LINE INTEGRAL

Definition: A **line integral** is a type of integral where a scalar or vector field is integrated along a curve. It calculates the accumulation of a quantity (such as work, mass, or heat) along a path in space.

Key Insights:

- **For Scalar Fields**: It measures the total value of the scalar field along the curve, weighted by the curve's length.
- **For Vector Fields**: It computes the work done by the vector field (e.g., force) along the path of the curve.

Easy Words:

A **line integral** adds up the values of a function (like force or temperature) along a path, taking into account the direction and length of the path.

Line Integrals

Definition:

A **line integral** (or path integral) is the integral of a function along a curve. If the function is a scalar function, it's a scalar line integral, and if the function is a vector field, it's a vector line integral.

In simple terms, a line integral measures how much a field (scalar or vector) accumulates along a path. For example, the work done by a force field when an object moves along a path can be computed using a line integral.

Formula:

For a **scalar function** $f(x, y, z)$ over a curve C:

$$\int_C f(x, y, z)\, ds$$

For a **vector field** $\mathbf{F} = (F_x, F_y, F_z)$, the line integral is:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C F_x\, dx + F_y\, dy + F_z\, dz$$

where $d\mathbf{r} = (dx, dy, dz)$ is the differential displacement along the curve C.

Real-Life Example:

- **Work Done by a Force:**
  If a force $\mathbf{F} = (F_x, F_y, F_z)$ acts on a particle and the particle moves along a path C, the work done is given by the line integral:

$$W = \int_C \mathbf{F} \cdot d\mathbf{r}$$

Numerical Example:

Let's calculate the work done by the force $\mathbf{F}(x, y) = (x^2, y^2)$ along the path $r(t) = (t, t^2)$ from $t = 0$ to $t = 1$.

Solution:

1. Parameterize the path $r(t) = (t, t^2)$.

   The velocity vector (derivative of position) is $\mathbf{r}'(t) = (1, 2t)$.

2. Compute the vector field at the path:

   $\mathbf{F}(r(t)) = (t^2, (t^2)^2) = (t^2, t^4)$.

3. Take the dot product of $\mathbf{F}(r(t))$ and $\mathbf{r}'(t)$:

$$\mathbf{F}(r(t)) \cdot \mathbf{r}'(t) = (t^2)(1) + (t^4)(2t) = t^2 + 2t^5$$

4. Now, integrate this expression from 0 to 1:

$$W = \int_0^1 (t^2 + 2t^5)\, dt = \left[\frac{t^3}{3} + \frac{t^6}{3}\right]_0^1 = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$$

So, the work done is $\frac{2}{3}$.

Explanation:

Problem:

Calculate the work done by the force $\mathbf{F}(x, y) = (x^2, y^2)$ along the path $r(t) = (t, t^2)$ from $t = 0$ to $t = 1$.

Step-by-Step Solution:

Step 1: Parameterize the curve

We are given a path $r(t)$, which is defined as:

$$r(t) = (t, t^2)$$

This represents a curve in 2D, where t is the parameter, and the points on the curve are (t, t^2). The curve starts at $t = 0$ (which gives the point $(0, 0)$) and ends at $t = 1$ (which gives the point $(1, 1)$).

Step 2: Find the velocity vector

The velocity vector is the derivative of the position vector $r(t)$, which tells us how the position is changing at each point. So, we compute:

$$r'(t) = \frac{d}{dt}(t, t^2) = (1, 2t)$$

This gives the rate of change of the position along the curve.

Step 3: Evaluate the force field at each point on the curve

We are also given the force field $\mathbf{F}(x, y) = (x^2, y^2)$. To evaluate the force field along the curve, we substitute the expressions for x and y from the parameterization $r(t)$:

$$\mathbf{F}(r(t)) = (x^2, y^2) = (t^2, (t^2)^2) = (t^2, t^4)$$

This is the value of the force at any point along the curve.

Step 4: Take the dot product of $\mathbf{F}(r(t))$ and $r'(t)$

Next, we compute the dot product of the force field $\mathbf{F}(r(t)) = (t^2, t^4)$ and the velocity vector $r'(t) = (1, 2t)$. The dot product is:

$$\mathbf{F}(r(t)) \cdot r'(t) = (t^2)(1) + (t^4)(2t) = t^2 + 2t^5$$

This expression represents the work done per unit length along the curve.

Step 5: Integrate this expression along the curve

Now, we integrate the dot product from $t = 0$ to $t = 1$ to find the total work done:

$$W = \int_0^1 (t^2 + 2t^5)\, dt$$

Breaking this down:

$$W = \int_0^1 t^2 \, dt + \int_0^1 2t^5 \, dt$$

Now, compute the integrals:

$$\int_0^1 t^2 \, dt = \left[\frac{t^3}{3}\right]_0^1 = \frac{1}{3}$$

$$\int_0^1 2t^5 \, dt = \left[\frac{t^6}{3}\right]_0^1 = \frac{1}{3}$$

Thus, the total work done is:

$$W = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$$

Conclusion:

The work done by the force field $\mathbf{F}(x, y) = (x^2, y^2)$ along the curve $r(t) = (t, t^2)$ from $t = 0$ to $t = 1$ is $\frac{2}{3}$.

Key Concepts Recap:

- A **line integral** measures the accumulation of a quantity (e.g., work, energy) along a path or curve.
- To solve it, we:
  1. Parametrize the curve.
  2. Find the derivative of the position (velocity vector).
  3. Evaluate the force field at each point on the curve.
  4. Take the dot product of the force and velocity.
  5. Integrate the result over the range of the curve.