SURFACE INTEGRAL

Definition: A **surface integral** is a type of integral used to calculate the total accumulation of a quantity (like flux, mass, or heat) across a surface in three-dimensional space. It generalizes the concept of a double integral to curved or flat surfaces.

Key Insights:

- **For Scalar Fields**: It measures the "total amount" of a scalar quantity (e.g., mass or charge density) over a surface.
- **For Vector Fields**: It computes the flux of a vector field (e.g., fluid flow, electric field) passing through a surface.

Easy Words:

A **surface integral** adds up quantities (like flow or heat) over a surface, taking into account its shape and orientation.

Real-Life Examples:

1. **Scalar Field Example**:
   - Calculating the mass of a curved metal sheet when its density varies across the surface.
2. **Vector Field Example**:
   - Finding the amount of air flowing through a curved surface (like the surface of a balloon) when subjected to a wind field.

Formula for Surface Integral:

1. **For Scalar Fields**:

$$\iint_S f(x,y,z)\,dS$$

Where $f(x,y,z)$ is the scalar field, and dS represents an infinitesimal piece of the surface.

2. **For Vector Fields (Flux)**:

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,dS$$

Where $\mathbf{F}$ is the vector field, $\mathbf{n}$ is the unit normal vector to the surface, and dS is the infinitesimal surface area.

Numerical Example:

Vector Field Flux Example:

Find the flux of the vector field $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ across the unit sphere $x^2 + y^2 + z^2 = 1$.

Solution:

1. **Surface Normal**: For a sphere, the outward unit normal vector is $\mathbf{n} = \frac{\mathbf{r}}{|\mathbf{r}|} = \mathbf{r}$ since $|\mathbf{r}| = 1$ on the unit sphere.
2. **Dot Product**: $\mathbf{F} \cdot \mathbf{n} = (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) \cdot (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) = x^2 + y^2 + z^2$.
3. **Simplify**: On the unit sphere, $x^2 + y^2 + z^2 = 1$.
4. **Integral**: $\iint_S 1 \, dS = \text{Surface area of the sphere} = 4\pi$.

Flux = 4π.

Analogy:

Think of the **surface integral** as measuring:

- **Scalar field**: Painting a wall where the paint thickness varies. The integral calculates the total paint used.
- **Vector field**: Air blowing through the wall. The integral calculates how much air passes through it.