## A Few Natural Connections to Linear Algebra

Let $V$ be the vector space (over $\mathbb{R}$) of all r.v.s defined on a common sample space $\Omega$ with a probability $P$ on it. In particular, each vector $X \in V$ is a map $X : \Omega \to \mathbb{R}$. We already know from the thm stated in Pg 151 (linearity of expectation, which holds for all r.v.s, not just jointly discrete or jointly cont ones) that $S_1 := \{X \in V : X \text{ has finite mean}\}$ is a vector subspace of $V$ (see Remark 1 of Pg 151).

<u>Exc:</u> Show that

$$S_2 := \{X \in V : X \text{ has finite } 2^{nd} \text{ moment}\}$$

is a vector subspace of $S_1$ and hence of $V$.

[<u>Hint:</u> You may use $(X+Y)^2 \leq 2(X^2+Y^2)$ and the Remark in Pg 163.]

<u>Exc:</u> Show that $\langle X, Y \rangle := E(XY)$ is a real valued symmetric $\mathbb{R}$-bilinear ~~form~~ form defined on $S_2 \times S_2$.

The last exc in Pg 200 means the following:

$$(X, Y) \longmapsto E(XY) \doteqdot \langle X, Y\rangle$$

is a map $S_2 \times S_2 \longrightarrow \mathbb{R}$ (note that this map is well-defined because of the inequality $|XY| \leqslant \frac{X^2+Y^2}{2}$ and the remark stated in Pg 163) satisfying

- <u>Symmetry</u>: $\langle X, Y\rangle = \langle Y, X\rangle$ $\forall\, X, Y \in S_2$.
- <u>$\mathbb{R}$-Bilinearity</u>: $\langle \alpha_1 X_1 + \alpha_2 X_2, Y\rangle = \alpha_1 \langle X_1, Y\rangle + \alpha_2 \langle X_2, Y\rangle$

$\forall\, X_1, X_2, Y \in S_2$ and $\forall\, \alpha_1, \alpha_2 \in \mathbb{R}$.

<u>Remark</u>: Clearly symmetry + $\mathbb{R}$-bilinearity in the first component (as above) $\Rightarrow$ $\mathbb{R}$-bilinearity in the second component.

<u>Question</u>: Is $\langle \cdot, \cdot \rangle$ defined above an ~~innear~~ inner product on $S_2$?

<u>Answer</u>: No.

Question: What prevents $\langle \cdot, \cdot \rangle$ defined in Pg 200 - 201 ~~to~~ from becoming an inner product on $S_2$?

Answer: Note that for all $X \in S_2$,

$\langle X, X \rangle = E(X^2) \geq 0$. However

$\langle X, X \rangle = E(X^2) = 0$ may not imply

$X$ is the zero map.

e.g. - Take $\Omega = \{1, 2, 3\}$ and

a prob $P$ on $\Omega$ defined by

$$P(\{1\}) = P(\{2\}) = \frac{1}{2} \text{ and } P(\{3\}) = 0.$$

Define $X : \Omega \to \mathbb{R}$ by

$X(1) = X(2) = 0$ but $X(3) = 1729$.

Then $E(X^2) = 0$ (i.e., $\langle X, X \rangle = 0$)

but $X$ is not the ~~zero~~ zero map.

Note: In this example, $P(X = 0) = 1$.

Algebraically speaking, $\langle \cdot, \cdot \rangle$ is a symmetric, $\mathbb{R}$-bilinear, nonnegative-definite (i.e, $\langle X, X \rangle \geq 0 \ \forall \ X \in S_2$) form on $S_2$ but it is not necessarily positive-definite (i.e., $\langle X, X \rangle = 0$ even though $X \neq 0$). This prevents $\langle \cdot, \cdot \rangle$ from being an inner product on $S_2$.

Question: How to turn $\langle \cdot, \cdot \rangle$ (defined in Pg 200 - 201) into an inner product?

Answer: Even though positive-definiteness fails to hold on $S_2$, the following result can always be shown to hold:

$$\langle X, X \rangle = E(X^2) = 0 \Rightarrow P(X = 0) = 1.$$

Therefore, informally speaking, we need to redefine the equality, which is "too stringent" on $S_2$. ~~The~~

**Question:** How to "redefine equality (on $S_2$)" so that $\langle \cdot , \cdot \rangle$ becomes an inner product?

**Answer:** We need to look at an appropriate quotient vector space.

**Exc:** Show that

$$T_2 := \{ X \in S_2 : P(X=0) = 1 \}$$

is a vector subspace of $S_2$.

[Hint: Observe that

$$(X_1 = 0) \cap (X_2 = 0) \subseteq (X_1 + X_2 = 0).]$$

**Defn:** Define $L^2(\Omega, P)$ to be the quotient vector space

$$L^2(\Omega, P) := S_2 / T_2 .$$

**Remarks:** ① Note that $L^2(\Omega, P)$ is the ~~spaces~~ space of all equivalence classes, ~~wh~~ under the equivalence relation $\sim$ defined on $S_2$ by

$$X \sim Y \quad \text{if and only if} \quad P(X=Y)=1.$$

In other words, elements of $L^2(\Omega, P)$ are not functions but equivalence classes of real valued functions (more specifically, r.v.s) defined on $\Omega$ with finite second moments.

② Informally, we shall understand $L^2(\Omega, P)$ to be the space of all (real valued) r.v.s defined on $\Omega$ with finite second moments such that the equality of two such r.v.s is understood (in the "almost sure sense" or) "with probability 1", i.e., we write $X=Y$ to mean $X \sim Y$ of ①, i.e.,

$X = Y$ on $L^2(\Omega, P)$ is same as saying $P(X=Y)=1$. We shall use this informal understanding of $L^2(\Omega, P)$ (and its equality) throughout this course.

<u>Question</u>: Is $\langle \cdot, \cdot \rangle$ a well-defined symmetric $\mathbb{R}$-bilinear form on the quotient vector space

$$L^2(\Omega, P) = S_2 / T_2 \ ?$$

In other words, in light of Remark 2 of Pg 205 - 206, the above ~~the~~ question can be restated as follows.

<u>Question</u>: Take $X, X', Y, Y' \in S_2$ such that $P(X=X') = P(Y=Y') = 1$. Then is it true that

$$E(XY) = E(X'Y') \ ?$$

Answer: YES!

Exc: ① Show that if $P(X = X') = 1$ and $P(Y = Y') = 1$, then

$$P(XY = X'Y') = 1.$$

(In other words, product of r.v.s honours the equality on ~~the~~ any $L^2$-spaces.)

[Hint: Observe that

$$(X = X') \cap (Y = Y') \subseteq (XY = X'Y')]$$

② Suppose $Z$ and $Z'$ are two r.v.s defined on the same sample space and $P(Z = Z') = 1$.

(a) If $Z$ is a discrete r.v., then show that $Z'$ is also discrete and $Z, Z'$ have the same pmf.

[Hint: Fix $z \in \mathbb{R}$. Show that $P(Z = z) \leq P(Z' = z)$.]

(b) If $Z$ is a cont r.v. with a pdf $h$, then $Z'$ is also a cont r.v. with a pdf $h$.

[Hint: Fix $u \in \mathbb{R}$. Show that $P(Z \leq u) \leq P(Z' \leq u)$.]

(c) Assume $Z$ is either discrete or cont. If $Z$ has finite mean, then show that $Z'$ also has finite mean and

$$E(Z) = E(Z').$$

Remark: Note that the conclusion of ~~Exc~~ Exc 2 (c) holds for any r.v. $Z$ (not necessarily discrete or cont) but the proof is beyond our scope simply because we have not even defined expectation for any r.v..

Exc: Using Exc 1 + above remark, show that $\langle X, Y \rangle := E(XY)$ is a well-defined real valued symmetric $\mathbb{R}$-bilinear form on the quotient vector space $L^2(\Omega, P) := S_2 / T_2$.