Motivation for Rigorous Probability

- study of Models that come from experiments
  when the model is fully known

Sample Space S

- is any non-empty set
- elements of the set $\equiv$ outcomes
  (listing of all possibilities that can occur)
- Experiment: process of selecting one of these outcomes

Case 1 :- $|S| < \infty$ or S - Countable

$|S| = m$ $\exists$ bijection $\phi : S \to \{1, \dots, m\}$ ($\exists$ bijection $\phi : S \to \mathbb{N}$)

Events :- $\mathcal{F}$ - $\mathcal{P}(S)$, i.e. power set of S.
$A \subseteq S$ then A is an event.

Probability :- $\mathbb{P} : \mathcal{F} \to [0,1]$

(i) $\mathbb{P}(S) = 1$

(ii) $\{E_k\}_{k\geq 1}$ are a sequence of disjoint ($E_k \cap E_l = \phi$, $k \neq l$) ($E_k \in \mathcal{F}$) events then

$$\mathbb{P}\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} \mathbb{P}(E_k)$$

$\{s \in S \mid s \in E_k \text{ for some } k \geq 1\} \subseteq S$

$\therefore \bigcup_{k=1}^{\infty} E_k \in \mathcal{F}$

$T_n = \sum_{k=1}^{n} \mathbb{P}(E_k)$

$T_n \longrightarrow \sum_{k=1}^{\infty} \mathbb{P}(E_k) \in [0,1]$

Example 1 (. $S = \{s_1, \dots, s_n\}$ Uniform(S)) $n \in \mathbb{N}$

- Equally likely outcomes

- $\mathcal{F} = \mathcal{P}(S)$
- $\mathbb{P}: \mathcal{F} \to [0,1]$ by

$$\mathbb{P}(E) = \frac{|E|}{n}$$

Ex: - $\mathbb{P}$ satisfies axiom 1 and 2

- $\mathbb{P}(\{s_i\}) = \frac{1}{n}$ $i = 1, \dots n$

Remark: "cannot" do the above "$n = \infty$"

Example 2 (Bernoulli Trials)

- Toss one coin ; Probability of heads in a toss $= p$. $0 \leq p \leq 1$

$S = \{H, T\}$ $\quad \mathcal{F} = \mathcal{P}(S)$

$P : \mathcal{F} \to [0,1]$

$P(\phi) = 0, \quad P(S) = 1$

$P(\{H\}) = p \quad \mathbb{P}(\{T\}) = 1-p$

Ex:- $\mathbb{P}$ satisfies axiom 1 and 2

- Toss a coin 3 times ; Probability of heads in a toss $= p$

- $\mathcal{F} = \mathcal{P}(S)$

- $P : \mathcal{F} \to [0,1]$

$P(A) = p^{\#\text{Heads in } A} \cdot (1-p)^{\#\text{Tails in } A}$

Ex:- $\mathbb{P}$ satisfies axiom 1 and 2

$A = \{\omega_1, \omega_2 \omega_3\}$

# of heads in $A := \#\{i : \omega_i = H\}$

• S	• $X =$ # of head in 3 tosses
HHH	3
HHT	2
HTH	2
HTT	1
THH	2
THT	1
TTH	1
TTT	0

$X : S \to T$ is a discrete random variable

Convention :- : $T \subseteq \mathbb{R}$

$T = \text{Range}(X)$

"A discrete random variable X - induce a distribution on its range."

$T \equiv \text{Range}(X) = \{3, 2, 1, 0\}$

$\mathcal{F}' = \mathcal{P}(T)$

$\Phi : \mathcal{F}' \to [0,1]$

$\Phi(\{k\}) = \mathbb{P}(X = k) \quad k = 0,1,2,3$

$$\boxed{\Phi(A) = \sum_{k \in A} \Phi(\{k\}) \text{ or } \mathbb{P}(X \in A)}$$

<u>$k=2$:</u>

$\Phi(\{2\}) = \mathbb{P}(X=2) = \mathbb{P}(\{HHT, THH, HTH\})$

$= \mathbb{P}(\{HHT\}) + \mathbb{P}(\{THH\}) + \mathbb{P}(\{HTH\})$

$= p^2(1-p) + p^2(1-p) + p^2(1-p)$

$= 3p^2(1-p)$

Ex:- S - sample space, $\mathcal{F} = \mathcal{P}(S)$, $\mathbb{P}$ - Probability on S

$X : S \to T$, $T \subseteq \mathbb{R}$

$B \subseteq T \qquad \Phi(B) = \mathbb{P}(X \in B)$

Show: Φ satisfies axiom 1 and 2 on $(T, \mathcal{P}(T))$

Example 3: $S = \{H, T\}^n \qquad n \in \mathbb{N}$

$\mathcal{F} = \mathcal{P}(S)$

$$\mathbb{P}(A) = \sum_{\substack{\omega \in A \\ \omega = (\omega_1, \ldots, \omega_n)}} p^{\#\{i : \omega_i = H\}} (1-p)^{\#\{i : \omega_i = T\}}$$

Ex:- $\mathbb{P}$ satisfies axiom 1 and 2

$X : S \to T \qquad X = \#$ of heads

Ex:- $\mathbb{P}(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$

[Say] $X \sim \text{Binomial}(n, p)$

$$X_n \sim \text{Binomial}\left(n, \frac{\lambda}{n}\right) \xrightarrow[n \to \infty]{\text{" "}} X$$

Example 4:-

$T := \text{Range}(X) = \{0, 1, 2, \ldots\}$, $\mathcal{F}' = \mathcal{P}(T)$

$\mathbb{Q} : \mathcal{F}' \to [0,1]$

$$\text{"}\mathbb{P}(X = k)\text{"} \equiv \mathbb{Q}(\{k\}) = \frac{e^{-\lambda} \lambda^k}{k!} \qquad k = 0, 1, \ldots$$

$$\mathbb{Q}(A) = \sum_{k \in A} \mathbb{Q}(\{k\})$$

$X \sim \text{Poisson}(\lambda)$

Case II : S – uncountable (E.g. $S = (0,1)$, $\mathbb{R}$, ..)

Experiment: $S = [0,1]$ – "choose a point at random from $[0,1]$"

Setup

" Probability of choosing a # x " $\equiv p$

Suppose: $\mathcal{F} = \mathcal{P}(S)$

$$E = \left\{ \frac{1}{k} : k \geq 1 \right\} = \bigcup_{k=1}^{\infty} \{1/k\}$$

• **Axiom 2**

$$\underbrace{\mathbb{P}(E)}_{\in [0,1]} = \sum_{k=1}^{\infty} \mathbb{P}(\{\tfrac{1}{k}\}) = \text{"} \underbrace{p + p + \cdots}_{} \text{"} \iff p = 0$$

$\Rightarrow$ $P(E) = 0$ & $p = 0$

" Every individual outcome has probability 0 "

$$P(S) = P\left(\bigcup_{s \in S} \{s\} \right) \overset{\otimes}{=} \sum_{s \in S} \mathbb{P}(\{s\}) = 0$$

$= 0 + \cdots$

not a countable union : Axiom 2 does not apply

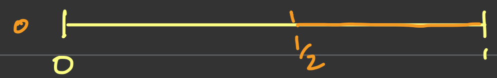
① In Uniform $[0,1]$:-

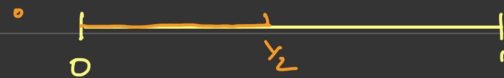
want :-

$\mathbb{P}([a,b]) = b - a$ ⌊ ①

" Probability of chosen point in $[a,b]$

Depends on length of [a,b]"





II

"like this"

$$P\left(\left[\tfrac{1}{4}, \tfrac{1}{2}\right] \cup \left[\tfrac{2}{3}, \tfrac{5}{6}\right]\right) = P\left(\left[\tfrac{1}{4}, \tfrac{1}{2}\right]\right) + P\left(\left[\tfrac{2}{3}, \tfrac{5}{6}\right]\right)$$

our definition

$\tfrac{1}{4} + \tfrac{1}{6}$

$\tfrac{5}{12}$

<u>want:</u> A & B disjoint subset of S

(1) $P(A \cup B) = P(A) + P(B)$ [finite additivity]

Axiom 2 requires:

$\{A_i\}_{i \geq 1}$ $A_i \cap A_j = \emptyset$ $i \neq j$

(2) —— $P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$ [countable additivity]

(length has this property if all the A_i's are intervals)

Requirement:

$A \subseteq [0,1]$ $r \in [0,1)$

- shift of A by r $\equiv$ $A \oplus r$

$$A \oplus r = \{a+r \mid a \in A, a+r \le 1\} \cup \{a+r-1 : a \in A, a+r > 1\}$$

- want : $\mathbb{P}(A) = \mathbb{P}(A \oplus r)$ —— ③

Q: Can we do this for all $A \subseteq \mathbb{R}$?

✓ A is an interval

? A = set of all roots of all polynomials with integer coefficients

$\tilde{Q}$: $S = [0,1)$ is $\mathcal{P}(S)$ the correct domain for "uniform" $\mathbb{P}$?

Proposition 1 :- There does not exists a definition of $\mathbb{P}(A)$ defined on all subsets of $A \subseteq [0,1]$, satisfying ①, ②, ③ Proof: [optional]

[Prove by Contradiction]

Suppose $\exists$ $P : \mathcal{P}(S) \to [0,1]$

satisfying ①, ②, ③

$\sim$ Equivalence relation: $x, y \in [0,1]$

$x \sim y$ iff $y - x$ is rational.

Ex:- $\sim$ partitions $[0,1]$ into disjoint equivalence classes.

e.g. $[x] = \{ y \in [0,1] \mid x - y \text{ is rational} \}$

$H \equiv$ (Axiom of choice) $\subseteq [0,1)$

- one member from each equivalence
- $0 \notin H$ [Convention], take $\frac{1}{2}$ (say)

Ex: • $\bigcup_{r \in \mathbb{Q} \cap [0,1]} H \oplus r = (0,1]$

• $\{ H \oplus r : r \in \mathbb{Q} \cap [0,1] \}$ is

a disjoint collection of sets in $[0,1]$

①, ②, ③ :- $\mathbb{P}$ Satisfies $\Rightarrow$

$$\mathbb{P}([0,1]) \overset{②}{=} \mathbb{P}\Big(\bigcup_{r \in \mathbb{Q} \cap [0,1]} H \oplus r\Big)$$

$$= \sum_{r \in \mathbb{Q} \cap [0,1]} \mathbb{P}(H \oplus r)$$

①

$$1 - 0 \overset{③}{=} \sum_{r \in \mathbb{Q} \cap [0,1]} \mathbb{P}(H)$$

$$1 \neq x - x \quad \mathbb{P}(H) = 0$$

(require)

Contradiction)

$\square$

$\mathcal{F}$: σ- algebra | σ-field

(i) $S \in \mathcal{F}$

(ii) $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$

(iii) $\forall k \geq 1 \; A_k \in \mathcal{F} \Rightarrow \bigcup_{k=1}^{\infty} A_k \in \mathcal{F}$

$S = (0,1)$ $\quad \mathcal{B}$ - Borel σ- algebra

- smallest σ- algebra that contains all intervals.

Example 1 (Uniform $(0,1)$)

[Probability density] $f: \mathbb{R} \to \mathbb{R}$

$$f(x) = \begin{cases} 1 & x \in (0,1) \\ 0 & \text{o.w.} \end{cases}$$

$A \in$ Borel σ-algebra of $(0,1)$

(one can show)

$$\mathbb{P}(A) = \int_A f(x)\,dx$$

$f \equiv$ p.d.f.

Example 2:

$X \sim \text{Poisson}(\lambda)$

$Y \sim \text{Normal}(\mu, \sigma^2)$

- $\mathbb{P}(X = k) = \dfrac{e^{-\lambda}\lambda^k}{k!}$ $\quad k = 0, 1, 2, \ldots$ (Discrete)
- $\mathbb{P}(Y \in B) = \displaystyle\int_B \frac{e^{-\frac{(x-\mu)^2}{2\sigma^2}}}{\sqrt{2\pi}\,\sigma}\,dx$ (absolute continuous)

$$Z = \begin{cases} X & \text{if coin comes up } H \\ Y & \text{if coin comes up } T \end{cases}$$

Ex: $\cdot\ \mathbb{P}(Z=k) > 0 \quad \forall\ k=0,1,2,\dots$

$\cdot\ P(Z \in [a,b] \cap \mathbb{Q}^c] > 0$

$\Rightarrow$ Z - neither discrete nor absolutely continuous.