(l) False. For instance, $A=\begin{bmatrix}0&0\\1&0\end{bmatrix}$ is not symmetric even though $A^2=\begin{bmatrix}0&0\\0&0\end{bmatrix}$ is.

(m) True. By Theorem 1.4.8(d), $(kA)^T=kA^T$. Since kA is symmetric, we also have $(kA)^T=kA$. For nonzero k the equality of the right hand sides $kA^T=kA$ implies $A^T=A$.

1.8 Matrix Transformations

1. **(a)** $T_A(\mathbf{x})=A\mathbf{x}$ maps any vector $\mathbf{x}$ in R^2 into a vector $\mathbf{w}=A\mathbf{x}$ in R^3.
The domain of T_A is R^2; the codomain is R^3.

(b) $T_A(\mathbf{x})=A\mathbf{x}$ maps any vector $\mathbf{x}$ in R^3 into a vector $\mathbf{w}=A\mathbf{x}$ in R^2.
The domain of T_A is R^3; the codomain is R^2.

(c) $T_A(\mathbf{x})=A\mathbf{x}$ maps any vector $\mathbf{x}$ in R^3 into a vector $\mathbf{w}=A\mathbf{x}$ in R^3.
The domain of T_A is R^3; the codomain is R^3.

(d) $T_A(\mathbf{x})=A\mathbf{x}$ maps any vector $\mathbf{x}$ in R^6 into a vector $\mathbf{w}=A\mathbf{x}$ in $R^1=R$.
The domain of T_A is R^6; the codomain is R.

3. **(a)** The transformation maps any vector $\mathbf{x}$ in R^2 into a vector $\mathbf{w}$ in R^2.
Its domain is R^2; the codomain is R^2.

(b) The transformation maps any vector $\mathbf{x}$ in R^2 into a vector $\mathbf{w}$ in R^3.
Its domain is R^2; the codomain is R^3.

5. **(a)** The transformation maps any vector $\mathbf{x}$ in R^3 into a vector in R^2.
Its domain is R^3; the codomain is R^2.

(b) The transformation maps any vector $\mathbf{x}$ in R^2 into a vector in R^3.
Its domain is R^2; the codomain is R^3.

7. **(a)** The transformation maps any vector $\mathbf{x}$ in R^2 into a vector in R^2.
Its domain is R^2; the codomain is R^2.

(b) The transformation maps any vector $\mathbf{x}$ in R^3 into a vector in R^2.
Its domain is R^3; the codomain is R^2.

9. The transformation maps any vector $\mathbf{x}$ in R^2 into a vector in R^3. Its domain is R^2; the codomain is R^3.

11. **(a)** The given equations can be expressed in matrix form as $\begin{bmatrix}w_1\\w_2\end{bmatrix}=\begin{bmatrix}2&-3&1\\3&5&-1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$

therefore the standard matrix for this transformation is $\begin{bmatrix}2&-3&1\\3&5&-1\end{bmatrix}$

(b) The given equations can be expressed in matrix form as $\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} 7 & 2 & -8 \\ 0 & -1 & 5 \\ 4 & 7 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

therefore the standard matrix for this transformation is $\begin{bmatrix} 7 & 2 & -8 \\ 0 & -1 & 5 \\ 4 & 7 & -1 \end{bmatrix}$.

13. **(a)** $T(x_1, x_2) = \begin{bmatrix} x_2 \\ -x_1 \\ x_1 + 3x_2 \\ x_1 - x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \\ 1 & 3 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$; the standard matrix is $\begin{bmatrix} 0 & 1 \\ -1 & 0 \\ 1 & 3 \\ 1 & -1 \end{bmatrix}$

(b) $T(x_1, x_2, x_3, x_4) = \begin{bmatrix} 7x_1 + 2x_2 - x_3 + x_4 \\ x_2 + x_3 \\ -x_1 \end{bmatrix} = \begin{bmatrix} 7 & 2 & -1 & 1 \\ 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$;

the standard matrix is $\begin{bmatrix} 7 & 2 & -1 & 1 \\ 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}$

(c) $T(x_1, x_2, x_3) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$; the standard matrix is $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

(d) $T(x_1, x_2, x_3, x_4) = \begin{bmatrix} x_4 \\ x_1 \\ x_3 \\ x_2 \\ x_1 - x_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$; the standard matrix is $\begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 \end{bmatrix}$

15. The given equations can be expressed in matrix form as $\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} 3 & 5 & -1 \\ 4 & -1 & 1 \\ 3 & 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ therefore the standard matrix for this operator is $\begin{bmatrix} 3 & 5 & -1 \\ 4 & -1 & 1 \\ 3 & 2 & -1 \end{bmatrix}$.

By directly substituting $(-1, 2, 4)$ for (x_1, x_2, x_3) into the given equation we obtain

$$w_1 = -(3)(1) + (5)(2) - (1)(4) = 3$$

$$w_2 = -(4)(1) - (1)(2) + (1)(4) = -2$$

$$w_3 = -(3)(1) + (2)(2) - (1)(4) = -3$$

By matrix multiplication, $\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} 3 & 5 & -1 \\ 4 & -1 & 1 \\ 3 & 2 & -1 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} -(3)(1)+(5)(2)-(1)(4) \\ -(4)(1)-(1)(2)+(1)(4) \\ -(3)(1)+(2)(2)-(1)(4) \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ -3 \end{bmatrix}$.

17. **(a)** $T(x_1, x_2) = \begin{bmatrix} -x_1 + x_2 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$; the standard matrix is $\begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}$.

$T(\mathbf{x}) = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 4 \end{bmatrix} = \begin{bmatrix} (1)(1)+(1)(4) \\ -(0)(1)+(1)(4) \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$ matches $T(-1,4) = (1+4,4) = (5,4)$.

(b) $T(x_1, x_2, x_3) = \begin{bmatrix} 2x_1 - x_2 + x_3 \\ x_2 + x_3 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$; the standard matrix is $\begin{bmatrix} 2 & -1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.

$T(\mathbf{x}) = \begin{bmatrix} 2 & -1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix} = \begin{bmatrix} (2)(2)-(1)(1)-(1)(3) \\ (0)(2)+(1)(1)-(1)(3) \\ (0)(2)+(0)(1)-(0)(3) \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix}$

matches $T(2,1,-3) = (4-1-3, 1-3, 0) = (0,-2,0)$.

19. **(a)** $T_A(\mathrm{x}) = A\mathrm{x} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

(b) $T_A(\mathrm{x}) = A\mathrm{x} = \begin{bmatrix} -1 & 2 & 0 \\ 3 & 1 & 5 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 13 \end{bmatrix}$

21. **(a)** If $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$ then

$$\begin{aligned} T(\mathbf{u}+\mathbf{v}) &= T(u_1+v_1, u_2+v_2) \\ &= \left(2(u_1+v_1)+(u_2+v_2), (u_1+v_1)-(u_2+v_2)\right) \\ &= (2u_1+u_2, u_1-u_2) + (2v_1+v_2, v_1-v_2) \\ &= T(\mathbf{u}) + T(\mathbf{v}) \end{aligned}$$

and $T(k\mathbf{u}) = T(ku_1, ku_2) = (2ku_1 + ku_2, ku_1 - ku_2) = k(2u_1 + u_2, u_1 - u_2) = kT(\mathbf{u})$.

(b) If $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$ then

$$\begin{aligned} T(\mathbf{u}+\mathbf{v}) &= T(u_1+v_1, u_2+v_2, u_3+v_3) \\ &= (u_1+v_1, u_3+v_3, u_1+v_1+u_2+v_2) \end{aligned}$$

$$=(u_1,u_3,u_1+u_2)+(v_1,v_3,v_1+v_2)$$

$$=T(\mathbf{u})+T(\mathbf{v})$$

and $T(k\mathbf{u})=T(ku_1,ku_2,ku_3)=(ku_1,ku_3,ku_1+ku_2)=k(u_1,u_3,u_1+u_2)=kT(\mathbf{u})$.

23. **(a)** The homogeneity property fails to hold since $T(kx,ky)=\left((kx)^2,ky\right)=\left(k^2x^2,ky\right)$ does not generally equal $kT(x,y)=k\left(x^2,y\right)=\left(kx^2,ky\right)$. (It can be shown that the additivity property fails to hold as well.)

(b) The homogeneity property fails to hold since $T(kx,ky,kz)=(kx,ky,kxkz)=\left(kx,ky,k^2xz\right)$ does not generally equal $kT(x,y,z)=k(x,y,xz)=(kx,ky,kxz)$. (It can be shown that the additivity property fails to hold as well.)

25. The homogeneity property fails to hold since for $b\neq 0$, $f(kx)=m(kx)+b$ does not generally equal $kf(x)=k(mx+b)=kmx+kb$. (It can be shown that the additivity property fails to hold as well.)
On the other hand, both properties hold for $b=0$: $f(x+y)=m(x+y)=mx+my=f(x)+f(y)$ and $f(kx)=m(kx)=k(mx)=kf(x)$.
Consequently, f is not a matrix transformation on R unless $b=0$.

27. By Formula (13), the standard matrix for T is $A=\left[\,T(\mathbf{e}_1)\;\middle|\;T(\mathbf{e}_2)\;\middle|\;T(\mathbf{e}_3)\,\right]$. Therefore

$$A=\begin{bmatrix}1&0&4\\3&0&-3\\0&1&-1\end{bmatrix} \text{ and } T(\mathbf{x})=A\mathbf{x}=\begin{bmatrix}(1)(2)+(0)(1)+(4)(0)\\(3)(2)+(0)(1)-(3)(0)\\(0)(2)+(1)(1)-(1)(0)\end{bmatrix}=\begin{bmatrix}2\\6\\1\end{bmatrix}.$$

29. **(a)** $\begin{bmatrix}1&0\\0&-1\end{bmatrix}\begin{bmatrix}-1\\2\end{bmatrix}=\begin{bmatrix}-1\\-2\end{bmatrix}$ **(b)** $\begin{bmatrix}-1&0\\0&1\end{bmatrix}\begin{bmatrix}-1\\2\end{bmatrix}=\begin{bmatrix}1\\2\end{bmatrix}$ **(c)** $\begin{bmatrix}0&1\\1&0\end{bmatrix}\begin{bmatrix}-1\\2\end{bmatrix}=\begin{bmatrix}2\\-1\end{bmatrix}$

31. **(a)** $\begin{bmatrix}1&0&0\\0&1&0\\0&0&-1\end{bmatrix}\begin{bmatrix}2\\-5\\3\end{bmatrix}=\begin{bmatrix}2\\-5\\-3\end{bmatrix}$ **(b)** $\begin{bmatrix}1&0&0\\0&-1&0\\0&0&1\end{bmatrix}\begin{bmatrix}2\\-5\\3\end{bmatrix}=\begin{bmatrix}2\\5\\3\end{bmatrix}$ **(c)** $\begin{bmatrix}-1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}2\\-5\\3\end{bmatrix}=\begin{bmatrix}-2\\-5\\3\end{bmatrix}$

33. **(a)** $\begin{bmatrix}1&0\\0&0\end{bmatrix}\begin{bmatrix}2\\-5\end{bmatrix}=\begin{bmatrix}2\\0\end{bmatrix}$ **(b)** $\begin{bmatrix}0&0\\0&1\end{bmatrix}\begin{bmatrix}2\\-5\end{bmatrix}=\begin{bmatrix}0\\-5\end{bmatrix}$

35. **(a)** $\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}\begin{bmatrix}-2\\1\\3\end{bmatrix}=\begin{bmatrix}-2\\1\\0\end{bmatrix}$ **(b)** $\begin{bmatrix}1&0&0\\0&0&0\\0&0&1\end{bmatrix}\begin{bmatrix}-2\\1\\3\end{bmatrix}=\begin{bmatrix}-2\\0\\3\end{bmatrix}$ **(c)** $\begin{bmatrix}0&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}-2\\1\\3\end{bmatrix}=\begin{bmatrix}0\\1\\3\end{bmatrix}$

37. **(a)** $\begin{bmatrix}\cos 30^\circ&-\sin 30^\circ\\\sin 30^\circ&\cos 30^\circ\end{bmatrix}\begin{bmatrix}3\\-4\end{bmatrix}=\begin{bmatrix}\frac{\sqrt{3}}{2}&-\frac{1}{2}\\\frac{1}{2}&\frac{\sqrt{3}}{2}\end{bmatrix}\begin{bmatrix}3\\-4\end{bmatrix}=\begin{bmatrix}\frac{3\sqrt{3}}{2}+2\\\frac{3}{2}-2\sqrt{3}\end{bmatrix}\approx\begin{bmatrix}4.60\\-1.96\end{bmatrix}$

(b) $\begin{bmatrix}\cos(-60^\circ)&-\sin(-60^\circ)\\\sin(-60^\circ)&\cos(-60^\circ)\end{bmatrix}\begin{bmatrix}3\\-4\end{bmatrix}=\begin{bmatrix}\frac{1}{2}&\frac{\sqrt{3}}{2}\\-\frac{\sqrt{3}}{2}&\frac{1}{2}\end{bmatrix}\begin{bmatrix}3\\-4\end{bmatrix}=\begin{bmatrix}\frac{3}{2}-2\sqrt{3}\\-\frac{3\sqrt{3}}{2}-2\end{bmatrix}\approx\begin{bmatrix}-1.96\\-4.60\end{bmatrix}$

(c) $\begin{bmatrix} \cos 45° & -\sin 45° \\ \sin 45° & \cos 45° \end{bmatrix}\begin{bmatrix} 3 \\ -4 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}\begin{bmatrix} 3 \\ -4 \end{bmatrix} = \begin{bmatrix} \frac{7\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \end{bmatrix} \approx \begin{bmatrix} 4.95 \\ -0.71 \end{bmatrix}$

(d) $\begin{bmatrix} \cos 90° & -\sin 90° \\ \sin 90° & \cos 90° \end{bmatrix}\begin{bmatrix} 3 \\ -4 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 3 \\ -4 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$

39. By Formula (13), the standard matrix for T is $A = \left[\, T(\mathbf{e}_1) \;\; | T(\mathbf{e}_2) \right]$. Therefore $A = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$ and $T(1,1) = A\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} a+c \\ b+d \end{bmatrix}$.

41. **(a)** $T_A(\mathbf{e}_1) = \begin{bmatrix} -1 \\ 2 \\ 4 \end{bmatrix}$, $T_A(\mathbf{e}_2) = \begin{bmatrix} 3 \\ 1 \\ 5 \end{bmatrix}$, $T_A(\mathbf{e}_3) = \begin{bmatrix} 0 \\ 2 \\ -3 \end{bmatrix}$.

(b) Since T_A is a matrix transformation,

$$T_A(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3) = T_A(\mathbf{e}_1) + T_A(\mathbf{e}_2) + T_A(\mathbf{e}_3) = \begin{bmatrix} -1 \\ 2 \\ 4 \end{bmatrix} + \begin{bmatrix} 3 \\ 1 \\ 5 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \\ -3 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix}.$$

(c) Since T_A is a matrix transformation, $T_A(7\boldsymbol{e}_3) = 7T_A(\mathbf{e}_3) = 7\begin{bmatrix} 0 \\ 2 \\ -3 \end{bmatrix} = \begin{bmatrix} 0 \\ 14 \\ -21 \end{bmatrix}$.

43. Reflection about the xy-plane: $T(1,2,3) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$.

Reflection about the xz-plane: $T(1,2,3) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$.

Reflection about the yz-plane: $T(1,2,3) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$.

45. The standard matrix for T is $A = \left[T(\mathbf{e}_1) \;\; | T(\mathbf{e}_2) \right]$. Observe that $\begin{bmatrix} 1 \\ 0 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 2 \\ 3 \end{bmatrix}$. Because T_A is a transformation, $T_A(\mathbf{e}_1) = T_A\left(3\begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 2 \\ 3 \end{bmatrix}\right) = 3T_A\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) - T_A\left(\begin{bmatrix} 2 \\ 3 \end{bmatrix}\right) = 3\begin{bmatrix} 1 \\ -2 \end{bmatrix} - \begin{bmatrix} -2 \\ 5 \end{bmatrix} = \begin{bmatrix} 5 \\ -11 \end{bmatrix}$.

Likewise, $\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} - 2\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ so we obtain

$$T_A(\mathbf{e}_2) = T_A\left(\begin{bmatrix} 2 \\ 3 \end{bmatrix} - 2\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = T_A\left(\begin{bmatrix} 2 \\ 3 \end{bmatrix}\right) - 2T_A\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -2 \\ 5 \end{bmatrix} - 2\begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} -4 \\ 9 \end{bmatrix}.$$