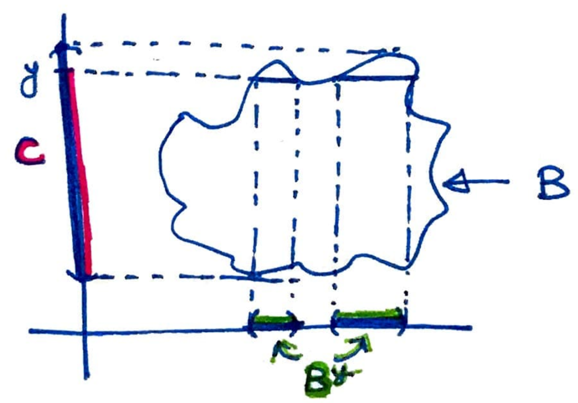
Double Integrals

Our next topic of discussion is bivariate (absolutely) continuous random vectors. For this topic, we will need double integrals. Suppose $f: \mathbb{R}^2 \to \mathbb{R}$ is a function and $B \subseteq \mathbb{R}^2$ is a "nice" set.

Question: What do we understand by $I = \iint_B f(x,y)\,dx\,dy$?

We shall understand this integral as a "repeated integral". ~~Fix~~ Let $C = \{y \in \mathbb{R} : (x,y) \in B \text{ for some } x\}$



be the projection of B to the vertical axis. We will deal with B nice enough so that C is a ctble union of intervals. Fix $y \in C$. Look at

$B^y = \{x \in \mathbb{R} : (x,y) \in B\}$, the section of B formed by a horizontal line through $(0, y)$. We will have nice enough B so that each B^y is a ctble union of intervals. Here singletons are also considered as intervals.

The integral I will be understood as follows. First ~~integral~~ integrate the function over x on each B^y treating y as a constant, and then integrate over y on C. Again, we shall only deal with functions f such that all of these integrals exist and are finite. Therefore, we shall first find

$$\cancel{F(y) =} \quad g(y) = \int_{B^y} f(x, y)\, dx$$

for each $y \in C$ and then compute the double integral $I := \int_C g(y)\, dy$. In other words,

$$I = \iint_B f(x, y)\, dx\, dy = \int_C \left[\int_{B^y} f(x, y)\, dx \right] dy$$

$$= \int_C \int_{B^y} f(x, y)\, dx\, dy .$$

Here we integrate w.r.t. x first and then integrated w.r.t y. What if we integrate wrt y first and then integrate wrt x? There is a very deep theorem (due to Fubini), that says that for all functions that we are going to work with, the order of the integration does not matter. For instance, Fubini's Thm states that for nonnegative real-valued functions, the order of the integrals can always be interchanged.

Examples:

① If $B = \mathbb{R}^2$, then $C = \mathbb{R}$ and for each $y \in C = \mathbb{R}$, $B^y = \mathbb{R}$. Therefore

$$\iint_{\mathbb{R}^2} f(x,y)\,dx\,dy = \int_{\mathbb{R}} \int_{\mathbb{R}} f(x,y)\,dx\,dy.$$

② If $B = (-\infty, u] \times (-\infty, v]$ for some $(u,v) \in \mathbb{R}^2$, then $C = (-\infty, v]$ and for each $y \in C = (-\infty, v]$, $B^y = (-\infty, u]$. Therefore

$$\iint_{(-\infty,u]\times(-\infty,v]} f(x,y)\,dx\,dy = \int_{-\infty}^{v} \int_{-\infty}^{u} f(x,y)\,dx\,dy.$$

More complicated examples will be given soon.

Bivariate (absolutely) continuous random vectors

Defn: A bivariate random vector (X, Y) is called (absolutely) continuous if $\exists$ a function $f_{X,Y} : \mathbb{R}^2 \longrightarrow [0, \infty)$ such that $\forall$ $(u, v) \in \mathbb{R}^2$,

$$F_{X,Y}(u,v) = P(X \leq u, Y \leq v) = \iint_{(-\infty,u]\times(-\infty,v]} f_{X,Y}(x,y)\, dx\, dy$$

$$= \int_{-\infty}^{v} \int_{-\infty}^{u} f_{X,Y}(x,y)\, dx\, dy.$$

In this case, $f_{X,Y}$ is called a joint pdf or joint density function of (X, Y). We also say that X and Y are jointly (absolutely) continuous.

It can be shown that for all "nice" $B \subseteq \mathbb{R}^2$,

$$(\star)\ldots P\left[(X,Y) \in B\right] = \iint_{B} f_{X,Y}(x,y)\, dx\, dy.$$

Proof of $(\star)$ is ~~beyound~~ beyond our scope. However, we shall use it.

Fact: If X and Y are jointly continuous with joint pdf $f_{X,Y}$, then marginally X and Y are both continuous r.v.s and their marginal pdfs are given by

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy, \quad x \in \mathbb{R} \qquad \text{and} \qquad f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx, \quad y \in \mathbb{R}$$

respectively.

Proof: To prove that X is a cont r.v. with (marginal) pdf f_X as above, we need to establish that $\forall\ u \in \mathbb{R}$,

$$F_X(u) = P(X \leq u) = \int_{-\infty}^{u} f_X(x)\,dx \ \ldots \ (1).$$

Note that $\forall\ u \in \mathbb{R}$,

$$P(X \leq u) = P(X \leq u, \cancel{Y < \infty}\ -\infty < Y < \infty)$$
$$= P((X,Y) \in (-\infty, u] \times (-\infty, \infty))$$
$$= \iint_{(-\infty,u] \times (-\infty,\infty)} f_{X,Y}(x,y)\,dx\,dy \qquad [\text{By } (\star)]$$

$$= \int_{-\infty}^{u} \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dy\, dx$$

$$= \int_{-\infty}^{u} f_X(x)\, dx$$

and this proves (1).

Similarly, we can establish that Y is a cont r.v. with (marginal) pdf f_Y.

Remarks: ① The above fact shows that if X and Y are jointly continuous, then they are both marginally continuous. However, the converse is not true as shown by the following exercise.

Exc: Take $X \sim \text{Unif}(0,1)$. Define $Y = X$. Then show that X and Y are not jointly cont even though marginally, ~~both~~ each is a cont r.v.

[Hint: Define $B = \{(x,y) \in (0,1)^2 : x = y\} \subseteq \mathbb{R}^2$. Using ($\bigstar$), show that $P[(X,Y) \in B] = 0$ if ~~However~~ ~~P[(X,Y)~~ X and Y are jointly cont.]

2 In parallel to the univariate case, one can show that ~~whenever~~ if (u,v) is a continuity point of $f_{X,Y}$ (this mean that $f_{X,Y}(u^{(n)}, v^{(n)}) \to f_{X,Y}(u,v)$ whenever $u^{(n)} \to u$ and $v^{(n)} \to v$), then

$$f_{X,Y}(u,v) = \frac{\partial}{\partial u}\frac{\partial}{\partial v} F_{X,Y}(u,v).$$

Here $\frac{\partial}{\partial v}$ refers to the "partial derivative" of $F_{X,Y}$ w.r.t. v, i.e., taking the derivative of $F_{X,Y}(u,v)$ w.r.t. v treating u as a constant. $\frac{\partial}{\partial u}$ has a similar meaning. Also, the order of the partial derivatives will not matter. Therefore, we shall use the following recipe for guessing a joint pdf of a cont random vector (X,Y) from its cdf:

$$f_{X,Y}(x,y) = \begin{cases} \frac{\partial}{\partial x}\frac{\partial}{\partial y} F_{X,Y}(x,y) & \text{if the partial derivatives exist} \\ 0 & \text{o.w.} \end{cases}$$

3 Whenever (u,v) is a continuity point of $f_{X,Y}$, we have

$$\lim_{\Delta u \to 0^+} \lim_{\Delta v \to 0^+} \frac{P(u < X \leq u+\Delta u,\ v < Y \leq v+\Delta v)}{\Delta u\, \Delta v}$$

$$= f_{X,Y}(u,v)$$

(again, this explains why $f_{X,Y}$ is called a joint probability "density" function)

$$\Rightarrow P\Big[(X,Y) \in (u, u+du] \times (v, v+dv]\Big] "=" f_{X,Y}(u,v)\, du\, dv.$$

4 Any joint density function $f_{X,Y}$ satisfies

I $f_{X,Y}(x,y) \geq 0 \quad \forall\ (x,y) \in \mathbb{R}^2$,

and

II $$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y) = 1.$$