Answer ~~The~~ to Question 2 of Pg 130

The role of $\frac{dg^{-1}(y)}{dy}$ played in the univariate change of density formula will be played in the bivariate case by the following one-dimensional summary of the two-dimensionl function g^{-1}:

$$\frac{dg^{-1}(\underset{\sim}{y})}{d\underset{\sim}{y}} := \det\left(J_{g^{-1}}(\underset{\sim}{y})\right), \quad \underset{\sim}{y} \in J$$

$$= \det\begin{pmatrix} \frac{\partial h_1}{\partial y_1} & \frac{\partial h_1}{\partial y_2} \\ \frac{\partial h_2}{\partial y_1} & \frac{\partial h_2}{\partial y_2} \end{pmatrix}, \quad \underset{\sim}{y} = (y_1, y_2) \in J$$

$$= \frac{\partial h_1}{\partial y_1}(\underset{\sim}{y})\frac{\partial h_2}{\partial y_2}(\underset{\sim}{y}) - \frac{\partial h_1}{\partial y_2}(\underset{\sim}{y})\frac{\partial h_2}{\partial y_1}(\underset{\sim}{y}),$$

$$\underset{\sim}{y} = (y_1, y_2) \in J.$$

Answer to Question 1 of Pg 130

We shall call g a "smooth" function if all the partial derivatives $\frac{\partial h_i}{\partial y_j}$, $i=1,2, j=1,2$ exist and are cont on J, and the determinant

$$\frac{dg^{-1}(\underset{\sim}{y})}{d\underset{\sim}{y}} = \det\left(J_{g^{-1}}(\underset{\sim}{y})\right) \neq 0 \quad \forall \underset{\sim}{y} \in J.$$

Thm (Change of Bivariate Joint Density Formula)

Suppose $I, J \subseteq \mathbb{R}^2$ are two open path-connected sets and $g: I \to J$ is a bijective and "smooth" (as described above) map. If $\underset{\sim}{X} = (X_1, X_2)$ is a cont random vector with a joint pdf $f_{\underset{\sim}{X}}$ that vanishes on I^c (this means $\text{Range}(\underset{\sim}{X}) \subseteq I$), then $\underset{\sim}{Y} = (Y_1, Y_2) := g(\underset{\sim}{X}) = g(X_1, X_2)$ is also a cont random vector with a joint pdf

$$f_{\underset{\sim}{Y}}(\underset{\sim}{y}) = \begin{cases} f_{\underset{\sim}{X}}(g^{-1}(\underset{\sim}{y}))\left|\frac{dg^{-1}(\underset{\sim}{y})}{d\underset{\sim}{y}}\right| & \text{if } \underset{\sim}{y} \in J, \\ 0 & \text{if } \underset{\sim}{y} \notin J. \end{cases}$$

Here $\frac{dg^{-1}(\underset{\sim}{y})}{d\underset{\sim}{y}}$ is the determinant defined in Pg 132.

Remarks: ① As in the univariate case, if $Range(\underset{\sim}{X}) = I$, then $Range(\underset{\sim}{Y}) = J$. This will indeed be the case in most examples.

② As we shall see, calculation of $\frac{dg^{-1}(\underset{\sim}{y})}{d\underset{\sim}{y}}$ can be computationally intensive. ~~but~~

③ The most challenging part of application of change of joint density formula is to figure out J and I correctly. While in most cases, $I = Range(\underset{\sim}{X})$, we may have to use a cleverly chosen joint density of $\underset{\sim}{X}$ (and thus ~~choosing~~ choose I accordingly) so that the change of joint density formula ~~can~~ becomes applicable. Finding out J correctly can be quite challenging.

④ The notations used in the change of joint density formula will be clearer when we go through a few examples. The proof needs bivariate change of variable formula for integrals and hence is skipped.

Example: Suppose $X_1, X_2 \overset{iid}{\sim} Unif(0,1)$. Find the joint distⁿ of $Y_1 := X_1 + X_2$ and $Y_2 := X_1 - X_2$. Using this, find the distⁿ of Y_1.

Note: The second part is Method ③ (see Pg 80) of finding the distⁿ of Y_1.

Solution: $X_1, X_2 \overset{iid}{\sim} Unif(0,1)$

$\Rightarrow$ a joint pdf of (X_1, X_2) is given by

$$f_{X_1,X_2}(x_1,x_2) = \begin{cases} 1 & \text{if } (x_1,x_2) \in (0,1)^2, \\ 0 & \text{if } (x_1,x_2) \notin (0,1)^2. \end{cases}$$

Exc: Check that $I \doteq Range(X_1, X_2) = (0,1)^2$ is open (either use the defn or use the thm stated in Pg 126) and path-connected (just check it visually).

Define $g : I \longrightarrow \mathbb{R}^2$ by

$$g(x_1, x_2) = (x_1 + x_2,\ x_1 - x_2).$$

Question: What is $g(I) \doteq J$?

Note that $g(I) \subseteq (0,2) \times (-1,1)$.

However, the above inclusion is actually strict.

~~Guesswork~~ We need to find

$$g(I) = \{(y_1,y_2) \in \mathbb{R}^2 : (y_1,y_2) = g(x_1,x_2) \text{ for some } (x_1,x_2) \in I\}.$$

$= \{(y_1, y_2) \in \mathbb{R}^2 : (y_1, y_2) = (x_1 + x_2, x_1 - x_2)$ for some $\cancel{(x_1, x_2)}$ $(x_1, x_2) \in I = (0,1)^2\}$

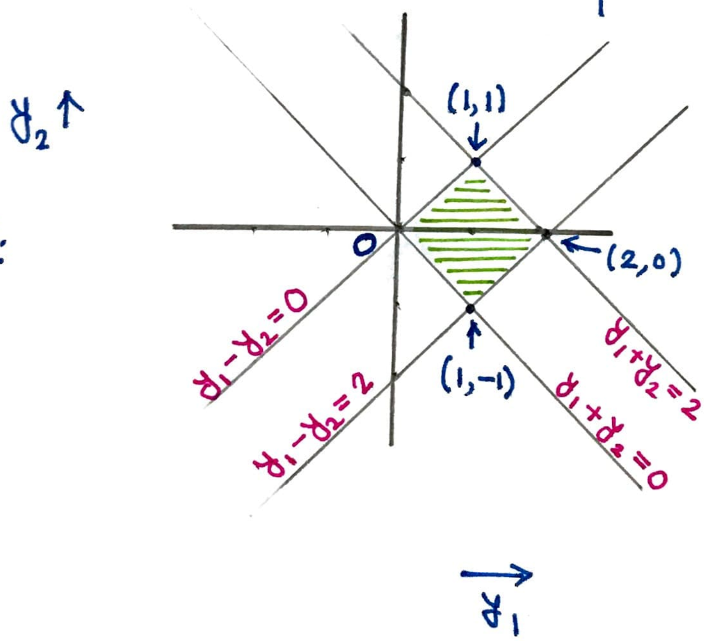
$= \{(y_1, y_2) \in \mathbb{R}^2 : (x_1, x_2) = \left(\frac{y_1 + y_2}{2}, \frac{y_1 - y_2}{2}\right) \in (0,1)^2\}$

$= \{(y_1, y_2) \in \mathbb{R}^2 : 0 < y_1 + y_2 < 2, \ 0 < y_1 - y_2 < 2\}.$

Therefore $J := \{(y_1, y_2) : 0 < y_1 + y_2 < 2, \ 0 < y_1 - y_2 < 2\}.$

Exc: Show that J is open and path-connected.

Picture of J:



Note that in the process, we have solved imp portions of the ~~the~~ following exercise:

Exc: Show that $g : I \to J$ is one-to-one and onto.

The inverse map $g^{-1} : J \to I$ is given by

$$g^{-1}(y_1, y_2) = \left(\frac{y_1 + y_2}{2}, \frac{y_1 - y_2}{2}\right), \quad (y_1, y_2) \in J.$$

Therefore, in the notations introduced in Pg 130 - 131, the maps $h_1 : J \to \mathbb{R}$ and $h_2 : J \to \mathbb{R}$ are given by

$$h_1(y_1, y_2) = \frac{y_1 + y_2}{2}, \quad (y_1, y_2) \in J,$$

$$h_2(y_1, y_2) = \frac{y_1 - y_2}{2}, \quad (y_1, y_2) \in J$$

so that

$$g^{-1}(y_1, y_2) = \left(h_1(y_1, y_2),\ h_2(y_1, y_2)\right), \quad (y_1, y_2) \in J.$$

Hence the Jacobian matrix of g^{-1} is given by

$$J_{g^{-1}}(y_1, y_2) = \begin{pmatrix} \frac{\partial h_1}{\partial y_1} & \frac{\partial h_1}{\partial y_2} \\ \frac{\partial h_2}{\partial y_1} & \frac{\partial h_2}{\partial y_2} \end{pmatrix}, \quad (y_1, y_2) \in J$$

$$= \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}, \quad (y_1, y_2) \in J$$

$$\Rightarrow \det(J_{g^{-1}}(\underset{\sim}{y})) = \left[\tfrac{1}{2} \times \left(-\tfrac{1}{2}\right)\right] - \left[\tfrac{1}{2} \times \tfrac{1}{2}\right], \quad (y_1, y_2) \in J$$

$$\Rightarrow \det\left(J_{g^{-1}}(y_1, y_2)\right) = -\frac{1}{2} \quad , \quad (y_1, y_2) \in J$$

In particular, this means that g is indeed "smooth" in the sense of Pg 133, i.e., the partial derivatives $\frac{\partial h_1}{\partial y_1}$, $\frac{\partial h_1}{\partial y_2}$, $\frac{\partial h_2}{\partial y_1}$ and $\frac{\partial h_2}{\partial y_2}$ ~~are~~ exist and are cont on J, and

~~the~~ $\cancel{\det\left(J_{g^{-1}}(y_1, y_2)\right)}$ $\frac{dg^{-1}(\underset{\sim}{y})}{d\underset{\sim}{y}} = \det\left(J_{g^{-1}}(\underset{\sim}{y})\right) \neq 0$
$(\equiv -\frac{1}{2})$

for all $\underset{\sim}{y} = (y_1, y_2) \in J$.

We have checked that all the assumptions of the bivariate change of joint density formula are satisfied. Therefore it follows that $\underset{\sim}{Y} = (Y_1, Y_2)$ is also a cont random vector with a joint pdf

$$f_{Y_1, Y_2}(y_1, y_2) = f_{X_1, X_2}\left(g^{-1}(y_1, y_2)\right) \left| \frac{dg^{-1}(\underset{\sim}{y})}{d\underset{\sim}{y}} \right| , \quad \underset{\sim}{y} \in J$$

$$= 1 \cdot \left| -\frac{1}{2} \right| , \quad \underset{\sim}{y} = (y_1, y_2) \in J$$

$$= \frac{1}{2} , \quad (y_1, y_2) \in J$$

$\Rightarrow \underset{\sim}{Y} \sim \text{Unif}(J)$, since $\text{Area}(J) = 2$.

We have proved: $(Y_1, Y_2) \sim \text{Unif}(J)$, i.e.,
Y_1 and Y_2 are jointly cont with a joint pdf

$$f_{Y_1,Y_2}(y_1,y_2) = \begin{cases} \frac{1}{2} & \text{if } 0<y_1+y_2<2,\ 0<y_1-y_2<2, \\ 0 & \text{otherwise.} \end{cases}$$

Exc: Using the above joint pdf, find ~~the~~ marginal pdfs of Y_1 and Y_2.
[This will finally solve the problem given in the example in Pg 80 using Method 3.]

Exc: Suppose $X_1, X_2 \overset{iid}{\sim} N(0,1)$. Show that $Y_1, Y_2 \overset{iid}{\sim} N(0,2)$, where
$Y_1 \coloneqq X_1 + X_2$ and $Y_2 \coloneqq X_1 - X_2$.

Exc: Suppose $X_1 \perp\!\!\!\perp X_2$, $X_1 \sim \text{Gamma}(\alpha_1, \lambda)$ and $X_2 \sim \text{Gamma}(\alpha_2, \lambda)$. Show that ~~$X_1+X_2 \sim$ Gamma~~
$X_1 + X_2 \sim \text{Gamma}(\alpha_1+\alpha_2, \lambda)$, $\frac{X_1}{X_1+X_2} \sim \text{Beta}(\alpha_1, \alpha_2)$,
and $X_1 + X_2 \perp\!\!\!\perp \frac{X_1}{X_1+X_2}$. [See Remark 2 of Pg 115.]