

Analysis of Several Variables

Homework 2

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Solution 1

Let, $\{e_i\}_{i=1}^n$ be the standard ordered basis of \mathbb{R}^n . Let, $a \in \mathbb{R}^n$. Fix, $i \in [n]$
Define, for some $\delta > 0$, $\eta_i : (-\delta, \delta) \rightarrow \mathbb{R}^n$ with $\eta_i(t) = a + te_i$
Then,

$$\begin{aligned}\lim_{t \rightarrow 0^+} \frac{f(\eta_i(t)) - f(a)}{t} &= \lim_{t \rightarrow 0^+} \frac{|a_i + t| - |a_i|}{t} \\ &= \begin{cases} 1 & \text{if } a_i > 0 \\ -1 & \text{if } a_i < 0 \\ 1 & \text{if } a_i = 0 \end{cases}\end{aligned}$$

And,

$$\begin{aligned}\lim_{t \rightarrow 0^-} \frac{f(\eta_i(t)) - f(a)}{t} &= \lim_{t \rightarrow 0^-} \frac{|a_i + t| - |a_i|}{t} \\ &= \begin{cases} 1 & \text{if } a_i > 0 \\ -1 & \text{if } a_i < 0 \\ -1 & \text{if } a_i = 0 \end{cases}\end{aligned}$$

Therefore, $\frac{\partial f}{\partial x_i}$ exist for every $a \in \mathbb{R}^n - \{x \in \mathbb{R}^n : x_i = 0\}$. Hence, f is differentiable only at $\mathbb{R}^n - \bigcup_{i=1}^n \{x \in \mathbb{R}^n : x_i = 0\}$ ■

Solution 2

Let, $\mathcal{O}_n \subseteq \mathbb{R}^n$ be a convex set. $f : \mathcal{O}_n \rightarrow \mathbb{R}^m$ be a differentiable function having bounded partial derivatives. Let, $f = (f_1, \dots, f_m)$. So, partial derivatives of each f_i is bounded, hence the norm of gradient of each f_i is bounded. Let, $M = 1 + \max_{i \in [n]} \{\sup_{\mathcal{O}_n} \|\nabla f_i\|\}$. Let, $a, b \in \mathcal{O}_n$. By Multivariate Mean Value Theorem, for each $i \exists c_i \in \{tb + (1-t)a : t \in [0, 1]\}$ such that $f_i(b) - f_i(a) =$

$(\nabla f_i)(c_i) \cdot (b - a)$. For, $\epsilon > 0$ take $\delta = \frac{\epsilon}{M}$. Then, $\|a - b\| < \delta$ yields,

$$\begin{aligned}
 \|f(b) - f(a)\| &= \sqrt{\sum_{i=1}^m (f_i(b) - f_i(a))^2} \\
 &= \sqrt{\sum_{i=1}^m ((\nabla f_i)(c_i) \cdot (b - a))^2} \\
 &\leq \sqrt{\sum_{i=1}^m \|(\nabla f_i)(c_i)\|^2 \|b - a\|^2} \quad (\text{By Cauchy-Schwarz inequality}) \\
 &= \|b - a\| \sqrt{\sum_{i=1}^m \|(\nabla f_i)(c_i)\|^2} \\
 &< \delta M = \epsilon
 \end{aligned}$$

Hence, f is uniformly continuous which is even true for $n = 2$. ■

Solution 3

Derivative of given f exists iff derivative of each component of f exists which is true because polynomial and exponential functions are differentiable on \mathbb{R} and differentiation is closed under multiplication. So, the derivative of f at $(x, y, z, w) \in \mathbb{R}^4$ is

$$\begin{aligned}
 [(Df)(x, y, z, w)] &= \begin{bmatrix} \frac{\partial}{\partial x}(xe^y) & \frac{\partial}{\partial y}(xe^y) & \frac{\partial}{\partial z}(xe^y) & \frac{\partial}{\partial w}(xe^y) \\ \frac{\partial}{\partial x}(ze^{-w}) & \frac{\partial}{\partial y}(ze^{-w}) & \frac{\partial}{\partial z}(ze^{-w}) & \frac{\partial}{\partial w}(ze^{-w}) \\ \frac{\partial}{\partial x}(e^x) & \frac{\partial}{\partial y}(e^x) & \frac{\partial}{\partial z}(e^x) & \frac{\partial}{\partial w}(e^x) \end{bmatrix} \\
 &= \begin{bmatrix} e^y & xe^y & 0 & 0 \\ 0 & 0 & e^{-w} & -ze^{-w} \\ e^x & 0 & 0 & 0 \end{bmatrix} \\
 \implies [(Df)(0, -\ln 6, \ln 2, 1)] &= \begin{bmatrix} \frac{1}{6} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{e} & -\frac{\ln 2}{e} \\ 1 & 0 & 0 & 0 \end{bmatrix}
 \end{aligned}$$
■

Solution 4

For, $\epsilon > 0$ take $\delta = \sqrt[4]{\epsilon}$. Then, $(x, y) \in D_\delta((0, 0)) \implies \left| \frac{x^2 y^4}{x^2 + y^2} - 0 \right| < y^4 < (x^2 + y^2)^2 < \delta^4 = \epsilon$. So, f is continuous at $(0, 0)$ and hence on \mathbb{R}^2 .

$$\lim_{(x, y) \rightarrow (0, 0)} \frac{|f(x, y) - f(0, 0) - (0, 0) \cdot (x, y)|}{\|(x, y)\|} = \lim_{(x, y) \rightarrow (0, 0)} \frac{\left| \frac{x^2 y^4}{x^2 + y^2} \right|}{\sqrt{x^2 + y^2}}$$

For $\epsilon > 0$ take $\delta = \sqrt[3]{\epsilon}$. So, $(x, y) \in D_\delta((0, 0)) \implies \left| \frac{x^2 y^4}{(x^2 + y^2)^{\frac{3}{2}}} \right| = x^2 |y| \left| \frac{|y|^3}{(x^2 + y^2)^{\frac{3}{2}}} \right| < x^2 |y| < \delta^3 = \epsilon$

Thus, the limit above is 0 and hence f is differentiable at $(0, 0)$ with $(Df)((0, 0)) = (0, 0)$.

Now define, $g : \mathbb{R}^2 \rightarrow \mathbb{R}$ and $h : \mathbb{R}^2 \rightarrow \mathbb{R}$ with

$$\begin{aligned}
 g(x, y) &= \begin{cases} \frac{2xy^6}{(x^2 + y^2)^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases} \\
 h(x, y) &= \begin{cases} \frac{2x^2 y^3 (2x^2 + y^2)}{(x^2 + y^2)^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}
 \end{aligned}$$

We will show that both g and h are continuous. We already have f, g being rational functions, are continuous on $\{(0, 0)\}^c$.

- For, $\epsilon_g > 0$ take $\delta_g = \sqrt[3]{\frac{\epsilon_g}{2}}$. Then, $(x, y) \in D_{\delta_g}((0, 0)) \implies \left| \frac{2xy^6}{(x^2+y^2)^2} - 0 \right| < 2|x|y^2 < 2\delta_g \cdot \delta_g^2 = \epsilon_g$. Thus, g is continuous at $(0, 0)$ and hence on \mathbb{R}^2 .
- For, $\epsilon_h > 0$ take $\delta_h = \sqrt[3]{\frac{\epsilon_h}{4}}$. Then, $(x, y) \in D_{\delta_h}((0, 0)) \implies \left| \frac{2x^2y^3(2x^2+y^2)}{(x^2+y^2)^2} - 0 \right| < 2|y|(2x^2 + y^2) < 2\delta_h(\delta_h^2 + \delta_h^2) = \epsilon_h$. Thus, h is continuous at $(0, 0)$ and hence on \mathbb{R}^2 .

Indeed, $g = f_x$ and $h = f_y$ which shows that $f \in C^1(\mathbb{R}^2)$. ■

Solution 5

(This question is wrong!)

$$\lim_{(x,y) \rightarrow (0,0)} \frac{|f(x,y) - f(0,0) - (0,0) \cdot (x,y)|}{\|(x,y)\|} = \lim_{(x,y) \rightarrow (0,0)} \frac{|x^3 \sin \frac{1}{x}|}{\sqrt{x^2 + y^2}}$$

For $\epsilon > 0$ take $\delta = \sqrt{\epsilon}$. So, $(x, y) \in D_\delta((0, 0)) \implies \frac{|x^3 \sin \frac{1}{x}|}{\sqrt{x^2 + y^2}} < x^2 < \delta^2 = \epsilon$. Thus, the limit above is 0 and hence f is differentiable at $(0, 0)$ with $(Df)((0, 0)) = (0, 0)$. Therefore,

$$f_x(x, y) = \begin{cases} 3x^2 \sin \frac{1}{x} - x \cos \frac{1}{x} & \text{if } (x, y) \neq 0 \\ 0 & \text{if } (x, y) = 0 \end{cases}$$

For $\epsilon' > 0$ take $\delta' = \frac{\sqrt{12\epsilon'+1}-1}{6} > 0$. So, $(x, y) \in D_{\delta'}((0, 0)) \implies |3x^2 \sin \frac{1}{x} - x \cos \frac{1}{x}| < 3x^2 + |x| < 3\delta'^2 + \delta' = \epsilon'$. Hence, f_x is continuous at $(0, 0)$. ■

Solution 6

(This question is wrong!)

Let $\mathcal{O}_n^{(1)}, \mathcal{O}_n^{(2)} \subseteq \mathbb{R}^n$ be two disjoint open sets. Take, $\mathcal{O}_n = \mathcal{O}_n^{(1)} \sqcup \mathcal{O}_n^{(2)}$, so, \mathcal{O}_n is open. Let, $f : \mathcal{O}_n \rightarrow \mathbb{R}$

$$\text{such that, } f(x) = \begin{cases} 1 & \text{if } x \in \mathcal{O}_n^{(1)} \\ 0 & \text{if } x \in \mathcal{O}_n^{(2)} \end{cases}$$

Clearly, for each i , $\frac{\partial f}{\partial x_i} = 0$. But f is not a constant function. ■

Solution 7

Let, $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a differentiable mapping whose total derivative is a diagonal matrix (w.r.t. standard ordered basis).

$\forall x, y \in \mathbb{R}^n, tx + (1-t)y \in \mathbb{R}^n \forall t \in [0, 1]$ which implies that \mathbb{R}^n is convex.

Fix, $j \in [n]$. Let, $a, b \in \mathbb{R}^n$. Then, by Multivariate Mean Value Theorem, $\exists c_j \in \{tb + (1-t)a : t \in [0, 1]\}$ such that $f_j(b) - f_j(a) = (\nabla f_j)(c_j) \cdot (b - a)$. Since, total derivative of f is a diagonal matrix, $\frac{\partial f_j}{\partial x_i}(c_j) = 0 \forall i \in [n] - \{j\}$ which implies $f_j(b) - f_j(a) = (b_j - a_j) \frac{\partial f_j}{\partial x_j}(c_j)$. For all such $a, b \in \mathbb{R}^n$ whose j^{th} coordinates are same, $f_i(b) - f_i(a) = 0$ which shows that, f_i is independent of $x_j \forall i \in [n] - \{j\}$. Therefore, each f_i is function of x_i , and hence the mappings of the form $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that $f(x) = (f_1(x_1), \dots, f_n(x_n))$ with each f_i differentiable, yields a diagonal matrix (w.r.t. standard ordered basis) in total derivative. ■

Solution 8

$$J_f(x, y, z) = \begin{bmatrix} 1 & 1 & 1 \\ 2x & 2y & 2z \\ 3x^2 & 3y^2 & 3z^2 \end{bmatrix}$$
$$\text{So, } \det(J_f(x, y, z)) = \begin{vmatrix} 1 & 1 & 1 \\ 2x & 2y & 2z \\ 3x^2 & 3y^2 & 3z^2 \end{vmatrix} = 12(x-y)(y-z)(z-x)$$

Clearly, $J_f(x, y, z)$ is non-singular unless two of the three variables are equal. And, the locus of the singularities $= \{(x, y, z) \in \mathbb{R}^3 : x = y \text{ or } y = z \text{ or } z = x\}$ which is just union of three planes, $x = y$, $y = z$ and $z = x$ in \mathbb{R}^3 . ■

Solution 9

False!

Let $\mathcal{O}_n^{(1)}, \mathcal{O}_n^{(2)} \subseteq \mathbb{R}^n$ be two disjoint open sets. Take, $\mathcal{O}_n = \mathcal{O}_n^{(1)} \sqcup \mathcal{O}_n^{(2)}$, so, \mathcal{O}_n is open. Let, $f : \mathcal{O}_n \rightarrow \mathbb{R}$ such that, $f(x) = \begin{cases} (1, 1, \dots, 1) & \text{if } x \in \mathcal{O}_n^{(1)} \\ (0, 0, \dots, 0) & \text{if } x \in \mathcal{O}_n^{(2)} \end{cases}$

Clearly, for each i, j $\frac{\partial f_i}{\partial x_j} = 0$. But f is not a constant function. ■

Solution 10

Let, $f = (f_1, \dots, f_n)$. So, $Df(t) = (f'_1, \dots, f'_n)$.

For $t \in (a, b)$,

$$\begin{aligned} \frac{d}{dt}(\|f(t)\|^2) &= \frac{d}{dt} \sum_{i=1}^n (f_i(t))^2 \\ &= \sum_{i=1}^n \frac{d}{dt} (f_i(t))^2 \\ &= 2 \sum_{i=1}^n f_i(t) f'_i(t) \\ &= 2 \langle f(t), Df(t) \rangle \end{aligned}$$

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