1. Random Walk on $\mathbb{Z}$ finite length N

Recall:

$(S, \mathcal{F}, P)$

Probability Space

S - sample space, $|S| < \infty$

S - Countable.. $\mathbb{N}, \mathbb{Z}$

$\mathcal{F}$ - Events $\quad \mathcal{F} = \mathcal{P}(S)$

$\mathbb{P} : \mathcal{F} \longrightarrow [0,1]$

(A1) $\mathbb{P}(S) = 1$

(A2) $\{E_k\}_{k \geq 1}$ $\quad E_i \cap E_j = \phi \quad i \neq j$

$$\mathbb{P}\left(\bigcup_{i=1}^{\infty} E_i \right) = \sum_{i=1}^{\infty} \mathbb{P}(E_i)$$

- A, B are independent iff

$$\mathbb{P}(A \cap B) = \mathbb{P}(A)\, \mathbb{P}(B)$$

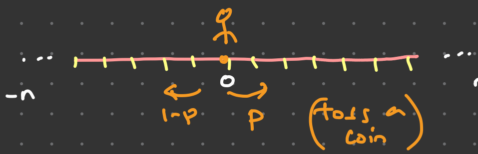
- A, B are two events $\quad \mathbb{P}(B) > 0$

$$\mathbb{P}(A \mid B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$$

Definition of Random Walk

Intuitive : R.W.

- Toss a coin at each step
- If head occurs move right
- If tail occurs move left



S_n = position at time n

Q :- • Typical position at time n?

- $P(S_n = 0) = ?$
- Find $x \in [-n, n]$ s.t $P(S_n = x)$ is maximum?

Notation : $\mathbb{N}$ — natural numbers

$\mathbb{N}_0$ — $\mathbb{N} \cup \{0\}$.

Simple random walk on $\mathbb{Z}$

Fix $N \in \mathbb{N}$.

$$\Omega_N = \{ \omega = (\omega_1, \dots, \omega_N) \mid \omega_i \in \{-1, +1\} \} \equiv \{-1, 1\}^N$$

Define for $1 \leq k \leq N$

$X_k : \Omega_N \to \{-1, 1\}$ by $X_k(\omega) = \omega_k$

(step of random walk at time k)

$$S_0(\omega) = 0 \ ; \quad S_n(\omega) = \sum_{k=1}^{n} X_k(\omega) \qquad 1 \leq n \leq N$$

(position of walk at time n)

$\mathcal{F}_N = \{ A \mid A \subseteq \Omega_N \}$

$\mathbb{P}_N : \mathcal{F}_N \to [0,1]$ (Uniform Probability) ($p = 1/2$ intuitive)

(1) $$\mathbb{P}_N(A) = \frac{|A|}{2^N} \quad \forall A \subseteq \Omega_N$$

(i.e. any sequence in $\{-1,1\}^N$ has the same probability)

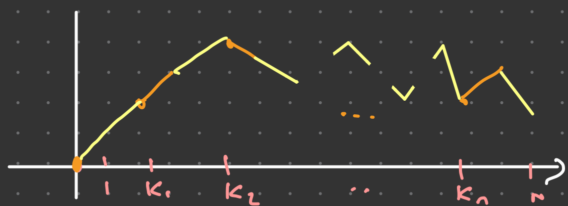
Definition: Sequence of random variable $\{S_n\}_{n=0}^{N}$ on Probability space $(\Omega_N, \mathcal{F}_N, \mathbb{P}_N)$ is called a simple symmetric random walk of length N starting at 0.

Observations:

Given:- $1 \le k_1 < k_2 \;..\; < k_n \le N$

$\underbrace{x_1 \quad x_2 \quad .. \quad x_n}_{\text{steps n position}}$ $\quad x_i \in \{-1,1\}$

(*) $$\mathbb{P}(X_{k_1} = x_1, \; \dots \;, X_{k_n} = x_n) = \; ? \; \overset{(1)}{=} \frac{2^{N-n}}{2^N} = \frac{1}{2^n}$$



e.g. $x_1 = 1$
$x_2 = -1$
$\vdots$
$x_n = 1$

$$|\{X_{k_1} = x_1 \;..\; , X_{k_n} = x_n\}| = |\{\omega \in \Omega_N \mid \omega_{k_1} = x_1 \;..\; \omega_{k_n} = x_n\}| = 2^{N-n}$$

- (Ex.) $\mathbb{P}(X_k = 1) = \frac{1}{2} = \mathbb{P}(X_k = -1)$

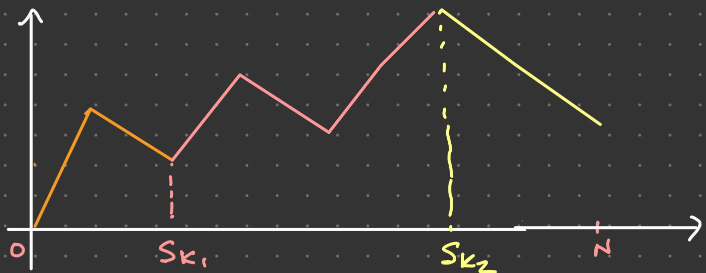
$\forall \ 1 \le k \le N$

$[\because \ n=1 \text{ in } *]$

$k \neq l$ $\quad \mathbb{P}(X_k = \pm 1, X_l = \pm 1) = \mathbb{P}(X_k = \pm 1)\,\mathbb{P}(X_l = \pm 1)$

(independence) $\quad k \neq l$

- (Ex)



Simple random walk has independent increments

$S_{k_1} - S_0, \quad S_{k_2} - S_{k_1}, \quad S_N - S_{k_2}$

are independent of each other.

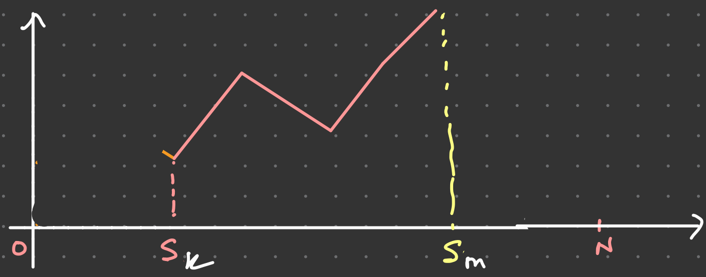
Note: $\cdot \ S_{k_1} = \sum_{n=0}^{k_1} X_n$ $\quad$ are dependent

$\cdot \ S_{k_2} = S_{k_1} + \sum_{n=k_1+1}^{k_2} X_n$

- $0 < k < m \le N$

$\forall a \in \mathbb{Z}$

$$\mathbb{P}(S_m - S_k = a) = \mathbb{P}(S_{m-k} = a)$$



- $S_k = \sum_{n=0}^{k} X_n$
- $S_m = S_k + \sum_{n=k+1}^{m} X_n$

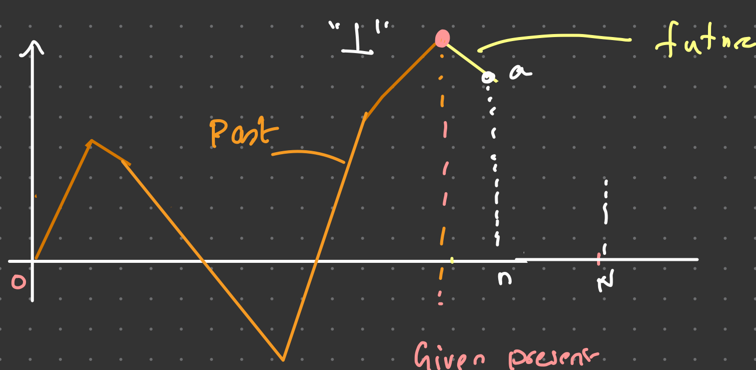
$\Big\}$ $S_m - S_k = \sum_{n=k+1}^{m} X_n$

(d) $\mathbb{P}(S_n = a \mid S_{n-1} = a_{n-1}, \dots, S_1 = a_1)$ — ⊕

$$= \mathbb{P}(S_n = a \mid S_{n-1} = a_{n-1})$$

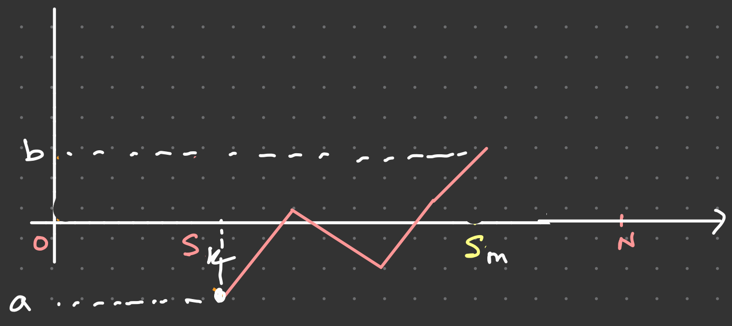
if $\mathbb{P}(S_{n-1} = a_{n-1}, \dots, S_1 = a_1) > 0$.

⊕ (Markov Property)



(8) Suppose $\mathbb{P}(S_k = a) > 0$. for $k \geq 1$, $a \in \mathbb{Z}$.
$m > k$ $b \in \mathbb{Z}$

$$\mathbb{P}(S_m = b \mid S_k = a) = \mathbb{P}(S_{m-k} = b-a)$$



<u>On an average where will the walk be:</u>

Expectation $E[S_n] = ?$ $Var[S_n] = ?$

$$E[X_k] = 1\,\mathbb{P}(X_k = 1) + (-1)\,\mathbb{P}(X_k = -1)$$

$$= 1 \cdot \frac{1}{2} - \frac{1}{2} = 0 \quad - \#$$

$$S_n = \sum_{k=1}^{n} X_k + 0$$

$$\Rightarrow E[S_n] = \sum_{k=1}^{n} E[X_k] = 0$$

$$Var[S_n] = E[S_n^2] - (E[S_n])^2$$

$$= E[S_n^2] = E\left(\left(\sum_{k=1}^{n} X_k\right)^2\right)$$

$$= E\left[\sum_{k=1}^{n} X_k^2 + \sum_{\substack{i,j \\ i\neq j}} X_i X_j\right]$$

$$= \sum_{k=1}^{n} E[X_k^2] + \sum_{\substack{i,j \\ i\neq j}} E[X_i X_j]$$

$\left(X_i \perp X_j \text{ independent},\ i \neq j \right)$

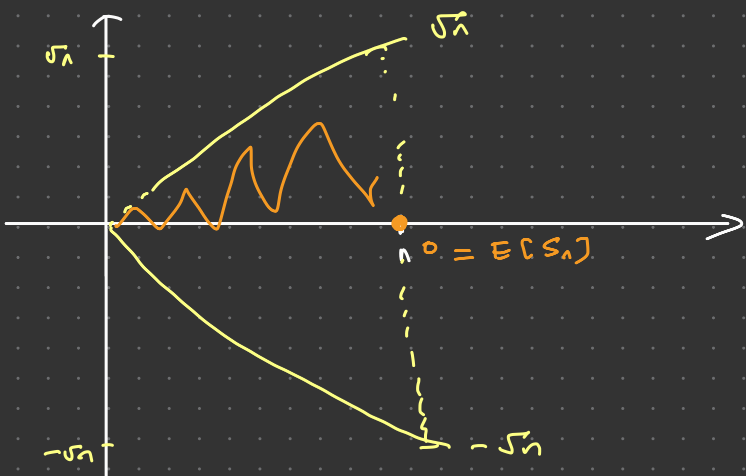
$$= \sum_{k=1}^{n} E[X_k^2] + \sum_{\substack{i,j \\ i\neq j}} E[X_i]E[X_j]$$

$\#$ 0

Now $E[X_k^2] = 1^2 P(X_k = 1) + (-1)^2 P(X_k = -1)$

$= 1 \cdot \frac{1}{2} + 1 \cdot \frac{1}{2} = 1 \quad \forall\, 1 \leq k \leq n$ $\#\#\#$

$\Rightarrow \operatorname{Var}[S_n] = n$, $SD[S_n] = \sqrt{n}$



Claim 1.1: $x \in \{-n, -n+2, \ldots, n-2, n\}$

$$\mathbb{P}(S_n = x) = \binom{n}{\frac{n+x}{2}} \frac{1}{2^n}$$

Prove claim 1.1 next class

Observations:

- Distribution of S_n - symmetric around 0.

$$\mathbb{P}(S_n = x) = \frac{n!}{\frac{n+x}{2}!\,\frac{n-x}{2}!} = \mathbb{P}(S_n = -x)$$

- (Ex) $\mathbb{P}(S_{2n} = 0) = \mathbb{P}(S_{2n-1} = 1) = \binom{2n}{n}\frac{1}{2^{2n}}$

(maximal weight)