

Physics I

Lecture 14

Recap.

$$J = \int_{x_1}^{x_2} f[y(x), y'(x); x] dx$$

the path that extremizes it, $y(x)$
is determined by the E-L eqn.

$$\frac{d}{dx} \left(\frac{\partial f}{\partial y'} \right) - \frac{\partial f}{\partial y} = 0$$

Generalization to multivariable case

$$f[y_1(x), y_1'(x), y_2(x), y_2'(x) \dots, x]$$

$$f[y_i(x), y_i'(x), x] \quad i = 1 \dots n$$

$$y_i(x) = y_i(0, x) + \alpha \eta_i(x)$$

following 1 variable derivation

$$\frac{\partial J}{\partial \alpha} = \int_{x_1}^{x_2} \sum_i \left(\frac{\partial f}{\partial y_i} - \frac{d}{dx} \left(\frac{\partial f}{\partial y_i'} \right) \right) \eta_i(x) dx$$

$$\eta_i \text{'s are independent} \quad \partial J / \partial \alpha = 0$$

$$\frac{\partial f}{\partial y_i} - \frac{d}{dx} \left(\frac{\partial f}{\partial y_i'} \right) = 0 \longrightarrow E-L \text{ eqns}$$

Hamilton's Principle & Lagrangian Dynamics

- experience shows that in inertial frames particle motion is correctly described by Newton's Laws
 $\vec{F} = \dot{\vec{p}}$
- Practical difficulties in applying Newton's Laws.
e.g. non Cartesian coordinates \leadsto motion on a sphere
projection of vector eqns on the sphere is complicated
- Constraints, example bead sliding on wire
forces of constraint are complicated and occasionally cannot get explicit expressions. \vec{F} includes all forces.

Alternative formulation of Newtonian dynamics

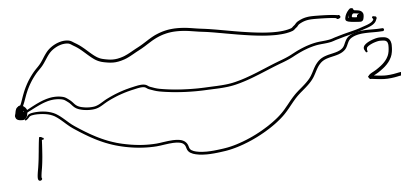
↙ Lagrange Eqs.
↕

Hamilton's Principle \equiv Newton's Laws .

() elegant, applicable to a wide variety of physical phenomena including field

We will stick to conservative systems .

Hamilton's Principle



The actual path which a particle follows between points 1 and 2 in a given time interval t_1 to t_2 is such that the action integral

$$S = \int_{t_1}^{t_2} L dt$$

is stationary when taken along the actual path

where $L = T - U$; L : Lagrangian

kinetic energy

potential energy.

Summary

In mechanics (conservative forces only):

Action S is a certain time integral which is "least*" for the true motion between initial positions at t_1 and final ones at t_2 . NOT +

In non-relativistic, no magnetic field case $S = \int (\text{Kinetic Energy} - \text{Potential Energy}) dt$

Eg. single particle, one dimension P.E. = $V(x)$; $S = \int_{t_1}^{t_2} \left[\frac{m}{2} \left(\frac{dx(t)}{dt} \right)^2 - V(x(t)) \right] dt$.

"least*" \rightarrow Not really least, just extremum \leftarrow means first order change = 0.

Prob: Find path $x(t)$ which makes $S = \int_1^2 \left[\frac{m}{2} \left(\frac{dx}{dt} \right)^2 - V(x) \right] dt$ least.

That is: for a path $x(t)$ & $x(t) + \eta(t)$ which differ from x by first order $\eta(t)$, the S must differ from $S(x)$ by zero to first order, for any $\eta(t)$ such that $\eta(t_1) = 0, \eta(t_2) = 0$



$$\begin{aligned} \frac{dx}{dt} &= \frac{d}{dt} (x + \eta) = \frac{dx}{dt} + \frac{d\eta}{dt} \\ S &= \int_1^2 \left[\frac{m}{2} \left(\frac{dx}{dt} + \frac{d\eta}{dt} \right)^2 - V(x + \eta) \right] dt \\ &= \int_1^2 \left[\frac{m}{2} \left(\frac{dx}{dt} \right)^2 - V(x) + m \frac{dx}{dt} \frac{d\eta}{dt} + \frac{m}{2} \left(\frac{d\eta}{dt} \right)^2 - \eta V'(x) \right] dt \end{aligned}$$

\therefore first order change in S is $S - S + \delta S = \int_1^2 \left[m \frac{dx}{dt} \frac{d\eta}{dt} - \eta V'(x) \right] dt$

General Rule: Must get into form $\int (\text{stuff}) \eta dt$, $\eta = 0$ at ends. Can do by integration by parts: $\int f(t) \frac{d\eta}{dt} dt = f \eta - \int \eta \frac{df}{dt} dt$; our case $f = m \frac{dx}{dt}$

$$\delta S = \left[m \frac{dx}{dt} \eta \right]_1^2 - \int_1^2 \frac{d}{dt} \left(m \frac{dx}{dt} \right) \eta dt - \int_1^2 \eta V'(x) dt$$

$= 0$ because $\eta = 0$ at t_1, t_2

$$\delta S = \int_1^2 \left(-m \frac{d^2 x}{dt^2} - V'(x) \right) \eta dt$$

any $\eta(t)$, conclude $-m \frac{d^2 x}{dt^2} - V'(x) = 0$

$$S = -mc^2 \int_1^2 \sqrt{1 - \frac{v^2}{c^2}} dt - \int_1^2 (q \mathbf{A} \cdot \mathbf{v} - \vec{v} \cdot \vec{A} q)$$

$\mathbf{v} = \frac{d\mathbf{x}}{dt}$



Prob. = $\int (\text{stuff})^2$
 To find S = sum of ampl for each path: η
 $\text{ampl} = \text{const} \propto \frac{1}{\hbar} S$

THE PRINCIPLE OF LEAST ACTION

FINAL EXAM 8⁰⁰ AM. M

SECTIONS ABCDE

SECTIONS F, G, H, J

Summary: In mechanics (conservative force)

action S is a certain time integral which

motion between initial position x_i and x_f

is independent of magnetic field case $S = \int_1^2$

Eg. single particle, one dimension, $V(x) = \frac{1}{2} kx^2$, $S = \int_1^2$

Least - Not really least, just extremum

$$S = \int_{t_1}^{t_2} L(x_i, \dot{x}_i; t) dt \quad \text{--- (1)}$$

Recall $J = \int_{x_1}^{x_2} f[y_i, y'_i; x] dx \quad \text{--- (2)}$

make the correspondence

$$x \rightarrow t$$

$$y_i(x) \rightarrow x_i(t)$$

$$y'_i(x) \rightarrow \dot{x}_i(t)$$

Euler-Lagrange eqn.

$$\frac{\partial L}{\partial x_i} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}_i} \right) = 0$$

Examples

① Free particle in 3D.

$$L = T - U, \quad U = 0.$$

$$L = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2).$$

$$\left. \begin{array}{l} L = T - U, \quad U = 0. \\ L = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2). \end{array} \right\} \begin{array}{l} \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}_i} \right) - \frac{\partial L}{\partial x_i} = 0 \\ \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}} \right) = 0 \end{array}$$

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{y}} \right) = 0, \quad \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{z}} \right) = 0$$

$$\frac{d}{dt} (m \dot{x}) = 0 = \frac{d}{dt} (m \dot{y}) = \frac{d}{dt} (m \dot{z})$$

$$\ddot{x} = \ddot{y} = \ddot{z} = 0.$$

Ex 2

1-d Harmonic Oscillator

$$L = T - U = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} k x^2 .$$

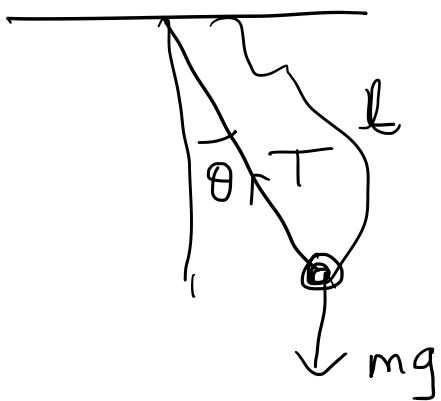
$$\frac{\partial L}{\partial x} = -kx \quad ; \quad \frac{\partial L}{\partial \dot{x}} = m \dot{x}$$

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}} \right) - \frac{\partial L}{\partial x} = 0 .$$

$$m \ddot{x} + kx = 0$$

Ex 3

Plane pendulum



$$L = \frac{1}{2} m l^2 \dot{\theta}^2 - m g l (1 - \cos \theta)$$

E-L eqn.

$$\frac{\partial L}{\partial \theta} = -m g l \sin \theta$$

$$\frac{\partial L}{\partial \dot{\theta}} = m l^2 \dot{\theta}$$

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\theta}} \right) - \frac{\partial L}{\partial \theta} = 0$$

$$\ddot{\theta} + \frac{g}{l} \sin \theta = 0$$

Identical to Newtonian result

Generalized Coordinates

Degrees of freedom : In general for n ^(free) particles is $3n$.

m constraints

degrees of freedom $\boxed{s = 3n - m}$

Need to choose s coordinates to describe motion
need not be Cartesian coordinates (can choose
curvilinear coordinates, spherical, cylindrical)
need not even have dimensions of length.

generalized coordinates $\{q_i\} \rightarrow$ not unique

generalized coordinates $\{q_j\}$ + generalized velocities $\{\dot{q}_j\}$

Coordinate transformations

$$\left\{ \begin{aligned} x_{\alpha,i} &= x_{\alpha,i}(q_1, \dots, q_s, t) \\ &= x_{\alpha,i}(q_j, t) \end{aligned} \right\}$$
$$\dot{x}_{\alpha,i} = \dot{x}_{\alpha,i}(q_j, \dot{q}_j, t)$$

where $\alpha = 1, \dots, n$

$i = 1, 2, 3$

$j = 1, \dots, s$

$m = 3n - s$

Inverse transform

$$\left\{ \begin{aligned} q_j &= q_j(x_{\alpha,i}, t) \\ \dot{q}_j &= \dot{q}_j(x_{\alpha,i}, \dot{x}_{\alpha,i}, t) \end{aligned} \right\}$$

+ constraint eqn

$$f_k(x_{\alpha,i}, t) = 0$$
$$k = 1, \dots, m$$