Cor: Suppose (X,Y) is a discrete random vector. Then $X \perp\!\!\!\perp Y$ if and only if

$$P(X\in C,\ Y\in D) = P(X\in C)\,P(Y\in D)$$

$\forall$ subsets $C, D \subseteq \mathbb{R}$.

Proof: Exc.

Remarks: ① The above corollary says that two discrete r.v.s X, Y are independent if and only if all pairs of events $(X\in C)$ and $(Y\in D)$ are independent as C, D runs over all subsets of $\mathbb{R}$.

② Note that using the theorem stated in Page 40, it becomes very easy to verify that X and Y are not independent for the random vectors (X,Y) introduced in Pages 9 and 15-16.

Question: Suppose X and Y are jointly continuous. How to check whether X and Y are independent?

45

Answer: Either do it from def^n (extremely tedious) or use the following theorem.

Thm: Suppose X and Y are continuous r.v.s with pdfs $f_X(x)$ and $f_Y(y)$, respectively. Then $X \perp\!\!\!\perp Y$ if and only if (X,Y) is a cont random vector with a joint pdf

$$g(x,y) = f_X(x) f_Y(y), \qquad (x,y) \in \mathbb{R}^2.$$

Proof: If part

Suppose (X,Y) is a cont random vector with a joint pdf $g(x,y) = f_X(x) f_Y(y)$, $(x,y) \in \mathbb{R}^2$.

To show: $X \perp\!\!\!\perp Y$

Take any $(u,v) \in \mathbb{R}^2$. Then

$$F_{X,Y}(u,v) = P(X \leq u, Y \leq v)$$

$$= \int_{-\infty}^{v}\int_{-\infty}^{u} g(x,y)\,dx\,dy$$

$$= \int_{-\infty}^{v}\int_{-\infty}^{u} f_X(x) f_Y(y)\,dx\,dy$$

$$= \left(\int_{-\infty}^{u} f_X(x)\, dx \right) \cdot \left(\int_{-\infty}^{v} f_Y(y)\, dy \right)$$

$$= P(X \leq u)\; P(Y \leq v)$$

$$= F_X(u)\, F_Y(v).$$

Since $(u,v) \in \mathbb{R}^2$ is arbitrary, the above calculation shows that $X \perp\!\!\!\perp Y$.

Only if part

Suppose X is a cont r.v. with a pdf f_X, Y is a cont r.v. with a pdf f_Y and $X \perp\!\!\!\perp Y$.

To show: X and Y are jointly cont with a joint pdf $g(x,y) = f_X(x)\, f_Y(y)$, $(x,y) \in \mathbb{R}^2$.

Take any $(u,v) \in \mathbb{R}^2$.

To show: $F_{X,Y}(u,v) = \int_{-\infty}^{v} \int_{-\infty}^{u} f_X(x)\, f_Y(y)\, dx\, dy.$

Using the independence of X and Y, we get

$$F_{X,Y}(u,v) = F_X(u)\, F_Y(v)$$

$$= \left(\int_{-\infty}^{u} f_X(x)\,dx\right)\cdot\left(\int_{-\infty}^{v} f_Y(y)\,dy\right)$$

$$= \int_{-\infty}^{u}\int_{-\infty}^{v} f_X(x)\, f_Y(y)\, dx\,dy\ ,$$

which completes the proof.

Remarks: ① Since joint and marginal pdfs are not unique, the above theorem should not be used to ~~show~~ verify that $X \perp\!\!\!\perp Y$. It is better to prove it from def$^{\underline{n}}$, i.e., by producing a pair $(u_0, v_0) \in \mathbb{R}^2$ such that

$$P(X \le u_0,\ Y \le v_0) \ne P(X \le u_0)\, P(Y \le v_0).$$

However, for establishing independence of two cont r.v.s, the theorem stated in Page 45 is very useful.

② We know that X and Y may not be jointly cont even if they are both marginally so. Under the assumption of independence, however, marginal continuity of the r.v.s X and Y implies joint continuity as seen in the above theorem.

3 It can be shown that if $X \perp\!\!\!\perp Y$, (for two cont r.v.s X, Y) then

$$P(X \in C, Y \in D) = P(X \in C) P(Y \in D)$$

for all "nice" subsets $C, D \subseteq \mathbb{R}$. Here "nice" means ctble union of intervals. This can be verified using (✱).

Example: Suppose $X \sim \text{Poi}(\lambda)$, $Y \sim \text{Poi}(\mu)$ and $X \perp\!\!\!\perp Y$. Compute $P(X+Y=10)$.

Solution: $P(X+Y=10)$

$$= \sum_{j=0}^{10} P(X=j, Y=10-j)$$

$$= \sum_{j=0}^{10} P(X=j) P(Y=10-j) \qquad [\because X \perp\!\!\!\perp Y]$$

$$= \sum_{j=0}^{10} e^{-\lambda} \frac{\lambda^j}{j!} e^{-\mu} \frac{\mu^{10-j}}{(10-j)!}$$

$$= e^{-(\lambda+\mu)} \frac{1}{10!} \sum_{j=0}^{10} \frac{10!}{j!(10-j)!} \lambda^j \mu^{10-j}$$

$$= e^{-(\lambda+\mu)} \frac{(\lambda+\mu)^{10}}{10!}.$$

The ~~an~~ answer is not at all surprising in light of the following result.

Propn: If $X \sim Poi(\lambda)$, $Y \sim Poi(\mu)$ and $X \perp\!\!\!\perp Y$, then $X+Y \sim Poi(\lambda+\mu)$.

Proof: $Range(X) = Range(Y) = \mathbb{N} \cup \{0\}$
$\Rightarrow Range(X+Y) \subseteq \mathbb{N} \cup \{0\}$.

Take $k \in \cancel{Range(X+Y) =} \mathbb{N} \cup \{0\}$. Then

$$P(X+Y=k) = e^{-(\lambda+\mu)} \frac{(\lambda+\mu)^k}{k!}$$

by the same calculation as in Page 48.

This shows $X+Y \sim Poi(\lambda+\mu)$.

Exc: If $X \sim Bin(m,p)$, $Y \sim Bin(n,p)$ and $X \perp\!\!\!\perp Y$, then show that

$$X+Y \sim Bin(m+n, p).$$

Note: The above result is expected.

The proposition and the exercise stated in Page 49 are examples of calculation of dist$^{\underline{n}}$ of scalar valued functions of random vectors. More generally, we will have a random vector (X, Y) and a function $T: \text{Range}(X,Y) \longrightarrow \mathbb{R}$, and we shall find the distribution of $T(X,Y)$.

Examples will include $T(x,y) = x \pm y$, $T(x,y) = xy$, $T(x,y) = \frac{x}{y}$ (provided $\text{Range}(X,Y) \subseteq \mathbb{R} \times (\mathbb{R} \setminus \{0\})$), etc.

<u>Example:</u> Suppose $X \sim \text{Exp}(\lambda)$, $Y \sim \text{Exp}(\mu)$ and $X \perp\!\!\!\perp Y$.

a) Calculate $P(X < Y)$.

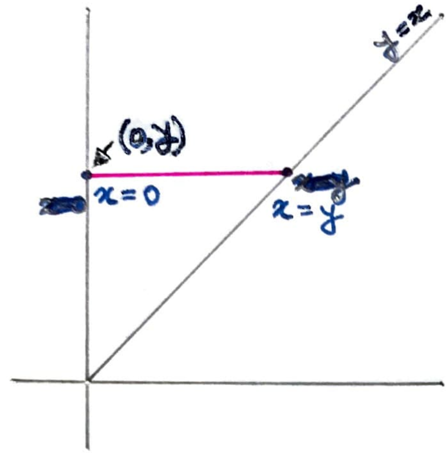
b) Assuming $\lambda = \mu$, compute $P(X \leq aY)$ for any $a > 0$. Using this, find the dist$^{\underline{n}}$ of $\frac{X}{Y}$ ~~when~~ in this case (i.e., when $\lambda = \mu$).

<u>Solution:</u> By the theorem is Pg 45, it follows that (X,Y) has a joint pdf

$$f_{X,Y}(x,y) = \lambda\mu\, e^{-(\lambda x + \mu y)}, \quad x > 0,\ y > 0.$$

(51)

a)



Using (☆), we have,

$$P(X<Y) = \iint_{x<y} f_{X,Y}(x,y)\,dx\,dy$$

$$= \iint_{x>0,\, y>0,\, x<y} \lambda\mu\, e^{-(\lambda x+\mu y)}\,dx\,dy$$

$$= \int_0^\infty \int_0^y \lambda\mu\, e^{-\lambda x} e^{-\mu y}\,dx\,dy$$

$$= \int_0^\infty \mu e^{-\mu y}\left(\int_0^y \lambda e^{-\lambda x}\,dx\right)dy$$

$$= \int_0^\infty \mu e^{-\mu y}\left(1-e^{-\lambda y}\right)dy.$$

Note that the last step used the following fact:

$$X \sim \text{Exp}(\lambda) \Rightarrow F_X(u) = P(X \leq u) = \begin{cases} 1 - e^{-\lambda u} & \text{if } u \geq 0, \\ 0 & \text{if } u < 0. \end{cases}$$

Therefore,

$$P(X<Y) = \int_0^\infty \mu e^{-\mu y} \left(1 - e^{-\lambda y}\right) dy$$

$$= \int_0^\infty \mu e^{-\mu y} dy - \int_0^\infty \mu e^{-(\mu+\lambda) y} dy$$

$$= 1 - \frac{\mu}{\mu+\lambda} \int_0^\infty (\mu+\lambda) e^{-(\mu+\lambda) y} dy$$

$$= 1 - \frac{\mu}{\mu+\lambda}$$

$$= \frac{\lambda}{\mu+\lambda}.$$

Remarks: ① When $\lambda = \mu$, then the above calculation yields

$$P(X<Y) = \frac{1}{2},$$

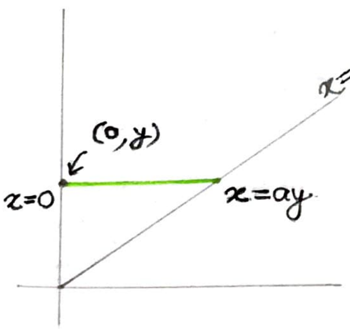
which can also be deduced from the symmetry of the joint pdf of (X, Y), i.e., the symmetry of

$$f_{X,Y}(x,y) = \lambda^2 e^{-\lambda(x+y)} \quad , \quad x>0,\ y>0$$

in parallel to the remark given in Pages 36 - 37.

2 Recall that $E(X)=\frac{1}{\lambda}$ and $E(Y)=\frac{1}{\mu}$, and hence when λ is quite large compared to μ, it is expected that X will be smaller ~~that~~ than Y with high probability. This is manifested in $P(X<Y)=\frac{\lambda}{\lambda+\mu}$.

b) We assume $\lambda=\mu$ and fix $a>0$.



By $(*)$, we have

$$P(X \leq aY)$$

$$= \iint_{x \leq ay} f_{X,Y}(x,y)\,dx\,dy$$

$$= \iint_{\substack{x>0,y>0,\\ x\leq ay}} \lambda^2 e^{-\lambda(x+y)}\,dx\,dy = \int_0^\infty \int_0^{ay} \lambda^2 e^{-\lambda(x+y)}\,dx\,dy$$

$$= \int_0^\infty \lambda e^{-\lambda y} \int_0^{ay} \lambda e^{-\lambda x}\, dx\, dy$$

$$= \int_0^\infty \lambda e^{-\lambda y} \left(1 - e^{-\lambda a y}\right) \cancel{dx}\, dy \qquad \left[\text{Using the form of cdf of } X\right]$$

$$= \int_0^\infty \lambda e^{-\lambda y}\, dy - \int_0^\infty \lambda e^{-\lambda(1+a)y}\, dy$$

$$= 1 - \frac{1}{1+a} \int_0^\infty \lambda(1+a)\, e^{-\lambda(1+a)y}\, dy$$

$$= 1 - \frac{1}{1+a} = \frac{a}{1+a}.$$

Define $Z = X/Y$.
Therefore, for any $a > 0$,

$$P[Z \leqslant a] = P[X \leqslant aY] = \frac{a}{1+a}.$$

On the other hand, for any $a \leqslant 0$, $P[Z \leqslant a] = 0$ since $\text{Range}(Z) \subseteq (0, \infty)$.

<u>Summary</u>: The cdf of $Z = \frac{X}{Y}$ is given by

$$F_Z(a) = P(Z \leqslant a) = \begin{cases} \frac{a}{1+a} & \text{if } a \geqslant 0, \\ 0 & \text{if } a < 0. \end{cases}$$