Implicit Differentiation

Implicit differentiation is a technique used to find the derivative of functions where y is not explicitly written as a function of x. Instead of solving for y in terms of x (or vice versa), we differentiate both sides of the equation **with respect to** x, treating y as a function of x (i.e., $y = y(x)$).

The Key Idea:

When differentiating y, we use the **Chain Rule** because y is considered a function of x. So:

$$\frac{d}{dx}[y] = \frac{dy}{dx}.$$

Steps for Implicit Differentiation:

1. **Differentiate both sides of the equation** with respect to x, applying the usual differentiation rules (product rule, chain rule, etc.).
2. Whenever you differentiate y, write $\frac{dy}{dx}$.
3. Solve for $\frac{dy}{dx}$ if needed.

Example 1: $x^2 + y^2 = 25$ (Equation of a circle)

1. Differentiate both sides with respect to x:

$$\frac{d}{dx}[x^2 + y^2] = \frac{d}{dx}[25].$$

2. Apply the Power Rule and the Chain Rule:

$$2x + 2y\frac{dy}{dx} = 0.$$

3. Solve for $\frac{dy}{dx}$:

$$2y\frac{dy}{dx} = -2x.$$

$$\frac{dy}{dx} = -\frac{x}{y}.$$

Example 2:

Use implicit differentiation to find dy/dx if $5y^2$ + siny = x^2

Step 1: Differentiate both sides with respect to x

We start with:

$$5y^2 + \sin y = x^2.$$

Differentiate both sides with respect to x, remembering that y is a function of x (so apply the Chain Rule when differentiating terms involving y):

$$\frac{d}{dx}[5y^2] + \frac{d}{dx}[\sin y] = \frac{d}{dx}[x^2].$$

Step 2: Differentiate term by term

1. For $5y^2$:
   Use the Chain Rule:

$$\frac{d}{dx}[5y^2] = 10y\frac{dy}{dx}.$$

2. For $\sin y$:
   Use the Chain Rule:

$$\frac{d}{dx}[\sin y] = \cos y\frac{dy}{dx}.$$

The **Chain Rule** in **this context** refers to the idea that y is a function of x, so when we differentiate terms that involve y, we need to account for the fact that y is changing with respect to x

3. For x^2:
   Use the Power Rule:

$$\frac{d}{dx}[x^2] = 2x.$$

So the equation becomes:

$$10y\frac{dy}{dx} + \cos y\frac{dy}{dx} = 2x.$$

Step 3: Factor out $\frac{dy}{dx}$

Group terms involving $\frac{dy}{dx}$:

$$\frac{dy}{dx}(10y + \cos y) = 2x.$$

Step 4: Solve for $\frac{dy}{dx}$

Isolate $\frac{dy}{dx}$:

$$\frac{dy}{dx} = \frac{2x}{10y + \cos y}.$$

Final Answer:

$$\boxed{\frac{dy}{dx} = \frac{2x}{10y + \cos y}}$$

Example 3:

Find the slope of the tangent lines to the curve $y^2 - x + 1 = 0$ at the point (2, −1) and (2, 1)

To find the slope of the tangent lines to the curve at the given points, we need to **implicitly differentiate** the equation $y^2 - x + 1 = 0$ with respect to x, and then plug in the given points to get the slopes.

Step 1: Differentiate the equation implicitly

The equation is:

$$y^2 - x + 1 = 0.$$

Differentiate both sides with respect to x:

1. The derivative of y^2 with respect to x is:

$$\frac{d}{dx}[y^2] = 2y \cdot \frac{dy}{dx} \quad \text{(by the Chain Rule)}.$$

2. The derivative of $-x$ with respect to x is:

$$\frac{d}{dx}[-x] = -1.$$

3. The derivative of 1 is 0.

So, the derivative of the equation $y^2 - x + 1 = 0$ is:

$$2y \cdot \frac{dy}{dx} - 1 = 0.$$

Step 2: Solve for $\frac{dy}{dx}$

Rearranging the equation to solve for $\frac{dy}{dx}$:

$$2y \cdot \frac{dy}{dx} = 1.$$

$$\frac{dy}{dx} = \frac{1}{2y}.$$

Step 3: Find the slopes at the given points

We are given two points: $(2, -1)$ and $(2, 1)$.

For the point $(2, -1)$:

At $y = -1$, substitute into the derivative formula:

$$\frac{dy}{dx} = \frac{1}{2(-1)} = \frac{1}{-2} = -\frac{1}{2}.$$

So, the slope of the tangent line at $(2, -1)$ is $-\frac{1}{2}$.

For the point $(2, 1)$:

At $y = 1$, substitute into the derivative formula:

$$\frac{dy}{dx} = \frac{1}{2(1)} = \frac{1}{2}.$$

So, the slope of the tangent line at $(2, 1)$ is $\frac{1}{2}$.

Final Answer:

- The slope of the tangent line at $(2, -1)$ is $-\frac{1}{2}$.
- The slope of the tangent line at $(2, 1)$ is $\frac{1}{2}$.