(b) True. $\det(B) = (4)(\frac{3}{4})\det(A) = 3\det(A)$.

(c) False. $\det(B) = \det(A)$.

(d) False. $\det(B) = n(n-1)\cdots 3\cdot 2\cdot 1\cdot \det(A) = (n!)\det(A)$.

(e) True. This follows from Theorem 2.2.5.

(f) True. Let B be obtained from A by adding the second row to the fourth row, so $\det(A) = \det(B)$. Since the fourth row and the sixth row of B are identical, by Theorem 2.2.5 $\det(B) = 0$.

2.3 Properties of Determinants; Cramer's Rule

1. $\det(2A) = \begin{vmatrix} -2 & 4 \\ 6 & 8 \end{vmatrix} = (-2)(8) - (4)(6) = -40$

$(2)^2 \det(A) = 4\begin{vmatrix} -1 & 2 \\ 3 & 4 \end{vmatrix} = 4((-1)(4) - (2)(3)) = (4)(-10) = -40$

2. $\det(-4A) = \begin{vmatrix} -8 & -8 \\ -20 & 8 \end{vmatrix} = (-8)(8) - (-8)(-20) = -224$

$(-4)^2 \det(A) = 16\begin{vmatrix} 2 & 2 \\ 5 & -2 \end{vmatrix} = 16((2)(-2) - (2)(5)) = (16)(-14) = -224$

3. We are using the arrow technique to evaluate both determinants.

$$\det(-2A) = \begin{vmatrix} -4 & 2 & -6 \\ -6 & -4 & -2 \\ -2 & -8 & -10 \end{vmatrix} = (-160 + 8 - 288) - (-48 - 64 + 120) = -448$$

$$(-2)^3 \det(A) = -8\begin{vmatrix} 2 & -1 & 3 \\ 3 & 2 & 1 \\ 1 & 4 & 5 \end{vmatrix} = (-8)((20 - 1 + 36) - (6 + 8 - 15)) = (-8)(56) = -448$$

4. We are using the cofactor expansion along the first column to evaluate both determinants.

$$\det(3A) = \begin{vmatrix} 3 & 3 & 3 \\ 0 & 6 & 9 \\ 0 & 3 & -6 \end{vmatrix} = 3\begin{vmatrix} 6 & 9 \\ 3 & -6 \end{vmatrix} = 3((6)(-6) - (9)(3)) = (3)(-63) = -189$$

$$3^3 \det(A) = 27\begin{vmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 0 & 1 & -2 \end{vmatrix} = (27)(1)\begin{vmatrix} 2 & 3 \\ 1 & -2 \end{vmatrix} = 27((2)(-2) - (3)(1)) = (27)(-7) = -189$$

5. We are using the arrow technique to evaluate the determinants in this problem.

$$\det(AB)=\begin{vmatrix}9&-1&8\\31&1&17\\10&0&2\end{vmatrix}=(18-170+0)-(80+0-62)=-170;$$

$$\det(BA)=\begin{vmatrix}-1&-3&6\\17&11&4\\10&5&2\end{vmatrix}=(-22-120+510)-(660-20-102)=-170;$$

$$\det(A+B)=\begin{vmatrix}3&0&3\\10&5&2\\5&0&3\end{vmatrix}=(45+0+0)-(75+0+0)=-30;$$

$\det(A)=(16+0+0)-(0+0+6)=10$;

$\det(B)=(1-10+0)-(15+0-7)=-17$;

$\det(A+B)\neq\det(A)+\det(B)$

6. We are using the arrow technique to evaluate the determinants in this problem.

$$\det(AB)=\begin{vmatrix}6&15&26\\2&-4&-3\\-2&10&12\end{vmatrix}=(-288+90+520)-(208-180+360)=-66;$$

$$\det(BA)=\begin{vmatrix}5&8&-3\\-6&14&7\\5&-2&-5\end{vmatrix}=(-350+280-36)-(-210-70+240)=-66;$$

$$\det(A+B)=\begin{vmatrix}1&7&-2\\2&1&2\\-2&5&1\end{vmatrix}=(1-28-20)-(4+10+14)=-75;$$

$\det(A)=(0+16+4)-(0+2+16)=2$;

$\det(B)=(-2+0-12)-(0+18+1)=-33$;

$\det(A+B)\neq\det(A)+\det(B)$;

7. $\det(A)=(-6+0-20)-(-10+0-15)=-1\neq 0$ therefore A is invertible by Theorem 2.3.3

8. $\det(A)=(-24+0+0)-(-18+0+0)=-6\neq 0$ therefore A is invertible by Theorem 2.3.3

9. $\det(A)=(2)(1)(2)=4\neq 0$ therefore A is invertible by Theorem 2.3.3

10. $\det(A)=0$ (second column contains only zeros) therefore A is not invertible by Theorem 2.3.3

11. $\det(A)=(24-24-16)-(24-16-24)=0$ therefore A is not invertible by Theorem 2.3.3

12. $\det(A)=(1+0-81)-(8+36+0)=-124\neq 0$ therefore A is invertible by Theorem 2.3.3

13. $\det(A)=(2)(1)(6)=12\neq 0$ therefore A is invertible by Theorem 2.3.3

14. $\det(A)=0$ (third column contains only zeros) therefore A is not invertible by Theorem 2.3.3

15. $\det(A)=(k-3)(k-2)-(-2)(-2)=k^2-5k+2=\left(k-\frac{5-\sqrt{17}}{2}\right)\left(k-\frac{5+\sqrt{17}}{2}\right)$. By Theorem 2.3.3, A is invertible if $k\neq\frac{5-\sqrt{17}}{2}$ and $k\neq\frac{5+\sqrt{17}}{2}$.

16. $\det(A)=k^2-4=(k-2)(k+2)$. By Theorem 2.3.3, A is invertible if $k\neq 2$ and $k\neq -2$.

17. $\det(A)=(2+12k+36)-(4k+18+12)=8+8k=8(1+k)$.
By Theorem 2.3.3, A is invertible if $k\neq -1$.

18. $\det(A)=(1+0+0)-(0+2k+2k)=1-4k$. By Theorem 2.3.3, A is invertible if $k\neq\frac{1}{4}$.

19. $\det(A)=(-6+0-20)-(-10+0-15)=-1\neq 0$ therefore A is invertible by Theorem 2.3.3.

The cofactors of A are:

$$C_{11}=\begin{vmatrix}-1&0\\4&3\end{vmatrix}=-3\quad C_{12}=-\begin{vmatrix}-1&0\\2&3\end{vmatrix}=3\quad C_{13}=\begin{vmatrix}-1&-1\\2&4\end{vmatrix}=-2$$
$$C_{21}=-\begin{vmatrix}5&5\\4&3\end{vmatrix}=5\quad C_{22}=\begin{vmatrix}2&5\\2&3\end{vmatrix}=-4\quad C_{23}=-\begin{vmatrix}2&5\\2&4\end{vmatrix}=2$$
$$C_{31}=\begin{vmatrix}5&5\\-1&0\end{vmatrix}=5\quad C_{32}=-\begin{vmatrix}2&5\\-1&0\end{vmatrix}=-5\quad C_{33}=\begin{vmatrix}2&5\\-1&-1\end{vmatrix}=3$$

The matrix of cofactors is $\begin{bmatrix}-3&3&-2\\5&-4&2\\5&-5&3\end{bmatrix}$ and the adjoint matrix is $\operatorname{adj}(A)=\begin{bmatrix}-3&5&5\\3&-4&-5\\-2&2&3\end{bmatrix}$.

From Theorem 2.3.6, we have $A^{-1}=\frac{1}{\det(A)}\operatorname{adj}(A)=\frac{1}{-1}\begin{bmatrix}-3&5&5\\3&-4&-5\\-2&2&3\end{bmatrix}=\begin{bmatrix}3&-5&-5\\-3&4&5\\2&-2&-3\end{bmatrix}$.

20. $\det(A)=(-24+0+0)-(-18+0+0)=-6\neq 0$ therefore A is invertible by Theorem 2.3.3.

The cofactors of A are:

$$C_{11}=\begin{vmatrix}3&2\\0&-4\end{vmatrix}=-12\quad C_{12}=-\begin{vmatrix}0&2\\-2&-4\end{vmatrix}=-4\quad C_{13}=\begin{vmatrix}0&3\\-2&0\end{vmatrix}=6$$
$$C_{21}=-\begin{vmatrix}0&3\\0&-4\end{vmatrix}=0\quad C_{22}=\begin{vmatrix}2&3\\-2&-4\end{vmatrix}=-2\quad C_{23}=-\begin{vmatrix}2&0\\-2&0\end{vmatrix}=0$$
$$C_{31}=\begin{vmatrix}0&3\\3&2\end{vmatrix}=-9\quad C_{32}=-\begin{vmatrix}2&3\\0&2\end{vmatrix}=-4\quad C_{33}=\begin{vmatrix}2&0\\0&3\end{vmatrix}=6$$

The matrix of cofactors is $\begin{bmatrix} -12 & -4 & 6 \\ 0 & -2 & 0 \\ -9 & -4 & 6 \end{bmatrix}$ and the adjoint matrix is $\operatorname{adj}(A)=\begin{bmatrix} -12 & 0 & -9 \\ -4 & -2 & -4 \\ 6 & 0 & 6 \end{bmatrix}$.

From Theorem 2.3.6, we have $A^{-1}=\frac{1}{\det(A)}\operatorname{adj}(A)=\frac{1}{-6}\begin{bmatrix} -12 & 0 & -9 \\ -4 & -2 & -4 \\ 6 & 0 & 6 \end{bmatrix}=\begin{bmatrix} 2 & 0 & \frac{3}{2} \\ \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ -1 & 0 & -1 \end{bmatrix}$.

21. $\det(A)=(2)(1)(2)=4\neq 0$ therefore A is invertible by Theorem 2.3.3.

The cofactors of A are:

$$C_{11}=\begin{vmatrix} 1 & -3 \\ 0 & 2 \end{vmatrix}=2 \qquad C_{12}=-\begin{vmatrix} 0 & -3 \\ 0 & 2 \end{vmatrix}=0 \qquad C_{13}=\begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix}=0$$
$$C_{21}=-\begin{vmatrix} -3 & 5 \\ 0 & 2 \end{vmatrix}=6 \qquad C_{22}=\begin{vmatrix} 2 & 5 \\ 0 & 2 \end{vmatrix}=4 \qquad C_{23}=-\begin{vmatrix} 2 & -3 \\ 0 & 0 \end{vmatrix}=0$$
$$C_{31}=\begin{vmatrix} -3 & 5 \\ 1 & -3 \end{vmatrix}=4 \qquad C_{32}=-\begin{vmatrix} 2 & 5 \\ 0 & -3 \end{vmatrix}=6 \qquad C_{33}=\begin{vmatrix} 2 & -3 \\ 0 & 1 \end{vmatrix}=2$$

The matrix of cofactors is $\begin{bmatrix} 2 & 0 & 0 \\ 6 & 4 & 0 \\ 4 & 6 & 2 \end{bmatrix}$ and the adjoint matrix is $\operatorname{adj}(A)=\begin{bmatrix} 2 & 6 & 4 \\ 0 & 4 & 6 \\ 0 & 0 & 2 \end{bmatrix}$.

From Theorem 2.3.6, we have $A^{-1}=\frac{1}{\det(A)}\operatorname{adj}(A)=\frac{1}{4}\begin{bmatrix} 2 & 6 & 4 \\ 0 & 4 & 6 \\ 0 & 0 & 2 \end{bmatrix}=\begin{bmatrix} \frac{1}{2} & \frac{3}{2} & 1 \\ 0 & 1 & \frac{3}{2} \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$.

22. $\det(A)=(2)(1)(6)=12$ is nonzero, therefore by Theorem 2.3.3, A is invertible.

The cofactors of A are:

$$C_{11}=\begin{vmatrix} 1 & 0 \\ 3 & 6 \end{vmatrix}=6 \qquad C_{12}=-\begin{vmatrix} 8 & 0 \\ -5 & 6 \end{vmatrix}=-48 \qquad C_{13}=\begin{vmatrix} 8 & 1 \\ -5 & 3 \end{vmatrix}=29$$
$$C_{21}=-\begin{vmatrix} 0 & 0 \\ 3 & 6 \end{vmatrix}=0 \qquad C_{22}=\begin{vmatrix} 2 & 0 \\ -5 & 6 \end{vmatrix}=12 \qquad C_{23}=-\begin{vmatrix} 2 & 0 \\ -5 & 3 \end{vmatrix}=-6$$
$$C_{31}=\begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix}=0 \qquad C_{32}=-\begin{vmatrix} 2 & 0 \\ 8 & 0 \end{vmatrix}=0 \qquad C_{33}=\begin{vmatrix} 2 & 0 \\ 8 & 1 \end{vmatrix}=2$$

The matrix of cofactors is $\begin{bmatrix} 6 & -48 & 29 \\ 0 & 12 & -6 \\ 0 & 0 & 2 \end{bmatrix}$ and the adjoint matrix is $\operatorname{adj}(A)=\begin{bmatrix} 6 & 0 & 0 \\ -48 & 12 & 0 \\ 29 & -6 & 2 \end{bmatrix}$.

From Theorem 2.3.6, we have $A^{-1}=\frac{1}{\det(A)}\operatorname{adj}(A)=\frac{1}{12}\begin{bmatrix} 6 & 0 & 0 \\ -48 & 12 & 0 \\ 29 & -6 & 2 \end{bmatrix}=\begin{bmatrix} \frac{1}{2} & 0 & 0 \\ -4 & 1 & 0 \\ \frac{29}{12} & -\frac{1}{2} & \frac{1}{6} \end{bmatrix}$.

23.

$$\begin{vmatrix} 1 & 3 & 1 & 1 \\ 2 & 5 & 2 & 2 \\ 1 & 3 & 8 & 9 \\ 1 & 3 & 2 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 3 & 1 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 7 & 8 \\ 0 & 0 & 1 & 1 \end{vmatrix}$$

← -2 times the first row was added to the second row; -1 times the first row was added to the third and fourth rows.

$$= -\begin{vmatrix} 1 & 3 & 1 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 7 & 8 \end{vmatrix}$$

← The third row and the fourth row were interchanged.

$$= -\begin{vmatrix} 1 & 3 & 1 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

← -7 times the third row was added to the fourth row

$$= -(1)(-1)(1)(1) = 1$$

The determinant of A is nonzero therefore by Theorem 2.3.3, A is invertible.

The cofactors of A are:

$$C_{11} = \begin{vmatrix} 5 & 2 & 2 \\ 3 & 8 & 9 \\ 3 & 2 & 2 \end{vmatrix} = (80+54+12)-(48+90+12) = -4$$

$$C_{12} = -\begin{vmatrix} 2 & 2 & 2 \\ 1 & 8 & 9 \\ 1 & 2 & 2 \end{vmatrix} = -\left[(32+18+4)-(16+36+4)\right] = 2$$

$$C_{13} = \begin{vmatrix} 2 & 5 & 2 \\ 1 & 3 & 9 \\ 1 & 3 & 2 \end{vmatrix} = (12+45+6)-(6+54+10) = -7$$

$$C_{14} = -\begin{vmatrix} 2 & 5 & 2 \\ 1 & 3 & 8 \\ 1 & 3 & 2 \end{vmatrix} = -\left[(12+40+6)-(6+48+10)\right] = 6$$

$$C_{21} = -\begin{vmatrix} 3 & 1 & 1 \\ 3 & 8 & 9 \\ 3 & 2 & 2 \end{vmatrix} = -\left[(48+27+6)-(24+54+6)\right] = 3$$

$$C_{22} = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 8 & 9 \\ 1 & 2 & 2 \end{vmatrix} = (16+9+2)-(8+18+2) = -1$$

$$C_{23} = -\begin{vmatrix} 1 & 3 & 1 \\ 1 & 3 & 9 \\ 1 & 3 & 2 \end{vmatrix} = -\left[(6+27+3)-(3+27+6)\right] = 0$$

$$C_{24} = \begin{vmatrix} 1 & 3 & 1 \\ 1 & 3 & 8 \\ 1 & 3 & 2 \end{vmatrix} = (6+24+3)-(3+24+6) = 0$$

$$C_{31} = \begin{vmatrix} 3 & 1 & 1 \\ 5 & 2 & 2 \\ 3 & 2 & 2 \end{vmatrix} = (12+6+10)-(6+12+10) = 0$$

$$C_{32} = -\begin{vmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 1 & 2 & 2 \end{vmatrix} = -\left[(4+2+4)-(2+4+4)\right] = 0$$

$$C_{33} = \begin{vmatrix} 1 & 3 & 1 \\ 2 & 5 & 2 \\ 1 & 3 & 2 \end{vmatrix} = (10+6+6)-(5+6+12) = -1$$

$$C_{34} = -\begin{vmatrix} 1 & 3 & 1 \\ 2 & 5 & 2 \\ 1 & 3 & 2 \end{vmatrix} = -\left[(10+6+6)-(5+6+12)\right] = 1$$

$$C_{41} = -\begin{vmatrix} 3 & 1 & 1 \\ 5 & 2 & 2 \\ 3 & 8 & 9 \end{vmatrix} = -\left[(54+6+40)-(6+48+45)\right] = -1$$

$$C_{42} = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 1 & 8 & 9 \end{vmatrix} = (18+2+16)-(2+16+18) = 0$$

$$C_{43} = -\begin{vmatrix} 1 & 3 & 1 \\ 2 & 5 & 2 \\ 1 & 3 & 9 \end{vmatrix} = -\left[(45+6+6)-(5+6+54)\right] = 8$$

$$C_{44} = \begin{vmatrix} 1 & 3 & 1 \\ 2 & 5 & 2 \\ 1 & 3 & 8 \end{vmatrix} = (40+6+6)-(5+6+48) = -7$$

The matrix of cofactors is $\begin{bmatrix} -4 & 2 & -7 & 6 \\ 3 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ -1 & 0 & 8 & -7 \end{bmatrix}$ and the adjoint matrix is $\operatorname{adj}(A) = \begin{bmatrix} -4 & 3 & 0 & -1 \\ 2 & -1 & 0 & 0 \\ -7 & 0 & -1 & 8 \\ 6 & 0 & 1 & -7 \end{bmatrix}$.

From Theorem 2.3.6, we have $A^{-1} = \frac{1}{\det(A)}\operatorname{adj}(A) = \frac{1}{1}\begin{bmatrix} -4 & 3 & 0 & -1 \\ 2 & -1 & 0 & 0 \\ -7 & 0 & -1 & 8 \\ 6 & 0 & 1 & -7 \end{bmatrix} = \begin{bmatrix} -4 & 3 & 0 & -1 \\ 2 & -1 & 0 & 0 \\ -7 & 0 & -1 & 8 \\ 6 & 0 & 1 & -7 \end{bmatrix}$.

24. $A = \begin{bmatrix} 7 & -2 \\ 3 & 1 \end{bmatrix}$, $A_1 = \begin{bmatrix} 3 & -2 \\ 5 & 1 \end{bmatrix}$, $A_2 = \begin{bmatrix} 7 & 3 \\ 3 & 5 \end{bmatrix}$; $x_1 = \frac{\det(A_1)}{\det(A)} = \frac{13}{13} = 1$, $x_2 = \frac{\det(A_2)}{\det(A)} = \frac{26}{13} = 2$

25. $\det(A)=\begin{vmatrix}4 & 5 & 0\\ 11 & 1 & 2\\ 1 & 5 & 2\end{vmatrix}=(8+10+0)-(0+40+110)=-132$,

$\det(A_1)=\begin{vmatrix}2 & 5 & 0\\ 3 & 1 & 2\\ 1 & 5 & 2\end{vmatrix}=(4+10+0)-(0+20+30)=-36$,

$\det(A_2)=\begin{vmatrix}4 & 2 & 0\\ 11 & 3 & 2\\ 1 & 1 & 2\end{vmatrix}=(24+4+0)-(0+8+44)=-24$,

$\det(A_3)=\begin{vmatrix}4 & 5 & 2\\ 11 & 1 & 3\\ 1 & 5 & 1\end{vmatrix}=(4+15+110)-(2+60+55)=12$;

$x=\frac{\det(A_1)}{\det(A)}=\frac{-36}{-132}=\frac{3}{11}$, $y=\frac{\det(A_2)}{\det(A)}=\frac{-24}{-132}=\frac{2}{11}$, $z=\frac{\det(A_3)}{\det(A)}=\frac{12}{-132}=-\frac{1}{11}$.

26. $\det(A)=\begin{vmatrix}1 & -4 & 1\\ 4 & -1 & 2\\ 2 & 2 & -3\end{vmatrix}=(3-16+8)-(-2+4+48)=-55$,

$\det(A_1)=\begin{vmatrix}6 & -4 & 1\\ -1 & -1 & 2\\ -20 & 2 & -3\end{vmatrix}=(18+160-2)-(20+24-12)=144$,

$\det(A_2)=\begin{vmatrix}1 & 6 & 1\\ 4 & -1 & 2\\ 2 & -20 & -3\end{vmatrix}=(3+24-80)-(-2-40-72)=61$,

$\det(A_3)=\begin{vmatrix}1 & -4 & 6\\ 4 & -1 & -1\\ 2 & 2 & -20\end{vmatrix}=(20+8+48)-(-12-2+320)=-230$;

$x=\frac{\det(A_1)}{\det(A)}=\frac{144}{-55}=-\frac{144}{55}$, $y=\frac{\det(A_2)}{\det(A)}=\frac{61}{-55}=-\frac{61}{55}$, $z=\frac{\det(A_3)}{\det(A)}=\frac{-230}{-55}=\frac{46}{11}$.

27. $\det(A)=\begin{vmatrix}1 & -3 & 1\\ 2 & -1 & 0\\ 4 & 0 & -3\end{vmatrix}=(3+0+0)-(-4+0+18)=-11$,

$\det(A_1)=\begin{vmatrix}4 & -3 & 1\\ -2 & -1 & 0\\ 0 & 0 & -3\end{vmatrix}=-3\begin{vmatrix}4 & -3\\ -2 & -1\end{vmatrix}=(-3)(-4-6)=30$,

$$\det(A_2) = \begin{vmatrix} 1 & 4 & 1 \\ 2 & -2 & 0 \\ 4 & 0 & -3 \end{vmatrix} = (6+0+0)-(-8+0-24) = 38\,,$$

$$\det(A_3) = \begin{vmatrix} 1 & -3 & 4 \\ 2 & -1 & -2 \\ 4 & 0 & 0 \end{vmatrix} = 4\begin{vmatrix} -3 & 4 \\ -1 & -2 \end{vmatrix} = (4)(6+4) = 40\,;$$

$x_1 = \frac{\det(A_1)}{\det(A)} = \frac{30}{-11} = -\frac{30}{11}$, $x_2 = \frac{\det(A_2)}{\det(A)} = \frac{38}{-11} = -\frac{38}{11}$, $x_3 = \frac{\det(A_3)}{\det(A)} = \frac{40}{-11} = -\frac{40}{11}$.

28. $$\det(A) = \begin{vmatrix} -1 & -4 & 2 & 1 \\ 2 & -1 & 7 & 9 \\ -1 & 1 & 3 & 1 \\ 1 & -2 & 1 & -4 \end{vmatrix}$$

$$= -1\begin{vmatrix} -1 & 7 & 9 \\ 1 & 3 & 1 \\ -2 & 1 & -4 \end{vmatrix} + 4\begin{vmatrix} 2 & 7 & 9 \\ -1 & 3 & 1 \\ 1 & 1 & -4 \end{vmatrix} + 2\begin{vmatrix} 2 & -1 & 9 \\ -1 & 1 & 1 \\ 1 & -2 & -4 \end{vmatrix} - 1\begin{vmatrix} 2 & -1 & 7 \\ -1 & 1 & 3 \\ 1 & -2 & 1 \end{vmatrix}$$

$$= -\left[(12-14+9)-(-54-1-28)\right] + 4\left[(-24+7-9)-(27+2+28)\right]$$
$$+2\left[(-8-1+18)-(9-4-4)\right] - \left[(2-3+14)-(7-12+1)\right]$$

$$= -90-332+16-17 = -423$$

$$\det(A_1) = \begin{vmatrix} -32 & -4 & 2 & 1 \\ 14 & -1 & 7 & 9 \\ 11 & 1 & 3 & 1 \\ -4 & -2 & 1 & -4 \end{vmatrix}$$

$$= -32\begin{vmatrix} -1 & 7 & 9 \\ 1 & 3 & 1 \\ -2 & 1 & -4 \end{vmatrix} + 4\begin{vmatrix} 14 & 7 & 9 \\ 11 & 3 & 1 \\ -4 & 1 & -4 \end{vmatrix} + 2\begin{vmatrix} 14 & -1 & 9 \\ 11 & 1 & 1 \\ -4 & -2 & -4 \end{vmatrix} - 1\begin{vmatrix} 14 & -1 & 7 \\ 11 & 1 & 3 \\ -4 & -2 & 1 \end{vmatrix}$$

$$= -32\left[(12-14+9)-(-54-1-28)\right] + 4\left[(-168-28+99)-(-108+14-308)\right]$$
$$+2\left[(-56+4-198)-(-36-28+44)\right] - \left[(14+12-154)-(-28-84-11)\right]$$

$$= -2880+1220-460+5 = -2115$$

$$\det(A_2) = \begin{vmatrix} -1 & -32 & 2 & 1 \\ 2 & 14 & 7 & 9 \\ -1 & 11 & 3 & 1 \\ 1 & -4 & 1 & -4 \end{vmatrix}$$

$$= -1\begin{vmatrix} 14 & 7 & 9 \\ 11 & 3 & 1 \\ -4 & 1 & -4 \end{vmatrix} + 32\begin{vmatrix} 2 & 7 & 9 \\ -1 & 3 & 1 \\ 1 & 1 & -4 \end{vmatrix} + 2\begin{vmatrix} 2 & 14 & 9 \\ -1 & 11 & 1 \\ 1 & -4 & -4 \end{vmatrix} - 1\begin{vmatrix} 2 & 14 & 7 \\ -1 & 11 & 3 \\ 1 & -4 & 1 \end{vmatrix}$$

$$= -\left[(-168-28+99)-(-108+14-308)\right] + 32\left[(-24+7-9)-(27+2+28)\right]$$

$$+2\left[(-88+14+36)-(99-8+56)\right]-\left[(22+42+28)-(77-24-14)\right]$$
$$=-305-2656-370-53=-3384$$

$$\det(A_3)=\begin{vmatrix}-1 & -4 & -32 & 1\\ 2 & -1 & 14 & 9\\ -1 & 1 & 11 & 1\\ 1 & -2 & -4 & -4\end{vmatrix}$$

$$=-1\begin{vmatrix}-1 & 14 & 9\\ 1 & 11 & 1\\ -2 & -4 & -4\end{vmatrix}+4\begin{vmatrix}2 & 14 & 9\\ -1 & 11 & 1\\ 1 & -4 & -4\end{vmatrix}-32\begin{vmatrix}2 & -1 & 9\\ -1 & 1 & 1\\ 1 & -2 & -4\end{vmatrix}-1\begin{vmatrix}2 & -1 & 14\\ -1 & 1 & 11\\ 1 & -2 & -4\end{vmatrix}$$

$$=-\left[(44-28-36)-(-198+4-56)\right]+4\left[(-88+14+36)-(99-8+56)\right]$$
$$-32\left[(-8-1+18)-(9-4-4)\right]-\left[(-8-11+28)-(14-44-4)\right]$$
$$=-230-740-256-43=-1269$$

$$\det(A_4)=\begin{vmatrix}-1 & -4 & 2 & -32\\ 2 & -1 & 7 & 14\\ -1 & 1 & 3 & 11\\ 1 & -2 & 1 & -4\end{vmatrix}$$

$$=-1\begin{vmatrix}-1 & 7 & 14\\ 1 & 3 & 11\\ -2 & 1 & -4\end{vmatrix}+4\begin{vmatrix}2 & 7 & 14\\ -1 & 3 & 11\\ 1 & 1 & -4\end{vmatrix}+2\begin{vmatrix}2 & -1 & 14\\ -1 & 1 & 11\\ 1 & -2 & -4\end{vmatrix}+32\begin{vmatrix}2 & -1 & 7\\ -1 & 1 & 3\\ 1 & -2 & 1\end{vmatrix}$$

$$=-\left[(12-154+14)-(-84-11-28)\right]+4\left[(-24+77-14)-(42+22+28)\right]$$
$$+2\left[(-8-11+28)-(14-44-4)\right]+32\left[(2-3+14)-(7-12+1)\right]$$
$$=5-212+86+544=423$$

$x_1=\frac{\det(A_1)}{\det(A)}=\frac{-2115}{-423}=5$, $\quad x_2=\frac{\det(A_2)}{\det(A)}=\frac{-3384}{-423}=8$,

$x_3=\frac{\det(A_3)}{\det(A)}=\frac{-1269}{-423}=3$, $\quad x_4=\frac{\det(A_4)}{\det(A)}=\frac{423}{-423}=-1$

29. $\det(A)=0$ therefore Cramer's rule does not apply.

30. $\det(A)=\cos^2\theta+\sin^2\theta=1$ is nonzero for all values of θ, therefore by Theorem 2.3.3, A is invertible.

The cofactors of A are:

$$\begin{array}{lll} C_{11}=\cos\theta & C_{12}=\sin\theta & C_{13}=0\\ C_{21}=-\sin\theta & C_{22}=\cos\theta & C_{23}=0\\ C_{31}=0 & C_{32}=0 & C_{33}=\cos^2\theta+\sin^2\theta=1\end{array}$$

The matrix of cofactors is

$$\begin{bmatrix}\cos\theta & \sin\theta & 0\\ -\sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{bmatrix}$$

and the adjoint matrix is

$$\operatorname{adj}(A)=\begin{bmatrix}\cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{bmatrix}$$

From Theorem 2.3.6, we have

$$A^{-1}=\tfrac{1}{\det(A)}\operatorname{adj}(A)=\frac{1}{1}\begin{bmatrix}\cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{bmatrix}=\begin{bmatrix}\cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{bmatrix}.$$

31. $\det(A)=\begin{vmatrix}4 & 1 & 1 & 1\\ 3 & 7 & -1 & 1\\ 7 & 3 & -5 & 8\\ 1 & 1 & 1 & 2\end{vmatrix}=-424$; $\det(A_2)=\begin{vmatrix}4 & 6 & 1 & 1\\ 3 & 1 & -1 & 1\\ 7 & -3 & -5 & 8\\ 1 & 3 & 1 & 2\end{vmatrix}=0$; $y=\tfrac{\det(A_2)}{\det(A)}=\tfrac{0}{-424}=0$

32. **(a)** $A=\begin{bmatrix}4 & 1 & 1 & 1\\ 3 & 7 & -1 & 1\\ 7 & 3 & -5 & 8\\ 1 & 1 & 1 & 2\end{bmatrix}$,

$A_1=\begin{bmatrix}6 & 1 & 1 & 1\\ 1 & 7 & -1 & 1\\ -3 & 3 & -5 & 8\\ 3 & 1 & 1 & 2\end{bmatrix}$, $A_2=\begin{bmatrix}4 & 6 & 1 & 1\\ 3 & 1 & -1 & 1\\ 7 & -3 & -5 & 8\\ 1 & 3 & 1 & 2\end{bmatrix}$, $A_3=\begin{bmatrix}4 & 1 & 6 & 1\\ 3 & 7 & 1 & 1\\ 7 & 3 & -3 & 8\\ 1 & 1 & 3 & 2\end{bmatrix}$, $A_4=\begin{bmatrix}4 & 1 & 1 & 6\\ 3 & 7 & -1 & 1\\ 7 & 3 & -5 & -3\\ 1 & 1 & 1 & 3\end{bmatrix}$;

$x=\tfrac{\det(A_1)}{\det(A)}=\tfrac{-424}{-424}=1$, $y=\tfrac{\det(A_2)}{\det(A)}=\tfrac{0}{-424}=0$, $z=\tfrac{\det(A_3)}{\det(A)}=\tfrac{-848}{-424}=2$, $w=\tfrac{\det(A_4)}{\det(A)}=\tfrac{0}{-424}=0$

(b) The augmented matrix of the system $\begin{bmatrix}4 & 1 & 1 & 1 & 6\\ 3 & 7 & -1 & 1 & 1\\ 7 & 3 & -5 & 8 & -3\\ 1 & 1 & 1 & 2 & 3\end{bmatrix}$ has the reduced row echelon form

$\begin{bmatrix}1 & 0 & 0 & 0 & 1\\ 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 2\\ 0 & 0 & 0 & 1 & 0\end{bmatrix}$ therefore the system has only one solution: $x=1$, $y=0$, $z=2$, and $w=0$.

(c) The method in part (b) requires fewer computations.

33. **(a)** $\det(3A)=3^3\det(A)=(27)(-7)=-189$ (using Formula (1))

(b) $\det\left(A^{-1}\right)=\tfrac{1}{\det(A)}=\tfrac{1}{-7}=-\tfrac{1}{7}$ (using Theorem 2.3.5)