1. In proof of Proposition 1 (class 4th august) where did we use the fact that $P([a, b]) = b - a$?

2. Suppose $\Omega = [0, 1]$ and $\mathcal{F}$ is the collection of all subsets of Ω. Can you find examples of Probability $P : \mathcal{F} \to [0, 1]$ so that

$$P(A \oplus r) = P(A),$$

where $r \in \mathbb{Q}$, $A \oplus r = \{x + r : x \in A, x + r \leq 1\} \cup \{x + r - 1 : x \in A, x + r > 1\}$.

3. Let $N > 0$ be a fixed natural number. Let

$$\Omega_N = \{\omega = (\omega_1, \omega_2, \ldots, \omega_N) : \omega_i \in \{-1, 1\}\} \equiv \{-1, 1\}^N$$

and $\mathcal{A}_N$ is the collection of all subsets of Ω_N. Define $P : \mathcal{A}_N \to [0, 1]$, by

$$P(A) = \frac{\mid A \mid}{2^N}.$$

For $1 \leq k \leq N$, let $X_k : \Omega_N \to \{-1, 1\}$ given by $X_k(\omega) = \omega_k$ denote the displacement in the k-th step of the walk and for $1 \leq n \leq N$ let

$$S_n(\omega) = \sum_{k=1}^{n} X_k(\omega),$$

denote the position of the random walk at time n.

(a) Show that P is a probability on $(\Omega_N \mathcal{A}_N)$.

(b) Show that $P(X_k = 1) = P(X_k = -1) = \frac{1}{2}$ for all $1 \leq k \leq n$ and that $X_1, X_2, \ldots X_N$ are independent.

(c) Suppose $0 < k_1 < k_2 < k_3 < N$. Show that $S_{k_2} - S_{k_1}$ and $S_{k_3} - S_{k_2}$ are independent.

(d) Suppose for $0 < k < m < N$, $a, b \in \mathbb{Z}$ we have $P(S_k = a) > 0$ then show that

$$P(S_m = b \mid S_k = a) = P(S_{m-k} = b - a).$$