We now present a very important application of (e).

<u>Thm</u>: Suppose $(X_1, X_2, \dots, X_k)$ is a (discrete or cont) random vector such that each X_i has finite mean. Then for all $\alpha_1, \alpha_2, \dots, \alpha_k \in \mathbb{R}$, the linear combination $\alpha_1 X_1 + \alpha_2 X_2 + \dots + \alpha_k X_k$ has finite mean and

$$E(\alpha_1 X_1 + \alpha_2 X_2 + \dots + \alpha_k X_k) = \alpha_1 E(X_1) + \alpha_2 E(X_2) + \dots + \alpha_k E(X_k).$$

<u>Remarks</u>: ① Note that the space V of all (real valued) random variables (defined on the same sample space) forms a vector space over $\mathbb{R}$. The above result (in its generality) says that

$$S = \{X \in V : X \text{ has finite mean}\}$$

is a linear subspace of V, and the expectation map $E : S \to \mathbb{R}$ defined by

$$E : X \longmapsto E(X)$$

is a linear map (i.e., a linear functional).

② The linearity of expectation is very useful - we shall see a few examples.

Proof of Thm (stated in Pg 151): Take any $\alpha_1 \in \mathbb{R}$. We shall first show that $\alpha_1 X_1$ ~~is~~ has finite mean and $E(\alpha_1 X_1) = \alpha_1 E(X_1)$. We give the proof when X_1 is discrete with pmf p_{X_1} (The cont case is left as an exercise.) Note that taking $h: \mathbb{R} \longrightarrow \mathbb{R}$ as $h(x) = \alpha_1 x$ and observing

$$\sum_{x \in Range(X_1)} |h(x)| p_{X_1}(x)$$

$$= \sum_{x \in Range(X_1)} |\alpha_1 x| p_{X_1}(x)$$

$$= |\alpha_1| \sum_{x \in Range(X_1)} |x| p_{X_1}(x) < \infty, \quad \left[\because X_1 \text{ has finite mean}\right]$$

we get that $h(X_1) = \alpha_1 X_1$ has finite mean. Therefore using (e) with the above choice of h, we obtain that

$$E(\alpha_1 X_1) = E(h(X_1))$$

$$= \sum_{x \in Range(X_1)} h(x) p_{X_1}(x) = \sum_{x \in Range(X_1)} \alpha_1 x \, p_{X_1}(x)$$

$$= \alpha_1 \sum_{x \in Range(X_1)} x\, p_{X_1}(x) = \alpha_1 E(X_1)$$

completing the proofof this step.

We now show the following: if X_1, X_2 have finite mean, then so does each linear combination $\alpha_1 X_1 + \alpha_2 X_2$ (here $\alpha_1, \alpha_2 \in \mathbb{R}$) and $E(\alpha_1 X_1 + \alpha_2 X_2) = \alpha_1 E(X_1) + \alpha_2 E(X_2)$.

Because of the previous ~~st~~ step, it is enough to show this step for $\alpha_1 = \alpha_2 = 1$. Therefore, we need to show the following: if X_1, X_2 have finite mean, then so does $X_1 + X_2$ and $E(X_1 + X_2) = E(X_1) + E(X_2)$. We shall show this when $\underset{\sim}{X} = (X_1, X_2)$ is a cont random vector. (The discrete case is left as an exercise.) with a joint pdf f_{X_1, X_2}.

Define $h: \mathbb{R}^2 \to \mathbb{R}$ by $h(x_1, x_2) = x_1 + x_2$. We have to verify that $h(X_1, X_2)$ has finite mean and compute the mean. To this end, observe that

$$\int_{\mathbb{R}^2} |h(\underset{\sim}{x})| f_{\underset{\sim}{X}}(\underset{\sim}{x})\, d\underset{\sim}{x}$$

$$= \int_{\mathbb{R}} \int_{\mathbb{R}} |h(x_1, x_2)|\, f_{X_1, X_2}(x_1, x_2)\, dx_1\, dx_2$$

$$= \int_{\mathbb{R}} \int_{\mathbb{R}} |x_1 + x_2|\, f_{X_1, X_2}(x_1, x_2)\, dx_1\, dx_2$$

$$\leqslant \int_{\mathbb{R}} \int_{\mathbb{R}} |x_1|\, f_{X_1, X_2}(x_1, x_2)\, dx_1\, dx_2$$

$$+ \int_{\mathbb{R}} \int_{\mathbb{R}} |x_2|\, f_{X_1, X_2}(x_1, x_2)\, dx_1\, dx_2$$

[Using triangle inequality]

$$\overset{\text{Fubini (1)}}{\underset{\text{Pg (88)}}{=}} \int_{\mathbb{R}} \int_{\mathbb{R}} |x_1|\, f_{X_1, X_2}(x_1, x_2)\, dx_2\, dx_1$$

$$+ \int_{\mathbb{R}} \int_{\mathbb{R}} |x_2|\, f_{X_1, X_2}(x_1, x_2)\, dx_1\, dx_2$$

$$= \int_{\mathbb{R}} |x_1| \int_{\mathbb{R}} f_{X_1,X_2}(x_1,x_2)\, dx_2\, dx_1$$

$$+ \int_{\mathbb{R}} |x_2| \int_{\mathbb{R}} f_{X_1,X_2}(x_1,x_2)\, dx_1\, dx_2$$

$$= \int_{\mathbb{R}} |x_1| f_{X_1}(x_1)\, dx_1 + \int_{\mathbb{R}} |x_2| f_{X_2}(x_2)\, dx_2$$

$< \infty$, $\quad$ [$\because$ X_1, X_2 have finite mean]

which shows $h(\underset{\sim}{X}) = h(X_1, X_2) = X_1 + X_2$ has finite mean. Using (e), we get

$$E(X_1+X_2) = E[h(X_1,X_2)]$$

$$= \int_{\mathbb{R}^2} h(\underset{\sim}{x}) f_{\underset{\sim}{X}}(\underset{\sim}{x})\, d\underset{\sim}{x}$$

$$= \int_{\mathbb{R}} \int_{\mathbb{R}} h(x_1,x_2) f_{X_1,X_2}(x_1,x_2)\, dx_1\, dx_2$$

$$= \int_{\mathbb{R}} \int_{\mathbb{R}} (x_1 + x_2)\, f_{X_1, X_2}(x_1, x_2)\, dx_1\, dx_2$$

$$= \int_{\mathbb{R}} \int_{\mathbb{R}} x_1\, f_{X_1, X_2}(x_1, x_2)\, dx_1\, dx_2 + \int_{\mathbb{R}} \int_{\mathbb{R}} x_2\, f_{X_1, X_2}(x_1, x_2)\, dx_1\, dx_2$$

(absolutely integrable by the previous calculation)

$$\overset{\text{Fubini (2)}}{\underset{\text{Pg } 88}{=}} \int_{\mathbb{R}} \int_{\mathbb{R}} x_1\, f_{X_1, X_2}(x_1, x_2)\, dx_2\, dx_1 + \int_{\mathbb{R}} \int_{\mathbb{R}} x_2\, f_{X_1, X_2}(x_1, x_2)\, dx_1\, dx_2$$

$$= \int_{\mathbb{R}} x_1 \int_{\mathbb{R}} f_{X_1, X_2}(x_1, x_2)\, dx_2\, dx_1 + \int_{\mathbb{R}} x_2 \int_{\mathbb{R}} f_{X_1, X_2}(x_1, x_2)\, dx_1\, dx_1$$

$$= \int_{\mathbb{R}} x_1 f_{X_1}(x_1)\, dx_1 + \int x_2 f_{X_2}(x_2)\, dx_2 = E(X_1) + E(X_2),$$

which finishes the proof of this step.

Exc: Using induction on k, show that under the hypothesis of the thm stated in Pg 151, $\alpha_1 X_1 + \alpha_2 X_2 + \cdots + \alpha_k X_k$ has finite mean and

$$E\left(\sum_{i=1}^{k} \alpha_i X_i\right) = \sum_{i=1}^{k} \alpha_i E(X_i).$$

[Thanks to the first step, it is enough to solve the above exercise for $\alpha_1 = \alpha_2 = \cdots = \alpha_k = 1$.]

The discrete case of the second step (which is left as an exercise) needs the following discrete version of Fubini's Thm.

Fubini's Thm: Suppose $h : \mathbb{N}^2 \longrightarrow \mathbb{R}$ is a double sequence. Then the following properties hold.

① If $h \geq 0$, then

$$(\text{int}) \ldots \sum_{i \in \mathbb{N}} \sum_{j \in \mathbb{N}} h(i,j) = \sum_{j \in \mathbb{N}} \sum_{i \in \mathbb{N}} h(i,j).$$

(Both LHS and RHS can be $+\infty$.)

② If either $\sum_{i \in \mathbb{N}} \sum_{j \in \mathbb{N}} |h(i,j)| < \infty$ or $\sum_{j \in \mathbb{N}} \sum_{i \in \mathbb{N}} |h(i,j)| < \infty$, then also (int) holds.