04/03/2022

$g(x) = x \equiv f(x) = 0$

Define $g(x) = x - f(x)$ ← given to ~~kno~~ us.

$\boxed{g(\alpha) = \alpha}$

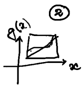
① $g(x)$ is continuous in $[a,b]$

$g : [a,b] \mapsto [a,b]$

$\Rightarrow \exists$ at least one f.p. $\in [a,b]$

② $g'(x)$ exists & continuous in $[a,b]$

$\lambda = \max_{x \in [a,b]} |g'(x)| < 1$



$\Rightarrow$ the f.p. in $[a,b]$ is unique

③ If the initial guess $x_0 \in [a,b]$ then $\lim_{n\to\infty} x_n = \alpha$, $g(\alpha) = \alpha$

$x_0 \in [a,b] \Rightarrow x_1 = g(x_0) \in [a,b]$.

Suppose $x_k \in [a,b]$ for some $k > 1$

$\Rightarrow x_{k+1} \in [a,b]$

$\Rightarrow x_m \in [a,b] \ \forall m \in \underbrace{\mathbb{Z}_+ \cup \{0\}}_{\overline{\mathbb{Z}_-}}$

$\alpha - x_{n+1} = g(\alpha) - g(x_n)$

$= g'(c_n)(\alpha - x_n)$ $c_n \in [\alpha, x_n]$ MVT

$\Rightarrow |\alpha - x_{n+1}| \leq \lambda |\alpha - x_n|$

$\Rightarrow |\alpha - x_n| \leq \lambda^n |\alpha - x_0|$

$\Rightarrow \lim_{n\to\infty} |\alpha - x_n| \to 0$

$|\alpha - x_n| \leq \frac{\lambda^n}{1-\lambda} |x_0 - x_1|$

$|\alpha - x_0| \leq |\alpha - x_1| + |x_1 - x_0|$

$\leq \lambda |\alpha - x_0| + |x_1 - x_0|$

$\Rightarrow (1-\lambda)|\alpha - x_0| \leq |x_1 - x_0|$

$\Rightarrow |\alpha - x_0| \leq \frac{1}{1-\lambda} |x_1 - x_0|$

$\Rightarrow$ use the λ^n inequality from the earlier proof.

$\lim_{n\to\infty} \frac{\alpha - x_{n+1}}{\alpha - x_n} = g'(\alpha)$

$\alpha - x_{n+1} = g'(c_n)(\alpha - x_n)$ $c_n \in [\alpha, x_n]$

$\Rightarrow \lim_{n\to\infty} \frac{\alpha - x_{n+1}}{\alpha - x_n} = \lim_{n\to\infty} g'(c_n)$

$c_n \in [\alpha, x_n]$ and $\lim_{n\to\infty} x_n \to \alpha$

$\Rightarrow \lim_{n\to\infty} c_n = \alpha$

$\Rightarrow \lim_{n\to\infty} g'(c_n) = g'(\alpha)$

$\Rightarrow \alpha - x_{n+1} \approx g'(\alpha)(\alpha - x_n)^p$

$|g'(\alpha)| < 1$ then linear convergence $p = 1$

—x—