# Chapter 6 : Direct methods for solving linear System

**CHAPTER OBJECTIVES**

1. Matrix (LU) factorization
2. CROUTE and Doolittle's Factorization
3. Diagonally dominant matrices
4. Positive definite matrices
5. Band matrix (CTS)
6. $LDL^t$ Factorization
7. Cholesky $LL^t$ Factorization

Exercise: 6.5 and 6.6

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# The Role of Linear Algebra in the Computer Science

Computer science has delivered extraordinary benefits over the last several decades. The breadth and depth of these contributions is accelerating as the world becomes globally connected. At the same time, the field of computer science has expanded to touch almost every facet of our lives. This places enormous pressure on the computer science curriculum to deliver a rigorous core while also allowing students to follow their interests into the many diverse and productive paths computer science can take them.

As science and engineering disciplines grow so the use of mathematics grows as new mathematical problems are encountered and new mathematical skills are required. In this respect, linear algebra has been particularly responsive to computer science as linear algebra plays a significant role in many important computer science undertakings.

A few well-known examples are:

- Internet search
- Graph analysis
- Machine learning
- Graphics
- Bioinformatics
- Scientific computing
- Data mining
- Computer vision
- Speech recognition
- Compilers
- Parallel computing

The broad utility of linear algebra to computer science reflects the deep connection that exists between the discrete nature of matrix mathematics and digital technology.

# Solution of linear system Ax=b:



1-Gauss Elimination
2-Gauss Jordan
3-Cramer's rule
4-Inversion method ($X = A^{-1}b$)

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n \end{aligned} \quad \Rightarrow \quad \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

**EXAMPLE 1** **Gauss Elimination. Partial Pivoting**

Solve the system

$$
\begin{aligned}
6x_1 + 2x_2 + 8x_3 &= 26 \\
3x_1 + 5x_2 + 2x_3 &= 8 \\
8x_2 + 2x_3 &= -7.
\end{aligned}
$$

$$
\left[\begin{array}{ccc|c} 6 & 2 & 8 & 26 \\ 3 & 5 & 2 & 8 \\ 0 & 8 & 2 & -7 \end{array}\right]. \Rightarrow \left[\begin{array}{ccc|c} 6 & 2 & 8 & 26 \\ 0 & 4 & -2 & -5 \\ 0 & 8 & 2 & -7 \end{array}\right]. \Rightarrow \left[\begin{array}{ccc|c} 6 & 2 & 8 & 26 \\ 0 & 8 & 2 & -7 \\ 0 & 4 & -2 & -5 \end{array}\right].
$$

$$
\left[\begin{array}{ccc|c} 6 & 2 & 8 & 26 \\ 0 & 8 & 2 & -7 \\ 0 & 0 & -3 & -\frac{3}{2} \end{array}\right]. \Rightarrow
\begin{aligned}
x_3 &= \tfrac{1}{2} \\
x_2 &= \tfrac{1}{8}(-7 - 2x_3) = -1 \\
x_1 &= \tfrac{1}{6}(26 - 2x_2 - 8x_3) = 4.
\end{aligned}
$$

Back substitution method

Solve the system of equations using Gauss-Jordan elimination method:

$$\left.\begin{aligned} x+2y+z&=8\\ 2x+3y+4z&=20\\ 4x+3y+2z&=16 \end{aligned}\right\}$$

$$\begin{bmatrix}1&2&1\\2&3&4\\4&3&2\end{bmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}8\\20\\16\end{pmatrix} \longrightarrow \begin{bmatrix}1&2&1\\0&-1&2\\0&-5&-2\end{bmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}8\\4\\-16\end{pmatrix} \longrightarrow \begin{bmatrix}1&0&5\\0&-1&2\\0&0&-12\end{bmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}16\\4\\-36\end{pmatrix}$$

$$\begin{bmatrix}1&0&5\\0&-1&2\\0&0&1\end{bmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}16\\4\\3\end{pmatrix} \longrightarrow \begin{bmatrix}1&0&0\\0&-1&0\\0&0&1\end{bmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}1\\-2\\3\end{pmatrix}$$

The solution is

x =1, y =2, z =3.

## Cramer's Rule

Problem Statement. Use Cramer's rule to solve

$$
\begin{aligned}
0.3x_1 + 0.52x_2 + \phantom{1.9}x_3 &= -0.01 \\
0.5x_1 + \phantom{0.52}x_2 + 1.9x_3 &= \phantom{-}0.67 \\
0.1x_1 + 0.3\ x_2 + 0.5x_3 &= -0.44
\end{aligned}
$$

$$x_1 = \frac{1}{D}\det\begin{bmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{bmatrix} \equiv \frac{D_1}{D}, \quad x_2 = \frac{1}{D}\det\begin{bmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{bmatrix} \equiv \frac{D_2}{D},$$

$$x_3 = \frac{1}{D}\det\begin{bmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{bmatrix} \equiv \frac{D_3}{D}, \quad \text{where} \quad D = \det\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}.$$

$$D = 0.3\begin{vmatrix} 1 & 1.9 \\ 0.3 & 0.5 \end{vmatrix} - 0.52\begin{vmatrix} 0.5 & 1.9 \\ 0.1 & 0.5 \end{vmatrix} + 1\begin{vmatrix} 0.5 & 1 \\ 0.1 & 0.3 \end{vmatrix} = -0.0022$$

$$x_1 = \frac{\begin{vmatrix} -0.01 & 0.52 & 1 \\ 0.67 & 1 & 1.9 \\ -0.44 & 0.3 & 0.5 \end{vmatrix}}{-0.0022} = \frac{0.03278}{-0.0022} = -14.9$$

$$x_2 = \frac{\begin{vmatrix} 0.3 & -0.01 & 1 \\ 0.5 & 0.67 & 1.9 \\ 0.1 & -0.44 & 0.5 \end{vmatrix}}{-0.0022} = \frac{0.0649}{-0.0022} = -29.5$$

$$x_3 = \frac{\begin{vmatrix} 0.3 & 0.52 & -0.01 \\ 0.5 & 1 & 0.67 \\ 0.1 & 0.3 & -0.44 \end{vmatrix}}{-0.0022} = \frac{-0.04356}{-0.0022} = 19.8$$

# EXERCISE SET 6.4

5. Find all values of $\alpha$ that make the following matrix singular.

$$A = \begin{bmatrix} 1 & -1 & \alpha \\ 2 & 2 & 1 \\ 0 & \alpha & -\frac{3}{2} \end{bmatrix}.$$

7. Find all values of $\alpha$ so that the following linear system has no solutions.

$$\begin{aligned} 2x_1 - \phantom{2}x_2 + 3x_3 &= 5, \\ 4x_1 + 2x_2 + 2x_3 &= 6, \\ -2x_1 + \alpha x_2 + 3x_3 &= 4. \end{aligned}$$

8. Find all values of $\alpha$ so that the following linear system has an infinite number of solutions.

$$\begin{aligned} 2x_1 - \phantom{2}x_2 + 3x_3 &= 5, \\ 4x_1 + 2x_2 + 2x_3 &= 6, \\ -2x_1 + \alpha x_2 + 3x_3 &= 1. \end{aligned}$$

**12.** The solution by **Cramer's rule** to the linear system

**a.** Find the solution to the linear system

$$\begin{aligned} 2x_1 + \ 3x_2 - \ x_3 &= 4, \\ x_1 - \ 2x_2 + \ x_3 &= 6, \\ x_1 - 12x_2 + 5x_3 &= 10, \end{aligned}$$

by Cramer's rule.

**b.** Show that the linear system

$$\begin{aligned} 2x_1 + \ 3x_2 - \ x_3 &= 4, \\ x_1 - \ 2x_2 + \ x_3 &= 6, \\ -x_1 - 12x_2 + 5x_3 &= 9 \end{aligned}$$

does not have a solution. Compute $D_1$, $D_2$, and $D_3$.

**c.** Show that the linear system

$$\begin{aligned} 2x_1 + \ 3x_2 - \ x_3 &= 4, \\ x_1 - \ 2x_2 + \ x_3 &= 6, \\ -x_1 - 12x_2 + 5x_3 &= 10 \end{aligned}$$

has an infinite number of solutions. Compute $D_1$, $D_2$, and $D_3$.

## 6.5 Matrix Factorization

# LU Factorization

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

$$A = LU$$

$$L = \begin{bmatrix} L_{11} & 0 & 0 \\ L_{21} & L_{22} & 0 \\ L_{31} & L_{32} & L_{33} \end{bmatrix} \qquad U = \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{bmatrix}$$

$$\begin{bmatrix} L_{11} & 0 & 0 \\ L_{21} & L_{22} & 0 \\ L_{31} & L_{32} & L_{33} \end{bmatrix} \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

$$\begin{bmatrix} L_{11}U_{11} & L_{11}U_{12} & L_{11}U_{13} \\ L_{21}U_{11} & L_{21}U_{12} + L_{22}U_{22} & L_{21}U_{13} + L_{22}U_{23} \\ L_{31}U_{11} & L_{31}U_{12} + L_{32}U_{22} & L_{31}U_{13} + L_{32}U_{23} + L_{33}U_{33} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

## Equating the elements of the First Row :-

$$L_{11}U_{11} = A_{11} \qquad L_{11}U_{12} = A_{12} \qquad L_{11}U_{13} = A_{13}$$

## Equating the elements of the 2nd Row :-

$$L_{21}U_{11} = A_{21} \qquad L_{21}U_{12} + L_{22}U_{22} = A_{22}$$

$$L_{21}U_{13} + L_{22}U_{23} = A_{23}$$

## Equating the elements of the 3rd Row :-

$$L_{31}U_{11} = A_{31} \qquad L_{31}U_{12} + L_{32}U_{22} = A_{32}$$

$$L_{31}U_{13} + L_{32}U_{23} + L_{33}U_{33} = A_{33}$$

We have 12 unknowns but only 9 equations. We need some sort of compromise.

## 1-Crout's Method

**Set** $$U_{11} = U_{22} = U_{33} = 1$$

## 2-Dolittle's Method

**Set** $$L_{11} = L_{22} = L_{33} = 1$$

# Different forms of LU factorization

- Doolittle form

  Obtained by

  Gaussian elimination

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

- Crout form

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

- $LDL^t$ form

$$A = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{21} & a_{22} & a_{32} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix} \begin{bmatrix} 1 & l_{21} & l_{31} \\ 0 & 1 & l_{32} \\ 0 & 0 & 1 \end{bmatrix}$$

## Matrix (LU) factorization

$$A = \begin{bmatrix} 1 & 3 & 8 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$$

Factorize the given matrix A = LU
Use Doolittle method

$$let\ LU = A$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \qquad U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

$$\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12} + u_{22} & l_{21}u_{13} + u_{23} \\ l_{31}u_{11} & l_{31}u_{12} + l_{32}u_{22} & l_{31}u_{13} + l_{32}u_{23} + u_{33} \end{bmatrix} = \begin{bmatrix} 1 & 3 & 8 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$$

$here$

$u_{11} = 1,\ u_{12} = 3 \quad u_{13} = 8$

$l_{21}u_{11} = -1 \ \Rightarrow l_{21} = 1$

$l_{31}u_{11} = 1 \quad \Rightarrow l_{31} = 1$

$$l_{21}u_{12} + u_{22} = 4 \quad \Rightarrow u_{22} =$$

$$l_{21}u_{13} + u_{23} = 3 \quad \Rightarrow u_{23} =$$

$$l_{31}u_{12} + l_{32}u_{22} = 3 \Rightarrow l_{32} = \frac{1}{u_{22}}[3 - l_{31}u_{12}] = 0$$

$$l_{31}u_{13} + l_{32}u_{23} + u_{33} \Rightarrow u_{33} = 4 - l_{31}u_{13} - l_{32}u_{23} = -4$$

$$let\ LU = A$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \qquad U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \qquad U = \begin{bmatrix} 1 & 3 & 8 \\ 0 & 1 & -5 \\ 0 & 0 & -4 \end{bmatrix}$$

## Crout's Method

$U_{1,1} := 1 \qquad U_{2,2} := 1 \qquad U_{3,3} := 1$

$$L_{1,1} := \frac{A_{1,1}}{U_{1,1}} \qquad U_{1,2} := \frac{A_{1,2}}{L_{1,1}} \qquad U_{1,3} := \frac{A_{1,3}}{L_{1,1}}$$

$$L_{2,1} := \frac{A_{2,1}}{U_{1,1}} \qquad L_{2,2} := \frac{A_{2,2} - L_{2,1} \cdot U_{1,2}}{U_{2,2}} \qquad U_{2,3} := \frac{A_{2,3} - L_{2,1} \cdot U_{1,3}}{L_{2,2}}$$

$$L_{3,1} := \frac{A_{3,1}}{U_{1,1}} \qquad L_{3,2} := \frac{A_{3,2} - L_{3,1} \cdot U_{1,2}}{U_{2,2}} \qquad L_{3,3} := \frac{A_{3,3} - L_{3,1} \cdot U_{1,3} - L_{3,2} \cdot U_{2,3}}{U_{3,3}}$$

# Crout general formula:

- First column of L is computed $l_{i1} = a_{i1}$
- Then first row of U is computed $u_{1j} = \dfrac{a_{1j}}{l_{11}}$
- The columns of L and rows of U are computed alternately

$$l_{ij} = a_{ij} - \sum_{k=1}^{j-1} l_{ik} u_{kj} \qquad j \le i, \quad i = 1,2,\ldots,n$$

$$u_{ij} = \frac{a_{ij} - \sum_{k=1}^{i-1} l_{ik} u_{kj}}{l_{ii}} \qquad i \le j, \quad j = 2,3,\ldots,n$$

## Using the $LU$ Factorization to solve $A\mathbf{x} = \mathbf{b}$

Once the matrix factorization is complete, the solution to a linear system of the form

$$A\mathbf{x} = LU\mathbf{x} = \mathbf{b}$$

is found by first letting

$$\mathbf{y} = U\mathbf{x}$$

and solving

$$L\mathbf{y} = \mathbf{b}$$

for $\mathbf{y}$.

Example: Solve the following system using an LU decomposition.

**Using CROUT method**

$$\begin{cases} x_1 + 2x_2 + 3x_3 = 5 \\ 2x_1 - 4x_2 + 6x_3 = 18 \\ 3x_1 - 9x_2 - 3x_3 = 6 \end{cases}$$

1. Set up the equation Ax = b.

$$\begin{cases} x_1 + 2x_2 + 3x_3 = 5 \\ 2x_1 - 4x_2 + 6x_3 = 18 \\ 3x_1 - 9x_2 - 3x_3 = 6 \end{cases} \rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 2 & -4 & 6 \\ 3 & -9 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 18 \\ 6 \end{bmatrix}$$

2. Find an LU decomposition for A. This will yield the equation $(LU)\mathbf{x} = \mathbf{b}$.

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & -4 & 6 \\ 3 & -9 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & -8 & 0 \\ 3 & -15 & -12 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & -4 & 6 \\ 3 & -9 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 18 \\ 6 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 2 & -8 & 0 \\ 3 & -15 & -12 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 18 \\ 6 \end{bmatrix}.$$

3. Let $\mathbf{y}$ = U$\mathbf{x}$. Then solve the equation L$\mathbf{y}$ = $\mathbf{b}$ for $\mathbf{y}$.

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & -8 & 0 \\ 3 & -15 & -12 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 18 \\ 6 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 2 & -8 & 0 \\ 3 & -15 & -12 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 18 \\ 6 \end{bmatrix}$$

Now solving for y gives the following values:

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & -8 & 0 \\ 3 & -15 & -12 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 18 \\ 6 \end{bmatrix} \rightarrow \begin{cases} y_1 = 5 \\ 2y_1 - 8y_2 = 18 \\ 3x_1 - 15y_2 - 12y_3 = 6 \end{cases} \rightarrow \begin{matrix} y_1 = 5 \\ y_2 = -1 \\ y_3 = 2 \end{matrix}$$

4. Take the values for y and solve the equation $\mathbf{y}$ = U$\mathbf{x}$ for $\mathbf{x}$.

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \rightarrow \begin{bmatrix} 5 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\rightarrow \begin{cases} x_1 + 2x_2 + 3x_3 = 5 \\ x_2 = -1 \\ x_3 = 2 \end{cases} \rightarrow \begin{matrix} x_1 = 1 \\ x_2 = -1 \\ x_3 = 2 \end{matrix}$$

**Summary**

**<u>Sol.of Linear Equation</u>**

1-write eqn in matrix form

2-Factorize A=LU

3-Solve LY=B

4-Solve UX=y

CROUT

Example:

Solve A X = b

$$A := \begin{pmatrix} 2 & 1 & 3 \\ 1 & 1 & -2 \\ 3 & -2 & 4 \end{pmatrix} \quad B := \begin{pmatrix} 13 \\ 7 \\ -5 \end{pmatrix}$$

Given that

$$L = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 0.5 & 0 \\ 3 & -3.5 & -25 \end{pmatrix} \qquad U = \begin{pmatrix} 1 & 0.5 & 1.5 \\ 0 & 1 & -7 \\ 0 & 0 & 1 \end{pmatrix}$$

**LUX = B** $\Rightarrow$ **LY = B**

$$\begin{bmatrix} 2 & 0 & 0 \\ 1 & 0.5 & 0 \\ 3 & -3.5 & -25 \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix} = \begin{bmatrix} 13 \\ 7 \\ -5 \end{bmatrix}$$

Use forward substitution method

$$Y_1 := \frac{B_1}{L_{1,1}} \qquad Y_2 := \frac{B_2 - L_{2,1} \cdot Y_1}{L_{2,2}}$$

$$Y_3 := \frac{B_3 - L_{3,1} \cdot Y_1 - L_{3,2} \cdot Y_2}{L_{3,3}} \qquad Y = \begin{pmatrix} 6.5 \\ 1 \\ 0.84 \end{pmatrix}$$

**UX = Y**

$$\begin{bmatrix} 1 & 0.5 & 1.5 \\ 0 & 1 & -7 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 6.5 \\ 1 \\ 0.84 \end{bmatrix}$$

Use backward substitution method

$$X_3 := \frac{Y_3}{U_{3,3}} \qquad X_2 := \frac{Y_2 - U_{2,3} \cdot X_3}{U_{2,2}}$$

$$X_1 := \frac{Y_1 - U_{1,2} \cdot X_2 - U_{1,3} \cdot X_3}{U_{1,1}} \qquad X = \begin{pmatrix} 1.8 \\ 6.88 \\ 0.84 \end{pmatrix}$$

**Definition 3.4.** The nonsingular matrix $A$ has a ***triangular factorization*** if it can be expressed as the product of a lower-triangular matrix $L$ and an upper-triangular matrix $U$:

$$A = LU. \tag{1}$$

## Doolittle's Factorization:

In matrix form, this is written as

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ m_{21} & 1 & 0 & 0 \\ m_{31} & m_{32} & 1 & 0 \\ m_{41} & m_{42} & m_{43} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} & u_{14} \\ 0 & u_{22} & u_{23} & u_{24} \\ 0 & 0 & u_{33} & u_{34} \\ 0 & 0 & 0 & u_{44} \end{bmatrix}.$$

## Doolittle's Method

**Doolittle method** are computed from

$$u_{1k} = a_{1k} \qquad k = 1, \cdots, n$$

$$m_{j1} = \frac{a_{j1}}{u_{11}} \qquad j = 2, \cdots, n$$

$$u_{jk} = a_{jk} - \sum_{s=1}^{j-1} m_{js} u_{sk} \qquad k = j, \cdots, n; \quad j \geqq 2$$

$$m_{jk} = \frac{1}{u_{kk}} \left( a_{jk} - \sum_{s=1}^{k-1} m_{js} u_{sk} \right) \qquad j = k + 1, \cdots, n; \quad k \geqq 2.$$

Solve the system

$$\begin{aligned} 3x_1 + 5x_2 + 2x_3 &= 8 \\ 8x_2 + 2x_3 &= -7 \\ 6x_1 + 2x_2 + 8x_3 &= 26. \end{aligned}$$

## Doolittle's Method

$$\mathbf{A} = [a_{jk}] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 3 & 5 & 2 \\ 0 & 8 & 2 \\ 6 & 2 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ m_{21} & 1 & 0 \\ m_{31} & m_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

| | | |
|---|---|---|
| $a_{11} = 3 = 1 \cdot u_{11} = u_{11}$ | $a_{12} = 5 = 1 \cdot u_{12} = u_{12}$ | $a_{13} = 2 = 1 \cdot u_{13} = u_{13}$ |
| $a_{21} = 0 = m_{21}u_{11}$ <br> $m_{21} = 0$ | $a_{22} = 8 = m_{21}u_{12} + u_{22}$ <br> $u_{22} = 8$ | $a_{23} = 2 = m_{21}u_{13} + u_{23}$ <br> $u_{23} = 2$ |
| $a_{31} = 6 = m_{31}u_{11}$ <br> $= m_{31} \cdot 3$ <br> $m_{31} = 2$ | $a_{32} = 2 = m_{31}u_{12} + m_{32}u_{22}$ <br> $= 2 \cdot 5 + m_{32} \cdot 8$ <br> $m_{32} = -1$ | $a_{33} = 8 = m_{31}u_{13} + m_{32}u_{23} + u_{33}$ <br> $= 2 \cdot 2 - 1 \cdot 2 + u_{33}$ <br> $u_{33} = 6$ |

$$\begin{bmatrix} 3 & 5 & 2 \\ 0 & 8 & 2 \\ 6 & 2 & 8 \end{bmatrix} = \mathbf{LU} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 5 & 2 \\ 0 & 8 & 2 \\ 0 & 0 & 6 \end{bmatrix}.$$

We first solve $\mathbf{Ly} = \mathbf{b}$,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 8 \\ -7 \\ 26 \end{bmatrix}. \qquad \text{Solution} \qquad \mathbf{y} = \begin{bmatrix} 8 \\ -7 \\ 3 \end{bmatrix}.$$

determining $y_1 = 8$, then $y_2 = -7$, then $y_3$ from $2y_1 - y_2 + y_3 = 16 + 7 + y_3 = 26$;

Then we solve $\mathbf{Ux} = \mathbf{y}$,

$$\begin{bmatrix} 3 & 5 & 2 \\ 0 & 8 & 2 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 8 \\ -7 \\ 3 \end{bmatrix}. \qquad \text{Solution} \qquad \mathbf{x} = \begin{bmatrix} 4 \\ -1 \\ \frac{1}{2} \end{bmatrix}.$$

**Example 5** Determine the Crout factorization of the symmetric tridiagonal matrix

$$\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix},$$

and use this factorization to solve the linear system

$$\begin{aligned} 2x_1 - x_2 \phantom{+ 2x_3 - x_4} &= 1, \\ -x_1 + 2x_2 - x_3 \phantom{- x_4} &= 0, \\ - x_2 + 2x_3 - x_4 &= 0, \\ - x_3 + 2x_4 &= 1. \end{aligned}$$

**Solution** The $LU$ factorization of $A$ has the form

$$A = \begin{bmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 \\ 0 & a_{32} & a_{33} & a_{34} \\ 0 & 0 & a_{43} & a_{44} \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 & 0 \\ l_{21} & l_{22} & 0 & 0 \\ 0 & l_{32} & l_{33} & 0 \\ 0 & 0 & l_{43} & l_{44} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & 0 & 0 \\ 0 & 1 & u_{23} & 0 \\ 0 & 0 & 1 & u_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 \\ 0 & a_{32} & a_{33} & a_{34} \\ 0 & 0 & a_{43} & a_{44} \end{bmatrix} = \begin{bmatrix} l_{11} & l_{11}u_{12} & 0 & 0 \\ l_{21} & l_{22} + l_{21}u_{12} & l_{22}u_{23} & 0 \\ 0 & l_{32} & l_{33} + l_{32}u_{23} & l_{33}u_{34} \\ 0 & 0 & l_{43} & l_{44} + l_{43}u_{34} \end{bmatrix}.$$

$$\begin{array}{llll}
a_{11}: & 2 = l_{11} \implies l_{11} = 2, & a_{12}: & -1 = l_{11}u_{12} \implies u_{12} = -\tfrac{1}{2}, \\
a_{21}: & -1 = l_{21} \implies l_{21} = -1, & a_{22}: & 2 = l_{22} + l_{21}u_{12} \implies l_{22} = -\tfrac{3}{2}, \\
a_{23}: & -1 = l_{22}u_{23} \implies u_{23} = -\tfrac{2}{3}, & a_{32}: & -1 = l_{32} \implies l_{32} = -1, \\
a_{33}: & 2 = l_{33} + l_{32}u_{23} \implies l_{33} = \tfrac{4}{3}, & a_{34}: & -1 = l_{33}u_{34} \implies u_{34} = -\tfrac{3}{4}, \\
a_{43}: & -1 = l_{43} \implies l_{43} = -1, & a_{44}: & 2 = l_{44} + l_{43}u_{34} \implies l_{44} = \tfrac{5}{4}.
\end{array}$$

This gives the Crout factorization

$$A = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 & 0 \\ -1 & \frac{3}{2} & 0 & 0 \\ 0 & -1 & \frac{4}{3} & 0 \\ 0 & 0 & -1 & \frac{5}{4} \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{2} & 0 & 0 \\ 0 & 1 & -\frac{2}{3} & 0 \\ 0 & 0 & 1 & -\frac{3}{4} \\ 0 & 0 & 0 & 1 \end{bmatrix} = LU.$$

Solving the system $Lz = b$

$$Lz = \begin{bmatrix} 2 & 0 & 0 & 0 \\ -1 & \frac{3}{2} & 0 & 0 \\ 0 & -1 & \frac{4}{3} & 0 \\ 0 & 0 & -1 & \frac{5}{4} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad \text{gives} \quad \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{3} \\ \frac{1}{4} \\ 1 \end{bmatrix},$$

and then solving $U\mathbf{x} = z$

$$U\mathbf{x} = \begin{bmatrix} 1 & -\frac{1}{2} & 0 & 0 \\ 0 & 1 & -\frac{2}{3} & 0 \\ 0 & 0 & 1 & -\frac{3}{4} \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{3} \\ \frac{1}{4} \\ 1 \end{bmatrix} \quad \text{gives} \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

The Crout Factorization Algorithm can be applied whenever $l_{ii} \neq 0$ for each $i = 1, 2, \ldots, n$. Two conditions, either of which ensure that this is true, are that the coefficient matrix of the system is positive definite or that it is strictly diagonally dominant. An ad-

In linear algebra, a *tridiagonal matrix* is a band matrix that has nonzero elements on the main diagonal,

## Example

(a) Determine the $LU$ factorization for matrix $A$ in the linear system $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} 1 & 1 & 0 & 3 \\ 2 & 1 & -1 & 1 \\ 3 & -1 & -1 & 2 \\ -1 & 2 & 3 & -1 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ -3 \\ 4 \end{bmatrix}$$

(b) Then use the factorization to solve the system

$$\begin{aligned} x_1 + x_2 \phantom{- x_3} + 3x_4 &= 8 \\ 2x_1 + x_2 - x_3 + x_4 &= 7 \\ 3x_1 - x_2 - x_3 + 2x_4 &= 14 \\ -x_1 + 2x_2 + 3x_3 - x_4 &= -7 \end{aligned}$$

Home Activity

**Solution:**

$x_4 = 2,\ x_3 = 0,\ x_2 = -1,\ x_1 = 3.$

## EXERCISE SET 6.5

**2.** Solve the following linear systems:

**a.** $$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 & -1 \\ 0 & 4 & 2 \\ 0 & 0 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -5 \end{bmatrix}$$

**b.** $$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -3 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \\ 8 \end{bmatrix}$$

**4.** Consider the following matrices. Find the permutation matrix $P$ so that $PA$ can be factored into the product $LU$, where $L$ is lower triangular with 1s on its diagonal and $U$ is upper triangular for these matrices.

**a.** $A = \begin{bmatrix} 0 & 2 & -1 \\ 1 & -1 & 2 \\ 1 & -1 & 4 \end{bmatrix}$ **b.** $A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 4 & 7 \\ -1 & 2 & 5 \end{bmatrix}$

**5.** Factor the following matrices into the $LU$ decomposition using the $LU$ Factorization Algorithm with $l_{ii} = 1$ for all $i$.

**a.** $\begin{bmatrix} 2 & -1 & 1 \\ 3 & 3 & 9 \\ 3 & 3 & 5 \end{bmatrix}$

**b.** $\begin{bmatrix} 1.012 & -2.132 & 3.104 \\ -2.132 & 4.096 & -7.013 \\ 3.104 & -7.013 & 0.014 \end{bmatrix}$

**7.** Modify the $LU$ Factorization Algorithm so that it can be used to solve a linear system, and then solve the following linear systems.

**a.**
$$\begin{aligned} 2x_1 - x_2 + x_3 &= -1, \\ 3x_1 + 3x_2 + 9x_3 &= 0, \\ 3x_1 + 3x_2 + 5x_3 &= 4. \end{aligned}$$

**b.**
$$\begin{aligned} 1.012x_1 - 2.132x_2 + 3.104x_3 &= 1.984, \\ -2.132x_1 + 4.096x_2 - 7.013x_3 &= -5.049, \\ 3.104x_1 - 7.013x_2 + 0.014x_3 &= -3.895. \end{aligned}$$

**c.**
$$\begin{aligned} 2x_1 \qquad\qquad\qquad\qquad &= 3, \\ x_1 + 1.5x_2 \qquad\qquad\quad &= 4.5, \\ -\ 3x_2 + 0.5x_3 \qquad &= -6.6, \\ 2x_1 - 2x_2 + x_3 + x_4 &= 0.8. \end{aligned}$$


ANY
Questions?