Recall :

Arc Sine law - last visit to origin

$\{S_n\}_{n\geq 0}^{2N}$ - Simple random walk of finite length $2N$

$$L = \max\{n \mid 0 \leq n \leq 2N, S_n = 0\}$$

(L- not a stopping time)

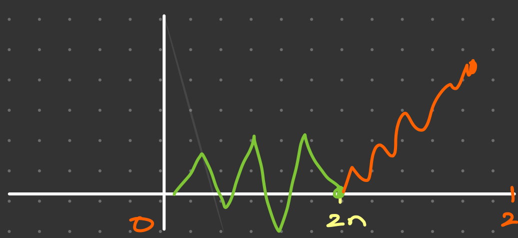
Theorem :-

$$\mathbb{P}(L = 2n) = \mathbb{P}(S_{2n} = 0)\,\mathbb{P}(S_{2N-2n} = 0)$$

$$= \frac{1}{2^{2N}} \binom{2n}{n} \binom{2N-2n}{N-n}$$

Proof :-

# of paths of length $2N$ with $L = 2n$



$$= \underbrace{\left(\text{\# of paths of length } 2n \text{ with } S_{2n} = 0\right)}_{1^{st}\ 2n \text{ tosses}} \times \underbrace{\left(\text{\# of paths of length } 2N-2n \text{ with } \sigma_0 > 2N-2n\right)}_{\text{next } 2N-2n \text{ tosses}}$$

$$\therefore \quad \mathbb{P}(L = 2n) = \mathbb{P}(S_{2n} = 0) \; \mathbb{P}(\sigma_0 > 2N - 2n)$$

$$\overset{\text{(lemma 4)}}{=} \mathbb{P}(S_{2n} = 0) \; \mathbb{P}(S_{2N-2n} = 0)$$

$$= \binom{2n}{n} \frac{1}{2^{2n}} \binom{2N-2n}{N-n} \frac{1}{2^{2N-2n}}$$

$$= \underbrace{\frac{1}{2^{2N}} \binom{2n}{n} \binom{2N-2n}{N-n}}_{\text{Discrete arc Sine law}} \qquad \square$$

<u>Sketch</u> (Ex: Make it precise)

$$\mathbb{P}(L = 2n) = \boxed{\begin{array}{c}\text{stirling's approximation} \\ n! \sim n^n e^{-n}\end{array}}$$

$$\overset{\text{Ex}}{\simeq} \frac{1}{\pi} \frac{1}{\sqrt{n(N-n)}}$$

$$= \frac{1}{\pi} \cdot \frac{1}{N} \frac{1}{\sqrt{\frac{n}{N}\left(1 - \frac{n}{N}\right)}}$$

$f : [0,1) \to \mathbb{R}$

$$f(x) = \frac{1}{\pi \sqrt{x(1-x)}}$$

will involve constants

$$\boxed{\mathbb{P}(L = 2n) \sim \frac{1}{N} f\left(\frac{n}{N}\right)}$$

Intuitions : $\{S_n\}_{n\geq 1}$ interpretation vis a vis game

L – time at which particular takes the lead for ever.

$$E[S_{2N}] = 0 \implies L \simeq 2N$$

Distribution function of scaled random variable

$$\mathbb{P}\left(\frac{L}{2N} \leq x\right) = \sum_{n \leq \lfloor Nx \rfloor} \mathbb{P}(L = 2Nn)$$

$$\sim \sum_{\substack{n \geq 1 \\ \frac{n}{N} \leq x}} \frac{1}{N} f\left(\frac{n}{N}\right)$$

Riemann approximation $\left(\xrightarrow[N\to\infty]{}\right)$ $\approx$ $\int_0^x f(y)\,dy$

$$= \int_0^x \frac{1}{\pi\sqrt{y(1-y)}}\,dy$$

$$= \frac{2}{\pi} \arcsin(\sqrt{x})$$

LLn.R (Law of large numbers)

$x = (-1, 1)$

```
runningmean = function (x,N){
  y = sample(x,N, replace=TRUE) # Sample with replacement
  c = cumsum(y) #cummulative
  n = 1:N
  c/n
}

u = runningmean(c(-1,1), 1000)
x=1:1000; plot(u~x, type="l");
x=1:1000; plot(u~x, type="l");

replicate(10, lines(runningmean(c(-1,1), 1000)~x, type="l",
col=rgb(runif(3),runif(3),runif(3))))
```

$y = (\underbrace{-1, 1, -1, \dots, \,}_{N=1000})$

$(y_1, y_1+y_2, \dots, y_1+y_2+\dots+y_{N})$

$(1, 2, \dots, n)$

$(y_1, \frac{y_1+y_2}{2}, \dots, \bar{y})$

$X_i \in \{-1, 1\}$

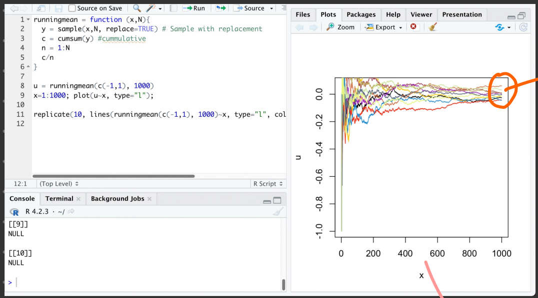
Chooses 1000 samples uniformly

$y = (X_1, \dots, X_{1000})$

$c = (S_1, \dots S_{1000})$

$\frac{c}{n} = \left(\frac{S_1}{1}, \frac{S_2}{2}, \dots, \frac{S_{1000}}{1000}\right)$

$S_n = \sum_{i=1}^{n} X_i \quad X_i \in \{-1, 1\}$



$n \approx 1000$ $\quad \boxed{\frac{S_n}{n} \approx 0}$

WLLN: $\frac{S_n}{n} \to 0$ as $n \to \infty$

10 realisations of

$\{\frac{S_n}{n} : 1 \leq n \leq 1000\}$

Central limit Theorem

$$X_i = \begin{cases} 1 & \text{w.p. } \frac{1}{2} \\ -1 & \text{w.p. } \frac{1}{2} \end{cases}$$

$$E[X_i] = 0$$

$$var[X_i] = 1$$

$$\frac{S_n - n0}{\sqrt{n} \cdot 1} \xrightarrow{d} N(0,1)$$

large n

Histogram: $\frac{S_n}{\sqrt{n}} \approx$ Histogram of $N(0,1)$

sample $\{-1,1\}$ 1000 times uniformly