Recall :

$X_1, X_2, \ldots$ i.i.d. random variables on $(\Omega, \mathcal{F}, \mathbb{P})$

$\mathbb{E}[X_1] = \mu$; $\operatorname{var}[X_1] = \sigma^2 < \infty$.

$S_n = X_1 + \cdots + X_n$

(SLLN) $$\mathbb{P}\left(\left\{\frac{1}{n} S_n \longrightarrow \mu \text{ as } n \to \infty\right\}\right) = 1$$

(WLLN) $\varepsilon > 0$ be given.

$$\mathbb{P}\left(\left|\frac{1}{n} S_n - \mu\right| > \varepsilon\right) \longrightarrow 0 \quad \text{as } n \to \infty.$$

(C.L.T.) $$\mathbb{P}\left(\frac{\sqrt{n}}{\sigma}\left(\frac{S_n}{n} - \mu\right) \leq x\right) \longrightarrow \int_{-\infty}^{x} \frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}}\, dy$$

$\forall\, x \in \mathbb{R}$.

Large Deviation Principle :

From above, we know $\frac{S_n}{n} \approx \mu$

Deviations beyond this are called "large"

e.g. $\{S_n \geq n(\mu + a)\}$ $\forall\, a > 0$

SLLN | WLLN : $\mathbb{P}(S_n \geq n(\mu + a)) \longrightarrow 0$ as $n \to \infty$

Show :- In many cases:

$$\frac{1}{n}\log \mathbb{P}(S_n \geq n(\mu+a)) \longrightarrow -I(a)$$

$\forall a>0$

Theorem:- $(X_i)_{i\geq 1}$ be i.i.d. such that

$$\mathbb{P}(X_i=0) = \mathbb{P}(X_i=1) = \frac{1}{2} \quad \left[\mathbb{E}[X_i] = \frac{1}{2}\right]$$

$$S_n = \sum_{i=1}^{n} X_i$$

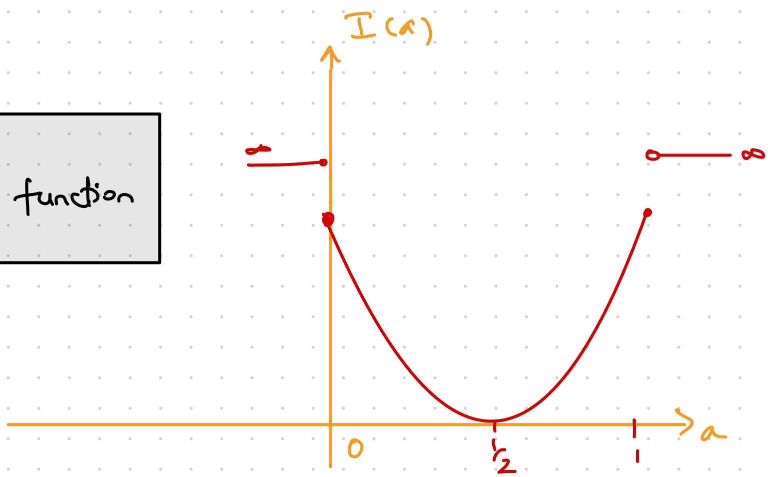
let $a > \frac{1}{2}$. Then

$$\frac{1}{n}\log \mathbb{P}(S_n \geq an) \longrightarrow -I(a)$$

as $n \to \infty$ where

$$I(z) = \begin{cases} \log 2 + z\log z + (1-z)\log(1-z) & \text{if } z\in[0,1] \\ \infty & \text{otherwise} \end{cases}$$

$\boxed{I(\cdot) \text{ rate function}}$



Proof :- $\cdot\ a > 1$ claim is trivial.

$\cdot\ a \in [\frac{1}{2}, 1]$

($S_n \sim \text{Binomial}(n, \frac{1}{2})$) $\mathbb{P}(S_n \geq an) = 2^{-n} \sum_{k \geq an} \binom{n}{k}$

$$\Rightarrow \quad 2^{-n} \max_{k \geq an} \binom{n}{k} \leq \mathbb{P}(S_n \geq an) \leq (n+1)\, 2^{-n} \max_{k \geq an} \binom{n}{k}$$

$\llcorner$ ⓪

. The $\max_{k \geq an} \binom{n}{k}$ is attained at $k = \lceil an \rceil$ (Smallest integer $\geq an$)

Ex:

$\cdot\ n! = n^n e^{-n} \sqrt{2\pi n} \left(1 + O\left(\frac{1}{n}\right)\right)$

$\star$ $\Rightarrow$ $\frac{1}{n} \log \max_{k \geq an} \binom{n}{k} \longrightarrow -a\log a - (1-a)\log(1-a)$ as $n \to \infty$ $\llcorner$ ①

Using ① we have

(i) $\frac{1}{n} \log \left(2^{-n} \max_{k \geq an} \binom{n}{k}\right) \xrightarrow{\text{as } n \to \infty} -\log 2 - a\log a - (1-a)\log(1-a)$

(ii) $\frac{1}{n} \log\left(2^{-n} (n+1) \max_{k \geq an} \binom{n}{k}\right)$

$$= -\log 2 + \frac{\log(n+1)}{n} + \frac{1}{n} \log\left(\max_{k \geq an} \binom{n}{k}\right)$$

$$\downarrow$$

$$-\log 2 \quad + 0 \qquad -a\log a - (1-a)\log(1-a)$$

Pluggings (i) and (ii) into ⓪ we have.

$$\frac{1}{n}\log \mathbb{P}(S_n \geq na) \longrightarrow -\log 2 - a\log a - (1-a)\log(1-a),$$
as $n\to\infty$

for $a \in (\frac{1}{2}, 1]$. $\square$

Ex:- $\underline{a < \frac{1}{2}}$: (By symmetry ; $I(1-z) = I(z)$)

- $\frac{1}{n}\log \mathbb{P}(S_n \leq an) \longrightarrow -I(a)$ as $n\to\infty$

- $I(\cdot)$ – Convex function
  - $z=\frac{1}{2}$, $I(\frac{1}{2}) = 0$ is the minimum.

[Cramer's Theorem]: $\{X_i\}_{i\geq 1}$ i.i.d random variables such that

$$\phi(t) = \mathbb{E}[e^{tX_1}] < \infty \quad \forall t \in \mathbb{R}$$

$S_n = \sum_{i=1}^{n} X_i$. Then for all $a > \mathbb{E}[X_1]$

$$\frac{1}{n}\log \mathbb{P}(S_n > na) \longrightarrow -I(a)$$

where

$$I(z) = \sup_{t\in\mathbb{R}} [zt - \log \phi(t$$

Kolmogorov 0-1 law : $(\Omega, \mathcal{F}, \mathbb{P})$

Independence :-

- $A, B \in \mathcal{F}$ are independent if

$$\mathbb{P}(A \cap B) = \mathbb{P}(A)\,\mathbb{P}(B)$$

- More generally $\{A_\alpha : \alpha \in I\}$ is independent if

$$\mathbb{P}(A_{\alpha_1} \cap A_{\alpha_2} \cdots \cap A_{\alpha_n}) = \prod_{i=1}^{n} \mathbb{P}(A_{\alpha_i})$$

$$\forall n \in \mathbb{N}, \quad \forall \alpha_1, \ldots, \alpha_n \in I$$

$\alpha_i \neq \alpha_j \quad i \neq j$

- [Caution] $\{A_\alpha : \alpha \in I\}$ is pairwise independent if

$$\mathbb{P}(A_\alpha \cap A_\beta) = \mathbb{P}(A_\alpha)\,\mathbb{P}(A_\beta)$$

$$\forall \alpha, \beta \in I$$

$\alpha \neq \beta$

Pairwise independence $\cancel{\Rightarrow}$ Independence

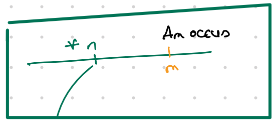
- let X and Y be two random variables. are independent iff

$$\mathbb{P}(X \leq x, Y \leq y) = \mathbb{P}(X \leq x)\,\mathbb{P}(Y \leq y)$$

$$\forall x, y \in \mathbb{R}$$

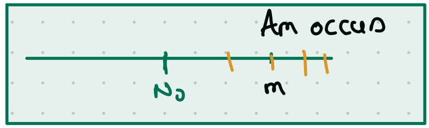
Borel Cantelli lemma :-

let $\{A_k\}_{k\geq 1}$ & $A_k \in \mathcal{F}$.



$$\overline{\lim_{n\to\infty}} A_n := \{A_n \text{ happens infinitely often}\} := \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} A_m$$

$$\underline{\lim_{n\to\infty}} A_n := \{A_n \text{ happens all but finitely many times}\} := \bigcup_{n=1}^{\infty} \bigcap_{m=n}^{\infty} A_m$$



Remarks :-

(i) $\overline{\lim_{n\to\infty}} A_n \in \mathcal{F}$, $\underline{\lim_{n\to\infty}} A_n \in \mathcal{F}$

(ii) $\underline{\lim_{n\to\infty}} A_n \subseteq \overline{\lim_{n\to\infty}} A_n$

(iii) $\left(\overline{\lim_{n\to\infty}} A_n\right)^c = \underline{\lim_{n\to\infty}} A_n^c$

(iv) $\mathbb{P}\left(\underline{\lim_{n\to\infty}} A_n\right) \leq \underline{\lim_{n\to\infty}} \mathbb{P}(A_n) \leq \overline{\lim_{n\to\infty}} \mathbb{P}(A_n) \leq \mathbb{P}\left(\overline{\lim_{n\to\infty}} A_n\right)$

(v) $\overline{\lim_{n\to\infty}} \mathbb{1}_{A_n} = \mathbb{1}_{\overline{\lim_{n\to\infty}} A_n}$

$$\underline{\lim_{n\to\infty}} \mathbb{1}_{A_n} = \mathbb{1}_{\underline{\lim_{n\to\infty}} A_n}$$

Borel Cantelli lemma

let $A_1, A_2, \ldots \in \mathcal{F}$

(i) If $\sum_{n=1}^{\infty} \mathbb{P}(A_n) < \infty$ then

$$\mathbb{P}(A_n \text{ infinitely often}) = 0$$

(ii) If $\sum_{n=1}^{\infty} \mathbb{P}(A_n) = \infty$, A_n's are independent,

$$\mathbb{P}(A_n \text{ infinitely often}) = 1$$

Proof:- (i) i.o. $\equiv$ infinitely often

$$\mathbb{P}(A_n \text{ i.o.}) = \mathbb{P}\left(\bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} A_m\right)$$

$$\left(\because \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} A_m \subseteq \bigcup_{m=n}^{\infty} A_m \quad \forall n \geq 1\right) \leq \sum_{m=n}^{\infty} \mathbb{P}(A_m) \qquad \forall n \geq 1$$

By hypothesis $\sum_{m=n}^{\infty} \mathbb{P}(A_m) \longrightarrow 0$ as $n \to \infty$

(tail sum)

$\Rightarrow$ result $\square$

(ii) $\mathbb{P}(\{A_n \text{ i.o.}\}^c) \underset{\text{Remark (iii)}}{\leq} \sum_{n=1}^{\infty} \mathbb{P}\left(\bigcap_{k=n}^{\infty} A_k^c\right)$ — (*)

$$\mathbb{P}\left(\bigcap_{k=n}^{\infty} A_k^c\right) \leq \mathbb{P}\left(\bigcap_{k=n}^{n+m} A_k^c\right)$$

$$\underset{\text{Independence}}{=} \prod_{k=n}^{n+m} \mathbb{P}(A_k^c)$$

$$= \prod_{k=n}^{n+m} (1 - \mathbb{P}(A_k))$$

$$\left(\underset{0<x<1}{1-x \leq e^{-x}}\right) \leq \prod_{k=n}^{n+m} e^{-\mathbb{P}(A_k)} = e^{-\sum_{k=n}^{n+m} \mathbb{P}(A_k)} \quad \forall m \geq 1$$

By hypothesis $\sum_{k=n}^{n+m} \mathbb{P}(A_k) \longrightarrow \infty$ as $m \to \infty$

$\therefore \mathbb{P}\left(\bigcap_{k=n}^{\infty} A_k^c\right) = 0 \quad \therefore$ (**)

$\therefore$ place (**) into (*) to get result $\square$