LINE INTEGRAL

1. Line Integral

What It Is

A **line integral** is a way to add up a function along a curve or path in space.

Think of walking along a path, and at every step, you measure something (like force, heat, or elevation). The line integral sums these measurements along the whole path.

Mathematical Definition

For a scalar function $f(x, y, z)$:

$$\int_C f(x, y, z)\, ds$$

- C: The curve or path.
- ds: A tiny piece of the curve (a differential length).

For a vector field $\mathbf{F}(x, y, z)$:

$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

- $d\mathbf{r}$: A small displacement vector along the path.
- $\mathbf{F} \cdot d\mathbf{r}$: The dot product (projects the vector field onto the path).

Example

Imagine a person pulling a sled along a path C with a force $\mathbf{F}$. The work done is:

$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

If the force is $\mathbf{F} = (2x, y, 0)$, and the path C is along $y = x^2$, you compute the integral along the curve.

Let's carefully work through this **line integral** step by step to find the work done by the vector field $\mathbf{F} = (x^2, y^2, 0)$ along the given path $\mathbf{r}(t) = (t, t^2, 0)$ for $t \in [0, 1]$.

Step 1: Line Integral Formula

The work done is given by the line integral:

$$\int_C \mathbf{F} \cdot d\mathbf{r}.$$

Here:

- $\mathbf{F} = (x^2, y^2, 0)$ is the vector field.
- $\mathbf{r}(t) = (t, t^2, 0)$ is the parameterized path.
- $d\mathbf{r}$ is the infinitesimal displacement along the path.

Step 2: Find $d\mathbf{r}$

The position vector is given by $\mathbf{r}(t) = (t, t^2, 0)$.

Now, differentiate $\mathbf{r}(t)$ with respect to t to get:

$$\frac{d\mathbf{r}}{dt} = \mathbf{r}'(t) = (1, 2t, 0).$$

This gives:

$$d\mathbf{r} = \mathbf{r}'(t)\, dt = (1, 2t, 0)\, dt.$$

Step 3: Substitute $\mathbf{r}(t)$ into $\mathbf{F}$

The vector field $\mathbf{F} = (x^2, y^2, 0)$ depends on x and y. Along the path, $x = t$ and $y = t^2$, so:

$$\mathbf{F}(\mathbf{r}(t)) = (x^2, y^2, 0) = (t^2, (t^2)^2, 0) = (t^2, t^4, 0).$$

Step 4: Compute $\mathbf{F} \cdot d\mathbf{r}$

Take the dot product of $\mathbf{F}(\mathbf{r}(t)) = (t^2, t^4, 0)$ and $\mathbf{r}'(t) = (1, 2t, 0)$:

$$\mathbf{F} \cdot d\mathbf{r} = (t^2, t^4, 0) \cdot (1, 2t, 0).$$

Using the dot product formula:

$$\mathbf{F} \cdot d\mathbf{r} = t^2(1) + t^4(2t) + 0(0) = t^2 + 2t^5.$$

Thus, the line integral becomes:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 (t^2 + 2t^5)\, dt.$$

Step 5: Solve the Integral

Separate the integral into two parts:

$$\int_0^1 (t^2 + 2t^5)\, dt = \int_0^1 t^2\, dt + \int_0^1 2t^5\, dt.$$

First Integral: $\int_0^1 t^2\, dt$

Use the power rule for integration ($\int t^n\, dt = \frac{t^{n+1}}{n+1}$):

$$\int_0^1 t^2\, dt = \left[\frac{t^3}{3}\right]_0^1 = \frac{1^3}{3} - \frac{0^3}{3} = \frac{1}{3}.$$

Second Integral: $\int_0^1 2t^5\, dt$

Factor out the constant 2 and use the power rule:

$$\int_0^1 2t^5\, dt = 2\int_0^1 t^5\, dt = 2\left[\frac{t^6}{6}\right]_0^1 = 2 \cdot \left(\frac{1^6}{6} - \frac{0^6}{6}\right) = 2 \cdot \frac{1}{6} = \frac{2}{6} = \frac{1}{3}.$$