8th September :

Expectation of a Random Variable

$(\Omega, \mathcal{F}, \mathbb{P})$ – Probability space

Definition 1 : $X : \Omega \to \mathbb{R}$ a random variable if

$$X^{-1}(B) \in \mathcal{F} \quad \forall \; B \in \mathcal{B}$$

where $\mathcal{B} = \sigma\{(a,b] \mid -\infty < a \leq b < \infty\}$

- A random variable X is called simple if range $\{X(\omega) : \omega \in \Omega\}$ is finite. say $\{x_1, \dots, x_n\}$ for some $n \geq 1$.

– $A_i = X^{-1}(\{x_i\}) \quad 1 \leq i \leq n$

$$\Rightarrow \quad X = \sum_{i=1}^{n} x_i \mathbb{1}_{A_i}$$

Definition 2 : Suppose X is a simple r.v. on $(\Omega, \mathcal{F}, \mathbb{P})$ s.t

$$X = \sum_{i=1}^{n} x_i \mathbb{1}_{A_i} \quad \text{for some}$$

$n \geq 1$, $A_i \in \mathcal{F}$, $x_i \in \mathbb{R}$. then

$$\mathbb{E}[X] = \sum_{i=1}^{n} x_i \mathbb{P}(A_i) = \sum_{i=1}^{n} x_i \mathbb{P}(X = x_i)$$

- If $A_i \cap A_j = \phi \; j \neq i$; $\bigcup_{i=1}^{n} A_i = \Omega$; Repetition is okay

x_i's are distinct

Properties of $\mathbb{E}[X]$ - X Simple

S-(1) $\mathbb{E}[c] = c \quad \forall\, c \in \mathbb{R}$

S-(2) $\mathbb{E}[\mathbb{1}_A] = \mathbb{P}(A) \quad \forall\, A \in \mathcal{F}$

S-(3) X, Y simple & $a, b \in \mathbb{R}$

$$\mathbb{E}[aX + bY] \overset{(Ex)}{=} a\mathbb{E}[X] + b\mathbb{E}[Y]$$

S-(4) $X \geq 0$ simple, $\mathbb{E}[X] \geq 0$

$\Rightarrow$ • $X \leq Y \quad \mathbb{E}[X] \leq \mathbb{E}[Y]$.

• $|\mathbb{E}[X]| \leq \mathbb{E}[|X|]$

S-(5) $\mathbb{E}[XY] = \mathbb{E}[X]\,\mathbb{E}[Y]$ if X, Y are independent (Converse only true if X, Y are Normal r.v.)

S-(6) X - simple $\quad X = \sum_{i=1}^{n} x_i \mathbb{1}_{A_i} \quad n \geq 1,\ A_i \in \mathcal{F}$
$x_i \in \mathbb{R}$

$f: \mathbb{R} \to \mathbb{R} \quad f(X) = \sum_{i=1}^{n} f(x_i) \mathbb{1}_{A_i}$

$$\Rightarrow \mathbb{E}[f(X)] = \sum_{i=1}^{n} f(x_i)\,\mathbb{P}(A_i)$$

$$= \sum_{i=1}^{n} f(x_i)\,\mathbb{P}(X = x_i)$$

Definition 3:- X is a non-negative random variable (not necessarily simple)

$$\mathbb{E}[X] = \sup \{ \mathbb{E}[Y] \mid Y\text{- simple} \quad Y \le X \}$$

Remarks:

- $0 \le \mathbb{E}[X] \le \infty$
- X - simple take $Y = X$

(Definition 3 and 2 are consistent)

Example:- $p \ge 1$ $\quad c_p^{-1} = \sum_{n=1}^{\infty} \frac{1}{n^p}$

Consider X $\quad \mathbb{P}(X = n) = \frac{c_p}{n^p}, \quad \forall n \ge 1$

Range $(X) = \mathbb{N}$ - not simple.

$$Y_N = \begin{cases} X & \text{if } X \le N \\ 0 & \text{otherwise} \end{cases}, \quad \begin{matrix} Y_N \text{ - simple} \\ Y_N \le X \end{matrix}$$

$$E[Y_N] = \sum_{k=1}^{N} k \cdot \frac{c_p}{k^p} = \sum_{k=1}^{N} \frac{c_p}{k^{p-1}}$$

Ex:-

- $p \in (1, 2]$; $\quad \mathbb{E}[Y_N] \longrightarrow \infty$ as $n \to \infty$
- $p > 2$ $\quad \mathbb{E}[Y_N] \longrightarrow c_p \sum_{k=1}^{\infty} \frac{1}{k^{p-1}}$
- $\mathbb{E}[X] = c_p \sum_{k=1}^{\infty} \frac{1}{k^{p-1}}$ for $p > 2$

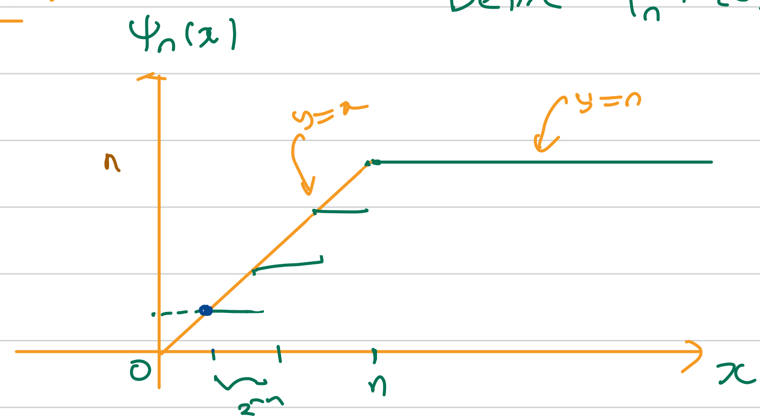
Remark: $\boxed{S-(1) \quad \dots \quad S-(6)}$ - Properties hold for non-negative random variables.

Proposition 1 : $X \geq 0$ random variable. Then we can construct $\{X_n\}_{n \geq 1}$ such that

- X_n – simple
- $X_n \leq X_{n+1}$ $\forall\, n \geq 1$
- $\forall\, \omega \in \Omega$ $\lim_{n \to \infty} X_n(\omega) = X(\omega)$

Proof: (sketch) : Define $\varphi_n : [0, \infty) \to \mathbb{R}$

$$\boxed{X_n := \varphi_n(X)}$$



$X_n \leq X_{n+1}$ & $X_n \uparrow X$ $\square$

Monotone Convergence Theorem let $(\Omega, \mathcal{F}, \mathbb{P})$ be the probability space.

$X \geq 0$ random variable $X_n \geq 0$ random variables

$X_n \uparrow X$ as $n \to \infty$

then $\mathbb{E}[X_n] \uparrow \mathbb{E}[X]$ as $n \to \infty$

(∞ is allowed)

Proof (sketch): • $X_n \le X_{n+1} \cdots \le X \quad \forall n \ge 1$

(Properties of $\mathbb{E}$) $\Rightarrow \quad \mathbb{E}[X_n] \le \mathbb{E}[X_{n+1}] \le \cdots \le \mathbb{E}[X] \quad \forall n \ge 1$

$\Rightarrow \{\mathbb{E}[X_n]\}_{n\ge1}$ is an increasing sequence in $[0,\infty]$

$\Rightarrow \lim_{n\to\infty} \mathbb{E}[X_n]$ (exists) & $\lim_{n\to\infty} \mathbb{E}[X_n] \le \mathbb{E}[X]$

E.T.S.: $\lim_{n\to\infty} \mathbb{E}[X_n] \ge \mathbb{E}[X]$

Take $Y = \sum_{i=1}^{N} y_i \mathbb{1}_{A_i}$ $\quad 0 \le Y \le X$

for some $N \ge 1$, $y_i \in [0,\infty)$ & $A_i \in \mathcal{F}$

Take: $0 < c < 1$ & $B_n = \{X_n \ge cY\} \in \mathcal{F}$

Ex: (need for c) $\quad = (X_n - cY)^{-1}\{[0,\infty)\}$

(*)₁ • $\ldots \subseteq B_n \subseteq B_{n+1} \subseteq \ldots$ & $\bigcup_{n=1}^{\infty} B_n = \Omega$

(*)₂ • $\mathbb{E}[X_m] \ge \mathbb{E}[X_n] \quad \forall m \ge n \ge 1$

$\Rightarrow$

$\lim_{m\to\infty} \mathbb{E}[X_m] \ge \mathbb{E}[X_n] \quad \forall n \ge 1$

• **observe:**

$$\mathbb{E}[X_n] \ge \mathbb{E}[X_n \mathbb{1}_{B_n}]$$

$X_n \ge X_n \mathbb{1}_{B_n}$

$$\geq \quad c \sum_{i=1}^{N} y_i \, \mathbb{P}(A_i \cap B_n) \quad - \circledast_3$$

$B_n = \{X_n \geq cY\}$

$$\circledast_1 \Rightarrow \mathbb{P}(A_i \cap B_n) \xrightarrow[n \to \infty]{} \mathbb{P}(A_i) \qquad 1 \leq i \leq N$$

$$\Rightarrow \sum_{i=1}^{N} y_i \, \mathbb{P}(A_i \cap B_n) \xrightarrow[n \to \infty]{(Ex)} \mathbb{E}[Y] \quad - \circledast_4$$

$\circledast_2, \ \circledast_3 \ \circledast_4 \Rightarrow$

$$\lim_{m \to \infty} \mathbb{E}[X_m] \geq c \, \mathbb{E}[Y]$$

$0 < c < 1$ is arbitrary

$$\Rightarrow \lim_{m \to \infty} \mathbb{E}[X_m] \geq \mathbb{E}[Y]$$

Y - simple $0 \leq Y \leq X$ was arbitrary

$$\Rightarrow \lim_{m \to \infty} \mathbb{E}[X_m] \geq \mathbb{E}[X]$$

$\square$