Due: Tuesday August 29th, 2023, 1030am

Let $N \in \mathbb{N}$. Consider

$$\Omega_N = \Big\{\omega = (\omega_1, \omega_2, \ldots, \omega_N) : \omega_i \in \{-1, +1\}\Big\}$$

equipped with the uniform distribution, denoted by $\mathbb{P} \equiv \mathbb{P}_N$.
For $1 \le k \le N$, let $X_K : \Omega_N \to \{-1, 1\}$ be given by $X_k(\omega) = \omega_k$ and
for $1 \le n \le N$, let $S_n : \Omega_N \to \mathbb{Z}$ be given by $S_n(\omega) = \sum_{k=1}^{n} X_k(\omega)$ and $S_0 = 0$.
Let $\mathcal{A}_n$ be the events that are observable by time n.

1. Let $\Omega \neq \emptyset$ and $S = \{A_1, A_2, \ldots, A_n\}$ be such that $A_i \cap A_j = \emptyset$ and $\cup_{i=1}^{k} A_i = \Omega$. Then show that

   $$\sigma(S) = \{\cup_{i=1}^{n} B_i : B_i \text{ is either } \emptyset \text{ or } A_i\}$$

   Describe $\sigma(T)$ when $T = \{C_i : 1 \le i \le n, C_i \subset \Omega\}$.

2. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a Probability space. Let $A_1, A_2, \ldots, A_k$ be in $\mathcal{F}$. Show that

   $$\sum_{i=1}^{n} \mathbb{P}(A_i) - \sum_{i<j} \mathbb{P}(A_i \cap A_j) \le \mathbb{P}(\cup_{i=1}^{n} A_i) \le \min\left\{\sum_{i=1}^{n} \mathbb{P}(A_i) - \sum_{k \neq i} \mathbb{P}(A_i \cap A_k). : 1 \le k \le n\right\}$$

Due: Friday September 1st, 2023, 5pm

1. *Ballot Theorem* Suppose L and W are candidates contesting the elections to be president of all students of the universe union. In the ballot, suppose each student voter is equally likely to vote for L and W, further L receives l votes and W receives w votes with $w > l$. Find the probability that W was leading through out the counting via the following steps.

   (a) Let $N = W + L$ and $\{S_k : 1 \ge k \ge N\}$ represents the number of votes received by W minus the number of votes received by L at the counting of k-th ballot. Show that S_k is a simple random walk of length N.

   (b) Show that

   $$\mathbb{P}(\text{ that } W \text{ was leading through out the counting and wins by } w - l \text{ votes}))$$
   $$=$$
   $$\mathbb{P}(S_k > 0, 1 \le k \le N \mid S_N = w - l)$$

   (c) Show that the number of valid random walk paths from $(1, 1)$ to $(N, w - l)$ that touch the $x - axis$ is equal to the number of paths valid random walk paths from $(1, -1)$ to $(N, w - l)$. *Hint:* Reflection Principle.

   (d) Show that the number of valid random walk paths from $(1, 1)$ to $(N, w - l)$ is $\binom{w+1-1}{l}$.

   (e) Show that the number of valid random walk paths from $(1, -1)$ to $(N, w - l)$ is $\binom{w+l-1}{l-1}$.

   (f) Using (b) (c) and (d) to conclude that the number of random walk paths from $(1, 1)$ to $(N, w-l)$ that do not touch the x-axis is given by $\frac{w-l}{w+l}\binom{w+l}{l}$.

   (g) Show that $\mathbb{P}(\text{ that } W \text{ was leading through out the counting and wins by } w - l \text{ votes})) = \frac{w-l}{w+l}$