Problem Collection

A collective effort of ISI Bangalore 2021-2024 batch

February 5, 2022

Contents

1 Introduction

2 Problems

2.1 Week 1

Problem 1. Let α be the greatest positive root of $x^3 - 3x^2 + 1 = 0$. Prove that

$$17 \mid \lfloor \alpha^{1996} \rfloor$$

(shared by Aaratrick Basu)

Problem 2. Find all n-tuples of positive integers $(a_1, a_2, \ldots, a_n)$ such that

$$(a_1! - 1)(a_2! - 1) \cdots (a_n! - 1) - 16$$

is a perfect square.

(shared by Soumya Dasgupta)

Problem 3. Find the angles A, B, C of all possible triangles ABC such that $\tan A, \tan B, \tan C \in \mathbb{N}$

(shared by Bikram Halder)

Problem 4. Find the fewest standard weights required to measure all integral weights from 1 to 40 units on a pan balannce.

(shared by Saraswata Sensarma)

Problem 5. Prove that for all positive integers $a > 1$ and n, $n \mid \varphi(a^n - 1)$.

(shared by Prognadipto Majumder)

Problem 6. Prove that if p is a prime, then $p^p - 1$ has a prime factor that is congruent to 1 modulo p.

(shared by Prognadipto Majumder)

Problem 7. For any integer a, set $n_a = 101a - 100 \cdot 2^a$. Show that for $0 \le a, b, c, d \le 99$,

$$n_a + n_b \equiv n_c + n_d \pmod{10100}$$

implies $\{a, b\} = \{c, d\}$.

(shared by Prognadipto Majumder)

Problem 8. Show that if $3 \le d \le 2^{n+1}$, then $d \nmid \left(a^{2^n} + 1\right)$ for all positive integers a.

(shared by Prognadipto Majumder)

2.2 Week 2

Problem 9. Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that $f(f(x)) = x$ for all real numbers x, and further it is given that f satisfies intermediate value property. Then show that set of fixed points of f is either the complete set $\mathbb{R}$ or contains an unique element.

(shared by Soumya Dasgupta)

Problem 10. Find the greatest integer k for which 1991^k divides $1990^{1991^{1992}} + 1992^{1991^{1990}}$.

(shared by Varun Balasubramanian)

2.3 Week 3

3 Solutions

3.1 Week 1

Solution. (Solution to **Problem** 1) ■

Solution. (Solution to **Problem** 2) ■

Solution. (Solution to **Problem** 3) ■

Solution. (Solution to **Problem** 4) ■

Solution. (Solution to **Problem** 5) ■

Solution. (Solution to **Problem** 6) ■

Solution. (Solution to **Problem** 7). We have

$$\begin{aligned}
&n_a + n_b \equiv n_c + n_d \pmod{10100} \\
&\Rightarrow 101(a+b-c-d) - 100(2^a + 2^b - 2^c - 2^d) \equiv 0 \pmod{10100} \\
&\Rightarrow a+b-c-d \equiv 0 \pmod{100}
\end{aligned}$$

Let $a+b-c-d = 100k$ where $k \in \mathbb{Z}$ then $2^a = 2^{100k+c+d-b} \equiv 2^{c+d-b} \pmod{101}$. Then we get

$$\begin{aligned}
&2^a + 2^b \equiv 2^c + 2^d \pmod{101} \\
&\Rightarrow 2^{c+d-b} + 2^b \equiv 2^c + 2^d \pmod{101} \\
&\Rightarrow (2^{c+d} - 2^c) + (2^{2b} - 2^{b+d}) \equiv 0 \pmod{101} \\
&\Rightarrow (2^b - 2^c)(2^b - 2^d) \equiv 0 \pmod{101}
\end{aligned}$$

WLOG assume $(2^b - 2^d) \equiv 0 \pmod{101}$ then we get $o_{101}(2) \mid (b-d) \Rightarrow 100 \mid (b-d) \leq 99$ (since 2 is a primitive root modulo 101) hence we must have $b = d$. But then we get $a - c \equiv 0 \pmod{100}$, but $a - c \leq 99$ hence we get $a = c$, thus $\{a, b\} = \{c, d\}$. The case when $(2^b - 2^c) \equiv 0 \pmod{101}$ is similar, just we will get $b = c$ and $a = d$. ■

Solution. (Solution to **Problem** 8) ■

3.2 Week 2

Solution. (Solution to **Problem** 9) ■

Solution. (Solution to **Problem** 10) ■

3.3 Week 3

4 Demo

This are all the environment styles available in this project. Feel free to add some more newenvironments, if you want. Just make sure to add them in the styles.tex file, and leave a comment so that everyone can see the changes.

Theorem 4.1. *This is a theorem.*

Problem 11. This is a problem.

Corollary 4.1.1. *This is a corollary from 4.1*

Lemma 4.2. *This is a lemma.*

Example 4.2.1. This is an example.

Solution. This is a solution. ■

Proof. This is a proof. □