Formulas. / Key concepts

(i) $F = ma$, $F = \Delta P / \Delta t$, $\Rightarrow F = m(\Delta v)/\Delta t$.

(ii) $\tan\theta = \frac{v^2}{rg} \Rightarrow v = \sqrt{\tan\theta\, rg}$

$\therefore \frac{N\sin\theta}{N\cos\theta} = \frac{\frac{mv^2}{r}}{\frac{mg}{1}} \Rightarrow \tan\theta = \frac{v^2}{rg}$

(iii) Rotors Concept: Objects in a spinning cylinder are held against the wall by friction forces.

- $N = \frac{mv^2}{r}$ (centripetal).
- friction opposes gravity $f = \mu_s N$. coefficient of friction
- Condition for staying on the wall.

$$\mu_s \geq \frac{gr}{v^2}$$

(iv) Simple Accelerometer:

- Concept: A pendulum tilts in response to the acceleration of its system

$$a = g\tan\theta.$$

- Derivation

Force components on the pendulum.

- Vertical $T\cos\theta = mg$
- Horizontal $T\sin\theta = ma$.

$$\frac{T\sin\theta}{T\cos\theta} = \frac{ma}{mg} \Rightarrow \tan\theta = \frac{a}{g}$$

Friction Forces

Static Friction: Prevents motions, $F_s \leq \mu_s N$

Kinetic Friction: Opposes motion, $F_K = \mu_K N$

A block of mass 10kg slides with $F_K = 20N$ and $N = 98N$.

$$F_K = \mu N$$

$$20 = \mu \; 98N$$

$$\mu = \frac{20}{98}$$

$$\mu \approx 0.2$$

(vi) Work-Energy theorem.

The work done on a system is equal to the change in its kinetic energy.

$$W = \Delta K.E = \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2$$

- If W is positive, kinetic energy increases (object speeds up)
- If W is negative, kinetic energy decreases (object slows down)

- A 2kg object initially moving at 3 m/s is accelerated to 5 m/s. Find work done

$$W = \Delta K.E = \frac{1}{2}(2)(5)^2 - \frac{1}{2}(2)(3^2)$$

$$W = 25 - 9 = 16J$$

(7) Conservation of Energy

Statement: The total mechanical energy (K.E + P.E) remains constant in the absence of non-conservative forces

- $K.E = \frac{1}{2} mv^2$
- $P.E = mgh$ (gravitational

→ A roller coaster descends from a height of 20 m (initial speed 0 m/s). Find its speed at h = 10 m.

Solution,

$$mgh_1 + \frac{1}{2} m v_i^2 = mgh_2 + \frac{1}{2} mv_2^2$$

$$\cancel{m}(gh_1 + \frac{1}{2} v_i^2) = \cancel{m}(gh_2 + \frac{1}{2} v_2^2)$$

$\therefore gh_1 = gh_2 + \frac{1}{2} v_2^2$

(8) ANGULAR MOMENTUM:

$L = I\omega$

$\therefore \omega$ = angular velocity

$\therefore I$ = moment of inert.

Angular momentum is conserved in isolated systems

→ A skater pulls her arms in, reducing I by half. If initial ω was 2 rad/s, find the new ω.

$L_{(initial)} = L_{(final)}$

$I_1 \omega_1 = I_2 \omega_2$

$1 \cdot (2) = \frac{1}{2} \omega_2$ ∴ final I is half of initial I

$\omega_2 = 4 \text{ rad/s}$

9) Center of Mass

Formula: For Discrete particles

$$x_{cm} = \frac{\sum m_i x_i}{\sum m_i}$$

→ Two masses $m_1 = 3kg$ at $x_1 = 2m$ and $m_2 = 2kg$ at $x_2 = 4m$.

$$x_{cm} = \frac{3(2) + 2(4)}{3+2}$$

$$= \frac{14}{5} = 2.8m$$

(10) Rotational Inertia (moment of inertia) (I)

- An objects resistance to changes in its rotational motion
- The farther the mass is distributed from the axis, the higher the rotational inertia.
- I depends on the shape and axis of rotation.

Question

Find the work done by the vector field $\vec{F}$
$\vec{F} = x^2\hat{i} + y^2\hat{j}$ along the path $\vec{r}(t) = t\hat{i} + t^2\hat{j}$
where $t \in [0, 1]$

Solution:

$\Rightarrow W = \int_C \vec{F} \cdot d\vec{r}$ ← Line integral formula

$\Rightarrow d\vec{r} = \frac{d\vec{r}}{dt} dt = (1\hat{i} + 2t\hat{j})\, dt$ $\because \vec{r}(t) = t\hat{i} + t^2\hat{j}$

Evaluate $\vec{F} \cdot d\vec{r}$

$\Rightarrow \vec{F}(t) = t^2\hat{i} + (t^2)^2\hat{j} = t^2\hat{i} + t^4\hat{j}$

$\Rightarrow \vec{F} \cdot d\vec{r} = t^2(1) + (t^4)(2t)$
$= t^2 + 2t^5$

Integrate:

$$W = \int_0^1 (t^2 + 2t^5)\, dt$$

$$= \int_0^1 t^2\, dt + \int_0^1 2t^5\, dt$$

$$\int_0^1 t^2\, dt = \left[\frac{t^3}{3}\right]_0^1 = \frac{1^3}{3} - \frac{0}{3} = \frac{1}{3}$$

First term

Second term

$$2\int_0^1 t^5\,dt = 2\left[\frac{t^6}{6}\right]_0^1$$

$$= 2\left(\frac{(1)^6}{6} - \frac{0^6}{6}\right)$$

$$= \cancel{2}1\left(\frac{1}{\cancel{6}3} - 0\right)$$

$$= \frac{1}{3}$$

Adding the results

$$w = \frac{1}{3} + \frac{1}{3}$$

$$\boxed{w = \frac{2}{3}}$$

BICREDA

Determine if the ~~vector~~ scalar field $F(x,y,z) = \nabla\phi$ where $\phi(x,y,z) = x^3 - 3y^2 + z^2$, is irrotational by computing its ~~divergence~~ curl

Solution:-

~~Formula~~: using F.

$$F(x,y,z) = \nabla\phi = \frac{\partial\phi}{\partial x} i + \frac{\partial\phi}{\partial y} j + \frac{\partial\phi}{\partial z} k$$

- Compute each component of F:

Finding F_x:

$$F_x = \frac{\partial\phi}{\partial x} = \frac{\partial(x^3 - 3y^2 + z^2)}{\partial x} = 3x^2$$

Partial derivative of ϕ with respect to x

- Finding F_y:

Partial Derivative of ϕ with respect to y

$$F_y = \frac{\partial\phi}{\partial y} = \frac{\partial(x^3 - 3y^2 + z^2)}{\partial x}$$

$$= -6y$$

- Finding F_z:

Partial Derivative of ϕ with respect to z. and treating x and y as constants

$$F_z = \frac{\partial\phi}{\partial z} = \frac{\partial(x^3 - 3y^2 + z^2)}{\partial z} = 2z$$

→ Now combining all of these into vector field

$$F(x,y,z) = F_x\hat{i} + F_y\hat{j} + F_z k.$$

$$F(x,y,z) = 3x^2\hat{i} - 6y\hat{j} + 2z k$$

Points to understand:
when we are given $\phi(x,y,z)$, its called a SCALAR FIELD. If we want to find the vector field $F(x,y,z)$ that comes from ϕ, we use the gradient of ϕ;

$$F(x,y,z) = \nabla\phi$$

This means we've to calculate partial derivatives of ϕ with respect to x,y,z and then put into the components F_x, F_y, F_z

→ To Find the rotation of this Vector we have to find the curl first

- Given $F = 3x^2\hat{i} - 6y\hat{j} + 2z k$

$$\nabla \times F = \begin{vmatrix} \hat{i} & \hat{j} & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3x^2 & -6y & 2z \end{vmatrix}$$

Expanding this determinant

$$= i\left(\frac{\partial(2z)}{\partial y} - \frac{\partial(-6y)}{\partial z}\right) - j\left(\frac{\partial(2z)}{\partial x} - \frac{\partial(3x^2)}{\partial x}\right)$$

$$+ k\left(\frac{\partial(-6y)}{\partial x} - \frac{\partial(3x^2)}{\partial y}\right)$$

For i:-

$$\frac{\partial(2z)}{\partial y} - 0 = 0 - 0 = 0$$

For j:

$$-\left(\frac{\partial(2z)}{\partial x} - \frac{\partial(3x^2)}{\partial x}\right)$$

$$= 0 - 0 = 0$$

For k:-

$$\frac{\partial(-6y)}{\partial x} - \frac{\partial(3x^2)}{\partial y} = 0 - 0 = 0$$

Result

$$\nabla \times F = 0.$$

Since the curl is zero, the vector Field F is irrotational

In the above all the solution, we were given a scalar field. We converted it into vector field by Gradient. Then we got the vector field, we had to find the rotation. To find the rotation, we calculated CURL.

EXAMPLE:

FIND THE divergence of a vector field in three dimensions. $F(x,y,z) = x^2 i + 2z j - y k$

Solution:

$$F(x,y,z) = x^2 i + 2z j - y k$$

As we know that,

$$\Rightarrow \quad \nabla \cdot F(x,y,z) = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$$\Rightarrow \quad \nabla \cdot F(x,y,z) = \frac{\partial (x^2)}{\partial x} + \frac{\partial (2z)}{\partial y} + \frac{\partial (-y)}{\partial z}$$

$$\nabla \cdot F = 2x + 0 + 0$$

$$\nabla \cdot F(x,y,z) = 2x$$

Q: Find the divergence of vector field
$F(x,y,z) = (yz)\hat{i} + (xz)\hat{j} + (xy)\hat{k}$.

Solutions -

Divergence Formula = $\nabla \cdot F = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$

Given

$F_x = yz$

$F_y = xz$

$F_z = xy$

$$\nabla \cdot F = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

$$= \frac{yz}{\partial x} + \frac{xz}{\partial y} + \frac{xy}{\partial z}$$

$$= 0 + 0 + 0$$

$$\nabla \cdot F = 0$$

The divergence being zero means that this vector field is divergence-free; There are No Sources or sinks

→ Relationship between Gradient, Divergence and Curl

1) Gradient acts on ~~vector~~ scalar fields to create a vector field:

$$\nabla F = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$$

∇f

2) Divergence acts on vector fields to create a scalar field.

$$\nabla \cdot F = \left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}\right)$$

$\nabla \cdot F$

direction is perpendicular

3) Curl acts on vector fields to create another vector field.

$\nabla \times F$.

$$\nabla F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}$$

→ Real world Analogy.

Gradient:
shows ~~Fells~~ you where to walk uphill and how steep it will be

Divergence: example
Checks if a balloon is at a point is inflating (positive divergence) or deflating (negative divergence)

Curl:
Determines if there's swirling motion.

APPLIED PHYSICS

- BANKING OF THE ROAD.

→ weight (mg) vertically downward
→ Normal force (F) . perp to the road
→ F_c : horizontally downward

$$\sum F_x = n \sin B = m a_{rad}$$

(1) The road is banked at an angel angle θ to the horizontal.

(2) velocity = v.

(3) ~~Negligible~~ No reliance on friction

(4)

The gravitational force mg acting vertically

→ Vertical equilibrium:

$$n \cos\theta = mg.$$

→ For horizontal motion:

$$n \sin\theta = \frac{mv^2}{r}$$

Derivation

$$\frac{N \sin\theta}{n \cos\theta} = \frac{\frac{mv^2}{r}}{\frac{mg}{1}} \Rightarrow \frac{\cancel{m}v^2}{r\cancel{m}g}$$

$$\boxed{\tan\theta = \frac{v^2}{rg}}$$

$$\theta = \tan^{-1}\left(\frac{v^2}{rg}\right).$$