Due: Friday, September 1, 3:30pm

There is an important inequality in Probability theory.

Markov's inequality: Let X be a random variable which takes on only non-negative values and suppose that X has a finite expected value $\mu = E[X]$. Then for any $c > 0$,

$$P(X \geq c) \leq \frac{\mu}{c}.$$

1. Let X be a random variable such that $g_X : \mathbb{R} \to [0, \infty)$ given by

$$g_X(r) = E[\exp(rX)]$$

   is well defined. Then show that

   (a) $P(Z \geq b) \leq g_X(r) \exp(-rb)$ for all $b \in \mathbb{R}, r > 0$

   (b) $P(Z \leq b) \leq g_X(r) \exp(-rb)$ for all $b \in \mathbb{R}, r < 0$

2. Suppose $S_n = \sum_{i=1}^{n} X_i$ where X_i are i.i.d. X such that $g_X(\cdot)$ is well defined. Show that

   (a) $P(S_n \geq na) \leq (g_X(r))^n \exp(-rna)$ for all $a \in \mathbb{R}, r > 0$

   (b) $P(S_n \leq na) \leq (g_X(r))^n \exp(-rna)$ for all $a \in \mathbb{R}, r < 0$

3. (Chernoff Bounds) Let S_n and $g_x(\cdot)$ be as above. Then show that

$$\mu_x(a) := \inf_{r \in \mathbb{R}} [\log(g_X(r)) - ra] < 0$$

   for all $a \neq E[X]$ and further,

   (a) $P(S_n \geq na) \leq \exp(-n\mu_X(a))$ for all $a > E[X]$

   (b) $P(S_n \leq na) \leq \exp(-n\mu_X(a))$ for all $a < E[X]$

4. Suppose $X \sim$ Bernoulli(p). Compute $\mu_X(\cdot)$.