(i) False. $p(I)=(a_0+a_1+a_2+\cdots+a_m)I$.

(j) True.
If the ith row vector of A is $\begin{bmatrix}0 & \cdots & 0\end{bmatrix}$ then it follows from Formula (9) in Section 1.3 that
ith row vector of $AB=\begin{bmatrix}0 & \cdots & 0\end{bmatrix}B=\begin{bmatrix}0 & \cdots & 0\end{bmatrix}$.

Consequently no matrix B can be found to make the product $AB=I$ thus A does not have an inverse.

If the jth column vector of A is $\begin{bmatrix}0\\ \vdots\\ 0\end{bmatrix}$ then it follows from Formula (8) in Section 1.3 that

the jth column vector of $BA=B\begin{bmatrix}0\\ \vdots\\ 0\end{bmatrix}=\begin{bmatrix}0\\ \vdots\\ 0\end{bmatrix}$.

Consequently no matrix B can be found to make the product $BA=I$ thus A does not have an inverse.

(k) False. E.g. I and $-I$ are both invertible but $I+(-I)=O$ is not.

1.5 Elementary Matrices and a Method for Finding A^{-1}

1. **(a)** Elementary matrix (corresponds to adding -5 times the first row to the second row)

(b) Not an elementary matrix

(c) Not an elementary matrix

(d) Not an elementary matrix

3. **(a)** Add 3 times the second row to the first row: $\begin{bmatrix}1 & 3\\ 0 & 1\end{bmatrix}$

(b) Multiply the first row by $-\frac{1}{7}$: $\begin{bmatrix}-\frac{1}{7} & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$

(c) Add 5 times the first row to the third row: $\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 5 & 0 & 1\end{bmatrix}$

(d) Interchange the first and third rows: $\begin{bmatrix}0 & 0 & 1 & 0\\ 0 & 1 & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$

5. **(a)** Interchange the first and second rows: $EA = \begin{bmatrix} 3 & -6 & -6 & -6 \\ -1 & -2 & 5 & -1 \end{bmatrix}$

(b) Add -3 times the second row to the third row: $EA = \begin{bmatrix} 2 & -1 & 0 & -4 & -4 \\ 1 & -3 & -1 & 5 & 3 \\ -1 & 9 & 4 & -12 & -10 \end{bmatrix}$

(c) Add 4 times the third row to the first row: $EA = \begin{bmatrix} 13 & 28 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$

7. **(a)** $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ (B was obtained from A by interchanging the first row and the third row)

(b) $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ (A was obtained from B by interchanging the first row and the third row)

(c) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$ (C was obtained from A by adding -2 times the first row to the third row)

(d) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$ (A was obtained from C by adding 2 times the first row to the third row)

9. **(a)** (Method I: using Theorem 1.4.5)

The determinant of A, $\det(A) = (1)(7) - (4)(2) = -1$, is nonzero. Therefore A is invertible and its inverse is

$$A^{-1} = \tfrac{1}{-1}\begin{bmatrix} 7 & -4 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} -7 & 4 \\ 2 & -1 \end{bmatrix}.$$

(Method II: using the inversion algorithm)

$\left[\begin{array}{cc|cc} 1 & 4 & 1 & 0 \\ 2 & 7 & 0 & 1 \end{array}\right]$ ← The identity matrix was adjoined to the given matrix.

$\left[\begin{array}{cc|cc} 1 & 4 & 1 & 0 \\ 0 & -1 & -2 & 1 \end{array}\right]$ ← -2 times the first row was added to the second row.

$\left[\begin{array}{cc|cc} 1 & 4 & 1 & 0 \\ 0 & 1 & 2 & -1 \end{array}\right]$ ← The second row was multiplied by -1.

$\left[\begin{array}{cc|cc} 1 & 0 & -7 & 4 \\ 0 & 1 & 2 & -1 \end{array}\right]$ ← -4 times the second row was added to the first row.

The inverse is $\begin{bmatrix} -7 & 4 \\ 2 & -1 \end{bmatrix}$.

(b) (Method I: using Theorem 1.4.5)
The determinant of A, $\det(A) = (2)(8) - (-4)(-4) = 0$. Therefore A is not invertible. (Method II: using the inversion algorithm)

$$\left[\begin{array}{rr|rr} 2 & -4 & 1 & 0 \\ -4 & 8 & 0 & 1 \end{array}\right] \longleftarrow \text{The identity matrix was adjoined to the given matrix.}$$

$$\left[\begin{array}{rr|rr} 2 & -4 & 1 & 0 \\ 0 & 0 & 2 & 1 \end{array}\right] \longleftarrow 2 \text{ times the first row was added to the second row.}$$

A row of zeros was obtained on the left side, therefore A is not invertible.

11. (a)

$$\left[\begin{array}{rrr|rrr} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 3 & 0 & 1 & 0 \\ 1 & 0 & 8 & 0 & 0 & 1 \end{array}\right] \longleftarrow \text{The identity matrix was adjoined to the given matrix.}$$

$$\left[\begin{array}{rrr|rrr} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & -2 & 5 & -1 & 0 & 1 \end{array}\right] \longleftarrow \begin{array}{l} -2 \text{ times the first row was added to the second row and} \\ -1 \text{ times the first row was added to the third row.} \end{array}$$

$$\left[\begin{array}{rrr|rrr} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & -1 & -5 & 2 & 1 \end{array}\right] \longleftarrow 2 \text{ times the second row was added to the third row.}$$

$$\left[\begin{array}{rrr|rrr} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right] \longleftarrow \text{The third row was multiplied by } -1.$$

$$\left[\begin{array}{rrr|rrr} 1 & 2 & 0 & -14 & 6 & 3 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right] \longleftarrow \begin{array}{l} 3 \text{ times the third row was added to the second row and} \\ -3 \text{ times the third row was added to the first row.} \end{array}$$

$$\left[\begin{array}{rrr|rrr} 1 & 0 & 0 & -40 & 16 & 9 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right] \longleftarrow -2 \text{ times the second row was added to the first row.}$$

The inverse is $\begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix}$.

(b) $\left[\begin{array}{ccc|ccc} -1 & 3 & -4 & 1 & 0 & 0 \\ 2 & 4 & 1 & 0 & 1 & 0 \\ -4 & 2 & -9 & 0 & 0 & 1 \end{array}\right]$ ← The identity matrix was adjoined to the given matrix.

$\left[\begin{array}{ccc|ccc} 1 & -3 & 4 & -1 & 0 & 0 \\ 2 & 4 & 1 & 0 & 1 & 0 \\ -4 & 2 & -9 & 0 & 0 & 1 \end{array}\right]$ ← The first row was multiplied by -1.

$\left[\begin{array}{ccc|ccc} 1 & -3 & 4 & -1 & 0 & 0 \\ 0 & 10 & -7 & 2 & 1 & 0 \\ 0 & -10 & 7 & -4 & 0 & 1 \end{array}\right]$ ← -2 times the first row was added to the second row and 4 times the first row was added to the third row.

$\left[\begin{array}{ccc|ccc} 1 & -3 & 4 & -1 & 0 & 0 \\ 0 & 10 & -7 & 2 & 1 & 0 \\ 0 & 0 & 0 & -2 & 1 & 1 \end{array}\right]$ ← The second row was added to the third row.

A row of zeros was obtained on the left side, therefore the matrix is not invertible.

13. $\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$ ← The identity matrix was adjoined to the given matrix.

$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & 0 & 1 \end{array}\right]$ ← -1 times the first row was added to the third row.

$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & -2 & -1 & -1 & 1 \end{array}\right]$ ← -1 times the second row was added to the third row.

$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{array}\right]$ ← The third row was multiplied by $-\frac{1}{2}$.

$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{array}\right]$ ← -1 times the third row was added to the second and -1 times the third row was added to the first row

The inverse is $\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$.

15.

$$\left[\begin{array}{ccc|ccc} 2 & 6 & 6 & 1 & 0 & 0 \\ 2 & 7 & 6 & 0 & 1 & 0 \\ 2 & 7 & 7 & 0 & 0 & 1 \end{array}\right] \longleftarrow \text{The identity matrix was adjoined to the given matrix.}$$

$$\left[\begin{array}{ccc|ccc} 2 & 6 & 6 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 & 1 \end{array}\right] \longleftarrow$$ -1 times the first row was added to the second and -1 times the first row was added to the third row

$$\left[\begin{array}{ccc|ccc} 2 & 6 & 6 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right] \longleftarrow$$ -1 times the second row was added to the third row.

$$\left[\begin{array}{ccc|ccc} 2 & 6 & 0 & 1 & 6 & -6 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right] \longleftarrow$$ -6 times the third row was added to the first row

$$\left[\begin{array}{ccc|ccc} 2 & 0 & 0 & 7 & 0 & -6 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right] \longleftarrow$$ -6 times the second row was added to the first row

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{7}{2} & 0 & -3 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right] \longleftarrow$$ The first row was multiplied by $\frac{1}{2}$.

The inverse is $\begin{bmatrix} \frac{7}{2} & 0 & -3 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$.

17.

$$\left[\begin{array}{cccc|cccc} 2 & -4 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 2 & 12 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & -4 & -5 & 0 & 0 & 0 & 1 \end{array}\right] \longleftarrow$$ The identity matrix was adjoined to the given matrix.

$$\left[\begin{array}{cccc|cccc} 1 & 2 & 12 & 0 & 0 & 1 & 0 & 0 \\ 2 & -4 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & -4 & -5 & 0 & 0 & 0 & 1 \end{array}\right] \longleftarrow$$ The first and second rows were interchanged.

$$\left[\begin{array}{cccc|cccc} 1 & 2 & 12 & 0 & 0 & 1 & 0 & 0 \\ 0 & -8 & -24 & 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & -4 & -5 & 0 & 0 & 0 & 1 \end{array}\right] \longleftarrow$$ -2 times the first row was added to the second.

$$\left[\begin{array}{cccc|cccc} 1 & 2 & 12 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & -4 & -5 & 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 0 & 0 & 0 & 1 & 0 \\ 0 & -8 & -24 & 0 & 1 & -2 & 0 & 0 \end{array}\right] \longleftarrow$$ The second and fourth rows were interchanged.

$$\left[\begin{array}{cccc|cccc} 1 & 2 & 12 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 4 & 5 & 0 & 0 & 0 & -1 \\ 0 & 0 & 2 & 0 & 0 & 0 & 1 & 0 \\ 0 & -8 & -24 & 0 & 1 & -2 & 0 & 0 \end{array}\right] \longleftarrow$$ The second row was multiplied by -1.

$$\left[\begin{array}{cccc|cccc} 1 & 2 & 12 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 4 & 5 & 0 & 0 & 0 & -1 \\ 0 & 0 & 2 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 8 & 40 & 1 & -2 & 0 & -8 \end{array}\right] \longleftarrow$$ 8 times the second row was added to the fourth.

$$\left[\begin{array}{cccc|cccc} 1 & 2 & 12 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 4 & 5 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 8 & 40 & 1 & -2 & 0 & -8 \end{array}\right] \longleftarrow$$ The third row was multiplied by $\frac{1}{2}$.

$$\left[\begin{array}{cccc|cccc} 1 & 2 & 12 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 4 & 5 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 40 & 1 & -2 & -4 & -8 \end{array}\right] \longleftarrow$$ -8 times the third row was added to the fourth row.

$$\left[\begin{array}{cccc|cccc} 1 & 2 & 12 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 4 & 5 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 & \frac{1}{40} & -\frac{1}{20} & -\frac{1}{10} & -\frac{1}{5} \end{array}\right] \longleftarrow$$ The fourth row was multiplied by $\frac{1}{40}$.

$$\left[\begin{array}{cccc|cccc} 1 & 2 & 12 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 & -\frac{1}{8} & \frac{1}{4} & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 & \frac{1}{40} & -\frac{1}{20} & -\frac{1}{10} & -\frac{1}{5} \end{array}\right] \longleftarrow$$ -5 times the fourth row was added to the second row.

$$\left[\begin{array}{cccc|cccc} 1 & 2 & 0 & 0 & 0 & 1 & -6 & 0 \\ 0 & 1 & 0 & 0 & -\frac{1}{8} & \frac{1}{4} & -\frac{3}{2} & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 & \frac{1}{40} & -\frac{1}{20} & -\frac{1}{10} & -\frac{1}{5} \end{array}\right] \longleftarrow$$ -4 times the third row was added to the second row and -12 times the third row was added to the first row.

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & \frac{1}{4} & \frac{1}{2} & -3 & 0 \\ 0 & 1 & 0 & 0 & -\frac{1}{8} & \frac{1}{4} & -\frac{3}{2} & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 & \frac{1}{40} & -\frac{1}{20} & -\frac{1}{10} & -\frac{1}{5} \end{array}\right]$$

$\leftarrow$ -2 times the second row was added to the first row.

The inverse is $\begin{bmatrix} \frac{1}{4} & \frac{1}{2} & -3 & 0 \\ -\frac{1}{8} & \frac{1}{4} & -\frac{3}{2} & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ \frac{1}{40} & -\frac{1}{20} & -\frac{1}{10} & -\frac{1}{5} \end{bmatrix}$.

19. **(a)** $\left[\begin{array}{cccc|cccc} k_1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & k_2 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & k_3 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & k_4 & 0 & 0 & 0 & 1 \end{array}\right]$ $\leftarrow$ The identity matrix was adjoined to the given matrix.

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & \frac{1}{k_1} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & \frac{1}{k_2} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{k_3} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{k_4} \end{array}\right]$$

$\leftarrow$ The first row was multiplied by $1/k_1$, the second row was multiplied by $1/k_2$, the third row was multiplied by $1/k_3$, and the fourth row was multiplied by $1/k_4$.

The inverse is $\begin{bmatrix} \frac{1}{k_1} & 0 & 0 & 0 \\ 0 & \frac{1}{k_2} & 0 & 0 \\ 0 & 0 & \frac{1}{k_3} & 0 \\ 0 & 0 & 0 & \frac{1}{k_4} \end{bmatrix}$.

(b) $\left[\begin{array}{cccc|cccc} k & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & k & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$ $\leftarrow$ The identity matrix was adjoined to the given matrix.

$$\left[\begin{array}{cccc|cccc} 1 & \frac{1}{k} & 0 & 0 & \frac{1}{k} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \frac{1}{k} & 0 & 0 & \frac{1}{k} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$\leftarrow$ First row and third row were both multiplied by $1/k$.