Common Properties Of Scalar And Vector Triple Products

Properties of Scalar Triple Product

1. **Cyclic Property**:

$$\vec{A} \cdot (\vec{B} \times \vec{C}) = \vec{B} \cdot (\vec{C} \times \vec{A}) = \vec{C} \cdot (\vec{A} \times \vec{B})$$

- **Explanation:** The scalar triple product is invariant under cyclic permutations of the vectors. This means that no matter how the vectors are arranged, the result of the scalar triple product remains the same. It is a key property in simplifying calculations.

2. **Antisymmetry with Respect to Exchange of Two Vectors**:

$$\vec{A} \cdot (\vec{B} \times \vec{C}) = -\vec{A} \cdot (\vec{C} \times \vec{B})$$

- **Explanation:** If we swap two vectors in the scalar triple product, the sign of the result changes. This property arises because the cross product itself is antisymmetric.

3. **Zero Scalar Triple Product (Coplanar Vectors)**:

$$\vec{A} \cdot (\vec{B} \times \vec{C}) = 0 \quad \text{if the vectors} \quad \vec{A}, \vec{B}, \vec{C} \quad \text{are coplanar.}$$

- **Explanation**: If three vectors are coplanar (lie in the same plane), the scalar triple product is zero. This happens because the volume of the parallelepiped formed by the three vectors is zero when the vectors lie in the same plane.

4. **Scalar Triple Product and Determinant**:

$$\vec{A} \cdot (\vec{B} \times \vec{C}) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$

- **Explanation**: The scalar triple product can be computed as the determinant of a 3x3 matrix formed by the components of the three vectors. This property is useful for direct calculation when the vectors are given in component form.

5. **Distributive Property (Over Addition of Vectors)**:

$$\vec{A} \cdot \left((\vec{B} + \vec{C}) \times \vec{D} \right) = \vec{A} \cdot (\vec{B} \times \vec{D}) + \vec{A} \cdot (\vec{C} \times \vec{D})$$

- **Explanation**: The scalar triple product is distributive over vector addition. This means that the scalar triple product of a vector with the sum of two other vectors is the sum of the scalar triple products with each vector individually.

Properties of Vector Triple Product

1. **BAC-CAB Rule (Vector Triple Product Identity):**

$$\vec{A} \times (\vec{B} \times \vec{C}) = (\vec{A} \cdot \vec{C})\vec{B} - (\vec{A} \cdot \vec{B})\vec{C}$$

   - **Explanation**: This is a key identity used to simplify the vector triple product. It allows the triple product to be expressed as a linear combination of the vectors $\vec{B}$ and $\vec{C}$, making it easier to compute or manipulate in calculations.

2. **Distributive Property (Over Addition of Vectors):**

$$\vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}$$

   - **Explanation:** The vector triple product is distributive over vector addition. This property means that the cross product of a vector with the sum of two vectors is the sum of the cross products of the vector with each of the other two vectors.

3. **Antisymmetry:**

$$\vec{A} \times (\vec{B} \times \vec{C}) = -(\vec{A} \times (\vec{C} \times \vec{B}))$$

   - **Explanation:** The vector triple product is antisymmetric with respect to the order of $\vec{B}$ and $\vec{C}$. If the order of the cross products is reversed, the result changes sign.

4. **Orthogonality:**
   - **Explanation**: The result of the vector triple product $\vec{A} \times (\vec{B} \times \vec{C})$ is always orthogonal to the vector $\vec{A}$, as shown by the **BAC-CAB rule**. The resulting vector lies in the plane formed by $\vec{B}$ and $\vec{C}$ and is perpendicular to $\vec{A}$.

Practice Problems and Hints

1. **Problem 1**: Compute the scalar triple product for vectors $\vec{A} = (1, 2, 3)$, $\vec{B} = (4, 5, 6)$, and $\vec{C} = (7, 8, 9)$.

   Hint: Use the determinant formula for the scalar triple product.

2. **Problem 2**: Show that if $\vec{A}$, $\vec{B}$, and $\vec{C}$ are coplanar, then $\vec{A} \cdot (\vec{B} \times \vec{C}) = 0$.

   Hint: Use the definition of the scalar triple product and the fact that the vectors being coplanar implies their volume is zero.

3. **Problem 3**: Verify the BAC-CAB rule for vectors $\vec{A} = (1, 2, 3)$, $\vec{B} = (4, 5, 6)$, and $\vec{C} = (7, 8, 9)$.

   Hint: Compute both sides of the equation separately: $\vec{A} \times (\vec{B} \times \vec{C})$ and $(\vec{A} \cdot \vec{C})\vec{B} - (\vec{A} \cdot \vec{B})\vec{C}$.

4. **Problem 4**: Prove that the vector $\vec{A} \times (\vec{B} \times \vec{C})$ is orthogonal to $\vec{A}$.