Markov chains – $\{X_n\}_{n\geq 1}$ (S, P, μ)

$\phi \neq S$ – finite set or countable set

μ – Probability on S

P – $P = [p_{ij}]_{i,j \in S}$;

$i \xrightarrow{p_{ij}} j$ $\sum_{j \in S} p_{ij} = 1$ $0 \leq p_{ij} \leq 1$

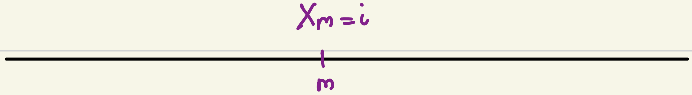
$(\Omega, \mathcal{F}, \mathbb{P})$ $X_n : \Omega \to S$ $\forall n \geq 1$ are a sequence of r.v.'s

$$\mathbb{P}(X_0 = i_0, \ldots, X_n = i_n) = \mu(\{i_0\})\, p_{i_0 i_1} \cdots p_{i_{n-1} i_n}$$

$\forall n \geq 1$

[Markov Property]

- $\forall m \geq 1$ $\forall n \geq 0$ $Y_n \overset{d}{=} X_{m+n} \mid X_m = i$



Then: $\{Y_n\}_{n \geq 0}$ is a Markov chain on S

with transition matrix P and $\mathbb{P}(Y_0 = i) = 1$

- $\{Y_n\}_{n \geq 0}$ is independent of the random variables $\{X_0, X_1, \dots, X_{n-1}\}$.

Compare:

If $\mathbb{P}(X_{n-1} = i, X_{n-2} = i_{n-2}, \dots, X_0 = i_0) > 0$

then $\mathbb{P}(X_n = j \mid X_{n-1} = i, X_{n-2} = i_{n-2}, \dots, X_0 = i_0)$

$= \mathbb{P}(X_n = j \mid X_{n-1} = i)$

Example M2: S - countable set - vertex

$E \subseteq \{ \{i,j\} \mid i,j \in S \}$ - Edges

Given: $\mu : E \to (0, \infty)$

$$\mu_{ji} = \mu_{ij} = \mu(\{i,j\}) \quad \text{for } \{i,j\} \in E$$
$$= 0 \quad \text{if } \{i,j\} \notin E$$

For $i \in S$ $\quad \mu_i = \sum_{j \in S} \mu_{ij} = \sum_{\{j \in S.\ \{i,j\} \in E\}} \mu(\{i,j\})$

Assume: $0 < \mu_i < \infty \quad \forall\, i \in S.$

Markov chain on S

let $\nu_0 \equiv$ probability on S

$$P = [p_{ij}]_{i,j \in S} \quad \text{with} \quad p_{ij} = \frac{\mu_{ij}}{\mu_i}$$

Ex:- P is a transition matrix.

$\{X_n\}_{n \geq 1}$ be a M.C. corresponding to (S, P, ν_0)

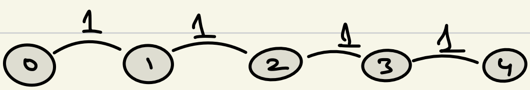
Random walk on a weighted graph.

(a) $S = \{0, 1, 2, 3, 4\}$

$E = \{ \{i,j\} \mid |i-j| = 1 \ \& \ i,j \in S \}$

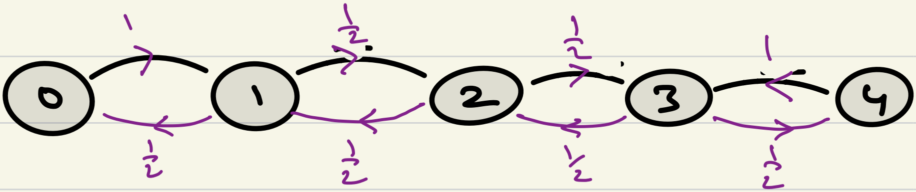
(natural weights) $\mu(\{i,j\}) = 1 \quad \forall\, \{i,j\} \in E$

(S,E) :



$$p_{ij} = \begin{cases} 1 & i=0,\ j=1 \\ 1 & i=4\ \ j=3 \\ \frac{1}{2} & i \in \{1,2,3\} \\ & \ \& \ j \in \{i+1, i-1\} \\ 0 & \text{o.w.} \end{cases}$$

$\{X_n\}_{n\geq 0}$ - Simple symmetric random walk on $\{0,1,2,3,4\}$ & is reflected at 0 and 4



(b) $S = \mathbb{Z}$ $\qquad E = \{ \{i,j\} \mid j = i+1, i-1 \}$
$\qquad\qquad\qquad\qquad i \in \mathbb{Z}$

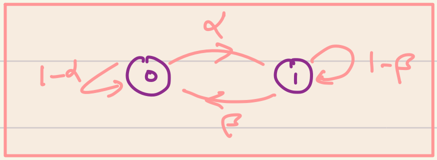
(natural weight) $\qquad \mu(\{i,j\}) = 1 \quad \forall \{i,j\} \in E$

$$p_{ij} = \begin{cases} \frac{1}{2} & \text{if } \ j = i+1 \text{ or } i-1 \\ & \qquad i \in \mathbb{Z} \\ 0 & \text{o.w.} \end{cases}$$

$\{X_n\}_{n\geq 0}$ - Simple symmetric random walk on $\mathbb{Z}$.

$\nu_0(\{0\}) = 1 \implies X_0 = 0$ w.p. 1

Example M3 : $\{X_n\}_{n\geq 1}$ is a Markov chain on



$S = \{0,1\}$

$$P = \begin{bmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{bmatrix}, \quad 0 < \alpha, \beta < 1$$

$$\mu_0 =: \mathbb{P}(X_0 = 0) \qquad \mu_0 + \mu_1 = 1$$
$$\mu_1 =: \mathbb{P}(X_0 = 1) \qquad 0 \leq \mu_0, \mu_1 \leq 1$$

Observations :

$$P^n = P^{n-1} P = \begin{bmatrix} p_{00}^{n-1} & p_{01}^{n-1} \\ p_{10}^{n-1} & p_{11}^{n-1} \end{bmatrix} \begin{bmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{bmatrix}$$

$$\Rightarrow \quad p_{00}^{n} = p_{00}^{n-1}(1-\alpha) + p_{01}^{n-1}\beta$$

$$\Rightarrow \quad \boxed{p_{00}^{n} = p_{00}^{n-1}(1-\alpha-\beta) + \beta} \quad (*)$$

$(p_{01}^{n-1} = 1 - p_{00}^{n-1})$

Convention: $P^0 = I$; $p_{00}^0 = 1$, $p_{11}^0 = 1$

To solve for p_{00}^n

Set

$n \geq 1;\quad x_n = p_{00}^n - \frac{\beta}{\alpha+\beta}\quad;\quad x_0 = 1 - \frac{\beta}{\alpha+\beta}$

$$\Rightarrow \quad x_n \overset{(*)}{=} p_{00}^{n-1}(1-\alpha-\beta) + \beta - \frac{\beta}{\alpha+\beta}$$

$$\Rightarrow \quad x_n = \left(p_{00}^{n-1} - \frac{\beta}{\alpha+\beta}\right)(1-\alpha-\beta)$$

$$\Rightarrow \quad x_n = x_{n-1}(1-\alpha-\beta)$$

inductively

$$\Rightarrow \quad x_n = (1-\alpha-\beta)^n x_0$$

As $\quad p_{00}^n = x_n + \frac{\beta}{\alpha+\beta}$

$$\Rightarrow \quad \boxed{p_{00}^n = \frac{\beta}{\alpha+\beta} + \frac{\alpha}{\alpha+\beta}(1-\alpha-\beta)^n} \quad (1)$$

(Ex.)

$$\boxed{p_{10}^{n} = \frac{\beta}{\alpha+\beta} - \frac{\beta}{\alpha+\beta}(1-\alpha-\beta)^{n}} \quad ②$$

$$\mathbb{P}(X_n=0) = \mathbb{P}(X_n=0 \mid X_0=0)\,\mathbb{P}(X_0=0) + \mathbb{P}(X_n=0 \mid X_0=1)\,\mathbb{P}(X_0=1)$$

$$= p_{00}^{n}\mu_0 + p_{10}^{n}\mu_1$$

$$= (p_{00}^{n} - p_{10}^{n})\mu_0 + p_{10}^{n}$$

substitute ① and ② above :

$$\mathbb{P}(X_n=0) = (1-\alpha-\beta)^{n}\mu_0 + \frac{\beta}{\alpha+\beta} - \frac{\beta}{\alpha+\beta}(1-\alpha-\beta)^{n}$$

$$\Rightarrow ③ \quad \therefore \left\{ \begin{aligned} &\mathbb{P}(X_n=0) = \frac{\beta}{\alpha+\beta} + (1-\alpha-\beta)^{n}\left(\mu_0 - \frac{\beta}{\alpha+\beta}\right) \\ &\mathbb{P}(X_n=1) = 1 - \mathbb{P}(X_n=0) \end{aligned} \right.$$

$1-\mu_1=\mu_0$

$$= \frac{\alpha}{\alpha+\beta} + (1-\alpha-\beta)^{n}\left(\mu_1 - \frac{\alpha}{\alpha+\beta}\right)$$

$1-\frac{\alpha}{\alpha+\beta} = \beta/\alpha+\beta$

Observation:

- $\alpha+\beta < 2 \Rightarrow |1-(\alpha+\beta)| < 1$

$n \to \infty$

For any μ_0, μ_1

$$\left.\begin{aligned} \mathbb{P}(X_n=0) &\longrightarrow \beta/\alpha+\beta \\ \mathbb{P}(X_n=1) &\longrightarrow \alpha/\alpha+\beta \end{aligned}\right\} \text{limiting distribution}$$

- $\mu_0 = \beta/\alpha+\beta \Rightarrow \mu_1 = \dfrac{\alpha}{\alpha+\beta}$

$\Rightarrow \mathbb{P}(X_n=0) = \mu_0$ $\quad \forall\, n \geq 0$ } stationary distribution

$\mathbb{P}(X_n=1) = \mu_1$

Definition: let S be countable set

π - a probability on S is called a stationary distribution for the markov chain $X_n \equiv (S, P, \mu)$ if

$$\pi(j) = \sum_{i \in S} \pi(i)\, p_{ij} \qquad \text{— (I)}$$

$$(\pi = \pi P)$$

Lemma M1: If $X_0 \overset{d}{=} \pi$ & π is stationary then

$$X_n \overset{d}{=} \pi \quad \forall n \geq 1$$

Proof:- $n=1$

$$\mathbb{P}(X_1 = i) = \sum_{k \in S} \mathbb{P}(X_1 = i, X_0 = k)$$

$$= \sum_{k \in S} \mathbb{P}(X_1 = i \mid X_0 = k)\, \mathbb{P}(X_0 = k)$$

$$= \sum_{k \in S} p_{ki}\, \pi(k)$$

$$\overset{(\ddagger)}{=} \pi(i)$$

By induction - $\mathbb{P}(X_n = i) = \pi(i)$

$\forall n \geq 1, \ i \in S$ $\square$