Lecture - 10/12/2020

- Gaussian elimination
- Pivoting
- Row echelon form
- Reduced row echelon
- Rank
- Nullspace/Nullity
- Nullspace basis

Gaussian Elimination (continued)

$$\begin{aligned} x_1 + x_2 + x_3 &= 1 \\ x_1 + x_2 + 2x_3 &= 2 \end{aligned}$$

$$\begin{aligned} 3x_1 + x_2 + 4x_3 &= 3 \\ x_1 + x_2 + x_3 &= 1 \\ x_1 + x_2 + 2x_3 &= 2 \end{aligned}$$

eq. : $x_1 + x_2 + x_3 = 1$, $x_3 = 1$, $x_1 + x_2 = 0$

$\Rightarrow$ Pivot/free = variable

single solution $\Rightarrow$ pivot variable

∞ $\Rightarrow$ free variable

$A \cdot x = b$

- $A = [a_1 | a_2 | \dots | a_n]$ $(m \times n)$
- $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ $(n \times 1)$
- $b = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$ $(m \times 1)$

if m^{th} variable,

$x_m = b^{th}$

$m > n$: more equations than variables

$m = n$: square system $(m \times n)$

$m < n$: more equations than variables

$\Rightarrow$ ∞ solutions

$x_1 + x_2 = 1$

$x_1 + x_2 = 2$

$\Rightarrow$ x_1 and x_2 are independent

$x_1 = \beta x_2 + \beta x_3$

$x_2 = \beta x_3$

$\Rightarrow$ single solution

$x = (\beta x_3, \beta x_3, x_3)$

- Multiple linear equations

$x_1 + x_2 = 1$

$x_1 + x_2 = 2$

$x_1 + x_2 = 3$

$\Rightarrow$ x_1 and x_2 are independent

$A \cdot x = b$

Gaussian (A^{-1}) $A \cdot x = b$

$\Rightarrow$ $A \cdot x = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ Check

$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ solution $x = x_0$

$m < n$

$\exists x_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\exists x_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

- Check with $x = x_0$
- Find linear combination of the variables

$\Rightarrow$ single solution

$x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$A \cdot x = b$

Rank : $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

- Subtract $A \cdot x = b$

$A \cdot x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$A \cdot x = \vec{b}$

$A \cdot x_2 = \vec{b}_2$

$A \cdot y = 1$ $[x_2, x_2, x_2, y]$

$[x_1] \cdot [x_2 \cup \vec{x}_2 \cup \vec{x}_2 \cup y]$

$= [x_1 \cup \vec{x}_2 \cup \vec{x}_2 \cup \vec{x}_2]$

- Invertible transformations

∞ Exchanging $\vec{x}_1$ and $\vec{x}_2$ $\quad y = b_1 \cup b_2$

$\alpha \cdot \vec{x}_1 + \beta \cdot \vec{x}_2$ $\quad \cup b_m \cup 2$

$\vec{x}_1 = \vec{x}_2 + b_1 \cup b_2 \cup 2$

(1d) $a^{\alpha}_{2,k} = \alpha \cdot a_{1,k} + \beta \cdot a_{2,k}$

$\Rightarrow g_{2,k} = \alpha \cdot g_{1,k} + \beta \cdot g_{2,k}$

(2d) Interchange $a^{\alpha}_{1} \leftrightarrow a^{\beta}_{2}$ $\quad$ $a_{1,k} = a_{2,k}$, $a^{\alpha}_{2,k} = a_{1,k}$

- $G - \bar{J}$ elimination $\vdash$ Reduce A to I by repeating (1), (2), (3) $\Rightarrow$ EIRR $\Rightarrow$ augmented solution

$A \cdot x = b$

a_{ik} : entry from $i \to j$

$\Rightarrow$ conta a_{ik} by a^{α}_{ik}

stem for array of mono a_{ik} multiplication $\left(\begin{array}{l} a_{ik} \cdot x_k = 0 \ \forall i, k \\ \Rightarrow \text{step } 2 \text{ of } i \to j \end{array} \right)$

$\Rightarrow$ $\dfrac{a_{ik}}{a_{jk}}$ $\Big\}$ common $a_{ik} \neq 0$

Reduce $a_{ik} = 0$ to zero

$a^{n}_{i0} = a_{i0}$

$\Rightarrow$ (a_{11}) $(a_{11} \neq 0)$

$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ a_{ii} : pivot

- partial pivoting
- full pivoting

$P_{ii} = I$ iff $(i = m)(i \neq n)$

$P_{mn} = P_{nm} = e_i$

$= 0$ o/w.

$PA \vdash$ Exchange m, n rows

$AP \vdash$ Exchange m, n columns

$A \cdot x = b$

$\underbrace{[\dots \cdot E_2 \cdot E_1 \cdot E_0 \cdot A]}_{I} \cdot \vec{x} = (E_2 \cdot E_1 \cdot E_0) \vec{b}$

$\vec{x} = (\dots E_2 \cdot E_1 \cdot E_0) \vec{b}$

$A \cdot \vec{x} = \vec{b}$

$(A \cdot a_0)(a_0^{-1} \cdot \vec{x}) = \vec{b}$

$\underbrace{(A \cdot a_0 \cdot a_1 \cdot a_2 \dots)}_{\parallel} (\dots a_2^{-1} \cdot a_1^{-1} \cdot a_0^{-1}) \cdot \vec{x} = \vec{b}$

$a_1 \dots a_2^{-1} \cdot a_1^{-1} \cdot a_0^{-1} \cdot \vec{x} = \vec{b}$

$\Rightarrow \vec{x} = (a_0 \cdot a_1 \cdot a_2 \dots) \vec{b}$