Let $N \in \mathbb{N}$. Consider

$$\Omega_N = \left\{ \omega = (\omega_1, \omega_2, \ldots, \omega_N) : \omega_i \in \{-1, +1\} \right\}$$

equipped with the uniform distribution, denoted by $\mathbb{P} \equiv \mathbb{P}_N$.
For $1 \leq k \leq N$, let $X_K : \Omega_N \to \{-1, 1\}$ be given by $X_k(\omega) = \omega_k$ and
for $1 \leq n \leq N$, let $S_n : \Omega_N \to \mathbb{Z}$ be given by $S_n(\omega) = \sum_{k=1}^{n} X_k(\omega)$ and $S_0 = 0$.
Let $\mathcal{A}_n$ be the events that are observable by time n.

1. For $1 \leq n \leq N$, show that the mode of S_n is $\{0, 1\}$ that is

$$\max\{\mathbb{P}(S_n = a) : a \in \mathbb{Z}\} = \begin{cases} \mathbb{P}(S_{2k} = 0) & \text{if } n = 2k, k \in \mathbb{N} \\ \mathbb{P}(S_{2k-1} = 1) & \text{if } n = 2k-1, k \in \mathbb{N} \end{cases} = \binom{2k}{k}\frac{1}{2^{2k}}$$

2. For $a < b, a, b \in \mathbb{Z}$, $1 \leq n \leq N$ show that

$$\mathbb{P}(a \leq S_n \leq b) \leq (b - a + 1)\mathbb{P}(S_n \in \{0, 1\})$$

   and conclude that $\lim_{N\to\infty} \mathbb{P}(a \leq S_N \leq b) = 0$.

3. Let $-\infty < a < 0 < b < \infty, a, b \in \mathbb{Z},,$

$$\sigma_a = \min\{k \geq 1 : S_k = a\} \qquad \text{and} \qquad \sigma_b = \min\{k \geq 1 : S_k = b\}.$$

   (a) Let $\tau_N = \min\{\sigma_a, \sigma_b, N\}$. Show that τ_N is a Stopping time.

   (b) Show that

$$\mathbb{E}(S_{\tau_N}) = a\mathbb{P}(\sigma_a < \sigma_b, \sigma_a \leq N) + b\mathbb{P}(\sigma_b < \sigma_a, \sigma_b \leq N) + \mathbb{E}(S_N 1(\min\{\sigma_a \sigma_b\} > N))$$

   and

$$\mathbb{E}(\tau_N) = a^2\mathbb{P}(\sigma_a < \sigma_b, \sigma_a \leq N) + b^2\mathbb{P}(\sigma_b < \sigma_a, \sigma_b \leq N) + \mathbb{E}(S_N^2 1(\min\{\sigma_a, \sigma_b\} > N)).$$

   (c) Show that

$$1 - \mathbb{P}(\sigma_a < \sigma_b, \sigma_a \leq N) - \mathbb{P}(\sigma_b < \sigma_a, \sigma_b \leq N) = \mathbb{P}(\min\{\sigma_a, \sigma_b\} > N)$$

   (d) Limits as $N \to \infty$.

      i. $\mathbb{P}(\min\{\sigma_a, \sigma_b\} > N) \to 0$ as $N \to \infty$.

      ii. $\mathbb{E}(S_N 1(\min\{\sigma_a, \sigma_b\} > N)) \to 0$ as $N \to \infty$.

      iii. Show that there exists $r_a^1, r_b^2 \in [0, 1]$ such that

$$r_a^1 = \lim_{n\to\infty} \mathbb{P}(\sigma_a < \sigma_b, \sigma_a \leq N) \qquad \text{and} \qquad r_b^2 = \lim_{n\to\infty} \mathbb{P}(\sigma_b < \sigma_a, \sigma_b \leq N).$$

      iv. Conclude that

$$r_a^1 + r_b^2 = 1 \qquad \text{and } ar_a^1 + br_b^2 = 0.$$

      Find r_a^1, r_b^2.

      v. $\mathbb{E}(\tau_N) \to -ab$ as $N \to \infty$.

   (e) Can you provide an interpretation to answers from (d)(iv) and (d) (v) ?

4. Let $\Omega \neq \emptyset$ and $\mathcal{S}$ be any collection of subsets of Ω, then show that the smallest $\sigma-$algebra containing $\mathcal{S}$ is given by

$$\cap\{\mathcal{B} : \mathcal{B} \text{ is a } \sigma\text{-algebra of subsets of } \Omega \text{ such that } \mathcal{S} \subseteq \mathcal{B}\}.$$

   We shall call it the σ-algebra generated by S.