Due: September 29th, 2023, 10am

1. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a Probability space and X be a random variable. Then assuming the formula

$$\mathbb{E}[\mid X \mid] = \int_0^\infty \mathbb{P}(\mid X \mid \geq t) dt$$

evaluate the expectation of the following random variables

   (a) $X \sim$ Poisson (λ)

   (b) $X \sim$ Exponential (λ)

2. Let X, X_1, X_2, $\ldots$ be i.i.d. random variables that are uniformly distributed over the interval $(0,1)$. Consider the first order statistic $X^n_{(1)} = \min\{X_1, \cdots, X_n\}$. Show that $\mathbb{P}(X^n_{(1)} \geq \epsilon) \to 0$ as $n \to \infty$.

3. Consider

$$\Omega = \left\{ \omega = (\omega_1, \omega_2, \ldots,) : \omega_i \in \{0,1\} \right\}$$

equipped with probability $\mathbb{P}$ such that

$$\mathbb{P}(X_k = 0) = \mathbb{P}(X_k = 1) = \frac{1}{2}$$

where for $1 \leq k$, $X_k : \Omega \to \{0,1\}$ be given by $X_k(\omega) = \omega_k$. Suppose for $1 \leq n$, let $S_n : \Omega \to \mathbb{Z}$ be given by $S_n(\omega) = \sum_{k=1}^n X_k(\omega)$ and $S_0 = 0$. Show that for $a < \frac{1}{2}$

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}(S_n \leq an) = \begin{cases} -\log(2) - a\log(a) - (1-a)\log(1-a) & \text{if } 0 < a < \frac{1}{2} \\ -\infty & \text{if } a < 0 \end{cases}$$

4. Let ($\Omega = [0,1]$, $\mathcal{F}$ be the Borel σ-algebra and $\mathbb{P}$) be the uniform Probability. Suppose $A_i = [0, \frac{1}{i})$ then

   (a) Describe the event $\limsup\limits_{i\to\infty} A_i$.

   (b) Find $\sum\limits_{i\geq 1} \mathbb{P}(A_i)$ and decide if it contradicts the Borel Cantelli Lemma.

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