Analysis of Several Variables

Homework 2

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Solution 1

Let, $\{e_i\}_{i=1}^{n}$ be the standard ordered basis of $\mathbb{R}^n$. Let, $a \in \mathbb{R}^n$. Fix, $i \in [n]$
Define, for some $\delta > 0$, $\eta_i : (-\delta, \delta) \to \mathbb{R}^n$ with $\eta_i(t) = a + te_i$
Then,

$$\begin{aligned} \lim_{t \to 0^+} \frac{f(\eta_i(t)) - f(a)}{t} &= \lim_{t \to 0^+} \frac{|a_i + t| - |a_i|}{t} \\ &= \begin{cases} 1 & \text{if } a_i > 0 \\ -1 & \text{if } a_i < 0 \\ 1 & \text{if } a_i = 0 \end{cases} \end{aligned}$$

And,

$$\begin{aligned} \lim_{t \to 0^-} \frac{f(\eta_i(t)) - f(a)}{t} &= \lim_{t \to 0^-} \frac{|a_i + t| - |a_i|}{t} \\ &= \begin{cases} 1 & \text{if } a_i > 0 \\ -1 & \text{if } a_i < 0 \\ -1 & \text{if } a_i = 0 \end{cases} \end{aligned}$$

Therefore, $\frac{\partial f}{\partial x_i}$ exist for every $a \in \mathbb{R}^n - \{x \in \mathbb{R}^n : x_i = 0\}$. Hence, f is differentiable only at $\mathbb{R}^n - \bigcup_{i=1}^{n} \{x \in \mathbb{R}^n : x_i = 0\}$ ■

Solution 2

Let, $\mathscr{O}_n \subseteq \mathbb{R}^n$ be a convex set. $f : \mathscr{O}_n \to \mathbb{R}^m$ be a differentiable function having bounded partial derivatives. Let, $f = (f_1, \ldots, f_m)$. So, partial derivatives of each f_i is bounded, hence the norm of gradient of each f_i is bounded. Let, $M = 1 + \max_{i \in [n]} \left\{ \sup_{\mathscr{O}_n} \|\nabla f_i\| \right\}$. Let, $a, b \in \mathscr{O}_n$. By Multivariate Mean Value Theorem, for each i $\exists\, c_i \in \{tb + (1-t)a : t \in [0, 1]\}$ such that $f_i(b) - f_i(a) =$

$(\nabla f_i)(c_i) \cdot (b-a)$. For, $\epsilon > 0$ take $\delta = \frac{\epsilon}{M}$. Then, $\|a-b\| < \delta$ yields,

$$\begin{aligned}
\|f(b)-f(a)\| &= \sqrt{\sum_{i=1}^{m} (f_i(b)-f_i(a))^2} \\
&= \sqrt{\sum_{i=1}^{m} ((\nabla f_i)(c_i)\cdot(b-a))^2} \\
&\leq \sqrt{\sum_{i=1}^{m} \|(\nabla f_i)(c_i)\|^2 \|b-a\|^2} && \text{(By Cauchy-Schwarz inequaity)} \\
&= \|b-a\| \sqrt{\sum_{i=1}^{m} \|(\nabla f_i)(c_i)\|^2} \\
&< \delta M = \epsilon
\end{aligned}$$

Hence, f is uniformly continuous which is even true for $n = 2$. ■

Solution 3

Derivative of given f exists iff derivative of each component of f exists which is true because polynomial and exponential functions are differentiable on $\mathbb{R}$ and differentiation is closed under multiplication. So, the derivative of f at $(x,y,z,w) \in \mathbb{R}^4$ is

$$\begin{aligned}
[(Df)(x,y,z,w)] &= \begin{bmatrix} \frac{\partial}{\partial x}(xe^y) & \frac{\partial}{\partial y}(xe^y) & \frac{\partial}{\partial z}(xe^y) & \frac{\partial}{\partial w}(xe^y) \\ \frac{\partial}{\partial x}(ze^{-w}) & \frac{\partial}{\partial y}(ze^{-w}) & \frac{\partial}{\partial z}(ze^{-w}) & \frac{\partial}{\partial w}(ze^{-w}) \\ \frac{\partial}{\partial x}(e^x) & \frac{\partial}{\partial y}(e^x) & \frac{\partial}{\partial z}(e^x) & \frac{\partial}{\partial w}(e^x) \end{bmatrix} \\
&= \begin{bmatrix} e^y & xe^y & 0 & 0 \\ 0 & 0 & e^{-w} & -ze^{-w} \\ e^x & 0 & 0 & 0 \end{bmatrix} \\
\implies [(Df)(0,-\ln 6, \ln 2, 1)] &= \begin{bmatrix} \frac{1}{6} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{e} & \frac{-\ln 2}{e} \\ 1 & 0 & 0 & 0 \end{bmatrix}
\end{aligned}$$

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Solution 4

For, $\epsilon > 0$ take $\delta = \sqrt[4]{\epsilon}$. Then, $(x,y) \in D_\delta((0,0)) \implies \left|\frac{x^2y^4}{x^2+y^2} - 0\right| < y^4 < (x^2+y^2)^2 < \delta^4 = \epsilon$. So, f is continuous at $(0,0)$ and hence on $\mathbb{R}^2$.

$$\lim_{(x,y)\to(0,0)} \frac{|f(x,y) - f(0,0) - (0,0)\cdot(x,y)|}{\|(x,y)\|} = \lim_{(x,y)\to(0,0)} \frac{\left|\frac{x^2y^4}{x^2+y^2}\right|}{\sqrt{x^2+y^2}}$$

For $\epsilon > 0$ take $\delta = \sqrt[3]{\epsilon}$. So, $(x,y) \in D_\delta((0,0)) \implies \left|\frac{x^2y^4}{(x^2+y^2)^{\frac{3}{2}}}\right| = x^2|y| \left|\frac{|y|^3}{(x^2+y^2)^{\frac{3}{2}}}\right| < x^2|y| < \delta^3 = \epsilon$

Thus, the limit above is 0 and hence f is differentiable at $(0,0)$ with $(Df)((0,0)) = (0,0)$.

Now define, $g : \mathbb{R}^2 \to \mathbb{R}$ and $h : \mathbb{R}^2 \to \mathbb{R}$ with

$$g(x,y) = \begin{cases} \frac{2xy^6}{(x^2+y^2)^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

$$h(x,y) = \begin{cases} \frac{2x^2y^3(2x^2+y^2)}{(x^2+y^2)^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

We will show that both g and h are continuous . We already have f, g being rational functions, are continuous on $\{(0,0)\}^c$.

- For, $\epsilon_g > 0$ take $\delta_g = \sqrt[3]{\frac{\epsilon_g}{2}}$. Then, $(x,y) \in D_{\delta_g}((0,0)) \implies \left|\frac{2xy^6}{(x^2+y^2)^2} - 0\right| < 2|x|y^2 < 2\delta_g \cdot \delta_g^2 = \epsilon$. Thus, g is continuous at $(0,0)$ and hence on $\mathbb{R}^2$.
- For, $\epsilon_h > 0$ take $\delta_h = \sqrt[3]{\frac{\epsilon_h}{4}}$. Then, $(x,y) \in D_{\delta_h}((0,0)) \implies \left|\frac{2x^2y^3(2x^2+y^2)}{(x^2+y^2)^2} - 0\right| < 2|y|(2x^2 + y^2) < 2\delta_h(\delta_h^2 + \delta_h^2) = \epsilon$. Thus, h is continuous at $(0,0)$ and hence on $\mathbb{R}^2$.

Indeed, $g = f_x$ and $h = f_y$ which shows that $f \in C^1(\mathbb{R}^2)$. ■

Solution 5
(This question is wrong!)

$$\lim_{(x,y)\to(0,0)} \frac{|f(x,y) - f(0,0) - (0,0)\cdot(x,y)|}{\|(x,y)\|} = \lim_{(x,y)\to(0,0)} \frac{\left|x^3 \sin\frac{1}{x}\right|}{\sqrt{x^2+y^2}}$$

For $\epsilon > 0$ take $\delta = \sqrt{\epsilon}$. So, $(x,y) \in D_\delta((0,0)) \implies \frac{\left|x^3 \sin\frac{1}{x}\right|}{\sqrt{x^2+y^2}} < x^2 < \delta^2 = \epsilon$. Thus, the limit above is 0 and hence f is differentiable at $(0,0)$ with $(Df)((0,0)) = (0,0)$. Therefore,

$$f_x(x,y) = \begin{cases} 3x^2 \sin\frac{1}{x} - x\cos\frac{1}{x} & \text{if } (x,y) \neq 0 \\ 0 & \text{if } (x,y) = 0 \end{cases}$$

For $\epsilon' > 0$ take $\delta' = \frac{\sqrt{12\epsilon'+1}-1}{6} > 0$. So, $(x,y) \in D_{\delta'}((0,0)) \implies \left|3x^2\sin\frac{1}{x} - x\cos\frac{1}{x}\right| < 3x^2 + |x| < 3\delta'^2 + \delta' = \epsilon'$. Hence, f_x is continuous at $(0,0)$. ■

Solution 6
(This question is wrong!)
Let $\mathscr{O}_n^{(1)}, \mathscr{O}_n^{(2)} \subseteq \mathbb{R}^n$ be two disjoint open sets. Take, $\mathscr{O}_n = \mathscr{O}_n^{(1)} \sqcup \mathscr{O}_n^{(2)}$, so, $\mathscr{O}_n$ is open. Let, $f : \mathscr{O}_n \to \mathbb{R}$ such that, $f(x) = \begin{cases} 1 & \text{if } x \in \mathscr{O}_n^{(1)} \\ 0 & \text{if } x \in \mathscr{O}_n^{(2)} \end{cases}$
Clearly, for each i, $\frac{\partial f}{\partial x_i} = 0$. But f is not a constant function. ■

Solution 7
Let, $f : \mathbb{R}^n \to \mathbb{R}^n$ be a differentiable mapping whose total derivative is a diagonal matrix (w.r.t. standard ordered basis).
$\forall\, x, y \in \mathbb{R}^n, tx + (1-t)y \in \mathbb{R}^n\ \forall t \in [0,1]$ which implies that $\mathbb{R}^n$ is convex.
Fix, $j \in [n]$. Let, $a, b \in \mathbb{R}^n$. Then, by Multivariate Mean Value Theorem, $\exists c_j \in \{tb + (1-t)a : t \in [0,1]\}$ such that $f_j(b) - f_j(a) = (\boldsymbol{\nabla} f_j)(c_j) \cdot (b-a)$. Since, total derivative of f is a diagonal matrix, $\frac{\partial f_j}{\partial x_i}(c_j) = 0\ \forall i \in [n] - \{j\}$ which implies $f_j(b) - f_j(a) = (b_j - a_j)\frac{\partial f_j}{\partial x_j}(c_j)$. For all such $a, b \in \mathbb{R}^n$ whose j^{th} coordinates are same, $f_i(b) - f_i(a) = 0$ which shows that, f_i is independent of $x_j\ \forall i \in [n] - \{j\}$.Therefore, each f_i is function of x_i, and hence the mappings of the form $f : \mathbb{R}^n \to \mathbb{R}^n$ such that $f(x) = (f_1(x_1), \ldots, f_n(x_n))$ with each f_i differentiable, yields a diagonal matrix (w.r.t. standard ordered basis) in total derivative. ■

Solution 8

$$J_f(x,y,z) = \begin{bmatrix} 1 & 1 & 1 \\ 2x & 2y & 2z \\ 3x^2 & 3y^2 & 3z^2 \end{bmatrix}$$

$$\text{So, } \det(J_f(x,y,z)) = \begin{vmatrix} 1 & 1 & 1 \\ 2x & 2y & 2z \\ 3x^2 & 3y^2 & 3z^2 \end{vmatrix} = 12(x-y)(y-z)(z-x)$$

Clearly, $J_f(x,y,z)$ is non-singular unless two of the three variables are equal. And, the locus of the singularities $= \{(x,y,z) \in \mathbb{R}^3 : x = y \text{ or } y = z \text{ or } z = x\}$ which is just union of three planes, $x = y, y = z$ and $z = x$ in $\mathbb{R}^3$. ■

Solution 9

False!

Let $\mathcal{O}_n^{(1)}, \mathcal{O}_n^{(2)} \subseteq \mathbb{R}^n$ be two disjoint open sets. Take, $\mathcal{O}_n = \mathcal{O}_n^{(1)} \sqcup \mathcal{O}_n^{(2)}$, so, $\mathcal{O}_n$ is open. Let, $f : \mathcal{O}_n \to \mathbb{R}$ such that, $f(x) = \begin{cases} (1,1,\ldots,1) & \text{if } x \in \mathcal{O}_n^{(1)} \\ (0,0,\ldots,0) & \text{if } x \in \mathcal{O}_n^{(2)} \end{cases}$

Clearly, for each i, j $\frac{\partial f_i}{\partial x_j} = 0$. But f is not a constant function. ■

Solution 10

Let, $f = (f_1, \ldots, f_n)$. So, $Df(t) = (f_1', \ldots, f_n')$.
For $t \in (a,b)$,

$$\begin{aligned} \frac{d}{dt}(\|f(t)\|^2) &= \frac{d}{dt}\sum_{i=1}^n (f_i(t))^2 \\ &= \sum_{i=1}^n \frac{d}{dt}(f_i(t))^2 \\ &= 2\sum_{i=1}^n f_i(t) f_i'(t) \\ &= 2\langle f(t), Df(t)\rangle \end{aligned}$$

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