Due: October 10th, 2023, 10am

Bond Percolation on Π_2*:* Let Π_2 be the binary tree. The vertex set is labelled as:

$$\pi_2^0 = \{0\}, \pi_2^n = \{0,1\}^n, \text{ for } n \geq 1, \Pi_2 = \cup_{n=0}^{\infty} \Pi_2^n.$$

For each $x \in \pi_2^n$, let $a(x) = (x_1, x_2, \ldots, x_{n-1}) \in \Pi_2^{n-1}$ be the ancestor of x in Π_2. The edge set is given by

$$E = \big\{\{x, a(x)\} : x \in \Pi_2 \setminus \Pi_2^0\big\}$$

Let $0 < p < 1$ and we make each edge open independently with probability p and closed with probability $1-p$.

1. For any $x, y \in \Pi^2$ show that there is a unique path between x and y in Π^2, i.e.

   there is a $n \geq 1$ and $x_k \in \Pi^2$ with $x_0 = x, x_n = y$ and $\{x_k, x_{k+1}\} \in E$ for $0 \leq k \leq n-1$.

2. Let $C_0 = \{x \in \Pi^2 : \text{ there is a path betweeen } x \text{ and } 0 \text{ that is open}\}$.

   (a) Show that $\sum_{x \in \Pi_2^n} \mathbb{P}(x \in C_0) = 2^n p^n$

   (b) Show that for any $x, y \in \Pi_2^n, x \neq y$ there is a unique $c \equiv c(x, y, 0) \in \Pi_2$ such that

   $$c \in \Pi_2^s,\ s \leq n-1,\ c \text{ lies on the path from } x \text{ to } 0 \text{ and } y \text{ to } x.$$

   Then conclude that

   $$\mathbb{P}(x \in C_0, y \in C_0) = p^{2n-s}$$

   (c) Show that

   $$\sum_{x,y \in \Pi_2^n, x \neq y} \mathbb{P}(x \in C_0, y \in C_0) = \frac{(2p)^{2n}}{2} \sum_{s=0}^{n-1} (2p)^{-s}$$

   (d) For each $n \geq 1$, define

   $$X_n = \sum_{x \in \Pi_2^n} 1(x \in C_0)$$

   Using the second moment method, show that if $\frac{1}{2} < p < 1$

   $$P(X_n > 0) \geq \frac{1}{\frac{p}{2p-1} + (2p)^{-n}}$$

3. Conclude that if $\frac{1}{2} < p < 1$ then $\mathbb{P}(| C_0 | = \infty) \geq \frac{2p-1}{p}$

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