(M) Markov chains — { Discrete time $n = 0, 1, 2, \ldots$
state space $S \equiv$ finite or Countable. }

History: · Andrey Markov – 1906 – Motivation

· Weak law of large numbers for i.i.d. sequences
(Does this hold without independence?)

Example: Eugene Onegin – Pushkin
alternation of consonants & vowels.

· Theory :– was challenging to develop
[Post] – A.N. Kolmogorov formulation of modern Probability

· Wide applications : Mathematics, physics, Biology

Notation :-

S – denote the set of all possible state of the system.
($|S| < \infty$ or S – Countable)

$\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ – denote time

$X_n \in S \equiv$ state of the system at time n.

Refer to as Markov chain
$\equiv$ (M.C.) or chain

Motivational Examples / Questions :

$\{X_n\}_{n\geq 0}$: – # of searches on a Google server in an hour ?

– # of tigers at Nagarhole national park ?

$$\underset{i}{X_n} \xrightarrow{\ ?\ } \underset{j}{X_{n+1}}$$

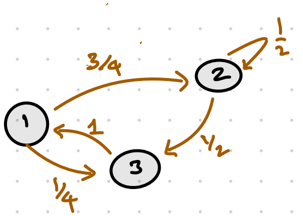
- At each time step n, the chain $\{X_n\}_{n\geq 1}$ moves from state $i \in S$ to another state $j \in S$ with probability p_{ij} ; [$i = j$ is also allowed]

Need :- $P = [p_{ij}]_{i\in S, j\in S}$ [one step transition matrix]

- $i \longrightarrow j \in S$ $\qquad \sum_{j\in S} p_{ij} = 1$, $p_{ij} \geq 0$.

Example M1 :- $S = \{1,2,3\}$

Intuitive way of defining M.C.



Directed, weighted graph

1 to 2	w.p.	3/4
1 to 3	w.p.	1/4
2 to 2	w.p	$\frac{1}{2}$
2 to 3	w.p.	$\frac{1}{2}$
3 to 1	w.p.	1

Prescribe transitions

$$P = \begin{pmatrix} 0 & \frac{3}{4} & \frac{1}{4} \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 \end{pmatrix}$$

Transition matrix

Let us define a Markov chain precisely.

Definition M.1: Let μ be a probability on S.

- Let $P = [p_{ij}]_{i,j \in S}$ be the matrix:

$\forall\, i,j \in S \qquad 0 \leq p_{ij} \leq 1$ and $\sum_{j \in S} p_{ij} = 1 \quad \forall\, i \in S.$

- Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space & $n \geq 0$. Let $X_n : \Omega \to S$ be a sequence of random variables whose joint distribution is given by

$$\mathbb{P}(X_0 = i_0, X_1 = i_1, \ldots, X_k = i_k)$$
$$= \mu(\{i_0\})\, p_{i_0 i_1}\, p_{i_1 i_2} \cdots p_{i_{k-1}, i_k} \qquad (*)$$

$\forall\, k \geq 0.$

$\{X_n\}_{n \geq 0}$ is said to be a Markov chain on a state space S with initial distribution μ and transition matrix P.

Properties of Markov chain X_n :

(a) **Distribution**

$$\mathbb{P}(X_0 = i_0) = \sum_j \mathbb{P}(X_0 = i_0, X_1 = j)$$

$$\overset{\text{Definition M.1}}{=} \sum_j \mu(\{i_0\})\, p_{i_0 j}$$

$$= \mu(\{i_0\}) \sum_j p_{i_0 j}$$

$$= \mu(\{i_0\})$$

$\boxed{X_0 \sim \mu}$ – initial distribution of X_0 is μ

$$\mathbb{P}(X_n = j) = \, ? \, = \mathbb{P}\Big(\overset{\in S}{i_0} \longrightarrow i_1 \cdots \longrightarrow \overset{X_n}{j} \Big)$$

$$= \mathbb{P}\Big(\bigcup_{\substack{i_k \in S \\ k=0,\dots,n-1}} \{X_0 = i_0, X_1 = i_1, \dots, X_n = j\} \Big)$$

$$= \sum_{\substack{i_k \in S \\ k=0,\dots,n-1}} \mathbb{P}(X_0 = i_0, X_1 = i_1, \dots, X_n = j)$$

$$= \sum_{\substack{i_k \in S \\ k=0,\dots,n-1}} \mu(\{i_0\}) \prod_{k=1}^{n-1} p_{i_{k-1} i_k} \cdot p_{i_{n-1} j}$$

The distribution of $\{X_n\}_{n \geq 0}$ is determined by μ & P.

(b) (Markov Property)

"Memoryless" $\equiv$ $\left\{ \begin{array}{l} \text{Past} \\ \text{future} \end{array} \right.$ & are independent given present

Fix: $n \in \mathbb{N}$ $\quad i_0, i_1, \ldots, i_{n-2}, i, j \in S.$ & assume

$$\mathbb{P}(X_0 = i_0, X_1 = i_1, \ldots, X_{n-2} = i_{n-2}, X_{n-1} = i) > 0$$

$$\mathbb{P}(\underbrace{X_n = j}_{\text{future}} \mid \underbrace{X_0 = i_0, X_1 = i_1, \ldots, X_{n-2} = i_{n-2},}_{\text{past}} \underbrace{X_{n-1} = i}_{\text{present}})$$

$$= \frac{\mathbb{P}(X_0 = i_0, X_1 = i_1, \ldots, X_{n-2} = i_{n-2}, X_{n-1} = i, X_n = j)}{\mathbb{P}(X_0 = i_0, X_1 = i_1, \ldots, X_{n-2} = i_{n-2}, X_{n-1} = i)}$$

$$= \frac{\mu(\{i_0\}) \prod_{k=1}^{n-2} p_{i_{k-1}, i_k} \; p_{i_{n-2} i} \; p_{ij}}{\mu(\{i_0\}) \prod_{k=1}^{n-2} p_{i_{k-1}, i_k} \; p_{i_{n-2} i}}$$

$$= p_{ij}$$

$$\mathbb{P}(X_n=j \mid X_{n-1}=i) = \frac{\mathbb{P}(X_n=j,\ X_{n-1}=i)}{\mathbb{P}(X_{n-1}=i)}$$

$$= \frac{\mathbb{P}\left(\bigcup_{\substack{i_k\in S\\ k=0,\dots,n-2}} \{X_0=i_0,\ X_1=i_1,\ \dots,\ X_{n-2}=i_{n-2},\ X_{n-1}=i,\ X_n=j\}\right)}{\mathbb{P}\left(\bigcup_{\substack{i_k\in S\\ k=0,\dots,n-2}} \{X_0=i_0,\ X_1=i_1,\ \dots,\ X_{n-2}=i_{n-2},\ X_{n-1}=i\}\right)}$$

$$= \frac{\sum_{\substack{i_k\in S\\ k=0,\dots,n-2}} \mu(\{i_0\}) \prod_{k=1}^{n-2} p_{i_{k-1},i_k}\ p_{i_{n-2}i}\ p_{ij}}{\sum_{\substack{i_k\in S\\ 0,\dots,n-2}} \mu(\{i_0\}) \prod_{k=1}^{n-2} p_{i_{k-1},i_k}\ p_{i_{n-2}i}}$$

$$= p_{ij} \frac{\sum_{\substack{i_k\in S\\ 0,\dots,n-2}} \cancel{\mu(\{i_0\}) \prod_{k=1}^{n-2}} p_{i_{k-1},i_k}\ p_{i_{n-2}i}}{\sum_{\substack{i_k\in S\\ 0,\dots,n-2}} \cancel{\mu(\{i_0\}) \prod_{k=1}^{n-2}} p_{i_{k-1},i_k}\ p_{i_{n-2}i}}$$

$$= p_{ij}$$

<u>We have shown:</u>

$$\mathbb{P}(X_n=j \mid X_0=i_0,\ X_1=i_1,\ \dots,\ X_{n-2}=i_{n-2},\ X_{n-1}=i)$$

$$=$$

$$\mathbb{P}(X_n=j \mid X_{n-1}=i)$$

(c) (nth step Transition matrix)

From (b), $n \geq 1$ $\quad \mathbb{P}(X_{n-1} = i) > 0$

$$\Rightarrow \quad \mathbb{P}(X_n = j \mid X_{n-1} = i) = p_{ij} \qquad i, j \in S.$$

$P = [p_{ij}]$ - one step transition matrix

$n \geq 1$ $\quad \mathbb{P}(X_0 = i) > 0$

$$\mathbb{P}(X_n = j \mid X_0 = i) = \frac{\mathbb{P}(X_n = j)}{\mathbb{P}(X_0 = i)}$$

$$= \frac{\displaystyle\sum_{\substack{i_k \in S \\ k = 1, \dots, n-1}} \mathbb{P}(X_0 = i, \dots, X_{n-1} = i_{n-1}, X_n = j)}{\mu(\{i\})}$$

$$= \frac{\displaystyle\sum_{\substack{i_k \in S \\ k = 1, \dots, n-1}} \mu(\{i\}) \, p_{i i_1} \prod_{k=2}^{n-1} p_{i_{k-1}, i_k} \; p_{i_{n-1} j}}{\mu(\{i\})}$$

$$= \sum_{\substack{i_k \in S \\ k = 1, \dots, n-1}} p_{i i_1} \prod_{k=2}^{n-1} p_{i_{k-1}, i_k} \; p_{i_{n-1} j}$$

observe $\equiv$ $(i,j)^{th}$ entry of P^n

$$\mathbb{P}(X_n = j \mid X_0 = i) = p^n_{ij}$$

nth step transition probability matrix.

(d) (Chapman–Kolmogorov equation)

$r, s, n \geq 1$ & $r + s = n$.

$$P^n = P^r P^s$$

Fact:
- $|S| < \infty$ = matrix multiplication
- holds for S countable

$$\Longrightarrow \quad p^n_{ij} = \sum_{k \in S} p^r_{ik} \, p^s_{kj}$$

$\Longrightarrow$

$$\boxed{\mathbb{P}(X_n = j \mid X_0 = i) = \sum_{k \in S} \mathbb{P}(X_r = k \mid X_0 = i)\, \mathbb{P}(X_s = j \mid X_0 = k)}$$

i to j in steps $\equiv$ i to $k \in S$ in r steps AND $k \in S$ to j in s steps