CHAPTER

12

Orthogonal Functions and Fourier Series

Our goal in Part 4 of this text is solve certain kinds of linear **partial differential equations** in an applied context. Although we do not solve any PDEs in this chapter, the material covered sets the stage for the procedures discussed later.

In calculus you saw that a sufficiently differentiable function f could often be expanded in a Taylor series, which essentially is an infinite series consisting of powers of x. The principal concept examined in this chapter also involves expanding a function in an infinite series. In the early 1800s, the French mathematician **Joseph Fourier** (1768–1830) advanced the idea of expanding a function f in a series of trigonometric functions. It turns out that **Fourier series** are just special cases of a more general type of series representation of a function using an infinite set of **orthogonal functions**. The notion of orthogonal functions leads us back to eigenvalues and the corresponding set of eigenfunctions. Since eigenvalues and eigenfunctions are the linchpins of the procedures in the two chapters that follow, you are encouraged to review Example 2 in Section 3.9.

CHAPTER CONTENTS

12.1 Orthogonal Functions

INTRODUCTION In certain areas of advanced mathematics, a function is considered to be a generalization of a vector. In this section we shall see how the two vector concepts of inner, or dot, product and orthogonality of vectors can be extended to functions. The remainder of the chapter is a practical application of this discussion.

Inner Product

Recall, if $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ are two vectors in R^3 or 3-space, then the inner product or dot product of $\mathbf{u}$ and $\mathbf{v}$ is a real number, called a **scalar**, defined as the sum of the products of their corresponding components:

In Chapter 7, the inner product was denoted by $\mathbf{u} \cdot \mathbf{v}$.

$$(\mathbf{u}, \mathbf{v}) = u_1v_1 + u_2v_2 + u_3v_3 = \sum_{k=1}^{3} u_kv_k.$$

The inner product $(\mathbf{u}, \mathbf{v})$ possesses the following properties:

(*i*) $(\mathbf{u}, \mathbf{v}) = (\mathbf{v}, \mathbf{u})$
(*ii*) $(k\mathbf{u}, \mathbf{v}) = k(\mathbf{u}, \mathbf{v})$, k a scalar
(*iii*) $(\mathbf{u}, \mathbf{u}) = 0$ if $\mathbf{u} = \mathbf{0}$ and $(\mathbf{u}, \mathbf{u}) > 0$ if $\mathbf{u} \neq \mathbf{0}$
(*iv*) $(\mathbf{u} + \mathbf{v}, \mathbf{w}) = (\mathbf{u}, \mathbf{w}) + (\mathbf{v}, \mathbf{w})$.

We expect any generalization of the inner product to possess these same properties.

Suppose that f_1 and f_2 are piecewise-continuous functions defined on an interval $[a, b]$.* Since a definite integral on the interval of the product $f_1(x)f_2(x)$ possesses properties (*i*)–(*iv*) of the inner product of vectors, whenever the integral exists we are prompted to make the following definition.

Definition 12.1.1 Inner Product of Functions

The **inner product** of two functions f_1 and f_2 on an interval $[a, b]$ is the number

$$(f_1, f_2) = \int_a^b f_1(x)f_2(x)\,dx.$$

Orthogonal Functions

Motivated by the fact that two vectors $\mathbf{u}$ and $\mathbf{v}$ are orthogonal whenever their inner product is zero, we define **orthogonal functions** in a similar manner.

Definition 12.1.2 Orthogonal Functions

Two functions f_1 and f_2 are said to be **orthogonal** on an interval $[a, b]$ if

$$(f_1, f_2) = \int_a^b f_1(x)f_2(x)\,dx = 0. \tag{1}$$

EXAMPLE 1 Orthogonal Functions

The functions $f_1(x) = x^2$ and $f_2(x) = x^3$ are orthogonal on the interval $[-1, 1]$. This fact follows from (1):

$$(f_1, f_2) = \int_{-1}^{1} x^2 \cdot x^3\,dx = \int_{-1}^{1} x^5\,dx = \frac{1}{6}x^6\Big]_{-1}^{1} = 0. \quad \equiv$$

Unlike vector analysis, where the word *orthogonal* is a synonym for *perpendicular*, in this present context the term *orthogonal* and condition (1) have no geometric significance.

Orthogonal Sets

We are primarily interested in infinite sets of orthogonal functions.

*The interval could also be $(-\infty, \infty)$, $[0, \infty)$, and so on.

Definition 12.1.3 Orthogonal Set

A set of real-valued functions $\{\phi_0(x), \phi_1(x), \phi_2(x), \ldots\}$ is said to be **orthogonal** on an interval $[a, b]$ if

$$(\phi_m, \phi_n) = \int_a^b \phi_m(x)\phi_n(x)\,dx = 0, \quad m \neq n. \tag{2}$$

Orthonormal Sets The norm, or length $\|\mathbf{u}\|$, of a vector $\mathbf{u}$ can be expressed in terms of the inner product. The expression $(\mathbf{u}, \mathbf{u}) = \|\mathbf{u}\|^2$ is called the square norm, and so the norm is $\|\mathbf{u}\| = \sqrt{(\mathbf{u}, \mathbf{u})}$. Similarly, the **square norm** of a function ϕ_n is $\|\phi_n(x)\|^2 = (\phi_n, \phi_n)$, and so the **norm**, or its generalized length, is $\|\phi_n(x)\| = \sqrt{(\phi_n, \phi_n)}$. In other words, the square norm and norm of a function ϕ_n in an orthogonal set $\{\phi_n(x)\}$ are, respectively,

$$\|\phi_n(x)\|^2 = \int_a^b \phi_n^2(x)\,dx \qquad \text{and} \qquad \|\phi_n(x)\| = \sqrt{\int_a^b \phi_n^2(x)\,dx}. \tag{3}$$

If $\{\phi_n(x)\}$ is an orthogonal set of functions on the interval $[a, b]$ with the property that $\|\phi_n(x)\| = 1$ for $n = 0, 1, 2, \ldots$, then $\{\phi_n(x)\}$ is said to be an **orthonormal set** on the interval.

EXAMPLE 2 Orthogonal Set of Functions

Show that the set $\{1, \cos x, \cos 2x, \ldots\}$ is orthogonal on the interval $[-\pi, \pi]$.

SOLUTION If we make the identification $\phi_0(x) = 1$ and $\phi_n(x) = \cos nx$, we must then show that $\int_{-\pi}^{\pi} \phi_0(x)\phi_n(x)\,dx = 0$, $n \neq 0$, and $\int_{-\pi}^{\pi} \phi_m(x)\phi_n(x)\,dx = 0$, $m \neq n$. We have, in the first case, for $n \neq 0$,

$$(\phi_0, \phi_n) = \int_{-\pi}^{\pi} \phi_0(x)\phi_n(x)\,dx = \int_{-\pi}^{\pi} \cos nx\,dx$$

$$= \frac{1}{n}\sin nx\Big]_{-\pi}^{\pi} = \frac{1}{n}[\sin n\pi - \sin(-n\pi)] = 0,$$

and in the second, for $m \neq n$,

$$(\phi_m, \phi_n) = \int_{-\pi}^{\pi} \phi_m(x)\phi_n(x)\,dx = \int_{-\pi}^{\pi} \cos mx \cos nx\,dx$$

$$= \frac{1}{2}\int_{-\pi}^{\pi} [\cos(m+n)x + \cos(m-n)x]\,dx \quad \leftarrow \text{trigonometric identity}$$

$$= \frac{1}{2}\left[\frac{\sin(m+n)x}{m+n} + \frac{\sin(m-n)x}{m-n}\right]_{-\pi}^{\pi} = 0.$$

≡

EXAMPLE 3 Norms

Find the norms of each function in the orthogonal set given in Example 2.

SOLUTION For $\phi_0(x) = 1$ we have from (3)

$$\|\phi_0(x)\|^2 = \int_{-\pi}^{\pi} dx = 2\pi$$

so that $\|\phi_0(x)\| = \sqrt{2\pi}$. For $\phi_n(x) = \cos nx$, $n > 0$, it follows that

$$\|\phi_n(x)\|^2 = \int_{-\pi}^{\pi} \cos^2 nx\,dx = \frac{1}{2}\int_{-\pi}^{\pi} [1 + \cos 2nx]\,dx = \pi.$$

Thus for $n > 0$, $\|\phi_n(x)\| = \sqrt{\pi}$.

≡

An orthogonal set can be made into an orthonormal set.

Any orthogonal set of nonzero functions $\{\phi_n(x)\}$, $n = 0, 1, 2, \ldots$, can be *normalized*—that is, made into an orthonormal set—by dividing each function by its norm. It follows from Examples 2 and 3 that the set

$$\left\{\frac{1}{\sqrt{2\pi}}, \frac{\cos x}{\sqrt{\pi}}, \frac{\cos 2x}{\sqrt{\pi}}, \ldots\right\}$$

is orthonormal on the interval $[-\pi, \pi]$.

Vector Analogy We shall make one more analogy between vectors and functions. Suppose $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are three mutually orthogonal nonzero vectors in 3-space. Such an orthogonal set can be used as a basis for 3-space; that is, any three-dimensional vector can be written as a linear combination

$$\mathbf{u} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3, \tag{4}$$

where the c_i, $i = 1, 2, 3$, are scalars called the components of the vector. Each component c_i can be expressed in terms of $\mathbf{u}$ and the corresponding vector $\mathbf{v}_i$. To see this we take the inner product of (4) with $\mathbf{v}_1$:

$$(\mathbf{u}, \mathbf{v}_1) = c_1(\mathbf{v}_1, \mathbf{v}_1) + c_2(\mathbf{v}_2, \mathbf{v}_1) + c_3(\mathbf{v}_3, \mathbf{v}_1) = c_1\|\mathbf{v}_1\|^2 + c_2 \cdot 0 + c_3 \cdot 0.$$

Hence
$$c_1 = \frac{(\mathbf{u}, \mathbf{v}_1)}{\|\mathbf{v}_1\|^2}.$$

In like manner we find that the components c_2 and c_3 are given by

$$c_2 = \frac{(\mathbf{u}, \mathbf{v}_2)}{\|\mathbf{v}_2\|^2} \quad \text{and} \quad c_3 = \frac{(\mathbf{u}, \mathbf{v}_3)}{\|\mathbf{v}_3\|^2}.$$

Hence (4) can be expressed as

$$\mathbf{u} = \frac{(\mathbf{u}, \mathbf{v}_1)}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 + \frac{(\mathbf{u}, \mathbf{v}_2)}{\|\mathbf{v}_2\|^2}\mathbf{v}_2 + \frac{(\mathbf{u}, \mathbf{v}_3)}{\|\mathbf{v}_3\|^2}\mathbf{v}_3 = \sum_{n=1}^{3}\frac{(\mathbf{u}, \mathbf{v}_n)}{\|\mathbf{v}_n\|^2}\mathbf{v}_n. \tag{5}$$

Orthogonal Series Expansion Suppose $\{\phi_n(x)\}$ is an infinite orthogonal set of functions on an interval $[a, b]$. We ask: If $y = f(x)$ is a function defined on the interval $[a, b]$, is it possible to determine a set of coefficients c_n, $n = 0, 1, 2, \ldots$, for which

$$f(x) = c_0\phi_0(x) + c_1\phi_1(x) + \cdots + c_n\phi_n(x) + \cdots? \tag{6}$$

As in the foregoing discussion on finding components of a vector, we can find the coefficients c_n by utilizing the inner product. Multiplying (6) by $\phi_m(x)$ and integrating over the interval $[a, b]$ gives

$$\begin{aligned}\int_a^b f(x)\phi_m(x)\,dx &= c_0\int_a^b \phi_0(x)\phi_m(x)\,dx + c_1\int_a^b \phi_1(x)\phi_m(x)\,dx + \cdots + c_n\int_a^b \phi_n(x)\phi_m(x)\,dx + \cdots \\ &= c_0(\phi_0, \phi_m) + c_1(\phi_1, \phi_m) + \cdots + c_n(\phi_n, \phi_m) + \cdots.\end{aligned}$$

By orthogonality, each term on the right-hand side of the last equation is zero *except* when $m = n$. In this case we have

$$\int_a^b f(x)\phi_n(x)\,dx = c_n\int_a^b \phi_n^2(x)\,dx.$$

It follows that the required coefficients c_n are given by

$$c_n = \frac{\int_a^b f(x)\phi_n(x)\,dx}{\int_a^b \phi_n^2(x)\,dx}, \quad n = 0, 1, 2, \ldots.$$

In other words, $$f(x) = \sum_{n=0}^{\infty} c_n \phi_n(x), \tag{7}$$

where $$c_n = \frac{\int_a^b f(x)\phi_n(x)\,dx}{\|\phi_n(x)\|^2}. \tag{8}$$

With inner product notation, (7) becomes

$$f(x) = \sum_{n=0}^{\infty} \frac{(f, \phi_n)}{\|\phi_n(x)\|^2} \phi_n(x). \tag{9}$$

Thus (9) is seen to be the function analogue of the vector result given in (5).

Definition 12.1.4 Orthogonal Set/Weight Function

A set of real-valued functions $\{\phi_0(x), \phi_1(x), \phi_2(x), \ldots\}$ is said to be **orthogonal with respect to a weight function** $w(x)$ on an interval $[a, b]$ if

$$\int_a^b w(x)\phi_m(x)\phi_n(x)\,dx = 0, \quad m \neq n.$$

The usual assumption is that $w(x) > 0$ on the interval of orthogonality $[a, b]$. The set $\{1, \cos x, \cos 2x, \ldots\}$ in Example 2 is orthogonal with respect to the weight function $w(x) = 1$ on the interval $[-\pi, \pi]$.

If $\{\phi_n(x)\}$ is orthogonal with respect to a weight function $w(x)$ on the interval $[a, b]$, then multiplying (6) by $w(x)\phi_n(x)$ and integrating yields

$$c_n = \frac{\int_a^b f(x)\,w(x)\,\phi_n(x)\,dx}{\|\phi_n(x)\|^2}, \tag{10}$$

where $$\|\phi_n(x)\|^2 = \int_a^b w(x)\,\phi_n^2(x)\,dx. \tag{11}$$

The series (7) with coefficients c_n given by either (8) or (10) is said to be an **orthogonal series expansion** of f or a **generalized Fourier series**.

Complete Sets The procedure outlined for determining the coefficients c_n was *formal*; that is, basic questions on whether an orthogonal series expansion such as (7) is actually possible were ignored. Also, to expand f in a series of orthogonal functions, it is certainly necessary that f not be orthogonal to each ϕ_n of the orthogonal set $\{\phi_n(x)\}$. (If f were orthogonal to every ϕ_n, then $c_n = 0$, $n = 0, 1, 2, \ldots$.) To avoid the latter problem we shall assume, for the remainder of the discussion, that an orthogonal set is **complete**. This means that the only continuous function orthogonal to each member of the set is the zero function.

REMARKS

Suppose that $\{f_0(x), f_1(x), f_2(x), \ldots\}$ is an infinite set of real-valued functions that are continuous on an interval $[a, b]$. If this set is *linearly independent* on $[a, b]$ (see page 357 for the definition of an infinite linearly independent set), then it can always be made into an orthogonal set and, as described earlier in this section, can be made into an orthonormal set. See Problem 27 in Exercises 12.1.

12.1 Exercises

Answers to selected odd-numbered problems begin on page ANS-29.

In Problems 1–6, show that the given functions are orthogonal on the indicated interval.

1. $f_1(x) = x, f_2(x) = x^2$; $[-2, 2]$
2. $f_1(x) = x^3, f_2(x) = x^2 + 1$; $[-1, 1]$
3. $f_1(x) = e^x, f_2(x) = xe^{-x} - e^{-x}$; $[0, 2]$
4. $f_1(x) = \cos x,\ f_2(x) = \sin^2 x$; $[0, \pi]$
5. $f_1(x) = x, f_2(x) = \cos 2x$; $[-\pi/2, \pi/2]$
6. $f_1(x) = e^x, f_2(x) = \sin x$; $[\pi/4, 5\pi/4]$

In Problems 7–12, show that the given set of functions is orthogonal on the indicated interval. Find the norm of each function in the set.

7. $\{\sin x,\ \sin 3x,\ \sin 5x, \ldots\}$; $[0, \pi/2]$
8. $\{\cos x,\ \cos 3x,\ \cos 5x, \ldots\}$; $[0, \pi/2]$
9. $\{\sin nx\}$, $n = 1, 2, 3, \ldots$; $[0, \pi]$
10. $\left\{\sin \frac{n\pi}{p}x\right\}$, $n = 1, 2, 3, \ldots$; $[0, p]$
11. $\left\{1, \cos \frac{n\pi}{p}x\right\}$, $n = 1, 2, 3, \ldots$; $[0, p]$
12. $\left\{1, \cos \frac{n\pi}{p}x, \sin \frac{m\pi}{p}x\right\}$, $n = 1, 2, 3, \ldots,$ $m = 1, 2, 3, \ldots$; $[-p, p]$

In Problems 13 and 14, verify by direct integration that the functions are orthogonal with respect to the indicated weight function on the given interval.

13. $H_0(x) = 1, H_1(x) = 2x, H_2(x) = 4x^2 - 2$; $w(x) = e^{-x^2}$, $(-\infty, \infty)$
14. $L_0(x) = 1, L_1(x) = -x + 1, L_2(x) = \frac{1}{2}x^2 - 2x + 1$; $w(x) = e^{-x}$, $[0, \infty)$
15. Let $\{\phi_n(x)\}$ be an orthogonal set of functions on $[a, b]$ such that $\phi_0(x) = 1$. Show that $\int_a^b \phi_n(x)\,dx = 0$ for $n = 1, 2, \ldots$.
16. Let $\{\phi_n(x)\}$ be an orthogonal set of functions on $[a, b]$ such that $\phi_0(x) = 1$ and $\phi_1(x) = x$. Show that $\int_a^b (\alpha x + \beta)\phi_n(x)\,dx = 0$ for $n = 2, 3, \ldots$ and any constants α and β.
17. Let $\{\phi_n(x)\}$ be an orthogonal set of functions on $[a, b]$. Show that $\|\phi_m(x) + \phi_n(x)\|^2 = \|\phi_m(x)\|^2 + \|\phi_n(x)\|^2$, $m \neq n$.
18. From Problem 1 we know that $f_1(x) = x$ and $f_2(x) = x^2$ are orthogonal on $[-2, 2]$. Find constants c_1 and c_2 such that $f_3(x) = x + c_1x^2 + c_2x^3$ is orthogonal to both f_1 and f_2 on the same interval.
19. The set of functions $\{\sin nx\}$, $n = 1, 2, 3, \ldots$, is orthogonal on the interval $[-\pi, \pi]$. Show that the set is not complete.
20. Suppose f_1, f_2, and f_3 are functions continuous on the interval $[a, b]$. Show that $(f_1 + f_2, f_3) = (f_1, f_3) + (f_2, f_3)$.

A real-valued function is said to be **periodic** with period $T \neq 0$ if $f(x + T) = f(x)$ for all x in the domain of f. If T is the smallest positive value for which $f(x + T) = f(x)$ holds, then T is called the **fundamental period** of f. In Problems 21–26, determine the fundamental period T of the given function.

21. $f(x) = \cos 2\pi x$
22. $f(x) = \sin \frac{4}{L}x, L > 0$
23. $f(x) = \sin x + \sin 2x$
24. $f(x) = \sin 2x + \cos 4x$
25. $f(x) = \sin 3x + \cos 2x$
26. $f(x) = \sin^2 \pi x$

Discussion Problems

27. The **Gram–Schmidt process** for constructing an orthogonal set that was discussed in Section 7.7 carries over to a linearly independent set $\{f_0(x), f_1(x), f_2(x), \ldots\}$ of real-valued functions continuous on an interval $[a, b]$. With the inner product $(f_n, \phi_n) = \int_a^b f_n(x)\phi_n(x)\,dx$, define the functions in the set $B' = \{\phi_0(x), \phi_1(x), \phi_2(x), \ldots\}$ to be

$$\phi_0(x) = f_0(x)$$

$$\phi_1(x) = f_1(x) - \frac{(f_1, \phi_0)}{(\phi_0, \phi_0)}\phi_0(x)$$

$$\phi_2(x) = f_2(x) - \frac{(f_2, \phi_0)}{(\phi_0, \phi_0)}\phi_0(x) - \frac{(f_2, \phi_1)}{(\phi_1, \phi_1)}\phi_1(x)$$

$$\vdots \qquad\qquad \vdots$$

and so on.

(a) Write out $\phi_3(x)$ in the set.

(b) By construction, the set $B' = \{\phi_0(x), \phi_1(x), \phi_2(x), \ldots\}$ is orthogonal on $[a, b]$. Demonstrate that $\phi_0(x)$, $\phi_1(x)$, and $\phi_2(x)$ are mutually orthogonal.

28. (a) Consider the set of functions $\{1, x, x^2, x^3, \ldots\}$ defined on the interval $[-1, 1]$. Apply the Gram–Schmidt process given in Problem 27 to this set and find $\phi_0(x), \phi_1(x), \phi_2(x)$, and $\phi_3(x)$ of the orthogonal set B'.

(b) Discuss: Do you recognize the orthogonal set?

29. Verify that the inner product (f_1, f_2) in Definition 12.1.1 satisfies properties $(i)-(iv)$ given on page 672.
30. In R^3, give an example of a set of orthogonal vectors that is not complete. Give a set of orthogonal vectors that is complete.
31. The function

$$f(x) = A_0 + \sum_{n=1}^{\infty}\left(A_n \cos \frac{n\pi}{p}x + B_n \sin \frac{n\pi}{p}x\right),$$

where the coefficients A_n and B_n depend only on n, is periodic. Find the period T of f.

12.2 Fourier Series

INTRODUCTION We have just seen in the preceding section that if $\{\phi_0(x), \phi_1(x), \phi_2(x), \ldots\}$ is a set of real-valued functions that is orthogonal on an interval $[a, b]$ and if f is a function defined on the same interval, then we can formally expand f in an orthogonal series $c_0\phi_0(x) + c_1\phi_1(x) + c_2\phi_2(x) + \cdots$. In this section we shall expand functions in terms of a special orthogonal set of trigonometric functions.

Trigonometric Series

In Problem 12 in Exercises 12.1, you were asked to show that the set of trigonometric functions

$$\left\{1, \cos\frac{\pi}{p}x, \cos\frac{2\pi}{p}x, \cos\frac{3\pi}{p}x, \ldots, \sin\frac{\pi}{p}x, \sin\frac{2\pi}{p}x, \sin\frac{3\pi}{p}x, \ldots\right\} \tag{1}$$

is orthogonal on the interval $[-p, p]$. This set will be of special importance later on in the solution of certain kinds of boundary-value problems involving linear partial differential equations. In those applications we will need to expand a function f defined on $[-p, p]$ in an orthogonal series consisting of the trigonometric functions in (1); that is,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi}{p}x + b_n\sin\frac{n\pi}{p}x\right). \tag{2}$$

The coefficients $a_0, a_1, a_2, \ldots, b_1, b_2, \ldots$, can be determined in exactly the same manner as in the general discussion of orthogonal series expansions on pages 674 and 675. Before proceeding, note that we have chosen to write the coefficient of 1 in the set (1) as $\frac{1}{2}a_0$ rather than a_0; this is for convenience only because the formula of a_n will then reduce to a_0 for $n = 0$.

This is why $\frac{1}{2}a_0$ is used instead of a_0.

Now integrating both sides of (2) from $-p$ to p gives

$$\int_{-p}^{p} f(x)\,dx = \frac{a_0}{2}\int_{-p}^{p} dx + \sum_{n=1}^{\infty}\left(a_n\int_{-p}^{p}\cos\frac{n\pi}{p}x\,dx + b_n\int_{-p}^{p}\sin\frac{n\pi}{p}x\,dx\right). \tag{3}$$

Since $\cos(n\pi x/p)$ and $\sin(n\pi x/p)$, $n \geq 1$, are orthogonal to 1 on the interval, the right side of (3) reduces to a single term:

$$\int_{-p}^{p} f(x)\,dx = \frac{a_0}{2}\int_{-p}^{p} dx = \frac{a_0}{2}x\,\Big]_{-p}^{p} = pa_0.$$

Solving for a_0 yields

$$a_0 = \frac{1}{p}\int_{-p}^{p} f(x)\,dx. \tag{4}$$

Now we multiply (2) by $\cos(m\pi x/p)$ and integrate:

$$\begin{aligned}\int_{-p}^{p} f(x)\cos\frac{m\pi}{p}x\,dx &= \frac{a_0}{2}\int_{-p}^{p}\cos\frac{m\pi}{p}x\,dx \\ &\quad + \sum_{n=1}^{\infty}\left(a_n\int_{-p}^{p}\cos\frac{m\pi}{p}x\cos\frac{n\pi}{p}x\,dx + b_n\int_{-p}^{p}\cos\frac{m\pi}{p}x\sin\frac{n\pi}{p}x\,dx\right).\end{aligned} \tag{5}$$

By orthogonality we have

$$\int_{-p}^{p}\cos\frac{m\pi}{p}x\,dx = 0, \quad m > 0, \qquad \int_{-p}^{p}\cos\frac{m\pi}{p}x\sin\frac{n\pi}{p}x\,dx = 0$$

and $$\int_{-p}^{p} \cos\frac{m\pi}{p}x \cos\frac{n\pi}{p}x\,dx = \begin{cases} 0, & m \neq n \\ p, & m = n. \end{cases}$$

Thus (5) reduces to $$\int_{-p}^{p} f(x)\cos\frac{n\pi}{p}x\,dx = a_n p,$$

and so $$a_n = \frac{1}{p}\int_{-p}^{p} f(x)\cos\frac{n\pi}{p}x\,dx. \tag{6}$$

Finally, if we multiply (2) by $\sin(m\pi x/p)$, integrate, and make use of the results

$$\int_{-p}^{p} \sin\frac{m\pi}{p}x\,dx = 0, \quad m > 0, \qquad \int_{-p}^{p} \sin\frac{m\pi}{p}x \cos\frac{n\pi}{p}x\,dx = 0$$

and $$\int_{-p}^{p} \sin\frac{m\pi}{p}x \sin\frac{n\pi}{p}x\,dx = \begin{cases} 0, & m \neq n \\ p, & m = n, \end{cases}$$

we find that $$b_n = \frac{1}{p}\int_{-p}^{p} f(x)\sin\frac{n\pi}{p}x\,dx. \tag{7}$$

The trigonometric series (2) with coefficients a_0, a_n, and b_n defined by (4), (6), and (7), respectively, is said to be the **Fourier series** of the function f. The coefficients obtained from (4), (6), and (7) are referred to as **Fourier coefficients** of f.

In finding the coefficients a_0, a_n, and b_n, we assumed that f was integrable on the interval and that (2), as well as the series obtained by multiplying (2) by $\cos(m\pi x/p)$, converged in such a manner as to permit term-by-term integration. Until (2) is shown to be convergent for a given function f, the equality sign is not to be taken in a strict or literal sense. Some texts use the symbol $\sim$ in place of $=$. In view of the fact that most functions in applications are of a type that guarantees convergence of the series, we shall use the equality symbol. We summarize the results:

Definition 12.2.1 Fourier Series

The **Fourier series** of a function f defined on the interval $(-p, p)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi}{p}x + b_n\sin\frac{n\pi}{p}x\right), \tag{8}$$

where $$a_0 = \frac{1}{p}\int_{-p}^{p} f(x)\,dx \tag{9}$$

$$a_n = \frac{1}{p}\int_{-p}^{p} f(x)\cos\frac{n\pi}{p}x\,dx \tag{10}$$

$$b_n = \frac{1}{p}\int_{-p}^{p} f(x)\sin\frac{n\pi}{p}x\,dx. \tag{11}$$

EXAMPLE 1 Expansion in a Fourier Series

Expand $$f(x) = \begin{cases} 0, & -\pi < x < 0 \\ \pi - x, & 0 \leq x < \pi \end{cases} \tag{12}$$

in a Fourier series.

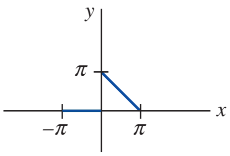


FIGURE 12.2.1 Function f in Example 1

SOLUTION The graph of f is given in **FIGURE 12.2.1**. With $p = \pi$ we have from (9) and (10) that

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} 0\,dx + \int_{0}^{\pi}(\pi - x)\,dx\right] = \frac{1}{\pi}\left[\pi x - \frac{x^2}{2}\right]_0^{\pi} = \frac{\pi}{2}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} 0\,dx + \int_{0}^{\pi}(\pi - x)\cos nx\,dx\right]$$

$$= \frac{1}{\pi}\left[(\pi - x)\frac{\sin nx}{n}\bigg|_0^{\pi} + \frac{1}{n}\int_0^{\pi}\sin nx\,dx\right]$$

$$= -\frac{1}{n\pi}\,\frac{\cos nx}{n}\bigg]_0^{\pi}$$

$$= \frac{-\cos n\pi + 1}{n^2\pi} \qquad \leftarrow \cos n\pi = (-1)^n$$

$$= \frac{1 - (-1)^n}{n^2\pi}.$$

In like manner we find from (11) that

$$b_n = \frac{1}{\pi}\int_0^{\pi}(\pi - x)\sin nx\,dx = \frac{1}{n}.$$

Note that $1 - (-1)^n = \begin{cases} 0, & n \text{ even} \\ 2, & n \text{ odd.} \end{cases}$

Therefore $$f(x) = \frac{\pi}{4} + \sum_{n=1}^{\infty}\left\{\frac{1 - (-1)^n}{n^2\pi}\cos nx + \frac{1}{n}\sin nx\right\}. \tag{13}$$ ≡

Note that a_n defined by (10) reduces to a_0 given by (9) when we set $n = 0$. But as Example 1 shows, this may not be the case *after* the integral for a_n is evaluated.

Convergence of a Fourier Series

The following theorem gives sufficient conditions for convergence of a Fourier series at a point.

Theorem 12.2.1 Conditions for Convergence

Let f and f' be piecewise continuous on the interval $[-p, p]$; that is, let f and f' be continuous except at a finite number of points in the interval and have only finite discontinuities at these points. Then for all x in the interval $(-p, p)$ the Fourier series of f converges to $f(x)$ at a point of continuity. At a point of discontinuity, the Fourier series converges to the average

$$\frac{f(x+) + f(x-)}{2},$$

where $f(x+)$ and $f(x-)$ denote the limit of f at x from the right and from the left, respectively.*

For a proof of this theorem you are referred to the classic text by Churchill and Brown.†

EXAMPLE 2 Convergence of a Point of Discontinuity

The function (12) in Example 1 satisfies the conditions of Theorem 12.2.1. Thus for every x in the interval $(-\pi, \pi)$, except at $x = 0$, the series (13) will converge to $f(x)$. At $x = 0$ the function is discontinuous, and so the series (13) will converge to

$$\frac{f(0+) + f(0-)}{2} = \frac{\pi + 0}{2} = \frac{\pi}{2}.$$ ≡

*In other words, for x a point in the interval and $h > 0$,

$$f(x+) = \lim_{h\to 0} f(x + h), \quad f(x-) = \lim_{h\to 0} f(x - h).$$

†Ruel V. Churchill and James Ward Brown, *Fourier Series and Boundary Value Problems* (New York: McGraw-Hill, 2000).

Periodic Extension Observe that each of the functions in the basic set (1) has a different fundamental period,* namely, $2p/n$, $n \geq 1$, but since a positive integer multiple of a period is also a period we see that all of the functions have in common the period $2p$ (verify). Hence the right-hand side of (2) is $2p$-periodic; indeed, $2p$ is the fundamental period of the sum. We conclude that a Fourier series not only represents the function on the interval $(-p, p)$ but also gives the **periodic extension** of f outside this interval. We can now apply Theorem 12.2.1 to the periodic extension of f, or we may assume from the outset that the given function is periodic with period $T = 2p$; that is, $f(x + T) = f(x)$. When f is piecewise continuous and the right- and left-hand derivatives exist at $x = -p$ and $x = p$, respectively, then the series (8) converges to the average $[f(p-) + f(-p+)]/2$ at these endpoints and to this value extended periodically to $\pm 3p$, $\pm 5p$, $\pm 7p$, and so on. The Fourier series in (13) converges to the periodic extension of (12) on the entire x-axis. At 0, $\pm 2\pi$, $\pm 4\pi$, …, and at $\pm\pi$, $\pm 3\pi$, $\pm 5\pi$, …, the series converges to the values

We may assume that the given function f is periodic.

$$\frac{f(0+) + f(0-)}{2} = \frac{\pi}{2} \qquad \text{and} \qquad \frac{f(\pi-) + f(-\pi+)}{2} = 0,$$

respectively. The solid dots in FIGURE 12.2.2 represent the value $\pi/2$.

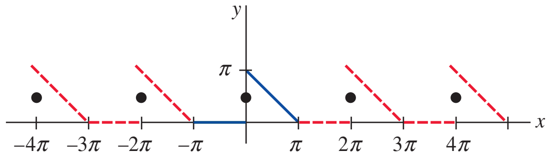
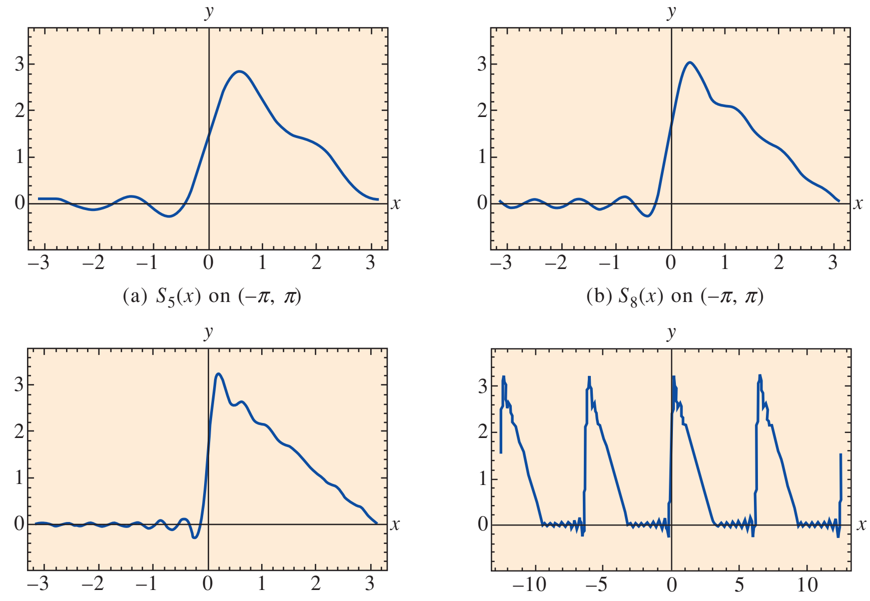


FIGURE 12.2.2 Periodic extension of the function f shown in Figure 12.2.1

Sequence of Partial Sums It is interesting to see how the sequence of partial sums $\{S_N(x)\}$ of a Fourier series approximates a function. For example, the first three partial sums of (13) are

$$S_1(x) = \frac{\pi}{4}, \quad S_2(x) = \frac{\pi}{4} + \frac{2}{\pi}\cos x + \sin x, \quad S_3(x) = \frac{\pi}{4} + \frac{2}{\pi}\cos x + \sin x + \frac{1}{2}\sin 2x.$$

In FIGURE 12.2.3 we have used a CAS to graph the partial sums $S_5(x)$, $S_8(x)$, and $S_{15}(x)$ of (13) on the interval $(-\pi, \pi)$. Figure 12.2.3(d) shows the periodic extension using $S_{15}(x)$ on $(-4\pi, 4\pi)$.



(a) $S_5(x)$ on $(-\pi, \pi)$

(b) $S_8(x)$ on $(-\pi, \pi)$

(c) $S_{15}(x)$ on $(-\pi, \pi)$

(d) $S_{15}(x)$ on $(-4\pi, 4\pi)$

FIGURE 12.2.3 Partial sums of a Fourier series

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* See Problems 21–26 in Exercises 12.1.

12.2 Exercises

Answers to selected odd-numbered problems begin on page ANS-29.

In Problems 1–16, find the Fourier series of the function f on the given interval. Give the number to which the Fourier series converges at a point of discontinuity of f.

1. $f(x) = \begin{cases} 0, & -\pi < x < 0 \\ 1, & 0 \le x < \pi \end{cases}$

2. $f(x) = \begin{cases} -1, & -\pi < x < 0 \\ 2, & 0 \le x < \pi \end{cases}$

3. $f(x) = \begin{cases} 1, & -1 < x < 0 \\ x, & 0 \le x < 1 \end{cases}$

4. $f(x) = \begin{cases} 0, & -1 < x < 0 \\ x, & 0 \le x < 1 \end{cases}$

5. $f(x) = \begin{cases} 0, & -\pi < x < 0 \\ x^2, & 0 \le x < \pi \end{cases}$

6. $f(x) = \begin{cases} \pi^2, & -\pi < x < 0 \\ \pi^2 - x^2, & 0 \le x < \pi \end{cases}$

7. $f(x) = x + \pi, \quad -\pi < x < \pi$

8. $f(x) = 3 - 2x, \quad -\pi < x < \pi$

9. $f(x) = \begin{cases} 0, & -\pi < x < 0 \\ \sin x, & 0 \le x < \pi \end{cases}$

10. $f(x) = \begin{cases} 0, & -\pi/2 < x < 0 \\ \cos x, & 0 \le x < \pi/2 \end{cases}$

11. $f(x) = \begin{cases} 0, & -2 < x < -1 \\ -2, & -1 \le x < 0 \\ 1, & 0 \le x < 1 \\ 0, & 1 \le x < 2 \end{cases}$

12. $f(x) = \begin{cases} 0, & -2 < x < 0 \\ x, & 0 \le x < 1 \\ 1, & 1 \le x < 2 \end{cases}$

13. $f(x) = \begin{cases} 1, & -5 < x < 0 \\ 1 + x, & 0 \le x < 5 \end{cases}$

14. $f(x) = \begin{cases} 2 + x, & -2 < x < 0 \\ 2, & 0 \le x < 2 \end{cases}$

15. $f(x) = e^x, \quad -\pi < x < \pi$

16. $f(x) = \begin{cases} 0, & -\pi < x < 0 \\ e^x - 1, & 0 \le x < \pi \end{cases}$

In Problems 17 and 18, sketch the periodic extension of the indicated function.

17. The function f in Problem 9

18. The function f in Problem 14

19. Use the result of Problem 5 to show

$$\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$$

and

$$\frac{\pi^2}{12} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots.$$

20. Use Problem 19 to find a series that gives the numerical value of $\pi^2/8$.

21. Use the result of Problem 7 to show

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots.$$

22. Use the result of Problem 9 to show

$$\frac{\pi}{4} = \frac{1}{2} + \frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} - \frac{1}{7\cdot 9} + \cdots.$$

23. The **root–mean–square value** of a function $f(x)$ defined over an interval (a, b) is given by

$$\text{RMS}(f) = \sqrt{\frac{\int_a^b f^2(x)\,dx}{b - a}}.$$

If the Fourier series expansion of f is given by (8), show that the RMS value of f over the interval $(-p, p)$ is given by

$$\text{RMS}(f) = \sqrt{\frac{1}{4}a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2)},$$

where a_0, a_n, and b_n are the Fourier coefficients in (9), (10), and (11), respectively.

12.3 Fourier Cosine and Sine Series

INTRODUCTION The effort expended in the evaluation of coefficients a_0, a_n, and b_n in expanding a function f in a Fourier series is reduced significantly when f is either an even or an odd function. A function f is said to be

even if $f(-x) = f(x)$ and **odd** if $f(-x) = -f(x)$.

On a symmetric interval such as $(-p, p)$, the graph of an even function possesses symmetry with respect to the y-axis, whereas the graph of an odd function possesses symmetry with respect to the origin.

Even and Odd Functions

It is likely the origin of the words *even* and *odd* derives from the fact that the graphs of polynomial functions that consist of all even powers of x are symmetric

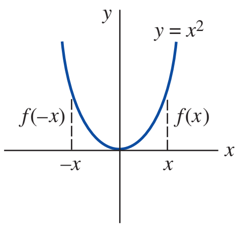


FIGURE 12.3.1 Even function

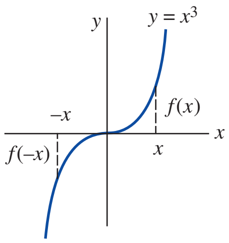


FIGURE 12.3.2 Odd function

with respect to the y-axis, whereas graphs of polynomials that consist of all odd powers of x are symmetric with respect to the origin. For example,

$$f(x) = x^{\overset{\text{even integer}}{\downarrow}2} \text{ is even since } f(-x) = (-x)^2 = x^2 = f(x)$$

$$f(x) = x^{\overset{\text{odd integer}}{\downarrow}3} \text{ is odd since } f(-x) = (-x)^3 = -x^3 = -f(x).$$

See **FIGURES 12.3.1** and **12.3.2**. The trigonometric cosine and sine functions are even and odd functions, respectively, since $\cos(-x) = \cos x$ and $\sin(-x) = -\sin x$. The exponential functions $f(x) = e^x$ and $f(x) = e^{-x}$ are neither even nor odd.

Properties

The following theorem lists some properties of even and odd functions.

Theorem 12.3.1 Properties of Even/Odd Functions

(a) The product of two even functions is even.
(b) The product of two odd functions is even.
(c) The product of an even function and an odd function is odd.
(d) The sum (difference) of two even functions is even.
(e) The sum (difference) of two odd functions is odd.
(f) If f is even, then $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$.
(g) If f is odd, then $\int_{-a}^{a} f(x)\,dx = 0$.

PROOF OF (b): Let us suppose that f and g are odd functions. Then we have $f(-x) = -f(x)$ and $g(-x) = -g(x)$. If we define the product of f and g as $F(x) = f(x)g(x)$, then

$$F(-x) = f(-x)g(-x) = (-f(x))(-g(x)) = f(x)g(x) = F(x).$$

This shows that the product F of two odd functions is an even function. The proofs of the remaining properties are left as exercises. See Problem 56 in Exercises 12.3. ≡

Cosine and Sine Series

If f is an even function on the interval $(-p, p)$, then in view of the foregoing properties, the coefficients (9), (10), and (11) of Section 12.2 become

$$a_0 = \frac{1}{p}\int_{-p}^{p} f(x)\,dx = \frac{2}{p}\int_0^p f(x)\,dx$$

$$a_n = \frac{1}{p}\int_{-p}^{p} \underbrace{f(x)\cos\frac{n\pi}{p}x}_{\text{even}}\,dx = \frac{2}{p}\int_0^p f(x)\cos\frac{n\pi}{p}x\,dx$$

$$b_n = \frac{1}{p}\int_{-p}^{p} \underbrace{f(x)\sin\frac{n\pi}{p}x}_{\text{odd}}\,dx = 0.$$

Similarly, when f is odd on the interval $(-p, p)$,

$$a_n = 0, \quad n = 0, 1, 2, \ldots, \qquad b_n = \frac{2}{p}\int_0^p f(x)\sin\frac{n\pi}{p}x\,dx.$$

We summarize the results in the following definition.

Definition 12.3.1 Fourier Cosine and Sine Series

(*i*) The Fourier series of an even function on the interval $(-p, p)$ is the **cosine series**

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi}{p}x, \tag{1}$$

where
$$a_0 = \frac{2}{p}\int_0^p f(x)\,dx \tag{2}$$

$$a_n = \frac{2}{p}\int_0^p f(x)\cos\frac{n\pi}{p}x\,dx. \tag{3}$$

(*ii*) The Fourier series of an odd function on the interval $(-p, p)$ is the **sine series**

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{p}x, \tag{4}$$

where
$$b_n = \frac{2}{p}\int_0^p f(x)\sin\frac{n\pi}{p}x\,dx. \tag{5}$$

EXAMPLE 1 Expansion in a Sine Series

Expand $f(x) = x,\ -2 < x < 2$, in a Fourier series.

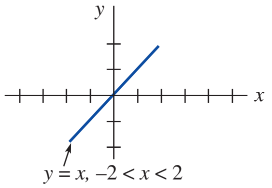


FIGURE 12.3.3 Odd function f in Example 1

SOLUTION Inspection of **FIGURE 12.3.3** shows that the given function is odd on the interval $(-2, 2)$, and so we expand f in a sine series. With the identification $2p = 4$, we have $p = 2$. Thus (5), after integration by parts, is

$$b_n = \int_0^2 x\sin\frac{n\pi}{2}x\,dx = \frac{4(-1)^{n+1}}{n\pi}.$$

Therefore
$$f(x) = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\frac{n\pi}{2}x. \tag{6}$$ ≡

The function in Example 1 satisfies the conditions of Theorem 12.2.1. Hence the series (6) converges to the function on $(-2, 2)$ and the periodic extension (of period 4) given in **FIGURE 12.3.4**.

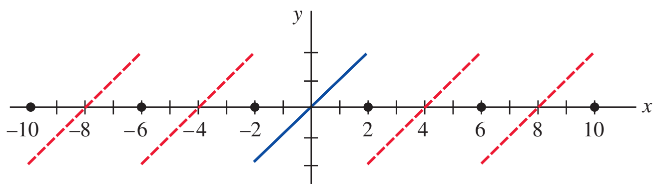


FIGURE 12.3.4 Periodic extension of the function f shown in Figure 12.3.3

EXAMPLE 2 Expansion in a Sine Series

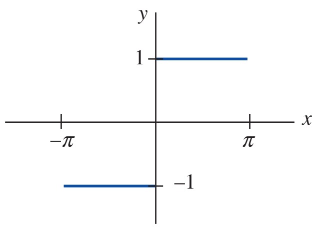


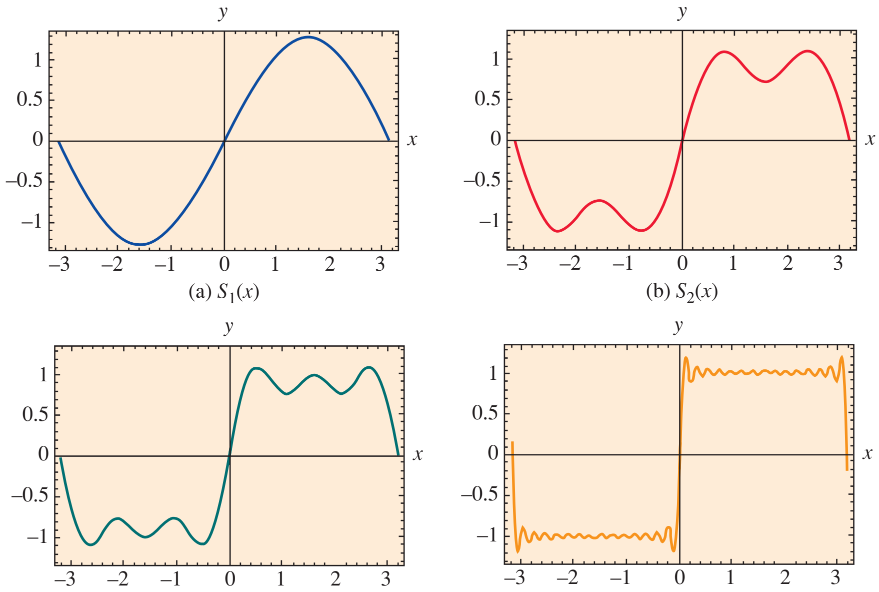
FIGURE 12.3.5 Odd function f in Example 2

The function $f(x) = \begin{cases} -1, & -\pi < x < 0 \\ 1, & 0 \le x < \pi \end{cases}$ shown in **FIGURE 12.3.5** is odd on the interval $(-\pi, \pi)$. With $p = \pi$ we have from (5)

$$b_n = \frac{2}{\pi}\int_0^{\pi}(1)\sin nx\,dx = \frac{2}{\pi}\frac{1-(-1)^n}{n},$$

and so
$$f(x) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n}\sin nx. \tag{7}$$ ≡

Gibbs Phenomenon With the aid of a CAS we have plotted in **FIGURE 12.3.6** the graphs $S_1(x)$, $S_2(x)$, $S_3(x)$, $S_{15}(x)$ of the partial sums of nonzero terms of (7). As seen in Figure 12.3.6(d) the graph of $S_{15}(x)$ has pronounced spikes near the discontinuities at $x = 0$, $x = \pi$, $x = -\pi$, and so on. This "overshooting" by the partial sums S_N from the function values near a point of discontinuity does not smooth out but remains fairly constant, even when the value N is taken to be large. This behavior of a Fourier series near a point at which f is discontinuous is known as the **Gibbs phenomenon**.



(a) $S_1(x)$ (b) $S_2(x)$ (c) $S_3(x)$ (d) $S_{15}(x)$

FIGURE 12.3.6 Partial sums of sine series (7) on the interval $(-\pi, \pi)$

The periodic extension of f in Example 2 onto the entire x-axis is a meander function (see page 247).

Half-Range Expansions Throughout the preceding discussion it was understood that a function f was defined on an interval with the origin as midpoint, that is, $(-p, p)$. However, in many instances we are interested in representing a function that is defined on an interval $(0, L)$ by a trigonometric series. This can be done in many different ways by supplying an arbitrary *definition* of the function on the interval $(-L, 0)$. For brevity we consider the three most important cases. If $y = f(x)$ is defined on the interval $(0, L)$, then:

(*i*) reflect the graph of the function about the y-axis onto $(-L, 0)$; the function is now even on the interval $(-L, L)$ (see **FIGURE 12.3.7**); or

(*ii*) reflect the graph of the function through the origin onto $(-L, 0)$; the function is now odd on the interval $(-L, L)$ (see **FIGURE 12.3.8**); or

(*iii*) define f on $(-L, 0)$ by $f(x) = f(x + L)$ (see **FIGURE 12.3.9**).

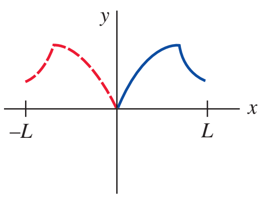


FIGURE 12.3.7 Even reflection

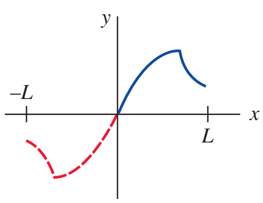


FIGURE 12.3.8 Odd reflection

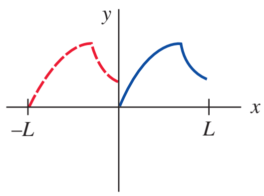


FIGURE 12.3.9 Identity reflection

Note that the coefficients of the series (1) and (4) utilize only the definition of the function on $(0, p)$, that is, for half of the interval $(-p, p)$. Hence in practice there is no actual need to make the reflections described in (*i*) and (*ii*). If f is defined on $(0, L)$, we simply identify the half-period as the length of the interval $p = L$. The coefficient formulas (2), (3), and (5) and the corresponding series yield either an even or an odd periodic extension of period $2L$ of the original function. The cosine and sine series obtained in this manner are known as **half-range expansions**. Last, in case (*iii*) we are defining the function values on the interval $(-L, 0)$ to be the same as the values on $(0, L)$. As in the previous two cases, there is no real need to do this. It can be shown that the set of functions in (1) of Section 12.2 is orthogonal on $[a, a + 2p]$ for any real number a. Choosing $a = -p$, we obtain the limits of integration in (9), (10), and (11) of that section. But for $a = 0$ the limits of integration are from $x = 0$ to $x = 2p$. Thus if f is defined over the interval $(0, L)$, we identify $2p = L$ or $p = L/2$. The resulting Fourier series will give the periodic extension of f with period L. In this manner the values to which the series converges will be the same on $(-L, 0)$ as on $(0, L)$.

EXAMPLE 3 Expansion in Three Series

Expand $f(x) = x^2$, $0 < x < L$, **(a)** in a cosine series, **(b)** in a sine series, **(c)** in a Fourier series.

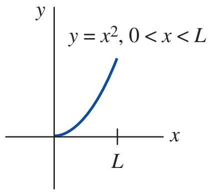


FIGURE 12.3.10 Function f in Example 3

SOLUTION The graph of the function is given in **FIGURE 12.3.10**.

(a) We have

$$a_0 = \frac{2}{L}\int_0^L x^2\,dx = \frac{2}{3}L^2, \qquad a_n = \frac{2}{L}\int_0^L x^2 \cos\frac{n\pi}{L}x\,dx = \frac{4L^2(-1)^n}{n^2\pi^2},$$

where integration by parts was used twice in the evaluation of a_n. Thus

$$f(x) = \frac{L^2}{3} + \frac{4L^2}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos\frac{n\pi}{L}x. \tag{8}$$

(b) In this case we must again integrate by parts twice:

$$b_n = \frac{2}{L}\int_0^L x^2 \sin\frac{n\pi}{L}x\,dx = \frac{2L^2(-1)^{n+1}}{n\pi} + \frac{4L^2}{n^3\pi^3}[(-1)^n - 1].$$

Hence

$$f(x) = \frac{2L^2}{\pi}\sum_{n=1}^{\infty}\left\{\frac{(-1)^{n+1}}{n} + \frac{2}{n^3\pi^2}[(-1)^n - 1]\right\}\sin\frac{n\pi}{L}x. \tag{9}$$

(c) With $p = L/2$, $1/p = 2/L$, and $n\pi/p = 2n\pi/L$, we have

$$a_0 = \frac{2}{L}\int_0^L x^2\,dx = \frac{2}{3}L^2, \qquad a_n = \frac{2}{L}\int_0^L x^2\cos\frac{2n\pi}{L}x\,dx = \frac{L^2}{n^2\pi^2}$$

and

$$b_n = \frac{2}{L}\int_0^L x^2 \sin\frac{2n\pi}{L}x\,dx = -\frac{L^2}{n\pi}.$$

Therefore

$$f(x) = \frac{L^2}{3} + \frac{L^2}{\pi}\sum_{n=1}^{\infty}\left\{\frac{1}{n^2\pi}\cos\frac{2n\pi}{L}x - \frac{1}{n}\sin\frac{2n\pi}{L}x\right\}. \tag{10}$$ ≡

The series (8), (9), and (10) converge to the $2L$-periodic even extension of f, the $2L$-periodic odd extension of f, and the L-periodic extension of f, respectively. The graphs of these periodic extensions are shown in **FIGURE 12.3.11**.

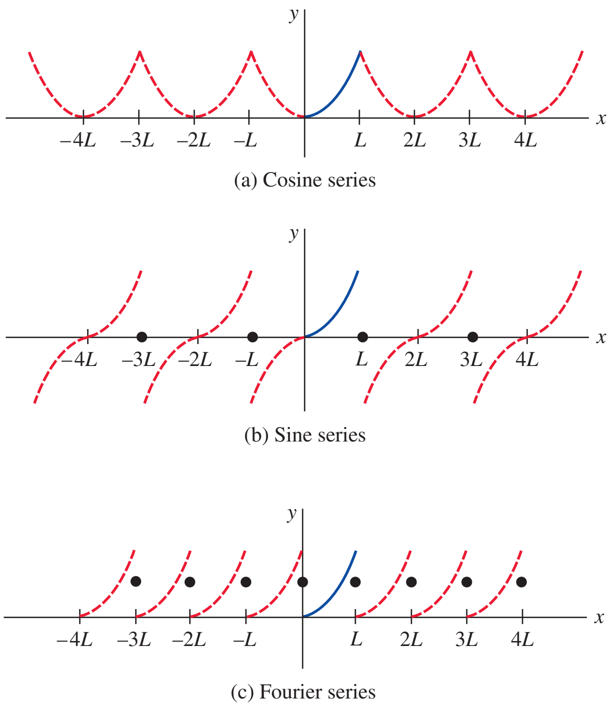


FIGURE 12.3.11 Different periodic extensions of the function f in Example 3

Periodic Driving Force

Fourier series are sometimes useful in determining a particular solution of a differential equation describing a physical system in which the input or driving force $f(t)$ is periodic. In the next example we find a particular solution of the differential equation

$$m\frac{d^2x}{dt^2} + kx = f(t) \tag{11}$$

by first representing f by a half-range sine expansion and then assuming a particular solution of the form

$$x_p(t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi}{p}t. \tag{12}$$

EXAMPLE 4 Particular Solution of a DE

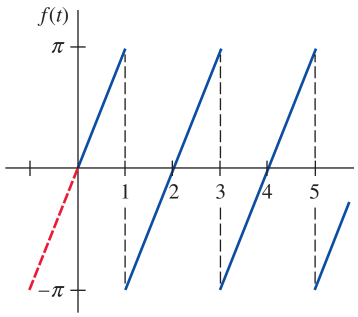


FIGURE 12.3.12 Periodic forcing function f in Example 4

An undamped spring/mass system, in which the mass $m = \frac{1}{16}$ slug and the spring constant $k = 4$ lb/ft, is driven by the 2-periodic external force $f(t)$ shown in **FIGURE 12.3.12**. Although the force $f(t)$ acts on the system for $t > 0$, note that if we extend the graph of the function in a 2-periodic manner to the negative t-axis, we obtain an odd function. In practical terms this means that we need only find the half-range sine expansion of $f(t) = \pi t,\ \ 0 < t < 1$. With $p = 1$ it follows from (5) and integration by parts that

$$b_n = 2\int_0^1 \pi t \sin n\pi t\, dt = \frac{2(-1)^{n+1}}{n}.$$

From (11) the differential equation of motion is seen to be

$$\frac{1}{16}\frac{d^2x}{dt^2} + 4x = \sum_{n=1}^{\infty}\frac{2(-1)^{n+1}}{n}\sin n\pi t. \tag{13}$$

To find a particular solution $x_p(t)$ of (13), we substitute the series (12) into the differential equation and equate coefficients of $\sin n\pi t$. This yields

$$\left(-\frac{1}{16}n^2\pi^2 + 4\right)B_n = \frac{2(-1)^{n+1}}{n} \quad \text{or} \quad B_n = \frac{32(-1)^{n+1}}{n(64 - n^2\pi^2)}.$$

Thus
$$x_p(t) = \sum_{n=1}^{\infty} \frac{32(-1)^{n+1}}{n(64 - n^2\pi^2)} \sin n\pi t. \qquad (14) \equiv$$

Observe in the solution (14) that there is no integer $n \geq 1$ for which the denominator $64 - n^2\pi^2$ of B_n is zero. In general, if there *is* a value of n, say, N, for which $N\pi/p = \omega$, where $\omega = \sqrt{k/m}$, then the system described by (11) is in a state of pure resonance. In other words, we have pure resonance if the Fourier series expansion of the driving force $f(t)$ contains a term $\sin(N\pi/L)t$ (or $\cos(N\pi/L)t$) that has the same frequency as the free vibrations.

Of course, if the $2p$-periodic extension of the driving force f onto the negative t-axis yields an even function, then we expand f in a cosine series.

12.3 Exercises

Answers to selected odd-numbered problems begin on page ANS-29.

In Problems 1–10, determine whether the given function is even, odd, or neither.

1. $f(x) = \sin 3x$

2. $f(x) = x \cos x$

3. $f(x) = x^2 + x$

4. $f(x) = x^3 - 4x$

5. $f(x) = e^{|x|}$

6. $f(x) = e^x - e^{-x}$

7. $f(x) = \begin{cases} x^2, & -1 < x < 0 \\ -x^2, & 0 \leq x < 1 \end{cases}$

8. $f(x) = \begin{cases} x + 5, & -2 < x < 0 \\ -x + 5, & 0 \leq x < 2 \end{cases}$

9. $f(x) = x^3,\ 0 \leq x \leq 2$

10. $f(x) = |x^5|$

In Problems 11–24, expand the given function in an appropriate cosine or sine series.

11. $f(x) = \begin{cases} \pi, & -1 < x < 0 \\ -\pi, & 0 \leq x < 1 \end{cases}$

12. $f(x) = \begin{cases} 1, & -2 < x < -1 \\ 0, & -1 < x < 1 \\ 1, & 1 < x < 2 \end{cases}$

13. $f(x) = |x|,\ -\pi < x < \pi$

14. $f(x) = x,\ -\pi < x < \pi$

15. $f(x) = x^2,\ -1 < x < 1$

16. $f(x) = x|x|,\ -1 < x < 1$

17. $f(x) = \pi^2 - x^2,\ -\pi < x < \pi$

18. $f(x) = x^3,\ -\pi < x < \pi$

19. $f(x) = \begin{cases} x - 1, & -\pi < x < 0 \\ x + 1, & 0 \leq x < \pi \end{cases}$

20. $f(x) = \begin{cases} x + 1, & -1 < x < 0 \\ x - 1, & 0 \leq x < 1 \end{cases}$

21. $f(x) = \begin{cases} 1, & -2 < x < -1 \\ -x, & -1 \leq x < 0 \\ x, & 0 \leq x < 1 \\ 1, & 1 \leq x < 2 \end{cases}$

22. $f(x) = \begin{cases} -\pi, & -2\pi < x < -\pi \\ x, & -\pi \leq x < \pi \\ \pi, & \pi \leq x < 2\pi \end{cases}$

23. $f(x) = |\sin x|,\ -\pi < x < \pi$

24. $f(x) = \cos x,\ -\pi/2 < x < \pi/2$

In Problems 25–34, find the half-range cosine and sine expansions of the given function.

25. $f(x) = \begin{cases} 1, & 0 < x < \frac{1}{2} \\ 0, & \frac{1}{2} \leq x < 1 \end{cases}$

26. $f(x) = \begin{cases} 0, & 0 < x < \frac{1}{2} \\ 1, & \frac{1}{2} \leq x < 1 \end{cases}$

27. $f(x) = \cos x,\ 0 < x < \pi/2$

28. $f(x) = \sin x,\ 0 < x < \pi$

29. $f(x) = \begin{cases} x, & 0 < x < \pi/2 \\ \pi - x, & \pi/2 \leq x < \pi \end{cases}$

30. $f(x) = \begin{cases} 0, & 0 < x < \pi \\ x - \pi, & \pi \leq x < 2\pi \end{cases}$

31. $f(x) = \begin{cases} x, & 0 < x < 1 \\ 1, & 1 \leq x < 2 \end{cases}$

32. $f(x) = \begin{cases} 1, & 0 < x < 1 \\ 2 - x, & 1 \leq x < 2 \end{cases}$

33. $f(x) = x^2 + x,\ 0 < x < 1$

34. $f(x) = x(2 - x),\ 0 < x < 2$

In Problems 35–38, expand the given function in a Fourier series.

35. $f(x) = x^2,\ 0 < x < 2\pi$

36. $f(x) = x,\ 0 < x < \pi$

37. $f(x) = x + 1,\ 0 < x < 1$

38. $f(x) = 2 - x,\ 0 < x < 2$

In Problems 39–42, suppose the function $y = f(x)$, $0 < x < L$, given in the figure is expanded in a cosine series, in a sine series, and in a Fourier series. Sketch the periodic extension to which each series converges.

39.

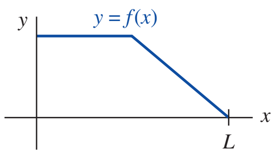


FIGURE 12.3.13 Graph for Problem 39

40.

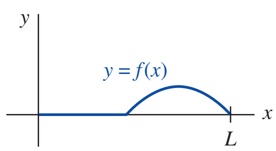


FIGURE 12.3.14 Graph for Problem 40

41.

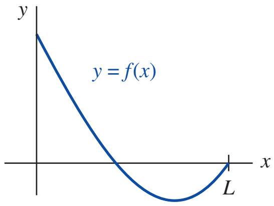


FIGURE 12.3.15 Graph for Problem 41

42.

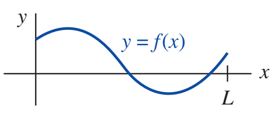


FIGURE 12.3.16 Graph for Problem 42