(b) True. This follows from Theorem 1.5.2.

(c) True. If A and B are row equivalent then there exist elementary matrices $E_1,\dots,E_p$ such that $B = E_p \cdots E_1 A$. Likewise, if B and C are row equivalent then there exist elementary matrices $E_1^*,\dots,E_q^*$ such that $C = E_q^* \cdots E_1^* B$. Combining the two equalities yields $C = E_q^* \cdots E_1^* E_p \cdots E_1 A$ therefore A and C are row equivalent.

(d) True. A homogeneous system $A\mathbf{x} = \mathbf{0}$ has either one solution (the trivial solution) or infinitely many solutions. If A is not invertible, then by Theorem 1.5.3 the system cannot have just one solution. Consequently, it must have infinitely many solutions.

(e) True. If the matrix A is not invertible then by Theorem 1.5.3 its reduced row echelon form is not I_n. However, the matrix resulting from interchanging two rows of A (an elementary row operation) must have the same reduced row echelon form as A does, so by Theorem 1.5.3 that matrix is not invertible either.

(f) True. Adding a multiple of the first row of a matrix to its second row is an elementary row operation. Denoting by E be the corresponding elementary matrix we can write $(EA)^{-1} = A^{-1}E^{-1}$ so the resulting matrix EA is invertible if A is.

(g) False. For instance, $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/3 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$.

1.6 More on Linear Systems and Invertible Matrices

1. The given system can be written in matrix form as $A\mathbf{x} = \mathbf{b}$, where $A = \begin{bmatrix} 1 & 1 \\ 5 & 6 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} 2 \\ 9 \end{bmatrix}$.

We begin by inverting the coefficient matrix A

$$\left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 5 & 6 & 0 & 1 \end{array}\right] \longleftarrow \text{The identity matrix was adjoined to the coefficient matrix.}$$

$$\left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 1 & -5 & 1 \end{array}\right] \longleftarrow -5 \text{ times the first row was added to the second row.}$$

$$\left[\begin{array}{cc|cc} 1 & 0 & 6 & -1 \\ 0 & 1 & -5 & 1 \end{array}\right] \longleftarrow -1 \text{ times the second row was added to the first row.}$$

Since $A^{-1} = \begin{bmatrix} 6 & -1 \\ -5 & 1 \end{bmatrix}$, Theorem 1.6.2 states that the system has exactly one solution $\mathbf{x} = A^{-1}\mathbf{b}$:

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6 & -1 \\ -5 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 9 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$, i.e., $x_1 = 3$, $x_2 = -1$.

3. The given system can be written in matrix form as $A\mathbf{x}=\mathbf{b}$, where $A=\begin{bmatrix}1&3&1\\2&2&1\\2&3&1\end{bmatrix}$, $\mathbf{x}=\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$, and $\mathbf{b}=\begin{bmatrix}4\\-1\\3\end{bmatrix}$. We begin by inverting the coefficient matrix A

$$\left[\begin{array}{ccc|ccc}1&3&1&1&0&0\\2&2&1&0&1&0\\2&3&1&0&0&1\end{array}\right]$$ ← The identity matrix was adjoined to the coefficient matrix.

$$\left[\begin{array}{ccc|ccc}1&3&1&1&0&0\\0&-4&-1&-2&1&0\\0&-3&-1&-2&0&1\end{array}\right]$$ ← -2 times the first row was added to the second and -2 times the first row was added to the third row.

$$\left[\begin{array}{ccc|ccc}1&3&1&1&0&0\\0&-4&-1&-2&1&0\\0&1&0&0&-1&1\end{array}\right]$$ ← -1 times the second row was added to the third row.

$$\left[\begin{array}{ccc|ccc}1&3&1&1&0&0\\0&1&0&0&-1&1\\0&-4&-1&-2&1&0\end{array}\right]$$ ← The second and third rows were interchanged.

$$\left[\begin{array}{ccc|ccc}1&3&1&1&0&0\\0&1&0&0&-1&1\\0&0&-1&-2&-3&4\end{array}\right]$$ ← 4 times the second row was added to the third row.

$$\left[\begin{array}{ccc|ccc}1&3&1&1&0&0\\0&1&0&0&-1&1\\0&0&1&2&3&-4\end{array}\right]$$ ← The third row was multiplied by -1.

$$\left[\begin{array}{ccc|ccc}1&3&0&-1&-3&4\\0&1&0&0&-1&1\\0&0&1&2&3&-4\end{array}\right]$$ ← -1 times the third row was added to the first row.

$$\left[\begin{array}{ccc|ccc}1&0&0&-1&0&1\\0&1&0&0&-1&1\\0&0&1&2&3&-4\end{array}\right]$$ ← -3 times the second row was added to the first row.

Since $A^{-1}=\begin{bmatrix}-1&0&1\\0&-1&1\\2&3&-4\end{bmatrix}$, Theorem 1.6.2 states that the system has exactly one solution $\mathbf{x}=A^{-1}\mathbf{b}$:

$$\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}-1&0&1\\0&-1&1\\2&3&-4\end{bmatrix}\begin{bmatrix}4\\-1\\3\end{bmatrix}=\begin{bmatrix}-1\\4\\-7\end{bmatrix}$$, i.e., $x_1=-1, x_2=4,$ and $x_3=-7$.

5. The given system can be written in matrix form as $A\mathbf{x}=\mathbf{b}$, where $A=\begin{bmatrix}1&1&1\\1&1&-4\\-4&1&1\end{bmatrix}$, $\mathbf{x}=\begin{bmatrix}x\\y\\z\end{bmatrix}$, and $\mathbf{b}=\begin{bmatrix}5\\10\\0\end{bmatrix}$. We begin by inverting the coefficient matrix A

$$\left[\begin{array}{ccc|ccc}1&1&1&1&0&0\\1&1&-4&0&1&0\\-4&1&1&0&0&1\end{array}\right] \longleftarrow \text{The identity matrix was adjoined to the coefficient matrix.}$$

$$\left[\begin{array}{ccc|ccc}1&1&1&1&0&0\\0&0&-5&-1&1&0\\0&5&5&4&0&1\end{array}\right] \longleftarrow \begin{array}{l}-1 \text{ times the first row was added to the second row and}\\ 4 \text{ times the first row was added to the third row.}\end{array}$$

$$\left[\begin{array}{ccc|ccc}1&1&1&1&0&0\\0&5&5&4&0&1\\0&0&-5&-1&1&0\end{array}\right] \longleftarrow \text{The second and third rows were interchanged.}$$

$$\left[\begin{array}{ccc|ccc}1&1&1&1&0&0\\0&1&1&\frac{4}{5}&0&\frac{1}{5}\\0&0&1&\frac{1}{5}&-\frac{1}{5}&0\end{array}\right] \longleftarrow \begin{array}{l}\text{The second row was multiplied by } \frac{1}{5} \text{ and}\\ \text{the third row was multiplied by } -\frac{1}{5}.\end{array}$$

$$\left[\begin{array}{ccc|ccc}1&1&0&\frac{4}{5}&\frac{1}{5}&0\\0&1&0&\frac{3}{5}&\frac{1}{5}&\frac{1}{5}\\0&0&1&\frac{1}{5}&-\frac{1}{5}&0\end{array}\right] \longleftarrow \begin{array}{l}-1 \text{ times the third row was added to the second row}\\ \text{and to the first row.}\end{array}$$

$$\left[\begin{array}{ccc|ccc}1&0&0&\frac{4}{5}&0&-\frac{1}{5}\\0&1&0&\frac{3}{5}&\frac{1}{5}&\frac{1}{5}\\0&0&1&\frac{1}{5}&-\frac{1}{5}&0\end{array}\right] \longleftarrow -1 \text{ times the second row was added to the first row.}$$

Since $A^{-1}=\begin{bmatrix}\frac{1}{5}&0&-\frac{1}{5}\\\frac{3}{5}&\frac{1}{5}&\frac{1}{5}\\\frac{1}{5}&-\frac{1}{5}&0\end{bmatrix}$, Theorem 1.6.2 states that the system has exactly one solution $\mathbf{x}=A^{-1}\mathbf{b}$:

$$\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}\frac{1}{5}&0&-\frac{1}{5}\\\frac{3}{5}&\frac{1}{5}&\frac{1}{5}\\\frac{1}{5}&-\frac{1}{5}&0\end{bmatrix}\begin{bmatrix}5\\10\\0\end{bmatrix}=\begin{bmatrix}1\\5\\-1\end{bmatrix}$$, i.e., $x=1, y=5$, and $z=-1$.

7. The given system can be written in matrix form as $A\mathbf{x}=\mathbf{b}$, where $A=\begin{bmatrix}3&5\\1&2\end{bmatrix}$, $\mathbf{x}=\begin{bmatrix}x_1\\x_2\end{bmatrix}$, and $\mathbf{b}=\begin{bmatrix}b_1\\b_2\end{bmatrix}$. We begin by inverting the coefficient matrix A

$$\left[\begin{array}{cc|cc} 3 & 5 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{array}\right] \longleftarrow \text{The identity matrix was adjoined to the coefficient matrix.}$$

$$\left[\begin{array}{cc|cc} 1 & 2 & 0 & 1 \\ 3 & 5 & 1 & 0 \end{array}\right] \longleftarrow \text{The first and second rows were interchanged.}$$

$$\left[\begin{array}{cc|cc} 1 & 2 & 0 & 1 \\ 0 & -1 & 1 & -3 \end{array}\right] \longleftarrow -3 \text{ times the first row was added to the second row.}$$

$$\left[\begin{array}{cc|cc} 1 & 2 & 0 & 1 \\ 0 & 1 & -1 & 3 \end{array}\right] \longleftarrow \text{The second row was multiplied by } -1 .$$

$$\left[\begin{array}{cc|cc} 1 & 0 & 2 & -5 \\ 0 & 1 & -1 & 3 \end{array}\right] \longleftarrow -2 \text{ times the second row was added to the first row.}$$

Since $A^{-1} = \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix}$, Theorem 1.6.2 states that the system has exactly one solution $\mathbf{x} = A^{-1}\mathbf{b}$:

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 2b_1 - 5b_2 \\ -b_1 + 3b_2 \end{bmatrix}$, i.e., $x_1 = 2b_1 - 5b_2$, $x_2 = -b_1 + 3b_2$.

9.

$$\left[\begin{array}{cc|c|c} 1 & -5 & 1 & -2 \\ 3 & 2 & 4 & 5 \end{array}\right] \longleftarrow \text{We augmented the coefficient matrix with two columns of constants on the right hand sides of the systems (i) and (ii) – refer to Example 2.}$$

$$\left[\begin{array}{cc|c|c} 1 & -5 & 1 & -2 \\ 0 & 17 & 1 & 11 \end{array}\right] \longleftarrow -3 \text{ times the first row was added to the second row.}$$

$$\left[\begin{array}{cc|c|c} 1 & -5 & 1 & -2 \\ 0 & 1 & \frac{1}{17} & \frac{11}{17} \end{array}\right] \longleftarrow \text{The second row was multiplied by } \tfrac{1}{17} .$$

$$\left[\begin{array}{cc|c|c} 1 & 0 & \frac{22}{17} & \frac{21}{17} \\ 0 & 1 & \frac{1}{17} & \frac{11}{17} \end{array}\right] \longleftarrow 5 \text{ times the second row was added to the first row.}$$

We conclude that the solutions of the two systems are:

(i) $x_1 = \frac{22}{17}$, $x_2 = \frac{1}{17}$ **(ii)** $x_1 = \frac{21}{17}$, $x_2 = \frac{11}{17}$

11.

$$\left[\begin{array}{cc|c|c|c|c} 4 & -7 & 0 & -4 & -1 & -5 \\ 1 & 2 & 1 & 6 & 3 & 1 \end{array}\right] \longleftarrow \text{We augmented the coefficient matrix with four columns of constants on the right hand sides of the systems (i), (ii), (iii), and (iv) – refer to Example 2.}$$

$$\left[\begin{array}{cc|c|c|c|c} 1 & 2 & 1 & 6 & 3 & 1 \\ 4 & -7 & 0 & -4 & -1 & -5 \end{array}\right] \longleftarrow \text{The first and second rows were interchanged.}$$

$$\left[\begin{array}{cc|c|c|c|c} 1 & 2 & 1 & 6 & 3 & 1 \\ 0 & -15 & -4 & -28 & -13 & -9 \end{array}\right] \longleftarrow -4 \text{ times the first row was added to the second row.}$$

$$\left[\begin{array}{cc|c|c|c|c} 1 & 2 & 1 & 6 & 3 & 1 \\ 0 & 1 & \frac{4}{15} & \frac{28}{15} & \frac{13}{15} & \frac{3}{5} \end{array}\right] \longleftarrow \text{The second row was multiplied by } -\tfrac{1}{15}.$$

$$\left[\begin{array}{cc|c|c|c|c} 1 & 0 & \frac{7}{15} & \frac{34}{15} & \frac{19}{15} & -\frac{1}{5} \\ 0 & 1 & \frac{4}{15} & \frac{28}{15} & \frac{13}{15} & \frac{3}{5} \end{array}\right] \longleftarrow -2 \text{ times the second row was added to the first row.}$$

We conclude that the solutions of the four systems are:

(i) $x_1 = \dfrac{7}{15},\ x_2 = \dfrac{4}{15}$ **(ii)** $x_1 = \dfrac{34}{15},\ x_2 = \dfrac{28}{15}$

(iii) $x_1 = \dfrac{19}{15},\ x_2 = \dfrac{13}{15}$ **(iv)** $x_1 = -\dfrac{1}{5},\ x_2 = \dfrac{3}{5}$

13.

$$\left[\begin{array}{cc|c} 1 & 3 & b_1 \\ -2 & 1 & b_2 \end{array}\right] \longleftarrow \text{The augmented matrix for the system.}$$

$$\left[\begin{array}{cc|c} 1 & 3 & b_1 \\ 0 & 7 & 2b_1 + b_2 \end{array}\right] \longleftarrow 2 \text{ times the first row was added to the second row.}$$

$$\left[\begin{array}{cc|c} 1 & 3 & b_1 \\ 0 & 1 & \frac{2}{7}b_1 + \frac{1}{7}b_2 \end{array}\right] \longleftarrow \text{The second row was multiplied by } \tfrac{1}{7}.$$

The system is consistent for all values of b_1 and b_2.

15.

$$\left[\begin{array}{ccc|c} 1 & -2 & 5 & b_1 \\ 4 & -5 & 8 & b_2 \\ -3 & 3 & -3 & b_3 \end{array}\right] \longleftarrow \text{The augmented matrix for the system.}$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 5 & b_1 \\ 0 & 3 & -12 & -4b_1 + b_2 \\ 0 & -3 & 12 & 3b_1 + b_3 \end{array}\right] \longleftarrow \begin{array}{l} -4 \text{ times the first row was added to the second row} \\ \text{and } 3 \text{ times the first row was added to the third row.} \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 5 & b_1 \\ 0 & 3 & -12 & -4b_1 + b_2 \\ 0 & 0 & 0 & -b_1 + b_2 + b_3 \end{array}\right] \longleftarrow \text{The second row was added to the third row.}$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 5 & b_1 \\ 0 & 1 & -4 & -\frac{4}{3}b_1 + \frac{1}{3}b_2 \\ 0 & 0 & 0 & -b_1 + b_2 + b_3 \end{array}\right] \longleftarrow \text{The second row was multiplied by } \tfrac{1}{3}.$$

The system is consistent if and only if $-b_1 + b_2 + b_3 = 0$, i.e. $b_1 = b_2 + b_3$.

17. $\left[\begin{array}{rrrr|l} 1 & -1 & 3 & 2 & b_1 \\ -2 & 1 & 5 & 1 & b_2 \\ -3 & 2 & 2 & -1 & b_3 \\ 4 & -3 & 1 & 3 & b_4 \end{array}\right]$ ⟵ The augmented matrix for the system.

$\left[\begin{array}{rrrr|r} 1 & -1 & 3 & 2 & b_1 \\ 0 & -1 & 11 & 5 & 2b_1+b_2 \\ 0 & -1 & 11 & 5 & 3b_1+b_3 \\ 0 & 1 & -11 & -5 & -4b_1+b_4 \end{array}\right]$ ⟵ 2 times the first row was added to the second row, 3 times the first row was added to the third row, and −4 times the first row was added to the fourth row.

$\left[\begin{array}{rrrr|r} 1 & -1 & 3 & 2 & b_1 \\ 0 & 1 & -11 & -5 & -2b_1-b_2 \\ 0 & -1 & 11 & 5 & 3b_1+b_3 \\ 0 & 1 & -11 & -5 & -4b_1+b_4 \end{array}\right]$ ⟵ The second row was multiplied by -1.

$\left[\begin{array}{rrrr|r} 1 & -1 & 3 & 2 & b_1 \\ 0 & 1 & -11 & -5 & -2b_1-b_2 \\ 0 & 0 & 0 & 0 & b_1-b_2+b_3 \\ 0 & 0 & 0 & 0 & -2b_1+b_2+b_4 \end{array}\right]$ ⟵ The second row was added to the third row and −1 times the second row was added to the fourth row.

The system is consistent for all values of b_1, b_2, b_3, and b_4 that satisfy the equations $b_1-b_2+b_3=0$ and $-2b_1+b_2+b_4=0$.

These equations form a linear system in the variables b_1, b_2, b_3, and b_4 whose augmented matrix $\left[\begin{array}{rrrrr} 1 & -1 & 1 & 0 & 0 \\ -2 & 1 & 0 & 1 & 0 \end{array}\right]$ has the reduced row echelon form $\left[\begin{array}{rrrrr} 1 & 0 & -1 & -1 & 0 \\ 0 & 1 & -2 & -1 & 0 \end{array}\right]$. Therefore the system is consistent if $b_1=b_3+b_4$ and $b_2=2b_3+b_4$.

19. $X=\begin{bmatrix} 1 & -1 & 1 \\ 2 & 3 & 0 \\ 0 & 2 & -1 \end{bmatrix}^{-1}\begin{bmatrix} 2 & -1 & 5 & 7 & 8 \\ 4 & 0 & -3 & 0 & 1 \\ 3 & 5 & -7 & 2 & 1 \end{bmatrix}$. Let us find $\begin{bmatrix} 1 & -1 & 1 \\ 2 & 3 & 0 \\ 0 & 2 & -1 \end{bmatrix}^{-1}$:

$\left[\begin{array}{rrr|rrr} 1 & -1 & 1 & 1 & 0 & 0 \\ 2 & 3 & 0 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right]$ ⟵ The identity matrix was adjoined to the matrix.

$\left[\begin{array}{rrr|rrr} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 5 & -2 & -2 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right]$ ⟵ −2 times the first row was added to the second row.

$$\left[\begin{array}{rrr|rrr} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & -2 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right] \longleftarrow \quad -2 \text{ times the third row was added to the second row.}$$

$$\left[\begin{array}{rrr|rrr} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & -2 \\ 0 & 0 & -1 & 4 & -2 & 5 \end{array}\right] \longleftarrow \quad -2 \text{ times the second row was added to the third row.}$$

$$\left[\begin{array}{rrr|rrr} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & -2 \\ 0 & 0 & 1 & -4 & 2 & -5 \end{array}\right] \longleftarrow \quad \text{The third row was multiplied by } -1 .$$

$$\left[\begin{array}{rrr|rrr} 1 & -1 & 0 & 5 & -2 & 5 \\ 0 & 1 & 0 & -2 & 1 & -2 \\ 0 & 0 & 1 & -4 & 2 & -5 \end{array}\right] \longleftarrow \quad -1 \text{ times the third row was added to the first row.}$$

$$\left[\begin{array}{rrr|rrr} 1 & 0 & 0 & 3 & -1 & 3 \\ 0 & 1 & 0 & -2 & 1 & -2 \\ 0 & 0 & 1 & -4 & 2 & -5 \end{array}\right] \longleftarrow \quad \text{The second row was added to the first row.}$$

Using $\begin{bmatrix} 1 & -1 & 1 \\ 2 & 3 & 0 \\ 0 & 2 & -1 \end{bmatrix}^{-1} = \begin{bmatrix} 3 & -1 & 3 \\ -2 & 1 & -2 \\ -4 & 2 & -5 \end{bmatrix}$ we obtain

$$X = \begin{bmatrix} 3 & -1 & 3 \\ -2 & 1 & -2 \\ -4 & 2 & -5 \end{bmatrix} \begin{bmatrix} 2 & -1 & 5 & 7 & 8 \\ 4 & 0 & -3 & 0 & 1 \\ 3 & 5 & -7 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 11 & 12 & -3 & 27 & 26 \\ -6 & -8 & 1 & -18 & -17 \\ -15 & -21 & 9 & -38 & -35 \end{bmatrix}$$

True-False Exercises

(a) True. By Theorem 1.6.1, if a system of linear equation has more than one solution then it must have infinitely many.

(b) True. If A is a square matrix such that $A\mathbf{x} = \mathbf{b}$ has a unique solution then the reduced row echelon form of A must be I. Consequently, $A\mathbf{x} = \mathbf{c}$ must have a unique solution as well.

(c) True. Since B is a square matrix then by Theorem 1.6.3(b) $AB = I_n$ implies $B = A^{-1}$.
Therefore, $BA = A^{-1}A = I_n$.

(d) True. Since A and B are row equivalent matrices, it must be possible to perform a sequence of elementary row operations on A resulting in B. Let E be the product of the corresponding elementary matrices, i.e., $EA = B$. Note that E must be an invertible matrix thus $A = E^{-1}B$.
Any solution of $A\mathbf{x} = \mathbf{0}$ is also a solution of $B\mathbf{x} = \mathbf{0}$ since $B\mathbf{x} = EA\mathbf{x} = E\mathbf{0} = \mathbf{0}$.
Likewise, any solution of $B\mathbf{x} = \mathbf{0}$ is also a solution of $A\mathbf{x} = \mathbf{0}$ since $A\mathbf{x} = E^{-1}B\mathbf{x} = E^{-1}\mathbf{0} = \mathbf{0}$.