Proof of claim 1.1 :

$$\{S_n = x\} \overset{Ex}{\equiv} \text{first } n \text{ components of } \omega \text{ take precisely } \boxed{k = \frac{n+x}{2}} \text{ times value } +1$$

i.e. $$S_n = \underbrace{k(+1)}_{\#\text{ up/right steps}} + (n-k)\underbrace{(-1)}_{\#\text{ down/left steps}} = \boxed{2k-n \equiv x}$$

# of elements ω : $S_n(\omega) = x$

$$:= \underbrace{|\{\omega \in \Omega \mid S_n(\omega) = x\}|} = \binom{n}{k} 2^{N-n}$$

we reach x in n steps

$$\mathbb{P}(S_n = x) = \frac{|\{\omega \in \Omega \mid S_n(\omega) = x\}|}{2^N}$$

$$= \frac{\binom{n}{k} 2^{N-n}}{2^N}$$

$$= \binom{n}{k} 2^{-n}$$

$$= \binom{n}{\frac{n+x}{2}} 2^{-n} \qquad \square$$

Observations (Contd.)

From Claim 1.1 :

claim 1.1 at 2n and with x=0

(maximal weight)

$$\mathbb{P}(S_{2n}=0) = \binom{2n}{n} 2^{-2n}$$

$$= \frac{2n!}{n!\,n!} 2^{-2n}$$

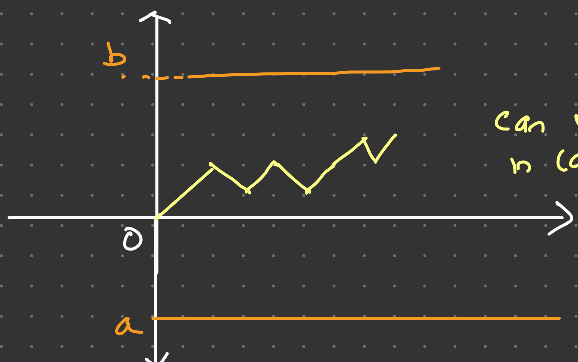
(Stirling's formula)

$$k! \underset{k\to\infty}{\sim} k^k e^{-k} \sqrt{2\pi k}$$

$$\sim \frac{(2n)^{2n} e^{-2n} \sqrt{2\pi 2n}}{(n^n e^{-n} \sqrt{2\pi n})^2} \cdot 2^{-2n}$$

$$\overset{\text{Ex.}}{=} \frac{1}{\sqrt{\pi n}} \quad (\text{as } n\to\infty)$$

Question



can walk S_n stay in (a,b) forever ?

$$0 \leq P(a \leq S_n \leq b) = \sum_{x\in[a,b]} \mathbb{P}(S_n = x)$$

$$\leq \sum_{x\in[a,b]} [\mathbb{P}(S_n=0) + \mathbb{P}(S_n=1)]$$

for some $c_1 > 0$ $\quad \leq (b-a+1)\dfrac{c_1}{\sqrt{n}}$ — (1)

By (1) $\Rightarrow \mathbb{P}(a \leq S_n \leq b) \longrightarrow 0$ as $n \to \infty$ $\square$

Mathematical issue:

- our $\mathbb{P} \equiv \mathbb{P}_N$ N - fixed. (length of walk)
  So $n \to \infty$ need $N \to \infty$ as well.
- Understanding $\mathbb{P}_N$ when $N \equiv \infty$ [needs work] come back to it later

1.2 Stopping times:

Interpretation: • $S_n \equiv$ represent "amount of capital" of the player after n rounds

• $X_k \equiv$ amount a player wins in round k.

* Expected "amount of capital" after n round

$= E[S_n] = 0 \quad 0 \leq n \leq N$

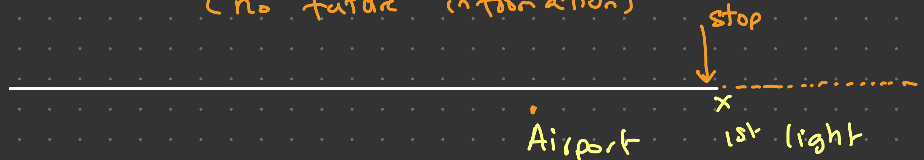
Question: Is it possible to stop the game in a favorable moment?

(clever stopping strategy ... to a positive expected gain)

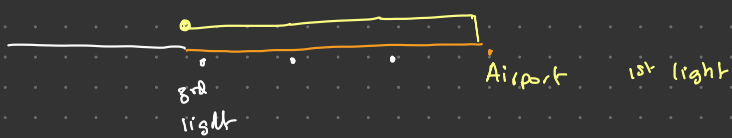
(no future). no - insider trading { decision to stop may (information) only depend on tosses till time n.

<u>Intuition</u>:

- stop at the first light after the airport
  (no future information)



- stop at the 3rd last light before the airport
  (use future information)



<u>Definition</u>: An event $A \subseteq \Omega$ is observable until time n when it can be written as a union of basic events of the form

$$\{\omega \in \Omega \mid \omega_1 = o_1, \ldots, \omega_n = o_n\} \quad o_i \in \{-1, 1\}, \ 1 \le i \le n$$

i.e. A – can be determined from the outcome of the 1st n tosses.

$\mathcal{A}_n :=$ class of event A that can be observed by time n. [include ϕ]

Notation: $A \subseteq \Omega$ (indicator of A)

$$\mathbb{1}_A(\omega) = \begin{cases} 1 & \omega \in A \\ 0 & \omega \notin A \end{cases}$$

Definition: A map $T : \Omega \to \{0, 1, \dots, N\} \cup \{\infty\}$ is called a stopping time if

$$\{T = n\} \equiv \{\omega \in \Omega \mid T(\omega) = n\} \in \mathcal{A}_n \quad 0 \le n \le N$$

i.e. $\{T = n\}$ is an event observable until time n.

Example: For $a \in \mathbb{Z}$ let

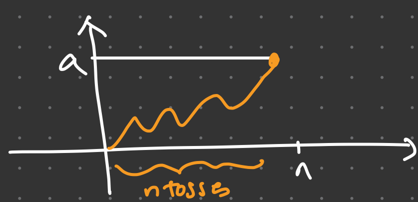
$$\sigma_a : \Omega \to \{0, 1, \dots, N\} \cup \{\infty\}$$

$$\sigma_a(\omega) = \min\{k \in [0, N] \mid S_k(\omega) = a\}$$

$\min(\phi) = \infty$

Ex:- $\sigma_a(\cdot)$ is a stopping time

$\{\omega \in \Omega : \sigma_a(\omega) = n\} = \dots$ "only depends on 1st n tosses"



Theorem 1 : For any stopping time

[Impossibility of a favorite stopping] $T : \Omega \longrightarrow \{0, 1, \dots, N\}$

$$E[S_T] = 0$$

$S_T(\omega) \equiv S_{T(\omega)}(\omega) \equiv$ outcome of the trajectory ω at stopping time $T(\omega)$