Due: October 17th, 2023, 10am

Problems Due for Introduction to Probability and Statistics: 1,2,3,4

Problems Due for Probability 3: 1,3,4,5,7

1. In Example M3 show that for $n \geq 1$, $p_{10}^{[n]} = p_{10}^{[n-1]}(1-\alpha-\beta)+\beta$, and use the technique for solving recurrence relations to show that $p_{10}^{[n]}$ is given by $p_{10}^{n} = \frac{\beta}{\alpha+\beta} - \frac{\beta}{\alpha+\beta}(1-\alpha-\beta)^n$.

2. The examples given below can be modeled by a Markov chain. Determine the state space, initial distribution and the transition matrix for each.

   (a) Consider a frog moving along a river bank according to the following random mechanism. At each time step, it tosses a coin; if heads is the result, it jumps 1 unit up and if tails is the result, it jumps 1 unit down. Let X_n denote the position of the frog at time n.

   (b) Suppose N black balls and N white balls are placed in two urns so that each urn contains N balls. At each step one ball is selected at random from each urn and the two balls interchange places. The state of the system at time $n \in \mathbb{N}$ is the number of white balls in the first urn after the n-th interchange.

   (c) Suppose a gambler starts out with a certain initial capital of N rupees and makes a series of 1 rupee bets against the gambling house until her captial runs out. Assume that she has probability p of winning each bet. Let the state of the system at time $n \in \mathbb{N}$ denote her capital at the n-th bet.

3. Meteorologist Chakrapani could not predict rainy days very well in the wet city of Cherapunjee. So they decided to use the following prediction model for rain. If it had rained yesterday and today, then it will rain tomorrow with probability 0.5. If it rained today but not yesterday, then it will rain tomorrow with probability 0.3. If it did not rain today, then it will rain tomorrow with probability 0.1. regardless of yesterday's weather. Let $X_n = R$ if it rained on day $n \in \mathbb{N}$ and $X_n = D$ if it was a dry day (no rain). Suppose $X_0 = D$ (so it didn't rain on day 0).

   (a) Show that $\{X_n : n \geq 0\}$ is not a Markov chain. (Hint: Show that the Markov property is violated).

   (b) Let $Y_n = (X_n, X_{n+1})$ for each $n \geq 0$. Show that Y_n is a Markov chain by writing down the state space, the initial distribution, and the transition matrix for the chain Y_n.

4. Consider a Markov chain X_n on state space $\{A, B, C\}$ with initial distribution μ and transition matrix given by

$$P = \begin{pmatrix} .2 & .4 & .4 \\ .4 & .4 & .2 \\ .4 & .6 & 0 \end{pmatrix}.$$

   (a) What is the probability of going from state A to state B in one step ?

   (b) What is the probability of going from state B to state C in exactly two steps ?

   (c) What is the probability of going from state C to state A in exactly two steps ?

   (e) What is the probability of going from state C to state A in exactly three steps ?

   (e) Calculate the second, third and fourth power of this matrix. Do you have a guess for P^n for large n

5. Let $X_n \xrightarrow{p} X$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$

(a) Show that there is a subsequence X_{n_k} such that

$$P(|X_{n_k} - X| > \frac{1}{k}) \leq \frac{1}{2^k}$$

(b) Let $A_k = \{|X_{n_k} - X| > \frac{1}{k}\}$. Show that $P(A_k \text{ occur i.o.}) = 0$

(c) Conclude that $X_{n_k} \xrightarrow{a.e.} X$.

6. Let Y_n be a sequence of independent and identically distributed random variables and let $X_n = \frac{Y_n}{n}$. Show that X_n converges in probability to 0. Decide whether X_n converges a.e. or not.

7. Let $\mathcal{X}$ be the set of all random variables on the probability space $(\Omega, \mathcal{B}, P)$. Define a function $\rho : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ by

$$\rho(X, Y) = E(\min(|X - Y|, 1),$$

for any $X, Y \in \mathcal{X}$. Show that $(\mathcal{X}, \rho)$ is a metric space. Further show that a sequence of random variables $\{X_n\}$ converges in probability to X if and only if $\rho(X_n, X) \to 0$.