The Chain Rule

Simple Definition: The **Chain Rule** is a method in calculus used to differentiate a function that is made up of one function inside another (a *composite function*). It allows us to break things down step by step. Here's how it works in the simplest way:

Step 1: What is a Composite Function?

A composite function is when one function is "inside" another, like this:

f(g(x))

Example:

If **f(u) = u^2** and **g(x) = 3x + 1**, then **f(g(x)) = $(3x + 1)^2$**.

Step 2: Goal of the Chain Rule

We want to differentiate the composite function **f(g(x))**. Instead of doing it all at once, the Chain Rule helps us by splitting the problem into two parts:

1. Differentiate the **outer function** (f).
2. Differentiate the **inner function** (g).

Step 3: The Formula

The Chain Rule says:

If y = f(g(x)), then dy/dx = f'(g(x)) × g'(x)

1. Differentiate the outer function **f** but leave the inner function **g(x)** as it is.
2. Multiply by the derivative of the inner function **g(x)**.

Step 4: Example

Let's apply the rule step by step.

Function: **y = (3x + 1)^2**
Here, the outer function is **f(u) = u^2**, and the inner function is **g(x) = 3x + 1**.

1. **Differentiate the outer function**:
   f'(u) = 2u
   (Replace **u** with **g(x)** later.)
2. **Differentiate the inner function**:
   g'(x) = 3
3. **Combine using the Chain Rule**:
   dy/dx = f'(g(x)) × g'(x)
   dy/dx = 2(3x + 1) × 3
4. **Simplify**:
   dy/dx = 6(3x + 1)

Step 5: Summary

- Break the function into outer and inner parts.
- Differentiate the outer part, keeping the inner part untouched.
- Multiply by the derivative of the inner part.

Example:
Find dw/dt if w = tan x and x = 4t^3 + t

Step 1: Recognize the relationship

We have a composite function:

- $w = \tan(x)$ (outer function).
- $x = 4t^3 + t$ (inner function).

We need to find $\frac{dw}{dt}$, which means differentiating w with respect to t.

Step 2: Apply the Chain Rule

The Chain Rule says:

$$\frac{dw}{dt} = \frac{dw}{dx} \cdot \frac{dx}{dt}.$$

Step 3: Differentiate step by step

1. **Differentiate the outer function $w = \tan(x)$:**

$$\frac{dw}{dx} = \sec^2(x).$$

2. **Differentiate the inner function $x = 4t^3 + t$:**

$$\frac{dx}{dt} = 12t^2 + 1.$$

Step 4: Combine results

Substitute the derivatives into the Chain Rule formula:

$$\frac{dw}{dt} = \sec^2(x) \cdot (12t^2 + 1).$$

Step 5: Substitute x back

Since $x = 4t^3 + t$, the final answer becomes:

$$\frac{dw}{dt} = \sec^2(4t^3 + t) \cdot (12t^2 + 1).$$

Final Answer:

$$\boxed{\frac{dw}{dt} = \sec^2(4t^3 + t) \cdot (12t^2 + 1)}.$$

Now look, the above may look confusing, am I right? But below thing might help you have “my” kind of confusion.

The Big Idea:

We're finding $\frac{dw}{dt}$, but w depends on x, and x depends on t.

So instead of jumping straight from w to t, the **Chain Rule** says we can find $\frac{dw}{dt}$ in two smaller steps:

1. First, figure out how w changes with x (i.e., $\frac{dw}{dx}$).
2. Then figure out how x changes with t (i.e., $\frac{dx}{dt}$).
3. Multiply them:

$$\frac{dw}{dt} = \frac{dw}{dx} \cdot \frac{dx}{dt}.$$

Why Does This Work?

Imagine w, x, and t as gears in a machine:

- When you turn t, it affects x (how fast x moves depends on $\frac{dx}{dt}$).
- Then, x affects w (how fast w moves depends on $\frac{dw}{dx}$).

By multiplying the two rates ($\frac{dw}{dx}$ and $\frac{dx}{dt}$), you get the overall rate of change of w with respect to t ($\frac{dw}{dt}$).

Step 2 explanation

Using Step 2 in This Problem

1. **Outer function**: Start with $w = \tan(x)$:
   - The rate of change of w with respect to x is $\frac{dw}{dx} = \sec^2(x)$.
   - This tells us how w responds to changes in x.
2. **Inner function**: Now, $x = 4t^3 + t$:
   - The rate of change of x with respect to t is $\frac{dx}{dt} = 12t^2 + 1$.
   - This tells us how x responds to changes in t.
3. **Combine them**: Since w depends on x, and x depends on t:

$$\frac{dw}{dt} = \frac{dw}{dx} \cdot \frac{dx}{dt}.$$

Substitute the values:

$$\frac{dw}{dt} = \sec^2(x) \cdot (12t^2 + 1).$$

Why Multiply?

Multiplying combines the effect of the two rates:

- $\frac{dw}{dx}$: How fast w changes when x changes.
- $\frac{dx}{dt}$: How fast x changes when t changes.

Together, they tell us how fast w changes as t changes directly.