Rolle's Theorem – UPDATE (1) (04 Feb 25)

Simple Definition: If a smooth curve starts and ends at the same height, there's at least one point in between where the curve is flat (horizontal tangent).

(Scroll Down Below For The Update Notice)

Rolle's Theorem: Overview

Definition

Rolle's theorem is a fundamental result in differential calculus that provides a condition under which there exists at least one point on a differentiable curve where the tangent is horizontal (slope = 0).

Statement

Let $f(x)$ be a function that satisfies the following conditions:

1. **Continuous** on the closed interval $[a, b]$.
2. **Differentiable** on the open interval (a, b).
3. $f(a) = f(b)$ (i.e., the function has the same value at the endpoints).

Then, there exists at least one point $c \in (a, b)$ such that:

$$f'(c) = 0$$

In simple terms, if a function starts and ends at the same height and is smooth and continuous, then somewhere in between, its slope (derivative) must be zero.

Definition of Critical Points

Critical points of a function $f(x)$ occur at $x = c$ if:

1. $f'(c) = 0$ (the derivative is zero), or
2. $f'(c)$ is undefined.

Critical points are where the function might have a local maximum, local minimum, or an inflection point.

How to Use Rolle's Theorem

1. Check the three conditions:
   - Continuity on $[a, b]$.
   - Differentiability on (a, b).
   - Equal function values at the endpoints $f(a) = f(b)$.
2. If all conditions are satisfied, find $f'(x)$.
3. Solve $f'(x) = 0$ to find the value(s) of c within (a, b).

Question:

Verify Rolle's Theorem for $f(x) = x^2 - 4x + 4$ on the interval $[2, 4]$.

"Correction Notice: In my notes on Rolle's Theorem, answer was mistakenly considered verified f(x) = x^2 −4x+ 4 for [2,4] even f(a) ≠ f(b). Since f(2) ≠ f(4), Rolle's Theorem does not apply. I've updated the notes accordingly." If you find other problems, kindly reach me out ASAP!

Solution:

Step 1: Check Continuity and Differentiability

Rolle's Theorem states that if a function $f(x)$ satisfies these three conditions:

1. **Continuous** on the closed interval $[a, b]$,
2. **Differentiable** on the open interval (a, b),
3. $f(a) = f(b)$ **(Equal function values at endpoints),**

then there exists some $c \in (a, b)$ such that $f'(c) = 0$.

The given function is:

$$f(x) = x^2 - 4x + 4$$

Since this is a **polynomial**, it is **continuous** and **differentiable** everywhere.

Step 2: Check if $f(a) = f(b)$

We evaluate the function at $x = 2$ and $x = 4$:

$$f(2) = (2)^2 - 4(2) + 4 = 4 - 8 + 4 = 0$$

$$f(4) = (4)^2 - 4(4) + 4 = 16 - 16 + 4 = 4$$

Since $f(2) \neq f(4)$ ($0 \neq 4$), the condition $f(a) = f(b)$ is **not satisfied**.

Conclusion:

Since one of the conditions of Rolle's Theorem is **not met**, **Rolle's Theorem does not apply** to this function on the given interval $[2, 4]$. 🚫

Example 1: Cubic Function

Verify Rolle's theorem for $f(x) = x^3 - 3x^2 + 2x$ on $[0, 2]$.

Solution

1. $f(x)$ is a polynomial, so it is continuous and differentiable everywhere.
2. $f(0) = 0^3 - 3(0)^2 + 2(0) = 0$, and $f(2) = 2^3 - 3(2)^2 + 2(2) = 0$. $f(0) = f(2)$.

Find $f'(x) : f'(x) = 3x^2 - 6x + 2$.
Set $f'(x) = 0$:

$$3x^2 - 6x + 2 = 0 \implies x = \frac{3 \pm \sqrt{3}}{3}.$$

The solutions $x \in (0, 2)$ confirm the theorem.

Example 2: Trigonometric Function

Verify Rolle's theorem for $f(x) = \sin(x)$ on $[0, \pi]$.

Solution

1. $f(x) = \sin(x)$ is continuous and differentiable.
2. $f(0) = \sin(0) = 0$, $f(\pi) = \sin(\pi) = 0$.
   $f(0) = f(\pi)$.

Find $f'(x) : f'(x) = \cos(x)$.
Set $f'(x) = 0$:

$$\cos(x) = 0 \implies x = \frac{\pi}{2}.$$

Since $\frac{\pi}{2} \in (0, \pi)$, the theorem is verified.

Example 3: Real-World Application (Vehicle Motion)

A car's position is modeled as $f(t) = t^3 - 6t^2 + 9t$, where t is time in seconds. Prove that at some moment between $t = 0$ and $t = 3$, the car's velocity is zero.

Solution

1. $f(t)$ is a polynomial, so it is continuous and differentiable.
2. $f(0) = 0^3 - 6(0)^2 + 9(0) = 0$, $f(3) = 3^3 - 6(3)^2 + 9(3) = 0$.
   $f(0) = f(3)$.

Find $f'(t) : f'(t) = 3t^2 - 12t + 9$.
Set $f'(t) = 0$:

$$3t^2 - 12t + 9 = 0 \implies t = 1, 3.$$

Since $t = 1 \in (0, 3)$, the car's velocity is zero at $t = 1$.

Question:

Verify **Rolle's Theorem** for the function $f(x) = x^2 - 4x + 3$ on the interval $[1, 3]$.

Step 1: Check if the conditions of Rolle's Theorem are satisfied.

1. **Continuity on the closed interval** $[1, 3]$:
   The function $f(x) = x^2 - 4x + 3$ is a polynomial, and polynomials are continuous on all real numbers, so it is continuous on $[1, 3]$. ✅
2. **Differentiability on the open interval** $(1, 3)$:
   The derivative of $f(x) = x^2 - 4x + 3$ is $f'(x) = 2x - 4$, which is also a polynomial and therefore differentiable everywhere. ✅
3. $f(1) = f(3)$:

$$f(1) = 1^2 - 4(1) + 3 = 1 - 4 + 3 = 0$$

$$f(3) = 3^2 - 4(3) + 3 = 9 - 12 + 3 = 0$$

So, $f(1) = f(3) = 0$, satisfying the condition that the function values at the endpoints are equal. ✅

Step 2: Find c such that $f'(c) = 0$.

From the derivative $f'(x) = 2x - 4$, we set $f'(x) = 0$:

$$2x - 4 = 0$$

$$2x = 4$$

$$x = 2$$

Thus, the critical point $c = 2$ is in the open interval $(1, 3)$.

Conclusion:

Since all the conditions of **Rolle's Theorem** are satisfied, we can conclude that there **exists a** $c = 2$ **where** $f'(c) = 0$ on the interval $[1, 3]$. ✅

Real-World Applications of Rolle's Theorem

While Rolle’s theorem is primarily a theoretical concept in calculus, its principles indirectly influence various real-world applications:

1. **Physics (Motion Analysis)**
   If a particle starts and ends at the same position over a certain period, Rolle's theorem guarantees that at some moment, the velocity of the particle is zero. This concept is useful in understanding oscillatory motion and equilibrium points.
2. **Economics (Optimization)**
   When analyzing profit or cost functions, if profit starts and ends at the same value over a range of prices or inputs, there must be a point where the marginal profit (derivative) is zero. This helps identify critical points for maximizing or minimizing profit.
3. **Traffic Flow**
   If traffic density at two points along a highway is the same, Rolle's theorem implies that there must be a point in between where the rate of change of traffic density (slope) is zero. This helps in traffic pattern analysis.
4. **Engineering (Structural Analysis)**
   In beam deflection problems, if the deflection of a beam is the same at two points, there must be a point where the slope of deflection is zero. Engineers use this to locate maximum deflection or stress points.
5. **Climate Studies**
   When temperature data is modeled as a continuous function, if temperatures are the same at two times in a day, Rolle’s theorem implies a time between those moments when the rate of temperature change is zero (i.e., temperature reaches a local maximum or minimum).

Key Definitions in a Nutshell:

- **Local Minimum (Lower Minima):**
  A point where the function's value is lower than at nearby points (a "valley").
- **Local Maximum (Upper Maxima):**
  A point where the function's value is higher than at nearby points (a "peak").
- **Global Minimum:**
  The lowest value of the function over its entire domain.
- **Global Maximum:**
  The highest value of the function over its entire domain.
- **Local Extrema:**
  Points where the function has either a local maximum or local minimum.
- **Global Extrema:**
  The highest or lowest points in the entire domain of the function.