Analysis of Several Variables

Homework 1

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Solution 1

For, $x, y \in \mathbb{R}^n$, $\|x-y\| = \|x+(-y)\| \leq \|x\| + \|-y\| = \|x\| + \|y\|$
Also, $\|y\| = \|x+y-x\| \leq \|x\| + \|y-x\| = \|x\| + \|x-y\|$

$$\text{,i.e., } -\|x-y\| \leq \|x\| - \|y\| \tag{1.1}$$

Again, $\|x\| = \|x-y+y\| \leq \|x-y\| + \|y\|$

$$\text{,i.e., } \|x\| - \|y\| \leq \|x-y\| \tag{1.2}$$

Thus, by 1.1 and 1.2, we get

$$|\|x\| - \|y\|| \leq \|x-y\| \tag{1.3}$$

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Solution 2

Denote,

- $L(S)$ = set of all limit points of S.
- $S_r^{n-1}(a) = \{x \in \mathbb{R}^n : \|x-a\| = r\}$

(i) For $x \in B_r(a)$, let $\{x_m\} \subseteq B_r(a)$ be a sequence with $x_m = \frac{x}{m} + \left(1 - \frac{1}{m}\right) a$ which converges to x. Since, x is arbitary, we get $B_r(a) \subset L(B_r(a))$. For $s \in S_r^{n-1}(a)$ the sequence $\{y_m\} \subseteq B_r(a)$ with $y_m = \frac{s}{m} + \left(1 - \frac{1}{m}\right) a$ converges to s which implies $S_r^{n-1}(a) \subset L(B_r(a))$.
But, for $x \in (B_r(a) \cup S_r^{n-1}(a))^c$ we have $\|x-a\| > r$. Assume that $\exists$ a sequence $\{x_m\} \subseteq B_r(a)$ converging to x. Take, $\epsilon = r$. Then, $\forall\ m$, with inequality (1.3)
$\|x - y_m\| = \|(x-a) - (y_m - a)\| \geq |\|x-a\| - \|y_m - a\|| > r + r > \epsilon$ **(contradiction!)**
Hence, $L(B_r(a)) = B_r(a) \cup S_r^{n-1}(a)$ ■

(ii) We claim the following:

> **Claim 2.1.** *For $p = (p_1, \ldots, p_n) \in S_1^{n-1}(0), \exists q \in S_1^{n-1}(0)$ such that $\langle p, q \rangle = 0$*

Proof of the claim. Since, $\|p\|^2 = \sum_{j=1}^n p_j^2 = 1$, $\exists k \in \{1, \ldots, n\}$ such that $p_k \neq 0$. For, $q = (q_1, \ldots, q_n)$ take $q_k = -\frac{1}{p_k} \sum_{\substack{j=1 \\ j \neq k}}^n p_j q_j$ then we get $\langle p, q \rangle = \sum_{j=1}^n p_j q_j = 0$ ■

For, $x \in S_1^{n-1}(0)$ take $x_\perp \in S_1^{n-1}(0)$ such that $\langle x, x_\perp \rangle = 0$. Define a sequence, $\{x_m\} \subset S_1^{n-1}(0)$ with $x_m = \left(\cos\left(\frac{1}{m}\right)\right) x + \left(\sin\left(\frac{1}{m}\right)\right) x_\perp$
$\Big($We should see, $\|x_m\|^2 = \|x\|^2 \cos^2\left(\frac{1}{m}\right) + \|x_\perp\|^2 \sin^2\left(\frac{1}{m}\right) + 2\left\langle \left(\cos\left(\frac{1}{m}\right)\right) x, \left(\sin\left(\frac{1}{m}\right)\right) x_\perp \right\rangle = 1\Big)$.

Clearly, $x_m \to x$. Thus, $S_1^{n-1}(0) \subset L(S_1^{n-1}(0))$
Now, take $x \in (S_1^{n-1}(0))^c$, so $\|x\| \neq 1$. Let, $\{x_m\} \subset S_1^{n-1}(0)$ be a sequence converging to x. Take, $\epsilon = \frac{|1-\|x\||}{2}$ Then, $\forall m$, with inequality 1.3, $\|x_m - x\| \geq |\|x_m\| - \|x\|| = |1 - \|x\|| > \epsilon$ **(contradiction!)**
Therefore, $S_1^{n-1}(0) = L(S_1^{n-1}(0))$ ∎

(iii) Let, $A^k := \mathbb{Q}^k \cap (0,1)^k, k \in \mathbb{N}$
For, $z = (x,y) \in (0,1)^2$, the sequence $\{z_m\} \subset A^2$ defined by, $z_m = \left(\frac{\lfloor 10^{m-1}x \rfloor}{10^{m-1}}, \frac{\lfloor 10^{m-1}y \rfloor}{10^{m-1}}\right)$ converges to z.
The sequences $\{z_m\} \subset A^2$ defined by,

- $z_m = \left(\frac{\lfloor 10^{m-1}x \rfloor}{10^{m-1}}, \frac{1}{m}\right)$ converges to $(x,0)$ for any $x \in A^1$
- $z_m = \left(\frac{\lfloor 10^{m-1}x \rfloor}{10^{m-1}}, 1 - \frac{1}{m}\right)$ converges to $(x,1)$ for any $x \in A^1$
- $z_m = \left(\frac{1}{m}, \frac{\lfloor 10^{m-1}y \rfloor}{10^{m-1}}\right)$ converges to $(0,y)$ for any $y \in A^1$
- $z_m = \left(1 - \frac{1}{m}, \frac{\lfloor 10^{m-1}y \rfloor}{10^{m-1}}\right)$ converges to $(1,y)$ for any $y \in A^1$
- $z_m = \left(\frac{1}{m}, \frac{1}{m}\right)$ converges to $(0,0)$
- $z_m = \left(1 - \frac{1}{m}, \frac{1}{m}\right)$ converges to $(1,0)$
- $z_m = \left(\frac{1}{m}, 1 - \frac{1}{m}\right)$ converges to $(0,1)$
- $z_m = \left(1 - \frac{1}{m}, 1 - \frac{1}{m}\right)$ converges to $(1,1)$

Therefore, $[0,1]^2 \subseteq L(A^2)$

Claim 2.2. *For, $z \in ([0,1]^2)^c$ and $\forall z' \in A^2$, $\exists\, \delta > 0$ such that $\|z - z'\| > \delta$*

Proof of the claim. Let, $z = (x,y) \in [0,1]^2$
For,

- $x > 1$ or $y > 1$, take $\delta = \min\left\{\frac{|x-1|}{2}, \frac{|y-1|}{2}\right\}$
- $x < 0$ or $y < 0$, take $\delta = \min\left\{\frac{|x|}{2}, \frac{|y|}{2}\right\}$

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So by the Claim (2.2) we conclude that no point in $([0,1]^2)^c$ is a limit point of A^2
Thus, $[0,1]^2 = L(A^2)$

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Solution 3

(i) Let $f(x,y) = \frac{x^2 y}{x^4 + y^2}$ $\forall (x,y) \neq (0,0)$, let $l_1 := \{0\} \times \mathbb{R}_{>0}$ and $l_2 := \{(x,y) \in \mathbb{R}^2_{>0} : x = y^2\}$. Then,

$$f\Big|_{l_1} = 0 \text{ and } f\Big|_{l_2} = \frac{1}{2}$$

So,

$$\lim_{\substack{(x,y)\to(0,0) \\ l_1}} f = 0 \neq \lim_{\substack{(x,y)\to(0,0) \\ l_2}} f = \frac{1}{2}$$

Hence, the limit doesn't exist!

(ii) Let $h(x,y) = \frac{\sin(x^2+y)}{x^2+y}$ $\forall (x,y) \in \{(x,y) \in \mathbb{R}^2 : x^2+y=0\}^c$. Then

$$\begin{aligned} \lim_{(x,y)\to(0,0)} h &= \lim_{z\to 0} \frac{\sin(z)}{z} \\ &= 1 && ((x,y)\to(0,0) \implies z = x^2+y \to 0) \end{aligned}$$

(iii) Let $g(x,y) = \frac{x^2y^3}{x^4+y^6}$ $\forall (x,y) \neq (0,0)$, let $l_1 := \{0\} \times \mathbb{R}$ and $l_2 := \{(x,y) \in \mathbb{R}^2 : x^2 = y^3\}$. Then,

$$f\Big|_{l_1} = 0 \text{ and } f\Big|_{l_2} = \frac{1}{2}$$

So,

$$\lim_{\substack{(x,y)\to(0,0)\\ l_1}} g = 0 \neq \lim_{\substack{(x,y)\to(0,0)\\ l_2}} g = \frac{1}{2}$$

Hence, the limit doesn't exist!

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Solution 4

Define, $h : \mathbb{R}^2 \to \mathbb{R}$ with $h(x,y) = xy$. Let $(a,b) \in \mathbb{R}^2$ for $\epsilon > 0$ take, $\delta = \sqrt{4\epsilon + (|a|+|b|)^2} - (|a|+|b|) > 0$. Then, $(x,y) \in B_{\frac{\delta}{2}}((a,b))$ which gives $|y-b| < \frac{\delta}{2}, |x-a| < \frac{\delta}{2}$. So,

$$\begin{aligned} |h(x,y) - h(a,b)| &= |xy - ab| \\ &= |xy - ab + xb - xb| \\ &\leq |x|\,|y-b| + |b|\,|x-a| \\ &= |x-a+a|\,|y-b| + |b|\,|x-a| \\ &\leq |x-a|\,|y-b| + |a|\,|y-b| + |b|\,|x-a| \\ &< \frac{\delta^2}{4} + (|a|+|b|)\frac{\delta}{2} = \epsilon \end{aligned}$$

Thus, h is continuous at (a,b). Since, $f = g \circ h$ and composition of 2 real valued continuous function is continuous , we get that f is continuous . ■

Solution 5

None!

(i) Let $l_1 := \{0\} \times \mathbb{R}_{>0}$ and $l_2 := \{(x,y) \in \mathbb{R}^2_{>0} : y = 2x\}$. Then,

$$f\Big|_{l_1} = 0 \text{ and } f\Big|_{l_2} = \frac{1-8x}{5}$$

So,

$$\lim_{\substack{(x,y)\to(0,0)\\ l_1}} f = 0 \neq \lim_{\substack{(x,y)\to(0,0)\\ l_2}} f = \frac{1}{5}$$

Hence, the limit doesn't exist at $(0,0)$

(ii) Let $l_1 := \{0\} \times \mathbb{R}_{>0}$ and $l_2 := \{(x,y) \in \mathbb{R}^2_{>0} : y^2 = x\}$. Then,

$$f\Big|_{l_1} = 0 \text{ and } f\Big|_{l_2} = \frac{1}{2}$$

So,

$$\lim_{\substack{(x,y)\to(0,0) \\ l_1}} f = 0 \neq \lim_{\substack{(x,y)\to(0,0) \\ l_2}} f = \frac{1}{2}$$

Hence, the limit doesn't exist at $(0,0)$

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Solution 6

Yes!

It is clear that f is continuous at every point on $\mathbb{R}^2 - \{(0,0)\}$

For $\epsilon > 0$ take $\delta = \epsilon$. For, $(x,y) \in D_\delta((0,0))$, $\left|\frac{xy}{|x|+|y|}\right| \leq \left|\frac{xy}{|x|}\right| \leq |y| < \|(x,y)\| < \delta = \epsilon$,i.e., $\lim_{(x,y)\to(0,0)} f = 0$. Therefore, f can be extended to a continuous function on $\mathbb{R}^2$ by defining, $f(0,0) = 0$ $\blacksquare$

Solution 7

It is clear that f is continuous at every point on $\mathbb{R}^2 - \{(0,0)\}$.

For $\epsilon > 0$, take $\delta = \epsilon$. For, $(x,y) \in D_\delta((0,0))$, $\left|\frac{x^3}{x^2+y^2}\right| \leq \left|\frac{x^3}{x^2}\right| \leq \|(x,y)\| < \delta = \epsilon$,i.e., $\lim_{(x,y)\to(0,0)} f = 0$

Therefore, f is continuous at $(0,0)$. Thus, f is continuous . $\blacksquare$

Solution 8

$\Longleftarrow$: Assume that each $\Pi_i f$ is uniformly continuous . Let, $a, b \in \mathbb{R}^n$. For, $\epsilon > 0$ take $\delta > 0$ such that $|\Pi_i f(a) - \Pi_i f(b)| < \frac{\epsilon}{\sqrt{n}}$ $\forall$ $\|a - b\| < \delta$ for every i . Then, $\|a - b\| < \delta$ implies $\|f(a) - f(b)\| = \sqrt{\sum_{i=1}^n (\Pi_i f(a) - \Pi_i f(b))^2} < \epsilon$ which implies, f is uniformly continuous .

$\Longrightarrow$: Assume that, f is uniformly continuous . Let, $a, b \in \mathbb{R}^n$. Then, for $\epsilon > 0, \exists \delta > 0$ such that $\|a - b\| < \delta$ gives $\|f(a) - f(b)\| < \epsilon$ which implies, $|\Pi_i f(a) - \Pi_i f(b)| < \epsilon$ $\forall i$,i.e., each $\Pi_i f$ is uniformly continuous .

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