Chapter 6 : Direct methods for solving linear System

CHAPTER OBJECTIVES

1. Matrix (LU) factorization
2. CROUTE and Doolittle's Factorization
3. Band matrix (CTS)
4. Diagonally dominant matrices
5. Positive definite matrices
6. LDL^t Factorization
7. Cholesky { LL^t Factorization }

Exercise: 6.5 and 6.6

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Exercise 6.6

Special Types of Matrices

Diagonally Dominant Matrices

Definition 6.20 The $n \times n$ matrix A is said to be **diagonally dominant** when

$$|a_{ii}| \geq \sum_{\substack{j=1,\\ j\neq i}}^{n} |a_{ij}| \quad \text{holds for each } i = 1, 2, \cdots, n.$$

Each main diagonal entry in a strictly diagonally dominant matrix has a magnitude that is strictly greater that the sum of the magnitudes of all the other entries in that row.

Consider the matrices

$$A = \begin{bmatrix} 7 & 2 & 0 \\ 3 & 5 & -1 \\ 0 & 5 & -6 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 6 & 4 & -3 \\ 4 & -2 & 0 \\ -3 & 0 & 1 \end{bmatrix}.$$

The nonsymmetric matrix A is strictly diagonally dominant because

$$|7| > |2| + |0|, \quad |5| > |3| + |-1|, \quad \text{and} \quad |-6| > |0| + |5|.$$

The symmetric matrix B is not strictly diagonally dominant

first row the absolute value of the diagonal element is $|6| < |4| + |-3| = 7$.

Positive Definite Matrices

Definition 6.22 A matrix A is **positive definite** if it is symmetric and if $\mathbf{x}^t A\mathbf{x} > 0$ for every n-dimensional vector $\mathbf{x} \neq \mathbf{0}$. ■

Example 1 Show that the matrix

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

is positive definite

Solution Suppose $\mathbf{x}$ is any three-dimensional column vector. Then

$$\mathbf{x}^t A\mathbf{x} = [x_1, x_2, x_3] \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= [x_1, x_2, x_3] \begin{bmatrix} 2x_1 & - & x_2 & & \\ -x_1 & + & 2x_2 & - & x_3 \\ -x_2 & + & 2x_3 & & \end{bmatrix}$$

Compute $x^T Ax$ by completing the square:

$$= 2x_1^2 - 2x_1x_2 + 2x_2^2 - 2x_2x_3 + 2x_3^2.$$

Rearranging the terms gives

$$\mathbf{x}^t A\mathbf{x} = x_1^2 + (x_1^2 - 2x_1x_2 + x_2^2) + (x_2^2 - 2x_2x_3 + x_3^2) + x_3^2$$

$$= x_1^2 + (x_1 - x_2)^2 + (x_2 - x_3)^2 + x_3^2,$$

which implies that

$$x_1^2 + (x_1 - x_2)^2 + (x_2 - x_3)^2 + x_3^2 > 0$$

EXAMPLE

Show that the matrix $A = \begin{bmatrix} 2 & 2 \\ 2 & 5 \end{bmatrix}$ is symmetric positive-definite.

Clearly A is symmetric. To show it is positive-definite, one applies the definition:

$$\begin{aligned} x^T A x &= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ &= 2x_1^2 + 4x_1x_2 + 5x_2^2 \\ &= 2(x_1 + x_2)^2 + 3x_2^2 \end{aligned}$$

EXAMPLE

Show that the symmetric matrix $A = \begin{bmatrix} 2 & 4 \\ 4 & 5 \end{bmatrix}$ is not positive-definite.

Compute $x^T Ax$ by completing the square:

$$\begin{aligned} x^T A x &= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ &= 2x_1^2 + 8x_1x_2 + 5x_2^2 \\ &= 2(x_1^2 + 4x_1x_2) + 5x_2^2 \\ &= 2(x_1 + 2x_2)^2 - 8x_2^2 + 5x_2^2 \\ &= 2(x_1 + 2x_2)^2 - 3x_2^2 \end{aligned}$$

Theorem 6.25 A symmetric matrix A is positive definite if and only if each of its leading principal submatrices has a positive determinant. ∎

Example 2 In Example 1 we used the definition to show that the symmetric matrix

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

is positive definite. Confirm this using Theorem 6.25.

Solution Note that

$$\det A_1 = \det[2] = 2 > 0,$$

$$\det A_2 = \det \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} = 4 - 1 = 3 > 0,$$

$$\det A_3 = \det \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} = 2 \det \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} - (-1) \det \begin{bmatrix} -1 & -1 \\ 0 & 2 \end{bmatrix}$$

$$= 2(4-1) + (-2+0) = 4 > 0.$$

Corollary 6.27 The matrix A is positive definite if and only if A can be factored in the form LDL^t, where L is lower triangular with 1s on its diagonal and D is a diagonal matrix with positive diagonal entries. ■

Corollary 6.28 The matrix A is positive definite if and only if A can be factored in the form LL^t, where L is lower triangular with nonzero diagonal entries. ■

NOTE:

If the coefficient matrix $[A]$ is symmetrical but not necessarily positive definite, then the above Cholesky algorithms will not be valid. In this case, the following LDL^T factorized algorithms can be employed

Cholesky and LDL^T Decomposition

Example 3 Determine the LDL^t factorization of the positive definite matrix

$$A = \begin{bmatrix} 4 & -1 & 1 \\ -1 & 4.25 & 2.75 \\ 1 & 2.75 & 3.5 \end{bmatrix}.$$

$$A = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{21} & a_{22} & a_{32} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix} \begin{bmatrix} 1 & l_{21} & l_{31} \\ 0 & 1 & l_{32} \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} d_1 & d_1 l_{21} & d_1 l_{31} \\ d_1 l_{21} & d_2 + d_1 l_{21}^2 & d_2 l_{32} + d_1 l_{21} l_{31} \\ d_1 l_{31} & d_1 l_{21} l_{31} + d_2 l_{32} & d_1 l_{31}^2 + d_2 l_{32}^2 + d_3 \end{bmatrix}$$

$$a_{11} : 4 = d_1 \implies d_1 = 4, \qquad a_{21} : -1 = d_1 l_{21} \implies l_{21} = -0.25$$

$$a_{31} : 1 = d_1 l_{31} \implies l_{31} = 0.25, \qquad a_{22} : 4.25 = d_2 + d_1 l_{21}^2 \implies d_2 = 4$$

$$a_{32} : 2.75 = d_1 l_{21} l_{31} + d_2 l_{32} \implies l_{32} = 0.75, \quad a_{33} : 3.5 = d_1 l_{31}^2 + d_2 l_{32}^2 + d_3 \implies d_3 = 1,$$

$$A = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{21} & a_{22} & a_{32} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix} \begin{bmatrix} 1 & l_{21} & l_{31} \\ 0 & 1 & l_{32} \\ 0 & 0 & 1 \end{bmatrix}$$

and we have

$$A = LDL^t = \begin{bmatrix} 1 & 0 & 0 \\ -0.25 & 1 & 0 \\ 0.25 & 0.75 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -0.25 & 0.25 \\ 0 & 1 & 0.75 \\ 0 & 0 & 1 \end{bmatrix}.$$

Procedure to solve Ax=b using LDL^t

If the coefficient matrix $[A]$ is symmetrical but not necessarily positive definite, then the above Cholesky algorithms will not be valid. In this case, the following LDL^T factorized algorithms can be employed

$$[A] = [L][D][L]^T$$

For example

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} d_{11} & 0 & 0 \\ 0 & d_{22} & 0 \\ 0 & 0 & d_{33} \end{bmatrix} \begin{bmatrix} 1 & l_{21} & l_{31} \\ 0 & 1 & l_{32} \\ 0 & 0 & 1 \end{bmatrix}$$

$$d_{jj} = a_{jj} - \sum_{k=1}^{j-1} l_{jk}^2 d_{kk}$$

$$l_{ij} = \left(a_{ij} - \sum_{k=1}^{j-1} l_{ik} d_{kk} l_{jk} \right) \times \left(\frac{1}{d_{jj}} \right)$$

the LDL^T algorithms can be summarized by the following step-by-step procedures.

Step1: Factorization phase

$$[A]=[L][D][L]^T$$

Step 2: Forward solution and diagonal scaling phase

$$[L][D][L]^T[x]=[b]$$

Let us define

$$[L]^T[x]=[y]$$

$$\begin{bmatrix} 1 & l_{21} & l_{31} \\ 0 & 1 & l_{32} \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \Rightarrow \quad x_i = y_i - \sum_{k=i+1}^{n} l_{ki}x_k;\ for\ i=n,n-1,\ldots,2,1$$

Also, define

$$[D][y]=[z]$$

$$\begin{bmatrix} d_{11} & 0 & 0 \\ 0 & d_{22} & 0 \\ 0 & 0 & d_{33} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \Rightarrow \quad y_i = \frac{z_i}{d_{ii}},\ for\ i = 1,2,3,\ \ldots\ldots, n$$

$$[L][z]=[b]$$

$$\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \Rightarrow \quad z_i = b_i - \sum_{k=1}^{i-1} L_{ik} z_k \quad for\ i = 1,2,3,\ \ldots\ldots, n$$

Step 3: Backward solution phase

Example: Using the LDL^T algorithm, solve the following system for the unknown vector $[x]$..

$[A][x] = [b]$ where

$$[A] = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \qquad [b] = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Solution

The factorized matrices $[D]$ and $[L]$ can be computed from

$$d_{jj} = a_{jj} - \sum_{k=1}^{j-1} l_{jk}^2 d_{kk}$$

$$l_{ij} = \left(a_{ij} - \sum_{k=1}^{j-1} l_{ik} d_{kk} l_{jk} \right) \times \left(\frac{1}{d_{jj}} \right)$$

We know that

$$[D] = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1.5 & 0 \\ 0 & 0 & 0.3333 \end{bmatrix} \qquad [L] = \begin{bmatrix} 1 & 0 & 0 \\ -0.5 & 1 & 0 \\ 0 & -0.6667 & 1 \end{bmatrix}$$

$$[L][z] = [b]$$

$$\begin{bmatrix} 1 & 0 & 0 \\ -0.5 & 1 & 0 \\ 0 & -0.667 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \quad Z = \begin{bmatrix} 1 \\ 0.5 \\ 0.3333 \end{bmatrix}$$

$$[D][y] = [z]$$

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1.5 & 0 \\ 0 & 0 & 0.3333 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0.5 \\ 0.3333 \end{bmatrix} \Rightarrow \quad Y = \begin{bmatrix} 0.5 \\ 0.333 \\ 1 \end{bmatrix}$$

$$[L]^T[x]=[y]$$

$$\begin{bmatrix} 1 & -0.5 & 0 \\ 0 & 1 & -0.667 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0.5 \\ 0.333 \\ 1 \end{bmatrix}$$

Hence

$$[x]=\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Summary

Sol.of Linear Equation

1-Factorize A=LDL^t

2-Solve $Lz=b$

3-Solve $DY=z$

4-Solve $L^tX=y$

Procedure to solve Ax = b using $\boldsymbol{lDl^t}$

$$Ax = b$$

$$LDL^t x = b$$

$$L\{DL^t x\} = b$$

$$Consider\ DL^t x = Z$$

$$LZ = b ------ -eqn1) \quad \rightarrow \quad find\ z$$

$$Consider\ L^t x = y$$

$$Dy = Z ------ -eqn2) \quad \rightarrow \quad find\ y$$

$$L^t X = y ------ -eqn3) \quad \rightarrow \quad find\ x$$

Cholesky's Method

For a *symmetric, positive definite* matrix $\mathbf{A}$ (thus $\mathbf{A} = \mathbf{A}^T, \mathbf{x}^T\mathbf{A}\mathbf{x} > 0$ for all $\mathbf{x} \neq \mathbf{0}$)

The popular method of solving $\mathbf{Ax} = \mathbf{b}$ based on this factorization $\mathbf{A} = \mathbf{LL}^T$ is called **Cholesky's method.**[3] In terms of the entries of $\mathbf{L} = [l_{jk}]$ the formulas for the factorization

Example:

$$\mathbf{A} = \begin{bmatrix} 4 & 2 & 14 \\ 2 & 17 & -5 \\ 14 & -5 & 83 \end{bmatrix} = \mathbf{LL}^T = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 4 & 0 \\ 7 & -3 & 5 \end{bmatrix} \begin{bmatrix} 2 & 1 & 7 \\ 0 & 4 & -3 \\ 0 & 0 & 5 \end{bmatrix}.$$

Procedure to find U_{ii} of ll^t

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} u_{11} & 0 & 0 \\ u_{12} & u_{22} & 0 \\ u_{13} & u_{23} & u_{33} \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

$$\left\{ \begin{array}{l} u_{11} = \sqrt{a_{11}}\,; u_{12} = \dfrac{a_{12}}{u_{11}}; u_{13} = \dfrac{a_{13}}{u_{11}} \\ u_{22} = \left(a_{22} - u_{12}^2\right)^{\frac{1}{2}}; u_{23} = \dfrac{a_{23} - u_{12}u_{13}}{u_{22}}; u_{33} = \left(a_{33} - u_{13}^2 - u_{23}^2\right)^{\frac{1}{2}} \end{array} \right.$$

Cholesky Factorization

Problem Statement. Compute the Cholesky factorization for the symmetric matrix

$$[A] = \begin{bmatrix} 6 & 15 & 55 \\ 15 & 55 & 225 \\ 55 & 225 & 979 \end{bmatrix}$$

Solution:

$$u_{11} = \sqrt{a_{11}} = \sqrt{6} = 2.44949$$

$$u_{12} = \frac{a_{12}}{u_{11}} = \frac{15}{2.44949} = 6.123724$$

$$u_{13} = \frac{a_{13}}{u_{11}} = \frac{55}{2.44949} = 22.45366$$

$$u_{22} = \sqrt{a_{22} - u_{12}^2} = \sqrt{55 - (6.123724)^2} = 4.1833$$

$$u_{23} = \frac{a_{23} - u_{12}u_{13}}{u_{22}} = \frac{225 - 6.123724(22.45366)}{4.1833} = 20.9165$$

$$u_{33} = \sqrt{a_{33} - u_{13}^2 - u_{23}^2} = \sqrt{979 - (22.45366)^2 - (20.9165)^2} = 6.110101$$

Thus, the Cholesky factorization yields

$$[U] = \begin{bmatrix} 2.44949 & 6.123724 & 22.45366 \\ & 4.1833 & 20.9165 \\ & & 6.110101 \end{bmatrix}$$

Example 4 Determine the Cholesky LL^t factorization of the positive definite matrix

$$A = \begin{bmatrix} 4 & -1 & 1 \\ -1 & 4.25 & 2.75 \\ 1 & 2.75 & 3.5 \end{bmatrix}.$$

$$A = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{21} & a_{22} & a_{32} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} l_{11} & l_{21} & l_{31} \\ 0 & l_{22} & l_{32} \\ 0 & 0 & l_{33} \end{bmatrix}$$

$$= \begin{bmatrix} l_{11}^2 & l_{11}l_{21} & l_{11}l_{31} \\ l_{11}l_{21} & l_{21}^2 + l_{22}^2 & l_{21}l_{31} + l_{22}l_{32} \\ l_{11}l_{31} & l_{21}l_{31} + l_{22}l_{32} & l_{31}^2 + l_{32}^2 + l_{33}^2 \end{bmatrix}$$

$$a_{11}: \quad 4 = l_{11}^2 \implies l_{11} = 2, \qquad a_{21}: \quad -1 = l_{11}l_{21} \implies l_{21} = -0.5$$

$$a_{31}: \quad 1 = l_{11}l_{31} \implies l_{31} = 0.5, \qquad a_{22}: \quad 4.25 = l_{21}^2 + l_{22}^2 \implies l_{22} = 2$$

$$a_{32}: \quad 2.75 = l_{21}l_{31} + l_{22}l_{32} \implies l_{32} = 1.5, \qquad a_{33}: \quad 3.5 = l_{31}^2 + l_{32}^2 + l_{33}^2 \implies l_{33} = 1,$$

Put all values then

$$\begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{21} & a_{22} & a_{32} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} l_{11} & l_{21} & l_{31} \\ 0 & l_{22} & l_{32} \\ 0 & 0 & l_{33} \end{bmatrix}$$

and we have

$$A = LL^t = \begin{bmatrix} 2 & 0 & 0 \\ -0.5 & 2 & 0 \\ 0.5 & 1.5 & 1 \end{bmatrix} \begin{bmatrix} 2 & -0.5 & 0.5 \\ 0 & 2 & 1.5 \\ 0 & 0 & 1 \end{bmatrix}.$$

Cholesky's Method

Solve by Cholesky's method:

$$\begin{aligned} 4x_1 + 2x_2 + 14x_3 &= 14 \\ 2x_1 + 17x_2 - 5x_3 &= -101 \\ 14x_1 - 5x_2 + 83x_3 &= 155. \end{aligned}$$

Solution:

$$A = L\,L^t$$

$$\begin{bmatrix} 4 & 2 & 14 \\ 2 & 17 & -5 \\ 14 & -5 & 83 \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} l_{11} & l_{21} & l_{31} \\ 0 & l_{22} & l_{32} \\ 0 & 0 & l_{33} \end{bmatrix}$$

we compute, in the given order,

$$l_{11} = \sqrt{a_{11}} = 2 \qquad l_{21} = \frac{a_{21}}{l_{11}} = \frac{2}{2} = 1 \qquad l_{31} = \frac{a_{31}}{l_{11}} = \frac{14}{2} = 7$$

$$l_{22} = \sqrt{a_{22} - l_{21}^2} = \sqrt{17 - 1} = 4$$

$$l_{32} = \frac{1}{l_{23}}(a_{32} - l_{31}l_{21}) = \frac{1}{4}(-5 - 7 \cdot 1) = -3$$

$$l_{33} = \sqrt{a_{33} - l_{31}^2 - l_{32}^2} = \sqrt{83 - 7^2 - (-3)^2} = 5.$$

We now have to solve $\mathbf{Ly} = \mathbf{b}$, that is,

$$\begin{bmatrix} 2 & 0 & 0 \\ 1 & 4 & 0 \\ 7 & -3 & 5 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 14 \\ -101 \\ 155 \end{bmatrix}. \qquad \text{Solution} \qquad \mathbf{y} = \begin{bmatrix} 7 \\ -27 \\ 5 \end{bmatrix}.$$

we have to solve $\mathbf{Ux} = \mathbf{L}^\mathsf{T}\mathbf{x} = \mathbf{y}$, that is,

$$\begin{bmatrix} 2 & 1 & 7 \\ 0 & 4 & -3 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ -27 \\ 5 \end{bmatrix}. \qquad \text{Solution} \qquad \mathbf{x} = \begin{bmatrix} 3 \\ -6 \\ 1 \end{bmatrix}.$$

Summary

<u>Sol.of Linear Equation</u>

Check symmetric (A=A^t) and positive definite then

1-Factorize A=ll^t

2-Solve LY=B

3-Solve l^tX=y

Solve Ax = b
using Cholesky method , where

$$[A]=\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \qquad [b]=\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Solution:

Cholesky factorization:

$$\begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{21} & a_{22} & a_{32} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} l_{11} & l_{21} & l_{31} \\ 0 & l_{22} & l_{32} \\ 0 & 0 & l_{33} \end{bmatrix}$$

$$l = \begin{bmatrix} 1.414 & 0 & 0 \\ -0.7071 & 1.225 & 0 \\ 0 & -0.8165 & 0.5774 \end{bmatrix} \qquad l^t = \begin{bmatrix} 1.414 & -0.7071 & 0 \\ 0 & 1.225 & -0.8165 \\ 0 & 0 & 0.5774 \end{bmatrix}$$

$solve\ LY = b$

$$\begin{bmatrix} 1.414 & 0 & 0 \\ -0.7071 & 1.225 & 0 \\ 0 & -0.8165 & 0.5774 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$Y = \begin{bmatrix} 0.7071 \\ 0.4082 \\ 0.5774 \end{bmatrix}$$

$solve\ L^t X = y$

$$\begin{bmatrix} 1.414 & -0.7071 & 0 \\ 0 & 1.225 & -0.8165 \\ 0 & 0 & 0.5774 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0.7071 \\ 0.4082 \\ 0.5774 \end{bmatrix}$$

Hence

$$[x] = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Home Activity:

Q#1 Solve the following linear system Ax=b using
a) Doolittle's b) Crout's

$$\begin{aligned} x_1 - x_2 \quad\quad &= 2, \\ 2x_1 + 2x_2 + 3x_3 &= -1, \\ -x_1 + 3x_2 + 2x_3 &= 4. \end{aligned}$$

Q#2 Check the coefficient matrix for positive definite then solve the given linear system use lDl^t and Cholesky factorization ll^t

$$\begin{aligned} 6x_1 + 2x_2 + x_3 - x_4 &= 0, \\ 2x_1 + 4x_2 + x_3 \quad\quad &= 7, \\ x_1 + x_2 + 4x_3 - x_4 &= -1, \\ -x_1 \quad\quad - x_3 + 3x_4 &= -2. \end{aligned}$$

Q#3 Use Crout factorization for tridiagonal system to solve linear system Ax=b

$$\begin{aligned} 2x_1 - x_2 \quad\quad &= 3, \\ x_1 + 2x_2 - x_3 \quad\quad &= 4, \\ x_2 - 2x_3 + x_4 &= 0, \\ x_3 + 2x_4 &= 6. \end{aligned}$$

EXERCISE SET 6.6

1. Determine which of the following matrices are (i) symmetric, (ii) singular, (iii) strictly diagonally dominant, (iv) positive definite.

   a. $\begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$

   b. $\begin{bmatrix} 2 & 1 & 0 \\ 0 & 3 & 0 \\ 1 & 0 & 4 \end{bmatrix}$

   c. $\begin{bmatrix} 4 & 2 & 6 \\ 3 & 0 & 7 \\ -2 & -1 & -3 \end{bmatrix}$

   d. $\begin{bmatrix} 4 & 0 & 0 & 0 \\ 6 & 7 & 0 & 0 \\ 9 & 11 & 1 & 0 \\ 5 & 4 & 1 & 1 \end{bmatrix}$

3. Use the LDL^t Factorization Algorithm to find a factorizaton of the form $A = LDL^t$ for the following matrices:

   a. $A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$

   b. $A = \begin{bmatrix} 4 & 1 & 1 & 1 \\ 1 & 3 & -1 & 1 \\ 1 & -1 & 2 & 0 \\ 1 & 1 & 0 & 2 \end{bmatrix}$

5. Use the Cholesky Algorithm to find a factorization of the form $A = LL^t$ for the matrices in Exercise 3.

9. Modify the Cholesky Algorithm as suggested in the text so that it can be used to solve linear systems, and use the modified algorithm to solve the linear systems in Exercise 7.

11. Use Crout factorization for tridiagonal systems to solve the following linear systems.

a.
$$\begin{aligned} x_1 - x_2 &= 0, \\ -2x_1 + 4x_2 - 2x_3 &= -1, \\ - x_2 + 2x_3 &= 1.5. \end{aligned}$$

b.
$$\begin{aligned} 3x_1 + x_2 &= -1, \\ 2x_1 + 4x_2 + x_3 &= 7, \\ 2x_2 + 5x_3 &= 9. \end{aligned}$$

c.
$$\begin{aligned} 2x_1 - x_2 &= 3, \\ -x_1 + 2x_2 - x_3 &= -3, \\ - x_2 + 2x_3 &= 1. \end{aligned}$$

d.
$$\begin{aligned} 0.5x_1 + 0.25x_2 &= 0.35, \\ 0.35x_1 + 0.8x_2 + 0.4x_3 &= 0.77, \\ 0.25x_2 + x_3 + 0.5x_4 &= -0.5, \\ x_3 - 2x_4 &= -2.25. \end{aligned}$$

12. Use Crout factorization for tridiagonal systems to solve the following linear systems.

a.
$$\begin{aligned} 2x_1 + x_2 &= 3, \\ x_1 + 2x_2 + x_3 &= -2, \\ 2x_2 + 3x_3 &= 0. \end{aligned}$$

b.
$$\begin{aligned} 2x_1 - x_2 &= 5, \\ -x_1 + 3x_2 + x_3 &= 4, \\ x_2 + 4x_3 &= 0. \end{aligned}$$

14. Modify the LDL^t factorization to factor a symmetric matrix A. [*Note:* The factorization may not always be possible.] Apply the new algorithm to the following matrices:

a. $A = \begin{bmatrix} 3 & -3 & 6 \\ -3 & 2 & -7 \\ 6 & -7 & 13 \end{bmatrix}$ **b.** $A = \begin{bmatrix} 3 & -6 & 9 \\ -6 & 14 & -20 \\ 9 & -20 & 29 \end{bmatrix}$

17. Find all α so that $A = \begin{bmatrix} 2 & \alpha & -1 \\ \alpha & 2 & 1 \\ -1 & 1 & 4 \end{bmatrix}$ is positive definite.

18. Find all α and $\beta > 0$ so that the matrix

$$A = \begin{bmatrix} 4 & \alpha & 1 \\ 2\beta & 5 & 4 \\ \beta & 2 & \alpha \end{bmatrix}$$


ANY
QUESTIONS