Chapter 10 One- and Two-Sample Tests of Hypotheses

Approach to Hypothesis Testing with Fixed Probability of Type I Error

Tests Concerning a Single Mean

1. State the null and alternative hypotheses.
2. Choose a fixed significance level α.
3. Choose an appropriate test statistic and establish the critical region based on α.
4. Reject H_0 if the computed test statistic is in the critical region. Otherwise, do not reject.
5. Draw scientific or engineering conclusions.

Hypothesis: A hypothesis is a statement that something is true.

For example, the statement "the mean weight of all bags of pretzels packaged differs from the advertised weight of 454 g" is a hypothesis.

In a typical study, the hypothesis test involves two types of hypotheses called "Null Hypothesis" and "Alternate Hypothesis"

Null hypothesis: A hypothesis to be tested. The symbol H_o to represent the null hypothesis.
H_o: $\mu = \mu_o$

Alternative hypothesis: A hypothesis to be considered as an alternative to the null hypothesis. The symbol H_a to represent the alternative hypothesis.
H_a: $\mu \neq \mu_o$ H_a: $\mu < \mu_o$ H_a: $\mu > \mu_o$

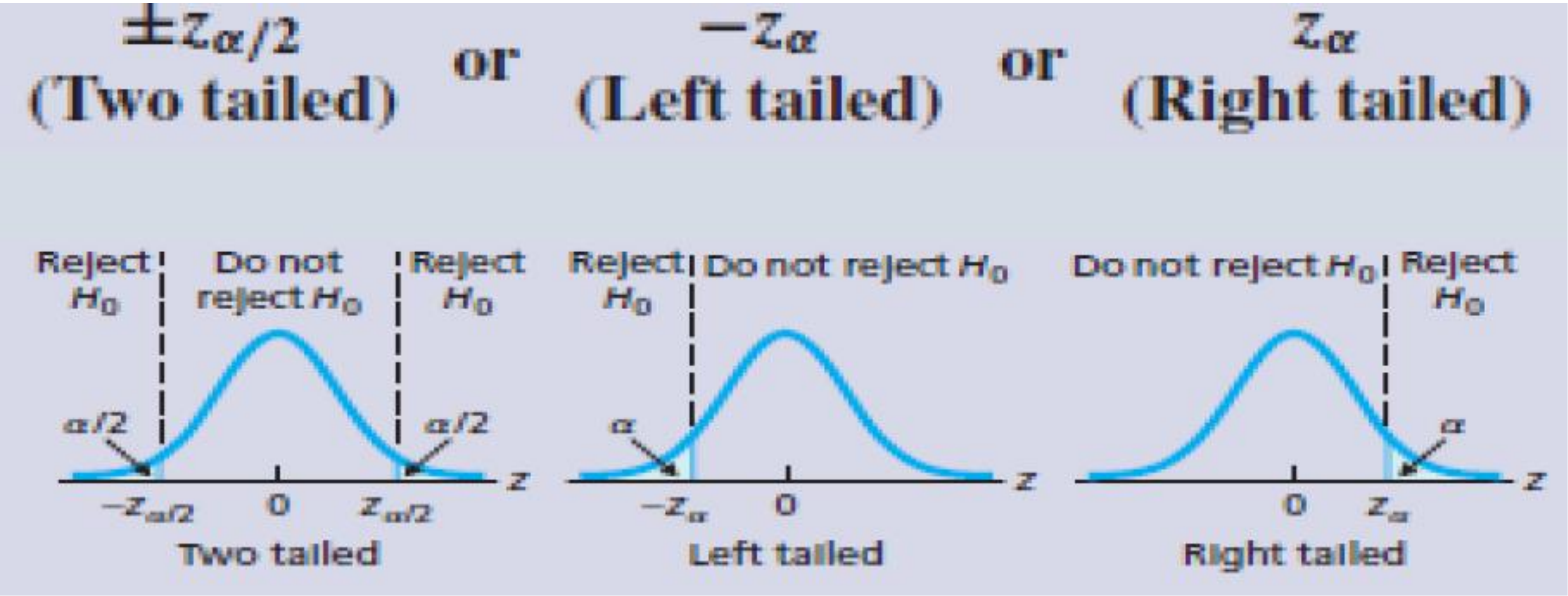
A hypothesis test is called a one-tailed test if it is either left tailed or right tailed.

Stepwise Procedure for Performing One Mean Z-test:

Step 1: The null hypothesis is H_0: $\mu = \mu_0$, and the alternative hypothesis is,
H_a: $\mu \neq \mu_0$ (Two tailed) or H_a: $\mu < \mu_0$ (Left tailed) or H_a: $\mu > \mu_0$ (Right tailed)
Step 2: Decide on the significance level, α.
Step 3: Compute the value of the test statistic,

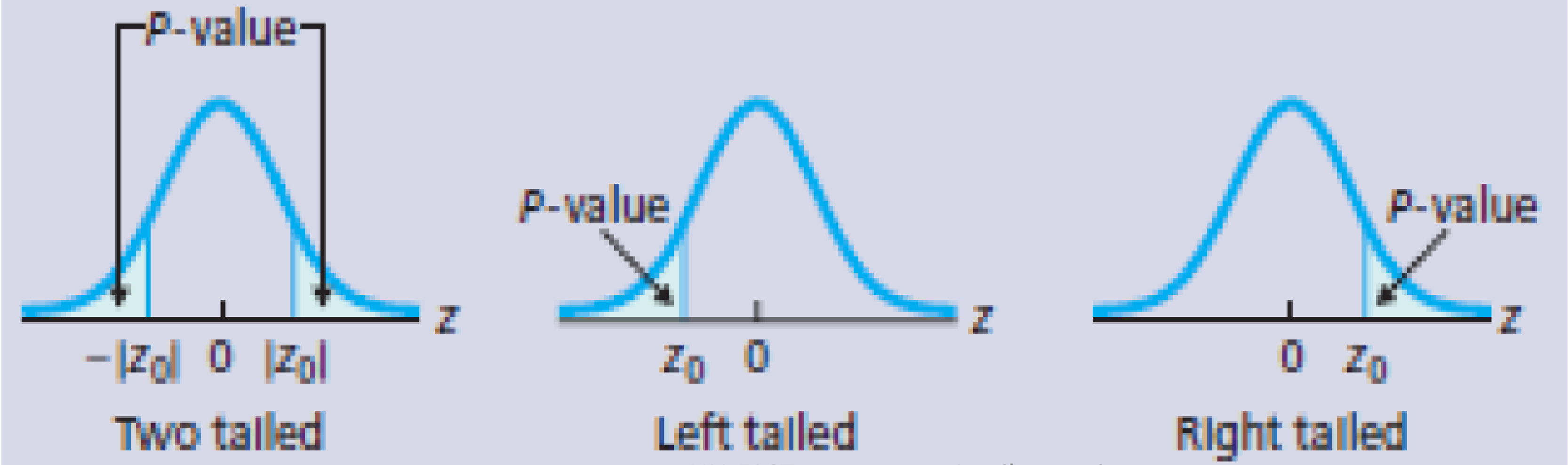
$$z = \frac{\bar{x} - \mu_o}{\sigma/\sqrt{n}}$$

and denote that value $\mathbf{z_0}$.
Step 4: Analyze the value
Approach 1: Critical Value Approach



Approach 2: P-Value Approach

Use Z-Table to find the critical value(s).



Step 5: If the value of the test statistic falls in the rejection region, reject H_0; otherwise, do not reject H_0.

OR

If $P \leq \alpha$, reject H_0; otherwise, do not reject H_0.
Step 6: Interpret the results of the hypothesis test.

Type I & Type II Error
In hypothesis testing, there are also two possible errors,
Type I error: Rejecting the null hypothesis when it is in fact true.
Type II error: Not rejecting the null hypothesis when it is in fact false.

	Truth		
Decision		H_o is true	H_o is false
	Do not reject H_o	Correct Decision	Type II Error
	Reject H_o	Type I Error	Correct Decision

Prices of History Books The R. R. Bowker Company collects information on the retail prices of books and publishes the data in *The Bowker Annual Library and Book Trade Almanac*. In 2005, the mean retail price of history books was \$78.01. Suppose that we want to perform a hypothesis test to decide whether this year's mean retail price of history books has increased from the 2005 mean.

a. Determine the null hypothesis for the hypothesis test.

b. Determine the alternative hypothesis for the hypothesis test.

c. Classify the hypothesis test as two tailed, left tailed, or right tailed.

Solution Let μ denote this year's mean retail price of history books.

a. The null hypothesis is that this year's mean retail price of history books *equals* the 2005 mean of \$78.01; that is, $H0{:}\mu = \$78.01$.

b. The alternative hypothesis is that this year's mean retail price of history books is *greater than* the 2005 mean of \$78.01; that is, $Ha{:}\ \mu > \$78.01$.

c. This hypothesis test is right tailed because a greater-than sign (>) appears in the alternative hypothesis.

Hypothesis Test with One Population Mean When σ is Known

Test Procedure for a Single Mean (Variance Known)

Assumptions

1. Simple random sample
2. Normal population or large sample
3. σ known

$$z = \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}} > z_{\alpha/2} \quad \text{or} \quad z = \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}} < -z_{\alpha/2}$$

If $-z_{\alpha/2} < z < z_{\alpha/2}$, do not reject H_0. Rejection of H_0, of course, implies acceptance of the alternative hypothesis $\mu \neq \mu_0$. With this definition of the critical region, it should be clear that there will be probability α of rejecting H_0 (falling into the critical region) when, indeed, $\mu = \mu_0$.

10.3: A random sample of 100 recorded deaths in the United States during the past year showed an average life span of 71.8 years. Assuming a population standard deviation of 8.9 years, does this seem to indicate that the mean life span today is greater than 70 years? Use a 0.05 level of significance.

Solution:

1. H_0: $\mu = 70$ years.
2. H_1: $\mu > 70$ years.
3. $\alpha = 0.05$.
4. Critical region: $z > 1.645$, where $z = \frac{\bar{x}-\mu_0}{\sigma/\sqrt{n}}$.
5. Computations: $\bar{x} = 71.8$ years, $\sigma = 8.9$ years, and hence $z = \frac{71.8-70}{8.9/\sqrt{100}} = 2.02$.
6. Decision: Reject H_0 and conclude that the mean life span today is greater than 70 years.

$$P = P(Z > 2.02) = 0.0217.$$

10.4: A manufacturer of sports equipment has developed a new synthetic fishing line that the company claims has a mean breaking strength of 8 kilograms with a standard deviation of 0.5 kilogram. Test the hypothesis that $\mu = 8$ kilograms against the alternative that $\mu \neq 8$ kilograms if a random sample of 50 lines is tested and found to have a mean breaking strength of 7.8 kilograms. Use a 0.01 level of significance.

Solution:

1. H_0: $\mu = 8$ kilograms.
2. H_1: $\mu \neq 8$ kilograms.
3. $\alpha = 0.01$.
4. Critical region: $z < -2.575$ and $z > 2.575$, where $z = \frac{\bar{x}-\mu_0}{\sigma/\sqrt{n}}$.
5. Computations: $\bar{x} = 7.8$ kilograms, $n = 50$, and hence $z = \frac{7.8-8}{0.5/\sqrt{50}} = -2.83$.
6. Decision: Reject H_0 and conclude that the average breaking strength is not equal to 8 but is, in fact, less than 8 kilograms.

$$P = P(|Z| > 2.83) = 2P(Z < -2.83) = 0.0046,$$

The t-Statistic for a Test on a Single Mean (Variance Unknown)

Hypothesis Tests for One Population Mean When σ Is Unknown

For the two-sided hypothesis

$$H_0\colon\ \mu = \mu_0,$$
$$H_1\colon\ \mu \neq \mu_0,$$

we reject H_0 at significance level α when the computed t-statistic

$$t = \frac{\bar{x} - \mu_0}{s/\sqrt{n}}$$

Example:

The Table shows the pH levels obtained by the researchers for 15 lakes. At the 5% significance level, do the data provide sufficient evidence to conclude that, on average, high mountain lakes in the Southern Alps are nonacidic? A lake is classified as nonacidic if it has a pH greater than 6.

pH levels for 15 lakes

7.2	7.3	6.1	6.9	6.6
7.3	6.3	5.5	6.3	6.5
5.7	6.9	6.7	7.9	5.8

Solution:

Step 1 State the null and alternative hypotheses
$H0$: $\mu = 6$ (on average, the lakes are acidic)
Ha: $\mu > 6$ (on average, the lakes are nonacidic).

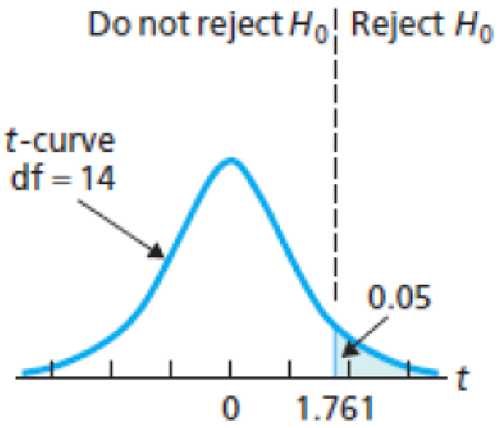
Step 2 Decide on the significance level, α.
We are to perform the test at the 5% significance level, so $\alpha = 0.05$.

Step 3 Compute the value of the test statistic
We have $\mu_0 = 6$ and $n = 15$ and calculate the mean and standard deviation of the sample data in given Table as 6.6 and 0.672, respectively. Hence the value of the test statistic is

$$t = \frac{6.6 - 6}{0.672/\sqrt{15}} = 3.458.$$

Step 4 The critical value for a right-tailed test is t_α with df = $n - 1$. Use T Table to find the critical value.

We have $n = 15$ and $\alpha = 0.05$. T Table shows that for df = 15 − 1 = 14, $t_{0.05} = 1.761$.



Step 5 If the value of the test statistic falls in the rejection region, reject H_0; otherwise, do not reject H_0.

The value of the test statistic, found in Step 3, is $t = 3.458$. It falls in the rejection region. Consequently, we reject $H0$. The test results are statistically significant at the 5% level.

Step 6 Interpret the results of the hypothesis test.

Interpretation At the 5% significance level, the data provide sufficient evidence to conclude that, on average, high mountain lakes in the Southern Alps are nonacidic.

10.5: The Edison Electric Institute has published figures on the number of kilowatt hours used annually by various home appliances. It is claimed that a vacuum cleaner uses an average of 46 kilowatt hours per year. If a random sample of 12 homes included in a planned study indicates that vacuum cleaners use an average of 42 kilowatt hours per year with a standard deviation of 11.9 kilowatt hours, does this suggest at the 0.05 level of significance that vacuum cleaners use, on average, less than 46 kilowatt hours annually? Assume the population of kilowatt hours to be normal.

Solution:

1. H_0: $\mu = 46$ kilowatt hours.
2. H_1: $\mu < 46$ kilowatt hours.
3. $\alpha = 0.05$.
4. Critical region: $t < -1.796$, where $t = \frac{\bar{x}-\mu_0}{s/\sqrt{n}}$ with 11 degrees of freedom.
5. Computations: $\bar{x} = 42$ kilowatt hours, $s = 11.9$ kilowatt hours, and $n = 12$. Hence,

$$t = \frac{42 - 46}{11.9/\sqrt{12}} = -1.16, \qquad P = P(T < -1.16) \approx 0.135.$$

6. Decision: Do not reject H_0

10.21 An electrical firm manufactures light bulbs that have a lifetime that is approximately normally distributed with a mean of 800 hours and a standard deviation of 40 hours. Test the hypothesis that $\mu = 800$ hours against the alternative, $\mu \neq 800$ hours, if a random sample of 30 bulbs has an average life of 788 hours. Use a P-value in your answer.

Solution:

The hypotheses are

$$H_0 : \mu = 40 \text{ months},$$
$$H_1 : \mu < 40 \text{ months}.$$

Now, $z = \frac{38-40}{5.8/\sqrt{64}} = -2.76$, and P-value$= P(Z < -2.76) = 0.0029$. Decision: reject H_0.

10.24 The average height of females in the freshman class of a certain college has historically been 162.5 centimeters with a standard deviation of 6.9 centimeters. Is there reason to believe that there has been a change in the average height if a random sample of 50 females in the present freshman class has an average height of 165.2 centimeters? Use a P-value in your conclusion. Assume the standard deviation remains the same.

Solution:

The hypotheses are

$$H_0 : \mu = 8,$$
$$H_1 : \mu > 8.$$

Now, $z = \frac{8.5-8}{2.25/\sqrt{225}} = 3.33$, and P-value$= P(Z > 3.33) = 0.0004$. Decision: Reject H_0 and conclude that men who use TM, on average, mediate more than 8 hours per week.

10.25 It is claimed that automobiles are driven on average more than 20,000 kilometers per year. To test this claim, 100 randomly selected automobile owners are asked to keep a record of the kilometers they travel. Would you agree with this claim if the random sample showed an average of 23,500 kilometers and a standard deviation of 3900 kilometers? Use a P-value in your conclusion.

Solution:

The hypotheses are

$$H_0 : \mu = 10,$$
$$H_1 : \mu \neq 10.$$

$\alpha = 0.01$ and $df = 9$.

Critical region: $t < -3.25$ or $t > 3.25$.

Computation: $t = \frac{10.06-10}{0.246/\sqrt{10}} = 0.77$.

Decision: Fail to reject H_0.

Two Samples: Tests on Two Means

Two-Sample Pooled t-Test

For the two-sided hypothesis

$$H_0\colon\ \mu_1 = \mu_2,$$
$$H_1\colon\ \mu_1 \neq \mu_2,$$

we reject H_0 at significance level α when the computed t-statistic

$$t = \frac{(\bar{x}_1 - \bar{x}_2) - d_0}{s_p\sqrt{1/n_1 + 1/n_2}},$$

where

$$s_p^2 = \frac{s_1^2(n_1 - 1) + s_2^2(n_2 - 1)}{n_1 + n_2 - 2}$$

exceeds $t_{\alpha/2, n_1+n_2-2}$ or is less than $-t_{\alpha/2, n_1+n_2-2}$.

Two Samples: Tests on Two Means

Two independent random samples of sizes n_1 and n_2, respectively, are drawn from two populations with means μ_1 and μ_2 and variances σ_1^2 and σ_2^2. We know that the random variable

$$Z = \frac{(\bar{X}_1 - \bar{X}_2) - (\mu_1 - \mu_2)}{\sqrt{\sigma_1^2/n_1 + \sigma_2^2/n_2}}$$

if we can assume that $\sigma_1 = \sigma_2 = \sigma$, $$Z = \frac{(\bar{X}_1 - \bar{X}_2) - (\mu_1 - \mu_2)}{\sigma\sqrt{1/n_1 + 1/n_2}}.$$

H_0: $\mu_1 - \mu_2 = d_0$. $$z = \frac{(\bar{x}_1 - \bar{x}_2) - d_0}{\sqrt{\sigma_1^2/n_1 + \sigma_2^2/n_2}},$$

with a two-tailed critical region in the case of a two-sided alternative. That is, reject H_0 in favor of H_1: $\mu_1 - \mu_2 \neq d_0$ if $z > z_{\alpha/2}$ or $z < -z_{\alpha/2}$. One-tailed critical regions are used in the case of the one-sided alternatives. The reader should, as before, study the test statistic and be satisfied that for, say, H_1: $\mu_1 - \mu_2 > d_0$, the signal favoring H_1 comes from large values of z. Thus, the upper-tailed critical region applies.

Hypothesis Tests for the Means of Two Populations

Pooled T-Test (Assume Equal Standard Deviations: σ1 = σ2)

Independent Samples Test

Assumptions

1. Simple random samples
2. Independent samples
3. Normal populations or large samples
4. Equal population standard deviations

Step 1 The null hypothesis is H_0: $\mu_1 = \mu_2$, and the alternative hypothesis is

Ha: $\mu_1 \neq \mu_2$	or	Ha: $\mu_1 < \mu_2$	or	Ha: $\mu_1 > \mu_2$
(Two tailed)		(Left tailed)		(Right tailed)

Step 2 Decide on the significance level, α.

Step 3 Compute the value of the test statistic

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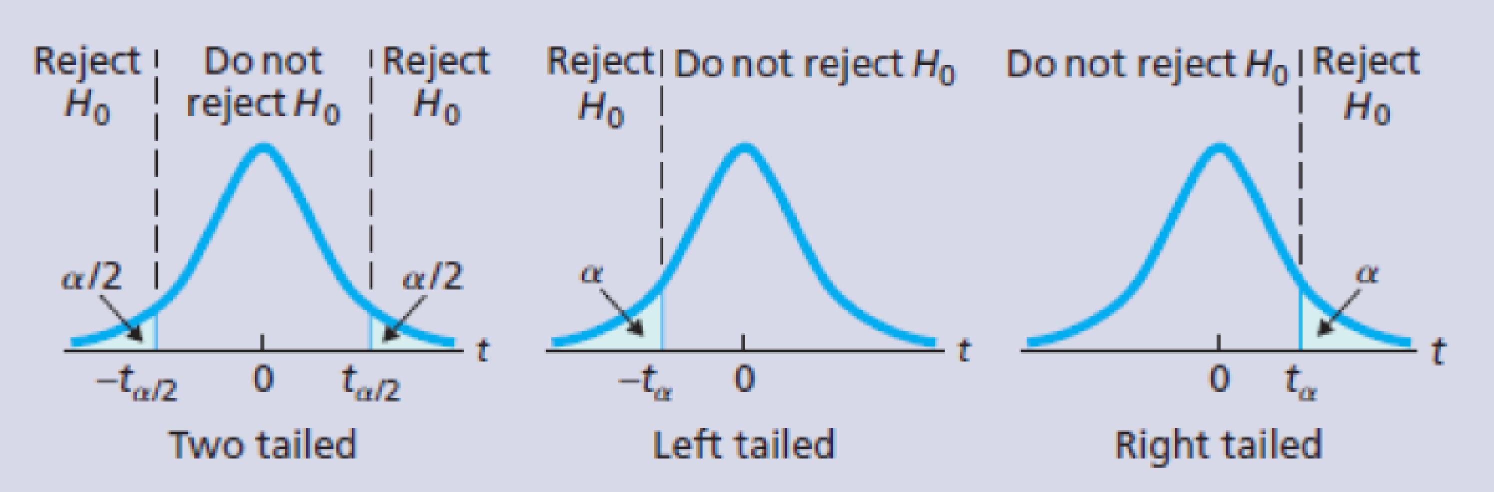
$$t = \frac{\bar{x}_1 - \bar{x}_2}{s_p\sqrt{(1/n_1) + (1/n_2)}}$$

Where $$s_p = \sqrt{\frac{(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2}{n_1 + n_2 - 2}}.$$

Step 4 The critical value(s) are

$\pm t_{\alpha/2}$	(Two tailed)	or
$-t_{\alpha}$	(Left tailed)	or
$+t_{\alpha}$	(Right tailed)	

with df = $n_1 + n_2 - 2$. Use T Table to find the critical value(s).



Step 5 If the value of the test statistic falls in the rejection region, reject H_0; otherwise, do not reject H_0.

Step 6 Interpret the results of the hypothesis test.

Example:

Annual salaries ($1000s) for 35 faculty members in private institutions and 30 faculty members in public institutions. At the 5% significance level, do the data provide sufficient evidence to conclude that mean salaries for faculty in private and public institutions differ?

Summary statistics for the samples	
Private institutions	**Public institutions**
$\overline{x_1} = 88.19$	$\overline{x_2} = 73.18$
$s_1 = 26.21$	$s_2 = 23.95$
$n_1 = 35$	$n_2 = 30$

Step 1 State the null and alternative hypotheses.
The null and alternative hypotheses are, respectively,
H_0: $\mu_1 = \mu_2$ (mean salaries are the same)
H_a: $\mu_1 \;\; = \mu_2$ (mean salaries are different),
where μ_1 and μ_2 are the mean salaries of all faculty in private and public institutions, respectively.
Note that the hypothesis test is two tailed.

Step 2 Decide on the significance level, α.
The test is to be performed at the 5% significance level, or $\alpha = 0.05$.

Step 3 Compute the value of the test statistic

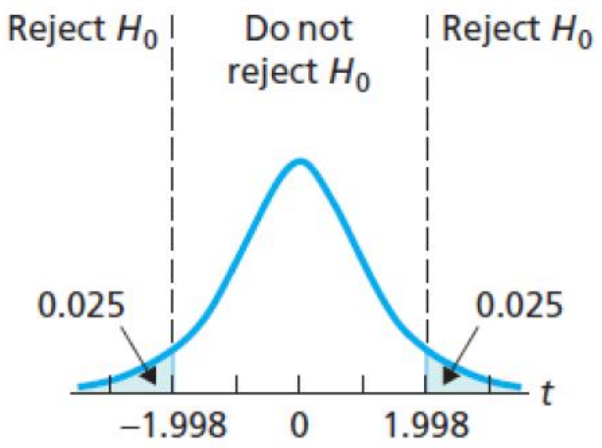
$$s_p = \sqrt{\frac{(35-1)\cdot(26.21)^2 + (30-1)\cdot(23.95)^2}{35+30-2}} = 25.19.$$

$$t = \frac{\bar{x}_1 - \bar{x}_2}{s_p\sqrt{(1/n_1)+(1/n_2)}} = \frac{88.19 - 73.18}{25.19\sqrt{(1/35)+(1/30)}} = 2.395.$$

Step 4 The critical values for a two-tailed test are $\pm t_{\alpha/2}$ with df = $n_1 + n_2 - 2$. Use T Table to find the critical values.

Since $n_1 = 35$ and $n_2 = 30$, so df = 35 + 30 − 2 = 63.

Also, from Step 2, we have $\alpha = 0.05$. Using T Table with df = 63, we find that the critical values are $\pm t_{\alpha/2} = \pm t_{0.05/2} = \pm t_{0.025} = \pm 1.998$, as shown in the Figure.



Step 5 If the value of the test statistic falls in the rejection region, reject H_0; otherwise, do not reject H_0.

From Step 3, the value of the test statistic is $t = 2.395$, which falls in the rejection region. Thus we reject H_0. The test results are statistically significant at the 5% level.

Step 6 Interpretation: At the 5% significance level, the data provide sufficient evidence to conclude that a difference exists between the mean salaries of faculty in private and public institutions.

Paired T Test: (Dependent Sample Test)

Inferences for Two Population Means, Using Paired Samples

A paired sample may be appropriate when the members of the two populations have a natural pairing.
Each pair in a **paired sample** consists of a member of one population and that member's corresponding member in the other population.

Paired t-Test

Purpose To perform a hypothesis test to compare two population means, μ_1 and μ_2

Assumptions
1. Simple random paired sample
2. Normal differences or large sample

Step 1 **The null hypothesis is H_0: $\mu_1 = \mu_2$, and the alternative hypothesis is**

H_a: $\mu_1 \neq \mu_2$ **(Two tailed)** **or** H_a: $\mu_1 < \mu_2$ **(Left tailed)** **or** H_a: $\mu_1 > \mu_2$ **(Right tailed)**

Step 2 **Decide on the significance level, α.**

Step 3 **Compute the value of the test statistic**

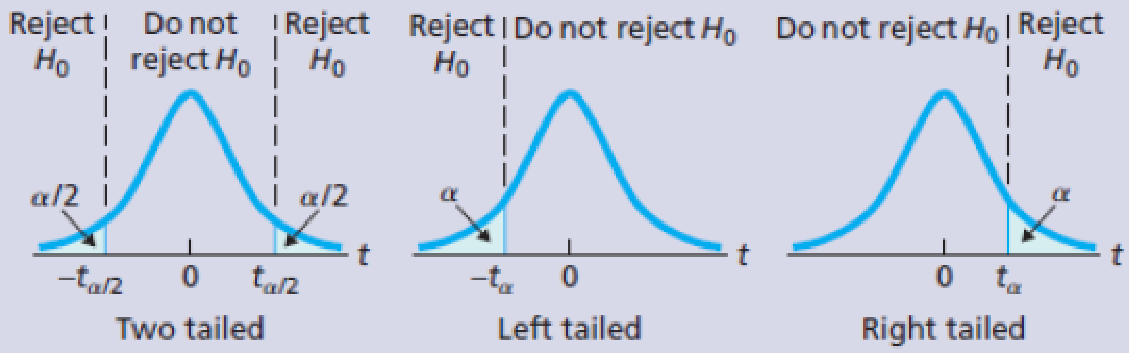
$$t = \frac{\bar{d}}{s_d/\sqrt{n}}$$

and denote that value t_0.

Step 4 The critical value(s) are

$\pm t_{\alpha/2}$ (Two tailed)	or	$-t_{\alpha}$ (Left tailed)	or	t_{α} (Right tailed)

with df $= n - 1$. Use Table IV to find the critical value(s).



Step 5 If the value of the test statistic falls in the rejection region, reject H_0; otherwise, do not reject H_0.

Step 6 Interpret the results of the hypothesis test.

Example

Ages of Married People The U.S. Census Bureau publishes information on the ages of married people in *Current Population Reports*. Suppose that we want to decide whether, in the United States, the mean age of married men differs from the mean age of married women.

Couple	Husband	Wife	Difference, d
1	59	53	6
2	21	22	−1
3	33	36	−3
4	78	74	4
5	70	64	6
6	33	35	−2
7	68	67	1
8	32	28	4
9	54	41	13
10	52	44	8
			36

We want to perform the hypothesis test

H_0: $\mu_1 = \mu_2$ (mean ages of married men and women are the same)

H_a: $\mu_1 \ = \mu_2$ (mean ages of married men and women differ).

Step 1 State the null and alternative hypotheses.

Let μ_1 denote the mean age of all married men, and let μ_2 denote the mean age of all married women. Then the null and alternative hypotheses are, respectively,

$H0$: $\mu_1 = \mu_2$ (mean ages are equal)

Ha: $\mu_1 \ = \mu_2$ (mean ages differ).

Note that the hypothesis test is two tailed.

Step 2 Decide on the significance level, α.

We are to perform the test at the 5% significance level, so $\alpha = 0.05$.

Step 3 Compute the value of the test statistic

The paired differences (d-values) of the sample pairs are shown in the last column of above Table. We need to determine the sample mean and sample standard deviation of those paired differences.

$$\bar{d} = \frac{\Sigma d_i}{n} = \frac{36}{10} = 3.6,$$

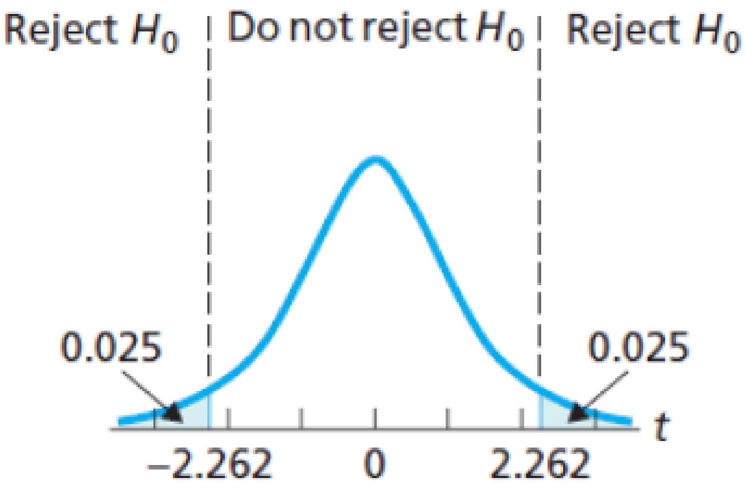
and

$$s_d = \sqrt{\frac{\Sigma d_i^2 - (\Sigma d_i)^2/n}{n-1}} = \sqrt{\frac{352 - (36)^2/10}{10-1}} = 4.97.$$

Consequently, the value of the test statistic is

$$t = \frac{\bar{d}}{s_d/\sqrt{n}} = \frac{3.6}{4.97/\sqrt{10}} = 2.291.$$

Step 4 The critical values for a two-tailed test are $\pm t_{\alpha/2}$ with df = n − 1. Use T Table to find the critical values.
We have $n = 10$ and $\alpha = 0.05$. for df = 10 − 1 = 9, $\pm t_{0.05/2} = \pm t_{0.025} = \pm 2.262$, as shown in Fig.



Step 5 If the value of the test statistic falls in the rejection region, reject $H0$; otherwise, do not reject $H0$.
From Step 3, the value of the test statistic is $t = 2.291$, which falls in the rejection region depicted in Fig. Thus we reject $H0$. The test results are statistically significant at the 5% level.

Step 6 Interpret the results of the hypothesis test.
Interpretation At the 5% significance level, the data provide sufficient evidence to conclude that the mean age of married men differs from the mean age of married women.

Table 10.3: Tests Concerning Means

H_0	Value of Test Statistic	H_1	Critical Region
$\mu = \mu_0$	$z = \dfrac{\bar{x} - \mu_0}{\sigma/\sqrt{n}}$; σ known	$\mu < \mu_0$ $\mu > \mu_0$ $\mu \neq \mu_0$	$z < -z_\alpha$ $z > z_\alpha$ $z < -z_{\alpha/2}$ or $z > z_{\alpha/2}$
$\mu = \mu_0$	$t = \dfrac{\bar{x} - \mu_0}{s/\sqrt{n}}$; $v = n - 1$, σ unknown	$\mu < \mu_0$ $\mu > \mu_0$ $\mu \neq \mu_0$	$t < -t_\alpha$ $t > t_\alpha$ $t < -t_{\alpha/2}$ or $t > t_{\alpha/2}$
$\mu_1 - \mu_2 = d_0$	$z = \dfrac{(\bar{x}_1 - \bar{x}_2) - d_0}{\sqrt{\sigma_1^2/n_1 + \sigma_2^2/n_2}}$; σ_1 and σ_2 known	$\mu_1 - \mu_2 < d_0$ $\mu_1 - \mu_2 > d_0$ $\mu_1 - \mu_2 \neq d_0$	$z < -z_\alpha$ $z > z_\alpha$ $z < -z_{\alpha/2}$ or $z > z_{\alpha/2}$
$\mu_1 - \mu_2 = d_0$	$t = \dfrac{(\bar{x}_1 - \bar{x}_2) - d_0}{s_p\sqrt{1/n_1 + 1/n_2}}$; $v = n_1 + n_2 - 2$, $\sigma_1 = \sigma_2$ but unknown, $s_p^2 = \dfrac{(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2}{n_1 + n_2 - 2}$	$\mu_1 - \mu_2 < d_0$ $\mu_1 - \mu_2 > d_0$ $\mu_1 - \mu_2 \neq d_0$	$t < -t_\alpha$ $t > t_\alpha$ $t < -t_{\alpha/2}$ or $t > t_{\alpha/2}$

Table 10.3: Tests Concerning Means

$\mu_1 - \mu_2 = d_0$	$t' = \dfrac{(\bar{x}_1 - \bar{x}_2) - d_0}{\sqrt{s_1^2/n_1 + s_2^2/n_2}}$; $v = \dfrac{(s_1^2/n_1 + s_2^2/n_2)^2}{\frac{(s_1^2/n_1)^2}{n_1-1} + \frac{(s_2^2/n_2)^2}{n_2-1}}$, $\sigma_1 \neq \sigma_2$ and unknown	$\mu_1 - \mu_2 < d_0$ $\mu_1 - \mu_2 > d_0$ $\mu_1 - \mu_2 \neq d_0$	$t' < -t_\alpha$ $t' > t_\alpha$ $t' < -t_{\alpha/2}$ or $t' > t_{\alpha/2}$
$\mu_D = d_0$ paired observations	$t = \dfrac{\bar{d} - d_0}{s_d/\sqrt{n}}$; $v = n - 1$	$\mu_D < d_0$ $\mu_D > d_0$ $\mu_D \neq d_0$	$t < -t_\alpha$ $t > t_\alpha$ $t < -t_{\alpha/2}$ or $t > t_{\alpha/2}$