Mean Value Theorem (05 Feb, 2025)

Mean Value Theorem (MVT) - Simple Explanation

The **Mean Value Theorem** helps us understand how a function behaves between two points. It connects the average rate of change of the function over an interval to its instantaneous rate of change at some point within that interval.

What You Need:

1. **Continuity:** The function must be continuous on the interval $[a, b]$ (no jumps or breaks).
2. **Differentiability:** The function must be smooth and have a derivative (i.e., no sharp turns) on the open interval (a, b).

The Rule:

If these two conditions are met, then there exists at least one point c between a and b where the slope of the tangent line (instantaneous rate of change) is the same as the slope of the secant line between a and b.

Mathematically:

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

- $f'(c)$: The derivative at point c (instantaneous rate of change).
- $\frac{f(b)-f(a)}{b-a}$: The average rate of change between a and b, which is the slope of the secant line connecting the points $(a, f(a))$ and $(b, f(b))$.

Geometric View:

- **Secant Line**: A line that connects two points on the curve $f(x)$. Its slope is the average rate of change between a and b.
- **Tangent Line**: A line that touches the curve at a single point and has the same slope as the curve at that point (instantaneous rate of change).

The MVT says there's at least one point where the tangent line is parallel to the secant line.

Example:

Let's say we have the function $f(x) = x^2$ on the interval $[1, 4]$.

1. Find the value of f at the endpoints:
   - $f(1) = 1^2 = 1$
   - $f(4) = 4^2 = 16$
2. The average rate of change (slope of the secant line) is:

$$\frac{f(4) - f(1)}{4 - 1} = \frac{16 - 1}{3} = 5$$

3. Now, find the derivative $f'(x) = 2x$.
4. Set $f'(c) = 5$ to find the point where the instantaneous rate of change equals the average rate of change:

$$2c = 5 \quad \Rightarrow \quad c = \frac{5}{2}$$

So, at $c = 2.5$, the slope of the tangent line (instantaneous rate of change) is 5, just like the slope of the secant line.

Key Points:

- **Secant Line**: The line connecting $(a, f(a))$ and $(b, f(b))$.
- **Tangent Line**: The line at a point on the curve with the same slope as the curve at that point.
- **MVT Conclusion**: At least one point in the interval has the same slope as the secant line, where the tangent line is parallel to it.