Modes of Convergence - Proof of strong law large numbers

(- Equivalence
- limit theorems)

[M: Keane - Proof]

Theorem (Strong law of large numbers)

Let $X, \{X_i\}_{i \geq 0}$ be an independent & identically distributed sequence of random variables (on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$).

- $\mu = E[X]$ and $E|X| < \infty$, $E|X|^2 < \infty$.

Then $\frac{1}{n} \sum_{i=0}^{n-1} X_i \longrightarrow \mu$ as $n \to \infty$ w.p. 1.

Definition: A sequence of random variables $\{Y_n\}_{n \geq 1}$ converge to Y with probability 1 if

$$\mathbb{P}(\lim_{n \to \infty} Y_n = Y) = 1$$

$\boxed{\text{w.p. 1} \quad \text{a.s.} \quad \text{a.e}}$

Proof: - (Bits and Pieces)

For special case: $X_i \in \{0,1\}$ $\mu = \mathbb{P}(X_i = 1)$.

$\therefore\ a_n \in \{0,1\}\ \forall n \geq 1 \xRightarrow{?} \frac{1}{n}\sum_{i=1}^{n} a_i$ converges to "*" as $n \to \infty$

[Check]

let $\omega \in \Omega$, $\varepsilon > 0$ be given. $\bar{S}(\omega) = \overline{\lim_{n\to\infty}} \frac{1}{n}\sum_{i=1}^{n} X_i(\omega)$

For each $k \geq 1$:

$$N_k = \min\left\{ n \geq 1 \;\middle|\; \frac{X_k + X_{k+1} + \cdots + X_{k+n-1}}{n} > \bar{S} - \varepsilon \right\}$$

Observations:
(Ex.)

- $0 \leq \bar{S} \leq 1$

(By definition of lim sup)

- $N_k < \infty$ w.p. 1

Case 1: Assume $\exists M > 0$: $N_k < M \quad \forall k \geq 1$

$$X_0 + X_1 + \cdots + X_{n-1} = \underbrace{X_0 + X_1 + \cdots X_{N_0}}_{\text{at most } M \text{ term}}$$

$$\underbrace{X_{N_0+1} + \cdots + \cdot + X_{N_{N_0}}}_{\text{at most } M \text{ term}}$$

$$\vdots \quad \cdots$$

$$+$$

$$\underbrace{X_{N_{N_k}} + \cdot\cdot + \cdots + X_n}_{\text{remainder}} \quad (\geq 0)$$

$$\underset{\text{Ex}}{\geq} (n - M)(\bar{S} - \varepsilon) \quad \text{—} (*)$$

$$\Rightarrow \quad \frac{X_0 + X_1 + \cdots + X_{n-1}}{n} \geq \left(1 - \frac{M}{n}\right)(\bar{S} - \varepsilon)$$

let $N \geq 1$ be such that $\frac{M}{n} < \varepsilon \quad \forall\, n \geq N$

$\Rightarrow \quad \forall\, n \geq N$

$$\frac{X_0 + X_1 + \cdots + X_{n-1}}{n} \geq (1-\varepsilon)(\bar{S} - \varepsilon)$$

$$\geq \bar{S} - 2\varepsilon$$

(Assume $\varepsilon > 0$ small enough)

General Case (Don't assume Case 1)

Observation: $N_k = \min\left\{ n \geq 1 \;\middle|\; \frac{X_k + X_{k+1} \cdots + X_{k+n-1}}{n} > \bar{S} - \varepsilon \right\}$

Ex • N_k have same distribution. $\forall\, k \geq 1$

($N_k < \infty$ w.p. 1) • $\exists\, M > 0$ st $\mathbb{P}(N_k > M) < \varepsilon$ — (+)

Define $$Y_k = \begin{cases} X_k & \text{if } N_k \leq M \\ 1 & \text{if } N_k > M \end{cases}$$

Observation: • $Y_k \geq X_k$

$$N_k^Y = \min\left\{ k \geq 1 \;\middle|\; \frac{Y_k + Y_{k+1} \cdots + Y_{k+n-1}}{n} \geq \bar{S} - \varepsilon \right\}$$

$\left(\overline{\lim_{n \to \infty}} \frac{1}{n} \sum_{i=0}^{n-1} X_i\right)$

$\circ \; \boxed{N_k^Y \leq M}$ – Crucial step

$\Rightarrow$ from Case 1 ⊛ argument

$$Y_0 + Y_1 + \dots + Y_{n-1} \geqslant (n-M)(\bar{S} - \varepsilon) \quad - (1)$$

Now $\cdot \; E X_i = \mu$, $E[X_i] \leq E[Y_i]$ – (2)

$$\cdot \; E Y_i = E(X_i \mathbb{1}_{N_i < M}) + \mathbb{P}(N_i > M)$$

Choose M as in (+)

$$\Rightarrow \quad E[Y_i] \leq \mu + \varepsilon \quad \forall i \geqslant 0 \quad - (3)$$

Taking Expectation in (1)

$$n\, E[Y_0] \geqslant (n-M)(E[\bar{S}] - \varepsilon)$$

Using (2)

$$n(\mu + \varepsilon) \geqslant (n-M)(E[\bar{S}] - \varepsilon)$$

$$\mu + \varepsilon \geqslant \left(1 - \frac{M}{n}\right)(E[\bar{S}] - \varepsilon)$$

$\forall\, n \geqslant 1$

let $n \to \infty$ and $\varepsilon \to 0$

$$\boxed{\mu \geqslant E[\bar{S}]} \quad - \text{(I)}$$

Next class: $\tilde{X}_k = 1 - X_k \quad k \geq 0 \qquad \tilde{\mu} = E[\tilde{X}_0]$

Repeat I to get

$$\boxed{\tilde{\mu} \geq E[\overline{\tilde{S}}]} \quad - \quad *_I$$

Observation: $\cdot \ \tilde{\mu} = 1 - \mu$

$\cdot \ \overline{\tilde{S}} = \limsup_{n \to \infty} \dfrac{\sum_{i=0}^{n-1} \tilde{X}_i}{n} \quad ; \quad \overline{\tilde{S}} = 1 - \underline{S}$

$$*_I \implies 1 - \mu \geq 1 - E[\underline{S}]$$

$$\implies \boxed{\mu \leq E[\underline{S}]} \quad - \quad \text{II}$$

$\cdot \quad \underline{S} \leq \overline{S} \quad \ldots \quad \text{III}$

But I and II $\implies E[\underline{S}] \geq \mu \geq E[\overline{S}]$ — IV

$$\text{III} \ \text{IV} \implies \underline{S} = \overline{S}$$

$$\implies \mathbb{P}\left(\frac{1}{n} \sum_{i=0}^{n-1} X_i \text{ converges as } n \to \infty \right) = 1$$