Recall:

Borel - Cantelli lemma

$(\Omega, \mathcal{F}, \mathbb{P})$ - Probability space

$\{A_k\}_{k\geq 1}$ $\quad A_k \in \mathcal{F}$

$$\limsup_{n\to\infty} A_n = \{A_n \text{ occurs infinitely often}\} := \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} \{A_m \text{ occurs}\}$$

$$\liminf_{n\to\infty} A_n = \{A_n \text{ happens all but finitely many times}\} := \bigcup_{n=1}^{\infty} \bigcap_{m=n}^{\infty} \{A_m \text{ occurs}\}$$

(i) If $\sum_{n=1}^{\infty} \mathbb{P}(A_n) < \infty$ then $\mathbb{P}(\{A_n \text{ occurs infinitely often}\}) = 0$

(ii) If $\sum_{n=1}^{\infty} \mathbb{P}(A_n) = \infty$, A_n's are independent, then

$$\mathbb{P}(\{A_n \text{ occurs infinitely often}\}) = 1$$

Kolmogorov 0-1 law :

Recall: $\{S_n\}_{n\geq 1}$ random walk.

- $A_n \equiv$ set of observable event upto time time n

$\circledast$ — $A_n = \{\omega \in \Omega \mid \omega_i = o_i \quad 1 \leq i \leq n\}$ $\quad o_i \in \{+1, -1\}$

• $\mathcal{A}_n :=$ take unions & complements of A_n

σ-algebra
(i) $\Omega \in \mathcal{A}_n$, $\phi \in \mathcal{A}_n$

(ii) $A \in \mathcal{A}_n$, $A^c \in \mathcal{A}_n$

(iii) $\{A_k\}_{k \geq 1} \in \mathcal{A}_n \Rightarrow \bigcup_{k=1}^{\infty} A_k \in \mathcal{A}_n$

$$\mathcal{A}_n := \sigma\{ A_n : A_n \text{ as in } \circledast \}$$

• $\mathcal{A}_n \subseteq \mathcal{A}_{n+1} \quad \forall n \geq 1$ ≡ (Filtration)

"observable by time n"

Tail σ-field (/σ-algebra) :-

"events that depend on tail of the sequence"

E.g:- $\left\{ \lim_{n\to\infty} \frac{S_n}{n} = 0 \right\}$ tail event

$\{ S_n \longrightarrow \alpha \text{ as } n \to \infty \}$ is not a tail event

let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space.

let $\{A_n\}_{n \geq 1}$ be any sequence of events

<u>Fix $n \geq 1$</u>:

$$\sigma\{A_n, A_{n+1}, \ldots\} \equiv \left\{ \begin{array}{l} \text{Smallest } \sigma\text{-algebra} \\ \text{that contains } A_n, A_{n+1} \ldots \end{array} \right.$$

$\mathcal{J} \equiv$ tail σ-field corresponding to $\{A_k\}_{k\geq 1}$ is

$$\bigcap_{n=1}^{\infty} \sigma\{A_n, A_{n+1}, \ldots\}$$

<u>Example</u>:

$\{A_n$ occurs i.o.$\} \in \mathcal{J}$

$\{A_n$ occurs all but finitely$\} \in \mathcal{J}$

<u>Kolmogorov 0-1 law</u>: let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space.

If $\{A_k\}_{k\geq 1} \in \mathcal{F}$ and independent and $E \in \mathcal{J}$ (tail σ-field w.r.t. $\{A_k\}_{k\geq 1}$)

$$\mathbb{P}(E) \in \{0, 1\}$$

<u>Proof</u>:

let $E \in \mathcal{J}$

$\Rightarrow E \in \sigma(A_n, A_{n+1}, \ldots) \quad \forall n \geq 1$

Note: $A_1, A_2, \ldots, A_{n-1}$ are independent of any event in $\sigma\{A_n, A_{n+1} \ldots \ldots\}$

$\Rightarrow$ $A_1, A_2, \ldots, A_{n-1}$ are independent of E.

$\forall n \geq 1$ (*)

Note: $E \in \sigma(A_1, A_2 \ldots)$

(*) E is independent of E

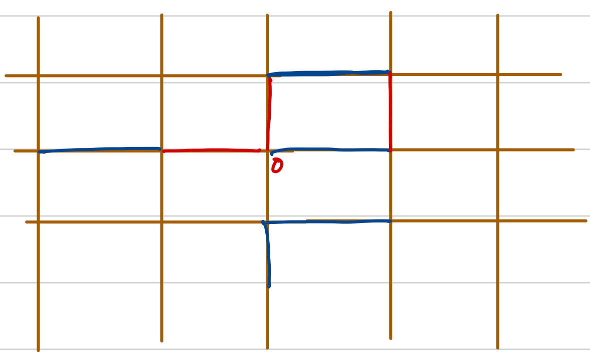
$\Rightarrow$ $P(E \cap E) = P(E)\, P(E)$

$\Rightarrow$ $P(E) = (P(E))^2$

$P(E) \in [0,1]$ $\Rightarrow$ $P(E) \in \{0, 1\}$ $\square$

Example: Percolation on $\mathbb{Z}^d$; $d \geq 2$

$d=2$



$p \in [0,1]$

Bonds are independently declared open w.p. p and closed w.p. $1-p$

We enumerate the bonds in some way.

let $A_i \equiv \{$ Bond i is open $\}$

- All $\{A_i\}_{i \geq 1}$ are independent.

Question of interest: For a given $p \in [0,1]$

$E = \{$ is there an infinite open cluster of bonds $\}$

If E happens then we say percolation occurs.

(Require proof) $$E \in \bigcap_{n=1}^{\infty} \sigma(A_n, A_{n+1}, \dots)$$

Kolmogorov 0-1 law

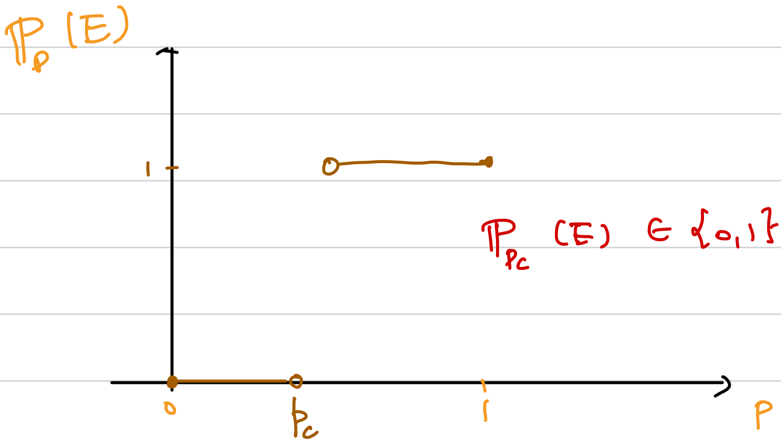
$$\Rightarrow \quad \mathbb{P}_p(E) \in \{0,1\} \, .$$

(Require proof)

$$p_1 < p_2 \quad \equiv \quad \underline{\text{Coupling}}$$

$$\mathbb{P}_{p_1}(E) \leq \mathbb{P}_{p_2}(E)$$

- $p \longrightarrow \mathbb{P}_p(E)$ is increasing in p



$\mathbb{P}_{p_c}(E) \in \{0,1\}$?

$d=2$; $\quad p_c = \frac{1}{2}$, $\quad \mathbb{P}_{p_c}(E) = 0$

⋮

$3 \leq d \leq 10$ — .. open

⋮

$d \geq 11$ .... $\quad \mathbb{P}_{p_c}(E) = 0$

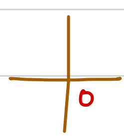
$$p_c = \frac{1}{2d} + \frac{1}{(2d)^2} + \cdots$$

as $d \to \infty$

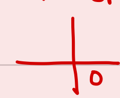
Question 2:

$\{0$ is the infinite open cluster$\}$

not a tail event



$0 \notin$ open infinite cluster

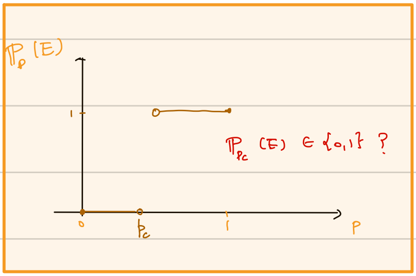


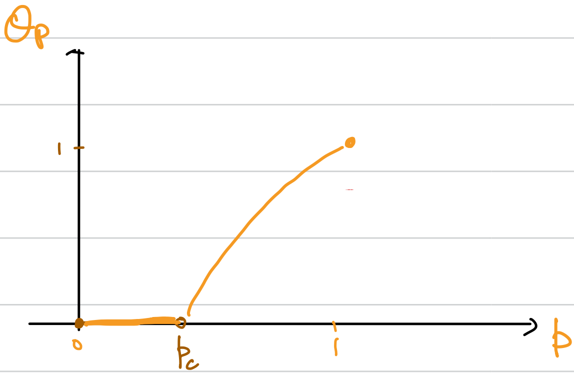
w.p $(1-p)^4$

$$\theta_p = \mathbb{P}_p(\, 0 \text{ is in the open cluster} \,)$$

$0 \longleftrightarrow \infty$ on open bonds

$$p_1 < p_2 \implies \theta_{p_1} \le \theta_{p_2}$$





Q:

- $\theta_{p_c} = 0$?
- $|\theta_p - \theta_{p_c}| \sim |p - p_c|^{\gamma}$

$\gamma = ?$