Recall :

- $\Omega = \{-1,1\}^N$, $\mathcal{F} = \mathcal{P}(\Omega)$
- $\mathbb{P}(A) = \frac{|A|}{2^n}$ $\quad A \subseteq \Omega$
- $\{S_n\}_{n \geq 0}^N$ $\quad S_0 = 0$ $\quad X_i : \Omega \to \{-1,1\}$, $\omega \mapsto \omega_i$ $\quad S_n = \sum_{i=1}^n X_i$
- $T : \Omega \to \{0,1,\dots,N\} \cup \{\infty\}$

  $\{T = k\} \in \mathcal{A}_k \equiv$ observable event upto time k

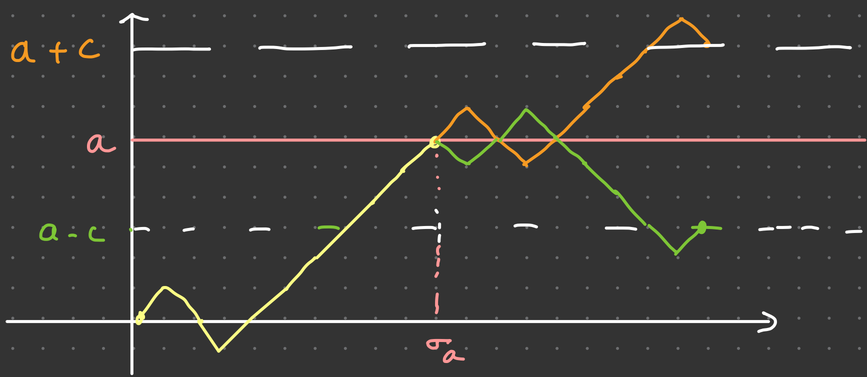
  $\equiv \{ (X_1, \dots, X_k)^{-1}(A) : A \subseteq \{-1,1\}^N \}$

- $E[S_T] = 0$, $E[S_T^2] = E[T]$

first hitting time of a :: $\sigma_a = \min\{k \geq 0 \mid S_k = a\}$

1.3 Reflection Principle

Picture Concept : $a \in \mathbb{N}$ $\quad c \in \mathbb{N} \cup \{0\}$



Ex: For every path of the random walk after σ_a:

Each orange path that lands at $a+c$

there is a "Reflected" path across $y=a$

green path that lands at $a-c$

$$|\{\omega\in\Omega : S_n = a+c\}|$$

$$= |\{\omega\in\Omega : \sigma_a \le n,\ S_n = a+c\}|$$

$$= |\{\omega\in\Omega : \sigma_a \le n,\ S_n = a-c\}|$$

Lemma 1 : $a, c \in \mathbb{N}\cup\{0\}$

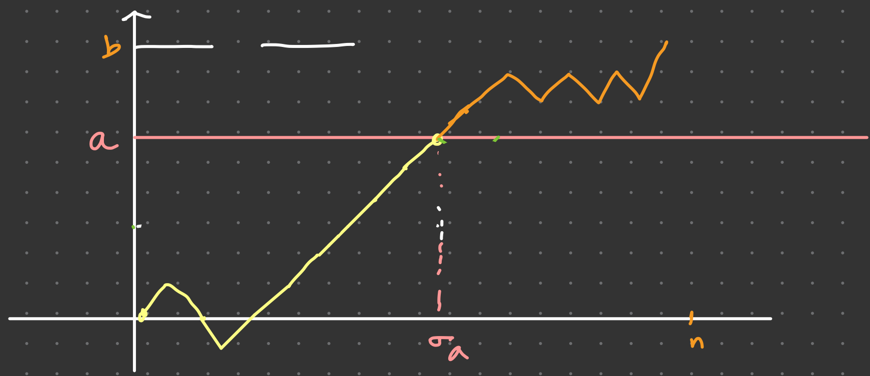
$$\mathbb{P}(S_n = a+c) = \mathbb{P}(\sigma_a \le n,\ S_n = a-c)$$

Theorem 2

(a) $\mathbb{P}(\sigma_a \le n) = \mathbb{P}(S_n \notin [-a, a-1])$

(b) $\mathbb{P}(\sigma_a = n) = \frac{1}{2}\left[\mathbb{P}(S_n = a-1) - \mathbb{P}(S_{n+1} = a+1)\right]$

Proof:- $\mathbb{P}(\sigma_a \le n) = \mathbb{P}(\sigma_a \le n,\ \bigcup_{b\in\mathbb{Z}}\{S_n = b\})$



1.c $\mathbb{P}(\sigma_a \le n) = \sum_{b \in \mathbb{Z}} \mathbb{P}(\sigma_a \le n, \; S_n = b)$

$$= \sum_{\substack{b \in \mathbb{Z} \\ b \ge a}} \mathbb{P}(\sigma_a \le n, \; S_n = b) + \sum_{\substack{b \in \mathbb{Z} \\ b < a}} \mathbb{P}(\sigma_a \le n, \; S_n = b)$$

Reflection principle

Apply lemma 1.3.1
$b = a - c$
($\Rightarrow$ $a + c = 2a - b$)

$$= \sum_{\substack{b \in \mathbb{Z} \\ b \ge a}} \mathbb{P}(\sigma_a \le n, \; S_n = b) + \sum_{\substack{b \in \mathbb{Z} \\ b < a}} \mathbb{P}(S_n = 2a - b)$$

(Ex)

$$= \sum_{\substack{b \in \mathbb{Z} \\ b \ge a}} \mathbb{P}(S_n = b) + \mathbb{P}(S_n > a)$$

$\{\sigma_a \le n\} \subseteq \{S_n = b\}$

$$= \mathbb{P}(S_n \ge a) + \mathbb{P}(S_n > a)$$

$$\overset{\text{(Symmetry)}}{=} \mathbb{P}(S_n \geq a) + \mathbb{P}(S_n < -a)$$

$$= \mathbb{P}(S_n \notin [-a, a-1])$$

ⓑ $$\mathbb{P}(\sigma_a = n) = \mathbb{P}(\sigma_a \leq n) - \mathbb{P}(\sigma_a \leq n-1)$$

$$\overset{\text{Ex}}{=} \frac{1}{2}\left[\mathbb{P}(S_{n-1} = a-1) - \mathbb{P}(S_{n-1} = a+1)\right]$$

(Apply ⓐ)

\+

(Symmetry)

$\square$

<u>Corollary 3</u> :- $a, n \in \mathbb{N}$

$$\mathbb{P}(\sigma_a = n) = \frac{a}{n} \mathbb{P}(S_n = a)$$

<u>Proof</u>: <u>Earlier class</u> :

$$\mathbb{P}(S_{n-1} = a-1) = \frac{1}{2^{n-1}} \binom{n-1}{\frac{(n-1)+(a-1)}{2}}$$

$$\mathbb{P}(S_{n-1} = a+1) = \frac{1}{2^{n-1}} \binom{n-1}{\frac{n-1+a+1}{2}}$$

$$\mathbb{P}(S_n = a) = \frac{1}{2^n} \binom{n}{\frac{n+a}{2}}$$

(Do the Combinatorics)

$$\binom{n}{k} = \frac{n}{k} \binom{n-1}{k-1}$$

$$\binom{n-1}{k} = \frac{n-k}{k} \binom{n}{k}$$

Thm 2 (b) $\Rightarrow$ $\mathbb{P}(\sigma_a = n) = \frac{a}{n}\,\mathbb{P}(S_n = a)$ $\square$

$n \in \mathbb{N}$ $\quad \sigma_a = \min\{n \geq 1 \mid S_n = a\}$

(hitting time of a) $a \in \mathbb{Z} / \{0\}$

(Return time) $\sigma_0 = \min\{n \geq 1 \mid S_n = 0\}$

lemma 4: (Escape from Origin)

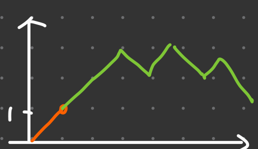
$$\mathbb{P}(\sigma_0 > 2n) = \mathbb{P}(S_{2n} = 0)$$

Proof:-

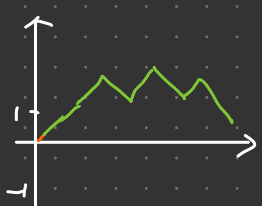
$$\mathbb{P}(\sigma_0 > 2n) = \mathbb{P}(S_1 \neq 0, S_2 \neq 0, \ldots, S_{2n} \neq 0)$$

(Reflection principle) $\xleftarrow{Er}$

$$= 2\,\mathbb{P}(S_1 > 0, S_2 > 0, \ldots, S_{2n} > 0)$$



$$= 2 \; \frac{\#\text{ paths start at 1 \& stay positive for } 2n-1 \text{ steps}}{2^{2n}}$$



$$= \frac{1}{2^{2n-1}} \; \underline{\#\text{ paths start at 0 \& stay above } -1 \text{ for } 2n-1 \text{ step}}$$

(Reflection Principle)

$$= \frac{1}{2^{2n-1}} \; \#\text{ paths start at 0 \& stay below 1 for } 2n-1 \text{ steps}$$

$$= \mathbb{P}(\sigma_1 > 2n-1)$$

$$= 1 - \mathbb{P}(\sigma_1 \le 2n-1)$$

Theorem 2 (a) with $a=1$

$$= 1 - \mathbb{P}(S_{2n-1} \notin [-1,0])$$

$$\overset{\text{Ex}}{=} \mathbb{P}(S_{2n-1} = -1) = \mathbb{P}(S_{2n} = 0) \quad \square$$

1.4 Arc Sine law - last visit to origin in $\frac{2N}{\text{time}}$

$$L = \max\{0 \le n \le 2N \mid S_n = 0\}$$

(not a stopping time)

"Order of L": time after which one player has gain in capital

$$L \approx 2N$$

Theorem 5: $n \in \mathbb{N} \cup \{0\}$ $n \le N$

$$\mathbb{P}(L = 2n) = \frac{1}{2^{2N}} \binom{2n}{n} \binom{2N-2n}{N-n}$$

Proof: (next class)

(next class). $\mathbb{P}(L=2n) \overset{\text{(Stirling formula)}}{=} \simeq \frac{1}{N} f\left(\frac{n}{N}\right)$

$n! \sim n^n e^{-n}$

where $f(x) = \dfrac{1}{\sqrt{x(1-x)}}$

To understand order

$$\mathbb{P}\left(\frac{L}{2N} \le x\right) = \mathbb{P}(L \le 2Nx)$$

$$\doteq \sum_{k=0}^{\text{"}Nx\text{"}} \mathbb{P}(L = 2k)$$

(Next class)

$$\overset{\text{Thm 5}}{=} \sum_{k=0}^{Nx} \frac{1}{N} f\left(\frac{k}{N}\right)$$

$\boxed{N \to \infty}$

$$\approx \int_0^x f(y)\, dy$$

$$= \int_0^x \frac{1}{\sqrt{y(1-y)}}\, dy$$

$$\boxed{\mathbb{P}\left(\frac{L}{2N} \le x\right) \sim \arcsin(\sqrt{x})}$$