Ex # 1.6

18. **(a)** The equation $Ax = x$ can be rewritten as $Ax = Ix$, which yields $Ax - Ix = 0$ and $(A - I)x = 0$.

This is a matrix form of a homogeneous linear system - to solve it, we reduce its augmented matrix to a row echelon form.

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 2 & 1 & -2 & 0 \\ 3 & 1 & 0 & 0 \end{array}\right] \longleftarrow \text{The augmented matrix for the homogeneous system } (A-I)x = 0.$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & -1 & -6 & 0 \\ 0 & -2 & -6 & 0 \end{array}\right] \longleftarrow -2 \text{ times the first row was added to the second row and } -3 \text{ times the first row was added to the third row.}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 1 & 6 & 0 \\ 0 & -2 & -6 & 0 \end{array}\right] \longleftarrow \text{The second row was multiplied by } -1.$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 1 & 6 & 0 \\ 0 & 0 & 6 & 0 \end{array}\right] \longleftarrow 2 \text{ times the second row was added to the third row.}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 1 & 6 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \longleftarrow \text{The third row was multiplied by } \tfrac{1}{6}.$$

Using back-substitution, we obtain the unique solution: $x_1 = x_2 = x_3 = 0$.

(b) As was done in part (a), the equation $A\mathrm{x} = 4\mathrm{x}$ can be rewritten as $(A-4I)\mathrm{x} = 0$. We solve the latter system by Gauss-Jordan elimination

$$\left[\begin{array}{rrr|r} -2 & 1 & 2 & 0 \\ 2 & -2 & -2 & 0 \\ 3 & 1 & -3 & 0 \end{array}\right]$$

The augmented matrix for the homogeneous system $(A-4I)\mathrm{x} = 0$.

$$\left[\begin{array}{rrr|r} 2 & -2 & -2 & 0 \\ -2 & 1 & 2 & 0 \\ 3 & 1 & -3 & 0 \end{array}\right]$$

The first and second rows were interchanged.

$$\left[\begin{array}{rrr|r} 1 & -1 & -1 & 0 \\ -2 & 1 & 2 & 0 \\ 3 & 1 & -3 & 0 \end{array}\right]$$

The first row was multiplied by $\frac{1}{2}$.

$$\left[\begin{array}{rrr|r} 1 & -1 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 4 & 0 & 0 \end{array}\right]$$

2 times the first row was added to the second row and -3 times the first row was added to the third row.

$$\left[\begin{array}{rrr|r} 1 & -1 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 4 & 0 & 0 \end{array}\right]$$

The second row was multiplied by -1.

$$\left[\begin{array}{rrr|r} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

-4 times the second row was added to the third row and the second row was added to the first row.

If we assign x_3 an arbitrary value t, the general solution is given by the formulas

$x_1 = t$, $x_2 = 0$, and $x_3 = t$.

19. $X=\begin{bmatrix}1&-1&1\\2&3&0\\0&2&-1\end{bmatrix}^{-1}\begin{bmatrix}2&-1&5&7&8\\4&0&-3&0&1\\3&5&-7&2&1\end{bmatrix}$. Let us find $\begin{bmatrix}1&-1&1\\2&3&0\\0&2&-1\end{bmatrix}^{-1}$:

$$\left[\begin{array}{ccc|ccc}1&-1&1&1&0&0\\2&3&0&0&1&0\\0&2&-1&0&0&1\end{array}\right] \longleftarrow \text{The identity matrix was adjoined to the matrix.}$$

$$\left[\begin{array}{ccc|ccc}1&-1&1&1&0&0\\0&5&-2&-2&1&0\\0&2&-1&0&0&1\end{array}\right] \longleftarrow -2 \text{ times the first row was added to the second row.}$$

$$\left[\begin{array}{ccc|ccc}1&-1&1&1&0&0\\0&1&0&-2&1&-2\\0&2&-1&0&0&1\end{array}\right] \longleftarrow -2 \text{ times the third row was added to the second row.}$$

$$\left[\begin{array}{ccc|ccc}1&-1&1&1&0&0\\0&1&0&-2&1&-2\\0&0&-1&4&-2&5\end{array}\right] \longleftarrow -2 \text{ times the second row was added to the third row.}$$

$$\left[\begin{array}{ccc|ccc}1&-1&1&1&0&0\\0&1&0&-2&1&-2\\0&0&1&-4&2&-5\end{array}\right] \longleftarrow \text{The third row was multiplied by } -1\text{ .}$$

$$\left[\begin{array}{ccc|ccc}1&-1&0&5&-2&5\\0&1&0&-2&1&-2\\0&0&1&-4&2&-5\end{array}\right] \longleftarrow -1 \text{ times the third row was added to the first row.}$$

$$\left[\begin{array}{ccc|ccc}1&0&0&3&-1&3\\0&1&0&-2&1&-2\\0&0&1&-4&2&-5\end{array}\right] \longleftarrow \text{The second row was added to the first row.}$$

Using $\begin{bmatrix}1&-1&1\\2&3&0\\0&2&-1\end{bmatrix}^{-1}=\begin{bmatrix}3&-1&3\\-2&1&-2\\-4&2&-5\end{bmatrix}$ we obtain

$$X=\begin{bmatrix}3&-1&3\\-2&1&-2\\-4&2&-5\end{bmatrix}\begin{bmatrix}2&-1&5&7&8\\4&0&-3&0&1\\3&5&-7&2&1\end{bmatrix}=\begin{bmatrix}11&12&-3&27&26\\-6&-8&1&-18&-17\\-15&-21&9&-38&-35\end{bmatrix}$$