Concepts in Function Analysis

In This document, I have compiled some important concepts used to analyse a function. This document provides a BASIC overview of key concepts in function analysis, including basic knowledge of critical points, intervals of increase and decrease, concavity, and points of inflection.

Main Topics

1. **Increasing and Decreasing Functions**
   - Understanding where a function is rising or falling based on its first derivative.
2. **Concavity**
   - Identifying how a function bends (concave up or concave down) based on the second derivative.
3. **Points of Inflection**
   - Locating where the concavity of a function changes.
4. **Critical Points**
   - Points where the first derivative $f'(x) = 0$ or is undefined.
5. **Second Derivative Test**
   - A method to determine whether critical points are maxima, minima, or points of inflection.
6. **Sign Charts**
   - A systematic way to evaluate the behavior of a function in different intervals.

LET'S BEGIN:

What are Increasing and Decreasing Functions?

1. Imagine you're climbing a hill. If you're going **up**, the slope is **positive**, and the function is **increasing**.
2. When you start going **down**, the slope is **negative**, and the function is **decreasing**.

In math terms:

- **Increasing**: $f'(x) > 0$ (Positive slope).
- **Decreasing**: $f'(x) < 0$ (Negative slope).

What is Concavity?

Concavity tells us how the curve bends:

1. **Concave Up:** The graph looks like a bowl ∪. It's bending upwards.
   - This happens when $f''(x) > 0$ (Second derivative is positive).
2. **Concave Down:** The graph looks like an upside-down bowl ∩. It's bending downwards.
   - This happens when $f''(x) < 0$ (Second derivative is negative).

What are Inflection Points?

An **inflection point** is where the graph **changes concavity**:

- From concave up (∪) to concave down (∩).
- Or from concave down (∩) to concave up (∪).

In math terms:

- Inflection points occur where $f''(x) = 0$ or $f''(x)$ changes sign.

Critical Points: Definition and Procedure

Definition:

A **critical point** of a function $f(x)$ is a point $x = c$ in the domain of $f(x)$ where:

1. $f'(c) = 0$, or
2. $f'(c)$ does not exist.

Critical points are potential locations of:

- Local maxima
- Local minima
- Points of inflection (not always extrema)

How to Find Critical Points:

1. **Find the first derivative:**
   Calculate $f'(x)$.
2. **Solve $f'(x) = 0$:**
   Identify where the derivative equals zero.
3. **Check where $f'(x)$ does not exist (optional):**
   Identify points where $f'(x)$ is undefined (if any).
4. **Confirm the nature of the critical points:**
   Use either the **First Derivative Test** or **Second Derivative Test** to classify the critical point as:
   - Local maximum
   - Local minimum
   - Neither (saddle point or other)

How to Use the Sign Chart

A **sign chart** is like a map that shows the behavior of the function:

1. **Look at $f'(x)$:**
   - $f'(x) > 0$: Function is increasing.
   - $f'(x) < 0$: Function is decreasing.
2. **Look at $f''(x)$:**
   - $f''(x) > 0$: Function is concave up.
   - $f''(x) < 0$: Function is concave down.

Real-World Analogy

Imagine a roller coaster:

1. **Increasing**: You're going up the hill ($f'(x) > 0$).
2. **Decreasing:** You're going down the hill ($f'(x) < 0$).
3. **Concave Up:** The track is curving up ($f''(x) > 0$), so it feels like you're being pushed into your seat.
4. **Concave Down:** The track is curving down ($f''(x) < 0$), so it feels like you're being lifted out of your seat.
5. **Inflection Points:** These are the transitions between curving up and curving down.

SOME OTHER BASIC DEFINITIONS YOU NEED TO KNOW:

Critical Point

A point $x = c$ where $f'(c) = 0$ or $f'(c)$ does not exist. Critical points are candidates for local maxima, minima, or saddle points.

Local Maxima and Minima

- **Local Maximum:** The value of $f(c)$ is greater than the values of $f(x)$ in a small neighborhood around c.
- **Local Minimum:** The value of $f(c)$ is less than the values of $f(x)$ in a small neighborhood around c.

Use the **First Derivative Test** or **Second Derivative Test** to classify these points:

- If $f'(x)$ changes from **positive to negative** at c, it's a local maximum.
- If $f'(x)$ changes from **negative to positive** at c, it's a local minimum.

Absolute Maxima and Minima

- The **highest** or **lowest** value of $f(x)$ on a given interval (including endpoints).

First Derivative Test

The first derivative helps to identify intervals of increase and decrease:

1. Compute $f'(x)$.
2. Find critical points where $f'(x) = 0$ or undefined.
3. Test the sign of $f'(x)$ in intervals around the critical points.

Second Derivative Test

A method to classify critical points:

1. Compute $f''(x)$.
2. At a critical point $x = c$ where $f'(c) = 0$:
   - If $f''(c) > 0$: $f(c)$ is a local minimum.
   - If $f''(c) < 0$: $f(c)$ is a local maximum.
   - If $f''(c) = 0$: The test is inconclusive (check further).

NOW I WANT YOU TO PONDER ON THE ABOVE THING BY ASKING, "WHAT? FIRST DERIVATIVE AND SECOND DERIVATIVE, BOTH ARE USED TO IDENTIFY LOCAL MINIMUM AND LOCAL MAXIMUM?" The answer is yes, but here's how:

1. First Derivative Test:

The **first derivative** $f'(x)$ is used to classify critical points by analyzing the sign change of $f'(x)$ around those points.

Steps:

1. **Find the critical points:** Solve $f'(x) = 0$ or where $f'(x)$ is undefined.
2. **Analyze the sign of $f'(x)$:**
   - If $f'(x)$ changes from **positive to negative** at a critical point, it's a **local maximum**.
   - If $f'(x)$ changes from **negative to positive**, it's a **local minimum**.
   - If $f'(x)$ does not change sign, the point is not a local extremum (it could be a saddle point).

2. Second Derivative Test:

The **second derivative** $f''(x)$ provides another way to classify critical points, but this method relies on the curvature (concavity) at those points.

Steps:

1. **Find the critical points:** Solve $f'(x) = 0$ or undefined.
2. **Check the second derivative at those critical points:**
   - If $f''(x) > 0$, the function is **concave up**, and the critical point is a **local minimum**.
   - If $f''(x) < 0$, the function is **concave down**, and the critical point is a **local maximum**.
   - If $f''(x) = 0$, the test is inconclusive—you need to use the First Derivative Test or further analysis.

Comparison of the Two Methods

Test	Uses	When to Use
First Derivative Test	Analyzes the sign change of $f'(x)$ around the critical point.	Use when you need to understand the function's behavior over an interval.
Second Derivative Test	Uses $f''(x)$ to directly determine whether the critical point is a max or min.	Use when $f''(x)$ is easy to compute or you are focusing only on the critical point.

Key Points to Remember

1. **First Derivative Test:** Based on the **sign change** of $f'(x)$.
2. **Second Derivative Test:** Based on the **concavity** determined by $f''(x)$.
3. **Inconclusive Cases:** If $f''(x) = 0$ at a critical point, the Second Derivative Test fails, and you must fall back to the First Derivative Test.

Okay, Got It, or not?

Look at this chart FOR THE UNDERSTANDING:

Step	What to Check	Purpose
Step 1: Critical Points	$f'(x) = 0$ or $f'(x)$ undefined	Identify possible maxima/minima
Step 2: Intervals of Increase/Decrease	Check the sign of $f'(x)$ in intervals	Determine where $f(x)$ is increasing/decreasing
Step 3: Concavity	$f''(x) > 0$ (concave up), $f''(x) < 0$ (concave down)	Understand the shape of the curve
Step 4: Inflection Points	$f''(x) = 0$ and sign change in $f''(x)$	Locate inflection points
Step 5: Local Extrema	Use the First or Second Derivative Test	Classify maxima or minima