Toy example 2

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Abstract

This document provides a brief description of the second toy example supplied with our code. In section 1, we illustrate the input Clifford+T quantum circuit, present its (exact) output distribution, and convert it to the appropriate adaptive Clifford circuit. Then, in section 2, we carry out the corresponding Pauli-based computation (PBC), explicitly computing all the possible PBC paths and demonstrating that the output distribution obtained from this procedure is the same as that of the original input circuit. Finally, in section 3, we discuss the results obtained when performing hybrid PBC with 1 virtual qubit using our Python code.

Note that since this is a very small example, from the point of view of the quantum resources used, there is no significant advantage in using PBC. Nevertheless, it is a good example to illustrate how the procedure works, and compare the results of the explicit calculations against those given by our code.

1 The input Clifford+T circuit

The unitary Clifford+T circuit we want to simulate using PBC is illustrated in figure 1. Here, it is relevant to point out that the second T gate, in being applied right before a measurement in the computational basis, is inconsequential for the final output distribution of this computation. This means that it could actually be removed from the quantum circuit. Nevertheless, we purposefully kept it in because, in that way, we must work with larger (4-qubit) Pauli operators and propagate more (4) measurements across the system, making this a more extensive and complete example.

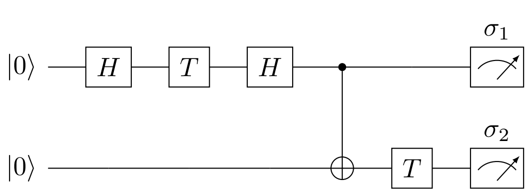

Figure 1: Depiction of the unitary Clifford+T circuit to be simulated using PBC.

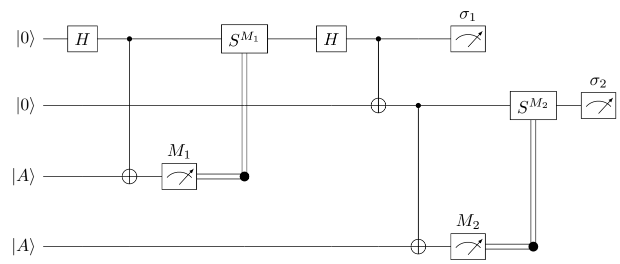

Figure 2: The adaptive Clifford circuit obtained by replacing each T gate with the T-gadget. Note that, just like the second T gate in the original circuit has no influence on the outcome of the computation, the same is true for S^{M_2}. That is, this gate can be removed from the adaptive Clifford circuit. This observation will simplify the explicit calculations performed in section 2.

The adaptive Clifford circuit obtained by replacing each T gate with the T-gadget is illustrated in figure 2. Here, it is useful to note that the presence or absence of the second S gate is irrelevant for the final output of the computation. For that reason, and for the purposes of the calculations carried out in the next section, its absence will be assumed for simplicity, and without loss of generality.

The output distribution of this circuit can be easily calculated and is:

$$\begin{cases} p(\sigma_1 = 0, \sigma_2 = 0) = \frac{1}{2} + \frac{\sqrt{2}}{4} \approx 0.8536 \\ p(\sigma_1 = 1, \sigma_2 = 1) = \frac{1}{2} - \frac{\sqrt{2}}{4} \approx 0.1464 \end{cases} . \tag{1}$$

2 Task 1: Compilation and weak simulation with Pauli-based computation

We start with the Z-measurement of the first auxiliary qubit and we propagate it to the beginning of the circuit obtaining

$$(I \otimes I \otimes Z \otimes I) \rightarrow P_1 = (X \otimes I \otimes Z \otimes I). \tag{2}$$

Since this operator anti-commutes with $(Z \otimes I \otimes I \otimes I)$, we can obtain its outcome M_1 classically, by making a random draw from the uniform distribution $\{0, 1\}$. Next,

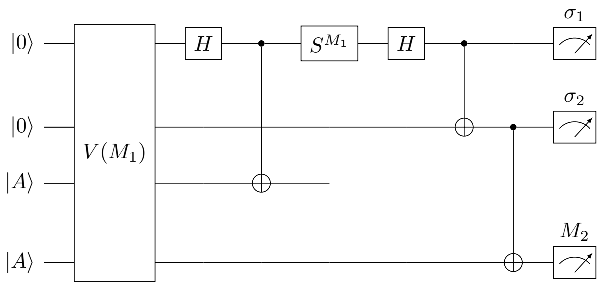

Figure 3: Quantum circuit after the first Pauli operator P_1 has been dealt with using the classical computer. The result of the coin toss corresponds to the value M_1 and fixes the next part of the circuit.

we need to place at the very beginning of the quantum circuit (before any of the other gates) the Clifford unitary

$$V(M_1) = \frac{(Z \otimes I \otimes I \otimes I) + (-1)^{M_1}(X \otimes I \otimes Z \otimes I)}{\sqrt{2}} .$$

The quantum circuit obtained after this step is depicted in figure 3.

2.1 Path 1

Assuming that the outcome of the draw is $M_1 = 0$, we find out there is no need for the S correction and thus fix the next part of the Clifford circuit. This allows us to now propagate the measurement corresponding to the second auxiliary qubit which yields:

$$(I \otimes I \otimes I \otimes Z) \rightarrow P_2 = (Z \otimes Z \otimes X \otimes Z). \tag{3}$$

To obtain the outcome of this operator we actually need to perform a quantum measurement. Following the PBC procedure, we can reduce this to the measurement of $(X \otimes Z)$ on the magic qubits. The measurement of such Pauli operator on the state $|A\rangle^{\otimes 2}$ yields $p(M_2 = 0) = p(M_2 = 1) = \frac{1}{2}$.

As commented above, the outcome of this measurement has no consequence over the next part of the circuit, so that regardless of having $M_2 = 0$ or $M_2 = 1$ the next steps are the same.

First we propagate the measurement of the first main qubit to the beginning of the quantum circuit and in doing so obtain the next Pauli operator in the generalized PBC:

$$(Z \otimes I \otimes I \otimes I) \rightarrow P_3 = (Z \otimes I \otimes X \otimes I). \tag{4}$$

This operator commutes with P_2, given in equation (3), and is independent from it so that this constitutes the second quantum measurement that needs to be carried out. Concretely, we measure $(X \otimes I)$ to get σ_1. Such Pauli measurement, performed on the state $|\varphi\rangle$ resulting from the previous measurement of $(X \otimes Z)$,

$$|\varphi\rangle \propto \frac{I \otimes I + (-1)^{M_2}(X \otimes Z)}{2} |A\rangle^{\otimes 2} , \tag{5}$$

yields $\sigma_1 = 0$ with $p(\sigma_1 = 0) = 1/2 + \sqrt{2}/4$, and $\sigma_1 = 1$ with $p(\sigma_1 = 1) = 1/2 - \sqrt{2}/4$.

Finally, we have to propagate the measurement of the second main qubit to the beginning of the circuit which yields

$$(I \otimes Z \otimes I \otimes I) \rightarrow P_4 = (Z \otimes Z \otimes X \otimes I). \tag{6}$$

Clearly, this Pauli operator is dependent on P_3, and we can obtain its outcome classically since it is obvious that σ_2 must be equal to σ_1: $\sigma_2 = \sigma_1$.

Thus, it is clear that this path on the PBC tree has the same probability distribution as the original circuit (equation (1)).

To summarize, the generalized PBC computation along this branch is:

$$\begin{gathered}(X \otimes I \otimes Z \otimes I) \xrightarrow{M_1=0} (Z \otimes Z \otimes X \otimes Z) \xrightarrow{M_2=0/1} \\ (Z \otimes I \otimes X \otimes I) \xrightarrow{\sigma_1=0/1} (Z \otimes Z \otimes X \otimes I) \xrightarrow{\sigma_2=\sigma_1} \\ y = \sigma_1\sigma_2 ,\end{gathered}$$

where $y = \sigma_1\sigma_2$ is the output bitstring and the probability distribution $p(y)$ corresponds to that of the original quantum circuit (equation (1)).

2.2 Path 2

Assuming that the outcome of the draw is $M_1 = 1$, we fix the next part of the Clifford circuit knowing that it requires the S gate. This allows us to now propagate the measurement corresponding to the second auxiliary qubit which yields:

$$(I \otimes I \otimes I \otimes Z) \rightarrow P_2 = (I \otimes Z \otimes Y \otimes Z). \tag{7}$$

This operator needs to be measured on a quantum processor. According to the PBC procedure we need only measure $(Y \otimes Z)$ acting on $|A\rangle^{\otimes 2}$, which will give the outcomes 0 and 1 with the same probability: $p(M_2 = 0) = p(M_2 = 1) = 1/2$.

Next, we work through the measurement of the first main qubit, which after being propagated to the beginning of the circuit yields the Pauli operator P_3:

$$(Z \otimes I \otimes I \otimes I) \rightarrow P_3 = (I \otimes I \otimes Y \otimes I) . \tag{8}$$

This operator commutes with P_2 (given in equation (7)) and is independent from it, so that it constitutes the second quantum measurement (which needs to be carried out in the quantum processor after the measurement of $(Y \otimes Z)$). Such measurement has outcome $\sigma_1 = 0$ with probability $p(\sigma_1 = 0) = 1/2+\sqrt{2}/4$, and outcome $\sigma_1 = 1$ with probability $p(\sigma_1 = 1) = 1/2 - \sqrt{2}/4$.

Finally, it remains only to propagate the measurement of the second main qubit, which gives:

$$(I \otimes Z \otimes I \otimes I) \rightarrow P_4 = (I \otimes Z \otimes Y \otimes I). \tag{9}$$

This operator depends on P_3 and so its outcome is deterministic and given by $\sigma_2 = \sigma_1$.

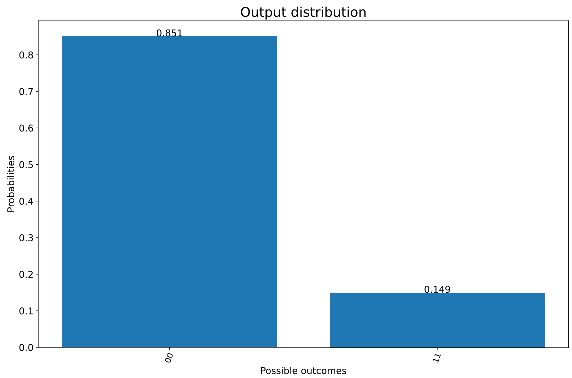

Figure 4: Output distribution obtained by applying our code (for a total of 1024 shots) to the input circuit depicted in figure 1.

To summarize, the generalized PBC computation along this branch is:

$$
\begin{gathered}
(X \otimes I \otimes Z \otimes I) \xrightarrow{M_1=1} (I \otimes Z \otimes Y \otimes Z) \xrightarrow{M_2=0/1} \\
(I \otimes I \otimes Y \otimes I) \xrightarrow{\sigma_1=0/1} (I \otimes Z \otimes Y \otimes I) \xrightarrow{\sigma_2=\sigma_1} \\
y = \sigma_1 \sigma_2 \,,
\end{gathered}
$$

where $y = \sigma_1 \sigma_2$ is the output bitstring, and the output probability distribution $p(y)$ corresponds to that of the original quantum circuit.

2.3 Code output

It is clear that the procedure described above (and implemented in our code) yields exactly the same output distribution as the original quantum circuit. This can be seen from the "Output_distribution-Toy2-input.pdf" output file which presents the histogram corresponding to 1024 shots (see figure 4). Additionally, it is also possible to check the different paths explored during the computation, which are stored in the file "Compilation_data.txt". These paths are consistent with the calculations above.

For more details, the interested user can explore the additional information accessible in the other output files.

3 Task 2: Hybrid computation

As seen above, the compilation procedure reduces the present problem to that of performing two consecutive quantum measurements of 2 independent and pairwise commuting 2-qubit Pauli operators (assisted by some classical processing). In general, the hybrid PBC procedure allows us to lessen the total number of real qubits required for the actual quantum computation by simulating a certain number k of virtual qubits, at the expense of having to compute 3^k smaller PBCs. Additionally, the task stops being a weak simulation task, and is transformed into the strong simulation problem of estimating the probability that a certain qubit outputs the value 1.

In the case of the present example, we can estimate the probability p that the first qubit outputs the value 1 up to a maximum relative error ϵ and using 1 virtual qubit (so that the quantum hardware needs only have 1 qubit).

The files output by our code can be found in the corresponding folder "output-hybrid". Analysing the results yields the plots presented in figure 5.

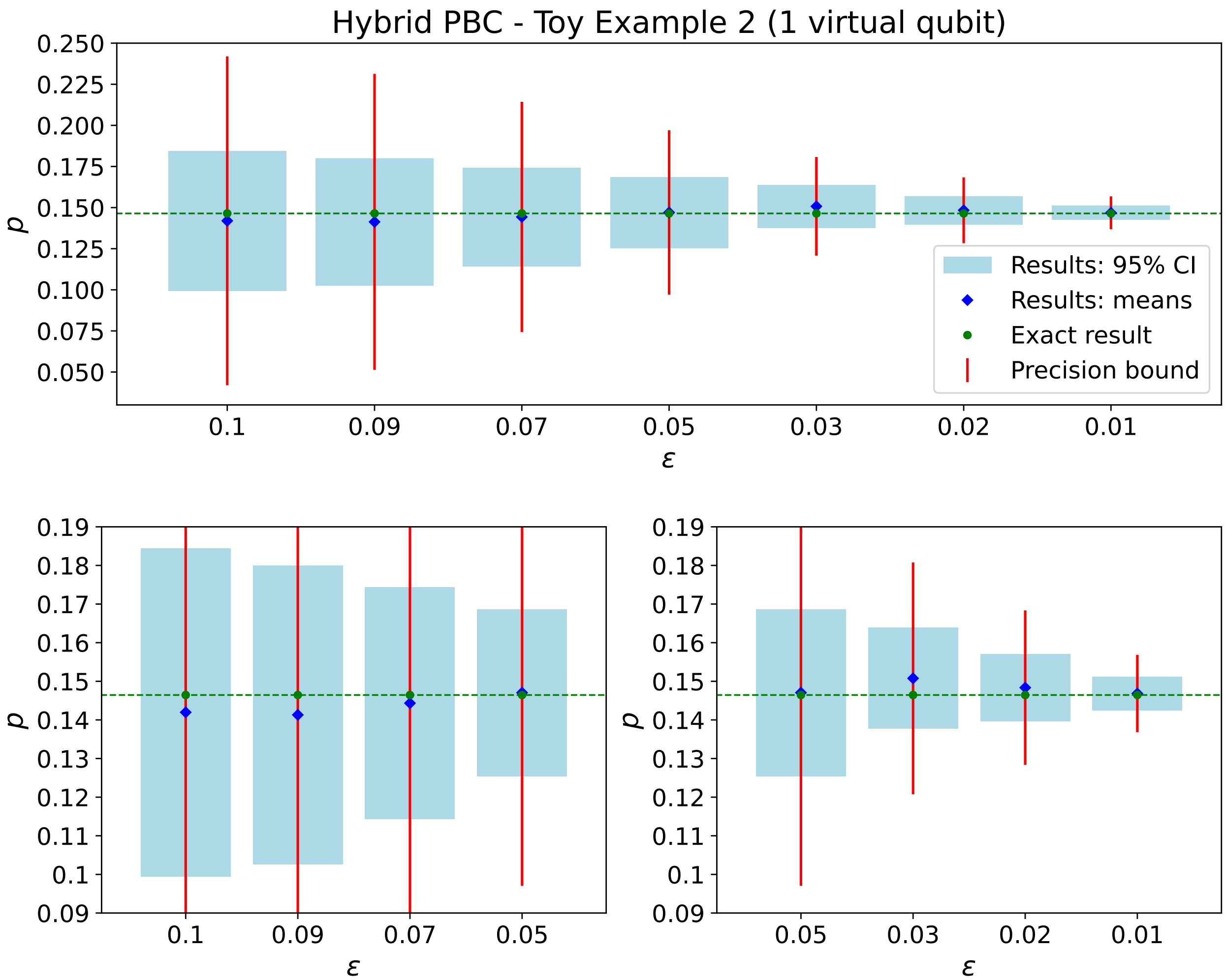

Figure 5: Estimate of the probability that the first qubit of the circuit depicted in figure 1 yields 1, as a function of different errors. We see that as we reduce the allowed maximum error the values expectation value converges to the exact value ($p = 1/2 - \sqrt{2}/4$); additionally, the confidence intervals become increasingly narrower.