Chapter 1, Part 2: Propositional Equivalences

The Foundations: Logic and Proofs

Propositional Logic Summary

- Logical Equivalences
  - Important Equivalences
  - Showing Equivalence
  - Satisfiability

Propositional Equivalences

Section 1.3

Section Summary

- Tautologies, Contradictions, and Contingencies.
- Logical Equivalence
  - Important Logical Equivalences
  - Showing Logical Equivalence
- Normal Forms (optional, covered in exercises in text)
  - Disjunctive Normal Form
  - Conjunctive Normal Form

Tautologies, Contradictions, and Contingencies

- A **tautology** is a proposition which is always true.
  - Example: $p \vee \neg p = \mathbf{T}$
- A **contradiction** is a proposition which is always false.
  - Example: $p \wedge \neg p = \mathbf{F}$
- A **contingency** is a proposition which is neither a tautology nor a contradiction, such as p

P	$\neg p$	$p \vee \neg p$	$p \wedge \neg p$
T	F	T	F
F	T	T	F

Logical Equivalences

- Compound propositions that have the same truth values in all possible cases are called logically equivalent.
- Two compound propositions p and q are logically equivalent if $p \leftrightarrow q$ is a tautology.
- We write this as $p \Leftrightarrow q$ or as $p \equiv q$ where p and q are compound propositions.
- Two compound propositions p and q are equivalent if and only if the columns in a truth table giving their truth values agree.
- This truth table show $\neg p \vee q$ is equivalent to $p \rightarrow q$.

p	q	$\neg p$	$\neg p \vee q$	$p \rightarrow q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

De Morgan’s Laws

Augustus De Morgan: 1806-1871

This truth table shows that De Morgan’s Second Law holds.
The following example follows that $\neg(p \lor q) \leftrightarrow (\neg p \land \neg q)$ is a tautology and that these compound propositions are logically equivalent.

p	q	$\neg p$	$\neg q$	$(p \lor q)$	$\neg(p \lor q)$	$\neg p \land \neg q$
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	F	T	T

Construct truth table and show that conditional statements is a tautology

- $(p \land q) \rightarrow p$
- $\neg(p \rightarrow q) \rightarrow p$
- $(p \land q) \rightarrow (p \rightarrow q)$
- $[\neg p \land (p \lor q)] \rightarrow q$
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Conditional Disjunction Equivalence

- Replace conditional statements with negations and disjunctions.
- Show that **$p \to q$ and $\neg p \vee q$** are logically equivalent. (This is known as the **conditional disjunction equivalence.**)
- Construct truth table:

Distributive Law of Disjunction over Conjunction

$p \lor (q \land r)$ and $(p \lor q) \land (p \lor r)$

Construct Truth Table

Equivalence Proofs

Example: Show that $\neg(p \vee (\neg p \wedge q))$ is logically equivalent to $\neg p \wedge \neg q$ and is a tautology.

Example: Show that $(p \wedge q) \rightarrow (p \vee q)$ is a tautology.

Example

Show that following statement is a **tautology** :

$$(P \to Q) \lor (Q \to P)$$

P	Q	$P \to Q$	$Q \to P$	$(P \to Q) \lor (Q \to P)$
T	T	T	T	T
T	F	F	T	T
F	T	T	F	T
F	F	T	T	T

TABLE 6 **Logical Equivalences.**

Equivalence	*Name*
$p \wedge \mathbf{T} \equiv p$ $p \vee \mathbf{F} \equiv p$	Identity laws
$p \vee \mathbf{T} \equiv \mathbf{T}$ $p \wedge \mathbf{F} \equiv \mathbf{F}$	Domination laws
$p \vee p \equiv p$ $p \wedge p \equiv p$	Idempotent laws
$\neg(\neg p) \equiv p$	Double negation law
$p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$	Commutative laws
$(p \vee q) \vee r \equiv p \vee (q \vee r)$ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	Associative laws
$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	Distributive laws
$\neg(p \wedge q) \equiv \neg p \vee \neg q$ $\neg(p \vee q) \equiv \neg p \wedge \neg q$	De Morgan's laws
$p \vee (p \wedge q) \equiv p$ $p \wedge (p \vee q) \equiv p$	Absorption laws
$p \vee \neg p \equiv \mathbf{T}$ $p \wedge \neg p \equiv \mathbf{F}$	Negation laws

More Logical Equivalences (conditional and bi-conditional)

TABLE 7 Logical Equivalences Involving Conditional Statements.
$p \to q \equiv \neg p \lor q$
$p \to q \equiv \neg q \to \neg p$
$p \lor q \equiv \neg p \to q$
$p \land q \equiv \neg(p \to \neg q)$
$\neg(p \to q) \equiv p \land \neg q$
$(p \to q) \land (p \to r) \equiv p \to (q \land r)$
$(p \to r) \land (q \to r) \equiv (p \lor q) \to r$
$(p \to q) \lor (p \to r) \equiv p \to (q \lor r)$
$(p \to r) \lor (q \to r) \equiv (p \land q) \to r$

TABLE 8 Logical Equivalences Involving Biconditional Statements.
$p \leftrightarrow q \equiv (p \to q) \land (q \to p)$
$p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$
$p \leftrightarrow q \equiv (p \land q) \lor (\neg p \land \neg q)$
$\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$

Constructing New Logical Equivalences

- Show that **$\neg(p \rightarrow q)$ and $p \wedge \neg q$** are logically equivalent.

1. $\neg(p \rightarrow q) \equiv \neg(\neg p \vee q)$ by the conditional-disjunction equivalence
2. $\equiv \neg(\neg p) \wedge \neg q$ by the second De Morgan law
3. $\equiv p \wedge \neg q$ by the double negation law

$(p \to q) \land (p \to r) \equiv p \to (q \land r)$

$(p \to q) \land (p \to r) \equiv (\neg p \lor q) \land (\neg p \lor r)$ **Substitution for $\to$, twice**

$\equiv \neg p \lor (q \land r)$ **Distribution law**

$\equiv p \to (q \land r)$ **Substitution for $\to$**

Constructing New Logical Equivalences

- Show that Show that $\neg(p \lor (\neg p \land q))$ and $\neg p \land \neg q$ are logically equivalent by developing a series of logical equivalences

1. $\neg(p \lor (\neg p \land q)) \equiv \neg p \land \neg(\neg p \land q)$ by the second De Morgan law
2. $\equiv \neg p \land [\neg(\neg p) \lor \neg q]$ by the first De Morgan law
3. $\equiv \neg p \land (p \lor \neg q)$ by the double negation law
4. $\equiv (\neg p \land p) \lor (\neg p \land \neg q)$ by the second distributive law
5. $\equiv \mathbf{F} \lor (\neg p \land \neg q)$ because $\neg p \land p \equiv \mathbf{F}$
6. $\equiv (\neg p \land \neg q) \lor \mathbf{F}$ by the commutative law for disjunction
7. $\equiv \neg p \land \neg q$ by the identity law for **F**

$$(p \land q) \to (p \lor q) \equiv T$$

Example

Show that $(\mathbf{p} \land \mathbf{q}) \rightarrow (\mathbf{p} \lor \mathbf{q})$ is a tautology.

Equivalence Proofs

- **Using logic equivalence laws:**

a. Show that $\neg(p \leftrightarrow q)$ and $p \leftrightarrow \neg q$ are logically equivalent.
b. Show that $\neg p \leftrightarrow q$ and $p \leftrightarrow \neg q$ are logically equivalent.
c. Show that $(p \rightarrow q) \land (p \rightarrow r)$ and $p \rightarrow (q \land r)$ are logically equivalent.
d. Show that $(p \rightarrow r) \lor (q \rightarrow r)$ and $(p \land q) \rightarrow r$ are logically equivalent.

Prove that: $[\neg p \wedge (p \vee q)] \rightarrow q$ is a tautology.

- By using truth table.
- By using logic equivalence laws.

Prove that: $[(p \to q) \land (q \to r)] \to [p \to r]$ is a tautology.

- By using truth table.
- By using logic equivalence laws