2. Infinite length random walks

2.1 Higher dimension finite length walks

$$\mathbb{Z}^d = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_d \end{bmatrix} \;\middle|\; x_i \in \mathbb{Z} \quad 1 \le i \le d \right\}$$

Fix $N \geq 1$, $\qquad |x| = \sqrt{\sum_{i=1}^{d} x_i^2}$

$$\Omega_N = \{ \omega = (\omega_1, \dots \omega_N) : \omega_k \in \mathbb{Z}^d, \; |\omega_k| = 1, \; 1 \le k \le N \}$$

$$X_k : \Omega_N \to \mathbb{Z}^d \qquad X_k(\omega) = \omega_k$$

$$S_n = \sum_{i=1}^{n} X_i(\omega) \quad ; \; 1 \le n \le N$$

$$S_n = \begin{bmatrix} S_n^{(1)} \\ \vdots \\ S_n^{(d)} \end{bmatrix} \qquad 0 \le n \le N, \quad S_n^{(j)} \in \mathbb{Z}, \; 1 \le j \le d$$

$$\mathbb{P}_N(A) = \frac{|A|}{(2d)^N}$$

$A \subseteq \Omega_N$

Ex:- (i) Can you characterize properties of $S_n^{(j)}$?

(ii) Suppose we take $\tilde{S}_n^{(i)}$ $1 \le i \le d$, $1 \le n \le N$ as d independent simple random walks on $\mathbb{Z}$ then

$$\tilde{S}_n = \begin{bmatrix} \tilde{S}_n^{(1)} \\ \vdots \\ \tilde{S}_n^{(d)} \end{bmatrix} \qquad 1 \le n \le N$$

How different / similar is $\tilde{S}_n$ to S_n above ?

2.2 Infinite length random walks : (length $N \equiv \infty$)

Observe that our construction of finite length walks of length N had the following property:

$0 < N < M$ $\qquad \omega = (\underbrace{\omega_1, \dots, \omega_N}_{\tilde{\omega} \in \Omega_N}, \omega_{N+1}, \dots, \omega_M) \in \Omega_M$

$\mathbb{P}_N(\cdot) \qquad \mathbb{P}_M(\cdot)$

$$\Pi_N : \Omega_M \longrightarrow \Omega_N$$
$$\omega \longrightarrow \tilde{\omega}$$

Fix $\tilde{\omega} \in \Omega_N$:

$$\mathbb{P}_M(\{\omega \in \Omega_M : \Pi_N(\omega) = \tilde{\omega}\})$$

$$= \frac{|\{\omega \in \Omega_M : \Pi_N(\omega) = \tilde{\omega}\}|}{(2d)^M} = \frac{(2d)^{M-N}}{(2d)^M}$$

$$= (2d)^{-N}$$

$\boxed{\text{CONSISTENCY}}$ $(*)_C$

$$\mathbb{P}_N(\{\tilde{\omega}\}) = \frac{1}{(2d)^N}$$

Construction for $N \equiv \infty$ case

$$\Omega_\infty = \{ (\omega_m)_{m=1}^{\infty} \mid \omega_m \in \mathbb{Z}^d, \ |\omega_m| = 1, \ 1 \leq m \}$$

$$X_k : \Omega_\infty \to \mathbb{Z}^d \qquad X_k(\omega) = \omega_k \qquad 1 \leq k$$

$$S_n = \sum_{k=1}^{n} X_k \qquad n \geq 1$$

define : $\mathbb{P}$ on $(\Omega_\infty, \mathcal{F}_\infty)$ $\quad \Big\{$ temporarily $\mathcal{F}_\infty = \mathcal{P}(\Omega_\infty)$

$\forall N \geq 1, \quad \tilde{\omega} \in \Omega_N$

$$\mathbb{P}\left(\{ \omega \in \Omega_\infty \mid \Pi_N(\omega) = \tilde{\omega} \} \right)$$

$$:= \frac{1}{(2d)^N} \equiv \mathbb{P}_N(\{\tilde{\omega}\})$$

Kolmogorov Consistency Theorem :

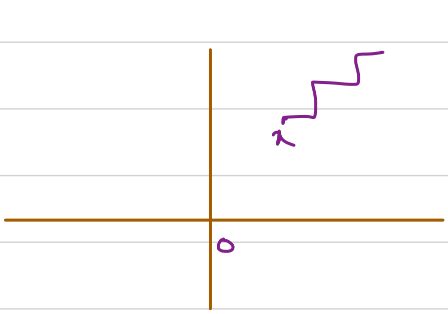
If $(\Omega_N, \mathcal{P}(\Omega_N), \mathbb{P}_N)$ satisfying $(*)_c$ Then $\exists$ $\mathbb{P}$ on $(\Omega_\infty, \mathcal{F}_\infty)$ such $\mathbb{P}|_{\Omega_N} = \mathbb{P}_N$

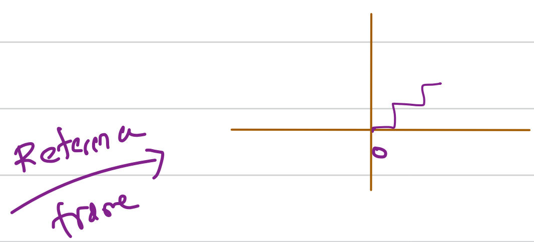
Notation / Convention :

- $(\Omega_\infty, \mathbb{P})$ and $\{S_n\}_{n \geq 1}$ as simple random walk starting at 0.

- $\{S_n\}_{n \geq 1}$ to start at x, define another Probability $\mathbb{P}_x(\cdot)$

$$\mathbb{P}_x(S_1 = s_1, \ldots, S_n = s_n) = \mathbb{P}(S_1 + x = s_1, \ldots, S_n + x = s_n)$$





$$\mathbb{E}_x[f(S_1, \ldots, S_n)] = \mathbb{E}[f(S_1 + x, \ldots, S_n + x)]$$

Generalisations : $\begin{cases} \cdot \text{ Each step is of size 1. / length 1} \\ \cdot \text{ Probability of each step} \equiv \text{same} \end{cases}$

Independence

- (Both length & equal. probabilities can be tweaked)

2.3 Tschebyschev's Inequality: X is a random variable on $(\Omega, \mathcal{F}, \mathbb{P})$

$E[X] = \mu$ and $Var[X] = \sigma^2$

$$\mathbb{P}(|X-\mu| \geq k\sigma) \leq \frac{1}{k^2} \quad \forall k \geq 1$$

d=1 $\{S_n\}_{n\geq 1}$ Simple random walk on $\mathbb{Z}$.

$$S_n = \sum_{k=1}^{n} X_i$$

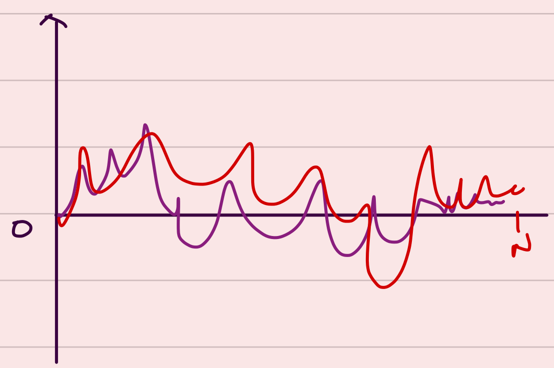
$$\mathbb{E}[S_n] = \sum_{i=1}^{n} \mathbb{E}[X_i] = 0$$

$$Var[S_n] = Var\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} Var[X_i]$$

$$= \sum_{i=1}^{n} E[X_i^2] = n$$

Recall (R-code)

"Law of large numbers"

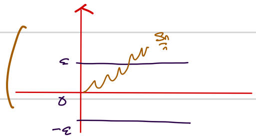


" $\frac{S_n}{n}$ "

Weak law of large numbers:

let $\varepsilon > 0$ be given

$$\mathbb{P}\left(\left| \frac{S_n}{n} - 0 \right| > \varepsilon \right) \quad \text{“} = \mathbb{P}\Big(\qquad \Big) \text{”}$$



$$= \mathbb{P}\left(| S_n - 0 | > n\varepsilon \right)$$

$\mu = E[S_n]$

$$= \mathbb{P}\left(| S_n - 0 | > \underbrace{(\sqrt{n}\,\varepsilon)}_{k} \sqrt{n} \right)$$

$\sigma = \sqrt{Var[S_n]}$

Tschebychev inequality

$$\leq \frac{1}{\varepsilon^2 n}$$

$$\Rightarrow \quad 0 \leq \mathbb{P}\left(\left| \frac{S_n}{n} \right| > \varepsilon \right) \leq \frac{1}{n\varepsilon^2}$$

$$\Rightarrow \quad \boxed{\begin{array}{l} \forall \varepsilon > 0 \\ \lim_{n \to \infty} \mathbb{P}\left(\left| \frac{S_n}{n} \right| > \varepsilon \right) = 0 \end{array}} \quad \text{— (WLLN)} \quad (*)_P$$

Probabilistic Terminology:

$\frac{S_n}{n}$ converges to 0 in Probability.

Strong law of large numbers:

$\{S_n\}_{n \geq 1}$ is a simple random walk on $\mathbb{Z}^d$.

$$A = \left\{ \omega \in \Omega \;\middle|\; \frac{S_n}{n} \longrightarrow 0 \text{ as } n \to \infty \right\}$$

then $\mathbb{P}(A) = 1.$ — $\circledast_s$

Ex:- SLLN stronger than WLLN

i.e. $\circledast_s \Rightarrow \circledast_p$

<u>Key Application</u> :- Speed of $\{S_n\}_{n\geq 1}$ approaches 0.

<u>2.4 Typical position of the walk</u>

<u>d=1</u> $\{S_n\}_{n\geq 1}$ is simple random walk on $\mathbb{Z}$.

<u>Central limit theorem</u>: Let $\{Y_i\}_{i\geq 1}$ be a collection of i.i.d. random variables (on $\mathbb{R}$) $E[Y_1] = \mu$, $Var[Y_1] = \sigma^2$

" $\sum_{i=1}^{n} Y_i \sim \text{Normal}(n\mu, n\sigma^2)$ "

$$\boxed{Z_n = \frac{\sum_{i=1}^{n} Y_i - n\mu}{\sqrt{n}\,\sigma} \equiv \quad E[Z_n] = 0 \quad Var[Z_n] = 1}$$

$$Z_n \sim \text{Normal}(0,1)$$

$$\mathbb{P}\left(\frac{\sum_{i=1}^{n} Y_i - n\mu}{\sqrt{n}\,\sigma} \leq x \right) \longrightarrow \int_{-\infty}^{x} \frac{e^{-y^2/2}}{\sqrt{2\pi}}\, dy$$

as $n \to \infty$

$$S_n = \sum_{i=1}^{n} X_i \,, \quad \begin{cases} X_i \text{ are i.i.d.} \\ \mathbb{P}(X_i = 1) = \mathbb{P}(X_i = -1) = \frac{1}{2} \\ E[X_i] = 0, \ \mathrm{var}[X_i] = 1 \end{cases}$$

C.L.T.

$$\mathbb{P}\left(\frac{S_n - n0}{\sqrt{n}\,1} \le x \right) \longrightarrow \int_{-\infty}^{x} \frac{e^{-y^2/2}}{\sqrt{2\pi}}\, dy$$

as $n \to \infty$

i.e. $$\mathbb{P}\left(\frac{S_n}{\sqrt{n}} \le x \right) \longrightarrow \int_{-\infty}^{x} \frac{e^{-y^2/2}}{\sqrt{2\pi}}\, dy$$

as $n \to \infty$

$x > 0$

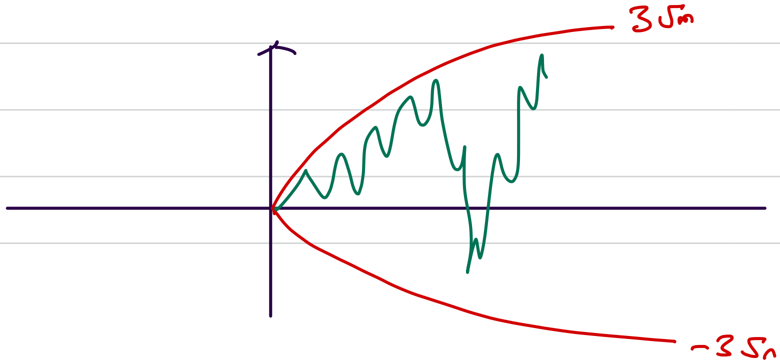
(Ex.) $\Rightarrow$ $$\mathbb{P}\left(\left| \frac{S_n}{\sqrt{n}} \right| \le x \right) \sim \underbrace{\int_{-x}^{x} \frac{e^{-y^2/2}}{\sqrt{2\pi}}\, dy}$$

$\boxed{\text{integral}}$ value

$x = 1$ say ≈ 0.75

$x = 3 \approx 0.99$

$\therefore \quad \mathbb{P}(-3\sqrt{n} \le S_n \le 3\sqrt{n}) \sim 0.99$



2.5 Probabilities of atypical events

e.g. $\mathbb{P}(S_n \geq an) \xrightarrow{?}$?

how does it go to 0

2.6 Time spent at each point in $\mathbb{Z}^d$.