Some derivations of Benney equations

10 décembre 2024

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1 Equations without control

1.1 Notations

Troughout all that document, I denote :
— dimensionnal quantities with a * (e.g x^*),
— non-dimensionnal one without anything (e.g x),
— derivation with a "," e.g. $f_{,xy} = \frac{\partial^2 f}{\partial x \partial y}$.

1.2 Quick Description of the System

I am studying a liquid flowing down an inclined 2D plane of angle θ from the horizontal. The system of axis (x^*, y^*) is such that x^* points downstream and y^* is normal to the plane going upward.

1.3 Governing equations

We study the Navier-Stokes equations :

$$\begin{cases} \rho(u^*_{,t^*} + u^* u^*_{,x^*} + v^* u^*_{,y^*}) & = -p^*_{,x^*} + \mu_l (u^*_{,x^* x^*} + u^*_{,y^* y^*}) + \rho sin(\theta) g \\ \rho(v^*_{,t^*} + u^* v^*_{,x^*} + v^* v^*_{,y^*}) & = -p^*_{,y^*} + \mu_l (v^*_{,x^* x^*} + v^*_{,y^* y^*}) - \rho cos(\theta) g \\ u^*_{,x^*} + v^*_{,y^*} & = 0 \quad \text{(incompressibility or continuity equation)} \end{cases}$$

We take a no slip boundary conditions at the wall so at y = 0 :

$$u^* = 0, v^* = 0$$

The Dynamic Boundary Condition or the dynamic stress balance at the interface is :

$$-p^*_l + \frac{2\mu_l}{1 + {h^*_{,x^*}}^2}(u^*_{,x^*} {h^*_{,x^*}}^2 - h^*_{,x^*}(u^*_{,y^*} + v^*_{,x^*}) + v^*_{,y^*}) = \frac{\gamma h^*_{,x^* x^*}}{(1 + {h^*_{,x^*}}^2)^{3/2}}$$

$$(v^*_{,x^*} + u^*_{,y^*})(1 - {h^*_{,x^*}}^2) + 2h^*_{,x^*}(v^*_{,y^*} - u^*_{,x^*}) = 0$$

The Kinematic Boundary condition is :

$$h^*_{,t^*} = v^* - u^* h^*_{,x^*}.$$

2 Reduced order models with and without Control

2.1 Derivation of the equations Without Control

2.1.1 Non-dimensionalisation, scaling and governing equations

We now make a change of variable in order to manipulate dimensionless quantities. We denote
— $\mathcal{L}^*$ characteristic horizontal length

— $\mathcal{H}^* = h_N^*$ typical fluid height (and typical vertical length) which is also the *Nusselt solution* i.e film height at steady state that we fix as we want.
— $\mathcal{U}^* := U_N = \frac{\rho g {h_N^*}^2 sin(\theta)}{2\mu_l}$ typical horizontal velocity and also velocity of the Nusselt solution.
— $\mathcal{P}^*$ typical pressure of the liquid

Moreover, we make the **Thin-Film Approximation** which is to take $\mathcal{H}^* \ll \mathcal{L}^*$ i.e the width of the film is small compared to its length. We now scale the variables of the problem with the **thin-film parameter**

$$\epsilon = \frac{\mathcal{H}^*}{\mathcal{L}^*}.$$

We have :

$$\begin{aligned} x^* &= \mathcal{L}^* x, \quad y^* = \epsilon \mathcal{L}^* y, \\ u^* &= \mathcal{U}^* u, \quad v^* = \epsilon \mathcal{U}^* v, \\ p_l^* &= \mathcal{P}^* p_l. \end{aligned}$$

We thus see here that the typical vertical velocity is chosen to be $\mathcal{V}^* := \epsilon \mathcal{U}^*$. We now take some convenient form for the typical pressure : $\mathcal{P}^* = \frac{\mu_l U^*}{h_N^*}$ and set the non dimensionalised quantities :

$$Re = \frac{\rho \mathcal{U}^* h_N}{\mu_l} \quad Ca = \frac{\mu_l \mathcal{U}^*}{\gamma}.$$

We choose the scaling $\underline{Re = O(1) \text{ and } Ca = O(\epsilon^2)}$. The first scaling allow to consider inertia and, the second, surface effect in the order 1 approximation in ϵ that we will carry out. Indeed, the effect of interest here are **viscosity**, **surface tension**, **gravity** and **inertia**. We would thus need 3 non-dimensionalised quantities (cf Birgmingham-Pi theorem ? to check) to describe them. However, as we already have some other fixed scales ($\mathcal{L}^*$,$\mathcal{U}^*$), we just need two which are e.g Ca and Re.

Just with the undimensionalisation, we have :

$$\begin{aligned} Re(u_{,t} + uu_{,x} + vu_{,y}) &= -p_{,x} + u_{,xx} + u_{,yy} + 2 \\ Re(u_{,t} + uu_{,x} + vu_{,y}) &= -p_{,x} + u_{,xx} + u_{,yy} + 2 + v_{,yy} - \rho cos(\theta) g \\ u_{,x} + v_{,y} &= 0 \end{aligned} \tag{1}$$

No slip :

$$u = 0, \quad v = 0 \quad \text{at y=0.} \tag{2}$$

Dynamic stress balance at the interface :

$$-p_l + \frac{2}{1 + {h_{,x}}^2}(u_{,x} {h_{,x}}^2 - h_{,x}(u_{,y} + v_{,x}) + v_{,y}) = \frac{1}{Ca} \frac{h_{,xx}}{(1 + {h_{,x}}^2)^{3/2}} \tag{3}$$

$$(v_{,x} + u_{,y})(1 - {h_{,x}}^2) + 2h_{,x}(v_{,y} - u_{,x}) = 0 \tag{4}$$

The Kinematic Boundary condition (KBC) is :

$$h_{,t} = v - uh_{,x}. \tag{5}$$

From the KBC, incompressibility and no slip equations, we can construct the 1D mass conservation equation :

$$h_{,t} + q_{,x} = 0 \tag{6}$$

with

$$q(x,t) = \int_{y=0}^{h_(x,t)} u(x,y,t)dy$$

the horizontal flux of the flowing liquid.

2.1.2 Benney equation

With the chosen scaling and some order 1 in ϵ asymptotics on the governing equations, we compute the order 1 flux to have the so called Benney equations :

$$\begin{aligned} h_{,t} + q_{,x} &= 0 \\ q(x,t) &= \frac{h^3}{3}(2 - 2h_{,x}cot(\theta) + \frac{h_{,xxx}}{Ca}) + Re\frac{8h^6h_{,x}}{15}. \end{aligned} \tag{7}$$

i.e

$$h_{,t}+h_{,x}h^2\left(2 - 2h_{,x}cot(\theta) + \frac{h_{,xxx}}{Ca}\right)+\frac{h^3}{3}\left(-2h_{,xx}cot(\theta) + \frac{h_{,xxxx}}{Ca}\right)+\frac{8Re}{15}\left(6h^5{h_{,x}}^2 + h^6h_{,xx}\right) = 0.$$

2.1.3 Weigthed residuals

2.2 Derivation of the equations with a control term

2.2.1 New governing equations

We want to stabilise the fluid by applying some air jet controled perturbation. However, we do not model the air and just take a specific form of the normal pressure (e.g. gaussian). Let us write

$$\underline{s_g} = -\underline{n}.\underline{\underline{\sigma_l}} \tag{8}$$

with $\underline{\underline{\sigma_l}}$ the tensor of constraint of the liquid, and $\underline{n}$ the normal from the liquid to the gas We take as normal and tangential vectors

$$\underline{n} = \frac{1}{\sqrt{1+h_{,x}^2}}(-h_{,x},1), \quad \sqrt{1+h_{,x}^2}(1,h_{,x}),$$

and$(\underline{n},\underline{t})$ is a normalized, orthogonal and direct couple.

We isolate the normal and tangential components of the external gas pressure :

$$(N_s, T_s) = \underline{s_g}(y=h).\underline{n} \tag{9}$$

As for the governing equations, only the KBC change :

$$-p_l + \frac{2}{1+{h_{,x}}^2}(u_{,x}{h_{,x}}^2 - h_{,x}(u_{,y}+v_{,x}) + v_{,y}) = \frac{1}{Ca}\frac{h_{,xx}}{(1+{h_{,x}}^2)^{3/2}} - N_s \tag{10}$$

$$\frac{(v_{,x}+u_{,y})(1-h_{,x}{}^2)+2h_{,x}(v_{,y}-u_{,x})}{1+h_{,x}^2}=-T_s. \tag{11}$$

Here, we will take $T_s = 0$ so only the normal KBC changes. This approximation has been made by the paper of D.Lunz & al. but it has been shown on another system studied by Radu.C & al. that the tangential component of the air jet pressure has non negligible effects on the profile of the liquid.

2.2.2 Benney equation

The computation of the pressure is different (cf Appendix) and it gives at order 1 in ϵ :

$$\begin{aligned} h_{,t}+q_{,x}&=0 \\ q(x,t)&=\frac{h^3}{3}(2-p_{l0,x})+Re\frac{8h^6h_{x,}}{15} \end{aligned} \tag{12}$$

with

$$p_{l0}=N_s+2(h-y)cot(\theta)-\frac{h_{,xx}}{Ca}$$

the pressure term approximated at order 0.

i.e

$$\begin{aligned} &h_{,t}+h_{,x}h^2\left(2-N_{s,x}-2h_{,x}cot(\theta)+\frac{h_{,xxx}}{Ca}\right)-\frac{h^3}{3}\left(N_{s,xx}+2h_{,xx}cot(\theta)-\frac{h_{,xxxx}}{Ca}\right) \\ &+\frac{8Re}{15}\left(6h^5h_{,x}{}^2+h^6h_{,xx}\right)=0 \end{aligned}$$

A The governing equations

The undimensionnalised Navier Stokes equation 1 are the Navier Stokes equation in the non-newtonian and incompressible case.

A.1 KBC and mass conservation

The KBC is

$$\frac{Df}{Dt} = \left[\frac{\partial}{\partial t} + (\underline{u}.\underline{\nabla})\right] f = 0 \text{ with } f(x,y,z) = z - h(x,y,t).$$

So it gives

$$h_{,t} = w - uh_{,x} - vh_{,y}.$$

$q_x(x,y,t) = \int_0^{h(x,y,t)} u(x,y,z,t)dz := I_x(h(x,y,t),x,t)$
$q_y(x,y,t) = \int_0^{h(x,y,t)} v(x,y,z,t)dz := I_y(h(x,y,t),x,t)$

So, as u(z=0) = v(z=0) = 0 (No slip),

$q_{x,x} = h_{,x}\partial_1 I_x + 1 \times \partial_2 Ix + 0 = h_{,x}[u(x,y,h,t) - 0] + \int_0^h u_{,x}(x,u,z,t)dz$
$q_{y,y} = h_{,y}\partial_1 I_y + 1 \times \partial_2 Iy + 0 = h_{,y}[v(x,y,h,t) - 0] + \int_0^h v_{,y}(x,u,z,t)dz$

So

$$\underline{\nabla}.\underline{q} = h_{,x}u + h_{,y}v + \int_0^h (u_{,x} + v_{,y})dz.$$

Finally (KBC + Incompressibility), $\underline{\nabla}.\underline{q} = w - h_{,t} - \int_0^h w_{,z}dz = w - h_{,t} - (w(x,y,h,t) - 0)$ so

$$h_{,t} + \underline{\nabla}.\underline{q} = 0.$$

A.2 Dynamic boundary conditions

Let us proove the Dynamic Boundary Conditions 3 and 4 in the 3D case and then return to the 2D one :

According to Kaliadasis book :

$$(\underline{\underline{\sigma_l}} - \underline{\underline{\sigma_g}}).\underline{n} = \gamma\kappa\underline{n} \tag{13}$$

with $\kappa = \underline{\nabla}.\underline{n}$ and $\underline{n} = \frac{1}{\sqrt{1+h_{,x}^2+h_{,y}^2}}(-h_{,x}, -h_{,y}, 1)$ the normal points<u>the gas from the liquid</u> which is the convention we're using in this document. Let's be carefull about that convention and somme potential sign errors as some paper takes the other direction (i.e gas->liquid) for the normal.

Hence, with the notation we have introduced in the paper,

$$\underline{\underline{\sigma_l}}.\underline{n} + \underline{s_g} = \gamma\kappa\underline{n}.$$

We define

$$N_s = \underline{s_g}.\underline{n}, \quad T_{si} = \underline{s_g}.\underline{t_i}, i \in \{x,y\}$$

with

$$\underline{t_x} = \frac{1}{\sqrt{1+h_{,x}^2}}(1,0,h_{,x}), \quad \underline{t_y} = \frac{1}{\sqrt{1+h_{,y}^2}}(0,1,h_{,y})$$

$(\underline{n}, \underline{t_x}, \underline{t_y})$ is normalized, orthogonal and direct (simple computation). The convention taken in Radu's paper are the same but with a different direction of $\underline{\mathrm{n}}$ so it wasn't direct in that case.

13.$\underline{\mathrm{n}}$ gives :

$$\begin{aligned}
\gamma\kappa|\underline{n}|^2 &= \underline{n}.\underline{\underline{\sigma_l}}.\underline{n} + \underline{n}.\underline{s_g} \\
&= \underline{n}.\left(-p_l\underline{\underline{I}} + \mu_l(\underline{\underline{\nabla}}^T u_l + \underline{\underline{\nabla}} u_l)\right).\underline{n} + N_s \\
&\overset{(u_l=u_g \text{ at } y=h)}{=} \frac{1}{1+h_{,x}^2+h_{,y}^2}(-h_{,x},-h_{,y},1)\left[-p_l\underline{\underline{I}} + \mu_l \begin{pmatrix} 2u_{,x} & u_{,y}+v_{,x} & u_{,z}+w_{,x} \\ \cdot & 2v_{,y} & v_{,z}+w_{,y} \\ \cdot & \cdot & w_{,z} \end{pmatrix}\right](-h_{,x},-h_{,y},1)^T \\
&+ N_s
\end{aligned}$$

And (cf Kaliadasis' book) :

$$\gamma\kappa|\underline{n}|^2 = \gamma\kappa = \frac{\gamma}{(1+h_{,y}^2+h_{,x}^2)^{3/2}}\left[h_{,yy}(1+h_{,x}^2) + h_{,xx}(1+h_{,y}^2) - 2h_{,y}h_{,x}h_{,yx}\right].$$

Hence (after non dimensionnalisation...)

$$\frac{\mu_l U_l}{h_l}\left[-p_l + \frac{2}{1+h_{,x}^2+h_{,y}^2}[\dots] + N_s\right] = \frac{\gamma}{h_l}\frac{1}{(1+h_{,y}^2+h_{,x}^2)^{3/2}}[\dots]$$

which gives finally, in 3D :

$$\begin{aligned}
-p_l + \frac{2}{1+h_{,x}^2+h_{,y}^2}&[h_{,x}^2 u_x + h_{,y}^2 + h_{,y}^2 v_y + w_{,z} + h_{,x}h_{,y}(v_{,x}+u_{,y}) - h_{,x}(w_{,x}+u_{,z}) - h_{,y}(w_{,y}+v_{,z})] \\
+ N_s = \frac{1}{Ca}&\frac{1}{(1+h_{,y}^2+h_{,x}^2)^{3/2}}[h_{,yy}(1+h_{,x}^2) + h_{,xx}(1+h_{,y}^2) - 2h_{,x}h_{,y}h_{,yx}].
\end{aligned} \tag{14}$$

In 2D (x<->x, y<->z, and y ignored) we retrieve 10 i.e :

$$-p_l + \frac{2}{1+h_{,x}{}^2}(u_{,x}h_{,x}{}^2 - h_{,x}(u_{,y}+v_{,x}) + v_{,y}) = \frac{1}{Ca}\frac{h_{,xx}}{(1+h_{,x}{}^2)^{3/2}} - N_s.$$

Tangential Case 13.$\underline{t_x}$ gives :

$$\underline{t_x}.(\underline{\underline{\sigma_l}} + \underline{s_g}).\underline{n} = \underline{t_x}.\underline{\underline{\sigma_l}}.\underline{n} + T_{sx} = \gamma\kappa\underline{t_x}.\underline{n} = 0 \quad \text{(orthogonal)}$$

$$0 = \frac{1}{\sqrt{1+h_{,x}^2+h_{,y}^2}\sqrt{1+h_{,x}^2}}(1,0,h_{,x})\left[-p_l\underline{\underline{I}} + \mu_l \begin{pmatrix} 2u_{,x} & u_{,y}+v_{,x} & u_{,z}+w_{,x} \\ \cdot & 2v_{,y} & v_{,z}+w_{,y} \\ \cdot & \cdot & w_{,z} \end{pmatrix}\right](-h_{,x},-h_{,y},1)^T + T_{sx}$$

So (undimensionnalisation & computation..)

$$\frac{1}{\sqrt{1+h_{,x}^2+h_{,y}^2}\sqrt{1+h_{,x}^2}}\left[2h_{,x}(w_{,z}-u_{,x})-h_{,y}(u_{,y}+v_{,x})\right.$$
$$\left.+(1-h_{,x}^2)(u_{,z}+w_{,x})-h_{,x}h_{,y}(v_{,z}+w_{,y})\right]=-T_{sx}.$$

Hence, in 2D, we retrieve 11 i.e :

$$\frac{(v_{,x}+u_{,y})(1-{h_{,x}}^2)+2h_{,x}(v_{,y}-u_{,x})}{1+h_{,x}^2}=-T_s.$$

It's the same for $-T_{sy}$ with some indices swapping. Let us underline that the choice of the direction of $\underline{n}$ can change the $\pm$ before the T_{sx} if we do not always take an direct orthonormal triplet (N_s, T_{sx}, T_{sy}). It's thus important to watch out to the significance these quantities (what does that mean if it's >0, <0).

B Derivation of the reduced order models

B.1 Benney equation

We present here only the scaling and the difference between the uncontrolled and the normal external pressure controlled cases. For more details, see the paper of "Falling liquid films with blowing and suction" from Alice.T and al. or the calculus on paper that I made.

The chosen scaling is

$$X=\epsilon x,\quad T=\epsilon t,\quad v=\epsilon w,\quad Ca=\epsilon^2\hat{Ca}.$$

The order 0 for the pressure are given by the y-axis projection of the NS equation and the tangential KBC. We take $Ca = O(\epsilon^2)$ and N_s of the same order as the liquid pressure p_l so that surface and external pressure effects appears in the order 0 pressure of the liquid. In the 0-order pressure is given by

$$p_{l0}=N_s+2(h-y)cot(\theta)-\frac{h_{,XX}}{\hat{Ca}}$$

with the scaled variables. The case without airjets is when $N_s = 0$.