Functions

Section 2.3

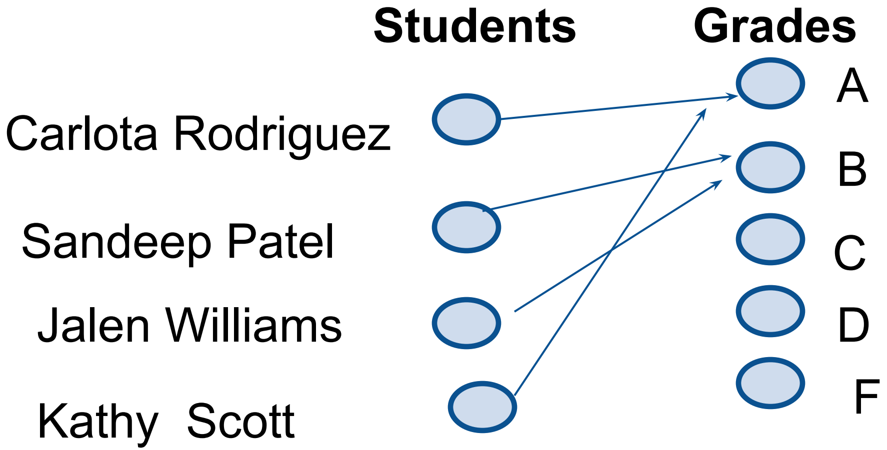
Section Summary

- Definition of a Function.
  - Domain, Cdomain
  - Image, Preimage
- Injection, Surjection, Bijection
- Inverse Function
- Function Composition

Functions

Definition: Let A and B be nonempty sets. A *function* f from A to B, denoted $f: A \to B$ is an assignment of each element of A to exactly one element of B. We write $f(a) = b$ if b is the unique element of B assigned by the function f to the element a of A.

- Functions are sometimes called *mappings* or *transformations*.



Functions

- A function $f: A \rightarrow B$ can also be defined as a subset of $A \times B$ (a relation). This subset is restricted to be a relation where no two elements of the relation have the same first element.
- Specifically, a function f from A to B contains one, and only one ordered pair (a, b) for every element $a \in A$.

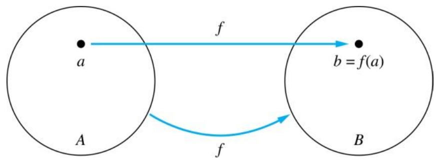
$$\forall x[x \in A \rightarrow \exists y[y \in B \wedge (x, y) \in f]]$$

and

$$\forall x, y_1, y_2[[(x, y_1) \in f \wedge (x, y_2)] \rightarrow y_1 = y_2]$$

Functions

Given a function f: $A \to B$:



- We say f *maps* A to B *or*

  f is a *mapping* from A to B.
- A is called the *domain* of f.
- B is called the *codomain* of f.
- If $f(a) = b$,
  - then b is called the *image* of a under f.
  - a is called the *preimage* of b.
- The range of f is the set of all images of points in **A** under f. We denote it by $f(A)$.
- Two functions are *equal* when they have the same domain, the same codomain and map each element of the domain to the same element of the codomain.

Representing Functions

- Functions may be specified in different ways:
  - An explicit statement of the assignment.

    Students and grades example.
  - A formula.

    $f(x) = x + 1$
  - A computer program.
    - A Java program that when given an integer *n*, produces the *n*th Fibonacci Number (covered in the next section and also inChapter 5).

Questions

f(a) = ?   z

The image of d is ? z

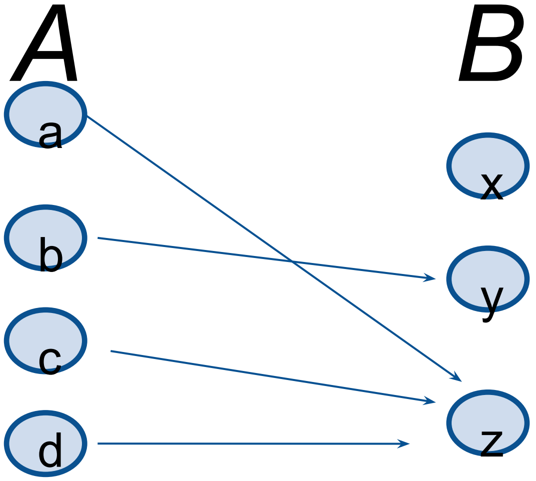
The domain of f is ? *A*

The codomain of f is   *B*
?

The preimage of y is b
?

$f(A)$ = ?

The preimage(s) of z is (are) ?   {a,c,d}



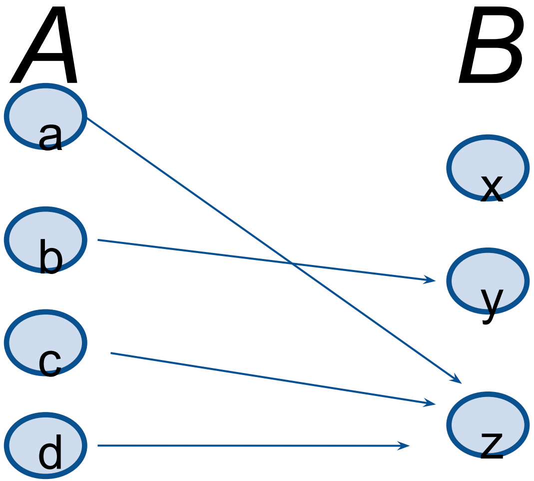
Question on Functions and Sets

- If $f : A \to B$ and S is a subset of A, then

$$f(S) = \{f(s) | s \in S\}$$

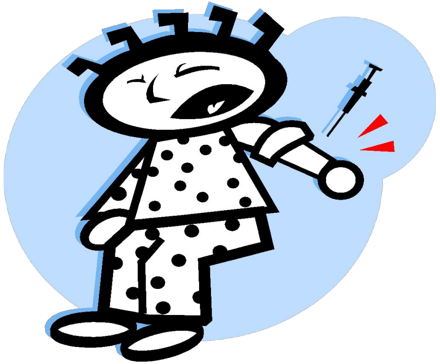
f {a,b,c,} is ? {y,z}

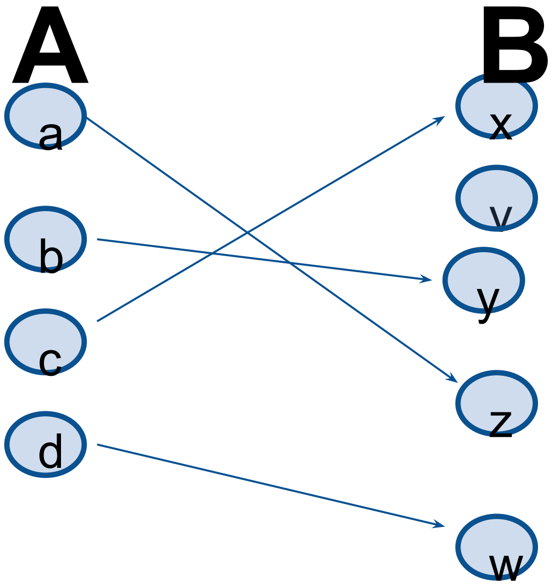
f {c,d} is ? {z}



Injections

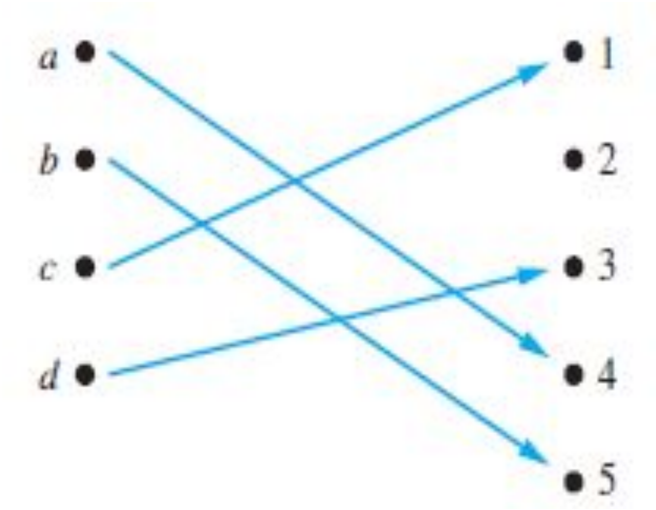
- **Definition**: A function *f* is said to be *one-to-one,* or an *injection,* if and only if *f* (*a*) = *f* (*b*) implies that *a* = *b* for all *a* and *b* in the domain of *f.* A function is said to be *injective* if it is one-to-one.





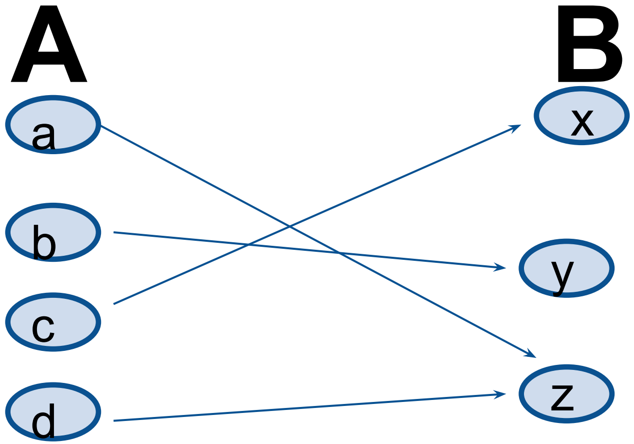
Example

Determine whether the function f from {a, b, c, d} to {1, 2, 3, 4, 5} with f (a) = 4, f (b) = 5, f (c) = 1, and f (d) = 3 is one-to-one.



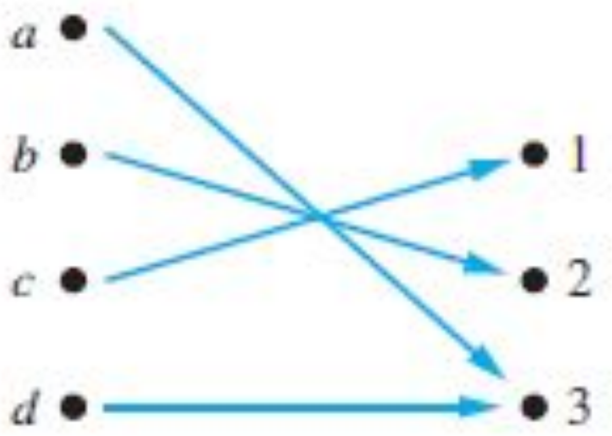
Surjections

- **Definition**:A function f from A to B is called *onto,* or a *surjection,* if and only if for every element $b \in B$ there is an element $a \in A$ with $f(a) = b$. A function f is called *surjective* if it is onto.



Examples

Example 1:Let f be the function from {a, b, c, d} to {1, 2, 3} defined by f (a) = 3, f (b) = 2, f (c) = 1, and f (d) = 3. Is f an onto function?

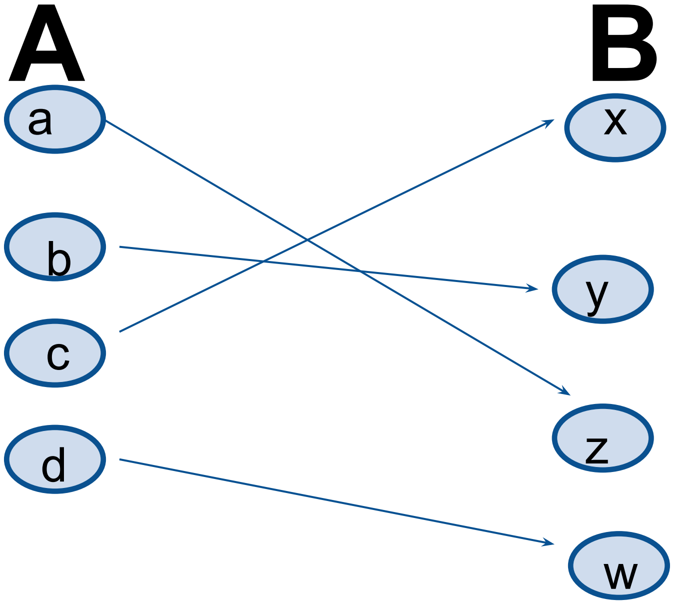


Example 2: Is the function $f(x) = x^2$ from the set of integers onto?

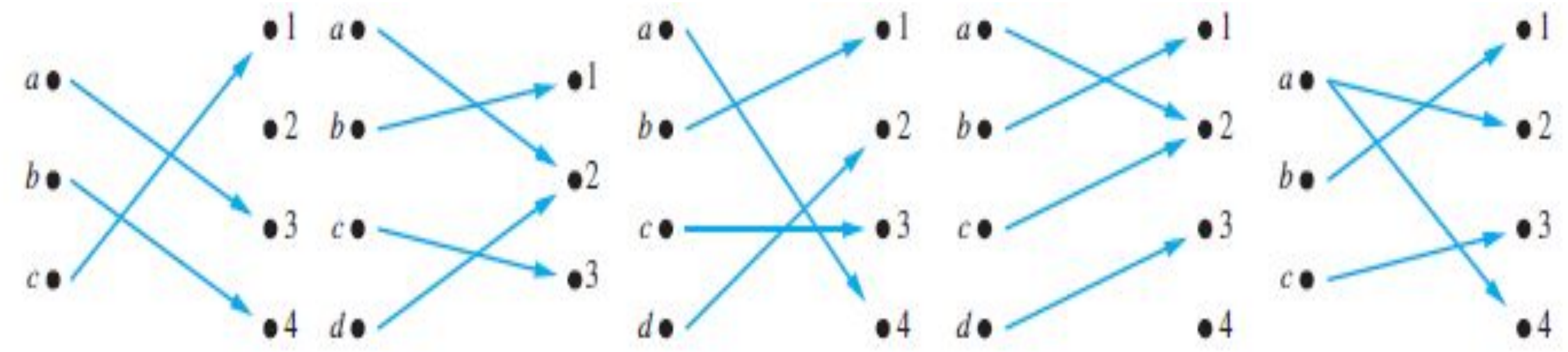
Solution: No, f is not onto because there is no integer x with $x^2 = -1$, for example.

Bijections

- **Definition**: The function *f* is a *one-to-one correspondence*, or a *bijection*, if it is both one-to-one and onto. We also say that such a function is *bijective*.



Identify one to one and onto or both

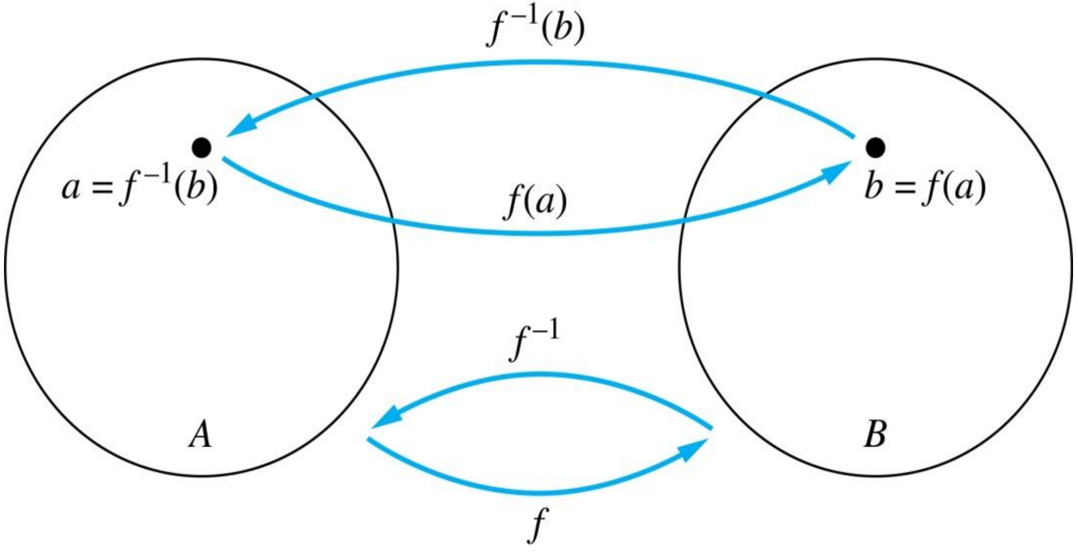


Example: Let **f** be the function from {a, b, c, d} to {1, 2, 3, 4} with f(a) = 4, f(b) = 2, f(c) = 1, and f(d) =3. Is f a bijection?

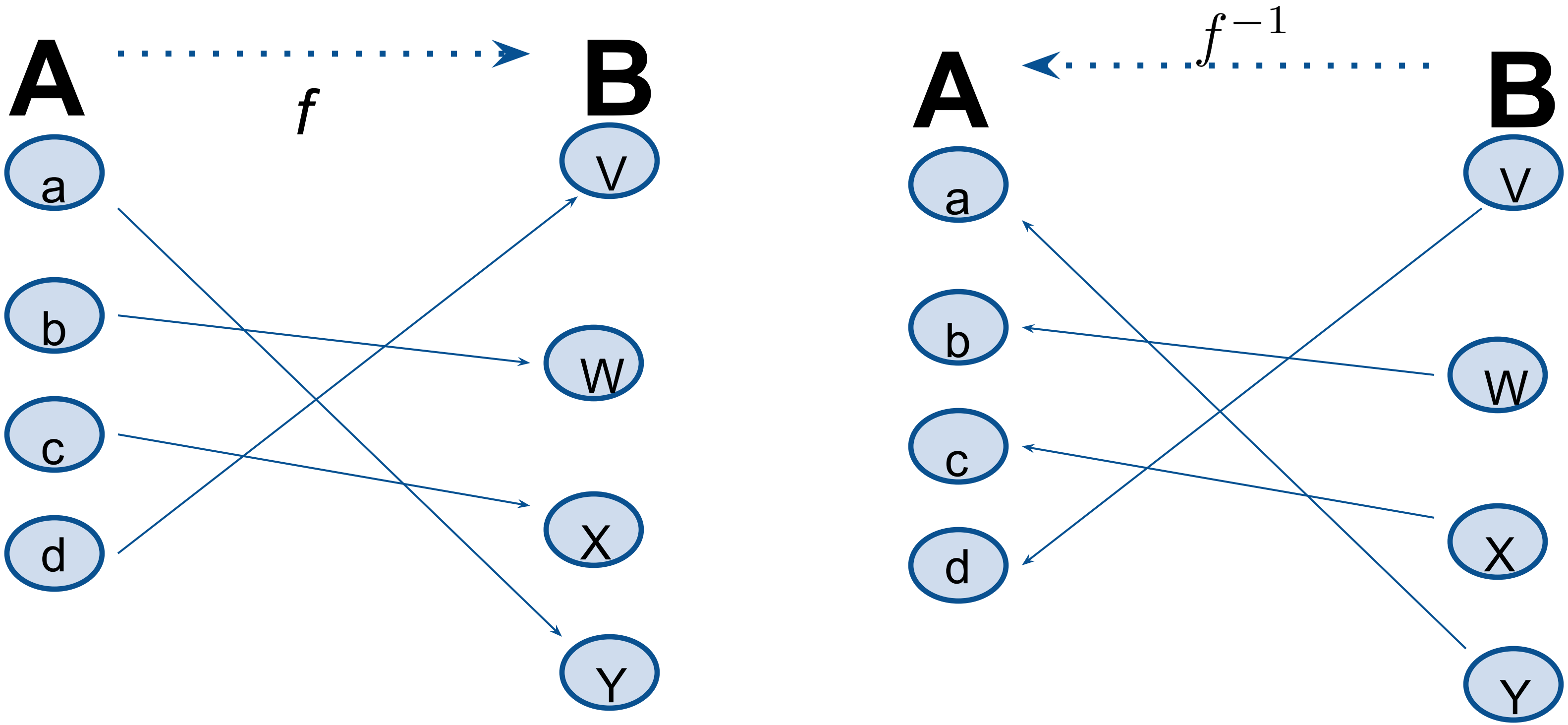
Solution: The function f is one-to-one and onto. It is one-to-one because no two values in the domain are assigned the same function value. It is onto because all four elements of the codomain are images of elements in the domain. Hence, f is a bijection.

Inverse Functions

- **Definition**: Let f be a one-to-one correspondence from the set A to the set B. The *inverse function* of f is the function that assigns to an element b belonging to B the unique element a in A such that $f(a) = b$. The inverse function of f is denoted by f −1. Hence, f −1(b) = a when $f(a) = b$.



Inverse Functions



Questions

Example 1: Let f be the function from $\{a,b,c\}$ to $\{1,2,3\}$ such that $f(a) = 2$, $f(b) = 3$, and $f(c) = 1$. Is f invertible and if so what is its inverse?

Solution: The function f is invertible because it is a one-to-one correspondence. The inverse function f^{-1} reverses the correspondence given by f, so $f^{-1}(1) = c$, $f^{-1}(2) = a$, and $f^{-1}(3) = b$.

Questions

Example 3: Let f be the function from **R** to **R** with $f(x) = x2$. Is f invertible?

- **Solution:** Because $f(-2) = f(2) = 4$, f is not one-to-one. If an inverse function were defined, it would have to assign two elements to 4. Hence, f is not invertible.

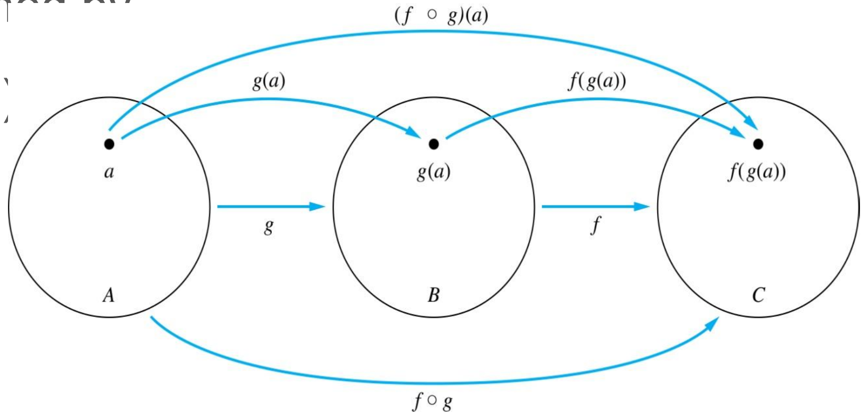
(Note we can also show that f is not invertible because it is not onto.)

Composition

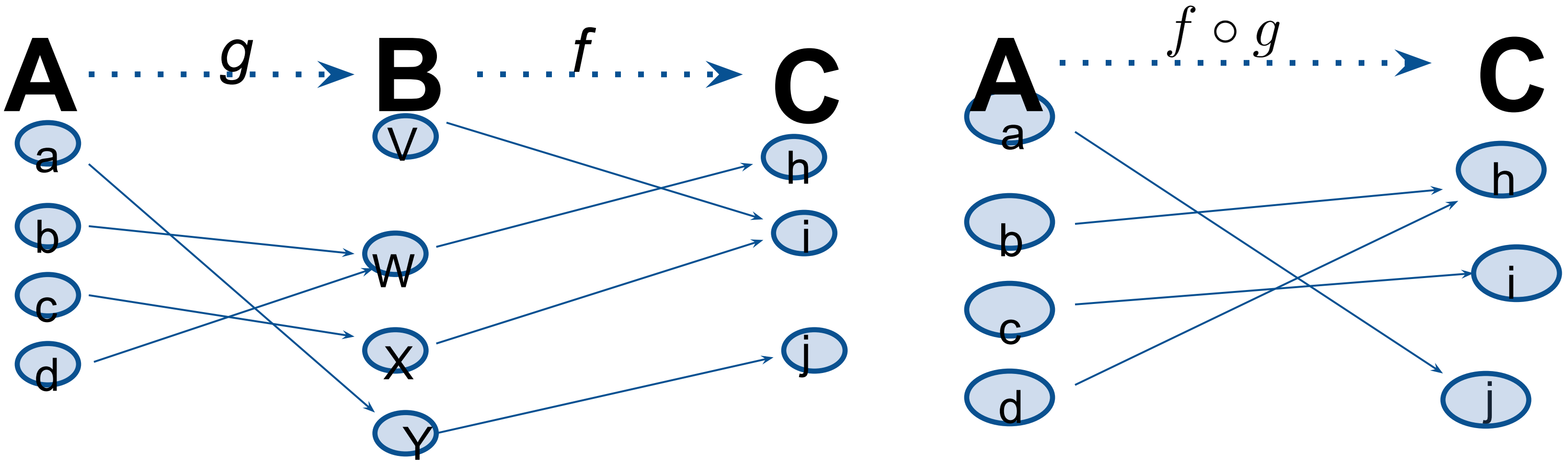
- **Definition**: Let g be a function from the set A to the set B and let f be a function from the set B to the the
- set C. The composition of the functions f and g, denoted for all a ∈ A by f ◦g, is the function

$$f \circ g(x) = f(g(x))$$

- from A to C defin
- (f ◦g)(a) = f (g(a
-



Composition



Composition

Example 1: If $f(x) = x^2$ and $g(x) = 2x + 1$, then

$$f(g(x)) = (2x + 1)^2$$

and

$$g(f(x)) = 2x^2 + 1$$

Composition Questions

Example 2: Let g be the function from the set $\{a,b,c\}$ to itself such that $g(a) = b$, $g(b) = c$, and $g(c) = a$. Let f be the function from the set $\{a,b,c\}$ to the set $\{1,2,3\}$ such that $f(a) = 3$, $f(b) = 2$, and $f(c) = 1$.

What is the composition of f and g, and what is the composition of g and f.

Solution: The composition $f \circ g$ is defined by

$f \circ g\ (a) = f(g(a)) = f(b) = 2.$

$f \circ g\ (b) = f(g(b)) = f(c) = 1.$

$f \circ g\ (c) = f(g(c)) = f(a) = 3.$

Note that $g \circ f$ is not defined, because the range of f is not a subset of the domain of g.

Composition Questions

Example 2: Let f and g be functions from the set of integers to the set of integers defined by $f(x) = 2x + 3$ and $g(x) = 3x + 2$.

What is the composition of *f* and *g*, and also the composition of *g* and *f* ?

Solution:

$f \circ g\ (x) = f(g(x)) = f(3x + 2) = 2(3x + 2) + 3 = 6x + 7$

$g \circ f\ (x) = g(f(x)) = g(2x + 3) = 3(2x + 3) + 2 = 6x + 11$

Quiz

1. Using law of logic, simplify the statement form

   a. $p \vee [\sim(\sim p \wedge q)]$

2. Using Laws of Logic, verify the logical equivalence

   a. $\sim(\sim p \wedge q) \wedge (p \vee q) \equiv p$

3. Express the following propositions as an *English sentence.*

   Let p, q, and r be the propositions:

   p = “you have the flu”

   q = “you miss the final exam”

   r = “you pass the course”

   a. $\sim q \rightarrow r$

   b. $\sim p \wedge \sim q \rightarrow r$