Note: $\mathcal{M} = \{ B \in \mathcal{M}(A) \mid E \in \mathcal{M}(A) \Rightarrow E \cup B \in \mathcal{M}(A) \}$

Clearly $\mathcal{M}$ is a monotone class.

$\Rightarrow$ $\mathcal{M}(A)$ is closed under union $\square$

September 18th, 2023

lemma A.1 & lemma A.2

- $\bigsqcup_{i=1}^{N} (a_i, b_i] \subseteq (a, b]$ $\quad N \in \mathbb{N} \cup \{\infty\}$ $\quad -\circledast_1$

$\Rightarrow \sum_{i=1}^{N} b_i - a_i \leq b - a$

- $(a, b] \subseteq \bigcup_{i=1}^{N} (a_i, b_i]$ $\quad -\circledast_2$

$\Rightarrow b - a \leq \sum_{i=1}^{N} b_i - a_i$.

Step 3: lemma A.1 and lemma A.2, definition of $\mathcal{A}(S)$ $\Rightarrow$

$\exists\, \mathbb{P} : \mathcal{A}(S) \to [0,1]$ satisfying ① and ② such that

(i) $\mathbb{P}([0,1]) = 1$

(ii) $E_k \in \mathcal{A}(S)$ $\quad E_l \cap E_m = \phi$

$\mathbb{P}\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} \mathbb{P}(E_k)$

— immediate if

$\mathbb{P}\left(\bigsqcup_{i=1}^{N} (a_i, b_i]\right) := \sum_{i=1}^{N} b_i - a_i$

$\Rightarrow$ $\mathbb{P}$ is a probability by $\circledast_1$ and $\circledast_2$

We now have a Probability P on $\mathcal{A}(S)$ satisfying the required properties.

$$\mathcal{A}(S) = \mathcal{A}$$

Carathéodory Extension Theorem

$$\exists \ \tilde{P} : \sigma(\mathcal{A}) \longrightarrow [0,1]$$

ⓐ $\tilde{\mathbb{P}}(S) = 1$

ⓑ $\{E_k\}_{k\geq 1}$ are disjoint

$$\tilde{\mathbb{P}}\Big(\bigcup_{k\geq 1} E_k\Big) = \sum_{k\geq 1} \tilde{\mathbb{P}}(E_k)$$

and agrees with $\mathbb{P}$ on $\mathcal{A}$. &

sketch of the proof:

Definition ⓪: Give $\mathbb{P}$ on $\mathcal{A}$ define

$$\mathbb{P}^*(E) := \inf\left\{ \sum_{i=1}^{n} \mathbb{P}(A_i) \;\middle|\; E \subseteq \bigcup_{i=1}^{\infty} A_i \quad A_i \in \mathcal{A} \ \forall n \geq 1 \right\}$$

$$\forall \ E \subseteq \Omega.$$

$\mathbb{P}^*(\cdot)$ outer Probability associated with $\mathbb{P}$.

strategy: (i) $\mathbb{P}^*\big|_{\mathcal{A}} = \mathbb{P}$

(ii) $\mathbb{P}^* : \sigma(\mathcal{A}) \longrightarrow [0,1]$ & satisfies

- $\mathbb{P}^*([0,1]) = 1$

- $\{E_k\}_{k \geq 1}$ and $E_l \cap E_m = \phi$ $l \neq m$ then

$$\mathbb{P}\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mathbb{P}(E_i)$$

(iii) $\mathbb{P}^*(A \oplus x) = \mathbb{P}^*(A) \quad \forall\ A \in \sigma(\mathcal{A})$.

(Ex)

Proof of (i) :

let $A \in \mathcal{A}$

- $\underset{\in \mathcal{A}}{A_1 = A}$, $A_k = \phi \ \forall k \geq 1 \Rightarrow A \subseteq \bigcup_{i=1}^{n} A_k \quad \forall n \geq 1$

$$\Rightarrow \mathbb{P}^*(A) \leq \mathbb{P}(A) \qquad - (1)$$

- $A_i \in \mathcal{A}$ such that $A \subseteq \bigcup_{i=1}^{\infty} A_i$

$$\Rightarrow \bigcup_{i=1}^{\infty} A \cap A_i = A$$

(countable subadditive)

(Ex.) $\mathbb{P}(A) \leq \sum_{i=1}^{\infty} \mathbb{P}(A \cap A_i)$

$$\leq \sum_{i=1}^{\infty} \mathbb{P}(A_i)$$

Since $\{A_i\}_{i \geq 1}$ were an arbitrary collection then (Ex)

$$\mathbb{P}(A) \leq \mathbb{P}^*(A) \quad -(2)$$

So by (1) and (2) $\Rightarrow$ $\mathbb{P}(A) = \mathbb{P}^*(A) \;\forall A \in \mathcal{A}$

Sketch of Proof of (ii)

$\mathcal{M}(\mathbb{P}) :=$ a collection of subsets of $[0,1]$ given by

$$\mathcal{M}(\mathbb{P}) := \left\{ E \subseteq [0,1] : \mathbb{P}^*(A) = \mathbb{P}^*(E \cap A) + \mathbb{P}^*(E^c \cap A) \;\; \forall A \subseteq [0,1]. \right\}$$

- Remark: $E = (E \cap A) \cup (E \cap A^c)$

  Ex: $\mathbb{P}^*(E) \leq \mathbb{P}^*(E \cap A) + \mathbb{P}^*(E \cap A^c) \;\; \forall A \subseteq \Omega.$

Key steps:

Step 4(a): $\mathcal{A} \subseteq \mathcal{M}(\mathbb{P})$.

Step 4(b): $\mathbb{P}^*\big|_{\mathcal{M}(\mathbb{P})}$ is a Probability

& $\mathcal{M}(\mathbb{P}) \supseteq \sigma(\mathcal{A})$.

$\boxed{\text{Ex: ?}}$

Step 4b(i) : $A, B \in \mathcal{M}(\mathbb{P})$, $A \subseteq B$

then $B \cap A^c \in \mathcal{M}(\mathbb{P})$.

(Extra reading)

Step 4b(ii) : $D = \bigcup_{i=1}^{n} D_i$ $\quad D_k \cap D_\ell = \phi$

& $D_k \in \mathcal{M}(\mathbb{P})$ then

$D \in \mathcal{M}(\mathbb{P})$.

Step 4b(iii) $\mathcal{M}(\mathbb{P})$ is a σ-algebra.

Proof of step 4(a) :

let $B \in \mathcal{A}$. & $F \subseteq \Omega$. let $\varepsilon > 0$ be given

let $\{A_n\}_{n \geq 1}$ $\quad A_n \in \mathcal{A}$ s.t

(Ex) $\begin{cases} \text{(i)} & F \subseteq \bigcup_{n=1}^{\infty} A_n \\ \text{(ii)} & \mathbb{P}^*(F) + \varepsilon \geq \sum_{i=1}^{\infty} \mathbb{P}(A_i) \end{cases}$

$B^c \in \mathcal{A}$, $A_i \in \mathcal{A}$ $\Rightarrow$ $B^c \cap A_i \in \mathcal{A}$

$\Rightarrow$ $F \cap B \subseteq \bigcup_{i=1}^{\infty} (B \cap A_i)$ & $F \cap B^c \subseteq \bigcup_{i=1}^{\infty} (B^c \cap A_i)$

$B \in \mathcal{A}$, $A_i \in \mathcal{A}$ $\Rightarrow$ $B \cap A_i \in \mathcal{A}$

$\mathbb{P}^*(F \cap B) + \mathbb{P}^*(F \cap B^c)$

$\overset{\text{Def}}{\leq} \sum_{i=1}^{\infty} \mathbb{P}(B \cap A_i) + \sum_{i=1}^{\infty} \mathbb{P}(B^c \cap A_i)$