CHAPTER 2: DETERMINANTS

2.1 Determinants by Cofactor Expansion

1.

$$M_{11} = \begin{vmatrix} 1 & -2 & 3 \\ 6 & 7 & -1 \\ -3 & 1 & 4 \end{vmatrix} = \begin{vmatrix} 7 & -1 \\ 1 & 4 \end{vmatrix} = 29 \qquad C_{11} = (-1)^{1+1} M_{11} = M_{11} = 29$$

$$M_{12} = \begin{vmatrix} 1 & -2 & 3 \\ 6 & 7 & -1 \\ -3 & 1 & 4 \end{vmatrix} = \begin{vmatrix} 6 & -1 \\ -3 & 4 \end{vmatrix} = 21 \qquad C_{12} = (-1)^{1+2} M_{12} = -M_{12} = -21$$

$$M_{13} = \begin{vmatrix} 1 & -2 & 3 \\ 6 & 7 & -1 \\ -3 & 1 & 4 \end{vmatrix} = \begin{vmatrix} 6 & 7 \\ -3 & 1 \end{vmatrix} = 27 \qquad C_{13} = (-1)^{1+3} M_{13} = M_{13} = 27$$

$$M_{21} = \begin{vmatrix} 1 & -2 & 3 \\ 6 & 7 & -1 \\ -3 & 1 & 4 \end{vmatrix} = \begin{vmatrix} -2 & 3 \\ 1 & 4 \end{vmatrix} = -11 \qquad C_{21} = (-1)^{2+1} M_{21} = -M_{21} = 11$$

$$M_{22} = \begin{vmatrix} 1 & -2 & 3 \\ 6 & 7 & -1 \\ -3 & 1 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 3 \\ -3 & 4 \end{vmatrix} = 13 \qquad C_{22} = (-1)^{2+2} M_{22} = M_{22} = 13$$

$$M_{23} = \begin{vmatrix} 1 & -2 & 3 \\ 6 & 7 & -1 \\ -3 & 1 & 4 \end{vmatrix} = \begin{vmatrix} 1 & -2 \\ -3 & 1 \end{vmatrix} = -5 \qquad C_{23} = (-1)^{2+3} M_{23} = -M_{23} = 5$$

$$M_{31} = \begin{vmatrix} 1 & -2 & 3 \\ 6 & 7 & -1 \\ -3 & 1 & 4 \end{vmatrix} = \begin{vmatrix} -2 & 3 \\ 7 & -1 \end{vmatrix} = -19 \qquad C_{31} = (-1)^{3+1} M_{31} = M_{31} = -19$$

$$M_{32} = \begin{vmatrix} 1 & -2 & 3 \\ 6 & 7 & -1 \\ -3 & 1 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 3 \\ 6 & -1 \end{vmatrix} = -19 \qquad C_{32} = (-1)^{3+2} M_{32} = -M_{32} = 19$$

$$M_{33} = \begin{vmatrix} 1 & -2 & 3 \\ 6 & 7 & -1 \\ -3 & 1 & 4 \end{vmatrix} = \begin{vmatrix} 1 & -2 \\ 6 & 7 \end{vmatrix} = 19 \qquad C_{33} = (-1)^{3+3} M_{33} = M_{33} = 19$$

3. **(a)** $M_{13} = \begin{vmatrix} 0 & 0 & 3 \\ 4 & 1 & 14 \\ 4 & 1 & 2 \end{vmatrix} = 0\begin{vmatrix} 1 & 14 \\ 1 & 2 \end{vmatrix} - 0\begin{vmatrix} 4 & 14 \\ 4 & 2 \end{vmatrix} + 3\begin{vmatrix} 4 & 1 \\ 4 & 1 \end{vmatrix}$ ← cofactor expansion along the first row

$= 0 - 0 + 3(0) = 0$

$C_{13} = (-1)^{1+3} M_{13} = M_{13} = 0$

(b) $M_{23} = \begin{vmatrix} 4 & -1 & 6 \\ 4 & 1 & 14 \\ 4 & 1 & 2 \end{vmatrix} = 4\begin{vmatrix} 1 & 14 \\ 1 & 2 \end{vmatrix} - (-1)\begin{vmatrix} 4 & 14 \\ 4 & 2 \end{vmatrix} + 6\begin{vmatrix} 4 & 1 \\ 4 & 1 \end{vmatrix}$ ← cofactor expansion along the first row

$= 4(-12) + 1(-48) + 6(0) = -96$

$C_{23} = (-1)^{2+3} M_{23} = -M_{23} = 96$

(c) $M_{22} = \begin{vmatrix} 4 & 1 & 6 \\ 4 & 0 & 14 \\ 4 & 3 & 2 \end{vmatrix} = -4\begin{vmatrix} 1 & 6 \\ 3 & 2 \end{vmatrix} + 0\begin{vmatrix} 4 & 6 \\ 4 & 2 \end{vmatrix} - 14\begin{vmatrix} 4 & 1 \\ 4 & 3 \end{vmatrix}$ ← cofactor expansion along the second row

$= -4(-16) + 0 - 14(8) = -48$

$C_{22} = (-1)^{2+2} M_{22} = M_{22} = -48$

(d) $M_{21} = \begin{vmatrix} -1 & 1 & 6 \\ 1 & 0 & 14 \\ 1 & 3 & 2 \end{vmatrix} = -1\begin{vmatrix} 1 & 6 \\ 3 & 2 \end{vmatrix} + 0\begin{vmatrix} -1 & 6 \\ 1 & 2 \end{vmatrix} - 14\begin{vmatrix} -1 & 1 \\ 1 & 3 \end{vmatrix}$ ← cofactor expansion along the second row

$= -1(-16) + 0 - 14(-4) = 72$

$C_{21} = (-1)^{2+1} M_{21} = -M_{21} = -72$

5. $\begin{vmatrix} 3 & 5 \\ -2 & 4 \end{vmatrix} = (3)(4) - (5)(-2) = 12 + 10 = 22 \neq 0$. Inverse: $\frac{1}{22}\begin{bmatrix} 4 & -5 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} \frac{2}{11} & \frac{-5}{22} \\ \frac{1}{11} & \frac{3}{22} \end{bmatrix}$

7. $\begin{vmatrix} -5 & 7 \\ -7 & -2 \end{vmatrix} = (-5)(-2) - (7)(-7) = 10 + 49 = 59 \neq 0$. Inverse: $\frac{1}{59}\begin{bmatrix} -2 & -7 \\ 7 & -5 \end{bmatrix} = \begin{bmatrix} \frac{-2}{59} & \frac{-7}{59} \\ \frac{7}{59} & \frac{-5}{59} \end{bmatrix}$

9. $\begin{vmatrix} a-3 & 5 \\ -3 & a-2 \end{vmatrix} = \begin{vmatrix} a-3 & 5 \\ -3 & a-2 \end{vmatrix} = (a-3)(a-2) - 5(-3) = a^2 - 5a + 6 + 15 = a^2 - 5a + 21$

11. $\begin{vmatrix} -2 & 1 & 4 \\ 3 & 5 & -7 \\ 1 & 6 & 2 \end{vmatrix} = \left|\begin{matrix} -2 & 1 & 4 \\ 3 & 5 & -7 \\ 1 & 6 & 2 \end{matrix}\right|\begin{matrix} -2 & 1 \\ 3 & 5 \\ 1 & 6 \end{matrix} = [-20-7+72]-[20+84+6] = -65$

13. $\begin{vmatrix} 3 & 0 & 0 \\ 2 & -1 & 5 \\ 1 & 9 & -4 \end{vmatrix} = \left|\begin{matrix} 3 & 0 & 0 \\ 2 & -1 & 5 \\ 1 & 9 & -4 \end{matrix}\right|\begin{matrix} 3 & 0 \\ 2 & -1 \\ 1 & 9 \end{matrix} = [12+0+0]-[0+135+0] = -123$

15. $\det(A) = \begin{vmatrix} \lambda-2 & 1 \\ -5 & \lambda+4 \end{vmatrix} = (\lambda-2)(\lambda+4)-(1)(-5) = \lambda^2+2\lambda-3 = (\lambda+3)(\lambda-1)$

The determinant is zero if $\lambda = -3$ or $\lambda = 1$.

17. $\det(A) = \begin{vmatrix} \lambda-1 & 0 \\ 2 & \lambda+1 \end{vmatrix} = (\lambda-1)(\lambda+1)$

The determinant is zero if $\lambda = 1$ or $\lambda = -1$.

19. **(a)** $3\begin{vmatrix} -1 & 5 \\ 9 & -4 \end{vmatrix} - 0 + 0 = 3(-41) = -123$

(b) $3\begin{vmatrix} -1 & 5 \\ 9 & -4 \end{vmatrix} - 2\begin{vmatrix} 0 & 0 \\ 9 & -4 \end{vmatrix} + 1\begin{vmatrix} 0 & 0 \\ -1 & 5 \end{vmatrix} = 3(-41)-2(0)+1(0) = -123$

(c) $-2\begin{vmatrix} 0 & 0 \\ 9 & -4 \end{vmatrix} + (-1)\begin{vmatrix} 3 & 0 \\ 1 & -4 \end{vmatrix} - 5\begin{vmatrix} 3 & 0 \\ 1 & 9 \end{vmatrix} = -2(0)-1(-12)-5(27) = -123$

(d) $-0+(-1)\begin{vmatrix} 3 & 0 \\ 1 & -4 \end{vmatrix} - 9\begin{vmatrix} 3 & 0 \\ 2 & 5 \end{vmatrix} = -1(-12)-9(15) = -123$

(e) $1\begin{vmatrix} 0 & 0 \\ -1 & 5 \end{vmatrix} - 9\begin{vmatrix} 3 & 0 \\ 2 & 5 \end{vmatrix} + (-4)\begin{vmatrix} 3 & 0 \\ 2 & -1 \end{vmatrix} = 1(0)-9(15)-4(-3) = -123$

(f) $0-5\begin{vmatrix} 3 & 0 \\ 1 & 9 \end{vmatrix} + (-4)\begin{vmatrix} 3 & 0 \\ 2 & -1 \end{vmatrix} = -5(27)-4(-3) = -123$

21. Calculate the determinant by a cofactor expansion along the second column:

$$-0+5\begin{vmatrix} -3 & 7 \\ -1 & 5 \end{vmatrix} - 0 = 5(-8) = -40$$

23. Calculate the determinant by a cofactor expansion along the first column:

$$1\begin{vmatrix} k & k^2 \\ k & k^2 \end{vmatrix} - 1\begin{vmatrix} k & k^2 \\ k & k^2 \end{vmatrix} + 1\begin{vmatrix} k & k^2 \\ k & k^2 \end{vmatrix} = 1(0)-1(0)+1(0) = 0$$

25. Calculate the determinant by a cofactor expansion along the third column:

$$\det(A) = 0 - 0 + (-3)\begin{vmatrix} 3 & 3 & 5 \\ 2 & 2 & -2 \\ 2 & 10 & 2 \end{vmatrix} - 3\begin{vmatrix} 3 & 3 & 5 \\ 2 & 2 & -2 \\ 4 & 1 & 0 \end{vmatrix}$$

Calculate the determinants in the third and fourth terms by a cofactor expansion along the first row:

$$\begin{vmatrix} 3 & 3 & 5 \\ 2 & 2 & -2 \\ 2 & 10 & 2 \end{vmatrix} = 3\begin{vmatrix} 2 & -2 \\ 10 & 2 \end{vmatrix} - 3\begin{vmatrix} 2 & -2 \\ 2 & 2 \end{vmatrix} + 5\begin{vmatrix} 2 & 2 \\ 2 & 10 \end{vmatrix} = 3(24) - 3(8) + 5(16) = 128$$

$$\begin{vmatrix} 3 & 3 & 5 \\ 2 & 2 & -2 \\ 4 & 1 & 0 \end{vmatrix} = 3\begin{vmatrix} 2 & -2 \\ 1 & 0 \end{vmatrix} - 3\begin{vmatrix} 2 & -2 \\ 4 & 0 \end{vmatrix} + 5\begin{vmatrix} 2 & 2 \\ 4 & 1 \end{vmatrix} = 3(2) - 3(8) + 5(-6) = -48$$

Therefore $\det(A) = 0 - 0 - 3(128) - 3(-48) = -240$.

27. By Theorem 2.1.2, determinant of a diagonal matrix is the product of the entries on the main diagonal: $\det(A) = (1)(-1)(1) = -1$.

29. By Theorem 2.1.2, determinant of a lower triangular matrix is the product of the entries on the main diagonal: $\det(A) = (0)(2)(3)(8) = 0$.

31. By Theorem 2.1.2, determinant of an upper triangular matrix is the product of the entries on the main diagonal: $\det(A) = (1)(1)(2)(3) = 6$.

33. **(a)** $\begin{vmatrix} \sin\theta & \cos\theta \\ -\cos\theta & \sin\theta \end{vmatrix} = (\sin\theta)(\sin\theta) - (\cos\theta)(-\cos\theta) = \sin^2\theta + \cos^2\theta = 1$

(b) Calculate the determinant by a cofactor expansion along the third column:

$0 - 0 + 1\begin{vmatrix} \sin\theta & \cos\theta \\ -\cos\theta & \sin\theta \end{vmatrix} = 0 - 0 + (1)(1) = 1$ (we used the result of part (a))

35. The minor M_{11} in both determinants is $\begin{vmatrix} 1 & f \\ 0 & 1 \end{vmatrix} = 1$. Expanding both determinants along the first row yields $d_1 + \lambda = d_2$.

43. Calculate the determinant by a cofactor expansion along the first column:

$$\begin{vmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{vmatrix} = \begin{vmatrix} x_2 & x_2^2 \\ x_3 & x_3^2 \end{vmatrix} + \begin{vmatrix} x_1 & x_1^2 \\ x_3 & x_3^2 \end{vmatrix} + \begin{vmatrix} x_1 & x_1^2 \\ x_2 & x_2^2 \end{vmatrix}$$

$$= (x_2x_3^2 - x_3x_2^2) - (x_1x_3^2 - x_3x_1^2) + (x_1x_2^2 - x_2x_1^2) = \left[x_3^2(x_2 - x_1) - x_3\left(x_2^2 - x_1^2\right)\right] + x_1x_2^2 - x_2x_1^2.$$

Factor out $(x_2 - x_1)$ to get $(x_2 - x_1)\left[x_3^2 - x_2x_3 - x_1x_3 + x_1x_2\right] = (x_2 - x_1)\left[x_3^2 - (x_2 + x_1)x_3 + x_1x_2\right]$.

Since $x_3^2 - (x_2 + x_1)x_3 + x_1x_2 = (x_3 - x_1)(x_3 - x_2)$, the determinant is $(x_2 - x_1)(x_3 - x_1)(x_3 - x_2)$.

True-False Exercises

(a) False. The determinant is $ad-bc$.

(b) False. E.g., $\det(I_2)=\det(I_3)=1$.

(c) True. If $i+j$ is even then $(-1)^{i+j}=1$ therefore $C_{ij}=(-1)^{i+j}M_{ij}=M_{ij}$.

(d) True. Let $A=\begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix}$.

Then $C_{12}=(-1)^{1+2}\begin{vmatrix} b & e \\ c & f \end{vmatrix}=-(bf-ec)$ and $C_{21}=(-1)^{2+1}\begin{vmatrix} b & c \\ e & f \end{vmatrix}=-(bf-ce)$ therefore $C_{12}=C_{21}$. In the same way, one can show $C_{13}=C_{31}$ and $C_{23}=C_{32}$.

(e) True. This follows from Theorem 2.1.1.

(f) True. In formulas (7) and (8), each cofactor C_{ij} is zero.

(g) False. The determinant of a lower triangular matrix is the *product* of the entries along the main diagonal.

(h) False. E.g. $\det(2I_2)=4\neq 2=2\det(I_2)$.

(i) False. E.g., $\det(I_2+I_2)=4\neq 2=\det(I_2)+\det(I_2)$.

(j) True. $\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}^2\right)=\begin{vmatrix} a^2+bc & ab+bd \\ ac+cd & bc+d^2 \end{vmatrix}=(a^2+bc)(bc+d^2)-(ab+bd)(ac+cd)$

$=a^2bc+a^2d^2+b^2c^2+bcd^2-a^2bc-abcd-abcd-bcd^2=a^2d^2+b^2c^2-2abcd$.

$\begin{vmatrix} a & b \\ c & d \end{vmatrix}^2=(ad-bc)^2=a^2d^2-2adbc+b^2c^2$ therefore $\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}^2\right)=\left(\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)\right)^2$.

2.2 Evaluating Determinants by Row Reduction

1. $\det(A)=\begin{vmatrix} -2 & 3 \\ 1 & 4 \end{vmatrix}=(-2)(4)-(3)(1)=-11$; $\det(A^T)=\begin{vmatrix} -2 & 1 \\ 3 & 4 \end{vmatrix}=(-2)(4)-(1)(3)=-11$

3. $\det(A)=\begin{vmatrix} 2 & -1 & 3 \\ 1 & 2 & 4 \\ 5 & -3 & 6 \end{vmatrix}=[24-20-9]-[30-24-6]=-5$;

$\det(A^T)=\begin{vmatrix} 2 & 1 & 5 \\ -1 & 2 & -3 \\ 3 & 4 & 6 \end{vmatrix}=[24-9-20]-[30-24-6]=-5$ (we used the arrow technique)

5. The third row of I_4 was multiplied by -5. By Theorem 2.2.4, the determinant equals -5.

7. The second and the third rows of I_4 were interchanged. By Theorem 2.2.4, the determinant equals -1.

9.

$$\begin{vmatrix} 3 & -6 & 9 \\ -2 & 7 & -2 \\ 0 & 1 & 5 \end{vmatrix} = 3\begin{vmatrix} 1 & -2 & 3 \\ -2 & 7 & -2 \\ 0 & 1 & 5 \end{vmatrix}$$

A common factor of 3 from the first row was taken through the determinant sign.

$$= 3\begin{vmatrix} 1 & -2 & 3 \\ 0 & 3 & 4 \\ 0 & 1 & 5 \end{vmatrix}$$

2 times the first row was added to the second row.

$$= 3(-1)\begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 0 & 3 & 4 \end{vmatrix}$$

The second and third rows were interchanged.

$$= (3)(-1)\begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 0 & 0 & -11 \end{vmatrix}$$

-3 times the second row was added to the third row.

$$= (3)(-1)(-11)\begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{vmatrix}$$

A common factor of -11 from the last row was taken through the determinant sign.

$$= (3)(-1)(-11)(1) = 33$$

Another way to evaluate the determinant would be to use cofactor expansion along the first column after the second step above:

$$\begin{vmatrix} 3 & -6 & 9 \\ -2 & 7 & -2 \\ 0 & 1 & 5 \end{vmatrix} = 3\begin{vmatrix} 1 & -2 & 3 \\ 0 & 3 & 4 \\ 0 & 1 & 5 \end{vmatrix} = 3\left[1\begin{vmatrix} 3 & 4 \\ 1 & 5 \end{vmatrix} - 0 + 0\right] = 3\big[(1)(11)\big] = 33 .$$

11.

$$\begin{vmatrix} 2 & 1 & 3 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 \end{vmatrix} = (-1)\begin{vmatrix} 1 & 0 & 1 & 1 \\ 2 & 1 & 3 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 \end{vmatrix}$$

The first and second rows were interchanged.

$$= (-1)\begin{vmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 \end{vmatrix}$$

-2 times the first row was added to the second row.

$$=(-1)\begin{vmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & -1 & 2 \\ 0 & 1 & 2 & 3 \end{vmatrix}$$ $\leftarrow$ -2 times the second row was added to the third row.

$$=(-1)\begin{vmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 1 & 4 \end{vmatrix}$$ $\leftarrow$ -1 times the second row was added to the fourth row.

$$=(-1)(-1)\begin{vmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 1 & 4 \end{vmatrix}$$ $\leftarrow$ A common factor of -1 from the third row was taken through the determinant sign.

$$=(-1)(-1)\begin{vmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 6 \end{vmatrix}$$ $\leftarrow$ -1 times the third row was added to the fourth row.

$$=(-1)(-1)(6)\begin{vmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$ $\leftarrow$ A common factor of 6 from the third row was taken through the determinant sign.

$$=(-1)(-1)(6)(1)=6$$

Another way to evaluate the determinant would be to use cofactor expansions along the first column after the fourth step above:

$$\begin{vmatrix} 2 & 1 & 3 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 \end{vmatrix} = (-1)\begin{vmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 1 & 4 \end{vmatrix} = (-1)(1)\begin{vmatrix} 1 & 1 & -1 \\ 0 & -1 & 2 \\ 0 & 1 & 4 \end{vmatrix} = (-1)(1)(1)\begin{vmatrix} -1 & 2 \\ 1 & 4 \end{vmatrix}$$

$$=(-1)(1)(1)(-6)=6 \; .$$

13. $$\begin{vmatrix} 1 & 3 & 1 & 5 & 3 \\ -2 & -7 & 0 & -4 & 2 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{vmatrix}$$

$$=\begin{vmatrix} 1 & 3 & 1 & 5 & 3 \\ 0 & -1 & 2 & 6 & 8 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{vmatrix} \longleftarrow \text{2 times the first row was added to the second row.}$$

$$=(-1)\begin{vmatrix} 1 & 3 & 1 & 5 & 3 \\ 0 & 1 & -2 & -6 & -8 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{vmatrix} \longleftarrow \text{A common factor of } -1 \text{ from the second row was taken through the determinant sign.}$$

$$=(-1)\begin{vmatrix} 1 & 3 & 1 & 5 & 3 \\ 0 & 1 & -2 & -6 & -8 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{vmatrix} \longleftarrow -2 \text{ times the third row was added to the fourth row.}$$

$$=(-1)\begin{vmatrix} 1 & 3 & 1 & 5 & 3 \\ 0 & 1 & -2 & -6 & -8 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 2 \end{vmatrix} \longleftarrow -1 \text{ times the fourth row was added to the fifth row.}$$

$$=(-1)(2)\begin{vmatrix} 1 & 3 & 1 & 5 & 3 \\ 0 & 1 & -2 & -6 & -8 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{vmatrix} \longleftarrow \text{A common factor of 2 from the fifth row was taken through the determinant sign.}$$

$$=(-1)(2)(1)=-2$$

Another way to evaluate the determinant would be to use cofactor expansions along the first column after the third step above:

$$\begin{vmatrix} 1 & 3 & 1 & 5 & 3 \\ -2 & -7 & 0 & -4 & 2 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{vmatrix} = (-1)\begin{vmatrix} 1 & 3 & 1 & 5 & 3 \\ 0 & 1 & -2 & -6 & -8 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{vmatrix} = (-1)(1)\begin{vmatrix} 1 & -2 & -6 & -8 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{vmatrix}$$

$$=(-1)(1)(1)\begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{vmatrix} = (-1)(1)(1)(1)\begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = (-1)(1)(1)(1)(2) = -2 .$$

15.

$$\begin{vmatrix} d & e & f \\ g & h & i \\ a & b & c \end{vmatrix} = (-1)\begin{vmatrix} a & b & c \\ g & h & i \\ d & e & f \end{vmatrix}$$

The first and third rows were interchanged.

$$= (-1)(-1)\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$$

The second and third rows were interchanged.

$$= (-1)(-1)(-6) = -6$$

17.

$$\begin{vmatrix} 3a & 3b & 3c \\ -d & -e & -f \\ 4g & 4h & 4i \end{vmatrix} = 3\begin{vmatrix} a & b & c \\ -d & -e & -f \\ 4g & 4h & 4i \end{vmatrix}$$

A common factor of 3 from the first row was taken through the determinant sign.

$$= 3(-1)\begin{vmatrix} a & b & c \\ d & e & f \\ 4g & 4h & 4i \end{vmatrix}$$

A common factor of -1 from the second row was taken through the determinant sign.

$$= 3(-1)(4)\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$$

A common factor of 4 from the third row was taken through the determinant sign.

$$= 3(-1)(4)(-6) = 72$$

19.

$$\begin{vmatrix} a+g & b+h & c+i \\ d & e & f \\ g & h & i \end{vmatrix} = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$$

-1 times the third row was added to the first row.

$$= -6$$

21.

$$\begin{vmatrix} -3a & -3b & -3c \\ d & e & f \\ g-4d & h-4e & i-4f \end{vmatrix}$$

$$= -3\begin{vmatrix} a & b & c \\ d & e & f \\ g-4d & h-4e & i-4f \end{vmatrix}$$

A common factor of -3 from the first row was taken through the determinant sign.

$$= -3\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$$

4 times the second row was added to the last row.

$$=(-3)(-6)=18$$

23. $$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & b-a & c-a \\ a^2 & b^2 & c^2 \end{vmatrix}$$

$-a$ times the first row was added to the second row.

$$= \begin{vmatrix} 1 & 1 & 1 \\ 0 & b-a & c-a \\ 0 & b^2-a^2 & c^2-a^2 \end{vmatrix}$$

$-a^2$ times the first row was added to the third row.

$$= \begin{vmatrix} 1 & 1 & 1 \\ 0 & b-a & c-a \\ 0 & 0 & c^2-a^2-(c-a)(b+a) \end{vmatrix}$$

$-(b+a)$ times the second row was added to the third row.

$$=(1)(b-a)(c-a)(c+a-b-a)$$

$$=(b-a)(c-a)(c-b)$$

25. $$\begin{vmatrix} a_1 & b_1 & a_1+b_1+c_1 \\ a_2 & b_2 & a_2+b_2+c_2 \\ a_3 & b_3 & a_3+b_3+c_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & b_1 & b_1+c_1 \\ a_2 & b_2 & b_2+c_2 \\ a_3 & b_3 & b_3+c_3 \end{vmatrix}$$

-1 times the first column was added to the third column.

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

-1 times the second column was added to the third column.

27. $$\begin{vmatrix} a_1+b_1 & a_1-b_1 & c_1 \\ a_2+b_2 & a_2-b_2 & c_2 \\ a_3+b_3 & a_3-b_3 & c_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1+b_1 & -2b_1 & c_1 \\ a_2+b_2 & -2b_2 & c_2 \\ a_3+b_3 & -2b_3 & c_3 \end{vmatrix}$$

-1 times the first column was added to the second column.

$$=-2\begin{vmatrix} a_1+b_1 & b_1 & c_1 \\ a_2+b_2 & b_2 & c_2 \\ a_3+b_3 & b_3 & c_3 \end{vmatrix}$$

A common factor of -2 from the second column was taken through the determinant sign.