(e) True. If $(S^{-1}AS)\mathbf{x}=\mathbf{b}$ then $SS^{-1}AS\mathbf{x}=A(S\mathbf{x})=S\mathbf{b}$. Consequently, $\mathbf{y}=S\mathbf{x}$ is a solution of $A\mathbf{y}=S\mathbf{b}$.

(f) True. $A\mathbf{x}=4\mathbf{x}$ is equivalent to $A\mathbf{x}=4I_n\mathbf{x}$, which can be rewritten as $(A-4I_n)\mathbf{x}=\mathbf{0}$. By Theorem 1.6.4, this homogeneous system has a unique solution (the trivial solution) if and only if its coefficient matrix $A-4I_n$ is invertible.

(g) True. If AB were invertible, then by Theorem 1.6.5 both A and B would be invertible.

1.7 Diagonal, Triangular, and Symmetric Matrices

1. **(a)** The matrix is upper triangular. It is invertible (its diagonal entries are both nonzero).

(b) The matrix is lower triangular. It is not invertible (its diagonal entries are zero).

(c) This is a diagonal matrix, therefore it is also both upper and lower triangular. It is invertible (its diagonal entries are all nonzero).

(d) The matrix is upper triangular. It is not invertible (its diagonal entries include a zero).

3. $$\begin{bmatrix}3&0&0\\0&-1&0\\0&0&2\end{bmatrix}\begin{bmatrix}2&1\\-4&1\\2&5\end{bmatrix}=\begin{bmatrix}(3)(2)&(3)(1)\\(-1)(-4)&(-1)(1)\\(2)(2)&(2)(5)\end{bmatrix}=\begin{bmatrix}6&3\\4&-1\\4&10\end{bmatrix}$$

5. $$\begin{bmatrix}5&0&0\\0&2&0\\0&0&-3\end{bmatrix}\begin{bmatrix}-3&2&0&4&-4\\1&-5&3&0&3\\-6&2&2&2&2\end{bmatrix}=\begin{bmatrix}(5)(-3)&(5)(2)&(5)(0)&(5)(4)&(5)(-4)\\(2)(1)&(2)(-5)&(2)(3)&(2)(0)&(2)(3)\\(-3)(-6)&(-3)(2)&(-3)(2)&(-3)(2)&(-3)(2)\end{bmatrix}$$
$$=\begin{bmatrix}-15&10&0&20&-20\\2&-10&6&0&6\\18&-6&-6&-6&-6\end{bmatrix}$$

7. $$A^2=\begin{bmatrix}1^2&0\\0&(-2)^2\end{bmatrix}=\begin{bmatrix}1&0\\0&4\end{bmatrix},\qquad A^{-2}=\begin{bmatrix}1^{-2}&0\\0&(-2)^{-2}\end{bmatrix}=\begin{bmatrix}1&0\\0&\frac{1}{4}\end{bmatrix},\quad A^{-k}=\begin{bmatrix}1^{-k}&0\\0&(-2)^{-k}\end{bmatrix}=\begin{bmatrix}1&0\\0&\frac{1}{(-2)^k}\end{bmatrix}$$

9. $$A^2=\begin{bmatrix}\left(\frac{1}{2}\right)^2&0&0\\0&\left(\frac{1}{3}\right)^2&0\\0&0&\left(\frac{1}{4}\right)^2\end{bmatrix}=\begin{bmatrix}\frac{1}{4}&0&0\\0&\frac{1}{9}&0\\0&0&\frac{1}{16}\end{bmatrix},\qquad A^{-2}=\begin{bmatrix}\left(\frac{1}{2}\right)^{-2}&0&0\\0&\left(\frac{1}{3}\right)^{-2}&0\\0&0&\left(\frac{1}{4}\right)^{-2}\end{bmatrix}=\begin{bmatrix}4&0&0\\0&9&0\\0&0&16\end{bmatrix},$$

$$A^{-k}=\begin{bmatrix}\left(\frac{1}{2}\right)^{-k}&0&0\\0&\left(\frac{1}{3}\right)^{-k}&0\\0&0&\left(\frac{1}{4}\right)^{-k}\end{bmatrix}=\begin{bmatrix}2^k&0&0\\0&3^k&0\\0&0&4^k\end{bmatrix}$$

11. $\begin{bmatrix} (1)(2)(0) & 0 & 0 \\ 0 & (0)(5)(2) & 0 \\ 0 & 0 & (3)(0)(1) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

13. $\begin{bmatrix} 1^{39} & 0 \\ 0 & (-1)^{39} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

15. **(a)** $\begin{bmatrix} au & av \\ bw & bx \\ cy & cz \end{bmatrix}$ **(b)** $\begin{bmatrix} ra & sb & tc \\ ua & vb & wc \\ xa & yb & zc \end{bmatrix}$

17. **(a)** $\begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix}$ **(b)** $\begin{bmatrix} 1 & 3 & 7 & 2 \\ 3 & 1 & -8 & -3 \\ 7 & -8 & 0 & 9 \\ 2 & -3 & 9 & 0 \end{bmatrix}$

19. From part (c) of Theorem 1.7.1, a triangular matrix is invertible if and only if its diagonal entries are all nonzero. Since this upper triangular matrix has a 0 on its diagonal, it is not invertible.

21. From part (c) of Theorem 1.7.1, a triangular matrix is invertible if and only if its diagonal entries are all nonzero. Since this lower triangular matrix has all four diagonal entries nonzero, it is invertible.

23. $AB = \begin{bmatrix} (3)(-1) & \times & \times \\ 0 & (1)(5) & \times \\ 0 & 0 & (-1)(6) \end{bmatrix}$. The diagonal entries of AB are: $-3, 5, -6$.

25. The matrix is symmetric if and only if $a+5=-3$. In order for A to be symmetric, we must have $a=-8$.

27. From part (c) of Theorem 1.7.1, a triangular matrix is invertible if and only if its diagonal entries are all nonzero. Therefore, the given upper triangular matrix is invertible for any real number x such that $x \neq 1$, $x \neq -2$, and $x \neq 4$.

29. By Theorem 1.7.1, A^{-1} is also an upper triangular or lower triangular invertible matrix. Its diagonal entries must all be nonzero - they are reciprocals of the corresponding diagonal entries of the matrix A.

31. $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

33. $AB = \begin{bmatrix} (-1)(2)+(2)(0)+(5)(0) & (-1)(-8)+(2)(2)+(5)(0) & (-1)(0)+(2)(1)+(5)(3) \\ (0)(2)+(1)(0)+(3)(0) & (0)(-8)+(1)(2)+(3)(0) & (0)(0)+(1)(1)+(3)(3) \\ (0)(2)+(0)(0)+(-4)(0) & (0)(-8)+(0)(2)+(-4)(0) & (0)(0)+(0)(1)+(-4)(3) \end{bmatrix}$

$= \begin{bmatrix} -2 & 12 & 17 \\ 0 & 2 & 10 \\ 0 & 0 & -12 \end{bmatrix}$. Since this is an upper triangular matrix, we have verified Theorem 1.7.1(b).

35. **(a)** $A^{-1} = \frac{1}{(2)(3)-(-1)(-1)} \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} \frac{3}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{2}{5} \end{bmatrix}$ is symmetric, therefore we verified Theorem 1.7.4.