Graph Theory

Assignment 2

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Question 1

Part i

First, we will prove by induction that a tree of n vertices contains $n - 1$ edges.

1. For $n = 1$, the graph will not contain 0 edges because any edge will result in a loop.
2. Suppose that the proposition holds for $n = k$ i.e. the tree contains $k - 1$ edges.
3. For $n = k + 1$, edges will be k. If we remove one vertex here and resulting edges are $k - 1$, then the assumption proposed in point 2 will be proved.
    a. Remove any leaf of the graph. The vertices are now k.
    b. Since leaves contribute only one edge in the tree, the total number of edges is reduced to $k - 1$
    c. Hence proved that total number of edges of a tree is $n - 1$

From the Handshaking Lemma, the total degree is twice the number of edges. Therefore, the total degree of a tree on n vertices is $2(n - 1) = 2n - 2$.

Part ii

Part iii

Let the maximum distance i.e. eccentricity of x from all vertices be a and that of y be b. Since x and y are adjacent, they are at a distance of 1 from each other. Therefore, all distances from x will be separated by a path of length one from all distances from y. Hence, proved that $a = b \pm 1$.

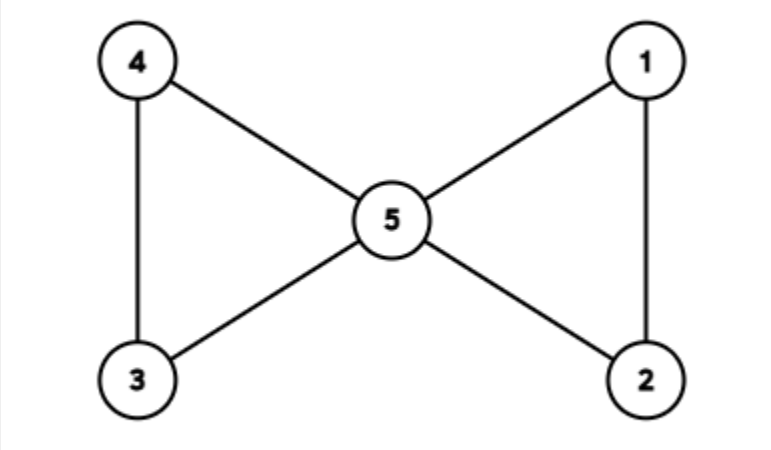
Question 2

Part a

Since all paths between graph components include the bridge, they will also include both of its endpoints. Therefore, by definition of the cut-vertex i.e. the vertex that is included in all paths, both endpoints of the bridge will be the cut-vertices.

Part b

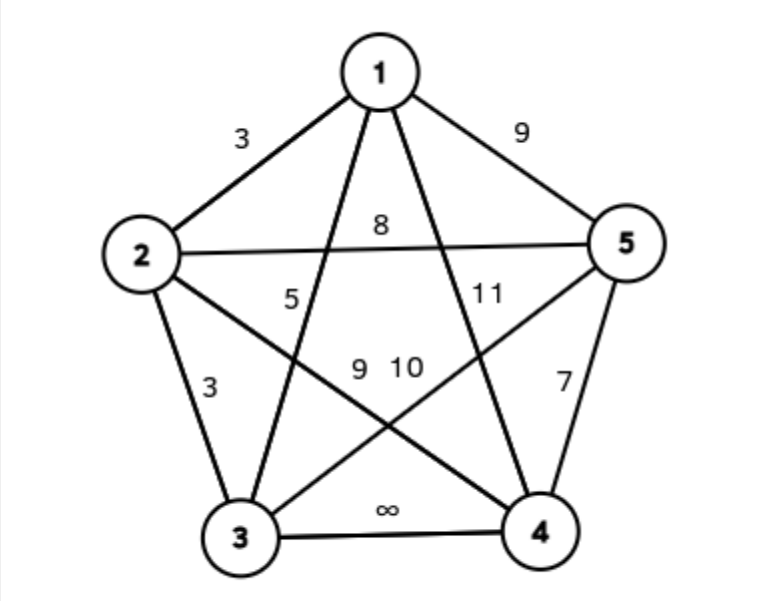
The converse is not true. Consider the following example



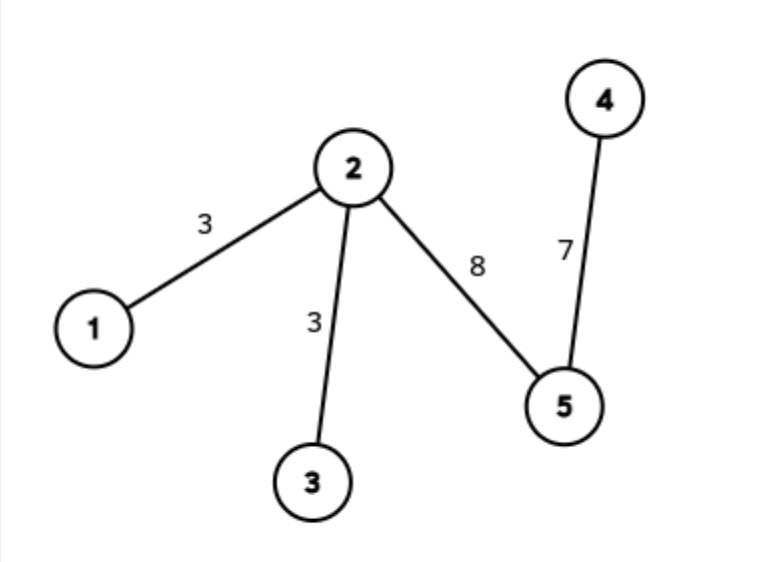
The graph contains vertex 5 as a cut-vertex but lacks any bridge because all edges are in a cycle and bridges are never contained in any cycle or circuit.

Question 3

We will determine the cost by using Kruskal's algorithm to find the minimum spanning tree. The given adjacency vertex represents the following graph



We get the following minimum spanning tree of cost 21



Question 4

Using Floyd-Warshall Algorithm:

Step | Travel Time | Intermediate Vertex (– means directly connected)

A^0

	1	2	3	4	5
1	0	10	20	∞	17
2	7	0	5	22	33
3	14	13	0	15	27
4	30	∞	17	0	10
5	∞	15	12	8	0

	1	2	3	4	5
1	–	–	–	–	–
2	–	–	–	–	–
3	–	–	–	–	–
4	–	–	–	–	–
5	–	–	–	–	–

A^1

	1	2	3	4	5
1	0	10	20	∞	17
2	7	0	5	22	**24**
3	14	13	0	15	27
4	30	**40**	17	0	10
5	∞	15	12	8	0

	1	2	3	4	5
1	–	–	–	–	–
2	–	–	–	–	1
3	–	–	–	–	–
4	–	1	–	–	–
5	–	–	–	–	–

A^2

	1	2	3	4	5
1	0	10	**15**	**32**	17
2	7	0	5	22	24
3	14	13	0	15	27
4	30	40	17	0	10
5	**22**	15	12	8	0

	1	2	3	4	5
1	–	–	2	2	–
2	–	–	–	–	1
3	–	–	–	–	–
4	–	1	–	–	–
5	2	–	–	–	–

A^3

	1	2	3	4	5
1	0	10	15	**30**	17
2	7	0	5	**20**	24
3	14	13	0	15	27
4	30	**30**	17	0	10
5	22	15	12	8	0

	1	2	3	4	5
1	–	–	2	3	–
2	–	–	–	3	1
3	–	–	–	–	–
4	–	3	–	–	–
5	2	–	–	–	–

A^4

	1	2	3	4	5
1	0	10	15	30	17
2	7	0	5	20	24
3	14	13	0	15	**25**
4	30	30	17	0	10
5	22	15	12	8	0

	1	2	3	4	5
1	–	–	2	3	–
2	–	–	–	3	1
3	–	–	–	–	4
4	–	3	–	–	–
5	2	–	–	–	–

A^5

	1	2	3	4	5
1	0	10	15	**25**	17
2	7	0	5	20	24
3	14	13	0	15	25
4	30	**25**	17	0	10
5	22	15	12	8	0

	1	2	3	4	5
1	–	–	2	5	–
2	–	–	–	3	1
3	–	–	–	–	4
4	–	5	–	–	–
5	2	–	–	–	–

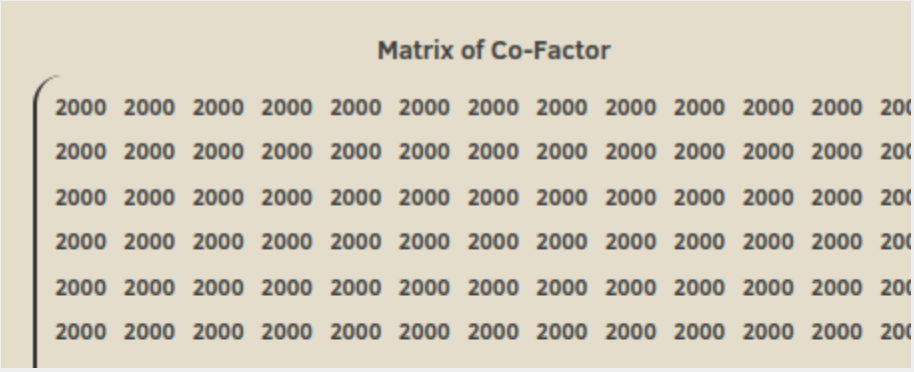
Question 5

For calculating the total numbers of spanning trees, we will use this method:
https://www.geeksforgeeks.org/total-number-spanning-trees-graph/

The matrix formed is:

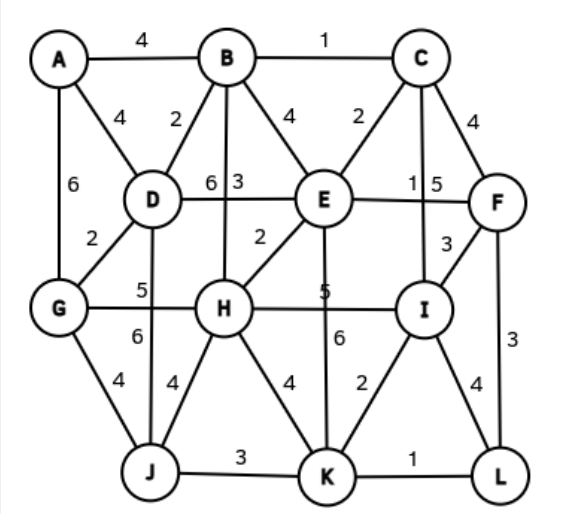
2 -1 0 0 0 0 0 0 0 0 0 0 0 0 -1
-1 2 -1 0 0 0 0 0 0 0 0 0 0 0 0
0 -1 2 -1 0 0 0 0 0 0 0 0 0 0 0
0 0 -1 4 -1 0 0 -1 0 0 0 0 0 0 -1
0 0 0 -1 2 -1 0 0 0 0 0 0 0 0 0
0 0 0 0 -1 2 -1 0 0 0 0 0 0 0 0
0 0 0 0 0 -1 2 -1 0 0 0 0 0 0 0
0 0 0 -1 0 0 -1 4 -1 0 0 -1 0 0 0
0 0 0 0 0 0 0 -1 2 -1 0 0 0 0 0
0 0 0 0 0 0 0 0 -1 2 -1 0 0 0 0
0 0 0 0 0 0 0 0 0 -1 2 -1 0 0 0
0 0 0 0 0 0 0 -1 0 0 -1 4 -1 0 0 -1
0 0 0 0 0 0 0 0 0 0 0 -1 2 -1 0 0
0 0 0 0 0 0 0 0 0 0 0 0 -1 2 -1 0
0 0 0 0 0 0 0 0 0 0 0 0 0 -1 2 -1
-1 0 0 -1 0 0 0 0 0 0 0 -1 0 0 -1 4

The cofactor matrix is calculated with *https://www.easycalculation.com/matrix/cofactor-matrix.php*

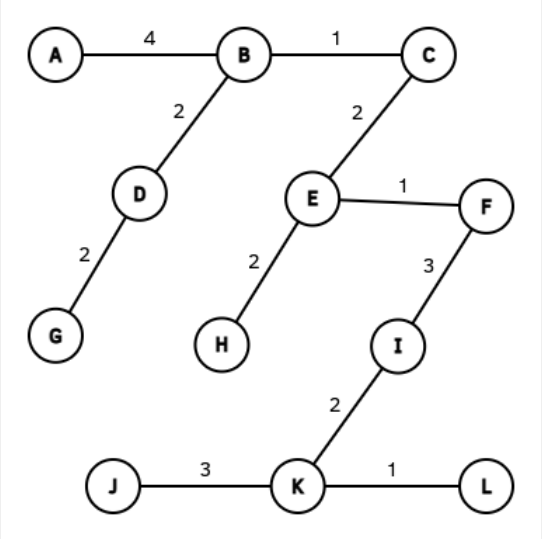


Therefore, the number of spanning trees is 2000

Question 6



Part a



Order of Selection: Cost: 23

bc, ef, kl, bd, dg, ce, eh, ki, kj, fi, ab,

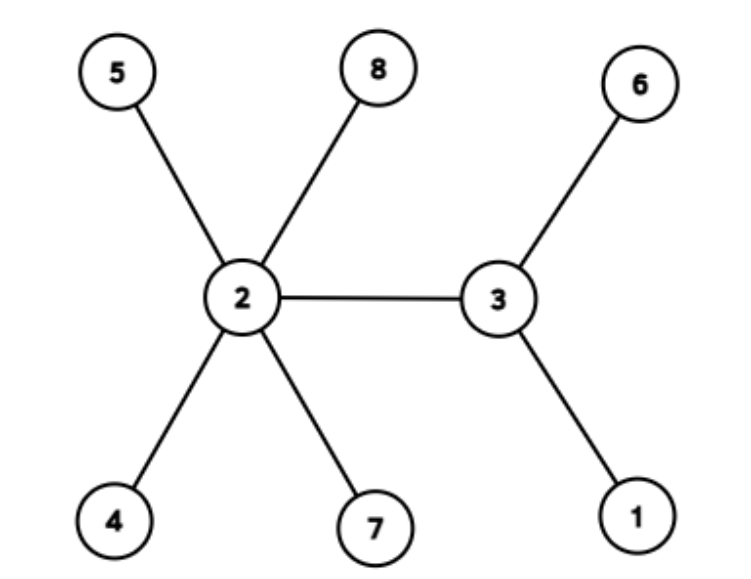
Part b

Same as Part a, except the order of selection when root is a:

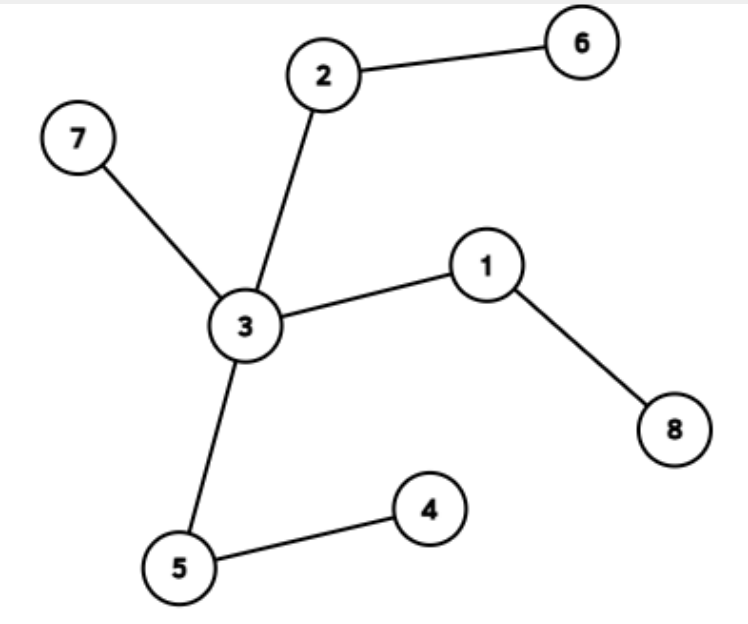
ab, bc, bd, ce, ef, dg, eh fi, ik, kl, kj

Question 7

Part i

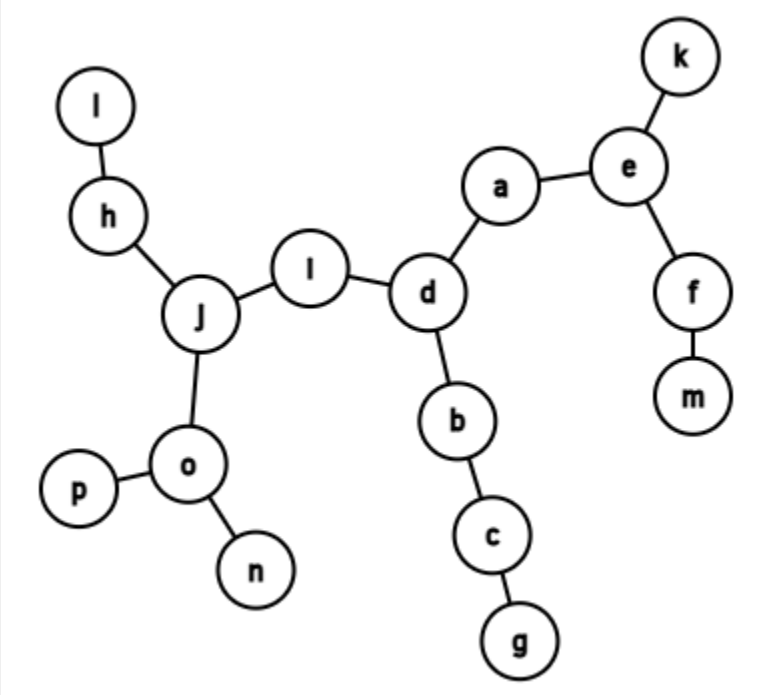


Part ii

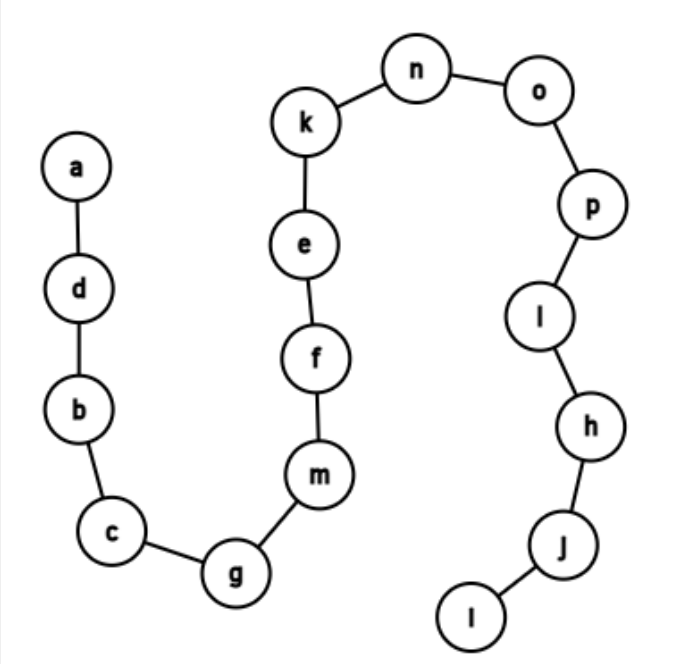


Question 8

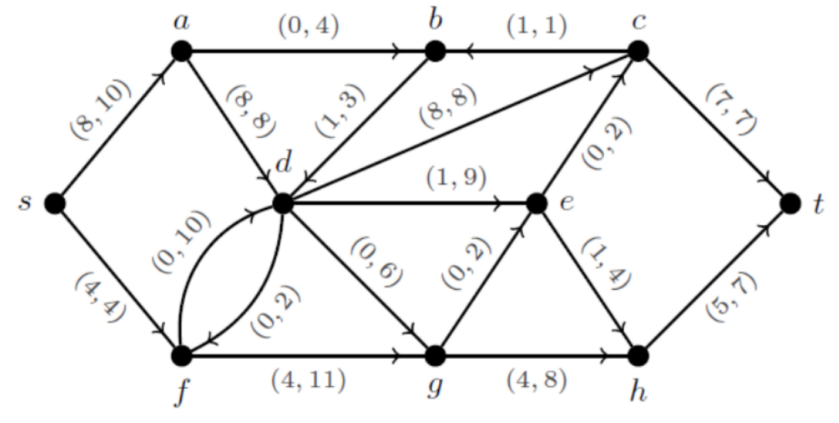
Part a

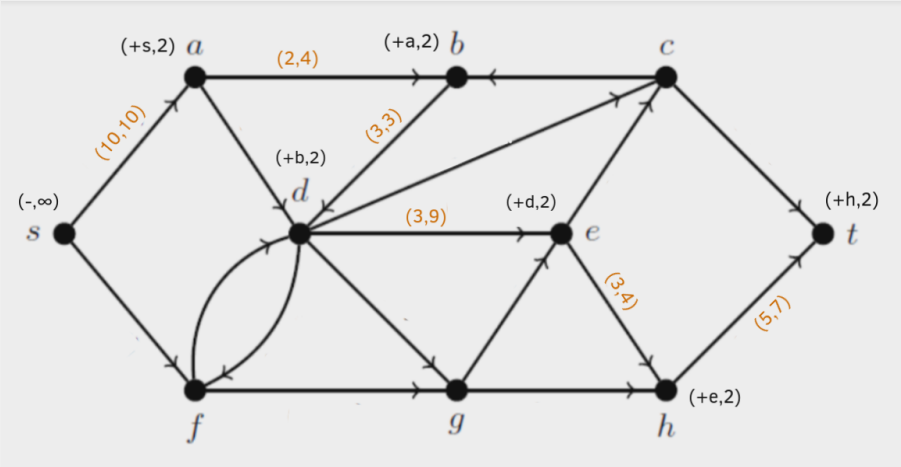


Part b



Question 9





In the 2nd iteration, we will be blocked at s. Hence, only s will be labeled and a flow of 2 is pumped. Thus, the total flow is 14.

Min-Cut Method

The Partitions are: $P = \{s\}$ and $\overline{P} = \{a, b, c, d, e, f, g, h, t\}$

Flow from P to $\overline{P}$: $10 + 4 = 14$

Graphing tool: *https://csacademy.com/app/graph_editor/*