Solutions of selected Questions of Ex # 1.2

Note: These solutions are just for your ease. You can compare your solutions with these answers for better understanding.

3. **(a)** The first three columns are pivot columns and all three rows are pivot rows. The linear system

$$\begin{array}{rcrcrcr} x & - & 3y & + & 4z & = & 7 \\ & & y & + & 2z & = & 2 \\ & & & & z & = & 5 \end{array} \qquad \text{can be rewritten as} \qquad \begin{array}{rcl} x & = & 7+3y-4z \\ y & = & 2-2z \\ z & = & 5 \end{array}$$

and solved by back-substitution:

$$\begin{aligned} &z=5 \\ &y=2-2(5)=-8 \\ &x=7+3(-8)-4(5)=-37 \end{aligned}$$

therefore the original linear system has a unique solution: $x=-37$, $y=-8$, $z=5$.

(b) The first three columns are pivot columns and all three rows are pivot rows. The linear system

$$\begin{array}{rcrcrcrcr} w & & & + & 8y & - & 5z & = & 6 \\ & & x & + & 4y & - & 9z & = & 3 \\ & & & & y & + & z & = & 2 \end{array} \qquad \text{can be rewritten as} \qquad \begin{array}{rcl} w & = & 6-8y+5z \\ x & = & 3-4y+9z \\ y & = & 2-z \end{array}$$

Let $z=t$. Then

$$\begin{aligned} &y=2-t \\ &x=3-4(2-t)+9t=-5+13t \\ &w=6-8(2-t)+5t=-10+13t \end{aligned}$$

therefore the original linear system has infinitely many solutions:

$$w=-10+13t\,,\ x=-5+13t\,,\ y=2-t\,,\ z=t$$

where t is an arbitrary value.

(c) Columns 1, 3, and 4 are pivot columns. The first three rows are pivot rows. The linear system

$$\begin{array}{rcrcrcrcrcr} x_1 & + & 7x_2 & - & 2x_3 & & & - & 8x_5 & = & -3 \\ & & & & x_3 & + & x_4 & + & 6x_5 & = & 5 \\ & & & & & & x_4 & + & 3x_5 & = & 9 \\ & & & & & & & & 0 & = & 0 \end{array}$$

can be rewritten: $x_1=-3-7x_2+2x_3+8x_5$, $x_3=5-x_4-6x_5$, $x_4=9-3x_5$.

Let $x_2=s$ and $x_5=t$. Then

$$x_4 = 9 - 3t$$
$$x_3 = 5 - (9 - 3t) - 6t = -4 - 3t$$
$$x_1 = -3 - 7s + 2(-4 - 3t) + 8t = -11 - 7s + 2t$$

therefore the original linear system has infinitely many solutions:

$$x_1 = -11 - 7s + 2t, \quad x_2 = s, \quad x_3 = -4 - 3t, \quad x_4 = 9 - 3t, \quad x_5 = t$$

where s and t are arbitrary values.

(d) The first two columns are pivot columns and the first two rows are pivot rows. The system is inconsistent since the third row of the augmented matrix corresponds to the equation

$$0x + 0y + 0z = 1.$$

4. **(a)** The first three columns are pivot columns and all three rows are pivot rows. A unique solution: $x = -3$, $y = 0$, $z = 7$.

(b) The first three columns are pivot columns and all three rows are pivot rows. Infinitely many solutions: $w = 8 + 7t$, $x = 2 - 3t$, $y = -5 - t$, $z = t$ where t is an arbitrary value.

(c) Columns 1, 3, and 4 are pivot columns. The first three rows are pivot rows. Infinitely many solutions: $v = -2 + 6s - 3t$, $w = s$, $x = 7 - 4t$, $y = 8 - 5t$, $z = t$ where s and t are arbitrary values.

(d) Columns 1 and 3 are pivot columns. The first two rows are pivot rows. The system is inconsistent since the third row of the augmented matrix corresponds to the equation

$$0x + 0y + 0z = 1.$$

6.

$$\begin{bmatrix} 2 & 2 & 2 & 0 \\ -2 & 5 & 2 & 1 \\ 8 & 1 & 4 & -1 \end{bmatrix} \longleftarrow \text{The augmented matrix for the system.}$$

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ -2 & 5 & 2 & 1 \\ 8 & 1 & 4 & -1 \end{bmatrix} \longleftarrow \text{The first row was multiplied by } \tfrac{1}{2}.$$

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 7 & 4 & 1 \\ 8 & 1 & 4 & -1 \end{bmatrix} \longleftarrow 2 \text{ times the first row was added to the second row.}$$

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 7 & 4 & 1 \\ 0 & -7 & -4 & -1 \end{bmatrix} \longleftarrow -8 \text{ times the first row was added to the third row.}$$

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & \tfrac{4}{7} & \tfrac{1}{7} \\ 0 & -7 & -4 & -1 \end{bmatrix} \longleftarrow \text{The second row was multiplied by } \tfrac{1}{7}.$$

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & \tfrac{4}{7} & \tfrac{1}{7} \\ 0 & 0 & 0 & 0 \end{bmatrix} \longleftarrow 7 \text{ times the second row was added to the third row.}$$

The system of equations corresponding to this augmented matrix in row echelon form is

$$\begin{array}{rcrcrcl} x_1 & + & x_2 & + & x_3 & = & 0 \\ & & x_2 & + & \frac{4}{7}x_3 & = & \frac{1}{7} \\ & & & & 0 & = & 0 \end{array}$$

Solve the equations for the leading variables

$$x_1 = -x_2 - x_3$$

$$x_2 = \frac{1}{7} - \frac{4}{7}x_3$$

then substitute the second equation into the first

$$x_1 = -\frac{1}{7} - \frac{3}{7}x_3$$
$$x_2 = \frac{1}{7} - \frac{4}{7}x_3$$

If we assign x_3 an arbitrary value t, the general solution is given by the formulas

$$x_1 = -\frac{1}{7} - \frac{3}{7}t, \quad x_2 = \frac{1}{7} - \frac{4}{7}t, \qquad x_3 = t$$

8.

$$\begin{bmatrix} 0 & -2 & 3 & 1 \\ 3 & 6 & -3 & -2 \\ 6 & 6 & 3 & 5 \end{bmatrix} \longleftarrow \text{The augmented matrix for the system.}$$

$$\begin{bmatrix} 3 & 6 & -3 & -2 \\ 0 & -2 & 3 & 1 \\ 6 & 6 & 3 & 5 \end{bmatrix} \longleftarrow \text{The first and second rows were interchanged.}$$

$$\begin{bmatrix} 1 & 2 & -1 & -\frac{2}{3} \\ 0 & -2 & 3 & 1 \\ 6 & 6 & 3 & 5 \end{bmatrix} \longleftarrow \text{The first row was multiplied by } \tfrac{1}{3}.$$

$$\begin{bmatrix} 1 & 2 & -1 & -\frac{2}{3} \\ 0 & -2 & 3 & 1 \\ 0 & -6 & 9 & 9 \end{bmatrix} \longleftarrow -6 \text{ times the first row was added to the third row.}$$

$$\begin{bmatrix} 1 & 2 & -1 & -\frac{2}{3} \\ 0 & 1 & -\frac{3}{2} & -\frac{1}{2} \\ 0 & -6 & 9 & 9 \end{bmatrix} \longleftarrow \text{The second row was multiplied by } -\tfrac{1}{2}.$$

$$\begin{bmatrix} 1 & 2 & -1 & -\frac{2}{3} \\ 0 & 1 & -\frac{3}{2} & -\frac{1}{2} \\ 0 & 0 & 0 & 6 \end{bmatrix} \longleftarrow \text{6 times the second row was added to the third row.}$$

$$\begin{bmatrix} 1 & 2 & -1 & -\frac{2}{3} \\ 0 & 1 & -\frac{3}{2} & -\frac{1}{2} \\ 0 & 0 & 0 & 1 \end{bmatrix} \longleftarrow \text{The third row was multiplied by } \tfrac{1}{6}.$$

The system of equations corresponding to this augmented matrix in row echelon form

$$\begin{aligned} a + 2b - c &= -\frac{2}{3} \\ b - \frac{3}{2}c &= -\frac{1}{2} \\ 0 &= 1 \end{aligned}$$

is clearly inconsistent.

10.

$$\begin{bmatrix} 2 & 2 & 2 & 0 \\ -2 & 5 & 2 & 1 \\ 8 & 1 & 4 & -1 \end{bmatrix} \longleftarrow \text{The augmented matrix for the system.}$$

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ -2 & 5 & 2 & 1 \\ 8 & 1 & 4 & -1 \end{bmatrix} \longleftarrow \text{The first row was multiplied by } \tfrac{1}{2}.$$

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 7 & 4 & 1 \\ 8 & 1 & 4 & -1 \end{bmatrix} \longleftarrow \text{2 times the first row was added to the second row.}$$

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 7 & 4 & 1 \\ 0 & -7 & -4 & -1 \end{bmatrix} \longleftarrow -8 \text{ times the first row was added to the third row.}$$

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & \frac{4}{7} & \frac{1}{7} \\ 0 & -7 & -4 & -1 \end{bmatrix} \longleftarrow \text{The second row was multiplied by } \tfrac{1}{7}.$$

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & \frac{4}{7} & \frac{1}{7} \\ 0 & 0 & 0 & 0 \end{bmatrix} \longleftarrow \text{7 times the second row was added to the third row.}$$

$$\begin{bmatrix} 1 & 0 & \frac{3}{7} & -\frac{1}{7} \\ 0 & 1 & \frac{4}{7} & \frac{1}{7} \\ 0 & 0 & 0 & 0 \end{bmatrix} \longleftarrow -1 \text{ times the second row was added to the first row.}$$

Infinitely many solutions: $x_1 = -\frac{1}{7} - \frac{3}{7}t$, $x_2 = \frac{1}{7} - \frac{4}{7}t$, $x_3 = t$ **where** t **is an arbitrary value.**

16. We present two different solutions.
Solution I uses Gauss-Jordan elimination

$$\begin{bmatrix} 2 & -1 & -3 & 0 \\ -1 & 2 & -3 & 0 \\ 1 & 1 & 4 & 0 \end{bmatrix} \longleftarrow \text{The augmented matrix for the system.}$$

$$\begin{bmatrix} 1 & -\frac{1}{2} & -\frac{3}{2} & 0 \\ -1 & 2 & -3 & 0 \\ 1 & 1 & 4 & 0 \end{bmatrix} \longleftarrow \text{The first row was multiplied by } \tfrac{1}{2}.$$

$$\begin{bmatrix} 1 & -\frac{1}{2} & -\frac{3}{2} & 0 \\ 0 & \frac{3}{2} & -\frac{9}{2} & 0 \\ 1 & 1 & 4 & 0 \end{bmatrix} \longleftarrow \text{The first row was added to the second row.}$$

$$\begin{bmatrix} 1 & -\frac{1}{2} & -\frac{3}{2} & 0 \\ 0 & \frac{3}{2} & -\frac{9}{2} & 0 \\ 0 & \frac{3}{2} & \frac{11}{2} & 0 \end{bmatrix} \longleftarrow -1 \text{ times the first row was added to the third row.}$$

$$\begin{bmatrix} 1 & -\frac{1}{2} & -\frac{3}{2} & 0 \\ 0 & 1 & -3 & 0 \\ 0 & \frac{3}{2} & \frac{11}{2} & 0 \end{bmatrix} \longleftarrow \text{The second row was multiplied by } \tfrac{2}{3}.$$

$$\begin{bmatrix} 1 & -\frac{1}{2} & -\frac{3}{2} & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 10 & 0 \end{bmatrix} \longleftarrow -\tfrac{3}{2} \text{ times the second row was added to the third row.}$$

$$\begin{bmatrix} 1 & -\frac{1}{2} & -\frac{3}{2} & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \longleftarrow \text{The third row was multiplied by } \tfrac{1}{10}.$$

$$\begin{bmatrix} 1 & -\frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \longleftarrow \begin{array}{l} 3 \text{ times the third row was added to the second row} \\ \text{and } \tfrac{3}{2} \text{ times the third row was added to the first row} \end{array}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \longleftarrow \tfrac{1}{2} \text{ times the second row was added to the first row.}$$

Unique solution: $x = 0$, $y = 0$, $z = 0$.

Solution II. This time, we shall choose the order of the elementary row operations differently in order to avoid introducing fractions into the computation. (Since every matrix has a unique reduced row echelon form, the exa sequence of elementary row operations being used does not matter – see part 1 of the discussion "Some Facts About Echelon Forms" in Section 1.2)

$$\begin{bmatrix} 2 & -1 & -3 & 0 \\ -1 & 2 & -3 & 0 \\ 1 & 1 & 4 & 0 \end{bmatrix} \longleftarrow$$ The augmented matrix for the system.

$$\begin{bmatrix} 1 & 1 & 4 & 0 \\ -1 & 2 & -3 & 0 \\ 2 & -1 & -3 & 0 \end{bmatrix} \longleftarrow$$ The first and third rows were interchanged (to avoid introducing fractions into the first row).

$$\begin{bmatrix} 1 & 1 & 4 & 0 \\ 0 & 3 & 1 & 0 \\ 2 & -1 & -3 & 0 \end{bmatrix} \longleftarrow$$ The first row was added to the second row.

$$\begin{bmatrix} 1 & 1 & 4 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & -3 & -11 & 0 \end{bmatrix} \longleftarrow$$ -2 times the first row was added to the third row.

$$\begin{bmatrix} 1 & 1 & 4 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & -10 & 0 \end{bmatrix} \longleftarrow$$ The second row was added to the third row.

$$\begin{bmatrix} 1 & 1 & 4 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \longleftarrow$$ The third row was multiplied by $-\frac{1}{10}$.

$$\begin{bmatrix} 1 & 1 & 4 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \longleftarrow$$ -1 times the third row was added to the second row.

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \longleftarrow$$ -4 times the third row was added to the first row.

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \longleftarrow$$ The second row was multiplied by $\frac{1}{3}$.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \longleftarrow$$ -1 times the second row was added to the first row.

Unique solution: $x=0$, $y=0$, $z=0$.

18. $\begin{bmatrix} 0 & 1 & 3 & -2 & 0 \\ 2 & 1 & -4 & 3 & 0 \\ 2 & 3 & 2 & -1 & 0 \\ -4 & -3 & 5 & -4 & 0 \end{bmatrix}$ ⟵ The augmented matrix for the system.

$\begin{bmatrix} 2 & 1 & -4 & 3 & 0 \\ 0 & 1 & 3 & -2 & 0 \\ 2 & 3 & 2 & -1 & 0 \\ -4 & -3 & 5 & -4 & 0 \end{bmatrix}$ ⟵ The first and second rows were interchanged.

$\begin{bmatrix} 1 & \frac{1}{2} & -2 & \frac{3}{2} & 0 \\ 0 & 1 & 3 & -2 & 0 \\ 2 & 3 & 2 & -1 & 0 \\ -4 & -3 & 5 & -4 & 0 \end{bmatrix}$ ⟵ The first row was multiplied by $\frac{1}{2}$.

$\begin{bmatrix} 1 & \frac{1}{2} & -2 & \frac{3}{2} & 0 \\ 0 & 1 & 3 & -2 & 0 \\ 0 & 2 & 6 & -4 & 0 \\ 0 & -1 & -3 & 2 & 0 \end{bmatrix}$ ⟵ -2 times the first row was added to the third row and 4 times the first row was added to the fourth row.

$\begin{bmatrix} 1 & \frac{1}{2} & -2 & \frac{3}{2} & 0 \\ 0 & 1 & 3 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ ⟵ -2 times the second row was added to the third row and the second row was added to the fourth row.

$\begin{bmatrix} 1 & 0 & -\frac{7}{2} & \frac{5}{2} & 0 \\ 0 & 1 & 3 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ ⟵ $-\frac{1}{2}$ times the second row was added to the first row.

If we assign w and x the arbitrary values s and t, respectively, the general solution is given by the formulas

$$u = \frac{7}{2}s - \frac{5}{2}t, \quad v = -3s + 2t, \quad w = s, \quad x = t.$$

23. **(a)** The system is consistent; it has a unique solution (back-substitution can be used to solve for all three unknowns).

(b) The system is consistent; it has infinitely many solutions (the third unknown can be assigned an arbitrary value t, then back-substitution can be used to solve for the first two unknowns).

(c) The system is inconsistent since the third equation $0 = 1$ is contradictory.

(d) There is insufficient information to decide whether the system is consistent as illustrated by these examples:

- For $\begin{bmatrix} 1 & * & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & * \end{bmatrix}$ the system is consistent with infinitely many solutions.
- For $\begin{bmatrix} 1 & * & * & * \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$ the system is inconsistent (the matrix can be reduced to $\begin{bmatrix} 1 & * & * & * \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$).

24. **(a)** The system is consistent; it has a unique solution (back-substitution can be used to solve for all three unknowns).

(b) The system is consistent; it has a unique solution (solve the first equation for the first unknown, then proceed to solve the second equation for the second unknown and solve the third equation last.)

(c) The system is inconsistent (adding -1 times the first row to the second yields $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & * & * & * \end{bmatrix}$; the second equation $0=1$ is contradictory).

(d) There is insufficient information to decide whether the system is consistent as illustrated by these examples:

- For $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}$ the system is consistent with infinitely many solutions.
- For $\begin{bmatrix} 1 & 0 & 0 & 2 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}$ the system is inconsistent (the matrix can be reduced to $\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$).

25. $\begin{bmatrix} 1 & 2 & -3 & 4 \\ 3 & -1 & 5 & 2 \\ 4 & 1 & a^2-14 & a+2 \end{bmatrix}$ ⟵ The augmented matrix for the system.

$\begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & -7 & 14 & -10 \\ 0 & -7 & a^2-2 & a-14 \end{bmatrix}$ ⟵ -3 times the first row was added to the second row and -4 times the first row was added to the third row.

$\begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & -7 & 14 & -10 \\ 0 & 0 & a^2-16 & a-4 \end{bmatrix}$ ⟵ -1 times the second row was added to the third row.

$\begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & 1 & -2 & \frac{10}{7} \\ 0 & 0 & a^2-16 & a-4 \end{bmatrix}$ ⟵ The second row was multiplied by $-\frac{1}{7}$.

The system has no solutions when $a=-4$ (since the third row of our last matrix would then correspond to a contradictory equation $0=-8$).

The system has infinitely many solutions when $a=4$ (since the third row of our last matrix would then correspond to the equation $0=0$).

For all remaining values of a (i.e., $a \neq -4$ and $a \neq 4$) the system has exactly one solution.