Rolle's Theorem

Simple Definition: If a smooth curve starts and ends at the same height, there's at least one point in between where the curve is flat (horizontal tangent).

Rolle's Theorem: Overview

Definition

Rolle's theorem is a fundamental result in differential calculus that provides a condition under which there exists at least one point on a differentiable curve where the tangent is horizontal (slope = 0).

Statement

Let $f(x)$ be a function that satisfies the following conditions:

1. **Continuous** on the closed interval $[a, b]$.
2. **Differentiable** on the open interval (a, b).
3. $f(a) = f(b)$ (i.e., the function has the same value at the endpoints).

Then, there exists at least one point $c \in (a, b)$ such that:

$$f'(c) = 0$$

In simple terms, if a function starts and ends at the same height and is smooth and continuous, then somewhere in between, its slope (derivative) must be zero.

Definition of Critical Points

Critical points of a function $f(x)$ occur at $x = c$ if:

1. $f'(c) = 0$ (the derivative is zero), or
2. $f'(c)$ is undefined.

Critical points are where the function might have a local maximum, local minimum, or an inflection point.

How to Use Rolle's Theorem

1. Check the three conditions:
   - Continuity on $[a, b]$.
   - Differentiability on (a, b).
   - Equal function values at the endpoints $f(a) = f(b)$.
2. If all conditions are satisfied, find $f'(x)$.
3. Solve $f'(x) = 0$ to find the value(s) of c within (a, b).

Example

Problem

Let $f(x) = x^2 - 4x + 4$ on the interval $[2, 4]$. Verify Rolle's theorem and find the value of c.

Solution

1. **Check Conditions:**
   - $f(x)$ is a polynomial, so it is continuous and differentiable on any interval.
   - $f(2) = 2^2 - 4(2) + 4 = 0$, $f(4) = 4^2 - 4(4) + 4 = 0$.
   - $f(2) = f(4)$.

   All conditions are satisfied.

2. **Find $f'(x)$:**
   $f'(x) = 2x - 4$.
3. **Solve $f'(x) = 0$:**
   $2x - 4 = 0$.
   $x = 2$.

Since $c = 3 \in (2, 4)$, Rolle's theorem is verified, and the value of c is 3.

Example 1: Cubic Function

Verify Rolle's theorem for $f(x) = x^3 - 3x^2 + 2x$ on $[0, 2]$.

Solution

1. $f(x)$ is a polynomial, so it is continuous and differentiable everywhere.
2. $f(0) = 0^3 - 3(0)^2 + 2(0) = 0$, and $f(2) = 2^3 - 3(2)^2 + 2(2) = 0$.
   $f(0) = f(2)$.

Find $f'(x)$: $f'(x) = 3x^2 - 6x + 2$.
Set $f'(x) = 0$:

$$3x^2 - 6x + 2 = 0 \implies x = \frac{3 \pm \sqrt{3}}{3}.$$

The solutions $x \in (0, 2)$ confirm the theorem.

Example 2: Trigonometric Function

Verify Rolle's theorem for $f(x) = \sin(x)$ on $[0, \pi]$.

Solution

1. $f(x) = \sin(x)$ is continuous and differentiable.
2. $f(0) = \sin(0) = 0$, $f(\pi) = \sin(\pi) = 0$.
   $f(0) = f(\pi)$.

Find $f'(x)$: $f'(x) = \cos(x)$.
Set $f'(x) = 0$:

$$\cos(x) = 0 \implies x = \frac{\pi}{2}.$$

Since $\frac{\pi}{2} \in (0, \pi)$, the theorem is verified.

Example 3: Real-World Application (Vehicle Motion)

A car's position is modeled as $f(t) = t^3 - 6t^2 + 9t$, where t is time in seconds. Prove that at some moment between $t = 0$ and $t = 3$, the car's velocity is zero.

Solution

1. $f(t)$ is a polynomial, so it is continuous and differentiable.
2. $f(0) = 0^3 - 6(0)^2 + 9(0) = 0$, $f(3) = 3^3 - 6(3)^2 + 9(3) = 0$.
   $f(0) = f(3)$.

Find $f'(t)$: $f'(t) = 3t^2 - 12t + 9$.
Set $f'(t) = 0$:

$$3t^2 - 12t + 9 = 0 \implies t = 1, 3.$$

Since $t = 1 \in (0, 3)$, the car's velocity is zero at $t = 1$.

Prove that the equation $f(x) = x^3 + 3x^2 - 9x + 1$ has exactly one critical point in $[-3, 1]$.

Solution

1. **Derivative:**
   Find $f'(x)$:
   $$f'(x) = 3x^2 + 6x - 9.$$
2. **Solve** $f'(x) = 0$:
   $$3x^2 + 6x - 9 = 0 \implies x^2 + 2x - 3 = 0.$$
   Factorize:
   $$(x + 3)(x - 1) = 0.$$
   Hence, $x = -3$ and $x = 1$.
3. **Critical Points in** $(-3, 1)$:
   Check the endpoints of the interval:
   - At $x = -3$, $f'(x) = 0$ (a critical point).
   - At $x = 1$, $f'(x) = 0$.

Conclusion: The critical points are at $x = -3$ and $x = 1$. Since the derivative changes sign at these points, they represent local extrema.

Real-World Applications of Rolle's Theorem

While Rolle’s theorem is primarily a theoretical concept in calculus, its principles indirectly influence various real-world applications:

1. **Physics (Motion Analysis)**
   If a particle starts and ends at the same position over a certain period, Rolle's theorem guarantees that at some moment, the velocity of the particle is zero. This concept is useful in understanding oscillatory motion and equilibrium points.
2. **Economics (Optimization)**
   When analyzing profit or cost functions, if profit starts and ends at the same value over a range of prices or inputs, there must be a point where the marginal profit (derivative) is zero. This helps identify critical points for maximizing or minimizing profit.
3. **Traffic Flow**
   If traffic density at two points along a highway is the same, Rolle's theorem implies that there must be a point in between where the rate of change of traffic density (slope) is zero. This helps in traffic pattern analysis.
4. **Engineering (Structural Analysis)**
   In beam deflection problems, if the deflection of a beam is the same at two points, there must be a point where the slope of deflection is zero. Engineers use this to locate maximum deflection or stress points.
5. **Climate Studies**
   When temperature data is modeled as a continuous function, if temperatures are the same at two times in a day, Rolle’s theorem implies a time between those moments when the rate of temperature change is zero (i.e., temperature reaches a local maximum or minimum).

Key Definitions in a Nutshell:

- **Local Minimum (Lower Minima):**
  A point where the function's value is lower than at nearby points (a "valley").
- **Local Maximum (Upper Maxima):**
  A point where the function's value is higher than at nearby points (a "peak").
- **Global Minimum:**
  The lowest value of the function over its entire domain.
- **Global Maximum:**
  The highest value of the function over its entire domain.
- **Local Extrema:**
  Points where the function has either a local maximum or local minimum.
- **Global Extrema:**
  The highest or lowest points in the entire domain of the function.