$

$$= P(X^2+Y^2 \leq a^2) = \iint_{x^2+y^2 \leq a^2} f_{X,Y}(x,y)\,dx\,dy$$

$$= \iint_{x^2+y^2 \le a^2} \frac{1}{\pi}\, dx\, dy$$

$$= \frac{1}{\pi} \text{ Area } (A). \qquad \left[\text{Here } A = \{(x,y) : x^2+y^2 \le a^2\}\right]$$

$$= \frac{\pi a^2}{\pi} = a^2.$$

Therefore, the cdf of Δ is

$$F_\Delta(a) = P(\Delta \le a) = \begin{cases} 0 & \text{if } a<0, \\ a^2 & \text{if } 0 \le a < 1, \\ 1 & \text{if } a \ge 1. \end{cases}$$

(d) **Exc:** Show that Δ is a cont r.v. with a pdf

$$f_\Delta(x) = 2x \quad \text{if} \quad 0<x<1.$$

In particular, establish that $E(\Delta) = \frac{2}{3}$.