# Recall: Stopping times & Games

**Interpretation:** • $S_n \equiv$ represent "amount of capital of the player after $n$ rounds

• $X_k \equiv$ amount a player wins in round $k$.

* Expected "amount of Capital" after $n$ rounds
$$= E[S_n] = 0 \qquad 0 \le n \le N$$

**Question:** Is it possible to stop the game in a favorable moment?

**Definition:** An event $A \subseteq \Omega$ is observable until time $n$ when it can be written as a union of basic events of the form
$$\{w \in \Omega \mid w_1 = a_1, \ldots, w_n = a_n\} \quad a_i \in \{-1, 1\}, \ 1 \le i \le n$$

i.e. $A$ - can be determined from the outcome of the 1st $n$ tosses.

$\mathcal{A}_n :=$ class of event $A$ that can be observed by time $n$. [include $\phi$]

<u>Definition</u>: A map $T : \Omega \to \{0, 1, \dots, N\} \cup \{\infty\}$ is called a stopping time if

$$\{T = n\} \equiv \{\omega \in \Omega \mid T(\omega) = n\} \in \mathcal{A}_n$$
$$0 \le n \le N$$

i.e. $\{T = n\}$ is an event observable until time $n$.

<u>Example</u>: For $a \in \mathbb{Z}$ let

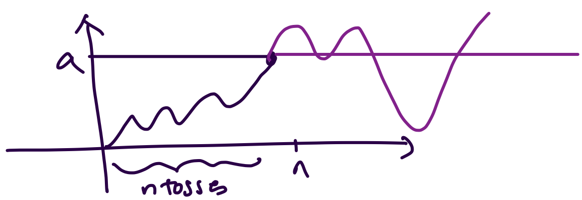
$$\sigma_a : \Omega \to \{0, 1, \dots, N\} \cup \{\infty\}$$

$$\sigma_a(\omega) = \min\{k \in [0, N] \mid S_k(\omega) = a\}$$

$$\min(\emptyset) = \infty$$

Ex:- $\sigma_a(\cdot)$ is a stopping time

$\{\omega \in \Omega : \sigma_a(\omega) = n\} = \dots$ "only depend on 1st $n$ tosses"



Theorem 1 :- For any stopping time

[ Impossibility of a favorite stopping ]

$$T : \Omega \longrightarrow \{0, 1, \ldots, N\}$$

$$E[S_T] = 0$$

$S_T(\omega) \equiv S_{T(\omega)}(\omega) \equiv$ outcome of the trajectory $\omega$ at stopping time $T(\omega)$

Proof :- T is a stopping time.

$k = 0, 1, 2, \ldots N$ $\qquad \{T \geq k\} \in \mathcal{A}_{k-1}$ — ①

$$\left[ \because \{T \geq k\} = \big( \underbrace{\{T=1\} \cup \ldots \{T = k-1\}}_{\in \mathcal{A}_{k-1}} \big)^c \right]$$

$\in \mathcal{A}_{k-1}$ & $\mathcal{A}_{k-1}$ is closed under compliments

$$S_T = \sum_{k=1}^{N} X_k \, \mathbb{1}(T \geq k) \quad \text{— ②}$$

$$\Big[ \because \quad S_T = \sum_{k=1}^{N} S_k \, \mathbb{1}(T = k)$$

$$= \sum_{k=1}^{N-1} S_k \left[ \mathbb{1}(T \geq k) - \mathbb{1}(T \geq k+1) \right] + S_N \, \mathbb{1}(T = N)$$

$$= \sum_{k=1}^{N-1} S_k \, \mathbb{1}(T \geq k) - \sum_{k=1}^{N-1} S_k \, \mathbb{1}(T \geq k+1) + S_N \, \mathbb{1}_{T=N}$$

$$= S_1 \mathbb{1}_{T\geq 1} + \sum_{k=2}^{N-1} S_k \mathbb{1}(T\geq k) + S_N \mathbb{1}_{T=N}$$

$$- \sum_{k=1}^{N-1} S_k \mathbb{1}(T\geq k+1)$$

$$= S_1 \mathbb{1}_{T\geq 1} + \sum_{m=1}^{N-2} S_{m+1} \mathbb{1}(T\geq m+1) + S_N \mathbb{1}_{T=N}$$

$$- \sum_{k=1}^{N-1} S_k \mathbb{1}(T\geq k+1)$$

$$= S_1 \mathbb{1}_{T\geq 1} + \sum_{l=1}^{N-2} (S_{l+1} - S_l) \mathbb{1}_{T\geq l+1}$$

$$+ (S_N - S_{N-1}) \mathbb{1}_{T=N}$$

$\{T=N\}$
$\equiv \{T\geq N\}$

$$= X_1 \mathbb{1}_{T\geq 1} + \sum_{l=1}^{N-2} X_l \mathbb{1}_{T\geq l+1} \quad X_N \mathbb{1}_{T=N} ]$$

$$E[S_T] \overset{(2)}{=} E\left[\sum_{k=1}^{N} X_k \mathbb{1}_{T\geq k}\right]$$

linearity of E

$$= \sum_{k=1}^{N} E[X_k \underbrace{\mathbb{1}_{T\geq k}}_{\in A_{k-1}}]$$

depend on the $k$th toss

$\omega_k$

" $\omega_1, \ldots, \omega_{k-1}$ "

independence $\xleftarrow{Ex}$

$$= \sum_{k=1}^{N} E[X_k] \, E[\mathbb{1}(T\geq k)]$$

$$= \sum_{k=1}^{N} 0 \, \mathbb{P}(T \geq k) = 0$$

$\square$

Game System: $V_1, \ldots, V_N : \Omega_N \to \mathbb{R}$

are random variables such that

$$\{V_k = c\} \in \mathcal{A}_{k-1} \quad c \in \mathbb{R}, \; k = 1, 2, \ldots N$$

- $V_k \equiv$ amount of money you will place as a bet in the $k^{th}$ round
- Result in the $k^{th}$ round $\equiv V_k X_k$

$$S_n^V = \sum_{k=1}^{N} V_k X_k$$

**Theorem 2:** For any $V_1, V_2, \ldots, V_N$ game system

then

$$E[S_N^V] = 0$$

**Proof:** like before

$$E[S_N^V] = \sum_{k=1}^{N} E[V_k X_k] \quad \text{—} \quad \#$$

It is enough to show $E[V_k X_k] = 0$.

$\mathcal{A}_{k-1}$ depends on $k$th toss

$$\mathrm{Range}(v_k) = \{c_1, c_2, \dots, c_m\}$$

$$\Rightarrow \quad X_k v_k = \sum_{i=1}^{m} c_i \, X_k \mathbb{1}(v_k = c_i)$$

$$E[X_k v_k] = \sum_{i=1}^{m} c_i \, E[X_k \mathbb{1}(v_k = c_i)]$$

"$w_k$" "$w_1, \dots w_{k-1}$"

$$= \sum_{i=1}^{m} c_i \, E[X_k] \, \mathbb{P}(v_k = c_i)$$

$$= \dots 0 \dots = 0.$$

$\hookrightarrow$ (x)

Place (x) into (#) $\Rightarrow$ to get result $\square$

Impact of Theorem 2:

$T: \Omega \to \{0, 1, \dots, N\}$ is a stopping time

$$v_k = \mathbb{1}_{\{T \geq k\}}$$

$$S_N^v = \sum_{k=1}^{N} X_k v_k = \sum_{k=1}^{N} X_k \mathbb{1}(T \geq k)$$

$$\overset{(2)}{=} S_T$$

[ Theorem 2 $\Rightarrow$ Theorem 1 ]

• $V_k = \underbrace{\underbrace{S_{k-1}}_{} \; \underbrace{1_{T \geq k}}_{①}}_{(\omega_1, \dots, \omega_{k-1})}$ (take this definition)

$\Rightarrow$ $V_1, V_2, \dots, V_N$ is a game system.

$$V_k X_k = S_{k-1} X_k 1_{T \geq k}$$

Observe: $S_k^2 = (S_{k-1} + X_k)^2$

$$= S_{k-1}^2 - 2 S_{k-1} X_k + X_k^2$$

$$\Rightarrow \quad V_k X_k = \frac{(S_k^2 - S_{k-1}^2 - X_k^2)}{2} 1_{T \geq k}$$

$$S_N^V = \sum_{k=1}^{N} V_k X_k$$

$$= \sum_{k=1}^{N} \frac{(S_k^2 - S_{k-1}^2 - X_k^2)}{2} 1_{T \geq k}$$

$(X_k^2 = 1)$

$$= \sum_{k=1}^{N} \frac{S_k^2 1_{T \geq k}}{2} - \sum_{k=1}^{N} \frac{S_{k-1}^2 1_{T \geq k}}{2} - \sum_{k=1}^{N} \frac{1_{T \geq k}}{2}$$

$$= \sum_{k=1}^{N} \frac{S_k^2 1_{T = k}}{2} - \frac{T}{2}$$

$$S_N^V = \frac{(S_T^2 - T)}{2} - \text{xxx}$$

Theorem 2 $\Rightarrow$ $E[S_N^V] = 0$

$\therefore$ By xxx for any $T: \Omega \to \{0, 1, \dots, N\}$

we have

$$0 = E[S_N^V] = E\left[\frac{S_T^2 - T}{2}\right]$$

$$\Rightarrow \quad E[S_T^2] = E[T] - \text{\#\#\#}$$

Theorem 1: $E[S_T] = 0$

$\Rightarrow$ $Var[S_T] = E[T]$

1.3 Ruin Problem:

Two players. Each with capital $\begin{cases} a - \mathrm{I} \\ b - \mathrm{II} \end{cases}$

Interpret

$S_n$ = gain of player 1

Ruin (of player 1)

$$\{\sigma_{-a} < \sigma_b ; \sigma_{-a} \le N\}$$

where, $\sigma_u = \min\{k : S_k = u\}$

<u>Define:</u> Ruin Probability

$$r'_N = \mathbb{P}(\sigma_{-a} \le \sigma_b, \ \sigma_{-a} \le N)$$

$$T_N = \min(\sigma_{-a}, \sigma_b, N)$$

Ex: $T_N$ is a stopping time

$\therefore$ <u>Theorem 1</u> $\Rightarrow$ $E[S_{T_N}] = 0$ — ③

$T_N = \sigma_{-a} \Rightarrow S_{T_N} = -a$

$T_N = \sigma_b \Rightarrow S_{T_N} = b$ — Use this in ③

$T_N = N \Rightarrow S_{T_N} = S_N$

$$0 = -a\,\mathbb{P}(T_N = \sigma_{-a}) + b\,\mathbb{P}(T_N = \sigma_b) + E[S_N \mathbb{1}_{T_N = N})$$

As $T_N = \min(\sigma_{-a}, \sigma_b, N)$

$$0 = -a\,\mathbb{P}(\sigma_{-a} < \sigma_b \ \& \ \sigma_{-a} \le N)$$

$$+ \; b \; \mathbb{P}(\sigma_b < \sigma_{-a} \;\&\; \sigma_b \leq N)$$

$$+ \; \mathbb{E}\, S_N \left(\mathbb{1}_{\min(\sigma_{-a},\, \sigma_b) \geq N}\right)$$

$$\Rightarrow \boxed{\begin{aligned} 0 = & -a \; \mathbb{P}_N(\text{Ruin of Player } 1) \\ & + b \; \mathbb{P}_N(\text{Ruin of Player } 2) \\ & + \mathbb{E}\left[ S_N \mathbb{1}_{\min(\sigma_{-a},\, \sigma_b) \geq N}\right] \quad (4) \end{aligned}}$$

- $r^1 = \lim_{N \to \infty} \mathbb{P}_N(\text{Ruin of Player } 1)$
- $r^2 = \lim_{N \to \infty} \mathbb{P}_N(\text{Ruin of Player } 2)$

$\Big\}$ Exercise

$$(4) \; \underset{\lim_{N\to\infty}}{\overset{\text{Ex}}{\Rightarrow}} \quad 0 = -a\, r^1 + b\, r^2 + 0$$

$$\Rightarrow \quad -a\, r^1 + b\, r^2 = 0 \quad — \; (5)$$

$$r^1 + r^2 = 1 \quad — \; (6)$$

Ex

$$(5) \;\&\; (6) \qquad r^1 = \frac{b}{a+b} \qquad r^2 = \frac{a}{a+b} \quad !$$

$\square$