Step 4: [Caratheodory Extension Theorem]

Theorem A2: If $\mathcal{A}$ is an algebra on $\Omega$ & $\mathbb{P} : \mathcal{A} \to [0,1]$ is a Probability then $\mathbb{P}$ extends Uniquely to $\tilde{\mathbb{P}}$ where $\tilde{\mathbb{P}} : \sigma(\mathcal{A}) \longrightarrow [0,1]$ is a Probability.

August 25th, 2023

# Proof of step 4

Start with:

$\exists \ \mathbb{P} : \mathcal{A}(\mathcal{S}) \to [0,1]$ satisfying

$$P((a,b]) = b-a \quad \forall \ (a,b] \in \mathcal{S}$$

$\llcorner$ 1

if $[a_i, b_i]$ such that $1 \le i \le n$

& $(a_i, b_i] \cap (a_j, b_j] = \phi$ $i \ne j$

then $$\mathbb{P}\left(\bigcup_{i=1}^{n} (a_i, b_i]\right) = \sum_{i=1}^{n} b_i - a_i$$ ②

such that

(i) $\mathbb{P}([0,1)) = 1$

(ii) $E_k \in \mathcal{A}(J)$ $E_l \cap E_m = \phi$

$$\mathbb{P}\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} \mathbb{P}(E_k)$$

(iii) ① holds ; $\mathbb{P}((a,b]) = b-a$

& $\mathbb{P}(A \oplus x) = \mathbb{P}(A)$

$x \in [0,1]$, $A \in \mathcal{A}(J)$

(Sweep under the Carpet Grand Ex.) Caratheodory Extension Theorem

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$\exists$ $\tilde{\mathbb{P}} : \sigma(\mathcal{A}) \longrightarrow [0,1]$

ⓐ $\tilde{\mathbb{P}}(S) = 1$

ⓑ $\{E_k\}_{k \ge 1}$ are disjoint

$$\tilde{\mathbb{P}}\left(\bigcup_{k \ge} E_k\right) = \sum_{k \ge 1} \tilde{\mathbb{P}}(E_k)$$

and agrees with $\mathbb{P}$ on $\mathcal{A}$. &

satisfies (3).

- We will show there is only such $\tilde{\mathbb{P}}$.

Suppose there is $\tilde{\mathbb{P}}_1$ & $\tilde{\mathbb{P}}_2$ that extend $\mathbb{P}$ to $\sigma(\mathcal{A})$.

Set
$$\mathcal{M} = \{ A \in \sigma(\mathcal{A}) \mid \tilde{\mathbb{P}}_1(A) = \tilde{\mathbb{P}}_2(A) \}$$

Show:- (i) $\mathcal{A} \subseteq \mathcal{M}$ [✓ $\tilde{\mathbb{P}}_1 \equiv \mathbb{P} \equiv \tilde{\mathbb{P}}_2$ on $\mathcal{A}$]

(ii) $\mathcal{M}$ is a $\sigma$-algebra. [Requires some work]

Proposition 3 : Suppose $\mathbb{Q}$ is a probability on $(S, \mathcal{A})$, $\mathcal{A}$ - being an algebra. Then

(i) If $A_i \in \mathcal{A}$ & $A_i \subseteq A_{i+1}$ $\forall\, 1 \le i$ & $\bigcup_{i=1}^{\infty} A_i \in \mathcal{A}$

$$\mathbb{Q}\left(\bigcup_{i=1}^{\infty} A_i\right) = \lim_{k \to \infty} \mathbb{Q}(A_k)$$

(ii) If $B_i \in \mathcal{A}$ & $B_i \supseteq B_{i+1}$ $\forall\, 1 \le i$ & $\bigcap_{i=1}^{\infty} B_i \in \mathcal{A}$

$$\mathbb{Q}\left(\bigcap_{i=1}^{\infty} B_i\right) = \lim_{k \to \infty} \mathbb{Q}(B_k)$$

Proof : Exercise. $\square$

Using Proposition 3, (with $\mathcal{A}$ by $\sigma(\mathcal{A})$)

I if $A_i \subseteq A_{i+1}$ $\forall i \geq 1$ & $A_i \in \mathcal{M}$

then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{M}$

if $B_i \supseteq B_{i+1}$ $\forall i \geq 1$ & $B_i \in \mathcal{M}$

then $\bigcap_{i=1}^{\infty} B_i \in \mathcal{M}$.

Assume property I

(i) If $\{E_k\}_{k \geq 1}$ & $E_k \in \mathcal{M}$

$\exists$ $\{A_k\}_{k \geq 1}$ $A_k \in \mathcal{M}$

$A_k \subseteq A_{k+1}$ & $\bigcup_{k=1}^{\infty} E_k = \bigcup_{k=1}^{\infty} A_k$.

$\overset{\mathrm{I}}{\Rightarrow}$ $\tilde{\mathbb{P}}_1 \left( \bigcup_{k=1}^{\infty} E_k \right) = \tilde{\mathbb{P}}_2 \left( \bigcup_{k=1}^{\infty} E_k \right)$

(ii) $A \in \mathcal{M}$ $\tilde{\mathbb{P}}_1(A) = \tilde{\mathbb{P}}_2(A)$

$\Rightarrow$ $\tilde{\mathbb{P}}_1(A^c) = \tilde{\mathbb{P}}_2(A^c)$

(i) & (ii) $\mathcal{M}$ is $\sigma$-algebra $\square$

# $\sigma$-Algebra & Monotone class

$\mathcal{M} \subseteq \mathcal{P}(S)$ is a Monotone class

if • $A_i \subseteq A_{i+1}$ $\forall\, i \geq 1$ & $A_i \in \mathcal{M}$

then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{M}$

• $B_i \supseteq B_{i+1}$ $\forall\, i \geq 1$ & $B_i \in \mathcal{M}$

then $\bigcap_{i=1}^{\infty} B_i \in \mathcal{M}$.

**Proposition 4** (Monotone class lemma) :

$\mathcal{A}$ is an algebra on any non-empty set $S$.

let $\sigma(\mathcal{A}) \equiv$ smallest $\sigma$-algebra containing $\mathcal{A}$

$\mathcal{M}(\mathcal{A}) \equiv$ smallest monotone class containing $\mathcal{A}$.

Then $\mathcal{M}(\mathcal{A}) = \sigma(\mathcal{A})$

**Proof**:-

Show • $\mathcal{M}(\mathcal{A})$ is Closed under Compliments

Define $\mathcal{M}_c = \{ E \in \mathcal{M}(\mathcal{A}) \mid E^c \in \mathcal{M}(\mathcal{A}) \}$

Ex: $\mathcal{M}_c$ is a monotone class.

$\Rightarrow \mathcal{M}_c = \mathcal{M}(\mathcal{A})$

Show: $\mathcal{M}$ is closed under unions

- (Ex) $\cdot$ $\mathcal{M}(\mathcal{A})$ is closed under countable unions.

$B \in \mathcal{A}$

Define: $\mathcal{M}_B = \{ E \in \mathcal{M}(\mathcal{A}) \mid E \cup B \in \mathcal{M}(\mathcal{A}) \}$

- $\mathcal{M}_B \supseteq \mathcal{A}$
- $\mathcal{M}_B$ is a monotone class

$\Rightarrow \mathcal{M}_B \supseteq \mathcal{M}(\mathcal{A})$

$\Rightarrow \forall B \in \mathcal{A} \quad \mathcal{M}_B = \mathcal{M}(\mathcal{A}) \quad - (*)$

Define: $\mathcal{M} = \{ B \in \mathcal{M}(\mathcal{A}) \mid \mathcal{M}_B = \mathcal{M}(\mathcal{A}) \}$

$(*) \Rightarrow \mathcal{M} \supseteq \mathcal{A}.$

Note: $\mathcal{M} = \{ B \in \mathcal{M}(A) \mid E \in \mathcal{M}(A) \Rightarrow E \cup B \in \mathcal{M}(A) \}$

Clearly $\mathcal{M}$ is a monotone class.

$\Rightarrow$ $\mathcal{M}(A)$ is closed under union $\square$