10 am - 11:00 am [Construct Uniform measure on $[0,1]$]

$\Omega = [0,1]$, $\mathcal{B} = --$, $\mathbb{P}(\cdot) \equiv$ uniform $[0,1]$

$\boxed{A}$ σ- Algebra as Events

$\Omega \neq \phi$ set.

Definition A.1 : A collection of subsets, $\mathcal{B}$, of Ω is called a $\underline{\sigma\text{- Algebra}}$ or $\underline{\sigma\text{-field}}$ if

(i) $\Omega \in \mathcal{B}$

closed

Compliment (ii) $E \in \mathcal{B}$ then $E^c \in \mathcal{B}$

countable unions (iii) $\{E_k\}_{k\geq 1}$ s.t. $E_k \in \mathcal{B}$ then $\bigcup_{k=1}^{\infty} E_k \in \mathcal{B}$.

Exercises :

- If $\{\mathcal{B}_i : i \in I\}$ are a collection of σ-algebras on Ω then

$$\bigcap_{i \in I} \mathcal{B}_i \equiv \{E \subseteq \Omega \mid E \in \mathcal{B}_i \ \forall\, i \in I\}$$

is also a σ- algebra

- S - any collection of subsets of Ω then $\exists$ a smallest σ- algebra that contains S. We shall denote it by $\underbrace{\sigma(S)}_{\substack{\sigma\text{-algebra} \\ \text{generated by } S}}$.

$[\ \mathcal{P}(\Omega) = \sigma\text{-algebra}$ ✓

$S \subseteq \mathcal{P}(\Omega)$ ✓

$\sigma(S) = \bigcap_{i \in I} \{ \mathcal{B}_i \mid \mathcal{B}_i \supseteq S,\ \mathcal{B}_i - \sigma\text{algebra} \}]$

Theorem A.1 :- • S - collection of closed intervals in $[0,1]$. i.e. $S = \{ [a,b] \mid 0 \le a \le b \le 1 \}$

• $\mathcal{B} = \sigma(S)$

• There exists a UNIQUE Probability $\mathbb{P} : \mathcal{B} \longrightarrow [0,1]$ satisfying:-

$$\mathbb{P}([a,b]) = b-a. \quad - (*)$$

& If $A \in \mathcal{B}$ and $x \in [0,1]$

$$\mathbb{P}(A \oplus x) = \mathbb{P}(A). \quad - (+)$$

Couple of Extensions:-

(i) The same theorem holds in n-dimensions

$$S = \left\{ \prod_{i=1}^{n} [a_i, b_i] : \begin{array}{c} 0 \le a_i \le b_i \le 1 \\ 1 \le i \le n \end{array} \right\}$$

$(*)$ $$\mathbb{P}\left(\prod_{i=1}^{n} [a_i, b_i] \right) = \prod_{i=1}^{n} (b_i - a_i)$$

$(+)$ remains the same

Definition A.2 :- A family $\mathcal{A}$ is said to be an Algebra if

Closed

Compliment

finite unions

(i) $\Omega \in \mathcal{A}$

(ii) $B \in \mathcal{A}$ then $B^c \in \mathcal{A}$

(iii) $B, C \in \mathcal{A} \Rightarrow B \cup C \in \mathcal{A}$

Proof of Theorem A.1 :-

$$\Omega = [0,1]$$

$$S = \{ (a,b] \mid 0 \leq a \leq b \leq 1 \}$$

$$\mathcal{A} :\overset{Ex}{=} \mathcal{A}(S) \equiv \left\{ \bigcup_{i=1}^{n} (a_i, b_i] \;\middle|\; \begin{array}{l} (a_i, b_i] \in S \\ 1 \leq i \leq n,\ n \geq 1 \end{array} \right\}$$

Algebra generated by S $\equiv$ Smallest algebra containing S

$$\mathcal{B} = \sigma(S) \overset{Ex}{=} \sigma(\mathcal{A}(S))$$

Disjoint collection: Countable $N = \infty$ or finite $N < \infty$

Lemma A.1 :- If $\bigcup_{n=1}^{N} (a_n, b_n] \subseteq (a,b]$ then

$$\sum_{n=1}^{N} (b_n - a_n) \leq b - a$$

$$\boxed{\begin{array}{l} 0 \leq a_n \leq b_n \leq 1 \\ 0 \leq a \leq b \leq 1 \end{array}}$$

• **lemma A.2**: If $(a,b] \subseteq \bigcup_{n=1}^{N} (a_n, b_n]$ then

$$b - a \leq \sum_{n=1}^{N} (b_n - a_n) \qquad (N \in \mathbb{N} \cup \{\infty\})$$

Define $P : \mathcal{A} \to [0,1]$ by

Step 1: $P((a,b]) = b-a \quad \forall (a,b] \in \mathcal{S}$ ①

step 2:
if $(a_i, b_i]$ such that $1 \leq i \leq n$
& $(a_i, b_i] \cap (a_j, b_j] = \phi \quad i \neq j$

then $$\mathbb{P}\left(\bigcup_{i=1}^{n} (a_i, b_i] \right) = \sum_{i=1}^{n} b_i - a_i$$ ②

Step3: lemma A.1 and lemma A.2, definition of $\mathcal{A}(\mathcal{S}) \Rightarrow$

$\exists \ \mathbb{P} : \mathcal{A}(\mathcal{S}) \to [0,1]$ satisfying ① and ② such that

(i) $P([0,1]) = 1$

(ii) $E_k \in \mathcal{A}(\mathcal{S}) \quad E_l \cap E_m = \phi$

$$\mathbb{P}\left(\bigcup_{k=1}^{\infty} E_k \right) = \sum_{k=1}^{\infty} \mathbb{P}(E_k)$$

(iii) ① holds ; $\mathbb{P}((a,b]) = b-a$

& $\mathbb{P}(A \oplus x) = \mathbb{P}(A)$

$x \in [0,1]$, $A \in \mathcal{A}(\mathcal{J})$

Step 4: [Caratheodory Extension Theorem]

Theorem A2: If $\mathcal{A}$ is an algebra on Ω

& $\mathbb{P} : \mathcal{A} \to [0,1]$ is a Probability

then $\mathbb{P}$ extends Uniquely to $\tilde{\mathbb{P}}$

where $\tilde{\mathbb{P}} : \sigma(\mathcal{A}) \longrightarrow [0,1]$ is a

Probability.