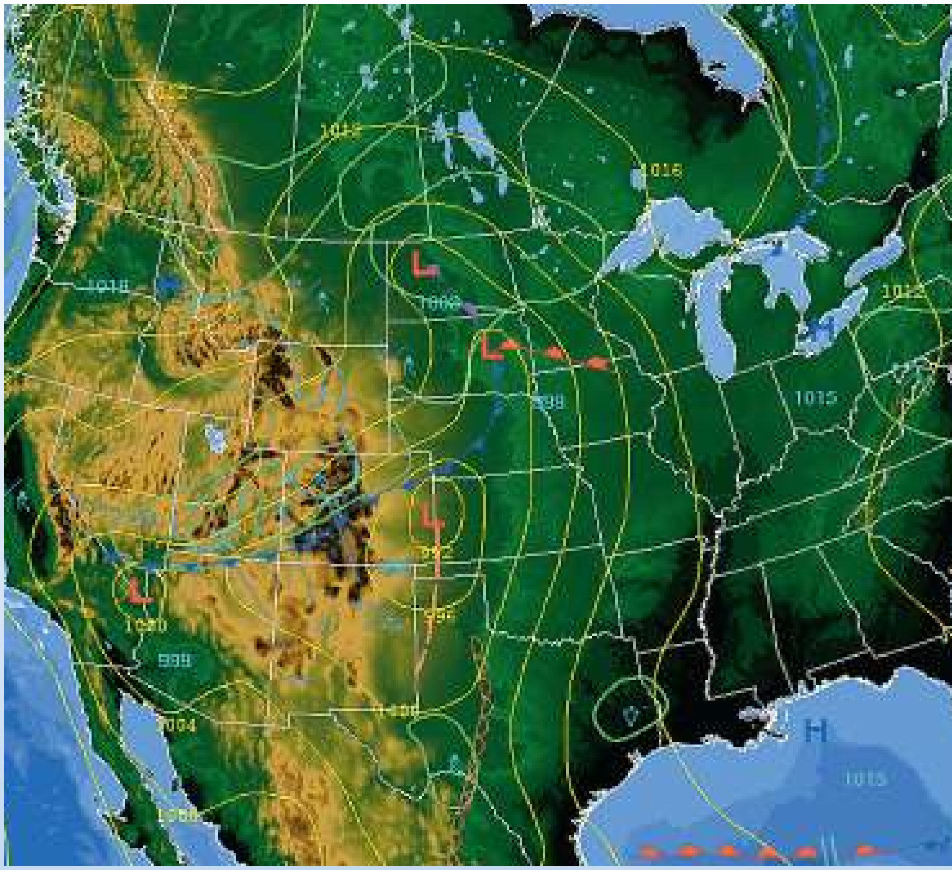
13 Functions of Several Variables

In this chapter, you will study functions of more than one independent variable. Many of the concepts presented are extensions of familiar ideas from earlier chapters.

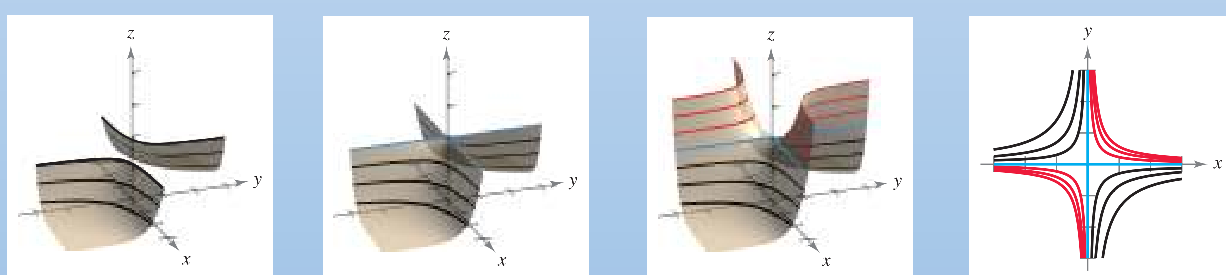
In this chapter, you should learn the following.

- How to sketch a graph, level curves, and level surfaces. (13.1)
- How to find a limit and determine continuity. (13.2)
- How to find and use a partial derivative. (13.3)
- How to find and use a total differential and determine differentiability. (13.4)
- How to use the Chain Rules and find a partial derivative implicitly. (13.5)
- How to find and use a directional derivative and a gradient. (13.6)
- How to find an equation of a tangent plane and an equation of a normal line to a surface, and how to find the angle of inclination of a plane. (13.7)
- How to find absolute and relative extrema. (13.8)
- How to solve an optimization problem, including constrained optimization using a Lagrange multiplier, and how to use the method of least squares. (13.9, 13.10)



NOAA

Meteorologists use maps that show curves of equal atmospheric pressure, called *isobars*, to predict weather patterns. How can you use pressure gradients to determine the area of the country that has the greatest wind speed? (See Section 13.6, Exercise 68.)



Many real-life quantities are functions of two or more variables. In Section 13.1, you will learn how to graph a function of two variables, like the one shown above. The first three graphs show cut-away views of the surface at various traces. Another way to visualize this surface is to project the traces onto the *xy*-plane, as shown in the fourth graph.

13.1 Introduction to Functions of Several Variables

- **Understand the notation for a function of several variables.**
- **Sketch the graph of a function of two variables.**
- **Sketch level curves for a function of two variables.**
- **Sketch level surfaces for a function of three variables.**
- **Use computer graphics to graph a function of two variables.**

Functions of Several Variables

EXPLORATION

Comparing Dimensions
Without using a graphing utility, describe the graph of each function of two variables.

a. $z = x^2 + y^2$
b. $z = x + y$
c. $z = x^2 + y$
d. $z = \sqrt{x^2 + y^2}$
e. $z = \sqrt{1 - x^2 + y^2}$

So far in this text, you have dealt only with functions of a single (independent) variable. Many familiar quantities, however, are functions of two or more variables. For instance, the work done by a force $(W = FD)$ and the volume of a right circular cylinder $(V = \pi r^2 h)$ are both functions of two variables. The volume of a rectangular solid $(V = lwh)$ is a function of three variables. The notation for a function of two or more variables is similar to that for a function of a single variable. Here are two examples.

$$z = f\underbrace{(x, y)}_{\text{2 variables}} = x^2 + xy \qquad \text{Function of two variables}$$

and

$$w = f\underbrace{(x, y, z)}_{\text{3 variables}} = x + 2y - 3z \qquad \text{Function of three variables}$$

DEFINITION OF A FUNCTION OF TWO VARIABLES

Let D be a set of ordered pairs of real numbers. If to each ordered pair (x, y) in D there corresponds a unique real number $f(x, y)$, then f is called a **function of x and y.** The set D is the **domain** of f, and the corresponding set of values for $f(x, y)$ is the **range** of f.

For the function given by $z = f(x, y)$, x and y are called the **independent variables** and z is called the **dependent variable.**

Similar definitions can be given for functions of three, four, or n variables, where the domains consist of ordered triples (x_1, x_2, x_3), quadruples (x_1, x_2, x_3, x_4), and n-tuples $(x_1, x_2, \ldots, x_n)$. In all cases, the range is a set of real numbers. In this chapter, you will study only functions of two or three variables.

As with functions of one variable, the most common way to describe a function of several variables is with an *equation*, and unless otherwise restricted, you can assume that the domain is the set of all points for which the equation is defined. For instance, the domain of the function given by

$$f(x, y) = x^2 + y^2$$

is assumed to be the entire xy-plane. Similarly, the domain of

$$f(x, y) = \ln xy$$

is the set of all points (x, y) in the plane for which $xy > 0$. This consists of all points in the first and third quadrants.

Archive Photos

MARY FAIRFAX SOMERVILLE (1780–1872)

Somerville was interested in the problem of creating geometric models for functions of several variables. Her most well-known book, *The Mechanics of the Heavens*, was published in 1831.

EXAMPLE 1 Domains of Functions of Several Variables

Find the domain of each function.

a. $f(x, y) = \dfrac{\sqrt{x^2 + y^2 - 9}}{x}$ **b.** $g(x, y, z) = \dfrac{x}{\sqrt{9 - x^2 - y^2 - z^2}}$

Solution

a. The function f is defined for all points (x, y) such that $x \neq 0$ and

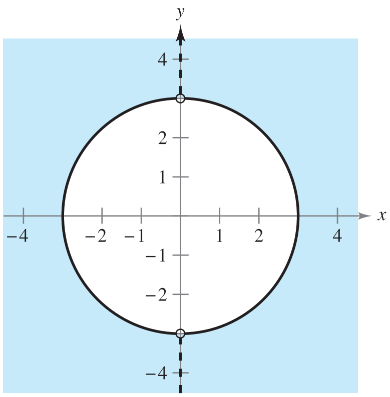
$$x^2 + y^2 \geq 9.$$

So, the domain is the set of all points lying on or outside the circle $x^2 + y^2 = 9$, *except* those points on the y-axis, as shown in Figure 13.1.

b. The function g is defined for all points (x, y, z) such that

$$x^2 + y^2 + z^2 < 9.$$

Consequently, the domain is the set of all points (x, y, z) lying inside a sphere of radius 3 that is centered at the origin. ■



Domain of

$f(x, y) = \dfrac{\sqrt{x^2 + y^2 - 9}}{x}$

Figure 13.1

Functions of several variables can be combined in the same ways as functions of single variables. For instance, you can form the sum, difference, product, and quotient of two functions of two variables as follows.

$(f \pm g)(x, y) = f(x, y) \pm g(x, y)$	Sum or difference
$(fg)(x, y) = f(x, y)g(x, y)$	Product
$\dfrac{f}{g}(x, y) = \dfrac{f(x, y)}{g(x, y)}, \quad g(x, y) \neq 0$	Quotient

You cannot form the composite of two functions of several variables. However, if h is a function of several variables and g is a function of a single variable, you can form the **composite** function $(g \circ h)(x, y)$ as follows.

$(g \circ h)(x, y) = g(h(x, y))$	Composition

The domain of this composite function consists of all (x, y) in the domain of h such that $h(x, y)$ is in the domain of g. For example, the function given by

$$f(x, y) = \sqrt{16 - 4x^2 - y^2}$$

can be viewed as the composite of the function of two variables given by $h(x, y) = 16 - 4x^2 - y^2$ and the function of a single variable given by $g(u) = \sqrt{u}$. The domain of this function is the set of all points lying on or inside the ellipse given by $4x^2 + y^2 = 16$.

A function that can be written as a sum of functions of the form $cx^m y^n$ (where c is a real number and m and n are nonnegative integers) is called a **polynomial function** of two variables. For instance, the functions given by

$$f(x, y) = x^2 + y^2 - 2xy + x + 2 \quad \text{and} \quad g(x, y) = 3xy^2 + x - 2$$

are polynomial functions of two variables. A **rational function** is the quotient of two polynomial functions. Similar terminology is used for functions of more than two variables.

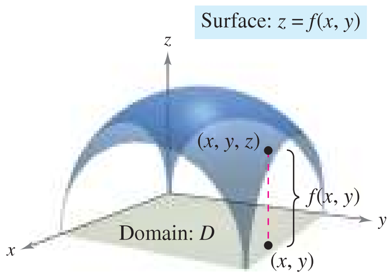


Figure 13.2

The Graph of a Function of Two Variables

As with functions of a single variable, you can learn a lot about the behavior of a function of two variables by sketching its graph. The **graph** of a function f of two variables is the set of all points (x, y, z) for which $z = f(x, y)$ and (x, y) is in the domain of f. This graph can be interpreted geometrically as a *surface in space*, as discussed in Sections 11.5 and 11.6. In Figure 13.2, note that the graph of $z = f(x, y)$ is a surface whose projection onto the xy-plane is D, the domain of f. To each point (x, y) in D there corresponds a point (x, y, z) on the surface, and, conversely, to each point (x, y, z) on the surface there corresponds a point (x, y) in D.

EXAMPLE 2 Describing the Graph of a Function of Two Variables

What is the range of $f(x, y) = \sqrt{16 - 4x^2 - y^2}$? Describe the graph of f.

Solution The domain D implied by the equation of f is the set of all points (x, y) such that $16 - 4x^2 - y^2 \geq 0$. So, D is the set of all points lying on or inside the ellipse given by

$$\frac{x^2}{4} + \frac{y^2}{16} = 1. \qquad \text{Ellipse in the } xy\text{-plane}$$

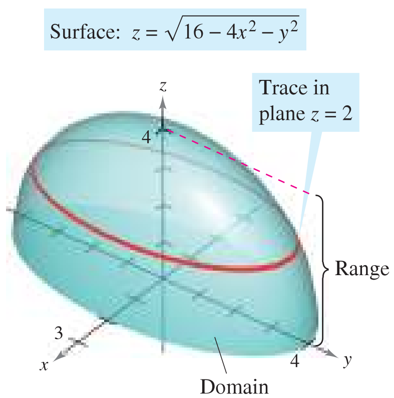
The range of f is all values $z = f(x, y)$ such that $0 \leq z \leq \sqrt{16}$ or

$$0 \leq z \leq 4. \qquad \text{Range of } f$$

A point (x, y, z) is on the graph of f if and only if

$$\begin{aligned} z &= \sqrt{16 - 4x^2 - y^2} \\ z^2 &= 16 - 4x^2 - y^2 \\ 4x^2 + y^2 + z^2 &= 16 \\ \frac{x^2}{4} + \frac{y^2}{16} + \frac{z^2}{16} &= 1, \qquad 0 \leq z \leq 4. \end{aligned}$$

From Section 11.6, you know that the graph of f is the upper half of an ellipsoid, as shown in Figure 13.3. ■



The graph of $f(x, y) = \sqrt{16 - 4x^2 - y^2}$ is the upper half of an ellipsoid.
Figure 13.3

To sketch a surface in space *by hand,* it helps to use traces in planes parallel to the coordinate planes, as shown in Figure 13.3. For example, to find the trace of the surface in the plane $z = 2$, substitute $z = 2$ in the equation $z = \sqrt{16 - 4x^2 - y^2}$ and obtain

$$2 = \sqrt{16 - 4x^2 - y^2} \quad \Rightarrow \quad \frac{x^2}{3} + \frac{y^2}{12} = 1.$$

So, the trace is an ellipse centered at the point $(0, 0, 2)$ with major and minor axes of lengths $4\sqrt{3}$ and $2\sqrt{3}$.

Traces are also used with most three-dimensional graphing utilities. For instance, Figure 13.4 shows a computer-generated version of the surface given in Example 2. For this graph, the computer took 25 traces parallel to the xy-plane and 12 traces in vertical planes.

If you have access to a three-dimensional graphing utility, use it to graph several surfaces.

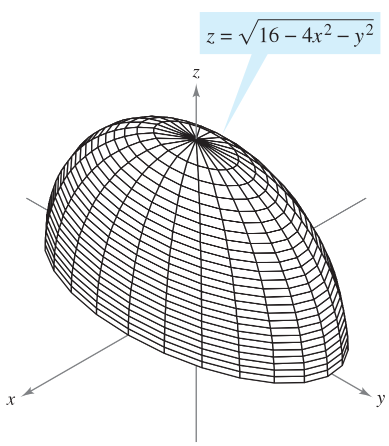
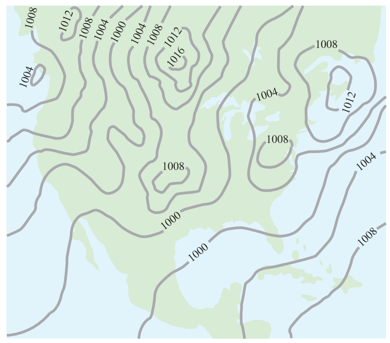


Figure 13.4

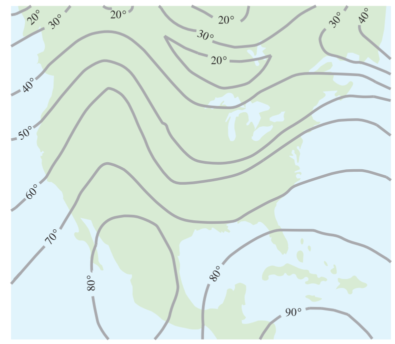
Level Curves

A second way to visualize a function of two variables is to use a **scalar field** in which the scalar $z = f(x, y)$ is assigned to the point (x, y). A scalar field can be characterized by **level curves** (or **contour lines**) along which the value of $f(x, y)$ is constant. For instance, the weather map in Figure 13.5 shows level curves of equal pressure called **isobars.** In weather maps for which the level curves represent points of equal temperature, the level curves are called **isotherms,** as shown in Figure 13.6. Another common use of level curves is in representing electric potential fields. In this type of map, the level curves are called **equipotential lines.**



Level curves show the lines of equal pressure (isobars) measured in millibars.

Figure 13.5



Level curves show the lines of equal temperature (isotherms) measured in degrees Fahrenheit.

Figure 13.6

Contour maps are commonly used to show regions on Earth's surface, with the level curves representing the height above sea level. This type of map is called a **topographic map.** For example, the mountain shown in Figure 13.7 is represented by the topographic map in Figure 13.8.

A contour map depicts the variation of z with respect to x and y by the spacing between level curves. Much space between level curves indicates that z is changing slowly, whereas little space indicates a rapid change in z. Furthermore, to produce a good three-dimensional illusion in a contour map, it is important to choose c-values that are *evenly spaced*.

Alfred B. Thomas/Earth Scenes

Figure 13.7

USGS

Figure 13.8

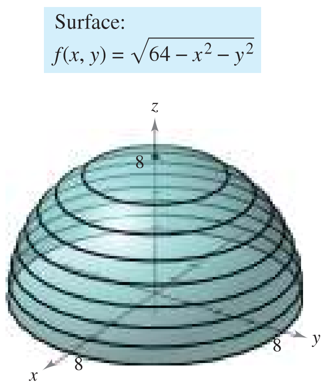
EXAMPLE 3 Sketching a Contour Map

The hemisphere given by $f(x, y) = \sqrt{64 - x^2 - y^2}$ is shown in Figure 13.9. Sketch a contour map of this surface using level curves corresponding to $c = 0, 1, 2, \ldots, 8$.

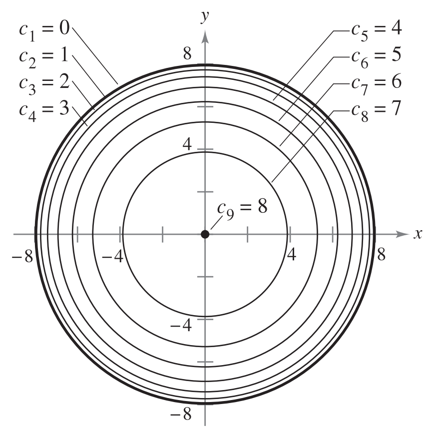
Solution For each value of c, the equation given by $f(x, y) = c$ is a circle (or point) in the xy-plane. For example, when $c_1 = 0$, the level curve is

$$x^2 + y^2 = 64 \qquad \text{Circle of radius 8}$$

which is a circle of radius 8. Figure 13.10 shows the nine level curves for the hemisphere.



Hemisphere
Figure 13.9



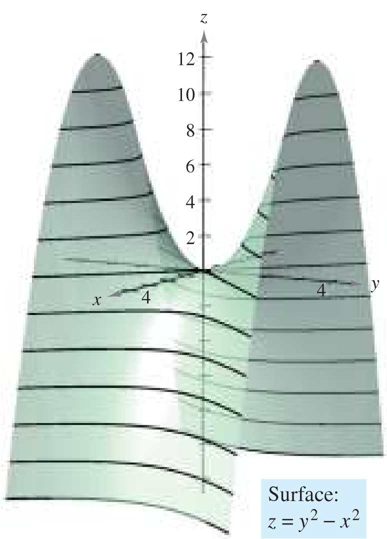
Contour map
Figure 13.10

EXAMPLE 4 Sketching a Contour Map

The hyperbolic paraboloid given by

$$z = y^2 - x^2$$

is shown in Figure 13.11. Sketch a contour map of this surface.



Hyperbolic paraboloid
Figure 13.11

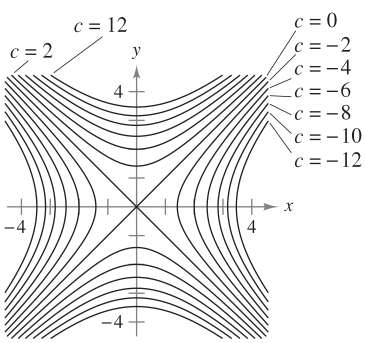
Solution For each value of c, let $f(x, y) = c$ and sketch the resulting level curve in the xy-plane. For this function, each of the level curves $(c \neq 0)$ is a hyperbola whose asymptotes are the lines $y = \pm x$. If $c < 0$, the transverse axis is horizontal. For instance, the level curve for $c = -4$ is given by

$$\frac{x^2}{2^2} - \frac{y^2}{2^2} = 1. \qquad \text{Hyperbola with horizontal transverse axis}$$

If $c > 0$, the transverse axis is vertical. For instance, the level curve for $c = 4$ is given by

$$\frac{y^2}{2^2} - \frac{x^2}{2^2} = 1. \qquad \text{Hyperbola with vertical transverse axis}$$

If $c = 0$, the level curve is the degenerate conic representing the intersecting asymptotes, as shown in Figure 13.12. ■



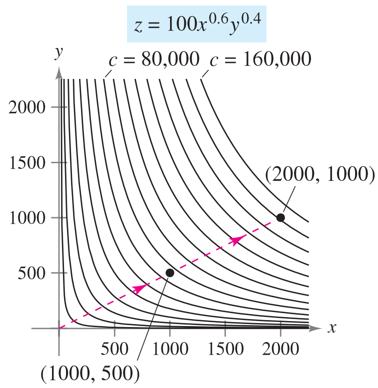
Hyperbolic level curves (at increments of 2)
Figure 13.12

The icon indicates that you will find a CAS Investigation on the book's website. The CAS Investigation is a collaborative exploration of this example using the computer algebra systems Maple *and* Mathematica.

One example of a function of two variables used in economics is the **Cobb-Douglas production function.** This function is used as a model to represent the numbers of units produced by varying amounts of labor and capital. If x measures the units of labor and y measures the units of capital, the number of units produced is given by

$$f(x, y) = Cx^a y^{1-a}$$

where C and a are constants with $0 < a < 1$.



Level curves (at increments of 10,000)
Figure 13.13

EXAMPLE 5 The Cobb-Douglas Production Function

A toy manufacturer estimates a production function to be $f(x, y) = 100x^{0.6}y^{0.4}$, where x is the number of units of labor and y is the number of units of capital. Compare the production level when $x = 1000$ and $y = 500$ with the production level when $x = 2000$ and $y = 1000$.

Solution When $x = 1000$ and $y = 500$, the production level is

$$f(1000, 500) = 100(1000^{0.6})(500^{0.4}) \approx 75{,}786.$$

When $x = 2000$ and $y = 1000$, the production level is

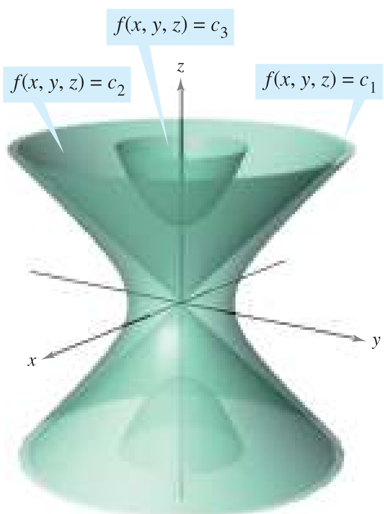
$$f(2000, 1000) = 100(2000^{0.6})(1000^{0.4}) = 151{,}572.$$

The level curves of $z = f(x, y)$ are shown in Figure 13.13. Note that by doubling both x and y, you double the production level (see Exercise 79). ■

Level Surfaces

The concept of a level curve can be extended by one dimension to define a **level surface.** If f is a function of three variables and c is a constant, the graph of the equation $f(x, y, z) = c$ is a **level surface** of the function f, as shown in Figure 13.14.

With computers, engineers and scientists have developed other ways to view functions of three variables. For instance, Figure 13.15 shows a computer simulation that uses color to represent the temperature distribution of fluid inside a pipe fitting.



Level surfaces of f
Figure 13.14

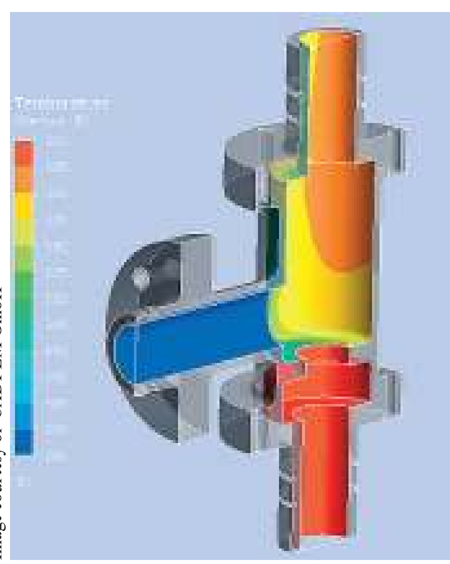


Image courtesy of CADFEM GmbH

One-way coupling of ANSYS CFX™ and ANSYS Mechanical™ for thermal stress analysis
Figure 13.15

EXAMPLE 6 Level Surfaces

Describe the level surfaces of the function

$$f(x, y, z) = 4x^2 + y^2 + z^2.$$

Solution Each level surface has an equation of the form

$$4x^2 + y^2 + z^2 = c. \qquad \text{Equation of level surface}$$

So, the level surfaces are ellipsoids (whose cross sections parallel to the yz-plane are circles). As c increases, the radii of the circular cross sections increase according to the square root of c. For example, the level surfaces corresponding to the values $c = 0$, $c = 4$, and $c = 16$ are as follows.

$$4x^2 + y^2 + z^2 = 0 \qquad \text{Level surface for } c = 0 \text{ (single point)}$$

$$\frac{x^2}{1} + \frac{y^2}{4} + \frac{z^2}{4} = 1 \qquad \text{Level surface for } c = 4 \text{ (ellipsoid)}$$

$$\frac{x^2}{4} + \frac{y^2}{16} + \frac{z^2}{16} = 1 \qquad \text{Level surface for } c = 16 \text{ (ellipsoid)}$$

These level surfaces are shown in Figure 13.16. ■

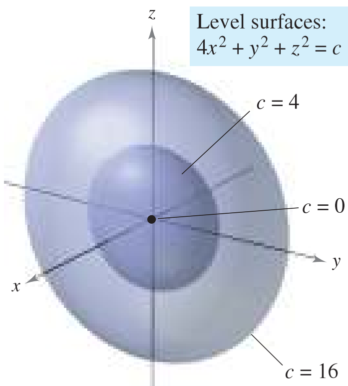


Figure 13.16

NOTE If the function in Example 6 represented the *temperature* at the point (x, y, z), the level surfaces shown in Figure 13.16 would be called **isothermal surfaces.** ■

Computer Graphics

The problem of sketching the graph of a surface in space can be simplified by using a computer. Although there are several types of three-dimensional graphing utilities, most use some form of trace analysis to give the illusion of three dimensions. To use such a graphing utility, you usually need to enter the equation of the surface, the region in the xy-plane over which the surface is to be plotted, and the number of traces to be taken. For instance, to graph the surface given by

$$f(x, y) = (x^2 + y^2)e^{1 - x^2 - y^2}$$

you might choose the following bounds for x, y, and z.

$$-3 \le x \le 3 \qquad \text{Bounds for } x$$

$$-3 \le y \le 3 \qquad \text{Bounds for } y$$

$$0 \le z \le 3 \qquad \text{Bounds for } z$$

Figure 13.17 shows a computer-generated graph of this surface using 26 traces taken parallel to the yz-plane. To heighten the three-dimensional effect, the program uses a "hidden line" routine. That is, it begins by plotting the traces in the foreground (those corresponding to the largest x-values), and then, as each new trace is plotted, the program determines whether all or only part of the next trace should be shown.

The graphs on page 893 show a variety of surfaces that were plotted by computer. If you have access to a computer drawing program, use it to reproduce these surfaces. Remember also that the three-dimensional graphics in this text can be viewed and rotated. These rotatable graphs are available in the premium eBook for this text.

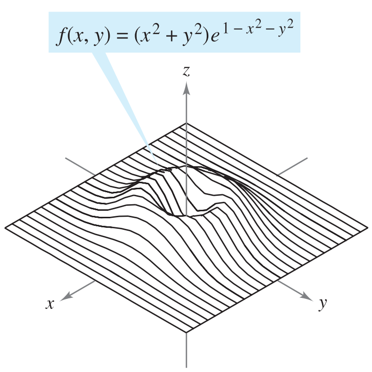
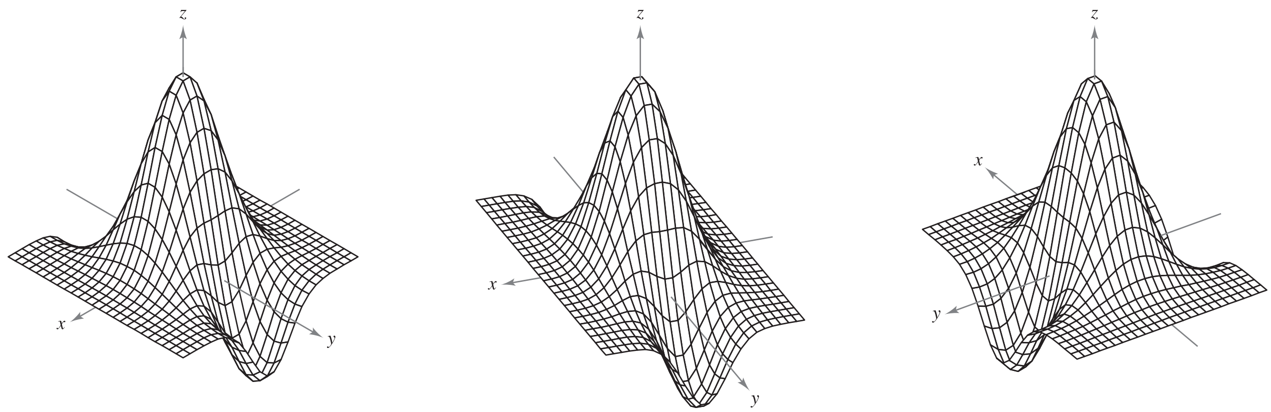
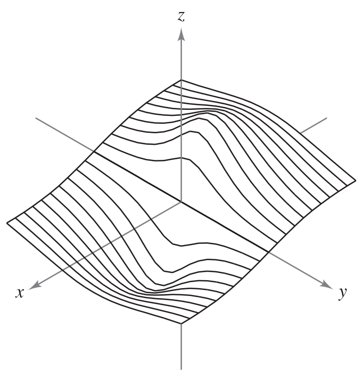


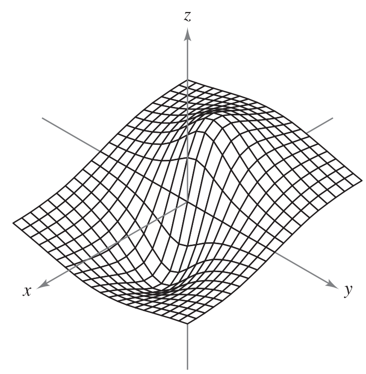
Figure 13.17



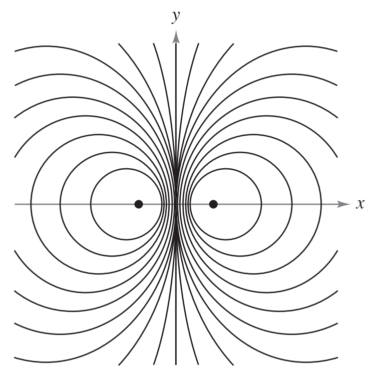
Three different views of the graph of $f(x, y) = (2 - y^2 + x^2)e^{1 - x^2 - (y^2/4)}$



Single traces

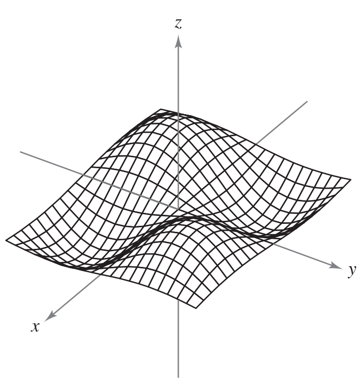


Double traces

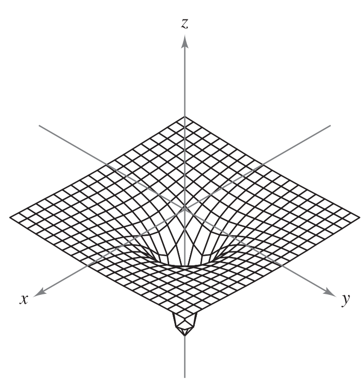


Level curves

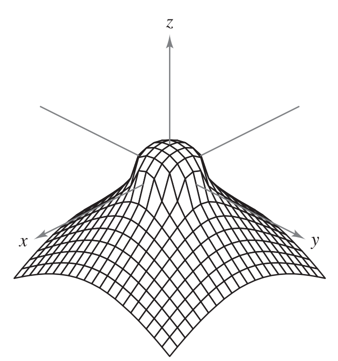
Traces and level curves of the graph of $f(x, y) = \dfrac{-4x}{x^2 + y^2 + 1}$



$f(x, y) = \sin x \sin y$



$f(x, y) = -\dfrac{1}{\sqrt{x^2 + y^2}}$



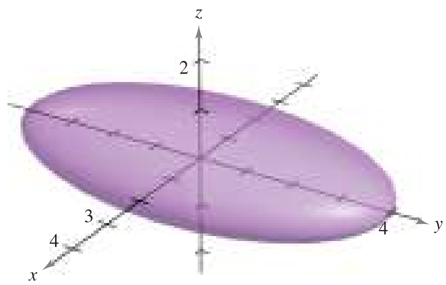
$f(x, y) = \dfrac{1 - x^2 - y^2}{\sqrt{|1 - x^2 - y^2|}}$

13.1 Exercises

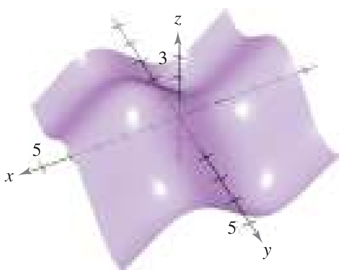
See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1 and 2, use the graph to determine whether z is a function of x and y. Explain.

1.



2.



In Exercises 3–6, determine whether z is a function of x and y.

3. $x^2z + 3y^2 - xy = 10$

4. $xz^2 + 2xy - y^2 = 4$

5. $\dfrac{x^2}{4} + \dfrac{y^2}{9} + z^2 = 1$

6. $z + x \ln y - 8yz = 0$

In Exercises 7–18, find and simplify the function values.

7. $f(x, y) = xy$
(a) $(3, 2)$ (b) $(-1, 4)$ (c) $(30, 5)$
(d) $(5, y)$ (e) $(x, 2)$ (f) $(5, t)$

8. $f(x, y) = 4 - x^2 - 4y^2$
(a) $(0, 0)$ (b) $(0, 1)$ (c) $(2, 3)$
(d) $(1, y)$ (e) $(x, 0)$ (f) $(t, 1)$

9. $f(x, y) = xe^y$
(a) $(5, 0)$ (b) $(3, 2)$ (c) $(2, -1)$
(d) $(5, y)$ (e) $(x, 2)$ (f) (t, t)

10. $g(x, y) = \ln|x + y|$
(a) $(1, 0)$ (b) $(0, -1)$ (c) $(0, e)$
(d) $(1, 1)$ (e) $(e, e/2)$ (f) $(2, 5)$

11. $h(x, y, z) = \dfrac{xy}{z}$
(a) $(2, 3, 9)$ (b) $(1, 0, 1)$ (c) $(-2, 3, 4)$ (d) $(5, 4, -6)$

12. $f(x, y, z) = \sqrt{x + y + z}$
(a) $(0, 5, 4)$ (b) $(6, 8, -3)$
(c) $(4, 6, 2)$ (d) $(10, -4, -3)$

13. $f(x, y) = x \sin y$
(a) $(2, \pi/4)$ (b) $(3, 1)$ (c) $(-3, \pi/3)$ (d) $(4, \pi/2)$

14. $V(r, h) = \pi r^2 h$
(a) $(3, 10)$ (b) $(5, 2)$ (c) $(4, 8)$ (d) $(6, 4)$

15. $g(x, y) = \displaystyle\int_x^y (2t - 3)\, dt$
(a) $(4, 0)$ (b) $(4, 1)$ (c) $\left(4, \frac{3}{2}\right)$ (d) $\left(\frac{3}{2}, 0\right)$

16. $g(x, y) = \displaystyle\int_x^y \frac{1}{t}\, dt$
(a) $(4, 1)$ (b) $(6, 3)$ (c) $(2, 5)$ (d) $\left(\frac{1}{2}, 7\right)$

17. $f(x, y) = 2x + y^2$
(a) $\dfrac{f(x + \Delta x, y) - f(x, y)}{\Delta x}$
(b) $\dfrac{f(x, y + \Delta y) - f(x, y)}{\Delta y}$

18. $f(x, y) = 3x^2 - 2y$
(a) $\dfrac{f(x + \Delta x, y) - f(x, y)}{\Delta x}$
(b) $\dfrac{f(x, y + \Delta y) - f(x, y)}{\Delta y}$

In Exercises 19–30, describe the domain and range of the function.

19. $f(x, y) = x^2 + y^2$

20. $f(x, y) = e^{xy}$

21. $g(x, y) = x\sqrt{y}$

22. $g(x, y) = \dfrac{y}{\sqrt{x}}$

23. $z = \dfrac{x + y}{xy}$

24. $z = \dfrac{xy}{x - y}$

25. $f(x, y) = \sqrt{4 - x^2 - y^2}$

26. $f(x, y) = \sqrt{4 - x^2 - 4y^2}$

27. $f(x, y) = \arccos(x + y)$

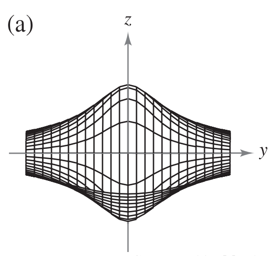
28. $f(x, y) = \arcsin(y/x)$

29. $f(x, y) = \ln(4 - x - y)$

30. $f(x, y) = \ln(xy - 6)$

31. ***Think About It*** The graphs labeled (a), (b), (c), and (d) are graphs of the function $f(x, y) = -4x/(x^2 + y^2 + 1)$. Match the four graphs with the points in space from which the surface is viewed. The four points are $(20, 15, 25)$, $(-15, 10, 20)$, $(20, 20, 0)$, and $(20, 0, 0)$.

(a)



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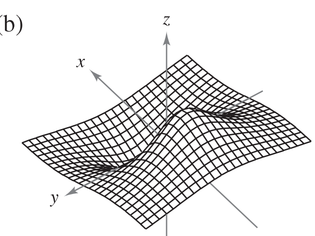
(b)



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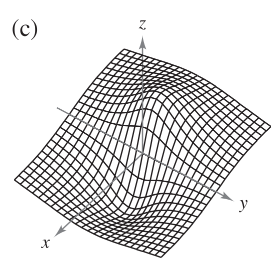
(c)



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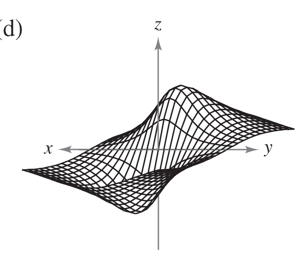
(d)



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32. ***Think About It*** Use the function given in Exercise 31.

(a) Find the domain and range of the function.

(b) Identify the points in the xy-plane at which the function value is 0.

(c) Does the surface pass through all the octants of the rectangular coordinate system? Give reasons for your answer.

In Exercises 33–40, sketch the surface given by the function.

33. $f(x, y) = 4$

34. $f(x, y) = 6 - 2x - 3y$

35. $f(x, y) = y^2$

36. $g(x, y) = \frac{1}{2}y$

37. $z = -x^2 - y^2$

38. $z = \frac{1}{2}\sqrt{x^2 + y^2}$

39. $f(x, y) = e^{-x}$

40. $f(x, y) = \begin{cases} xy, & x \geq 0, y \geq 0 \\ 0, & x < 0 \text{ or } y < 0 \end{cases}$

CAS **In Exercises 41–44, use a computer algebra system to graph the function.**

41. $z = y^2 - x^2 + 1$

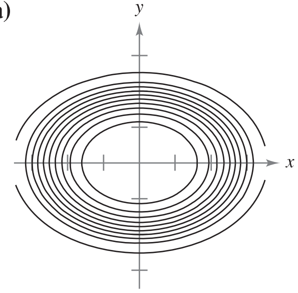
42. $z = \frac{1}{12}\sqrt{144 - 16x^2 - 9y^2}$

43. $f(x, y) = x^2e^{(-xy/2)}$

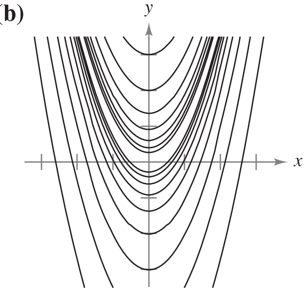
44. $f(x, y) = x \sin y$

In Exercises 45–48, match the graph of the surface with one of the contour maps. [The contour maps are labeled (a), (b), (c), and (d).]

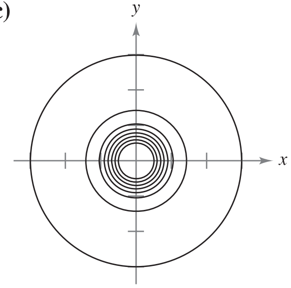
(a)



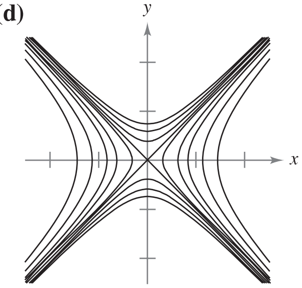
(b)



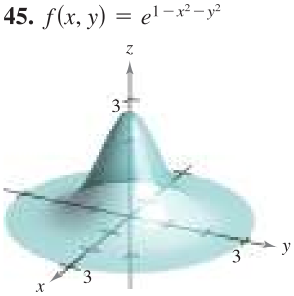
(c)



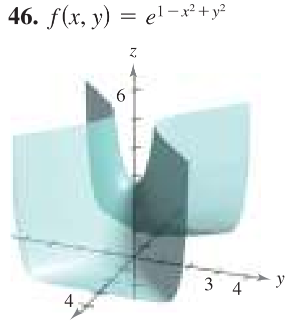
(d)



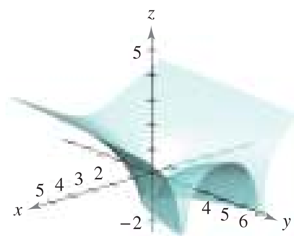
45. $f(x, y) = e^{1 - x^2 - y^2}$



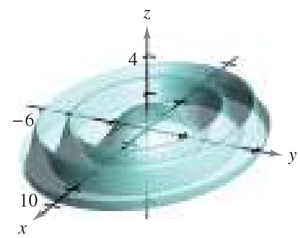
46. $f(x, y) = e^{1 - x^2 + y^2}$



47. $f(x, y) = \ln|y - x^2|$



48. $f(x, y) = \cos\left(\frac{x^2 + 2y^2}{4}\right)$



In Exercises 49–56, describe the level curves of the function. Sketch the level curves for the given c-values.

49. $z = x + y, \quad c = -1, 0, 2, 4$

50. $z = 6 - 2x - 3y, \quad c = 0, 2, 4, 6, 8, 10$

51. $z = x^2 + 4y^2, \quad c = 0, 1, 2, 3, 4$

52. $f(x, y) = \sqrt{9 - x^2 - y^2}, \quad c = 0, 1, 2, 3$

53. $f(x, y) = xy, \quad c = \pm1, \pm2, \ldots, \pm6$

54. $f(x, y) = e^{xy/2}, \quad c = 2, 3, 4, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}$

55. $f(x, y) = x/(x^2 + y^2), \quad c = \pm\frac{1}{2}, \pm1, \pm\frac{3}{2}, \pm2$

56. $f(x, y) = \ln(x - y), \quad c = 0, \pm\frac{1}{2}, \pm1, \pm\frac{3}{2}, \pm2$

In Exercises 57–60, use a graphing utility to graph six level curves of the function.

57. $f(x, y) = x^2 - y^2 + 2$

58. $f(x, y) = |xy|$

59. $g(x, y) = \dfrac{8}{1 + x^2 + y^2}$

60. $h(x, y) = 3\sin(|x| + |y|)$

WRITING ABOUT CONCEPTS

61. What is a graph of a function of two variables? How is it interpreted geometrically? Describe level curves.

62. All of the level curves of the surface given by $z = f(x, y)$ are concentric circles. Does this imply that the graph of f is a hemisphere? Illustrate your answer with an example.

63. Construct a function whose level curves are lines passing through the origin.

CAPSTONE

64. Consider the function $f(x, y) = xy$, for $x \geq 0$ and $y \geq 0$.

(a) Sketch the graph of the surface given by f.

(b) Make a conjecture about the relationship between the graphs of f and $g(x, y) = f(x, y) - 3$. Explain your reasoning.

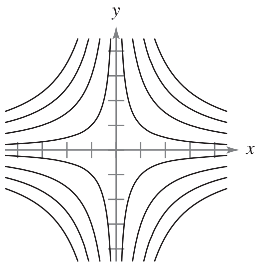
(c) Make a conjecture about the relationship between the graphs of f and $g(x, y) = -f(x, y)$. Explain your reasoning.

(d) Make a conjecture about the relationship between the graphs of f and $g(x, y) = \frac{1}{2}f(x, y)$. Explain your reasoning.

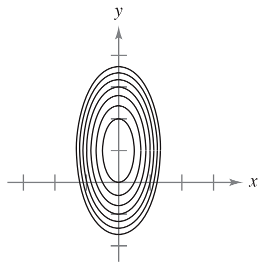
(e) On the surface in part (a), sketch the graph of $z = f(x, x)$.

Writing **In Exercises 65 and 66, use the graphs of the level curves (c-values evenly spaced) of the function f to write a description of a possible graph of f. Is the graph of f unique? Explain.**

65.



66.



67. ***Investment*** In 2009, an investment of \$1000 was made in a bond earning 6% compounded annually. Assume that the buyer pays tax at rate R and the annual rate of inflation is I. In the year 2019, the value V of the investment in constant 2009 dollars is

$$V(I, R) = 1000\left[\frac{1 + 0.06(1 - R)}{1 + I}\right]^{10}.$$

Use this function of two variables to complete the table.

	Inflation Rate		
Tax Rate	**0**	**0.03**	**0.05**
0			
0.28			
0.35			

68. ***Investment*** A principal of \$5000 is deposited in a savings account that earns interest at a rate of r (written as a decimal), compounded continuously. The amount $A(r, t)$ after t years is $A(r, t) = 5000e^{rt}$. Use this function of two variables to complete the table.

	Number of Years			
Rate	**5**	**10**	**15**	**20**
0.02				
0.03				
0.04				
0.05				

In Exercises 69–74, sketch the graph of the level surface $f(x, y, z) = c$ at the given value of c.

69. $f(x, y, z) = x - y + z, \quad c = 1$

70. $f(x, y, z) = 4x + y + 2z, \quad c = 4$

71. $f(x, y, z) = x^2 + y^2 + z^2, \quad c = 9$

72. $f(x, y, z) = x^2 + \frac{1}{4}y^2 - z, \quad c = 1$

73. $f(x, y, z) = 4x^2 + 4y^2 - z^2, \quad c = 0$

74. $f(x, y, z) = \sin x - z, \quad c = 0$

75. ***Forestry*** The **Doyle Log Rule** is one of several methods used to determine the lumber yield of a log (in board-feet) in terms of its diameter d (in inches) and its length L (in feet). The number of board-feet is

$$N(d, L) = \left(\frac{d - 4}{4}\right)^2 L.$$

(a) Find the number of board-feet of lumber in a log 22 inches in diameter and 12 feet in length.

(b) Find $N(30, 12)$.

76. ***Queuing Model*** The average length of time that a customer waits in line for service is

$$W(x, y) = \frac{1}{x - y}, \quad x > y$$

where y is the average arrival rate, written as the number of customers per unit of time, and x is the average service rate, written in the same units. Evaluate each of the following.

(a) $W(15, 9)$ (b) $W(15, 13)$ (c) $W(12, 7)$ (d) $W(5, 2)$

77. ***Temperature Distribution*** The temperature T (in degrees Celsius) at any point (x, y) in a circular steel plate of radius 10 meters is $T = 600 - 0.75x^2 - 0.75y^2$, where x and y are measured in meters. Sketch some of the isothermal curves.

78. ***Electric Potential*** The electric potential V at any point (x, y) is

$$V(x, y) = \frac{5}{\sqrt{25 + x^2 + y^2}}.$$

Sketch the equipotential curves for $V = \frac{1}{2}$, $V = \frac{1}{3}$, and $V = \frac{1}{4}$.

79. ***Cobb-Douglas Production Function*** Use the Cobb-Douglas production function (see Example 5) to show that if the number of units of labor and the number of units of capital are doubled, the production level is also doubled.

80. ***Cobb-Douglas Production Function*** Show that the Cobb-Douglas production function $z = Cx^a y^{1-a}$ can be rewritten as

$$\ln \frac{z}{y} = \ln C + a \ln \frac{x}{y}.$$

81. ***Construction Cost*** A rectangular box with an open top has a length of x feet, a width of y feet, and a height of z feet. It costs \$1.20 per square foot to build the base and \$0.75 per square foot to build the sides. Write the cost C of constructing the box as a function of x, y, and z.

82. ***Volume*** A propane tank is constructed by welding hemispheres to the ends of a right circular cylinder. Write the volume V of the tank as a function of r and l, where r is the radius of the cylinder and hemispheres, and l is the length of the cylinder.

83. ***Ideal Gas Law*** According to the Ideal Gas Law, $PV = kT$, where P is pressure, V is volume, T is temperature (in Kelvins), and k is a constant of proportionality. A tank contains 2000 cubic inches of nitrogen at a pressure of 26 pounds per square inch and a temperature of 300 K.

(a) Determine k.

(b) Write P as a function of V and T and describe the level curves.

84. ***Modeling Data*** The table shows the net sales x (in billions of dollars), the total assets y (in billions of dollars), and the shareholder's equity z (in billions of dollars) for Wal-Mart for the years 2002 through 2007. (*Source: 2007 Annual Report for Wal-Mart*)

Year	2002	2003	2004	2005	2006	2007
x	201.2	226.5	252.8	281.5	208.9	345.0
y	79.3	90.2	102.5	117.1	135.6	151.2
z	35.2	39.5	43.6	49.4	53.2	61.6

A model for these data is

$$z = f(x, y) = 0.026x + 0.316y + 5.04.$$

(a) Use a graphing utility and the model to approximate z for the given values of x and y.

(b) Which of the two variables in this model has the greater influence on shareholder's equity?

(c) Simplify the expression for $f(x, 95)$ and interpret its meaning in the context of the problem.

85. ***Meteorology*** Meteorologists measure the atmospheric pressure in millibars. From these observations they create weather maps on which the curves of equal atmospheric pressure (isobars) are drawn (see figure). On the map, the closer the isobars the higher the wind speed. Match points A, B, and C with (a) highest pressure, (b) lowest pressure, and (c) highest wind velocity.

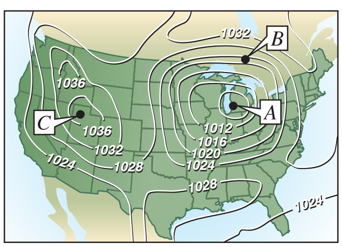


Figure for 85

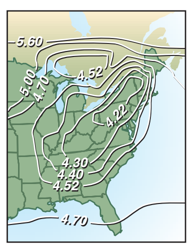
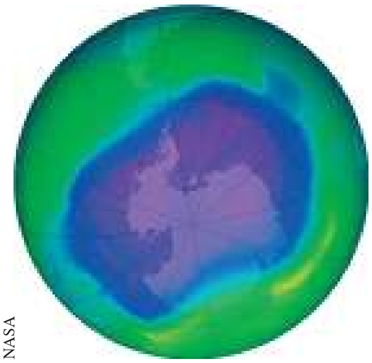


Figure for 86

86. ***Acid Rain*** The acidity of rainwater is measured in units called pH. A pH of 7 is neutral, smaller values are increasingly acidic, and larger values are increasingly alkaline. The map shows curves of equal pH and gives evidence that downwind of heavily industrialized areas the acidity has been increasing. Using the level curves on the map, determine the direction of the prevailing winds in the northeastern United States.

87. ***Atmosphere*** The contour map shown in the figure was computer generated using data collected by satellite instrumentation. Color is used to show the "ozone hole" in Earth's atmosphere. The purple and blue areas represent the lowest levels of ozone and the green areas represent the highest levels. (*Source: National Aeronautics and Space Administration*)



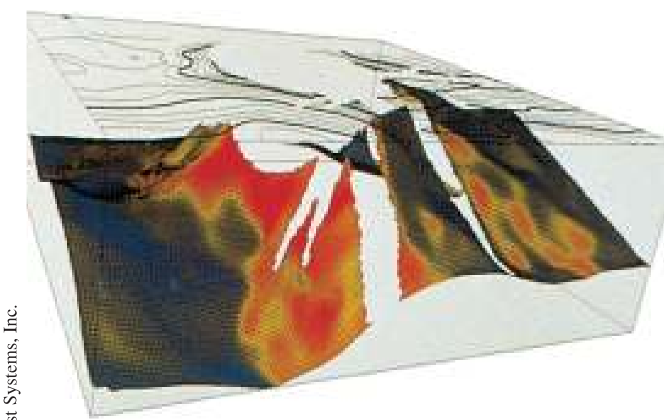
NASA

Figure for 87

(a) Do the level curves correspond to equally spaced ozone levels? Explain.

(b) Describe how to obtain a more detailed contour map.

88. ***Geology*** The contour map in the figure represents color-coded seismic amplitudes of a fault horizon and a projected contour map, which is used in earthquake studies. (*Source: Adapted from Shipman/Wilson/Todd,* An Introduction to Physical Science, *Tenth Edition*)



GeoQuest Systems, Inc.

(a) Discuss the use of color to represent the level curves.

(b) Do the level curves correspond to equally spaced amplitudes? Explain.

True or False? **In Exercises 89–92, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

89. If $f(x_0, y_0) = f(x_1, y_1)$, then $x_0 = x_1$ and $y_0 = y_1$.

90. If f is a function, then $f(ax, ay) = a^2 f(x, y)$.

91. A vertical line can intersect the graph of $z = f(x, y)$ at most once.

92. Two different level curves of the graph of $z = f(x, y)$ can intersect.

13.2 Limits and Continuity

- Understand the definition of a neighborhood in the plane.
- Understand and use the definition of the limit of a function of two variables.
- Extend the concept of continuity to a function of two variables.
- Extend the concept of continuity to a function of three variables.

The Granger Collection

SONYA KOVALEVSKY (1850–1891)

Much of the terminology used to define limits and continuity of a function of two or three variables was introduced by the German mathematician Karl Weierstrass (1815–1897). Weierstrass's rigorous approach to limits and other topics in calculus gained him the reputation as the "father of modern analysis." Weierstrass was a gifted teacher. One of his best-known students was the Russian mathematician Sonya Kovalevsky, who applied many of Weierstrass's techniques to problems in mathematical physics and became one of the first women to gain acceptance as a research mathematician.

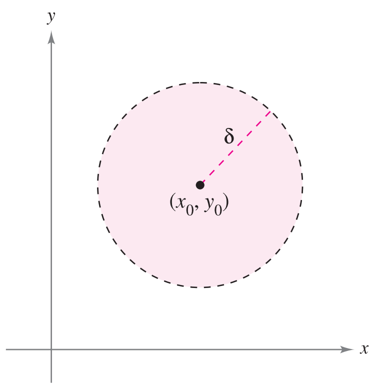
Neighborhoods in the Plane

In this section, you will study limits and continuity involving functions of two or three variables. The section begins with functions of two variables. At the end of the section, the concepts are extended to functions of three variables.

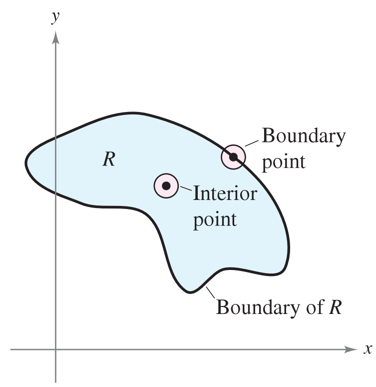
We begin our discussion of the limit of a function of two variables by defining a two-dimensional analog to an interval on the real number line. Using the formula for the distance between two points (x, y) and (x_0, y_0) in the plane, you can define the **δ-neighborhood** about (x_0, y_0) to be the **disk** centered at (x_0, y_0) with radius $\delta > 0$

$$\left\{(x, y): \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta\right\} \qquad \text{Open disk}$$

as shown in Figure 13.18. When this formula contains the *less than* inequality sign, $<$, the disk is called **open,** and when it contains the *less than or equal to* inequality sign, $\leq$, the disk is called **closed.** This corresponds to the use of $<$ and $\leq$ to define open and closed intervals.



An open disk
Figure 13.18



The boundary and interior points of a region R
Figure 13.19

A point (x_0, y_0) in a plane region R is an **interior point** of R if there exists a δ-neighborhood about (x_0, y_0) that lies entirely in R, as shown in Figure 13.19. If every point in R is an interior point, then R is an **open region.** A point (x_0, y_0) is a **boundary point** of R if every open disk centered at (x_0, y_0) contains points inside R *and* points outside R. By definition, a region must contain its interior points, but it need not contain its boundary points. If a region contains all its boundary points, the region is **closed.** A region that contains some but not all of its boundary points is neither open nor closed.

FOR FURTHER INFORMATION For more information on Sonya Kovalevsky, see the article "S. Kovalevsky: A Mathematical Lesson" by Karen D. Rappaport in *The American Mathematical Monthly*. To view this article, go to the website *www.matharticles.com*.

Limit of a Function of Two Variables

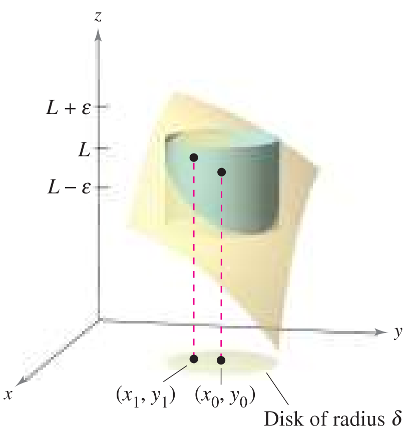
DEFINITION OF THE LIMIT OF A FUNCTION OF TWO VARIABLES

Let f be a function of two variables defined, except possibly at (x_0, y_0), on an open disk centered at (x_0, y_0), and let L be a real number. Then

$$\lim_{(x, y)\to(x_0, y_0)} f(x, y) = L$$

if for each $\varepsilon > 0$ there corresponds a $\delta > 0$ such that

$$|f(x, y) - L| < \varepsilon \quad \text{whenever} \quad 0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta.$$



For any (x, y) in the disk of radius δ, the value $f(x, y)$ lies between $L + \varepsilon$ and $L - \varepsilon$.
Figure 13.20

NOTE Graphically, this definition of a limit implies that for any point $(x, y) \neq (x_0, y_0)$ in the disk of radius δ, the value $f(x, y)$ lies between $L + \varepsilon$ and $L - \varepsilon$, as shown in Figure 13.20. ■

The definition of the limit of a function of two variables is similar to the definition of the limit of a function of a single variable, yet there is a critical difference. To determine whether a function of a single variable has a limit, you need only test the approach from two directions—from the right and from the left. If the function approaches the same limit from the right and from the left, you can conclude that the limit exists. However, for a function of two variables, the statement

$$(x, y) \to (x_0, y_0)$$

means that the point (x, y) is allowed to approach (x_0, y_0) from any direction. If the value of

$$\lim_{(x, y)\to(x_0, y_0)} f(x, y)$$

is not the same for all possible approaches, or **paths,** to (x_0, y_0), the limit does not exist.

EXAMPLE 1 Verifying a Limit by the Definition

Show that

$$\lim_{(x, y)\to(a, b)} x = a.$$

Solution Let $f(x, y) = x$ and $L = a$. You need to show that for each $\varepsilon > 0$, there exists a δ-neighborhood about (a, b) such that

$$|f(x, y) - L| = |x - a| < \varepsilon$$

whenever $(x, y) \neq (a, b)$ lies in the neighborhood. You can first observe that from

$$0 < \sqrt{(x - a)^2 + (y - b)^2} < \delta$$

it follows that

$$\begin{aligned} |f(x, y) - a| &= |x - a| \\ &= \sqrt{(x - a)^2} \\ &\leq \sqrt{(x - a)^2 + (y - b)^2} \\ &< \delta. \end{aligned}$$

So, you can choose $\delta = \varepsilon$, and the limit is verified. ■

Limits of functions of several variables have the same properties regarding sums, differences, products, and quotients as do limits of functions of single variables. (See Theorem 1.2 in Section 1.3.) Some of these properties are used in the next example.

EXAMPLE 2 Verifying a Limit

Evaluate $\displaystyle\lim_{(x, y)\to(1, 2)} \frac{5x^2y}{x^2 + y^2}$.

Solution By using the properties of limits of products and sums, you obtain

$$\begin{aligned}\lim_{(x, y)\to(1, 2)} 5x^2y &= 5(1^2)(2)\\ &= 10\end{aligned}$$

and

$$\begin{aligned}\lim_{(x, y)\to(1, 2)} (x^2 + y^2) &= (1^2 + 2^2)\\ &= 5.\end{aligned}$$

Because the limit of a quotient is equal to the quotient of the limits (and the denominator is not 0), you have

$$\begin{aligned}\lim_{(x, y)\to(1, 2)} \frac{5x^2y}{x^2 + y^2} &= \frac{10}{5}\\ &= 2.\end{aligned}$$

EXAMPLE 3 Verifying a Limit

Evaluate $\displaystyle\lim_{(x, y)\to(0, 0)} \frac{5x^2y}{x^2 + y^2}$.

Solution In this case, the limits of the numerator and of the denominator are both 0, and so you cannot determine the existence (or nonexistence) of a limit by taking the limits of the numerator and denominator separately and then dividing. However, from the graph of f in Figure 13.21, it seems reasonable that the limit might be 0. So, you can try applying the definition to $L = 0$. First, note that

$$|y| \le \sqrt{x^2 + y^2} \qquad \text{and} \qquad \frac{x^2}{x^2 + y^2} \le 1.$$

Then, in a δ-neighborhood about $(0, 0)$, you have $0 < \sqrt{x^2 + y^2} < \delta$, and it follows that, for $(x, y) \ne (0, 0)$,

$$\begin{aligned}|f(x, y) - 0| &= \left|\frac{5x^2y}{x^2 + y^2}\right|\\ &= 5|y|\left(\frac{x^2}{x^2 + y^2}\right)\\ &\le 5|y|\\ &\le 5\sqrt{x^2 + y^2}\\ &< 5\delta.\end{aligned}$$

So, you can choose $\delta = \varepsilon/5$ and conclude that

$$\lim_{(x, y)\to(0, 0)} \frac{5x^2y}{x^2 + y^2} = 0.$$

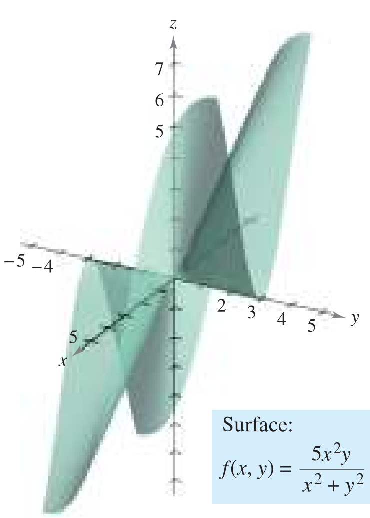


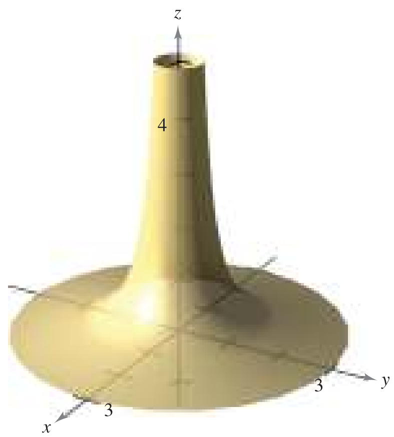
Figure 13.21

For some functions, it is easy to recognize that a limit does not exist. For instance, it is clear that the limit

$$\lim_{(x, y)\to(0, 0)} \frac{1}{x^2 + y^2}$$

does not exist because the values of $f(x, y)$ increase without bound as (x, y) approaches $(0, 0)$ along *any* path (see Figure 13.22).

For other functions, it is not so easy to recognize that a limit does not exist. For instance, the next example describes a limit that does not exist because the function approaches different values along different paths.



$\displaystyle\lim_{(x, y)\to(0, 0)} \frac{1}{x^2 + y^2}$ does not exist.

Figure 13.22

EXAMPLE 4 A Limit That Does Not Exist

Show that the following limit does not exist.

$$\lim_{(x, y)\to(0, 0)} \left(\frac{x^2 - y^2}{x^2 + y^2}\right)^2$$

Solution The domain of the function given by

$$f(x, y) = \left(\frac{x^2 - y^2}{x^2 + y^2}\right)^2$$

consists of all points in the xy-plane except for the point $(0, 0)$. To show that the limit as (x, y) approaches $(0, 0)$ does not exist, consider approaching $(0, 0)$ along two different "paths," as shown in Figure 13.23. Along the x-axis, every point is of the form $(x, 0)$, and the limit along this approach is

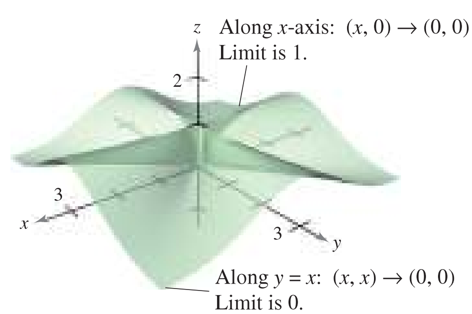
$$\lim_{(x, 0)\to(0, 0)} \left(\frac{x^2 - 0^2}{x^2 + 0^2}\right)^2 = \lim_{(x, 0)\to(0, 0)} 1^2 = 1. \qquad \text{Limit along } x\text{-axis}$$

However, if (x, y) approaches $(0, 0)$ along the line $y = x$, you obtain

$$\lim_{(x, x)\to(0, 0)} \left(\frac{x^2 - x^2}{x^2 + x^2}\right)^2 = \lim_{(x, x)\to(0, 0)} \left(\frac{0}{2x^2}\right)^2 = 0. \qquad \text{Limit along line } y = x$$

NOTE In Example 4, you could conclude that the limit does not exist because you found two approaches that produced different limits. If two approaches had produced the same limit, you still could not have concluded that the limit exists. To form such a conclusion, you must show that the limit is the same along *all* possible approaches.

This means that in any open disk centered at $(0, 0)$, there are points (x, y) at which f takes on the value 1, and other points at which f takes on the value 0. For instance, $f(x, y) = 1$ at the points $(1, 0)$, $(0.1, 0)$, $(0.01, 0)$, and $(0.001, 0)$, and $f(x, y) = 0$ at the points $(1, 1)$, $(0.1, 0.1)$, $(0.01, 0.01)$, and $(0.001, 0.001)$. So, f does not have a limit as $(x, y) \to (0, 0)$.



$\displaystyle\lim_{(x, y)\to(0, 0)} \left(\frac{x^2 - y^2}{x^2 + y^2}\right)^2$ does not exist.

Figure 13.23

Continuity of a Function of Two Variables

Notice in Example 2 that the limit of $f(x, y) = 5x^2y/(x^2 + y^2)$ as $(x, y) \to (1, 2)$ can be evaluated by direct substitution. That is, the limit is $f(1, 2) = 2$. In such cases the function f is said to be **continuous** at the point $(1, 2)$.

NOTE This definition of continuity can be extended to *boundary points* of the open region R by considering a special type of limit in which (x, y) is allowed to approach (x_0, y_0) along paths lying in the region R. This notion is similar to that of one-sided limits, as discussed in Chapter 1.

DEFINITION OF CONTINUITY OF A FUNCTION OF TWO VARIABLES

A function f of two variables is **continuous at a point (x_0, y_0)** in an open region R if $f(x_0, y_0)$ is equal to the limit of $f(x, y)$ as (x, y) approaches (x_0, y_0). That is,

$$\lim_{(x, y) \to (x_0, y_0)} f(x, y) = f(x_0, y_0).$$

The function f is **continuous in the open region R** if it is continuous at every point in R.

In Example 3, it was shown that the function

$$f(x, y) = \frac{5x^2y}{x^2 + y^2}$$

is not continuous at $(0, 0)$. However, because the limit at this point exists, you can remove the discontinuity by defining f at $(0, 0)$ as being equal to its limit there. Such a discontinuity is called **removable.** In Example 4, the function

$$f(x, y) = \left(\frac{x^2 - y^2}{x^2 + y^2}\right)^2$$

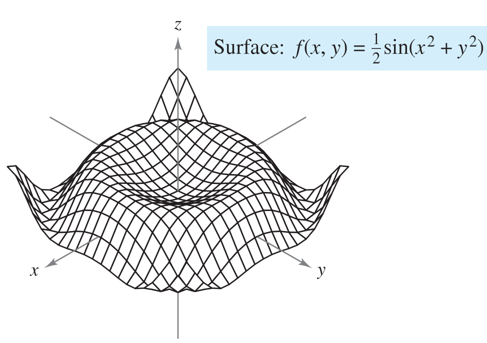
was also shown not to be continuous at $(0, 0)$, but this discontinuity is **nonremovable.**

THEOREM 13.1 CONTINUOUS FUNCTIONS OF TWO VARIABLES

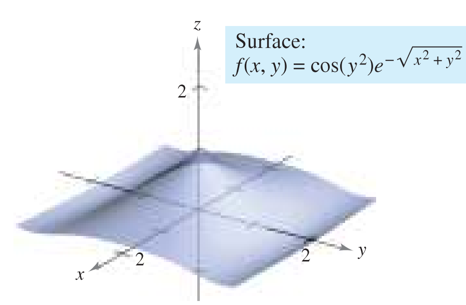
If k is a real number and f and g are continuous at (x_0, y_0), then the following functions are continuous at (x_0, y_0).

1. Scalar multiple: kf
2. Sum and difference: $f \pm g$
3. Product: fg
4. Quotient: f/g, if $g(x_0, y_0) \neq 0$

Theorem 13.1 establishes the continuity of *polynomial* and *rational* functions at every point in their domains. Furthermore, the continuity of other types of functions can be extended naturally from one to two variables. For instance, the functions whose graphs are shown in Figures 13.24 and 13.25 are continuous at every point in the plane.



The function f is continuous at every point in the plane.
Figure 13.24



The function f is continuous at every point in the plane.
Figure 13.25

EXPLORATION

Hold a spoon a foot or so from your eyes. Look at your image in the spoon. It should be upside down. Now, move the spoon closer and closer to one eye. At some point, your image will be right side up. Could it be that your image is being continuously deformed? Talk about this question and the general meaning of continuity with other members of your class. (This exploration was suggested by Irvin Roy Hentzel, Iowa State University.)

The next theorem states conditions under which a composite function is continuous.

THEOREM 13.2 CONTINUITY OF A COMPOSITE FUNCTION

If h is continuous at (x_0, y_0) and g is continuous at $h(x_0, y_0)$, then the composite function given by $(g \circ h)(x, y) = g(h(x, y))$ is continuous at (x_0, y_0). That is,

$$\lim_{(x, y)\to(x_0, y_0)} g(h(x, y)) = g(h(x_0, y_0)).$$

NOTE Note in Theorem 13.2 that h is a function of two variables and g is a function of one variable. ■

EXAMPLE 5 Testing for Continuity

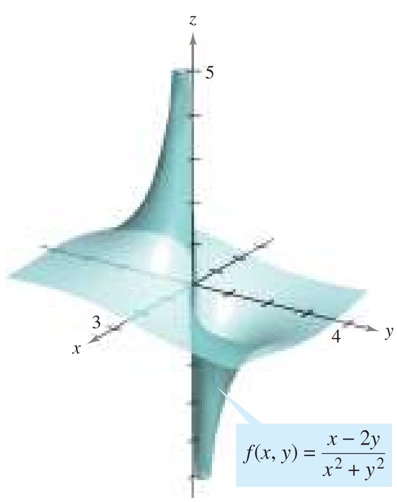
Discuss the continuity of each function.

a. $f(x, y) = \dfrac{x - 2y}{x^2 + y^2}$ **b.** $g(x, y) = \dfrac{2}{y - x^2}$

Solution

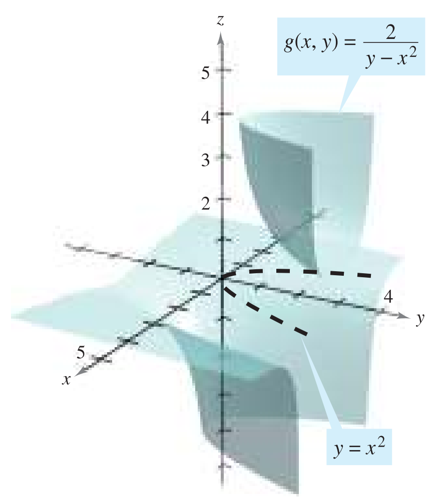
a. Because a rational function is continuous at every point in its domain, you can conclude that f is continuous at each point in the xy-plane except at $(0, 0)$, as shown in Figure 13.26.

b. The function given by $g(x, y) = 2/(y - x^2)$ is continuous except at the points at which the denominator is 0, $y - x^2 = 0$. So, you can conclude that the function is continuous at all points except those lying on the parabola $y = x^2$. Inside this parabola, you have $y > x^2$, and the surface represented by the function lies above the xy-plane, as shown in Figure 13.27. Outside the parabola, $y < x^2$, and the surface lies below the xy-plane.



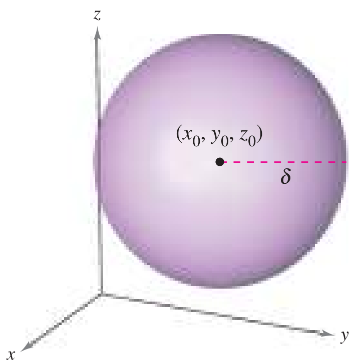
The function f is not continuous at $(0, 0)$.

Figure 13.26



The function g is not continuous on the parabola $y = x^2$.

Figure 13.27 ■



Open sphere in space
Figure 13.28

Continuity of a Function of Three Variables

The preceding definitions of limits and continuity can be extended to functions of three variables by considering points (x, y, z) within the *open sphere*

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 < \delta^2. \quad \text{Open sphere}$$

The radius of this sphere is δ, and the sphere is centered at (x_0, y_0, z_0), as shown in Figure 13.28. A point (x_0, y_0, z_0) in a region R in space is an **interior point** of R if there exists a δ-sphere about (x_0, y_0, z_0) that lies entirely in R. If every point in R is an interior point, then R is called **open.**

DEFINITION OF CONTINUITY OF A FUNCTION OF THREE VARIABLES

A function f of three variables is **continuous at a point (x_0, y_0, z_0)** in an open region R if $f(x_0, y_0, z_0)$ is defined and is equal to the limit of $f(x, y, z)$ as (x, y, z) approaches (x_0, y_0, z_0). That is,

$$\lim_{(x, y, z)\to(x_0, y_0, z_0)} f(x, y, z) = f(x_0, y_0, z_0).$$

The function f is **continuous in the open region R** if it is continuous at every point in R.

EXAMPLE 6 Testing Continuity of a Function of Three Variables

The function

$$f(x, y, z) = \frac{1}{x^2 + y^2 - z}$$

is continuous at each point in space except at the points on the paraboloid given by $z = x^2 + y^2$.

13.2 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–4, use the definition of the limit of a function of two variables to verify the limit.

1. $\lim_{(x, y)\to(1, 0)} x = 1$
2. $\lim_{(x, y)\to(4, -1)} x = 4$
3. $\lim_{(x, y)\to(1, -3)} y = -3$
4. $\lim_{(x, y)\to(a, b)} y = b$

In Exercises 5–8, find the indicated limit by using the limits

$$\lim_{(x, y)\to(a, b)} f(x, y) = 4 \quad \text{and} \quad \lim_{(x, y)\to(a, b)} g(x, y) = 3.$$

5. $\lim_{(x, y)\to(a, b)} [f(x, y) - g(x, y)]$
6. $\lim_{(x, y)\to(a, b)} \left[\frac{5f(x, y)}{g(x, y)}\right]$
7. $\lim_{(x, y)\to(a, b)} [f(x, y)g(x, y)]$
8. $\lim_{(x, y)\to(a, b)} \left[\frac{f(x, y) + g(x, y)}{f(x, y)}\right]$

In Exercises 9–22, find the limit and discuss the continuity of the function.

9. $\lim_{(x, y)\to(2, 1)} (2x^2 + y)$
10. $\lim_{(x, y)\to(0, 0)} (x + 4y + 1)$
11. $\lim_{(x, y)\to(1, 2)} e^{xy}$
12. $\lim_{(x, y)\to(2, 4)} \frac{x + y}{x^2 + 1}$
13. $\lim_{(x, y)\to(0, 2)} \frac{x}{y}$
14. $\lim_{(x, y)\to(-1, 2)} \frac{x + y}{x - y}$
15. $\lim_{(x, y)\to(1, 1)} \frac{xy}{x^2 + y^2}$
16. $\lim_{(x, y)\to(1, 1)} \frac{x}{\sqrt{x + y}}$
17. $\lim_{(x, y)\to(\pi/4, 2)} y \cos xy$
18. $\lim_{(x, y)\to(2\pi, 4)} \sin \frac{x}{y}$
19. $\lim_{(x, y)\to(0, 1)} \frac{\arcsin xy}{1 - xy}$
20. $\lim_{(x, y)\to(0, 1)} \frac{\arccos (x/y)}{1 + xy}$
21. $\lim_{(x, y, z)\to(1, 3, 4)} \sqrt{x + y + z}$
22. $\lim_{(x, y, z)\to(-2, 1, 0)} xe^{yz}$

In Exercises 23–36, find the limit (if it exists). If the limit does not exist, explain why.

23. $\displaystyle\lim_{(x,y)\to(1,1)} \frac{xy-1}{1+xy}$

24. $\displaystyle\lim_{(x,y)\to(1,-1)} \frac{x^2y}{1+xy^2}$

25. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{1}{x+y}$

26. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{1}{x^2y^2}$

27. $\displaystyle\lim_{(x,y)\to(2,2)} \frac{x^2-y^2}{x-y}$

28. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^4-4y^4}{x^2+2y^2}$

29. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x-y}{\sqrt{x}-\sqrt{y}}$

30. $\displaystyle\lim_{(x,y)\to(2,1)} \frac{x-y-1}{\sqrt{x-y}-1}$

31. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x+y}{x^2+y}$

32. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x}{x^2-y^2}$

33. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2}{(x^2+1)(y^2+1)}$

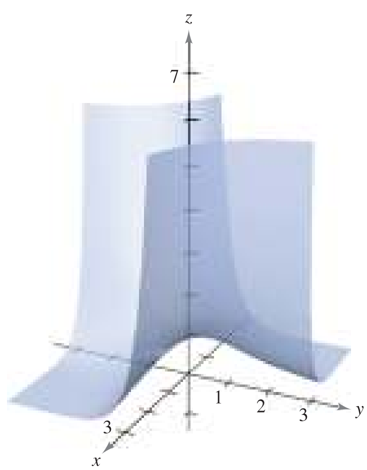
34. $\displaystyle\lim_{(x,y)\to(0,0)} \ln(x^2+y^2)$

35. $\displaystyle\lim_{(x,y,z)\to(0,0,0)} \frac{xy+yz+xz}{x^2+y^2+z^2}$

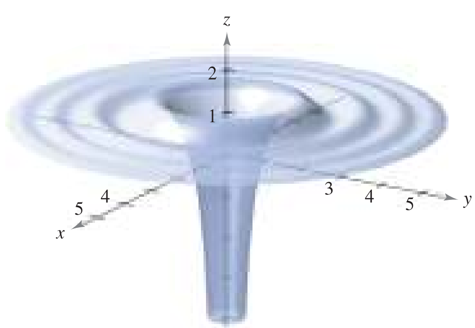
36. $\displaystyle\lim_{(x,y,z)\to(0,0,0)} \frac{xy+yz^2+xz^2}{x^2+y^2+z^2}$

In Exercises 37 and 38, discuss the continuity of the function and evaluate the limit of $f(x, y)$ (if it exists) as $(x, y) \to (0, 0)$.

37. $f(x, y) = e^{xy}$

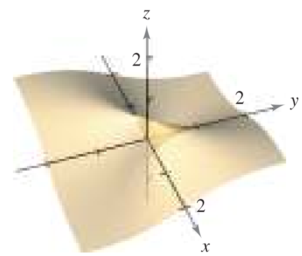


38. $f(x, y) = 1 - \dfrac{\cos(x^2+y^2)}{x^2+y^2}$



In Exercises 39–42, use a graphing utility to make a table showing the values of $f(x, y)$ at the given points for each path. Use the result to make a conjecture about the limit of $f(x, y)$ as $(x, y) \to (0, 0)$. Determine whether the limit exists analytically and discuss the continuity of the function.

39. $f(x, y) = \dfrac{xy}{x^2+y^2}$



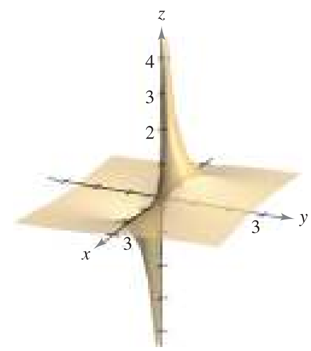
Path: $y = 0$

Points: $(1, 0)$, $(0.5, 0)$, $(0.1, 0)$, $(0.01, 0)$, $(0.001, 0)$

Path: $y = x$

Points: $(1, 1)$, $(0.5, 0.5)$, $(0.1, 0.1)$, $(0.01, 0.01)$, $(0.001, 0.001)$

40. $f(x, y) = \dfrac{y}{x^2+y^2}$



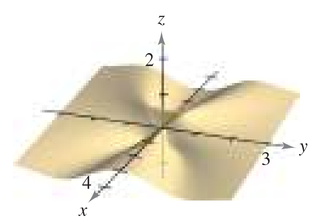
Path: $y = 0$

Points: $(1, 0)$, $(0.5, 0)$, $(0.1, 0)$, $(0.01, 0)$, $(0.001, 0)$

Path: $y = x$

Points: $(1, 1)$, $(0.5, 0.5)$, $(0.1, 0.1)$, $(0.01, 0.01)$, $(0.001, 0.001)$

41. $f(x, y) = -\dfrac{xy^2}{x^2+y^4}$



Path: $x = y^2$

Points: $(1, 1)$, $(0.25, 0.5)$, $(0.01, 0.1)$, $(0.0001, 0.01)$, $(0.000001, 0.001)$

Path: $x = -y^2$

Points: $(-1, 1)$, $(-0.25, 0.5)$, $(-0.01, 0.1)$, $(-0.0001, 0.01)$, $(-0.000001, 0.001)$

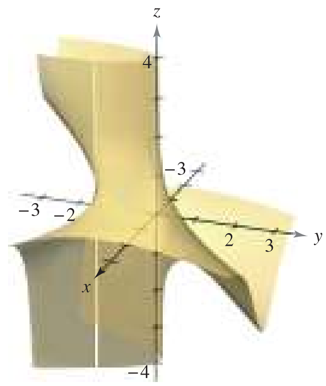
42. $f(x, y) = \dfrac{2x - y^2}{2x^2 + y}$

Path: $y = 0$

Points: $(1, 0)$, $(0.25, 0)$, $(0.01, 0)$, $(0.001, 0)$, $(0.000001, 0)$

Path: $y = x$

Points: $(1, 1)$, $(0.25, 0.25)$, $(0.01, 0.01)$, $(0.001, 0.001)$, $(0.0001, 0.0001)$



In Exercises 43–46, discuss the continuity of the functions f and g. Explain any differences.

43. $f(x, y) = \begin{cases} \dfrac{x^4 - y^4}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$

$g(x, y) = \begin{cases} \dfrac{x^4 - y^4}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 1, & (x, y) = (0, 0) \end{cases}$

44. $f(x, y) = \begin{cases} \dfrac{4x^4 - y^4}{2x^2 + y^2}, & (x, y) \neq (0, 0) \\ -1, & (x, y) = (0, 0) \end{cases}$

$g(x, y) = \begin{cases} \dfrac{4x^4 - y^4}{2x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$

45. $f(x, y) = \begin{cases} \dfrac{4x^2y^2}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$

$g(x, y) = \begin{cases} \dfrac{4x^2y^2}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 2, & (x, y) = (0, 0) \end{cases}$

46. $f(x, y) = \begin{cases} \dfrac{x^2 + 2xy^2 + y^2}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$

$g(x, y) = \begin{cases} \dfrac{x^2 + 2xy^2 + y^2}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 1, & (x, y) = (0, 0) \end{cases}$

CAS **In Exercises 47–52, use a computer algebra system to graph the function and find $\lim\limits_{(x, y) \to (0, 0)} f(x, y)$ (if it exists).**

47. $f(x, y) = \sin x + \sin y$

48. $f(x, y) = \sin \dfrac{1}{x} + \cos \dfrac{1}{x}$

49. $f(x, y) = \dfrac{x^2y}{x^4 + 2y^2}$

50. $f(x, y) = \dfrac{x^2 + y^2}{x^2y}$

51. $f(x, y) = \dfrac{5xy}{x^2 + 2y^2}$

52. $f(x, y) = \dfrac{6xy}{x^2 + y^2 + 1}$

In Exercises 53–58, use polar coordinates to find the limit. [*Hint:* Let $x = r\cos\theta$ and $y = r\sin\theta$, and note that $(x, y) \to (0, 0)$ implies $r \to 0$.]

53. $\lim\limits_{(x, y) \to (0, 0)} \dfrac{xy^2}{x^2 + y^2}$

54. $\lim\limits_{(x, y) \to (0, 0)} \dfrac{x^3 + y^3}{x^2 + y^2}$

55. $\lim\limits_{(x, y) \to (0, 0)} \dfrac{x^2y^2}{x^2 + y^2}$

56. $\lim\limits_{(x, y) \to (0, 0)} \dfrac{x^2 - y^2}{\sqrt{x^2 + y^2}}$

57. $\lim\limits_{(x, y) \to (0, 0)} \cos(x^2 + y^2)$

58. $\lim\limits_{(x, y) \to (0, 0)} \sin\sqrt{x^2 + y^2}$

In Exercises 59–62, use polar coordinates and L'Hôpital's Rule to find the limit.

59. $\lim\limits_{(x, y) \to (0, 0)} \dfrac{\sin\sqrt{x^2 + y^2}}{\sqrt{x^2 + y^2}}$

60. $\lim\limits_{(x, y) \to (0, 0)} \dfrac{\sin(x^2 + y^2)}{x^2 + y^2}$

61. $\lim\limits_{(x, y) \to (0, 0)} \dfrac{1 - \cos(x^2 + y^2)}{x^2 + y^2}$

62. $\lim\limits_{(x, y) \to (0, 0)} (x^2 + y^2)\ln(x^2 + y^2)$

In Exercises 63–68, discuss the continuity of the function.

63. $f(x, y, z) = \dfrac{1}{\sqrt{x^2 + y^2 + z^2}}$

64. $f(x, y, z) = \dfrac{z}{x^2 + y^2 - 4}$

65. $f(x, y, z) = \dfrac{\sin z}{e^x + e^y}$

66. $f(x, y, z) = xy\sin z$

67. $f(x, y) = \begin{cases} \dfrac{\sin xy}{xy}, & xy \neq 0 \\ 1, & xy = 0 \end{cases}$

68. $f(x, y) = \begin{cases} \dfrac{\sin(x^2 - y^2)}{x^2 - y^2}, & x^2 \neq y^2 \\ 1, & x^2 = y^2 \end{cases}$

In Exercises 69–72, discuss the continuity of the composite function $f \circ g$.

69. $f(t) = t^2$

$g(x, y) = 2x - 3y$

70. $f(t) = \dfrac{1}{t}$

$g(x, y) = x^2 + y^2$

71. $f(t) = \dfrac{1}{t}$

$g(x, y) = 2x - 3y$

72. $f(t) = \dfrac{1}{1 - t}$

$g(x, y) = x^2 + y^2$

In Exercises 73–78, find each limit.

(a) $\lim\limits_{\Delta x \to 0} \dfrac{f(x + \Delta x, y) - f(x, y)}{\Delta x}$

(b) $\lim\limits_{\Delta y \to 0} \dfrac{f(x, y + \Delta y) - f(x, y)}{\Delta y}$

73. $f(x, y) = x^2 - 4y$

74. $f(x, y) = x^2 + y^2$

75. $f(x, y) = \dfrac{x}{y}$

76. $f(x, y) = \dfrac{1}{x + y}$

77. $f(x, y) = 3x + xy - 2y$

78. $f(x, y) = \sqrt{y}\,(y + 1)$

True or False? **In Exercises 79–82, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

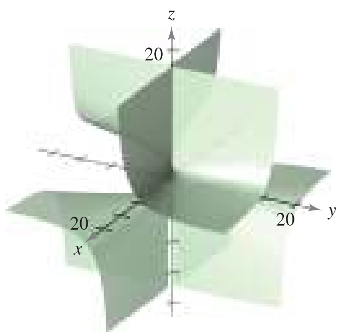
79. If $\lim_{(x,y)\to(0,0)} f(x, y) = 0$, then $\lim_{x\to 0} f(x, 0) = 0$.

80. If $\lim_{(x,y)\to(0,0)} f(0, y) = 0$, then $\lim_{(x,y)\to(0,0)} f(x, y) = 0$.

81. If f is continuous for all nonzero x and y, and $f(0, 0) = 0$, then $\lim_{(x,y)\to(0,0)} f(x, y) = 0$.

82. If g and h are continuous functions of x and y, and $f(x, y) = g(x) + h(y)$, then f is continuous.

83. Consider $\lim_{(x,y)\to(0,0)} \dfrac{x^2 + y^2}{xy}$ (see figure).

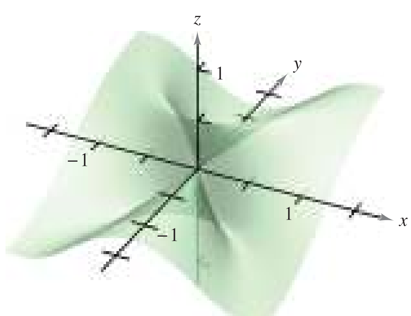


(a) Determine (if possible) the limit along any line of the form $y = ax$.

(b) Determine (if possible) the limit along the parabola $y = x^2$.

(c) Does the limit exist? Explain.

84. Consider $\lim_{(x,y)\to(0,0)} \dfrac{x^2y}{x^4 + y^2}$ (see figure).



(a) Determine (if possible) the limit along any line of the form $y = ax$.

(b) Determine (if possible) the limit along the parabola $y = x^2$.

(c) Does the limit exist? Explain.

In Exercises 85 and 86, use spherical coordinates to find the limit. [*Hint:* Let $x = \rho \sin\phi \cos\theta$, $y = \rho \sin\phi \sin\theta$, and $z = \rho \cos\phi$, and note that $(x, y, z)\to(0, 0, 0)$ implies $\rho\to 0^+$.]

85. $\lim_{(x,y,z)\to(0,0,0)} \dfrac{xyz}{x^2 + y^2 + z^2}$

86. $\lim_{(x,y,z)\to(0,0,0)} \tan^{-1}\left[\dfrac{1}{x^2 + y^2 + z^2}\right]$

87. Find the following limit.

$$\lim_{(x,y)\to(0,1)} \tan^{-1}\left[\frac{x^2 + 1}{x^2 + (y - 1)^2}\right]$$

88. For the function

$$f(x, y) = xy\left(\frac{x^2 - y^2}{x^2 + y^2}\right)$$

define $f(0, 0)$ such that f is continuous at the origin.

89. Prove that

$$\lim_{(x,y)\to(a,b)} [f(x, y) + g(x, y)] = L_1 + L_2$$

where $f(x, y)$ approaches L_1 and $g(x, y)$ approaches L_2 as $(x, y) \to (a, b)$.

90. Prove that if f is continuous and $f(a, b) < 0$, there exists a δ-neighborhood about (a, b) such that $f(x, y) < 0$ for every point (x, y) in the neighborhood.

WRITING ABOUT CONCEPTS

91. Define the limit of a function of two variables. Describe a method for showing that

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y)$$

does not exist.

92. State the definition of continuity of a function of two variables.

93. Determine whether each of the following statements is true or false. Explain your reasoning.

(a) If $\lim_{(x,y)\to(2,3)} f(x, y) = 4$, then $\lim_{x\to 2} f(x, 3) = 4$.

(b) If $\lim_{x\to 2} f(x, 3) = 4$, then $\lim_{(x,y)\to(2,3)} f(x, y) = 4$.

(c) If $\lim_{x\to 2} f(x, 3) = \lim_{y\to 3} f(2, y) = 4$, then $\lim_{(x,y)\to(2,3)} f(x, y) = 4$.

(d) If $\lim_{(x,y)\to(0,0)} f(x, y) = 0$, then for any real number k, $\lim_{(x,y)\to(0,0)} f(kx, y) = 0$.

CAPSTONE

94. (a) If $f(2, 3) = 4$, can you conclude anything about $\lim_{(x,y)\to(2,3)} f(x, y)$? Give reasons for your answer.

(b) If $\lim_{(x,y)\to(2,3)} f(x, y) = 4$, can you conclude anything about $f(2, 3)$? Give reasons for your answer.

13.3 Partial Derivatives

- Find and use partial derivatives of a function of two variables.
- Find and use partial derivatives of a function of three or more variables.
- Find higher-order partial derivatives of a function of two or three variables.

Mary Evans Picture Library

JEAN LE ROND D'ALEMBERT (1717–1783)

The introduction of partial derivatives followed Newton's and Leibniz's work in calculus by several years. Between 1730 and 1760, Leonhard Euler and Jean Le Rond d'Alembert separately published several papers on dynamics, in which they established much of the theory of partial derivatives. These papers used functions of two or more variables to study problems involving equilibrium, fluid motion, and vibrating strings.

Partial Derivatives of a Function of Two Variables

In applications of functions of several variables, the question often arises, "How will the value of a function be affected by a change in one of its independent variables?" You can answer this by considering the independent variables one at a time. For example, to determine the effect of a catalyst in an experiment, a chemist could conduct the experiment several times using varying amounts of the catalyst, while keeping constant other variables such as temperature and pressure. You can use a similar procedure to determine the rate of change of a function f with respect to one of its several independent variables. This process is called **partial differentiation,** and the result is referred to as the **partial derivative** of f with respect to the chosen independent variable.

DEFINITION OF PARTIAL DERIVATIVES OF A FUNCTION OF TWO VARIABLES

If $z = f(x, y)$, then the **first partial derivatives** of f with respect to x and y are the functions f_x and f_y defined by

$$f_x(x, y) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}$$

$$f_y(x, y) = \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y}$$

provided the limits exist.

This definition indicates that if $z = f(x, y)$, then to find f_x you *consider y constant* and differentiate with respect to x. Similarly, to find f_y, you *consider x constant* and differentiate with respect to y.

EXAMPLE 1 Finding Partial Derivatives

Find the partial derivatives f_x and f_y for the function

$$f(x, y) = 3x - x^2y^2 + 2x^3y.$$

Solution Considering y to be constant and differentiating with respect to x produces

$$f(x, y) = 3x - x^2y^2 + 2x^3y \qquad \text{Write original function.}$$

$$f_x(x, y) = 3 - 2xy^2 + 6x^2y. \qquad \text{Partial derivative with respect to } x$$

Considering x to be constant and differentiating with respect to y produces

$$f(x, y) = 3x - x^2y^2 + 2x^3y \qquad \text{Write original function.}$$

$$f_y(x, y) = -2x^2y + 2x^3. \qquad \text{Partial derivative with respect to } y$$

■

NOTATION FOR FIRST PARTIAL DERIVATIVES

For $z = f(x, y)$, the partial derivatives f_x and f_y are denoted by

$$\frac{\partial}{\partial x} f(x, y) = f_x(x, y) = z_x = \frac{\partial z}{\partial x}$$

and

$$\frac{\partial}{\partial y} f(x, y) = f_y(x, y) = z_y = \frac{\partial z}{\partial y}.$$

The first partials evaluated at the point (a, b) are denoted by

$$\left.\frac{\partial z}{\partial x}\right|_{(a,b)} = f_x(a, b) \quad \text{and} \quad \left.\frac{\partial z}{\partial y}\right|_{(a,b)} = f_y(a, b).$$

EXAMPLE 2 Finding and Evaluating Partial Derivatives

For $f(x, y) = xe^{x^2 y}$, find f_x and f_y, and evaluate each at the point $(1, \ln 2)$.

Solution Because

$$f_x(x, y) = xe^{x^2 y}(2xy) + e^{x^2 y} \qquad \text{Partial derivative with respect to } x$$

the partial derivative of f with respect to x at $(1, \ln 2)$ is

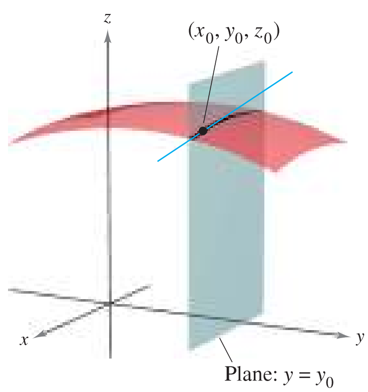
$$\begin{aligned} f_x(1, \ln 2) &= e^{\ln 2}(2 \ln 2) + e^{\ln 2} \\ &= 4 \ln 2 + 2. \end{aligned}$$

Because

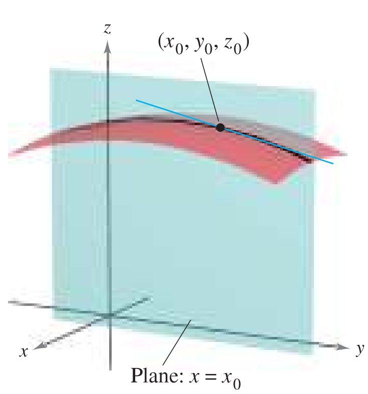
$$\begin{aligned} f_y(x, y) &= xe^{x^2 y}(x^2) \\ &= x^3 e^{x^2 y} \qquad \text{Partial derivative with respect to } y \end{aligned}$$

the partial derivative of f with respect to y at $(1, \ln 2)$ is

$$\begin{aligned} f_y(1, \ln 2) &= e^{\ln 2} \\ &= 2. \end{aligned}$$



$\frac{\partial f}{\partial x}$ = slope in x-direction
Figure 13.29



$\frac{\partial f}{\partial y}$ = slope in y-direction
Figure 13.30

The partial derivatives of a function of two variables, $z = f(x, y)$, have a useful geometric interpretation. If $y = y_0$, then $z = f(x, y_0)$ represents the curve formed by intersecting the surface $z = f(x, y)$ with the plane $y = y_0$, as shown in Figure 13.29. Therefore,

$$f_x(x_0, y_0) = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x, y_0) - f(x_0, y_0)}{\Delta x}$$

represents the slope of this curve at the point $(x_0, y_0, f(x_0, y_0))$. Note that both the curve and the tangent line lie in the plane $y = y_0$. Similarly,

$$f_y(x_0, y_0) = \lim_{\Delta y \to 0} \frac{f(x_0, y_0 + \Delta y) - f(x_0, y_0)}{\Delta y}$$

represents the slope of the curve given by the intersection of $z = f(x, y)$ and the plane $x = x_0$ at $(x_0, y_0, f(x_0, y_0))$, as shown in Figure 13.30.

Informally, the values of $\partial f/\partial x$ and $\partial f/\partial y$ at the point (x_0, y_0, z_0) denote the **slopes of the surface in the x- and y-directions,** respectively.

EXAMPLE 3 Finding the Slopes of a Surface in the *x*- and *y*-Directions

Find the slopes in the x-direction and in the y-direction of the surface given by

$$f(x, y) = -\frac{x^2}{2} - y^2 + \frac{25}{8}$$

at the point $\left(\frac{1}{2}, 1, 2\right)$.

Solution The partial derivatives of f with respect to x and y are

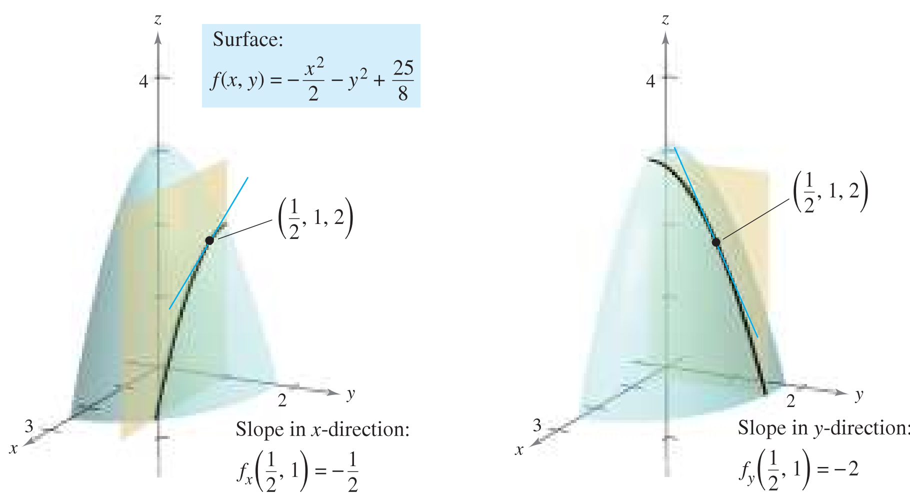
$$f_x(x, y) = -x \quad \text{and} \quad f_y(x, y) = -2y. \qquad \text{Partial derivatives}$$

So, in the x-direction, the slope is

$$f_x\left(\frac{1}{2}, 1\right) = -\frac{1}{2} \qquad \text{Figure 13.31(a)}$$

and in the y-direction, the slope is

$$f_y\left(\frac{1}{2}, 1\right) = -2. \qquad \text{Figure 13.31(b)}$$



(a) **(b)**

Figure 13.31

EXAMPLE 4 Finding the Slopes of a Surface in the *x*- and *y*-Directions

Find the slopes of the surface given by

$$f(x, y) = 1 - (x - 1)^2 - (y - 2)^2$$

at the point $(1, 2, 1)$ in the x-direction and in the y-direction.

Solution The partial derivatives of f with respect to x and y are

$$f_x(x, y) = -2(x - 1) \quad \text{and} \quad f_y(x, y) = -2(y - 2). \qquad \text{Partial derivatives}$$

So, at the point $(1, 2, 1)$, the slopes in the x- and y-directions are

$$f_x(1, 2) = -2(1 - 1) = 0 \quad \text{and} \quad f_y(1, 2) = -2(2 - 2) = 0$$

as shown in Figure 13.32.

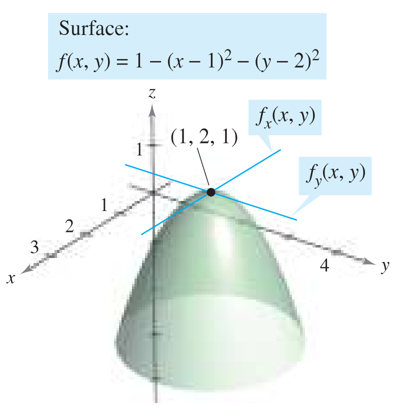
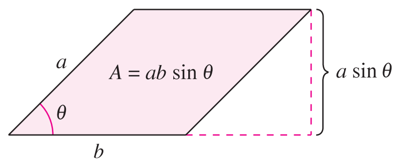


Figure 13.32

No matter how many variables are involved, partial derivatives can be interpreted as *rates of change*.

EXAMPLE 5 Using Partial Derivatives to Find Rates of Change



The area of the parallelogram is $ab \sin \theta$.
Figure 13.33

The area of a parallelogram with adjacent sides a and b and included angle θ is given by $A = ab \sin \theta$, as shown in Figure 13.33.

a. Find the rate of change of A with respect to a for $a = 10$, $b = 20$, and $\theta = \dfrac{\pi}{6}$.

b. Find the rate of change of A with respect to θ for $a = 10$, $b = 20$, and $\theta = \dfrac{\pi}{6}$.

Solution

a. To find the rate of change of the area with respect to a, hold b and θ constant and differentiate with respect to a to obtain

$$\frac{\partial A}{\partial a} = b \sin \theta \qquad \text{Find partial with respect to } a.$$

$$\frac{\partial A}{\partial a} = 20 \sin \frac{\pi}{6} = 10. \qquad \text{Substitute for } b \text{ and } \theta.$$

b. To find the rate of change of the area with respect to θ, hold a and b constant and differentiate with respect to θ to obtain

$$\frac{\partial A}{\partial \theta} = ab \cos \theta \qquad \text{Find partial with respect to } \theta.$$

$$\frac{\partial A}{\partial \theta} = 200 \cos \frac{\pi}{6} = 100\sqrt{3}. \qquad \text{Substitute for } a, b, \text{ and } \theta.$$

■

Partial Derivatives of a Function of Three or More Variables

The concept of a partial derivative can be extended naturally to functions of three or more variables. For instance, if $w = f(x, y, z)$, there are three partial derivatives, each of which is formed by holding two of the variables constant. That is, to define the partial derivative of w with respect to x, consider y and z to be constant and differentiate with respect to x. A similar process is used to find the derivatives of w with respect to y and with respect to z.

$$\frac{\partial w}{\partial x} = f_x(x, y, z) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y, z) - f(x, y, z)}{\Delta x}$$

$$\frac{\partial w}{\partial y} = f_y(x, y, z) = \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y, z) - f(x, y, z)}{\Delta y}$$

$$\frac{\partial w}{\partial z} = f_z(x, y, z) = \lim_{\Delta z \to 0} \frac{f(x, y, z + \Delta z) - f(x, y, z)}{\Delta z}$$

In general, if $w = f(x_1, x_2, \ldots, x_n)$, there are n partial derivatives denoted by

$$\frac{\partial w}{\partial x_k} = f_{x_k}(x_1, x_2, \ldots, x_n), \quad k = 1, 2, \ldots, n.$$

To find the partial derivative with respect to one of the variables, hold the other variables constant and differentiate with respect to the given variable.

EXAMPLE 6 Finding Partial Derivatives

a. To find the partial derivative of $f(x, y, z) = xy + yz^2 + xz$ with respect to z, consider x and y to be constant and obtain

$$\frac{\partial}{\partial z}[xy + yz^2 + xz] = 2yz + x.$$

b. To find the partial derivative of $f(x, y, z) = z \sin(xy^2 + 2z)$ with respect to z, consider x and y to be constant. Then, using the Product Rule, you obtain

$$\begin{aligned}\frac{\partial}{\partial z}[z \sin(xy^2 + 2z)] &= (z)\frac{\partial}{\partial z}[\sin(xy^2 + 2z)] + \sin(xy^2 + 2z)\frac{\partial}{\partial z}[z] \\ &= (z)[\cos(xy^2 + 2z)](2) + \sin(xy^2 + 2z) \\ &= 2z \cos(xy^2 + 2z) + \sin(xy^2 + 2z).\end{aligned}$$

c. To find the partial derivative of $f(x, y, z, w) = (x + y + z)/w$ with respect to w, consider x, y, and z to be constant and obtain

$$\frac{\partial}{\partial w}\left[\frac{x + y + z}{w}\right] = -\frac{x + y + z}{w^2}.$$

Higher-Order Partial Derivatives

As is true for ordinary derivatives, it is possible to take second, third, and higher-order partial derivatives of a function of several variables, provided such derivatives exist. Higher-order derivatives are denoted by the order in which the differentiation occurs. For instance, the function $z = f(x, y)$ has the following second partial derivatives.

1. Differentiate twice with respect to x:

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = f_{xx}.$$

2. Differentiate twice with respect to y:

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = f_{yy}.$$

3. Differentiate first with respect to x and then with respect to y:

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = f_{xy}.$$

4. Differentiate first with respect to y and then with respect to x:

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = f_{yx}.$$

The third and fourth cases are called **mixed partial derivatives.**

NOTE Note that the two types of notation for mixed partials have different conventions for indicating the order of differentiation.

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} \quad \text{Right-to-left order}$$

$$(f_x)_y = f_{xy} \quad \text{Left-to-right order}$$

You can remember the order by observing that in both notations you differentiate first with respect to the variable "nearest" f.

EXAMPLE 7 Finding Second Partial Derivatives

Find the second partial derivatives of $f(x, y) = 3xy^2 - 2y + 5x^2y^2$, and determine the value of $f_{xy}(-1, 2)$.

Solution Begin by finding the first partial derivatives with respect to x and y.

$$f_x(x, y) = 3y^2 + 10xy^2 \qquad \text{and} \qquad f_y(x, y) = 6xy - 2 + 10x^2y$$

Then, differentiate each of these with respect to x and y.

$$f_{xx}(x, y) = 10y^2 \qquad \text{and} \qquad f_{yy}(x, y) = 6x + 10x^2$$
$$f_{xy}(x, y) = 6y + 20xy \qquad \text{and} \qquad f_{yx}(x, y) = 6y + 20xy$$

At $(-1, 2)$, the value of f_{xy} is $f_{xy}(-1, 2) = 12 - 40 = -28$. ■

NOTE Notice in Example 7 that the two mixed partials are equal. Sufficient conditions for this occurrence are given in Theorem 13.3. ■

THEOREM 13.3 EQUALITY OF MIXED PARTIAL DERIVATIVES

If f is a function of x and y such that f_{xy} and f_{yx} are continuous on an open disk R, then, for every (x, y) in R,

$$f_{xy}(x, y) = f_{yx}(x, y).$$

Theorem 13.3 also applies to a function f of *three or more variables* so long as all second partial derivatives are continuous. For example, if $w = f(x, y, z)$ and all the second partial derivatives are continuous in an open region R, then at each point in R the order of differentiation in the mixed second partial derivatives is irrelevant. If the third partial derivatives of f are also continuous, the order of differentiation of the mixed third partial derivatives is irrelevant.

EXAMPLE 8 Finding Higher-Order Partial Derivatives

Show that $f_{xz} = f_{zx}$ and $f_{xzz} = f_{zxz} = f_{zzx}$ for the function given by

$$f(x, y, z) = ye^x + x \ln z.$$

Solution

First partials:

$$f_x(x, y, z) = ye^x + \ln z, \qquad f_z(x, y, z) = \frac{x}{z}$$

Second partials (note that the first two are equal):

$$f_{xz}(x, y, z) = \frac{1}{z}, \qquad f_{zx}(x, y, z) = \frac{1}{z}, \qquad f_{zz}(x, y, z) = -\frac{x}{z^2}$$

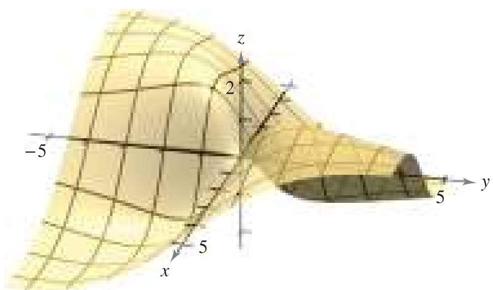
Third partials (note that all three are equal):

$$f_{xzz}(x, y, z) = -\frac{1}{z^2}, \qquad f_{zxz}(x, y, z) = -\frac{1}{z^2}, \qquad f_{zzx}(x, y, z) = -\frac{1}{z^2}$$ ■

13.3 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Think About It **In Exercises 1–4, use the graph of the surface to determine the sign of the indicated partial derivative.**



1. $f_x(4, 1)$
2. $f_y(-1, -2)$
3. $f_y(4, 1)$
4. $f_x(-1, -1)$

In Exercises 5–8, explain whether or not the Quotient Rule should be used to find the partial derivative. Do not differentiate.

5. $\dfrac{\partial}{\partial y}\left(\dfrac{x-y}{x^2+1}\right)$
6. $\dfrac{\partial}{\partial x}\left(\dfrac{x-y}{x^2+1}\right)$
7. $\dfrac{\partial}{\partial x}\left(\dfrac{xy}{x^2+1}\right)$
8. $\dfrac{\partial}{\partial y}\left(\dfrac{xy}{x^2+1}\right)$

In Exercises 9–40, find both first partial derivatives.

9. $f(x, y) = 2x - 5y + 3$
10. $f(x, y) = x^2 - 2y^2 + 4$
11. $f(x, y) = x^2y^3$
12. $f(x, y) = 4x^3y^{-2}$
13. $z = x\sqrt{y}$
14. $z = 2y^2\sqrt{x}$
15. $z = x^2 - 4xy + 3y^2$
16. $z = y^3 - 2xy^2 - 1$
17. $z = e^{xy}$
18. $z = e^{x/y}$
19. $z = x^2e^{2y}$
20. $z = ye^{y/x}$
21. $z = \ln\dfrac{x}{y}$
22. $z = \ln\sqrt{xy}$
23. $z = \ln(x^2 + y^2)$
24. $z = \ln\dfrac{x+y}{x-y}$
25. $z = \dfrac{x^2}{2y} + \dfrac{3y^2}{x}$
26. $z = \dfrac{xy}{x^2+y^2}$
27. $h(x, y) = e^{-(x^2+y^2)}$
28. $g(x, y) = \ln\sqrt{x^2+y^2}$
29. $f(x, y) = \sqrt{x^2+y^2}$
30. $f(x, y) = \sqrt{2x+y^3}$
31. $z = \cos xy$
32. $z = \sin(x + 2y)$
33. $z = \tan(2x - y)$
34. $z = \sin 5x \cos 5y$
35. $z = e^y \sin xy$
36. $z = \cos(x^2 + y^2)$
37. $z = \sinh(2x + 3y)$
38. $z = \cosh xy^2$
39. $f(x, y) = \displaystyle\int_x^y (t^2 - 1)\, dt$
40. $f(x, y) = \displaystyle\int_x^y (2t + 1)\, dt + \int_y^x (2t - 1)\, dt$

In Exercises 41–44, use the limit definition of partial derivatives to find $f_x(x, y)$ and $f_y(x, y)$.

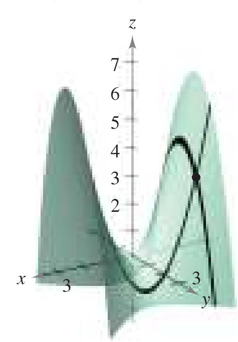
41. $f(x, y) = 3x + 2y$
42. $f(x, y) = x^2 - 2xy + y^2$
43. $f(x, y) = \sqrt{x+y}$
44. $f(x, y) = \dfrac{1}{x+y}$

In Exercises 45–52, evaluate f_x and f_y at the given point.

45. $f(x, y) = e^y \sin x$, $(\pi, 0)$
46. $f(x, y) = e^{-x} \cos y$, $(0, 0)$
47. $f(x, y) = \cos(2x - y)$, $\left(\dfrac{\pi}{4}, \dfrac{\pi}{3}\right)$
48. $f(x, y) = \sin xy$, $\left(2, \dfrac{\pi}{4}\right)$
49. $f(x, y) = \arctan\dfrac{y}{x}$, $(2, -2)$
50. $f(x, y) = \arccos xy$, $(1, 1)$
51. $f(x, y) = \dfrac{xy}{x-y}$, $(2, -2)$
52. $f(x, y) = \dfrac{2xy}{\sqrt{4x^2+5y^2}}$, $(1, 1)$

In Exercises 53 and 54, find the slopes of the surface in the x- and y-directions at the given point.

53. $g(x, y) = 4 - x^2 - y^2$
$(1, 1, 2)$
54. $h(x, y) = x^2 - y^2$
$(-2, 1, 3)$



CAS **In Exercises 55–58, use a computer algebra system to graph the curve formed by the intersection of the surface and the plane. Find the slope of the curve at the given point.**

	Surface	*Plane*	*Point*
55.	$z = \sqrt{49 - x^2 - y^2}$	$x = 2$	$(2, 3, 6)$
56.	$z = x^2 + 4y^2$	$y = 1$	$(2, 1, 8)$
57.	$z = 9x^2 - y^2$	$y = 3$	$(1, 3, 0)$
58.	$z = 9x^2 - y^2$	$x = 1$	$(1, 3, 0)$

In Exercises 59–64, find the first partial derivatives with respect to x, y, and z.

59. $H(x, y, z) = \sin(x + 2y + 3z)$

60. $f(x, y, z) = 3x^2y - 5xyz + 10yz^2$

61. $w = \sqrt{x^2 + y^2 + z^2}$

62. $w = \dfrac{7xz}{x + y}$

63. $F(x, y, z) = \ln\sqrt{x^2 + y^2 + z^2}$

64. $G(x, y, z) = \dfrac{1}{\sqrt{1 - x^2 - y^2 - z^2}}$

In Exercises 65–70, evaluate f_x, f_y, and f_z at the given point.

65. $f(x, y, z) = x^3yz^2$, $(1, 1, 1)$

66. $f(x, y, z) = x^2y^3 + 2xyz - 3yz$, $(-2, 1, 2)$

67. $f(x, y, z) = \dfrac{x}{yz}$, $(1, -1, -1)$

68. $f(x, y, z) = \dfrac{xy}{x + y + z}$, $(3, 1, -1)$

69. $f(x, y, z) = z\sin(x + y)$, $\left(0, \dfrac{\pi}{2}, -4\right)$

70. $f(x, y, z) = \sqrt{3x^2 + y^2 - 2z^2}$, $(1, -2, 1)$

In Exercises 71–80, find the four second partial derivatives. Observe that the second mixed partials are equal.

71. $z = 3xy^2$

72. $z = x^2 + 3y^2$

73. $z = x^2 - 2xy + 3y^2$

74. $z = x^4 - 3x^2y^2 + y^4$

75. $z = \sqrt{x^2 + y^2}$

76. $z = \ln(x - y)$

77. $z = e^x \tan y$

78. $z = 2xe^y - 3ye^{-x}$

79. $z = \cos xy$

80. $z = \arctan\dfrac{y}{x}$

In Exercises 81–88, for $f(x, y)$, find all values of x and y such that $f_x(x, y) = 0$ and $f_y(x, y) = 0$ simultaneously.

81. $f(x, y) = x^2 + xy + y^2 - 2x + 2y$

82. $f(x, y) = x^2 - xy + y^2 - 5x + y$

83. $f(x, y) = x^2 + 4xy + y^2 - 4x + 16y + 3$

84. $f(x, y) = x^2 - xy + y^2$

85. $f(x, y) = \dfrac{1}{x} + \dfrac{1}{y} + xy$

86. $f(x, y) = 3x^3 - 12xy + y^3$

87. $f(x, y) = e^{x^2 + xy + y^2}$

88. $f(x, y) = \ln(x^2 + y^2 + 1)$

CAS **In Exercises 89–92, use a computer algebra system to find the first and second partial derivatives of the function. Determine whether there exist values of x and y such that $f_x(x, y) = 0$ and $f_y(x, y) = 0$ simultaneously.**

89. $f(x, y) = x\sec y$

90. $f(x, y) = \sqrt{25 - x^2 - y^2}$

91. $f(x, y) = \ln\dfrac{x}{x^2 + y^2}$

92. $f(x, y) = \dfrac{xy}{x - y}$

In Exercises 93–96, show that the mixed partial derivatives f_{xyy}, f_{yxy}, and f_{yyx} are equal.

93. $f(x, y, z) = xyz$

94. $f(x, y, z) = x^2 - 3xy + 4yz + z^3$

95. $f(x, y, z) = e^{-x}\sin yz$

96. $f(x, y, z) = \dfrac{2z}{x + y}$

Laplace's Equation **In Exercises 97–100, show that the function satisfies Laplace's equation $\partial^2z/\partial x^2 + \partial^2z/\partial y^2 = 0$.**

97. $z = 5xy$

98. $z = \frac{1}{2}(e^y - e^{-y})\sin x$

99. $z = e^x \sin y$

100. $z = \arctan\dfrac{y}{x}$

Wave Equation **In Exercises 101–104, show that the function satisfies the wave equation $\partial^2z/\partial t^2 = c^2(\partial^2z/\partial x^2)$.**

101. $z = \sin(x - ct)$

102. $z = \cos(4x + 4ct)$

103. $z = \ln(x + ct)$

104. $z = \sin \omega ct \sin \omega x$

Heat Equation **In Exercises 105 and 106, show that the function satisfies the heat equation $\partial z/\partial t = c^2(\partial^2z/\partial x^2)$.**

105. $z = e^{-t}\cos\dfrac{x}{c}$

106. $z = e^{-t}\sin\dfrac{x}{c}$

In Exercises 107 and 108, determine whether or not there exists a function $f(x, y)$ with the given partial derivatives. Explain your reasoning. If such a function exists, give an example.

107. $f_x(x, y) = -3\sin(3x - 2y)$, $f_y(x, y) = 2\sin(3x - 2y)$

108. $f_x(x, y) = 2x + y$, $f_y(x, y) = x - 4y$

In Exercises 109 and 110, find the first partial derivative with respect to x.

109. $f(x, y, z) = (\tan y^2z)e^{z^2 + y^{-2}\sqrt{z}}$

110. $f(x, y, z) = x\left(\sinh\dfrac{y}{z}\right)^{(y^2 - 2\sqrt{y-1})z}$

WRITING ABOUT CONCEPTS

111. Let f be a function of two variables x and y. Describe the procedure for finding the first partial derivatives.

112. Sketch a surface representing a function f of two variables x and y. Use the sketch to give geometric interpretations of $\partial f/\partial x$ and $\partial f/\partial y$.

113. Sketch the graph of a function $z = f(x, y)$ whose derivative f_x is always negative and whose derivative f_y is always positive.

114. Sketch the graph of a function $z = f(x, y)$ whose derivatives f_x and f_y are always positive.

115. If f is a function of x and y such that f_{xy} and f_{yx} are continuous, what is the relationship between the mixed partial derivatives? Explain.

CAPSTONE

116. Find the four second partial derivatives of the function given by $f(x, y) = \sin(x - 2y)$. Show that the second mixed partial derivatives f_{xy} and f_{yx} are equal.

117. ***Marginal Revenue*** A pharmaceutical corporation has two plants that produce the same over-the-counter medicine. If x_1 and x_2 are the numbers of units produced at plant 1 and plant 2, respectively, then the total revenue for the product is given by $R = 200x_1 + 200x_2 - 4x_1^2 - 8x_1x_2 - 4x_2^2$. When $x_1 = 4$ and $x_2 = 12$, find (a) the marginal revenue for plant 1, $\partial R/\partial x_1$, and (b) the marginal revenue for plant 2, $\partial R/\partial x_2$.

118. ***Marginal Costs*** A company manufactures two types of wood-burning stoves: a freestanding model and a fireplace-insert model. The cost function for producing x freestanding and y fireplace-insert stoves is

$$C = 32\sqrt{xy} + 175x + 205y + 1050.$$

(a) Find the marginal costs ($\partial C/\partial x$ and $\partial C/\partial y$) when $x = 80$ and $y = 20$.

(b) When additional production is required, which model of stove results in the cost increasing at a higher rate? How can this be determined from the cost model?

119. ***Psychology*** Early in the twentieth century, an intelligence test called the *Stanford-Binet Test* (more commonly known as the *IQ test*) was developed. In this test, an individual's mental age M is divided by the individual's chronological age C and the quotient is multiplied by 100. The result is the individual's IQ.

$$IQ(M, C) = \frac{M}{C} \times 100$$

Find the partial derivatives of IQ with respect to M and with respect to C. Evaluate the partial derivatives at the point $(12, 10)$ and interpret the result. *(Source: Adapted from Bernstein/Clark-Stewart/Roy/Wickens, Psychology, Fourth Edition)*

120. ***Marginal Productivity*** Consider the Cobb-Douglas production function $f(x, y) = 200x^{0.7}y^{0.3}$. When $x = 1000$ and $y = 500$, find (a) the marginal productivity of labor, $\partial f/\partial x$, and (b) the marginal productivity of capital, $\partial f/\partial y$.

121. ***Think About It*** Let N be the number of applicants to a university, p the charge for food and housing at the university, and t the tuition. N is a function of p and t such that $\partial N/\partial p < 0$ and $\partial N/\partial t < 0$. What information is gained by noticing that both partials are negative?

122. ***Investment*** The value of an investment of \$1000 earning 6% compounded annually is

$$V(I, R) = 1000\left[\frac{1 + 0.06(1 - R)}{1 + I}\right]^{10}$$

where I is the annual rate of inflation and R is the tax rate for the person making the investment. Calculate $V_I(0.03, 0.28)$ and $V_R(0.03, 0.28)$. Determine whether the tax rate or the rate of inflation is the greater "negative" factor in the growth of the investment.

123. ***Temperature Distribution*** The temperature at any point (x, y) in a steel plate is $T = 500 - 0.6x^2 - 1.5y^2$, where x and y are measured in meters. At the point $(2, 3)$, find the rates of change of the temperature with respect to the distances moved along the plate in the directions of the x- and y-axes.

124. ***Apparent Temperature*** A measure of how hot weather feels to an average person is the Apparent Temperature Index. A model for this index is

$$A = 0.885t - 22.4h + 1.20th - 0.544$$

where A is the apparent temperature in degrees Celsius, t is the air temperature, and h is the relative humidity in decimal form. *(Source: The UMAP Journal)*

(a) Find $\partial A/\partial t$ and $\partial A/\partial h$ when $t = 30°$ and $h = 0.80$.

(b) Which has a greater effect on A, air temperature or humidity? Explain.

125. ***Ideal Gas Law*** The Ideal Gas Law states that $PV = nRT$, where P is pressure, V is volume, n is the number of moles of gas, R is a fixed constant (the gas constant), and T is absolute temperature. Show that

$$\frac{\partial T}{\partial P}\frac{\partial P}{\partial V}\frac{\partial V}{\partial T} = -1.$$

126. ***Marginal Utility*** The utility function $U = f(x, y)$ is a measure of the utility (or satisfaction) derived by a person from the consumption of two products x and y. Suppose the utility function is $U = -5x^2 + xy - 3y^2$.

(a) Determine the marginal utility of product x.

(b) Determine the marginal utility of product y.

(c) When $x = 2$ and $y = 3$, should a person consume one more unit of product x or one more unit of product y? Explain your reasoning.

CAS (d) Use a computer algebra system to graph the function. Interpret the marginal utilities of products x and y graphically.

127. ***Modeling Data*** Per capita consumptions (in gallons) of different types of milk in the United States from 1999 through 2005 are shown in the table. Consumption of flavored milk, plain reduced-fat milk, and plain light and skim milks are represented by the variables x, y, and z, respectively. *(Source: U.S. Department of Agriculture)*

Year	1999	2000	2001	2002	2003	2004	2005
x	1.4	1.4	1.4	1.6	1.6	1.7	1.7
y	7.3	7.1	7.0	7.0	6.9	6.9	6.9
z	6.2	6.1	5.9	5.8	5.6	5.5	5.6

A model for the data is given by $z = -0.92x + 1.03y + 0.02$.

(a) Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$.

(b) Interpret the partial derivatives in the context of the problem.

128. ***Modeling Data*** The table shows the public medical expenditures (in billions of dollars) for worker's compensation x, public assistance y, and Medicare z from 2000 through 2005. *(Source: Centers for Medicare and Medicaid Services)*

Year	2000	2001	2002	2003	2004	2005
x	24.9	28.1	30.1	31.4	32.1	33.5
y	207.5	233.2	258.4	281.9	303.2	324.9
z	224.3	247.7	265.7	283.5	312.8	342.0

A model for the data is given by

$$z = -1.2225x^2 + 0.0096y^2 + 71.381x - 4.121y - 354.65.$$

(a) Find $\dfrac{\partial^2 z}{\partial x^2}$ and $\dfrac{\partial^2 z}{\partial y^2}$.

(b) Determine the concavity of traces parallel to the xz-plane. Interpret the result in the context of the problem.

(c) Determine the concavity of traces parallel to the yz-plane. Interpret the result in the context of the problem.

True or False? **In Exercises 129–132, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

129. If $z = f(x, y)$ and $\partial z/\partial x = \partial z/\partial y$, then $z = c(x + y)$.

130. If $z = f(x)g(y)$, then $(\partial z/\partial x) + (\partial z/\partial y) = f'(x)g(y) + f(x)g'(y)$.

131. If $z = e^{xy}$, then $\dfrac{\partial^2 z}{\partial y \partial x} = (xy + 1)e^{xy}$.

132. If a cylindrical surface $z = f(x, y)$ has rulings parallel to the y-axis, then $\partial z/\partial y = 0$.

133. Consider the function defined by

$$f(x, y) = \begin{cases} \dfrac{xy(x^2 - y^2)}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}.$$

(a) Find $f_x(x, y)$ and $f_y(x, y)$ for $(x, y) \neq (0, 0)$.

(b) Use the definition of partial derivatives to find $f_x(0, 0)$ and $f_y(0, 0)$.

$$\left[\textit{Hint: } f_x(0, 0) = \lim_{\Delta x \to 0} \frac{f(\Delta x, 0) - f(0, 0)}{\Delta x}.\right]$$

(c) Use the definition of partial derivatives to find $f_{xy}(0, 0)$ and $f_{yx}(0, 0)$.

(d) Using Theorem 13.3 and the result of part (c), what can be said about f_{xy} or f_{yx}?

134. Let $f(x, y) = \displaystyle\int_x^y \sqrt{1 + t^3}\, dt$. Find $f_x(x, y)$ and $f_y(x, y)$.

135. Consider the function $f(x, y) = (x^3 + y^3)^{1/3}$.

(a) Find $f_x(0, 0)$ and $f_y(0, 0)$.

(b) Determine the points (if any) at which $f_x(x, y)$ or $f_y(x, y)$ fails to exist.

136. Consider the function $f(x, y) = (x^2 + y^2)^{2/3}$. Show that

$$f_x(x, y) = \begin{cases} \dfrac{4x}{3(x^2 + y^2)^{1/3}}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}.$$

FOR FURTHER INFORMATION For more information about this problem, see the article "A Classroom Note on a Naturally Occurring Piecewise Defined Function" by Don Cohen in *Mathematics and Computer Education*.

SECTION PROJECT

Moiré Fringes

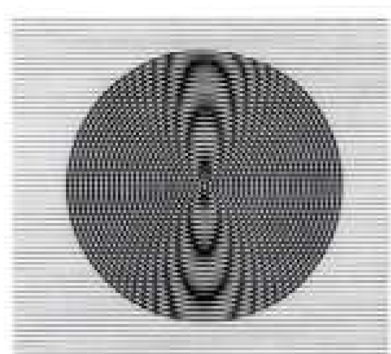
Read the article "Moiré Fringes and the Conic Sections" by Mike Cullen in *The College Mathematics Journal*. The article describes how two families of level curves given by

$$f(x, y) = a \qquad \text{and} \qquad g(x, y) = b$$

can form Moiré patterns. After reading the article, write a paper explaining how the expression

$$\frac{\partial f}{\partial x} \cdot \frac{\partial g}{\partial x} + \frac{\partial f}{\partial y} \cdot \frac{\partial g}{\partial y}$$

is related to the Moiré patterns formed by intersecting the two families of level curves. Use one of the following patterns as an example in your paper.



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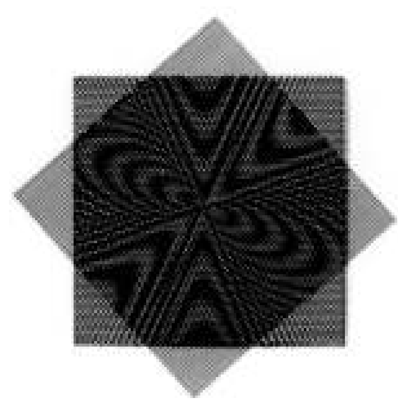


Mike Cullen

13.4 Differentials

- Understand the concepts of increments and differentials.
- Extend the concept of differentiability to a function of two variables.
- Use a differential as an approximation.

Increments and Differentials

In this section, the concepts of increments and differentials are generalized to functions of two or more variables. Recall from Section 3.9 that for $y = f(x)$, the differential of y was defined as

$$dy = f'(x)\,dx.$$

Similar terminology is used for a function of two variables, $z = f(x, y)$. That is, Δx and Δy are the **increments of x and y,** and the **increment of z** is given by

$$\Delta z = f(x + \Delta x, y + \Delta y) - f(x, y). \qquad \text{Increment of } z$$

DEFINITION OF TOTAL DIFFERENTIAL

If $z = f(x, y)$ and Δx and Δy are increments of x and y, then the **differentials** of the independent variables x and y are

$$dx = \Delta x \qquad \text{and} \qquad dy = \Delta y$$

and the **total differential** of the dependent variable z is

$$dz = \frac{\partial z}{\partial x}\,dx + \frac{\partial z}{\partial y}\,dy = f_x(x, y)\,dx + f_y(x, y)\,dy.$$

This definition can be extended to a function of three or more variables. For instance, if $w = f(x, y, z, u)$, then $dx = \Delta x$, $dy = \Delta y$, $dz = \Delta z$, $du = \Delta u$, and the total differential of w is

$$dw = \frac{\partial w}{\partial x}\,dx + \frac{\partial w}{\partial y}\,dy + \frac{\partial w}{\partial z}\,dz + \frac{\partial w}{\partial u}\,du.$$

EXAMPLE 1 Finding the Total Differential

Find the total differential for each function.

a. $z = 2x \sin y - 3x^2y^2$ **b.** $w = x^2 + y^2 + z^2$

Solution

a. The total differential dz for $z = 2x \sin y - 3x^2y^2$ is

$$\begin{aligned} dz &= \frac{\partial z}{\partial x}\,dx + \frac{\partial z}{\partial y}\,dy && \text{Total differential } dz \\ &= (2 \sin y - 6xy^2)\,dx + (2x \cos y - 6x^2y)\,dy. \end{aligned}$$

b. The total differential dw for $w = x^2 + y^2 + z^2$ is

$$\begin{aligned} dw &= \frac{\partial w}{\partial x}\,dx + \frac{\partial w}{\partial y}\,dy + \frac{\partial w}{\partial z}\,dz && \text{Total differential } dw \\ &= 2x\,dx + 2y\,dy + 2z\,dz. \end{aligned}$$

Differentiability

In Section 3.9, you learned that for a *differentiable* function given by $y = f(x)$, you can use the differential $dy = f'(x)\,dx$ as an approximation (for small Δx) to the value $\Delta y = f(x + \Delta x) - f(x)$. When a similar approximation is possible for a function of two variables, the function is said to be **differentiable.** This is stated explicitly in the following definition.

DEFINITION OF DIFFERENTIABILITY

A function f given by $z = f(x, y)$ is **differentiable** at (x_0, y_0) if Δz can be written in the form

$$\Delta z = f_x(x_0, y_0)\,\Delta x + f_y(x_0, y_0)\,\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

where both ε_1 and $\varepsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0, 0)$. The function f is **differentiable in a region R** if it is differentiable at each point in R.

EXAMPLE 2 Showing That a Function Is Differentiable

Show that the function given by

$$f(x, y) = x^2 + 3y$$

is differentiable at every point in the plane.

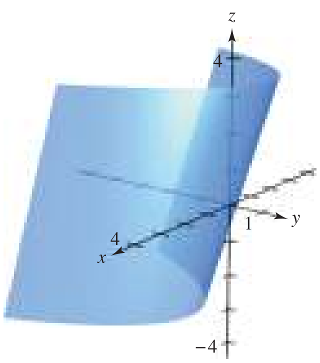


Figure 13.34

Solution Letting $z = f(x, y)$, the increment of z at an arbitrary point (x, y) in the plane is

$$\begin{aligned}
\Delta z &= f(x + \Delta x, y + \Delta y) - f(x, y) && \text{Increment of } z\\
&= (x^2 + 2x\Delta x + \Delta x^2) + 3(y + \Delta y) - (x^2 + 3y)\\
&= 2x\Delta x + \Delta x^2 + 3\Delta y\\
&= 2x(\Delta x) + 3(\Delta y) + \Delta x(\Delta x) + 0(\Delta y)\\
&= f_x(x, y)\,\Delta x + f_y(x, y)\,\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y
\end{aligned}$$

where $\varepsilon_1 = \Delta x$ and $\varepsilon_2 = 0$. Because $\varepsilon_1 \to 0$ and $\varepsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0, 0)$, it follows that f is differentiable at every point in the plane. The graph of f is shown in Figure 13.34. ■

Be sure you see that the term “differentiable” is used differently for functions of two variables than for functions of one variable. A function of one variable is differentiable at a point if its derivative exists at the point. However, for a function of two variables, the existence of the partial derivatives f_x and f_y does not guarantee that the function is differentiable (see Example 5). The following theorem gives a *sufficient* condition for differentiability of a function of two variables. A proof of Theorem 13.4 is given in Appendix A.

THEOREM 13.4 SUFFICIENT CONDITION FOR DIFFERENTIABILITY

If f is a function of x and y, where f_x and f_y are continuous in an open region R, then f is differentiable on R.

Approximation by Differentials

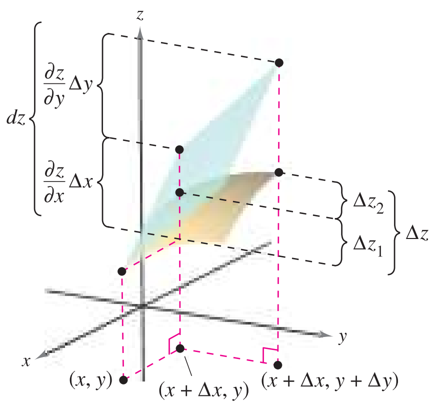
Theorem 13.4 tells you that you can choose $(x + \Delta x, y + \Delta y)$ close enough to (x, y) to make $\varepsilon_1 \Delta x$ and $\varepsilon_2 \Delta y$ insignificant. In other words, for small Δx and Δy, you can use the approximation

$$\Delta z \approx dz.$$

This approximation is illustrated graphically in Figure 13.35. Recall that the partial derivatives $\partial z/\partial x$ and $\partial z/\partial y$ can be interpreted as the slopes of the surface in the x- and y-directions. This means that

$$dz = \frac{\partial z}{\partial x}\Delta x + \frac{\partial z}{\partial y}\Delta y$$

represents the change in height of a plane that is tangent to the surface at the point $(x, y, f(x, y))$. Because a plane in space is represented by a linear equation in the variables x, y, and z, the approximation of Δz by dz is called a **linear approximation.** You will learn more about this geometric interpretation in Section 13.7.



The exact change in z is Δz. This change can be approximated by the differential dz.
Figure 13.35

EXAMPLE 3 Using a Differential as an Approximation

Use the differential dz to approximate the change in $z = \sqrt{4 - x^2 - y^2}$ as (x, y) moves from the point $(1, 1)$ to the point $(1.01, 0.97)$. Compare this approximation with the exact change in z.

Solution Letting $(x, y) = (1, 1)$ and $(x + \Delta x, y + \Delta y) = (1.01, 0.97)$ produces $dx = \Delta x = 0.01$ and $dy = \Delta y = -0.03$. So, the change in z can be approximated by

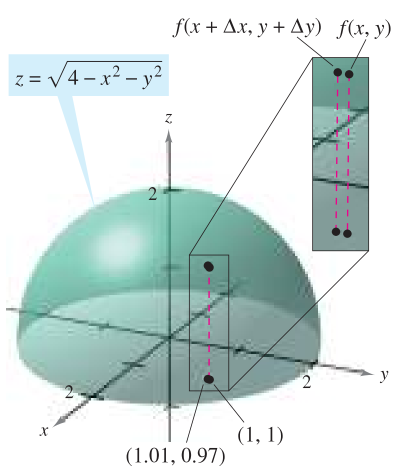
$$\Delta z \approx dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = \frac{-x}{\sqrt{4 - x^2 - y^2}}\Delta x + \frac{-y}{\sqrt{4 - x^2 - y^2}}\Delta y.$$

When $x = 1$ and $y = 1$, you have

$$\Delta z \approx -\frac{1}{\sqrt{2}}(0.01) - \frac{1}{\sqrt{2}}(-0.03) = \frac{0.02}{\sqrt{2}} = \sqrt{2}(0.01) \approx 0.0141.$$

In Figure 13.36, you can see that the exact change corresponds to the difference in the heights of two points on the surface of a hemisphere. This difference is given by

$$\begin{aligned}\Delta z &= f(1.01, 0.97) - f(1, 1)\\ &= \sqrt{4 - (1.01)^2 - (0.97)^2} - \sqrt{4 - 1^2 - 1^2} \approx 0.0137.\end{aligned}$$



As (x, y) moves from $(1, 1)$ to the point $(1.01, 0.97)$, the value of $f(x, y)$ changes by about 0.0137.
Figure 13.36

A function of three variables $w = f(x, y, z)$ is called **differentiable** at (x, y, z) provided that

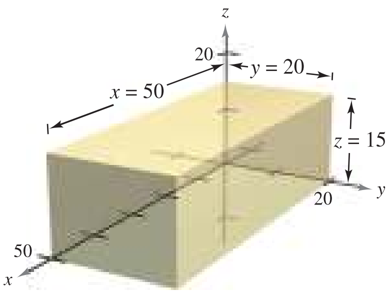
$$\Delta w = f(x + \Delta x, y + \Delta y, z + \Delta z) - f(x, y, z)$$

can be written in the form

$$\Delta w = f_x \Delta x + f_y \Delta y + f_z \Delta z + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y + \varepsilon_3 \Delta z$$

where ε_1, ε_2, and $\varepsilon_3 \to 0$ as $(\Delta x, \Delta y, \Delta z) \to (0, 0, 0)$. With this definition of differentiability, Theorem 13.4 has the following extension for functions of three variables: If f is a function of x, y, and z, where f, f_x, f_y, and f_z are continuous in an open region R, then f is differentiable on R.

In Section 3.9, you used differentials to approximate the propagated error introduced by an error in measurement. This application of differentials is further illustrated in Example 4.



Volume $= xyz$
Figure 13.37

EXAMPLE 4 Error Analysis

The possible error involved in measuring each dimension of a rectangular box is ± 0.1 millimeter. The dimensions of the box are $x = 50$ centimeters, $y = 20$ centimeters, and $z = 15$ centimeters, as shown in Figure 13.37. Use dV to estimate the propagated error and the relative error in the calculated volume of the box.

Solution The volume of the box is given by $V = xyz$, and so

$$dV = \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz$$
$$= yz\,dx + xz\,dy + xy\,dz.$$

Using 0.1 millimeter = 0.01 centimeter, you have $dx = dy = dz = \pm 0.01$, and the propagated error is approximately

$$dV = (20)(15)(\pm 0.01) + (50)(15)(\pm 0.01) + (50)(20)(\pm 0.01)$$
$$= 300(\pm 0.01) + 750(\pm 0.01) + 1000(\pm 0.01)$$
$$= 2050(\pm 0.01) = \pm 20.5 \text{ cubic centimeters.}$$

Because the measured volume is

$$V = (50)(20)(15) = 15{,}000 \text{ cubic centimeters,}$$

the relative error, $\Delta V/V$, is approximately

$$\frac{\Delta V}{V} \approx \frac{dV}{V} = \frac{20.5}{15{,}000} \approx 0.14\%.$$

As is true for a function of a single variable, if a function in two or more variables is differentiable at a point, it is also continuous there.

THEOREM 13.5 DIFFERENTIABILITY IMPLIES CONTINUITY

If a function of x and y is differentiable at (x_0, y_0), then it is continuous at (x_0, y_0).

PROOF Let f be differentiable at (x_0, y_0), where $z = f(x, y)$. Then

$$\Delta z = [f_x(x_0, y_0) + \varepsilon_1]\,\Delta x + [f_y(x_0, y_0) + \varepsilon_2]\,\Delta y$$

where both ε_1 and $\varepsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0, 0)$. However, by definition, you know that Δz is given by

$$\Delta z = f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0).$$

Letting $x = x_0 + \Delta x$ and $y = y_0 + \Delta y$ produces

$$f(x, y) - f(x_0, y_0) = [f_x(x_0, y_0) + \varepsilon_1]\,\Delta x + [f_y(x_0, y_0) + \varepsilon_2]\,\Delta y$$
$$= [f_x(x_0, y_0) + \varepsilon_1](x - x_0) + [f_y(x_0, y_0) + \varepsilon_2](y - y_0).$$

Taking the limit as $(x, y) \to (x_0, y_0)$, you have

$$\lim_{(x, y) \to (x_0, y_0)} f(x, y) = f(x_0, y_0)$$

which means that f is continuous at (x_0, y_0).

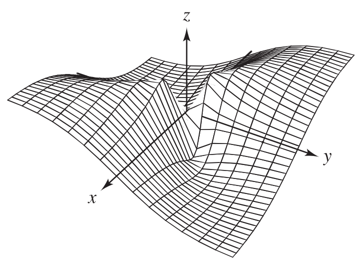
Remember that the existence of f_x and f_y is not sufficient to guarantee differentiability, as illustrated in the next example.

EXAMPLE 5 A Function That Is Not Differentiable

Show that $f_x(0, 0)$ and $f_y(0, 0)$ both exist, but that f is not differentiable at $(0, 0)$ where f is defined as

$$f(x, y) = \begin{cases} \dfrac{-3xy}{x^2 + y^2}, & \text{if } (x, y) \neq (0, 0) \\ 0, & \text{if } (x, y) = (0, 0) \end{cases}.$$

TECHNOLOGY Use a graphing utility to graph the function given in Example 5. For instance, the graph shown below was generated by *Mathematica*.



Generated by Mathematica

Solution You can show that f is not differentiable at $(0, 0)$ by showing that it is not continuous at this point. To see that f is not continuous at $(0, 0)$, look at the values of $f(x, y)$ along two different approaches to $(0, 0)$, as shown in Figure 13.38. Along the line $y = x$, the limit is

$$\lim_{(x, x)\to(0, 0)} f(x, y) = \lim_{(x, x)\to(0, 0)} \frac{-3x^2}{2x^2} = -\frac{3}{2}$$

whereas along $y = -x$ you have

$$\lim_{(x, -x)\to(0, 0)} f(x, y) = \lim_{(x, -x)\to(0, 0)} \frac{3x^2}{2x^2} = \frac{3}{2}.$$

So, the limit of $f(x, y)$ as $(x, y) \to (0, 0)$ does not exist, and you can conclude that f is not continuous at $(0, 0)$. Therefore, by Theorem 13.5, you know that f is not differentiable at $(0, 0)$. On the other hand, by the definition of the partial derivatives f_x and f_y, you have

$$f_x(0, 0) = \lim_{\Delta x\to 0} \frac{f(\Delta x, 0) - f(0, 0)}{\Delta x} = \lim_{\Delta x\to 0} \frac{0 - 0}{\Delta x} = 0$$

and

$$f_y(0, 0) = \lim_{\Delta y\to 0} \frac{f(0, \Delta y) - f(0, 0)}{\Delta y} = \lim_{\Delta y\to 0} \frac{0 - 0}{\Delta y} = 0.$$

So, the partial derivatives at $(0, 0)$ exist.

$$f(x, y) = \begin{cases} \dfrac{-3xy}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$

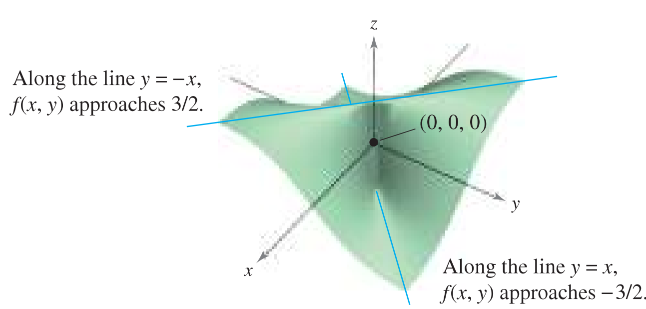


Figure 13.38

13.4 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–10, find the total differential.

1. $z = 2x^2y^3$
2. $z = \dfrac{x^2}{y}$
3. $z = \dfrac{-1}{x^2 + y^2}$
4. $w = \dfrac{x + y}{z - 3y}$
5. $z = x \cos y - y \cos x$
6. $z = \frac{1}{2}(e^{x^2+y^2} - e^{-x^2-y^2})$
7. $z = e^x \sin y$
8. $w = e^y \cos x + z^2$
9. $w = 2z^3y \sin x$
10. $w = x^2yz^2 + \sin yz$

In Exercises 11–16, (a) evaluate $f(2, 1)$ and $f(2.1, 1.05)$ and calculate Δz, and (b) use the total differential dz to approximate Δz.

11. $f(x, y) = 2x - 3y$
12. $f(x, y) = x^2 + y^2$
13. $f(x, y) = 16 - x^2 - y^2$
14. $f(x, y) = \dfrac{y}{x}$
15. $f(x, y) = ye^x$
16. $f(x, y) = x \cos y$

In Exercises 17–20, find $z = f(x, y)$ and use the total differential to approximate the quantity.

17. $(2.01)^2(9.02) - 2^2 \cdot 9$
18. $\sqrt{(5.05)^2 + (3.1)^2} - \sqrt{5^2 + 3^2}$
19. $\dfrac{1 - (3.05)^2}{(5.95)^2} - \dfrac{1 - 3^2}{6^2}$
20. $\sin[(1.05)^2 + (0.95)^2] - \sin(1^2 + 1^2)$

WRITING ABOUT CONCEPTS

21. Define the total differential of a function of two variables.
22. Describe the change in accuracy of dz as an approximation of Δz as Δx and Δy increase.
23. What is meant by a linear approximation of $z = f(x, y)$ at the point $P(x_0, y_0)$?
24. When using differentials, what is meant by the terms *propagated error* and *relative error*?

25. ***Area*** The area of the shaded rectangle in the figure is $A = lh$. The possible errors in the length and height are Δl and Δh, respectively. Find dA and identify the regions in the figure whose areas are given by the terms of dA. What region represents the difference between ΔA and dA?

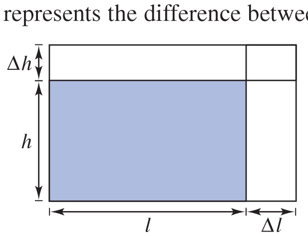


Figure for 25

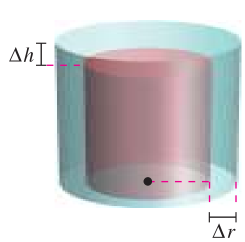


Figure for 26

26. ***Volume*** The volume of the red right circular cylinder in the figure is $V = \pi r^2 h$. The possible errors in the radius and the height are Δr and Δh, respectively. Find dV and identify the solids in the figure whose volumes are given by the terms of dV. What solid represents the difference between ΔV and dV?

27. ***Numerical Analysis*** A right circular cone of height $h = 8$ and radius $r = 4$ is constructed, and in the process errors Δr and Δh are made in the radius and height, respectively. Complete the table to show the relationship between ΔV and dV for the indicated errors.

Δr	Δh	dV or dS	ΔV or ΔS	$\Delta V - dV$ or $\Delta S - dS$
0.1	0.1			
0.1	-0.1			
0.001	0.002			
-0.0001	0.0002			

28. ***Numerical Analysis*** The height and radius of a right circular cone are measured as $h = 16$ meters and $r = 6$ meters. In the process of measuring, errors Δr and Δh are made. S is the lateral surface area of the cone. Complete the table above to show the relationship between ΔS and dS for the indicated errors.

29. ***Modeling Data*** Per capita consumptions (in gallons) of different types of plain milk in the United States from 1999 through 2005 are shown in the table. Consumption of flavored milk, plain reduced-fat milk, and plain light and skim milks are represented by the variables x, y, and z, respectively. *(Source: U.S. Department of Agriculture)*

Year	1999	2000	2001	2002	2003	2004	2005
x	1.4	1.4	1.4	1.6	1.6	1.7	1.7
y	7.3	7.1	7.0	7.0	6.9	6.9	6.9
z	6.2	6.1	5.9	5.8	5.6	5.5	5.6

A model for the data is given by $z = -0.92x + 1.03y + 0.02$.

(a) Find the total differential of the model.

(b) A dairy industry forecast for a future year is that per capita consumption of flavored milk will be 1.9 ± 0.25 gallons and that per capita consumption of plain reduced-fat milk will be 7.5 ± 0.25 gallons. Use dz to estimate the maximum possible propagated error and relative error in the prediction for the consumption of plain light and skim milks.

30. ***Rectangular to Polar Coordinates*** A rectangular coordinate system is placed over a map, and the coordinates of a point of interest are $(7.2, 2.5)$. There is a possible error of 0.05 in each coordinate. Approximate the maximum possible error in measuring the polar coordinates of the point.

31. ***Volume*** The radius r and height h of a right circular cylinder are measured with possible errors of 4% and 2%, respectively. Approximate the maximum possible percent error in measuring the volume.

32. ***Area*** A triangle is measured and two adjacent sides are found to be 3 inches and 4 inches long, with an included angle of $\pi/4$. The possible errors in measurement are $\frac{1}{16}$ inch for the sides and 0.02 radian for the angle. Approximate the maximum possible error in the computation of the area.

33. ***Wind Chill*** The formula for wind chill C (in degrees Fahrenheit) is given by

$$C = 35.74 + 0.6215T - 35.75v^{0.16} + 0.4275Tv^{0.16}$$

where v is the wind speed in miles per hour and T is the temperature in degrees Fahrenheit. The wind speed is 23 ± 3 miles per hour and the temperature is $8^\circ \pm 1^\circ$. Use dC to estimate the maximum possible propagated error and relative error in calculating the wind chill. *(Source: National Oceanic and Atmospheric Administration)*

34. ***Acceleration*** The centripetal acceleration of a particle moving in a circle is $a = v^2/r$, where v is the velocity and r is the radius of the circle. Approximate the maximum percent error in measuring the acceleration due to errors of 3% in v and 2% in r.

35. ***Volume*** A trough is 16 feet long (see figure). Its cross sections are isosceles triangles with each of the two equal sides measuring 18 inches. The angle between the two equal sides is θ.

(a) Write the volume of the trough as a function of θ and determine the value of θ such that the volume is a maximum.

(b) The maximum error in the linear measurements is one-half inch and the maximum error in the angle measure is 2°. Approximate the change in the maximum volume.

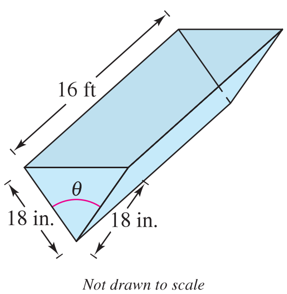


Figure for 35

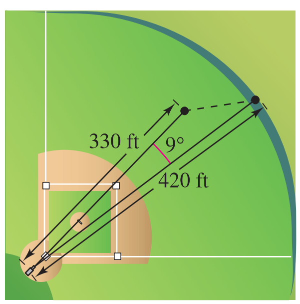


Figure for 36

36. ***Sports*** A baseball player in center field is playing approximately 330 feet from a television camera that is behind home plate. A batter hits a fly ball that goes to the wall 420 feet from the camera (see figure).

(a) The camera turns 9° to follow the play. Approximate the number of feet that the center fielder has to run to make the catch.

(b) The position of the center fielder could be in error by as much as 6 feet and the maximum error in measuring the rotation of the camera is 1°. Approximate the maximum possible error in the result of part (a).

37. ***Power*** Electrical power P is given by $P = E^2/R$, where E is voltage and R is resistance. Approximate the maximum percent error in calculating power if 120 volts is applied to a 2000-ohm resistor and the possible percent errors in measuring E and R are 3% and 4%, respectively.

38. ***Resistance*** The total resistance R of two resistors connected in parallel is given by

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}.$$

Approximate the change in R as R_1 is increased from 10 ohms to 10.5 ohms and R_2 is decreased from 15 ohms to 13 ohms.

39. ***Inductance*** The inductance L (in microhenrys) of a straight nonmagnetic wire in free space is

$$L = 0.00021\left(\ln\frac{2h}{r} - 0.75\right)$$

where h is the length of the wire in millimeters and r is the radius of a circular cross section. Approximate L when $r = 2 \pm \frac{1}{16}$ millimeters and $h = 100 \pm \frac{1}{100}$ millimeters.

40. ***Pendulum*** The period T of a pendulum of length L is $T = 2\pi\sqrt{L/g}$, where g is the acceleration due to gravity. A pendulum is moved from the Canal Zone, where $g = 32.09$ feet per second per second, to Greenland, where $g = 32.23$ feet per second per second. Because of the change in temperature, the length of the pendulum changes from 2.5 feet to 2.48 feet. Approximate the change in the period of the pendulum.

In Exercises 41–44, show that the function is differentiable by finding values of ε_1 and ε_2 as designated in the definition of differentiability, and verify that both ε_1 and $\varepsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0, 0)$.

41. $f(x, y) = x^2 - 2x + y$

42. $f(x, y) = x^2 + y^2$

43. $f(x, y) = x^2y$

44. $f(x, y) = 5x - 10y + y^3$

In Exercises 45 and 46, use the function to show that $f_x(0, 0)$ and $f_y(0, 0)$ both exist, but that f is not differentiable at $(0, 0)$.

45. $f(x, y) = \begin{cases} \dfrac{3x^2y}{x^4 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$

46. $f(x, y) = \begin{cases} \dfrac{5x^2y}{x^3 + y^3}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$

47. Show that if $f(x, y)$ is differentiable at (x_0, y_0), then $f(x, y_0)$ is differentiable at $x = x_0$. Use this result to prove that $f(x, y) = \sqrt{x^2 + y^2}$ is not differentiable at $(0, 0)$.

CAPSTONE

48. Consider the function $f(x, y) = \sqrt{x^2 + y^2}$.

(a) Evaluate $f(3, 1)$ and $f(3.05, 1.1)$.

(b) Use the results of part (a) to calculate Δz.

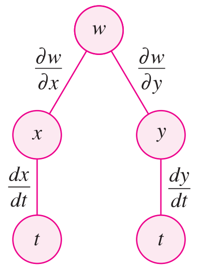
(c) Use the total differential dz to approximate Δz. Compare your result with that of part (b).

13.5 Chain Rules for Functions of Several Variables

- **Use the Chain Rules for functions of several variables.**
- **Find partial derivatives implicitly.**

Chain Rules for Functions of Several Variables

Your work with differentials in the preceding section provides the basis for the extension of the Chain Rule to functions of two variables. There are two cases—the first case involves w as a function of x and y, where x and y are functions of a single independent variable t, as shown in Theorem 13.6. (A proof of this theorem is given in Appendix A.)



Chain Rule: one independent variable w is a function of x and y, which are each functions of t. This diagram represents the derivative of w with respect to t.
Figure 13.39

THEOREM 13.6 CHAIN RULE: ONE INDEPENDENT VARIABLE

Let $w = f(x, y)$, where f is a differentiable function of x and y. If $x = g(t)$ and $y = h(t)$, where g and h are differentiable functions of t, then w is a differentiable function of t, and

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}.$$

See Figure 13.39.

EXAMPLE 1 Using the Chain Rule with One Independent Variable

Let $w = x^2y - y^2$, where $x = \sin t$ and $y = e^t$. Find dw/dt when $t = 0$.

Solution By the Chain Rule for one independent variable, you have

$$\begin{aligned}\frac{dw}{dt} &= \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}\\ &= 2xy(\cos t) + (x^2 - 2y)e^t\\ &= 2(\sin t)(e^t)(\cos t) + (\sin^2 t - 2e^t)e^t\\ &= 2e^t \sin t \cos t + e^t \sin^2 t - 2e^{2t}.\end{aligned}$$

When $t = 0$, it follows that

$$\frac{dw}{dt} = -2.$$

The Chain Rules presented in this section provide alternative techniques for solving many problems in single-variable calculus. For instance, in Example 1, you could have used single-variable techniques to find dw/dt by first writing w as a function of t,

$$\begin{aligned}w &= x^2y - y^2\\ &= (\sin t)^2(e^t) - (e^t)^2\\ &= e^t \sin^2 t - e^{2t}\end{aligned}$$

and then differentiating as usual.

$$\frac{dw}{dt} = 2e^t \sin t \cos t + e^t \sin^2 t - 2e^{2t}$$

The Chain Rule in Theorem 13.6 can be extended to any number of variables. For example, if each x_i is a differentiable function of a single variable t, then for

$$w = f(x_1, x_2, \ldots, x_n)$$

you have

$$\frac{dw}{dt} = \frac{\partial w}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial w}{\partial x_2}\frac{dx_2}{dt} + \cdots + \frac{\partial w}{\partial x_n}\frac{dx_n}{dt}.$$

EXAMPLE 2 An Application of a Chain Rule to Related Rates

Two objects are traveling in elliptical paths given by the following parametric equations.

$x_1 = 4 \cos t$ and $y_1 = 2 \sin t$ First object

$x_2 = 2 \sin 2t$ and $y_2 = 3 \cos 2t$ Second object

At what rate is the distance between the two objects changing when $t = \pi$?

Solution From Figure 13.40, you can see that the distance s between the two objects is given by

$$s = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

and that when $t = \pi$, you have $x_1 = -4$, $y_1 = 0$, $x_2 = 0$, $y_2 = 3$, and

$$s = \sqrt{(0 + 4)^2 + (3 - 0)^2} = 5.$$

When $t = \pi$, the partial derivatives of s are as follows.

$$\frac{\partial s}{\partial x_1} = \frac{-(x_2 - x_1)}{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}} = -\frac{1}{5}(0 + 4) = -\frac{4}{5}$$

$$\frac{\partial s}{\partial y_1} = \frac{-(y_2 - y_1)}{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}} = -\frac{1}{5}(3 - 0) = -\frac{3}{5}$$

$$\frac{\partial s}{\partial x_2} = \frac{(x_2 - x_1)}{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}} = \frac{1}{5}(0 + 4) = \frac{4}{5}$$

$$\frac{\partial s}{\partial y_2} = \frac{(y_2 - y_1)}{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}} = \frac{1}{5}(3 - 0) = \frac{3}{5}$$

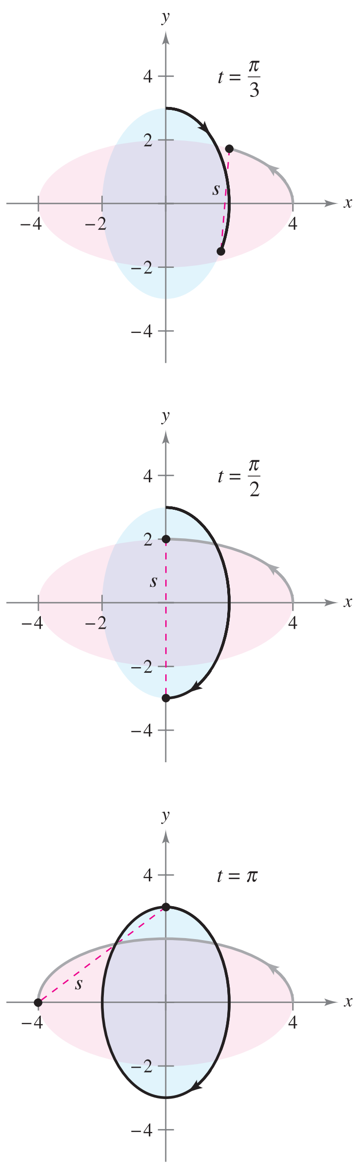
When $t = \pi$, the derivatives of x_1, y_1, x_2, and y_2 are

$$\frac{dx_1}{dt} = -4 \sin t = 0 \qquad \frac{dy_1}{dt} = 2 \cos t = -2$$

$$\frac{dx_2}{dt} = 4 \cos 2t = 4 \qquad \frac{dy_2}{dt} = -6 \sin 2t = 0.$$

So, using the appropriate Chain Rule, you know that the distance is changing at a rate of

$$\begin{aligned}\frac{ds}{dt} &= \frac{\partial s}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial s}{\partial y_1}\frac{dy_1}{dt} + \frac{\partial s}{\partial x_2}\frac{dx_2}{dt} + \frac{\partial s}{\partial y_2}\frac{dy_2}{dt} \\ &= \left(-\frac{4}{5}\right)(0) + \left(-\frac{3}{5}\right)(-2) + \left(\frac{4}{5}\right)(4) + \left(\frac{3}{5}\right)(0) \\ &= \frac{22}{5}.\end{aligned}$$



Paths of two objects traveling in elliptical orbits
Figure 13.40

In Example 2, note that s is the function of four *intermediate* variables, x_1, y_1, x_2, and y_2, each of which is a function of a single variable t. Another type of composite function is one in which the intermediate variables are themselves functions of more than one variable. For instance, if $w = f(x, y)$, where $x = g(s, t)$ and $y = h(s, t)$, it follows that w is a function of s and t, and you can consider the partial derivatives of w with respect to s and t. One way to find these partial derivatives is to write w as a function of s and t explicitly by substituting the equations $x = g(s, t)$ and $y = h(s, t)$ into the equation $w = f(x, y)$. Then you can find the partial derivatives in the usual way, as demonstrated in the next example.

EXAMPLE 3 Finding Partial Derivatives by Substitution

Find $\partial w/\partial s$ and $\partial w/\partial t$ for $w = 2xy$, where $x = s^2 + t^2$ and $y = s/t$.

Solution Begin by substituting $x = s^2 + t^2$ and $y = s/t$ into the equation $w = 2xy$ to obtain

$$w = 2xy = 2(s^2 + t^2)\left(\frac{s}{t}\right) = 2\left(\frac{s^3}{t} + st\right).$$

Then, to find $\partial w/\partial s$, hold t constant and differentiate with respect to s.

$$\begin{aligned}\frac{\partial w}{\partial s} &= 2\left(\frac{3s^2}{t} + t\right)\\ &= \frac{6s^2 + 2t^2}{t}\end{aligned}$$

Similarly, to find $\partial w/\partial t$, hold s constant and differentiate with respect to t to obtain

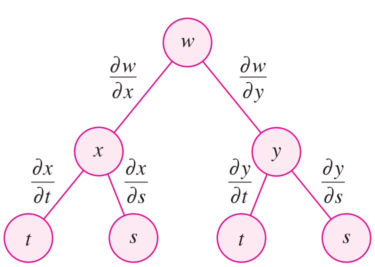
$$\begin{aligned}\frac{\partial w}{\partial t} &= 2\left(-\frac{s^3}{t^2} + s\right)\\ &= 2\left(\frac{-s^3 + st^2}{t^2}\right)\\ &= \frac{2st^2 - 2s^3}{t^2}.\end{aligned}$$

Theorem 13.7 gives an alternative method for finding the partial derivatives in Example 3, without explicitly writing w as a function of s and t.

THEOREM 13.7 CHAIN RULE: TWO INDEPENDENT VARIABLES

Let $w = f(x, y)$, where f is a differentiable function of x and y. If $x = g(s, t)$ and $y = h(s, t)$ such that the first partials $\partial x/\partial s$, $\partial x/\partial t$, $\partial y/\partial s$, and $\partial y/\partial t$ all exist, then $\partial w/\partial s$ and $\partial w/\partial t$ exist and are given by

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} \quad \text{and} \quad \frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t}.$$



Chain Rule: two independent variables
Figure 13.41

PROOF To obtain $\partial w/\partial s$, hold t constant and apply Theorem 13.6 to obtain the desired result. Similarly, for $\partial w/\partial t$, hold s constant and apply Theorem 13.6.

NOTE The Chain Rule in Theorem 13.7 is shown schematically in Figure 13.41.

EXAMPLE 4 The Chain Rule with Two Independent Variables

Use the Chain Rule to find $\partial w/\partial s$ and $\partial w/\partial t$ for

$$w = 2xy$$

where $x = s^2 + t^2$ and $y = s/t$.

Solution Note that these same partials were found in Example 3. This time, using Theorem 13.7, you can hold t constant and differentiate with respect to s to obtain

$$\begin{aligned}
\frac{\partial w}{\partial s} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} \\
&= 2y(2s) + 2x\left(\frac{1}{t}\right) \\
&= 2\left(\frac{s}{t}\right)(2s) + 2(s^2 + t^2)\left(\frac{1}{t}\right) && \text{Substitute } (s/t) \text{ for } y \text{ and } s^2 + t^2 \text{ for } x. \\
&= \frac{4s^2}{t} + \frac{2s^2 + 2t^2}{t} \\
&= \frac{6s^2 + 2t^2}{t}.
\end{aligned}$$

Similarly, holding s constant gives

$$\begin{aligned}
\frac{\partial w}{\partial t} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t} \\
&= 2y(2t) + 2x\left(\frac{-s}{t^2}\right) \\
&= 2\left(\frac{s}{t}\right)(2t) + 2(s^2 + t^2)\left(\frac{-s}{t^2}\right) && \text{Substitute } (s/t) \text{ for } y \text{ and } s^2 + t^2 \text{ for } x. \\
&= 4s - \frac{2s^3 + 2st^2}{t^2} \\
&= \frac{4st^2 - 2s^3 - 2st^2}{t^2} \\
&= \frac{2st^2 - 2s^3}{t^2}.
\end{aligned}$$

The Chain Rule in Theorem 13.7 can also be extended to any number of variables. For example, if w is a differentiable function of the n variables $x_1, x_2, \ldots, x_n$, where each x_i is a differentiable function of the m variables $t_1, t_2, \ldots, t_m$, then for

$$w = f(x_1, x_2, \ldots, x_n)$$

you obtain the following.

$$\begin{aligned}
\frac{\partial w}{\partial t_1} &= \frac{\partial w}{\partial x_1}\frac{\partial x_1}{\partial t_1} + \frac{\partial w}{\partial x_2}\frac{\partial x_2}{\partial t_1} + \cdots + \frac{\partial w}{\partial x_n}\frac{\partial x_n}{\partial t_1} \\
\frac{\partial w}{\partial t_2} &= \frac{\partial w}{\partial x_1}\frac{\partial x_1}{\partial t_2} + \frac{\partial w}{\partial x_2}\frac{\partial x_2}{\partial t_2} + \cdots + \frac{\partial w}{\partial x_n}\frac{\partial x_n}{\partial t_2} \\
&\vdots \\
\frac{\partial w}{\partial t_m} &= \frac{\partial w}{\partial x_1}\frac{\partial x_1}{\partial t_m} + \frac{\partial w}{\partial x_2}\frac{\partial x_2}{\partial t_m} + \cdots + \frac{\partial w}{\partial x_n}\frac{\partial x_n}{\partial t_m}
\end{aligned}$$

EXAMPLE 5 The Chain Rule for a Function of Three Variables

Find $\partial w/\partial s$ and $\partial w/\partial t$ when $s = 1$ and $t = 2\pi$ for the function given by

$$w = xy + yz + xz$$

where $x = s\cos t$, $y = s\sin t$, and $z = t$.

Solution By extending the result of Theorem 13.7, you have

$$\begin{aligned}\frac{\partial w}{\partial s} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s}\\ &= (y + z)(\cos t) + (x + z)(\sin t) + (y + x)(0)\\ &= (y + z)(\cos t) + (x + z)(\sin t).\end{aligned}$$

When $s = 1$ and $t = 2\pi$, you have $x = 1$, $y = 0$, and $z = 2\pi$. So, $\partial w/\partial s = (0 + 2\pi)(1) + (1 + 2\pi)(0) = 2\pi$. Furthermore,

$$\begin{aligned}\frac{\partial w}{\partial t} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial t}\\ &= (y + z)(-s\sin t) + (x + z)(s\cos t) + (y + x)(1)\end{aligned}$$

and for $s = 1$ and $t = 2\pi$, it follows that

$$\begin{aligned}\frac{\partial w}{\partial t} &= (0 + 2\pi)(0) + (1 + 2\pi)(1) + (0 + 1)(1)\\ &= 2 + 2\pi.\end{aligned}$$

■

Implicit Partial Differentiation

This section concludes with an application of the Chain Rule to determine the derivative of a function defined *implicitly*. Suppose that x and y are related by the equation $F(x, y) = 0$, where it is assumed that $y = f(x)$ is a differentiable function of x. To find dy/dx, you could use the techniques discussed in Section 2.5. However, you will see that the Chain Rule provides a convenient alternative. If you consider the function given by

$$w = F(x, y) = F(x, f(x))$$

you can apply Theorem 13.6 to obtain

$$\frac{dw}{dx} = F_x(x, y)\frac{dx}{dx} + F_y(x, y)\frac{dy}{dx}.$$

Because $w = F(x, y) = 0$ for all x in the domain of f, you know that $dw/dx = 0$ and you have

$$F_x(x, y)\frac{dx}{dx} + F_y(x, y)\frac{dy}{dx} = 0.$$

Now, if $F_y(x, y) \neq 0$, you can use the fact that $dx/dx = 1$ to conclude that

$$\frac{dy}{dx} = -\frac{F_x(x, y)}{F_y(x, y)}.$$

A similar procedure can be used to find the partial derivatives of functions of several variables that are defined implicitly.

THEOREM 13.8 CHAIN RULE: IMPLICIT DIFFERENTIATION

If the equation $F(x, y) = 0$ defines y implicitly as a differentiable function of x, then

$$\frac{dy}{dx} = -\frac{F_x(x, y)}{F_y(x, y)}, \qquad F_y(x, y) \neq 0.$$

If the equation $F(x, y, z) = 0$ defines z implicitly as a differentiable function of x and y, then

$$\frac{\partial z}{\partial x} = -\frac{F_x(x, y, z)}{F_z(x, y, z)} \quad \text{and} \quad \frac{\partial z}{\partial y} = -\frac{F_y(x, y, z)}{F_z(x, y, z)}, \quad F_z(x, y, z) \neq 0.$$

This theorem can be extended to differentiable functions defined implicitly with any number of variables.

EXAMPLE 6 Finding a Derivative Implicitly

Find dy/dx, given $y^3 + y^2 - 5y - x^2 + 4 = 0$.

Solution Begin by defining a function F as

$$F(x, y) = y^3 + y^2 - 5y - x^2 + 4.$$

Then, using Theorem 13.8, you have

$$F_x(x, y) = -2x \quad \text{and} \quad F_y(x, y) = 3y^2 + 2y - 5$$

and it follows that

$$\frac{dy}{dx} = -\frac{F_x(x, y)}{F_y(x, y)} = \frac{-(-2x)}{3y^2 + 2y - 5} = \frac{2x}{3y^2 + 2y - 5}.$$

NOTE Compare the solution of Example 6 with the solution of Example 2 in Section 2.5.

EXAMPLE 7 Finding Partial Derivatives Implicitly

Find $\partial z/\partial x$ and $\partial z/\partial y$, given $3x^2z - x^2y^2 + 2z^3 + 3yz - 5 = 0$.

Solution To apply Theorem 13.8, let

$$F(x, y, z) = 3x^2z - x^2y^2 + 2z^3 + 3yz - 5.$$

Then

$$F_x(x, y, z) = 6xz - 2xy^2$$
$$F_y(x, y, z) = -2x^2y + 3z$$
$$F_z(x, y, z) = 3x^2 + 6z^2 + 3y$$

and you obtain

$$\frac{\partial z}{\partial x} = -\frac{F_x(x, y, z)}{F_z(x, y, z)} = \frac{2xy^2 - 6xz}{3x^2 + 6z^2 + 3y}$$

$$\frac{\partial z}{\partial y} = -\frac{F_y(x, y, z)}{F_z(x, y, z)} = \frac{2x^2y - 3z}{3x^2 + 6z^2 + 3y}.$$

13.5 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–4, find dw/dt using the appropriate Chain Rule.

1. $w = x^2 + y^2$
   $x = 2t, \quad y = 3t$
2. $w = \sqrt{x^2 + y^2}$
   $x = \cos t, \quad y = e^t$
3. $w = x \sin y$
   $x = e^t, \quad y = \pi - t$
4. $w = \ln \dfrac{y}{x}$
   $x = \cos t, \quad y = \sin t$

In Exercises 5–10, find dw/dt (a) by using the appropriate Chain Rule and (b) by converting w to a function of t before differentiating.

5. $w = xy, \quad x = e^t, \quad y = e^{-2t}$
6. $w = \cos(x - y), \quad x = t^2, \quad y = 1$
7. $w = x^2 + y^2 + z^2, \quad x = \cos t, \quad y = \sin t, \quad z = e^t$
8. $w = xy \cos z, \quad x = t, \quad y = t^2, \quad z = \arccos t$
9. $w = xy + xz + yz, \quad x = t - 1, \quad y = t^2 - 1, \quad z = t$
10. $w = xy^2 + x^2z + yz^2, \quad x = t^2, \quad y = 2t, \quad z = 2$

Projectile Motion **In Exercises 11 and 12, the parametric equations for the paths of two projectiles are given. At what rate is the distance between the two objects changing at the given value of t?**

11. $x_1 = 10 \cos 2t, \; y_1 = 6 \sin 2t$ First object
    $x_2 = 7 \cos t, \; y_2 = 4 \sin t$ Second object
    $t = \pi/2$
12. $x_1 = 48\sqrt{2}\,t, \; y_1 = 48\sqrt{2}\,t - 16t^2$ First object
    $x_2 = 48\sqrt{3}\,t, \; y_2 = 48t - 16t^2$ Second object
    $t = 1$

In Exercises 13 and 14, find d^2w/dt^2 using the appropriate Chain Rule. Evaluate d^2w/dt^2 at the given value of t.

13. $w = \ln(x + y), \quad x = e^t, \quad y = e^{-t}, \quad t = 0$
14. $w = \dfrac{x^2}{y}, \quad x = t^2, \quad y = t + 1, \quad t = 1$

In Exercises 15–18, find $\partial w/\partial s$ and $\partial w/\partial t$ using the appropriate Chain Rule, and evaluate each partial derivative at the given values of s and t.

	Function	*Point*
15.	$w = x^2 + y^2$; $x = s + t, \quad y = s - t$	$s = 1, \quad t = 0$
16.	$w = y^3 - 3x^2y$; $x = e^s, \quad y = e^t$	$s = -1, \quad t = 2$
17.	$w = \sin(2x + 3y)$; $x = s + t, \quad y = s - t$	$s = 0, \quad t = \dfrac{\pi}{2}$
18.	$w = x^2 - y^2$; $x = s \cos t, \quad y = s \sin t$	$s = 3, \quad t = \dfrac{\pi}{4}$

In Exercises 19–22, find $\partial w/\partial r$ and $\partial w/\partial \theta$ (a) by using the appropriate Chain Rule and (b) by converting w to a function of r and θ before differentiating.

19. $w = \dfrac{yz}{x}, \quad x = \theta^2, \quad y = r + \theta, \quad z = r - \theta$
20. $w = x^2 - 2xy + y^2, \quad x = r + \theta, \quad y = r - \theta$
21. $w = \arctan \dfrac{y}{x}, \quad x = r \cos \theta, \quad y = r \sin \theta$
22. $w = \sqrt{25 - 5x^2 - 5y^2}, \quad x = r \cos \theta, \quad y = r \sin \theta$

In Exercises 23–26, find $\partial w/\partial s$ and $\partial w/\partial t$ by using the appropriate Chain Rule.

23. $w = xyz, \quad x = s + t, \quad y = s - t, \quad z = st^2$
24. $w = x^2 + y^2 + z^2, \quad x = t \sin s, \quad y = t \cos s, \quad z = st^2$
25. $w = ze^{xy}, \quad x = s - t, \quad y = s + t, \quad z = st$
26. $w = x \cos yz, \quad x = s^2, \quad y = t^2, \quad z = s - 2t$

In Exercises 27–30, differentiate implicitly to find dy/dx.

27. $x^2 - xy + y^2 - x + y = 0$
28. $\sec xy + \tan xy + 5 = 0$
29. $\ln\sqrt{x^2 + y^2} + x + y = 4$
30. $\dfrac{x}{x^2 + y^2} - y^2 = 6$

In Exercises 31–38, differentiate implicitly to find the first partial derivatives of z.

31. $x^2 + y^2 + z^2 = 1$
32. $xz + yz + xy = 0$
33. $x^2 + 2yz + z^2 = 1$
34. $x + \sin(y + z) = 0$
35. $\tan(x + y) + \tan(y + z) = 1$
36. $z = e^x \sin(y + z)$
37. $e^{xz} + xy = 0$
38. $x \ln y + y^2z + z^2 = 8$

In Exercises 39–42, differentiate implicitly to find the first partial derivatives of w.

39. $xy + yz - wz + wx = 5$
40. $x^2 + y^2 + z^2 - 5yw + 10w^2 = 2$
41. $\cos xy + \sin yz + wz = 20$
42. $w - \sqrt{x - y} - \sqrt{y - z} = 0$

Homogeneous Functions **A function f is *homogeneous of degree n* if $f(tx, ty) = t^n f(x, y)$. In Exercises 43–46, (a) show that the function is homogeneous and determine n, and (b) show that $xf_x(x, y) + yf_y(x, y) = nf(x, y)$.**

43. $f(x, y) = \dfrac{xy}{\sqrt{x^2 + y^2}}$
44. $f(x, y) = x^3 - 3xy^2 + y^3$
45. $f(x, y) = e^{x/y}$
46. $f(x, y) = \dfrac{x^2}{\sqrt{x^2 + y^2}}$

47. Let $w = f(x, y)$, $x = g(t)$, and $y = h(t)$, where f, g, and h are differentiable. Use the appropriate Chain Rule to find dw/dt when $t = 2$ given the following table of values.

$g(2)$	$h(2)$	$g'(2)$	$h'(2)$	$f_x(4, 3)$	$f_y(4, 3)$
4	3	-1	6	-5	7

48. Let $w = f(x, y)$, $x = g(s, t)$, and $y = h(s, t)$, where f, g, and h are differentiable. Use the appropriate Chain Rule to find $w_s(1, 2)$ and $w_t(1, 2)$ given the following table of values.

$g(1, 2)$	$h(1, 2)$	$g_s(1, 2)$	$h_s(1, 2)$
4	3	-3	5
$g_t(1, 2)$	$h_t(1, 2)$	$f_x(4, 3)$	$f_y(4, 3)$
-2	8	-5	7

WRITING ABOUT CONCEPTS

49. Let $w = f(x, y)$ be a function in which x and y are functions of a single variable t. Give the Chain Rule for finding dw/dt.

50. Let $w = f(x, y)$ be a function in which x and y are functions of two variables s and t. Give the Chain Rule for finding $\partial w/\partial s$ and $\partial w/\partial t$.

51. If $f(x, y) = 0$, give the rule for finding dy/dx implicitly. If $f(x, y, z) = 0$, give the rule for finding $\partial z/\partial x$ and $\partial z/\partial y$ implicitly.

CAPSTONE

52. Consider the function $f(x, y, z) = xyz$, where $x = t^2$, $y = 2t$, and $z = e^{-t}$.

(a) Use the appropriate Chain Rule to find df/dt.

(b) Write f as a function of t and then find df/dt. Explain why this result is the same as that of part (a).

53. ***Volume and Surface Area*** The radius of a right circular cylinder is increasing at a rate of 6 inches per minute, and the height is decreasing at a rate of 4 inches per minute. What are the rates of change of the volume and surface area when the radius is 12 inches and the height is 36 inches?

54. ***Volume and Surface Area*** Repeat Exercise 53 for a right circular cone.

55. ***Ideal Gas Law*** The Ideal Gas Law is $pV = mRT$, where R is a constant, m is a constant mass, and p and V are functions of time. Find dT/dt, the rate at which the temperature changes with respect to time.

56. ***Area*** Let θ be the angle between equal sides of an isosceles triangle and let x be the length of these sides. x is increasing at $\frac{1}{2}$ meter per hour and θ is increasing at $\pi/90$ radian per hour. Find the rate of increase of the area when $x = 6$ and $\theta = \pi/4$.

57. ***Moment of Inertia*** An annular cylinder has an inside radius of r_1 and an outside radius of r_2 (see figure). Its moment of inertia is $I = \frac{1}{2}m(r_1^2 + r_2^2)$, where m is the mass. The two radii are increasing at a rate of 2 centimeters per second. Find the rate at which I is changing at the instant the radii are 6 centimeters and 8 centimeters. (Assume mass is a constant.)

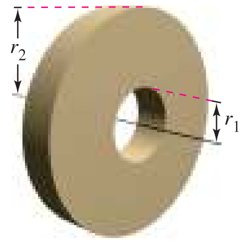


Figure for 57

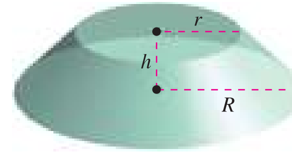


Figure for 58

58. ***Volume and Surface Area*** The two radii of the frustum of a right circular cone are increasing at a rate of 4 centimeters per minute, and the height is increasing at a rate of 12 centimeters per minute (see figure). Find the rates at which the volume and surface area are changing when the two radii are 15 centimeters and 25 centimeters, and the height is 10 centimeters.

59. Show that $(\partial w/\partial u) + (\partial w/\partial v) = 0$ for $w = f(x, y)$, $x = u - v$, and $y = v - u$.

60. Demonstrate the result of Exercise 59 for

$w = (x - y)\sin(y - x)$.

61. Consider the function $w = f(x, y)$, where $x = r\cos\theta$ and $y = r\sin\theta$. Verify each of the following.

(a) $$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial r}\cos\theta - \frac{\partial w}{\partial \theta}\frac{\sin\theta}{r}$$

$$\frac{\partial w}{\partial y} = \frac{\partial w}{\partial r}\sin\theta + \frac{\partial w}{\partial \theta}\frac{\cos\theta}{r}$$

(b) $$\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2 = \left(\frac{\partial w}{\partial r}\right)^2 + \left(\frac{1}{r^2}\right)\left(\frac{\partial w}{\partial \theta}\right)^2$$

62. Demonstrate the result of Exercise 61(b) for $w = \arctan(y/x)$.

63. ***Cauchy-Riemann Equations*** Given the functions $u(x, y)$ and $v(x, y)$, verify that the **Cauchy-Riemann differential equations**

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

can be written in polar coordinate form as

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad \text{and} \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}.$$

64. Demonstrate the result of Exercise 63 for the functions

$$u = \ln\sqrt{x^2 + y^2} \quad \text{and} \quad v = \arctan\frac{y}{x}.$$

65. Show that if $f(x,y)$ is homogeneous of degree n, then

$xf_x(x, y) + yf_y(x, y) = nf(x, y)$.

[*Hint:* Let $g(t) = f(tx, ty) = t^n f(x, y)$. Find $g'(t)$ and then let $t = 1$.]

13.6 Directional Derivatives and Gradients

- Find and use directional derivatives of a function of two variables.
- Find the gradient of a function of two variables.
- Use the gradient of a function of two variables in applications.
- Find directional derivatives and gradients of functions of three variables.

Directional Derivative

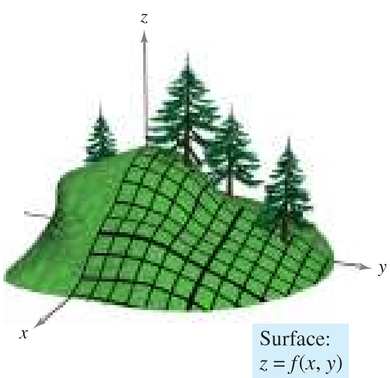


Figure 13.42

You are standing on the hillside pictured in Figure 13.42 and want to determine the hill's incline toward the z-axis. If the hill were represented by $z = f(x, y)$, you would already know how to determine the slopes in two different directions—the slope in the y-direction would be given by the partial derivative $f_y(x, y)$, and the slope in the x-direction would be given by the partial derivative $f_x(x, y)$. In this section, you will see that these two partial derivatives can be used to find the slope in *any* direction.

To determine the slope at a point on a surface, you will define a new type of derivative called a **directional derivative.** Begin by letting $z = f(x, y)$ be a *surface* and $P(x_0, y_0)$ be a *point* in the domain of f, as shown in Figure 13.43. The "direction" of the directional derivative is given by a unit vector

$$\mathbf{u} = \cos\theta\mathbf{i} + \sin\theta\mathbf{j}$$

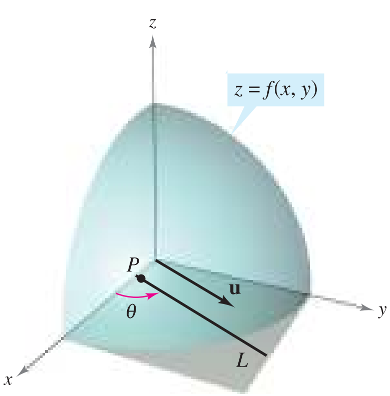


Figure 13.43

where θ is the angle the vector makes with the positive x-axis. To find the desired slope, reduce the problem to two dimensions by intersecting the surface with a vertical plane passing through the point P and parallel to $\mathbf{u}$, as shown in Figure 13.44. This vertical plane intersects the surface to form a curve C. The slope of the surface at $(x_0, y_0, f(x_0, y_0))$ in the direction of $\mathbf{u}$ is defined as the slope of the curve C at that point.

Informally, you can write the slope of the curve C as a limit that looks much like those used in single-variable calculus. The vertical plane used to form C intersects the xy-plane in a line L, represented by the parametric equations

$$x = x_0 + t\cos\theta$$

and

$$y = y_0 + t\sin\theta$$

so that for any value of t, the point $Q(x, y)$ lies on the line L. For each of the points P and Q, there is a corresponding point on the surface.

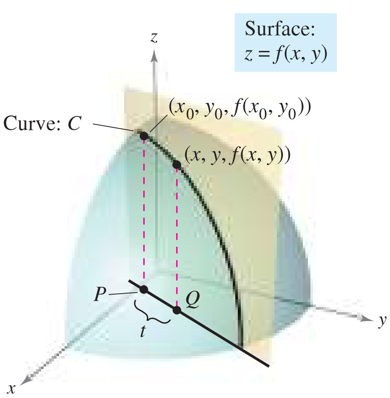


Figure 13.44

$(x_0, y_0, f(x_0, y_0))$ Point above P

$(x, y, f(x, y))$ Point above Q

Moreover, because the distance between P and Q is

$$\sqrt{(x - x_0)^2 + (y - y_0)^2} = \sqrt{(t\cos\theta)^2 + (t\sin\theta)^2} = |t|$$

you can write the slope of the secant line through $(x_0, y_0, f(x_0, y_0))$ and $(x, y, f(x, y))$ as

$$\frac{f(x, y) - f(x_0, y_0)}{t} = \frac{f(x_0 + t\cos\theta, y_0 + t\sin\theta) - f(x_0, y_0)}{t}.$$

Finally, by letting t approach 0, you arrive at the following definition.

DEFINITION OF DIRECTIONAL DERIVATIVE

Let f be a function of two variables x and y and let $\mathbf{u} = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}$ be a unit vector. Then the **directional derivative of f in the direction of u,** denoted by $D_{\mathbf{u}}f$, is

$$D_{\mathbf{u}}f(x, y) = \lim_{t\to 0} \frac{f(x + t\cos\theta, y + t\sin\theta) - f(x, y)}{t}$$

provided this limit exists.

Calculating directional derivatives by this definition is similar to finding the derivative of a function of one variable by the limit process (given in Section 2.1). A simpler "working" formula for finding directional derivatives involves the partial derivatives f_x and f_y.

THEOREM 13.9 DIRECTIONAL DERIVATIVE

If f is a differentiable function of x and y, then the directional derivative of f in the direction of the unit vector $\mathbf{u} = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}$ is

$$D_{\mathbf{u}}f(x, y) = f_x(x, y)\cos\theta + f_y(x, y)\sin\theta.$$

PROOF For a fixed point (x_0, y_0), let $x = x_0 + t\cos\theta$ and let $y = y_0 + t\sin\theta$. Then, let $g(t) = f(x, y)$. Because f is differentiable, you can apply the Chain Rule given in Theorem 13.6 to obtain

$$g'(t) = f_x(x, y)x'(t) + f_y(x, y)y'(t) = f_x(x, y)\cos\theta + f_y(x, y)\sin\theta.$$

If $t = 0$, then $x = x_0$ and $y = y_0$, so

$$g'(0) = f_x(x_0, y_0)\cos\theta + f_y(x_0, y_0)\sin\theta.$$

By the definition of $g'(t)$, it is also true that

$$\begin{aligned} g'(0) &= \lim_{t\to 0}\frac{g(t) - g(0)}{t} \\ &= \lim_{t\to 0}\frac{f(x_0 + t\cos\theta, y_0 + t\sin\theta) - f(x_0, y_0)}{t}. \end{aligned}$$

Consequently, $D_{\mathbf{u}}f(x_0, y_0) = f_x(x_0, y_0)\cos\theta + f_y(x_0, y_0)\sin\theta$. ■

There are infinitely many directional derivatives of a surface at a given point—one for each direction specified by **u**, as shown in Figure 13.45. Two of these are the partial derivatives f_x and f_y.

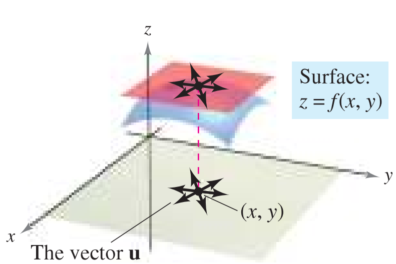


Figure 13.45

1. Direction of positive x-axis ($\theta = 0$): $\mathbf{u} = \cos 0\,\mathbf{i} + \sin 0\,\mathbf{j} = \mathbf{i}$

$$D_{\mathbf{i}}f(x, y) = f_x(x, y)\cos 0 + f_y(x, y)\sin 0 = f_x(x, y)$$

2. Direction of positive y-axis ($\theta = \pi/2$): $\mathbf{u} = \cos\frac{\pi}{2}\,\mathbf{i} + \sin\frac{\pi}{2}\,\mathbf{j} = \mathbf{j}$

$$D_{\mathbf{j}}f(x, y) = f_x(x, y)\cos\frac{\pi}{2} + f_y(x, y)\sin\frac{\pi}{2} = f_y(x, y)$$

EXAMPLE 1 Finding a Directional Derivative

Find the directional derivative of

$$f(x, y) = 4 - x^2 - \tfrac{1}{4}y^2 \qquad \text{Surface}$$

at $(1, 2)$ in the direction of

$$\mathbf{u} = \left(\cos \frac{\pi}{3}\right)\mathbf{i} + \left(\sin \frac{\pi}{3}\right)\mathbf{j}. \qquad \text{Direction}$$

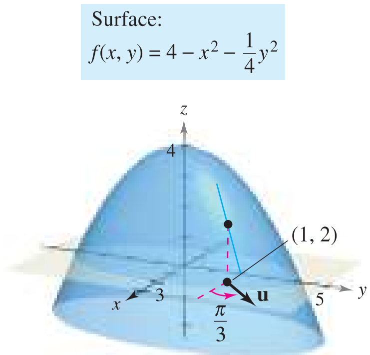


Figure 13.46

Solution Because f_x and f_y are continuous, f is differentiable, and you can apply Theorem 13.9.

$$\begin{aligned} D_{\mathbf{u}} f(x, y) &= f_x(x, y) \cos \theta + f_y(x, y) \sin \theta \\ &= (-2x) \cos \theta + \left(-\frac{y}{2}\right) \sin \theta \end{aligned}$$

Evaluating at $\theta = \pi/3$, $x = 1$, and $y = 2$ produces

$$\begin{aligned} D_{\mathbf{u}} f(1, 2) &= (-2)\left(\frac{1}{2}\right) + (-1)\left(\frac{\sqrt{3}}{2}\right) \\ &= -1 - \frac{\sqrt{3}}{2} \\ &\approx -1.866. \end{aligned}$$

See Figure 13.46. ■

NOTE Note in Figure 13.46 that you can interpret the directional derivative as giving the slope of the surface at the point $(1, 2, 2)$ in the direction of the unit vector $\mathbf{u}$. ■

You have been specifying direction by a unit vector $\mathbf{u}$. If the direction is given by a vector whose length is not 1, you must normalize the vector before applying the formula in Theorem 13.9.

EXAMPLE 2 Finding a Directional Derivative

Find the directional derivative of

$$f(x, y) = x^2 \sin 2y \qquad \text{Surface}$$

at $(1, \pi/2)$ in the direction of

$$\mathbf{v} = 3\mathbf{i} - 4\mathbf{j}. \qquad \text{Direction}$$

Solution Because f_x and f_y are continuous, f is differentiable, and you can apply Theorem 13.9. Begin by finding a unit vector in the direction of $\mathbf{v}$.

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j} = \cos \theta \mathbf{i} + \sin \theta \mathbf{j}$$

Using this unit vector, you have

$$\begin{aligned} D_{\mathbf{u}} f(x, y) &= (2x \sin 2y)(\cos \theta) + (2x^2 \cos 2y)(\sin \theta) \\ D_{\mathbf{u}} f\left(1, \frac{\pi}{2}\right) &= (2 \sin \pi)\left(\frac{3}{5}\right) + (2 \cos \pi)\left(-\frac{4}{5}\right) \\ &= (0)\left(\frac{3}{5}\right) + (-2)\left(-\frac{4}{5}\right) \\ &= \frac{8}{5}. \end{aligned}$$

See Figure 13.47. ■

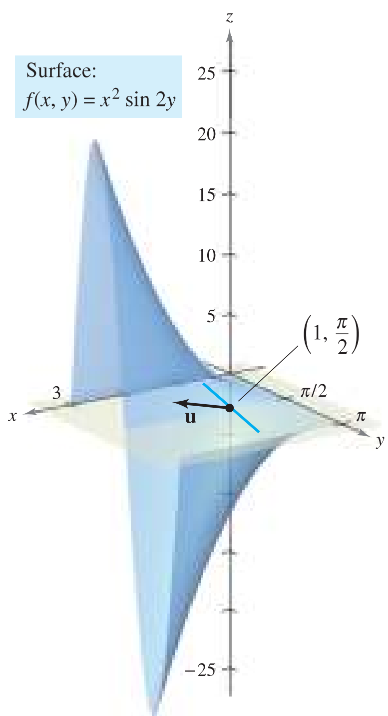
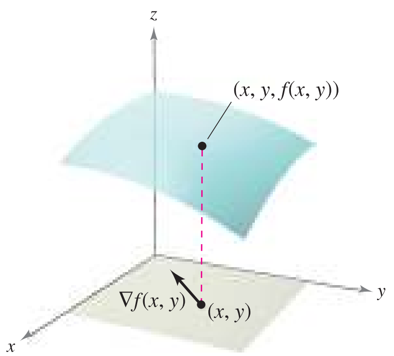


Figure 13.47

The Gradient of a Function of Two Variables



The gradient of f is a vector in the xy-plane.
Figure 13.48

The **gradient** of a function of two variables is a vector-valued function of two variables. This function has many important uses, some of which are described later in this section.

DEFINITION OF GRADIENT OF A FUNCTION OF TWO VARIABLES

Let $z = f(x, y)$ be a function of x and y such that f_x and f_y exist. Then the **gradient of *f*,** denoted by $\nabla f(x, y)$, is the vector

$$\nabla f(x, y) = f_x(x, y)\mathbf{i} + f_y(x, y)\mathbf{j}.$$

∇f is read as "del f." Another notation for the gradient is **grad** $f(x, y)$. In Figure 13.48, note that for each (x, y), the gradient $\nabla f(x, y)$ is a vector in the plane (not a vector in space).

NOTE No value is assigned to the symbol ∇ by itself. It is an operator in the same sense that d/dx is an operator. When ∇ operates on $f(x, y)$, it produces the vector $\nabla f(x, y)$. ■

EXAMPLE 3 Finding the Gradient of a Function

Find the gradient of $f(x, y) = y \ln x + xy^2$ at the point $(1, 2)$.

Solution Using

$$f_x(x, y) = \frac{y}{x} + y^2 \qquad \text{and} \qquad f_y(x, y) = \ln x + 2xy$$

you have

$$\nabla f(x, y) = \left(\frac{y}{x} + y^2\right)\mathbf{i} + (\ln x + 2xy)\mathbf{j}.$$

At the point $(1, 2)$, the gradient is

$$\begin{aligned}\nabla f(1, 2) &= \left(\frac{2}{1} + 2^2\right)\mathbf{i} + [\ln 1 + 2(1)(2)]\mathbf{j}\\ &= 6\mathbf{i} + 4\mathbf{j}.\end{aligned}$$

■

Because the gradient of f is a vector, you can write the directional derivative of f in the direction of $\mathbf{u}$ as

$$D_{\mathbf{u}} f(x, y) = [f_x(x, y)\mathbf{i} + f_y(x, y)\mathbf{j}] \cdot [\cos\theta\mathbf{i} + \sin\theta\mathbf{j}].$$

In other words, the directional derivative is the dot product of the gradient and the direction vector. This useful result is summarized in the following theorem.

THEOREM 13.10 ALTERNATIVE FORM OF THE DIRECTIONAL DERIVATIVE

If f is a differentiable function of x and y, then the directional derivative of f in the direction of the unit vector $\mathbf{u}$ is

$$D_{\mathbf{u}} f(x, y) = \nabla f(x, y) \cdot \mathbf{u}.$$

EXAMPLE 4 Using $\nabla f(x, y)$ to Find a Directional Derivative

Find the directional derivative of

$$f(x, y) = 3x^2 - 2y^2$$

at $\left(-\frac{3}{4}, 0\right)$ in the direction from $P\left(-\frac{3}{4}, 0\right)$ to $Q(0, 1)$.

Solution Because the partials of f are continuous, f is differentiable and you can apply Theorem 13.10. A vector in the specified direction is

$$\overrightarrow{PQ} = \mathbf{v} = \left(0 + \frac{3}{4}\right)\mathbf{i} + (1 - 0)\mathbf{j}$$

$$= \frac{3}{4}\mathbf{i} + \mathbf{j}$$

and a unit vector in this direction is

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j}.$$ Unit vector in direction of $\overrightarrow{PQ}$

Because $\nabla f(x, y) = f_x(x, y)\mathbf{i} + f_y(x, y)\mathbf{j} = 6x\mathbf{i} - 4y\mathbf{j}$, the gradient at $\left(-\frac{3}{4}, 0\right)$ is

$$\nabla f\left(-\frac{3}{4}, 0\right) = -\frac{9}{2}\mathbf{i} + 0\mathbf{j}.$$ Gradient at $\left(-\frac{3}{4}, 0\right)$

Consequently, at $\left(-\frac{3}{4}, 0\right)$ the directional derivative is

$$D_{\mathbf{u}} f\left(-\frac{3}{4}, 0\right) = \nabla f\left(-\frac{3}{4}, 0\right) \cdot \mathbf{u}$$

$$= \left(-\frac{9}{2}\mathbf{i} + 0\mathbf{j}\right) \cdot \left(\frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j}\right)$$

$$= -\frac{27}{10}.$$ Directional derivative at $\left(-\frac{3}{4}, 0\right)$

See Figure 13.49.

Surface:
$f(x, y) = 3x^2 - 2y^2$

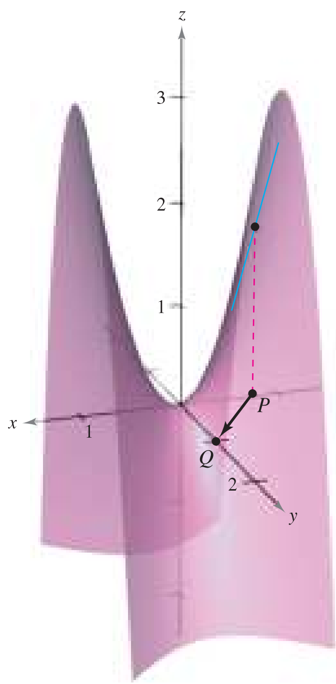


Figure 13.49

Applications of the Gradient

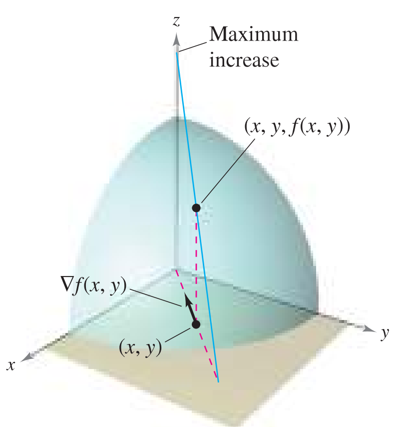
You have already seen that there are many directional derivatives at the point (x, y) on a surface. In many applications, you may want to know in which direction to move so that $f(x, y)$ increases most rapidly. This direction is called the direction of steepest ascent, and it is given by the gradient, as stated in the following theorem.

THEOREM 13.11 PROPERTIES OF THE GRADIENT

Let f be differentiable at the point (x, y).

1. If $\nabla f(x, y) = \mathbf{0}$, then $D_{\mathbf{u}} f(x, y) = 0$ for all $\mathbf{u}$.
2. The direction of *maximum* increase of f is given by $\nabla f(x, y)$. The maximum value of $D_{\mathbf{u}} f(x, y)$ is $\|\nabla f(x, y)\|$.
3. The direction of *minimum* increase of f is given by $-\nabla f(x, y)$. The minimum value of $D_{\mathbf{u}} f(x, y)$ is $-\|\nabla f(x, y)\|$.

NOTE Part 2 of Theorem 13.11 says that at the point (x, y), f increases most rapidly in the direction of the gradient, $\nabla f(x, y)$.



The gradient of f is a vector in the xy-plane that points in the direction of maximum increase on the surface given by $z = f(x, y)$.
Figure 13.50

PROOF If $\nabla f(x, y) = \mathbf{0}$, then for any direction (any $\mathbf{u}$), you have

$$\begin{aligned} D_{\mathbf{u}} f(x, y) &= \nabla f(x, y) \cdot \mathbf{u} \\ &= (0\mathbf{i} + 0\mathbf{j}) \cdot (\cos\theta\mathbf{i} + \sin\theta\mathbf{j}) \\ &= 0. \end{aligned}$$

If $\nabla f(x, y) \neq \mathbf{0}$, then let ϕ be the angle between $\nabla f(x, y)$ and a unit vector $\mathbf{u}$. Using the dot product, you can apply Theorem 11.5 to conclude that

$$\begin{aligned} D_{\mathbf{u}} f(x, y) &= \nabla f(x, y) \cdot \mathbf{u} \\ &= \|\nabla f(x, y)\| \, \|\mathbf{u}\| \cos\phi \\ &= \|\nabla f(x, y)\| \cos\phi \end{aligned}$$

and it follows that the maximum value of $D_{\mathbf{u}} f(x, y)$ will occur when $\cos\phi = 1$. So, $\phi = 0$, and the maximum value of the directional derivative occurs when $\mathbf{u}$ has the same direction as $\nabla f(x, y)$. Moreover, this largest value of $D_{\mathbf{u}} f(x, y)$ is precisely

$$\|\nabla f(x, y)\| \cos\phi = \|\nabla f(x, y)\|.$$

Similarly, the minimum value of $D_{\mathbf{u}} f(x, y)$ can be obtained by letting $\phi = \pi$ so that $\mathbf{u}$ points in the direction opposite that of $\nabla f(x, y)$, as shown in Figure 13.50. ■

To visualize one of the properties of the gradient, imagine a skier coming down a mountainside. If $f(x, y)$ denotes the altitude of the skier, then $-\nabla f(x, y)$ indicates the *compass direction* the skier should take to ski the path of steepest descent. (Remember that the gradient indicates direction in the xy-plane and does not itself point up or down the mountainside.)

As another illustration of the gradient, consider the temperature $T(x, y)$ at any point (x, y) on a flat metal plate. In this case, $\nabla T(x, y)$ gives the direction of greatest temperature increase at the point (x, y), as illustrated in the next example.

EXAMPLE 5 Finding the Direction of Maximum Increase

The temperature in degrees Celsius on the surface of a metal plate is

$$T(x, y) = 20 - 4x^2 - y^2$$

where x and y are measured in centimeters. In what direction from $(2, -3)$ does the temperature increase most rapidly? What is this rate of increase?

Solution The gradient is

$$\begin{aligned} \nabla T(x, y) &= T_x(x, y)\mathbf{i} + T_y(x, y)\mathbf{j} \\ &= -8x\,\mathbf{i} - 2y\,\mathbf{j}. \end{aligned}$$

It follows that the direction of maximum increase is given by

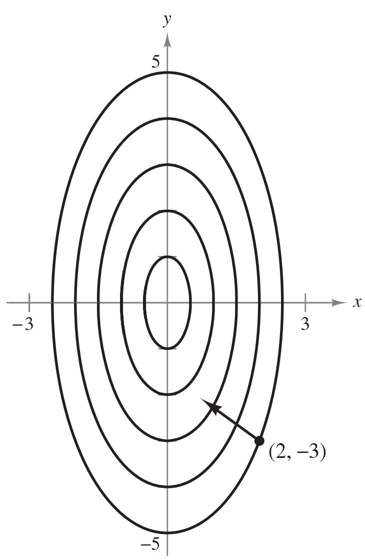
$$\nabla T(2, -3) = -16\mathbf{i} + 6\mathbf{j}$$

as shown in Figure 13.51, and the rate of increase is

$$\begin{aligned} \|\nabla T(2, -3)\| &= \sqrt{256 + 36} \\ &= \sqrt{292} \\ &\approx 17.09° \text{ per centimeter.} \end{aligned}$$

■

Level curves:
$T(x, y) = 20 - 4x^2 - y^2$



The direction of most rapid increase in temperature at $(2, -3)$ is given by $-16\mathbf{i} + 6\mathbf{j}$.
Figure 13.51

The solution presented in Example 5 can be misleading. Although the gradient points in the direction of maximum temperature increase, it does not necessarily point toward the hottest spot on the plate. In other words, the gradient provides a local solution to finding an increase relative to the temperature at the point $(2, -3)$. *Once you leave that position, the direction of maximum increase may change.*

EXAMPLE 6 Finding the Path of a Heat-Seeking Particle

A heat-seeking particle is located at the point $(2, -3)$ on a metal plate whose temperature at (x, y) is

$$T(x, y) = 20 - 4x^2 - y^2.$$

Find the path of the particle as it continuously moves in the direction of maximum temperature increase.

Solution Let the path be represented by the position function

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}.$$

A tangent vector at each point $(x(t), y(t))$ is given by

$$\mathbf{r}'(t) = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j}.$$

Because the particle seeks maximum temperature increase, the directions of $\mathbf{r}'(t)$ and $\nabla T(x, y) = -8x\mathbf{i} - 2y\mathbf{j}$ are the same at each point on the path. So,

$$-8x = k\frac{dx}{dt} \quad \text{and} \quad -2y = k\frac{dy}{dt}$$

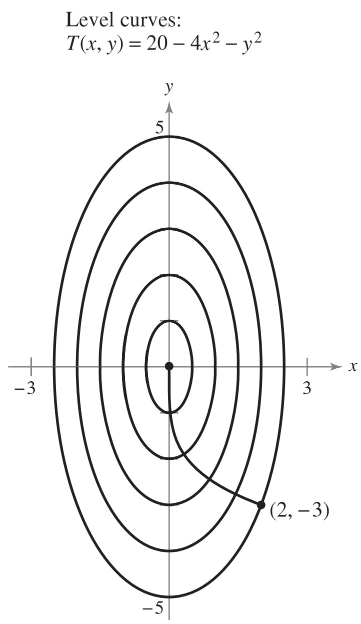
where k depends on t. By solving each equation for dt/k and equating the results, you obtain

$$\frac{dx}{-8x} = \frac{dy}{-2y}.$$

The solution of this differential equation is $x = Cy^4$. Because the particle starts at the point $(2, -3)$, you can determine that $C = 2/81$. So, the path of the heat-seeking particle is

$$x = \frac{2}{81}y^4.$$

The path is shown in Figure 13.52. ■



Path followed by a heat-seeking particle
Figure 13.52

In Figure 13.52, the path of the particle (determined by the gradient at each point) appears to be orthogonal to each of the level curves. This becomes clear when you consider that the temperature $T(x, y)$ is constant along a given level curve. So, at any point (x, y) on the curve, the rate of change of T in the direction of a unit tangent vector $\mathbf{u}$ is 0, and you can write

$$\nabla f(x, y) \cdot \mathbf{u} = D_{\mathbf{u}}T(x, y) = 0. \qquad \mathbf{u} \text{ is a unit tangent vector.}$$

Because the dot product of $\nabla f(x, y)$ and $\mathbf{u}$ is 0, you can conclude that they must be orthogonal. This result is stated in the following theorem.

THEOREM 13.12 GRADIENT IS NORMAL TO LEVEL CURVES

If f is differentiable at (x_0, y_0) and $\nabla f(x_0, y_0) \neq \mathbf{0}$, then $\nabla f(x_0, y_0)$ is normal to the level curve through (x_0, y_0).

EXAMPLE 7 Finding a Normal Vector to a Level Curve

Sketch the level curve corresponding to $c = 0$ for the function given by

$$f(x, y) = y - \sin x$$

and find a normal vector at several points on the curve.

Solution The level curve for $c = 0$ is given by

$$0 = y - \sin x$$
$$y = \sin x$$

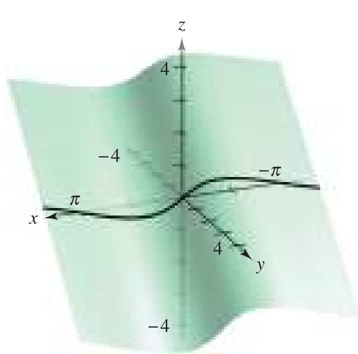
as shown in Figure 13.53(a). Because the gradient vector of f at (x, y) is

$$\begin{aligned}\nabla f(x, y) &= f_x(x, y)\mathbf{i} + f_y(x, y)\mathbf{j}\\ &= -\cos x\mathbf{i} + \mathbf{j}\end{aligned}$$

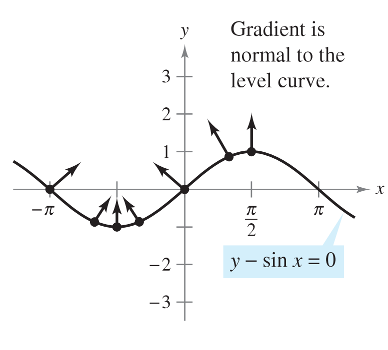
you can use Theorem 13.12 to conclude that $\nabla f(x, y)$ is normal to the level curve at the point (x, y). Some gradient vectors are

$$\nabla f(-\pi, 0) = \mathbf{i} + \mathbf{j}$$
$$\nabla f\left(-\frac{2\pi}{3}, -\frac{\sqrt{3}}{2}\right) = \frac{1}{2}\mathbf{i} + \mathbf{j}$$
$$\nabla f\left(-\frac{\pi}{2}, -1\right) = \mathbf{j}$$
$$\nabla f\left(-\frac{\pi}{3}, -\frac{\sqrt{3}}{2}\right) = -\frac{1}{2}\mathbf{i} + \mathbf{j}$$
$$\nabla f(0, 0) = -\mathbf{i} + \mathbf{j}$$
$$\nabla f\left(\frac{\pi}{3}, \frac{\sqrt{3}}{2}\right) = -\frac{1}{2}\mathbf{i} + \mathbf{j}$$
$$\nabla f\left(\frac{\pi}{2}, 1\right) = \mathbf{j}.$$

These are shown in Figure 13.53(b).



(a) The surface is given by $f(x, y) = y - \sin x$.



(b) The level curve is given by $f(x, y) = 0$.

Figure 13.53

Functions of Three Variables

The definitions of the directional derivative and the gradient can be extended naturally to functions of three or more variables. As often happens, some of the geometric interpretation is lost in the generalization from functions of two variables to those of three variables. For example, you cannot interpret the directional derivative of a function of three variables to represent slope.

The definitions and properties of the directional derivative and the gradient of a function of three variables are given in the following summary.

DIRECTIONAL DERIVATIVE AND GRADIENT FOR THREE VARIABLES

Let f be a function of x, y, and z, with continuous first partial derivatives. The **directional derivative of f** in the direction of a unit vector $\mathbf{u} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ is given by

$$D_{\mathbf{u}} f(x, y, z) = af_x(x, y, z) + bf_y(x, y, z) + cf_z(x, y, z).$$

The **gradient of f** is defined as

$$\nabla f(x, y, z) = f_x(x, y, z)\mathbf{i} + f_y(x, y, z)\mathbf{j} + f_z(x, y, z)\mathbf{k}.$$

Properties of the gradient are as follows.

1. $D_{\mathbf{u}} f(x, y, z) = \nabla f(x, y, z) \cdot \mathbf{u}$
2. If $\nabla f(x, y, z) = \mathbf{0}$, then $D_{\mathbf{u}} f(x, y, z) = 0$ for all $\mathbf{u}$.
3. The direction of *maximum* increase of f is given by $\nabla f(x, y, z)$. The maximum value of $D_{\mathbf{u}} f(x, y, z)$ is

   $$\|\nabla f(x, y, z)\|. \qquad \text{Maximum value of } D_{\mathbf{u}} f(x, y, z)$$

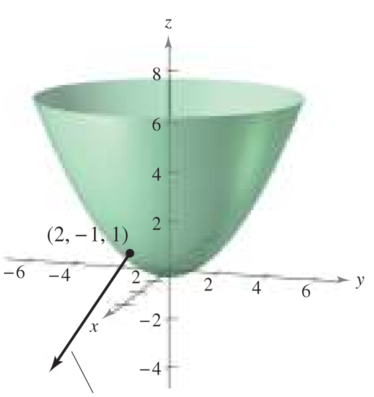
4. The direction of *minimum* increase of f is given by $-\nabla f(x, y, z)$. The minimum value of $D_{\mathbf{u}} f(x, y, z)$ is

   $$-\|\nabla f(x, y, z)\|. \qquad \text{Minimum value of } D_{\mathbf{u}} f(x, y, z)$$

NOTE You can generalize Theorem 13.12 to functions of three variables. Under suitable hypotheses,

$$\nabla f(x_0, y_0, z_0)$$

is normal to the level surface through (x_0, y_0, z_0). ■



Level surface and gradient vector at $(2, -1, 1)$ for $f(x, y, z) = x^2 + y^2 - 4z$
Figure 13.54

EXAMPLE 8 Finding the Gradient for a Function of Three Variables

Find $\nabla f(x, y, z)$ for the function given by

$$f(x, y, z) = x^2 + y^2 - 4z$$

and find the direction of maximum increase of f at the point $(2, -1, 1)$.

Solution The gradient vector is given by

$$\begin{aligned}\nabla f(x, y, z) &= f_x(x, y, z)\mathbf{i} + f_y(x, y, z)\mathbf{j} + f_z(x, y, z)\mathbf{k}\\ &= 2x\,\mathbf{i} + 2y\,\mathbf{j} - 4\mathbf{k}.\end{aligned}$$

So, it follows that the direction of maximum increase at $(2, -1, 1)$ is

$$\nabla f(2, -1, 1) = 4\,\mathbf{i} - 2\,\mathbf{j} - 4\,\mathbf{k}. \qquad \text{See Figure 13.54.}$$ ■

13.6 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–12, find the directional derivative of the function at P in the direction of $\mathbf{v}$.

1. $f(x, y) = 3x - 4xy + 9y,\quad P(1, 2),\ \mathbf{v} = \frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j}$
2. $f(x, y) = x^3 - y^3,\quad P(4, 3),\ \mathbf{v} = \frac{\sqrt{2}}{2}(\mathbf{i} + \mathbf{j})$
3. $f(x, y) = xy,\quad P(0, -2),\ \mathbf{v} = \frac{1}{2}(\mathbf{i} + \sqrt{3}\mathbf{j})$
4. $f(x, y) = \dfrac{x}{y},\quad P(1, 1),\ \mathbf{v} = -\mathbf{j}$
5. $h(x, y) = e^x \sin y,\quad P\left(1, \frac{\pi}{2}\right),\ \mathbf{v} = -\mathbf{i}$
6. $g(x, y) = \arccos xy,\quad P(1, 0),\ \mathbf{v} = \mathbf{j}$
7. $g(x, y) = \sqrt{x^2 + y^2},\quad P(3, 4),\ \mathbf{v} = 3\mathbf{i} - 4\mathbf{j}$
8. $h(x, y) = e^{-(x^2+y^2)},\quad P(0, 0),\ \mathbf{v} = \mathbf{i} + \mathbf{j}$
9. $f(x, y, z) = x^2 + y^2 + z^2,\quad P(1, 1, 1),\ \mathbf{v} = \frac{\sqrt{3}}{3}(\mathbf{i} - \mathbf{j} + \mathbf{k})$
10. $f(x, y, z) = xy + yz + xz,\quad P(1, 2, -1),\ \mathbf{v} = 2\mathbf{i} + \mathbf{j} - \mathbf{k}$
11. $h(x, y, z) = xyz,\quad P(2, 1, 1),\ \mathbf{v} = \langle 2, 1, 2\rangle$
12. $h(x, y, z) = x \arctan yz,\quad P(4, 1, 1),\ \mathbf{v} = \langle 1, 2, -1\rangle$

In Exercises 13–16, find the directional derivative of the function in the direction of the unit vector $\mathbf{u} = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}$.

13. $f(x, y) = x^2 + y^2,\quad \theta = \dfrac{\pi}{4}$
14. $f(x, y) = \dfrac{y}{x + y},\quad \theta = -\dfrac{\pi}{6}$
15. $f(x, y) = \sin(2x + y),\quad \theta = \dfrac{\pi}{3}$
16. $g(x, y) = xe^y,\quad \theta = \dfrac{2\pi}{3}$

In Exercises 17–20, find the directional derivative of the function at P in the direction of Q.

17. $f(x, y) = x^2 + 3y^2,\quad P(1, 1),\ Q(4, 5)$
18. $f(x, y) = \cos(x + y),\quad P(0, \pi),\ Q\left(\frac{\pi}{2}, 0\right)$
19. $g(x, y, z) = xye^z,\quad P(2, 4, 0),\ Q(0, 0, 0)$
20. $h(x, y, z) = \ln(x + y + z),\quad P(1, 0, 0),\ Q(4, 3, 1)$

In Exercises 21–26, find the gradient of the function at the given point.

21. $f(x, y) = 3x + 5y^2 + 1,\quad (2, 1)$
22. $g(x, y) = 2xe^{y/x},\quad (2, 0)$
23. $z = \ln(x^2 - y),\quad (2, 3)$
24. $z = \cos(x^2 + y^2),\quad (3, -4)$
25. $w = 3x^2 - 5y^2 + 2z^2,\quad (1, 1, -2)$
26. $w = x \tan(y + z),\quad (4, 3, -1)$

In Exercises 27–30, use the gradient to find the directional derivative of the function at P in the direction of Q.

27. $g(x, y) = x^2 + y^2 + 1,\quad P(1, 2),\ Q(2, 3)$
28. $f(x, y) = 3x^2 - y^2 + 4,\quad P(-1, 4),\ Q(3, 6)$
29. $f(x, y) = e^y \sin x,\quad P(0, 0),\ Q(2, 1)$
30. $f(x, y) = \sin 2x \cos y,\quad P(\pi, 0),\ Q\left(\frac{\pi}{2}, \pi\right)$

In Exercises 31–40, find the gradient of the function and the maximum value of the directional derivative at the given point.

	Function	*Point*
31.	$f(x, y) = x^2 + 2xy$	$(1, 0)$
32.	$f(x, y) = \dfrac{x + y}{y + 1}$	$(0, 1)$
33.	$h(x, y) = x \tan y$	$\left(2, \frac{\pi}{4}\right)$
34.	$h(x, y) = y \cos(x - y)$	$\left(0, \frac{\pi}{3}\right)$
35.	$g(x, y) = ye^{-x}$	$(0, 5)$
36.	$g(x, y) = \ln \sqrt[3]{x^2 + y^2}$	$(1, 2)$
37.	$f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$	$(1, 4, 2)$
38.	$w = \dfrac{1}{\sqrt{1 - x^2 - y^2 - z^2}}$	$(0, 0, 0)$
39.	$w = xy^2z^2$	$(2, 1, 1)$
40.	$f(x, y, z) = xe^{yz}$	$(2, 0, -4)$

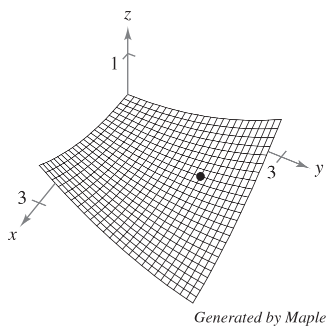
In Exercises 41–46, consider the function $f(x, y) = 3 - \dfrac{x}{3} - \dfrac{y}{2}$.

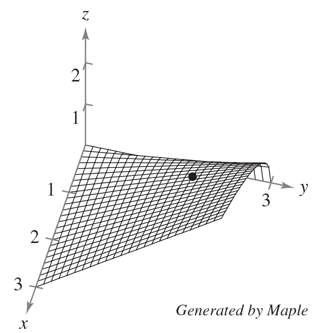
41. Sketch the graph of f in the first octant and plot the point $(3, 2, 1)$ on the surface.
42. Find $D_{\mathbf{u}}f(3, 2)$, where $\mathbf{u} = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}$, using each given value of θ.
    (a) $\theta = \dfrac{\pi}{4}$ (b) $\theta = \dfrac{2\pi}{3}$
    (c) $\theta = \dfrac{4\pi}{3}$ (d) $\theta = -\dfrac{\pi}{6}$
43. Find $D_{\mathbf{u}}f(3, 2)$, where $\mathbf{u} = \dfrac{\mathbf{v}}{\|\mathbf{v}\|}$, using each given vector $\mathbf{v}$.
    (a) $\mathbf{v} = \mathbf{i} + \mathbf{j}$
    (b) $\mathbf{v} = -3\mathbf{i} - 4\mathbf{j}$
    (c) $\mathbf{v}$ is the vector from $(1, 2)$ to $(-2, 6)$.
    (d) $\mathbf{v}$ is the vector from $(3, 2)$ to $(4, 5)$.
44. Find $\nabla f(x, y)$.
45. Find the maximum value of the directional derivative at $(3, 2)$.
46. Find a unit vector $\mathbf{u}$ orthogonal to $\nabla f(3, 2)$ and calculate $D_{\mathbf{u}}f(3, 2)$. Discuss the geometric meaning of the result.

Investigation **In Exercises 47 and 48, (a) use the graph to estimate the components of the vector in the direction of the maximum rate of increase in the function at the given point. (b) Find the gradient at the point and compare it with your estimate in part (a). (c) In what direction would the function be decreasing at the greatest rate? Explain.**

47. $f(x, y) = \frac{1}{10}(x^2 - 3xy + y^2)$, $(1, 2)$

48. $f(x, y) = \frac{1}{2}y\sqrt{x}$, $(1, 2)$





CAS **49.** ***Investigation*** Consider the function

$$f(x, y) = x^2 - y^2$$

at the point $(4, -3, 7)$.

(a) Use a computer algebra system to graph the surface represented by the function.

(b) Determine the directional derivative $D_{\mathbf{u}}f(4, -3)$ as a function of θ, where $\mathbf{u} = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}$. Use a computer algebra system to graph the function on the interval $[0, 2\pi)$.

(c) Approximate the zeros of the function in part (b) and interpret each in the context of the problem.

(d) Approximate the critical numbers of the function in part (b) and interpret each in the context of the problem.

(e) Find $\|\nabla f(4, -3)\|$ and explain its relationship to your answers in part (d).

(f) Use a computer algebra system to graph the level curve of the function f at the level $c = 7$. On this curve, graph the vector in the direction of $\nabla f(4, -3)$, and state its relationship to the level curve.

50. ***Investigation*** Consider the function

$$f(x, y) = \frac{8y}{1 + x^2 + y^2}.$$

(a) Analytically verify that the level curve of $f(x, y)$ at the level $c = 2$ is a circle.

(b) At the point $(\sqrt{3}, 2)$ on the level curve for which $c = 2$, sketch the vector showing the direction of the greatest rate of increase of the function. (To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.)

(c) At the point $(\sqrt{3}, 2)$ on the level curve, sketch a vector such that the directional derivative is 0.

CAS (d) Use a computer algebra system to graph the surface to verify your answers in parts (a)–(c).

In Exercises 51–54, find a normal vector to the level curve $f(x, y) = c$ at P.

51. $f(x, y) = 6 - 2x - 3y$
$c = 6, \quad P(0, 0)$

52. $f(x, y) = x^2 + y^2$
$c = 25, \quad P(3, 4)$

53. $f(x, y) = xy$
$c = -3, \quad P(-1, 3)$

54. $f(x, y) = \dfrac{x}{x^2 + y^2}$
$c = \frac{1}{2}, \quad P(1, 1)$

In Exercises 55–58, (a) find the gradient of the function at P, (b) find a unit normal vector to the level curve $f(x, y) = c$ at P, (c) find the tangent line to the level curve $f(x, y) = c$ at P, and (d) sketch the level curve, the unit normal vector, and the tangent line in the xy-plane.

55. $f(x, y) = 4x^2 - y$
$c = 6, P(2, 10)$

56. $f(x, y) = x - y^2$
$c = 3, P(4, -1)$

57. $f(x, y) = 3x^2 - 2y^2$
$c = 1, P(1, 1)$

58. $f(x, y) = 9x^2 + 4y^2$
$c = 40, P(2, -1)$

WRITING ABOUT CONCEPTS

59. Define the derivative of the function $z = f(x, y)$ in the direction $\mathbf{u} = \cos\theta\mathbf{i} + \sin\theta\mathbf{j}$.

60. Write a paragraph describing the directional derivative of the function f in the direction $\mathbf{u} = \cos\theta\mathbf{i} + \sin\theta\mathbf{j}$ when (a) $\theta = 0°$ and (b) $\theta = 90°$.

61. Define the gradient of a function of two variables. State the properties of the gradient.

62. Sketch the graph of a surface and select a point P on the surface. Sketch a vector in the xy-plane giving the direction of steepest ascent on the surface at P.

63. Describe the relationship of the gradient to the level curves of a surface given by $z = f(x, y)$.

CAPSTONE

64. Consider the function $f(x, y) = 9 - x^2 - y^2$.

(a) Sketch the graph of f in the first octant and plot the point $(1, 2, 4)$ on the surface.

(b) Find $D_{\mathbf{u}}f(1, 2)$, where $\mathbf{u} = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}$, for $\theta = -\pi/4$.

(c) Repeat part (b) for $\theta = \pi/3$.

(d) Find $\nabla f(1, 2)$ and $\|\nabla f(1, 2)\|$.

(e) Find a unit vector $\mathbf{u}$ orthogonal to $\nabla f(1, 2)$ and calculate $D_{\mathbf{u}}f(1, 2)$. Discuss the geometric meaning of the result.

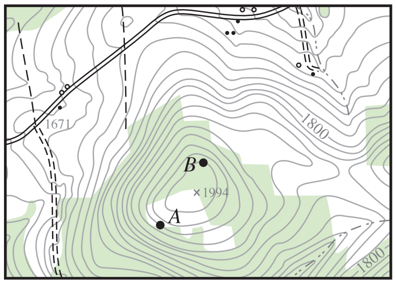
65. ***Temperature Distribution*** The temperature at the point (x, y) on a metal plate is

$$T = \frac{x}{x^2 + y^2}.$$

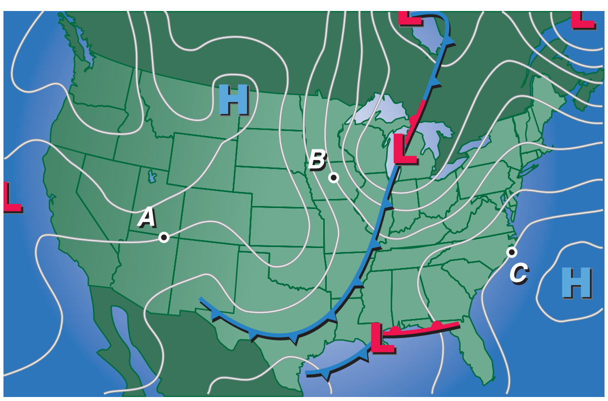
Find the direction of greatest increase in heat from the point $(3, 4)$.

66. ***Topography*** The surface of a mountain is modeled by the equation $h(x, y) = 5000 - 0.001x^2 - 0.004y^2$. A mountain climber is at the point $(500, 300, 4390)$. In what direction should the climber move in order to ascend at the greatest rate?

67. ***Topography*** The figure shows a topographic map carried by a group of hikers. Sketch the paths of steepest descent if the hikers start at point A and if they start at point B. (To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.)



68. ***Meteorology*** Meteorologists measure the atmospheric pressure in units called millibars. From these observations they create weather maps on which the curves of equal atmospheric pressure (isobars) are drawn (see figure). These are level curves to the function $P(x, y)$ yielding the pressure at any point. Sketch the gradients to the isobars at the points A, B, and C. Although the magnitudes of the gradients are unknown, their lengths relative to each other can be estimated. At which of the three points is the wind speed greatest if the speed increases as the pressure gradient increases? (To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.)



Heat-Seeking Path **In Exercises 69 and 70, find the path of a heat-seeking particle placed at point P on a metal plate with a temperature field $T(x, y)$.**

	Temperature Field	*Point*
69.	$T(x, y) = 400 - 2x^2 - y^2$	$P(10, 10)$
70.	$T(x, y) = 100 - x^2 - 2y^2$	$P(4, 3)$

71. ***Temperature*** The temperature at the point (x, y) on a metal plate is modeled by $T(x, y) = 400e^{-(x^2+y)/2}$, $x \ge 0$, $y \ge 0$.

CAS (a) Use a computer algebra system to graph the temperature distribution function.

(b) Find the directions of no change in heat on the plate from the point $(3, 5)$.

(c) Find the direction of greatest increase in heat from the point $(3, 5)$.

CAS **72.** ***Investigation*** A team of oceanographers is mapping the ocean floor to assist in the recovery of a sunken ship. Using sonar, they develop the model

$$D = 250 + 30x^2 + 50 \sin \frac{\pi y}{2}, \quad 0 \le x \le 2, 0 \le y \le 2$$

where D is the depth in meters, and x and y are the distances in kilometers.

(a) Use a computer algebra system to graph the surface.

(b) Because the graph in part (a) is showing depth, it is not a map of the ocean floor. How could the model be changed so that the graph of the ocean floor could be obtained?

(c) What is the depth of the ship if it is located at the coordinates $x = 1$ and $y = 0.5$?

(d) Determine the steepness of the ocean floor in the positive x-direction from the position of the ship.

(e) Determine the steepness of the ocean floor in the positive y-direction from the position of the ship.

(f) Determine the direction of the greatest rate of change of depth from the position of the ship.

True or False? **In Exercises 73–76, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

73. If $f(x, y) = \sqrt{1 - x^2 - y^2}$, then $D_{\mathbf{u}}f(0, 0) = 0$ for any unit vector $\mathbf{u}$.

74. If $f(x, y) = x + y$, then $-1 \le D_{\mathbf{u}}f(x, y) \le 1$.

75. If $D_{\mathbf{u}}f(x, y)$ exists, then $D_{\mathbf{u}}f(x, y) = -D_{-\mathbf{u}}f(x, y)$.

76. If $D_{\mathbf{u}}f(x_0, y_0) = c$ for any unit vector $\mathbf{u}$, then $c = 0$.

77. Find a function f such that $\nabla f = e^x \cos y\,\mathbf{i} - e^x \sin y\,\mathbf{j} + z\,\mathbf{k}$.

78. Consider the function

$$f(x, y) = \begin{cases} \dfrac{4xy}{x^2 + y^2}, & (x, y) \ne (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$

and the unit vector $\mathbf{u} = \dfrac{1}{\sqrt{2}}(\mathbf{i} + \mathbf{j})$.

Does the directional derivative of f at $P(0, 0)$ in the direction of $\mathbf{u}$ exist? If $f(0, 0)$ were defined as 2 instead of 0, would the directional derivative exist?

79. Consider the function $f(x, y) = \sqrt[3]{xy}$.

(a) Show that f is continuous at the origin.

(b) Show that f_x and f_y exist at the origin, but that the directional derivatives at the origin in all other directions do not exist.

CAS (c) Use a computer algebra system to graph f near the origin to verify your answers in parts (a) and (b). Explain.

13.7 Tangent Planes and Normal Lines

- Find equations of tangent planes and normal lines to surfaces.
- Find the angle of inclination of a plane in space.
- Compare the gradients $\nabla f(x, y)$ and $\nabla F(x, y, z)$.

Tangent Plane and Normal Line to a Surface

So far you have represented surfaces in space primarily by equations of the form

$$z = f(x, y). \qquad \text{Equation of a surface } S$$

In the development to follow, however, it is convenient to use the more general representation $F(x, y, z) = 0$. For a surface S given by $z = f(x, y)$, you can convert to the general form by defining F as

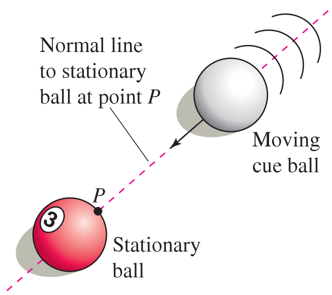
$$F(x, y, z) = f(x, y) - z.$$

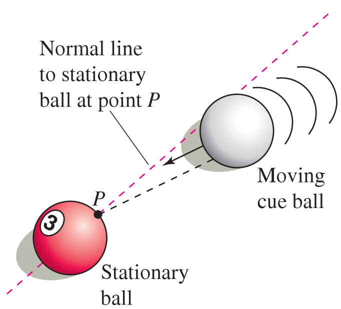
Because $f(x, y) - z = 0$, you can consider S to be the level surface of F given by

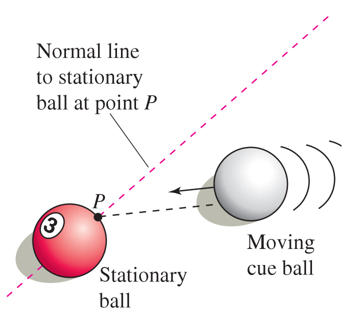
$$F(x, y, z) = 0. \qquad \text{Alternative equation of surface } S$$

EXPLORATION

Billiard Balls and Normal Lines
In each of the three figures below, the cue ball is about to strike a stationary ball at point P. Explain how you can use the normal line to the stationary ball at point P to describe the resulting motion of each of the two balls. Assuming that each cue ball has the same speed, which stationary ball will acquire the greatest speed? Which will acquire the least? Explain your reasoning.







EXAMPLE 1 Writing an Equation of a Surface

For the function given by

$$F(x, y, z) = x^2 + y^2 + z^2 - 4$$

describe the level surface given by $F(x, y, z) = 0$.

Solution The level surface given by $F(x, y, z) = 0$ can be written as

$$x^2 + y^2 + z^2 = 4$$

which is a sphere of radius 2 whose center is at the origin. ■

You have seen many examples of the usefulness of normal lines in applications involving curves. Normal lines are equally important in analyzing surfaces and solids. For example, consider the collision of two billiard balls. When a stationary ball is struck at a point P on its surface, it moves along the **line of impact** determined by P and the center of the ball. The impact can occur in *two* ways. If the cue ball is moving along the line of impact, it stops dead and imparts all of its momentum to the stationary ball, as shown in Figure 13.55. If the cue ball is not moving along the line of impact, it is deflected to one side or the other and retains part of its momentum. That part of the momentum that is transferred to the stationary ball occurs along the line of impact, *regardless* of the direction of the cue ball, as shown in Figure 13.56. This line of impact is called the **normal line** to the surface of the ball at the point P.

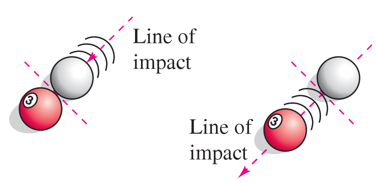


Figure 13.55

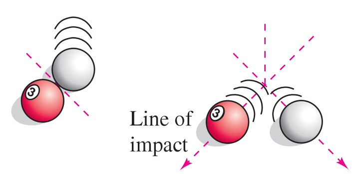


Figure 13.56

In the process of finding a normal line to a surface, you are also able to solve the problem of finding a **tangent plane** to the surface. Let S be a surface given by

$$F(x, y, z) = 0$$

and let $P(x_0, y_0, z_0)$ be a point on S. Let C be a curve on S through P that is defined by the vector-valued function

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}.$$

Then, for all t,

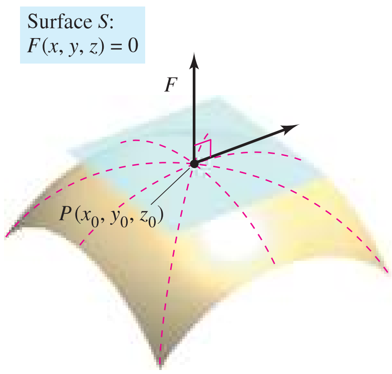
$$F(x(t),\ y(t),\ z(t)) = 0.$$

If F is differentiable and $x'(t)$, $y'(t)$, and $z'(t)$ all exist, it follows from the Chain Rule that

$$\begin{aligned} 0 &= F'(t) \\ &= F_x(x, y, z)x'(t) + F_y(x, y, z)y'(t) + F_z(x, y, z)z'(t). \end{aligned}$$

At (x_0, y_0, z_0), the equivalent vector form is

$$0 = \underbrace{\nabla F(x_0, y_0, z_0)}_{\text{Gradient}} \cdot \underbrace{\mathbf{r}'(t_0)}_{\text{Tangent vector}}.$$



Tangent plane to surface S at P
Figure 13.57

This result means that the gradient at P is orthogonal to the tangent vector of every curve on S through P. So, all tangent lines on S lie in a plane that is normal to $\nabla F(x_0, y_0, z_0)$ and contains P, as shown in Figure 13.57.

DEFINITIONS OF TANGENT PLANE AND NORMAL LINE

Let F be differentiable at the point $P(x_0, y_0, z_0)$ on the surface S given by $F(x, y, z) = 0$ such that $\nabla F(x_0, y_0, z_0) \neq \mathbf{0}$.

1. The plane through P that is normal to $\nabla F(x_0, y_0, z_0)$ is called the **tangent plane to S at P.**
2. The line through P having the direction of $\nabla F(x_0, y_0, z_0)$ is called the **normal line to S at P.**

NOTE In the remainder of this section, assume $\nabla F(x_0, y_0, z_0)$ to be nonzero unless stated otherwise. ■

To find an equation for the tangent plane to S at (x_0, y_0, z_0), let (x, y, z) be an arbitrary point in the tangent plane. Then the vector

$$\mathbf{v} = (x - x_0)\mathbf{i} + (y - y_0)\mathbf{j} + (z - z_0)\mathbf{k}$$

lies in the tangent plane. Because $\nabla F(x_0, y_0, z_0)$ is normal to the tangent plane at (x_0, y_0, z_0), it must be orthogonal to every vector in the tangent plane, and you have $\nabla F(x_0, y_0, z_0) \cdot \mathbf{v} = 0$, which leads to the following theorem.

THEOREM 13.13 EQUATION OF TANGENT PLANE

If F is differentiable at (x_0, y_0, z_0), then an equation of the tangent plane to the surface given by $F(x, y, z) = 0$ at (x_0, y_0, z_0) is

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0.$$

EXAMPLE 2 Finding an Equation of a Tangent Plane

Find an equation of the tangent plane to the hyperboloid given by

$$z^2 - 2x^2 - 2y^2 = 12$$

at the point $(1, -1, 4)$.

Solution Begin by writing the equation of the surface as

$$z^2 - 2x^2 - 2y^2 - 12 = 0.$$

Then, considering

$$F(x, y, z) = z^2 - 2x^2 - 2y^2 - 12$$

you have

$$F_x(x, y, z) = -4x, \qquad F_y(x, y, z) = -4y, \quad \text{and} \quad F_z(x, y, z) = 2z.$$

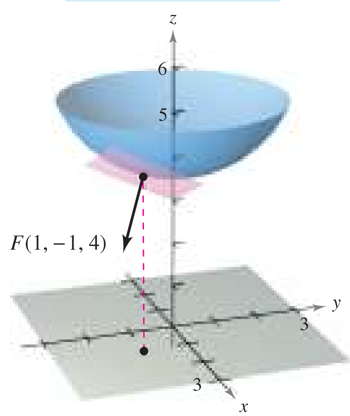
At the point $(1, -1, 4)$ the partial derivatives are

$$F_x(1, -1, 4) = -4, \qquad F_y(1, -1, 4) = 4, \quad \text{and} \quad F_z(1, -1, 4) = 8.$$

So, an equation of the tangent plane at $(1, -1, 4)$ is

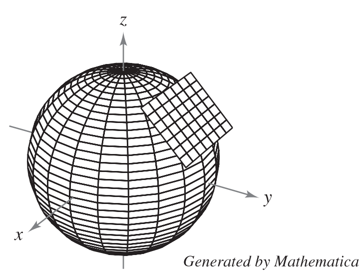
$$\begin{aligned} -4(x - 1) + 4(y + 1) + 8(z - 4) &= 0 \\ -4x + 4 + 4y + 4 + 8z - 32 &= 0 \\ -4x + 4y + 8z - 24 &= 0 \\ x - y - 2z + 6 &= 0. \end{aligned}$$

Figure 13.58 shows a portion of the hyperboloid and tangent plane. ■

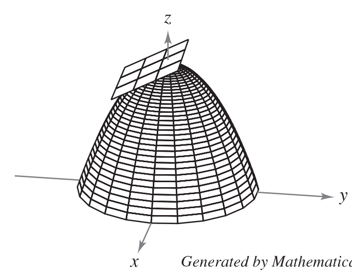


Tangent plane to surface
Figure 13.58

TECHNOLOGY Some three-dimensional graphing utilities are capable of graphing tangent planes to surfaces. Two examples are shown below.



Sphere: $x^2 + y^2 + z^2 = 1$



Paraboloid: $z = 2 - x^2 - y^2$

To find the equation of the tangent plane at a point on a surface given by $z = f(x, y)$, you can define the function F by

$$F(x, y, z) = f(x, y) - z.$$

Then S is given by the level surface $F(x, y, z) = 0$, and by Theorem 13.13 an equation of the tangent plane to S at the point (x_0, y_0, z_0) is

$$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) - (z - z_0) = 0.$$

EXAMPLE 3 Finding an Equation of the Tangent Plane

Find the equation of the tangent plane to the paraboloid

$$z = 1 - \frac{1}{10}(x^2 + 4y^2)$$

at the point $\left(1, 1, \frac{1}{2}\right)$.

Solution From $z = f(x, y) = 1 - \frac{1}{10}(x^2 + 4y^2)$, you obtain

$$f_x(x, y) = -\frac{x}{5} \quad \Rightarrow \quad f_x(1, 1) = -\frac{1}{5}$$

and

$$f_y(x, y) = -\frac{4y}{5} \quad \Rightarrow \quad f_y(1, 1) = -\frac{4}{5}.$$

So, an equation of the tangent plane at $\left(1, 1, \frac{1}{2}\right)$ is

$$f_x(1, 1)(x - 1) + f_y(1, 1)(y - 1) - \left(z - \frac{1}{2}\right) = 0$$

$$-\frac{1}{5}(x - 1) - \frac{4}{5}(y - 1) - \left(z - \frac{1}{2}\right) = 0$$

$$-\frac{1}{5}x - \frac{4}{5}y - z + \frac{3}{2} = 0.$$

This tangent plane is shown in Figure 13.59. ■

Surface: $z = 1 - \frac{1}{10}(x^2 + 4y^2)$

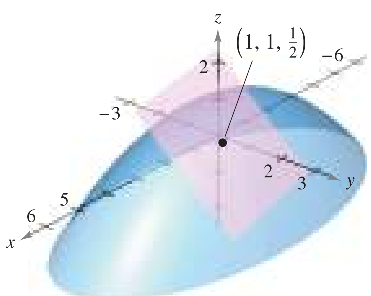


Figure 13.59

The gradient $\nabla F(x, y, z)$ provides a convenient way to find equations of normal lines, as shown in Example 4.

EXAMPLE 4 Finding an Equation of a Normal Line to a Surface

Find a set of symmetric equations for the normal line to the surface given by $xyz = 12$ at the point $(2, -2, -3)$.

Solution Begin by letting

$$F(x, y, z) = xyz - 12.$$

Then, the gradient is given by

$$\begin{aligned}\nabla F(x, y, z) &= F_x(x, y, z)\mathbf{i} + F_y(x, y, z)\mathbf{j} + F_z(x, y, z)\mathbf{k} \\ &= yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}\end{aligned}$$

and at the point $(2, -2, -3)$ you have

$$\begin{aligned}\nabla F(2, -2, -3) &= (-2)(-3)\mathbf{i} + (2)(-3)\mathbf{j} + (2)(-2)\mathbf{k} \\ &= 6\mathbf{i} - 6\mathbf{j} - 4\mathbf{k}.\end{aligned}$$

The normal line at $(2, -2, -3)$ has direction numbers 6, -6, and -4, and the corresponding set of symmetric equations is

$$\frac{x - 2}{6} = \frac{y + 2}{-6} = \frac{z + 3}{-4}.$$

See Figure 13.60. ■

Surface: $xyz = 12$

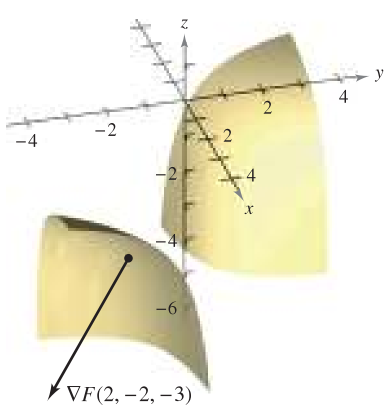


Figure 13.60

Knowing that the gradient $\nabla F(x, y, z)$ is normal to the surface given by $F(x, y, z) = 0$ allows you to solve a variety of problems dealing with surfaces and curves in space.

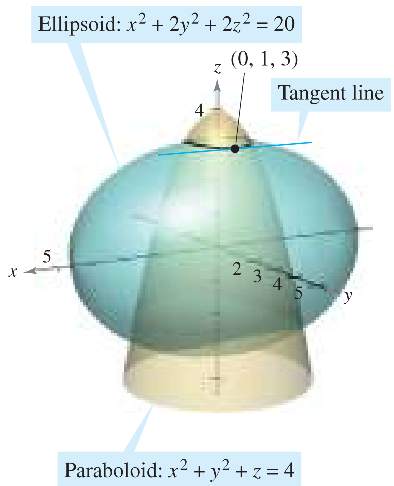


Figure 13.61

EXAMPLE 5 Finding the Equation of a Tangent Line to a Curve

Describe the tangent line to the curve of intersection of the surfaces

$$x^2 + 2y^2 + 2z^2 = 20 \qquad \text{Ellipsoid}$$

$$x^2 + y^2 + z = 4 \qquad \text{Paraboloid}$$

at the point $(0, 1, 3)$, as shown in Figure 13.61.

Solution Begin by finding the gradients to both surfaces at the point $(0, 1, 3)$.

Ellipsoid

$$F(x, y, z) = x^2 + 2y^2 + 2z^2 - 20$$
$$\nabla F(x, y, z) = 2x\mathbf{i} + 4y\mathbf{j} + 4z\mathbf{k}$$
$$\nabla F(0, 1, 3) = 4\mathbf{j} + 12\mathbf{k}$$

Paraboloid

$$G(x, y, z) = x^2 + y^2 + z - 4$$
$$\nabla G(x, y, z) = 2x\mathbf{i} + 2y\mathbf{j} + \mathbf{k}$$
$$\nabla G(0, 1, 3) = 2\mathbf{j} + \mathbf{k}$$

The cross product of these two gradients is a vector that is tangent to both surfaces at the point $(0, 1, 3)$.

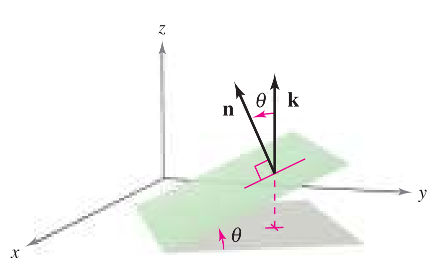
$$\nabla F(0, 1, 3) \times \nabla G(0, 1, 3) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 4 & 12 \\ 0 & 2 & 1 \end{vmatrix} = -20\mathbf{i}$$

So, the tangent line to the curve of intersection of the two surfaces at the point $(0, 1, 3)$ is a line that is parallel to the x-axis and passes through the point $(0, 1, 3)$. ■

The Angle of Inclination of a Plane

Another use of the gradient $\nabla F(x, y, z)$ is to determine the angle of inclination of the tangent plane to a surface. The **angle of inclination** of a plane is defined as the angle θ $(0 \leq \theta \leq \pi/2)$ between the given plane and the xy-plane, as shown in Figure 13.62. (The angle of inclination of a horizontal plane is defined as zero.) Because the vector $\mathbf{k}$ is normal to the xy-plane, you can use the formula for the cosine of the angle between two planes (given in Section 11.5) to conclude that the angle of inclination of a plane with normal vector $\mathbf{n}$ is given by

$$\cos\theta = \frac{|\mathbf{n} \cdot \mathbf{k}|}{\|\mathbf{n}\|\,\|\mathbf{k}\|} = \frac{|\mathbf{n} \cdot \mathbf{k}|}{\|\mathbf{n}\|}. \qquad \text{Angle of inclination of a plane}$$



The angle of inclination
Figure 13.62

EXAMPLE 6 Finding the Angle of Inclination of a Tangent Plane

Find the angle of inclination of the tangent plane to the ellipsoid given by

$$\frac{x^2}{12} + \frac{y^2}{12} + \frac{z^2}{3} = 1$$

at the point $(2, 2, 1)$.

Solution If you let

$$F(x, y, z) = \frac{x^2}{12} + \frac{y^2}{12} + \frac{z^2}{3} - 1$$

the gradient of F at the point $(2, 2, 1)$ is given by

$$\nabla F(x, y, z) = \frac{x}{6}\mathbf{i} + \frac{y}{6}\mathbf{j} + \frac{2z}{3}\mathbf{k}$$

$$\nabla F(2, 2, 1) = \frac{1}{3}\mathbf{i} + \frac{1}{3}\mathbf{j} + \frac{2}{3}\mathbf{k}.$$

Because $\nabla F(2, 2, 1)$ is normal to the tangent plane and $\mathbf{k}$ is normal to the xy-plane, it follows that the angle of inclination of the tangent plane is given by

$$\cos\theta = \frac{|\nabla F(2, 2, 1) \cdot \mathbf{k}|}{\|\nabla F(2, 2, 1)\|} = \frac{2/3}{\sqrt{(1/3)^2 + (1/3)^2 + (2/3)^2}} = \sqrt{\frac{2}{3}}$$

which implies that

$$\theta = \arccos\sqrt{\frac{2}{3}} \approx 35.3°,$$

as shown in Figure 13.63. ■

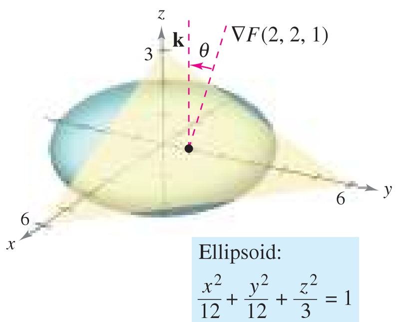


Figure 13.63

NOTE A special case of the procedure shown in Example 6 is worth noting. The angle of inclination θ of the tangent plane to the surface $z = f(x, y)$ at (x_0, y_0, z_0) is given by

$$\cos\theta = \frac{1}{\sqrt{[f_x(x_0, y_0)]^2 + [f_y(x_0, y_0)]^2 + 1}}.$$

Alternative formula for angle of inclination (See Exercise 77.) ■

A Comparison of the Gradients $\nabla f(x, y)$ and $\nabla F(x, y, z)$

This section concludes with a comparison of the gradients $\nabla f(x, y)$ and $\nabla F(x, y, z)$. In the preceding section, you saw that the gradient of a function f of two variables is normal to the level curves of f. Specifically, Theorem 13.12 states that if f is differentiable at (x_0, y_0) and $\nabla f(x_0, y_0) \neq \mathbf{0}$, then $\nabla f(x_0, y_0)$ is normal to the level curve through (x_0, y_0). Having developed normal lines to surfaces, you can now extend this result to a function of three variables. The proof of Theorem 13.14 is left as an exercise (see Exercise 78).

THEOREM 13.14 GRADIENT IS NORMAL TO LEVEL SURFACES

If F is differentiable at (x_0, y_0, z_0) and $\nabla F(x_0, y_0, z_0) \neq \mathbf{0}$, then $\nabla F(x_0, y_0, z_0)$ is normal to the level surface through (x_0, y_0, z_0).

When working with the gradients $\nabla f(x, y)$ and $\nabla F(x, y, z)$, be sure you remember that $\nabla f(x, y)$ is a vector in the xy-plane and $\nabla F(x, y, z)$ is a vector in space.

13.7 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–4, describe the level surface $F(x, y, z) = 0$.

1. $F(x, y, z) = 3x - 5y + 3z - 15$
2. $F(x, y, z) = x^2 + y^2 + z^2 - 25$
3. $F(x, y, z) = 4x^2 + 9y^2 - 4z^2$
4. $F(x, y, z) = 16x^2 - 9y^2 + 36z$

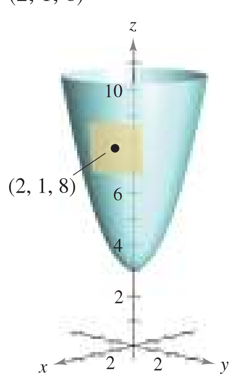
In Exercises 5–16, find a unit normal vector to the surface at the given point. [*Hint:* Normalize the gradient vector $\nabla F(x, y, z)$.]

	Surface	*Point*
5.	$3x + 4y + 12z = 0$	$(0, 0, 0)$
6.	$x + y + z = 4$	$(2, 0, 2)$
7.	$x^2 + y^2 + z^2 = 6$	$(1, 1, 2)$
8.	$z = \sqrt{x^2 + y^2}$	$(3, 4, 5)$
9.	$z = x^3$	$(2, -1, 8)$
10.	$x^2y^4 - z = 0$	$(1, 2, 16)$
11.	$x^2 + 3y + z^3 = 9$	$(2, -1, 2)$
12.	$x^2y^3 - y^2z + 2xz^3 = 4$	$(-1, 1, -1)$
13.	$\ln\left(\dfrac{x}{y - z}\right) = 0$	$(1, 4, 3)$
14.	$ze^{x^2-y^2} - 3 = 0$	$(2, 2, 3)$
15.	$z - x \sin y = 4$	$\left(6, \dfrac{\pi}{6}, 7\right)$
16.	$\sin(x - y) - z = 2$	$\left(\dfrac{\pi}{3}, \dfrac{\pi}{6}, -\dfrac{3}{2}\right)$

In Exercises 17–30, find an equation of the tangent plane to the surface at the given point.

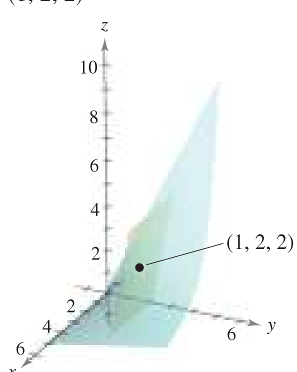
17. $z = x^2 + y^2 + 3$

$(2, 1, 8)$



18. $f(x, y) = \dfrac{y}{x}$

$(1, 2, 2)$



19. $z = \sqrt{x^2 + y^2}$, $(3, 4, 5)$
20. $g(x, y) = \arctan \dfrac{y}{x}$, $(1, 0, 0)$
21. $g(x, y) = x^2 + y^2$, $(1, -1, 2)$
22. $f(x, y) = x^2 - 2xy + y^2$, $(1, 2, 1)$
23. $z = 2 - \frac{2}{3}x - y$, $(3, -1, 1)$
24. $z = e^x(\sin y + 1)$, $\left(0, \dfrac{\pi}{2}, 2\right)$
25. $h(x, y) = \ln \sqrt{x^2 + y^2}$, $(3, 4, \ln 5)$
26. $h(x, y) = \cos y$, $\left(5, \dfrac{\pi}{4}, \dfrac{\sqrt{2}}{2}\right)$
27. $x^2 + 4y^2 + z^2 = 36$, $(2, -2, 4)$
28. $x^2 + 2z^2 = y^2$, $(1, 3, -2)$
29. $xy^2 + 3x - z^2 = 8$, $(1, -3, 2)$
30. $x = y(2z - 3)$, $(4, 4, 2)$

In Exercises 31–40, find an equation of the tangent plane and find symmetric equations of the normal line to the surface at the given point.

31. $x + y + z = 9$, $(3, 3, 3)$
32. $x^2 + y^2 + z^2 = 9$, $(1, 2, 2)$
33. $x^2 + y^2 + z = 9$, $(1, 2, 4)$
34. $z = 16 - x^2 - y^2$, $(2, 2, 8)$
35. $z = x^2 - y^2$, $(3, 2, 5)$
36. $xy - z = 0$, $(-2, -3, 6)$
37. $xyz = 10$, $(1, 2, 5)$
38. $z = ye^{2xy}$, $(0, 2, 2)$
39. $z = \arctan \dfrac{y}{x}$, $\left(1, 1, \dfrac{\pi}{4}\right)$
40. $y \ln xz^2 = 2$, $(e, 2, 1)$

In Exercises 41–46, (a) find symmetric equations of the tangent line to the curve of intersection of the surfaces at the given point, and (b) find the cosine of the angle between the gradient vectors at this point. State whether or not the surfaces are orthogonal at the point of intersection.

41. $x^2 + y^2 = 2$, $z = x$, $(1, 1, 1)$
42. $z = x^2 + y^2$, $z = 4 - y$, $(2, -1, 5)$
43. $x^2 + z^2 = 25$, $y^2 + z^2 = 25$, $(3, 3, 4)$
44. $z = \sqrt{x^2 + y^2}$, $5x - 2y + 3z = 22$, $(3, 4, 5)$
45. $x^2 + y^2 + z^2 = 14$, $x - y - z = 0$, $(3, 1, 2)$
46. $z = x^2 + y^2$, $x + y + 6z = 33$, $(1, 2, 5)$

In Exercises 47–50, find the angle of inclination θ of the tangent plane to the surface at the given point.

47. $3x^2 + 2y^2 - z = 15$, $(2, 2, 5)$
48. $2xy - z^3 = 0$, $(2, 2, 2)$
49. $x^2 - y^2 + z = 0$, $(1, 2, 3)$
50. $x^2 + y^2 = 5$, $(2, 1, 3)$

In Exercises 51–56, find the point(s) on the surface at which the tangent plane is horizontal.

51. $z = 3 - x^2 - y^2 + 6y$

52. $z = 3x^2 + 2y^2 - 3x + 4y - 5$

53. $z = x^2 - xy + y^2 - 2x - 2y$

54. $z = 4x^2 + 4xy - 2y^2 + 8x - 5y - 4$

55. $z = 5xy$

56. $z = xy + \frac{1}{x} + \frac{1}{y}$

In Exercises 57 and 58, show that the surfaces are tangent to each other at the given point by showing that the surfaces have the same tangent plane at this point.

57. $x^2 + 2y^2 + 3z^2 = 3$, $x^2 + y^2 + z^2 + 6x - 10y + 14 = 0$, $(-1, 1, 0)$

58. $x^2 + y^2 + z^2 - 8x - 12y + 4z + 42 = 0$, $x^2 + y^2 + 2z = 7$, $(2, 3, -3)$

In Exercises 59 and 60, (a) show that the surfaces intersect at the given point, and (b) show that the surfaces have perpendicular tangent planes at this point.

59. $z = 2xy^2$, $8x^2 - 5y^2 - 8z = -13$, $(1, 1, 2)$

60. $x^2 + y^2 + z^2 + 2x - 4y - 4z - 12 = 0$, $4x^2 + y^2 + 16z^2 = 24$, $(1, -2, 1)$

61. Find a point on the ellipsoid $x^2 + 4y^2 + z^2 = 9$ where the tangent plane is perpendicular to the line with parametric equations

$x = 2 - 4t$, $y = 1 + 8t$, and $z = 3 - 2t$.

62. Find a point on the hyperboloid $x^2 + 4y^2 - z^2 = 1$ where the tangent plane is parallel to the plane $x + 4y - z = 0$.

WRITING ABOUT CONCEPTS

63. Give the standard form of the equation of the tangent plane to a surface given by $F(x, y, z) = 0$ at (x_0, y_0, z_0).

64. For some surfaces, the normal lines at any point pass through the same geometric object. What is the common geometric object for a sphere? What is the common geometric object for a right circular cylinder? Explain.

65. Discuss the relationship between the tangent plane to a surface and approximation by differentials.

CAPSTONE

66. Consider the elliptic cone given by

$x^2 - y^2 + z^2 = 0.$

(a) Find an equation of the tangent plane at the point $(5, 13, -12)$.

(b) Find symmetric equations of the normal line at the point $(5, 13, -12)$.

67. ***Investigation*** Consider the function

$$f(x, y) = \frac{4xy}{(x^2 + 1)(y^2 + 1)}$$

on the intervals $-2 \le x \le 2$ and $0 \le y \le 3$.

(a) Find a set of parametric equations of the normal line and an equation of the tangent plane to the surface at the point $(1, 1, 1)$.

(b) Repeat part (a) for the point $\left(-1, 2, -\frac{4}{5}\right)$.

CAS (c) Use a computer algebra system to graph the surface, the normal lines, and the tangent planes found in parts (a) and (b).

68. ***Investigation*** Consider the function

$$f(x, y) = \frac{\sin y}{x}$$

on the intervals $-3 \le x \le 3$ and $0 \le y \le 2\pi$.

(a) Find a set of parametric equations of the normal line and an equation of the tangent plane to the surface at the point $\left(2, \frac{\pi}{2}, \frac{1}{2}\right)$.

(b) Repeat part (a) for the point $\left(-\frac{2}{3}, \frac{3\pi}{2}, \frac{3}{2}\right)$.

CAS (c) Use a computer algebra system to graph the surface, the normal lines, and the tangent planes found in parts (a) and (b).

69. Consider the functions

$f(x, y) = 6 - x^2 - y^2/4$ and $g(x, y) = 2x + y$.

(a) Find a set of parametric equations of the tangent line to the curve of intersection of the surfaces at the point $(1, 2, 4)$, and find the angle between the gradient vectors.

CAS (b) Use a computer algebra system to graph the surfaces. Graph the tangent line found in part (a).

70. Consider the functions

$$f(x, y) = \sqrt{16 - x^2 - y^2 + 2x - 4y}$$

and

$$g(x, y) = \frac{\sqrt{2}}{2}\sqrt{1 - 3x^2 + y^2 + 6x + 4y}.$$

CAS (a) Use a computer algebra system to graph the first-octant portion of the surfaces represented by f and g.

(b) Find one first-octant point on the curve of intersection and show that the surfaces are orthogonal at this point.

(c) These surfaces are orthogonal along the curve of intersection. Does part (b) prove this fact? Explain.

In Exercises 71 and 72, show that the tangent plane to the quadric surface at the point (x_0, y_0, z_0) can be written in the given form.

71. Ellipsoid: $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

Plane: $\frac{x_0 x}{a^2} + \frac{y_0 y}{b^2} + \frac{z_0 z}{c^2} = 1$

72. Hyperboloid: $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 1$

Plane: $\dfrac{x_0 x}{a^2} + \dfrac{y_0 y}{b^2} - \dfrac{z_0 z}{c^2} = 1$

73. Show that any tangent plane to the cone

$z^2 = a^2x^2 + b^2y^2$

passes through the origin.

74. Let f be a differentiable function and consider the surface $z = xf(y/x)$. Show that the tangent plane at any point $P(x_0, y_0, z_0)$ on the surface passes through the origin.

75. ***Approximation*** Consider the following approximations for a function $f(x, y)$ centered at $(0, 0)$.

Linear approximation:

$P_1(x, y) = f(0, 0) + f_x(0, 0)x + f_y(0, 0)y$

Quadratic approximation:

$$P_2(x, y) = f(0, 0) + f_x(0, 0)x + f_y(0, 0)y + \tfrac{1}{2}f_{xx}(0, 0)x^2 + f_{xy}(0, 0)xy + \tfrac{1}{2}f_{yy}(0, 0)y^2$$

[Note that the linear approximation is the tangent plane to the surface at $(0, 0, f(0, 0))$.]

(a) Find the linear approximation of $f(x, y) = e^{(x-y)}$ centered at $(0, 0)$.

(b) Find the quadratic approximation of $f(x, y) = e^{(x-y)}$ centered at $(0, 0)$.

(c) If $x = 0$ in the quadratic approximation, you obtain the second-degree Taylor polynomial for what function? Answer the same question for $y = 0$.

(d) Complete the table.

x	y	$f(x, y)$	$P_1(x, y)$	$P_2(x, y)$
0	0			
0	0.1			
0.2	0.1			
0.2	0.5			
1	0.5			

CAS (e) Use a computer algebra system to graph the surfaces $z = f(x, y)$, $z = P_1(x, y)$, and $z = P_2(x, y)$.

76. ***Approximation*** Repeat Exercise 75 for the function $f(x, y) = \cos(x + y)$.

77. Prove that the angle of inclination θ of the tangent plane to the surface $z = f(x, y)$ at the point (x_0, y_0, z_0) is given by

$$\cos\theta = \frac{1}{\sqrt{[f_x(x_0, y_0)]^2 + [f_y(x_0, y_0)]^2 + 1}}.$$

78. Prove Theorem 13.14.

SECTION PROJECT

Wildflowers

The diversity of wildflowers in a meadow can be measured by counting the numbers of daisies, buttercups, shooting stars, and so on. If there are n types of wildflowers, each with a proportion p_i of the total population, it follows that $p_1 + p_2 + \cdots + p_n = 1$. The measure of diversity of the population is defined as

$$H = -\sum_{i=1}^{n} p_i \log_2 p_i.$$

In this definition, it is understood that $p_i \log_2 p_i = 0$ when $p_i = 0$. The tables show proportions of wildflowers in a meadow in May, June, August, and September.

May

Flower type	1	2	3	4
Proportion	$\frac{5}{16}$	$\frac{5}{16}$	$\frac{5}{16}$	$\frac{1}{16}$

June

Flower type	1	2	3	4
Proportion	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$

August

Flower type	1	2	3	4
Proportion	$\frac{1}{4}$	0	$\frac{1}{4}$	$\frac{1}{2}$

September

Flower type	1	2	3	4
Proportion	0	0	0	1

(a) Determine the wildflower diversity for each month. How would you interpret September's diversity? Which month had the greatest diversity?

(b) If the meadow contains 10 types of wildflowers in roughly equal proportions, is the diversity of the population greater than or less than the diversity of a similar distribution of 4 types of flowers? What type of distribution (of 10 types of wildflowers) would produce maximum diversity?

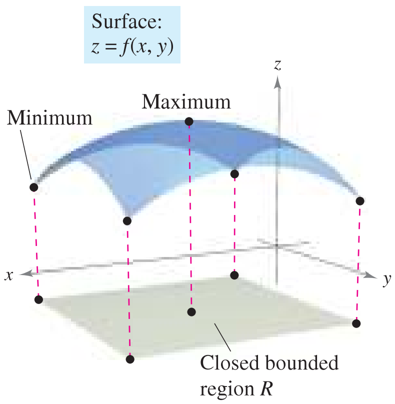
(c) Let H_n represent the maximum diversity of n types of wildflowers. Does H_n approach a limit as $n \to \infty$?

FOR FURTHER INFORMATION Biologists use the concept of diversity to measure the proportions of different types of organisms within an environment. For more information on this technique, see the article "Information Theory and Biological Diversity" by Steven Kolmes and Kevin Mitchell in the *UMAP Modules*.

13.8 Extrema of Functions of Two Variables

- Find absolute and relative extrema of a function of two variables.
- Use the Second Partials Test to find relative extrema of a function of two variables.

Absolute Extrema and Relative Extrema



R contains point(s) at which $f(x, y)$ is a minimum and point(s) at which $f(x, y)$ is a maximum.
Figure 13.64

In Chapter 3, you studied techniques for finding the extreme values of a function of a single variable. In this section, you will extend these techniques to functions of two variables. For example, in Theorem 13.15 below, the Extreme Value Theorem for a function of a single variable is extended to a function of two variables.

Consider the continuous function f of two variables, defined on a closed bounded region R. The values $f(a, b)$ and $f(c, d)$ such that

$$f(a, b) \leq f(x, y) \leq f(c, d) \qquad (a, b) \text{ and } (c, d) \text{ are in } R.$$

for all (x, y) in R are called the **minimum** and **maximum** of f in the region R, as shown in Figure 13.64. Recall from Section 13.2 that a region in the plane is *closed* if it contains all of its boundary points. The Extreme Value Theorem deals with a region in the plane that is both closed and *bounded*. A region in the plane is called **bounded** if it is a subregion of a closed disk in the plane.

THEOREM 13.15 EXTREME VALUE THEOREM

Let f be a continuous function of two variables x and y defined on a closed bounded region R in the xy-plane.

1. There is at least one point in R at which f takes on a minimum value.
2. There is at least one point in R at which f takes on a maximum value.

A minimum is also called an **absolute minimum** and a maximum is also called an **absolute maximum.** As in single-variable calculus, there is a distinction made between absolute extrema and **relative extrema.**

DEFINITION OF RELATIVE EXTREMA

Let f be a function defined on a region R containing (x_0, y_0).

1. The function f has a **relative minimum** at (x_0, y_0) if

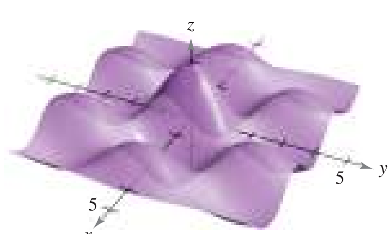
$$f(x, y) \geq f(x_0, y_0)$$

for all (x, y) in an *open* disk containing (x_0, y_0).

2. The function f has a **relative maximum** at (x_0, y_0) if

$$f(x, y) \leq f(x_0, y_0)$$

for all (x, y) in an *open* disk containing (x_0, y_0).



Relative extrema
Figure 13.65

To say that f has a relative maximum at (x_0, y_0) means that the point (x_0, y_0, z_0) is at least as high as all nearby points on the graph of $z = f(x, y)$. Similarly, f has a relative minimum at (x_0, y_0) if (x_0, y_0, z_0) is at least as low as all nearby points on the graph. (See Figure 13.65.)

The Granger Collection

KARL WEIERSTRASS (1815–1897)

Although the Extreme Value Theorem had been used by earlier mathematicians, the first to provide a rigorous proof was the German mathematician Karl Weierstrass. Weierstrass also provided rigorous justifications for many other mathematical results already in common use. We are indebted to him for much of the logical foundation on which modern calculus is built.

To locate relative extrema of f, you can investigate the points at which the gradient of f is $\mathbf{0}$ or the points at which one of the partial derivatives does not exist. Such points are called **critical points** of f.

DEFINITION OF CRITICAL POINT

Let f be defined on an open region R containing (x_0, y_0). The point (x_0, y_0) is a **critical point** of f if one of the following is true.

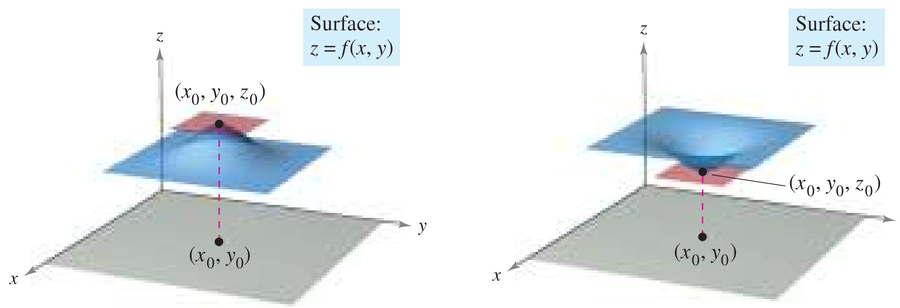
1. $f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$

2. $f_x(x_0, y_0)$ or $f_y(x_0, y_0)$ does not exist.

Recall from Theorem 13.11 that if f is differentiable and

$$\begin{aligned}\nabla f(x_0, y_0) &= f_x(x_0, y_0)\mathbf{i} + f_y(x_0, y_0)\mathbf{j}\\ &= 0\mathbf{i} + 0\mathbf{j}\end{aligned}$$

then every directional derivative at (x_0, y_0) must be 0. This implies that the function has a horizontal tangent plane at the point (x_0, y_0), as shown in Figure 13.66. It appears that such a point is a likely location of a relative extremum. This is confirmed by Theorem 13.16.



Relative maximum

Relative minimum

Figure 13.66

THEOREM 13.16 RELATIVE EXTREMA OCCUR ONLY AT CRITICAL POINTS

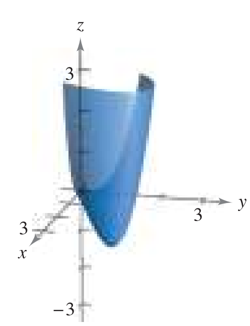
If f has a relative extremum at (x_0, y_0) on an open region R, then (x_0, y_0) is a critical point of f.

EXPLORATION

Use a graphing utility to graph

$$z = x^3 - 3xy + y^3$$

using the bounds $0 \le x \le 3$, $0 \le y \le 3$, and $-3 \le z \le 3$. This view makes it appear as though the surface has an absolute minimum. But does it?



EXAMPLE 1 Finding a Relative Extremum

Determine the relative extrema of

$$f(x, y) = 2x^2 + y^2 + 8x - 6y + 20.$$

Solution Begin by finding the critical points of f. Because

$$f_x(x, y) = 4x + 8 \qquad \text{Partial with respect to } x$$

and

$$f_y(x, y) = 2y - 6 \qquad \text{Partial with respect to } y$$

are defined for all x and y, the only critical points are those for which both first partial derivatives are 0. To locate these points, set $f_x(x, y)$ and $f_y(x, y)$ equal to 0, and solve the equations

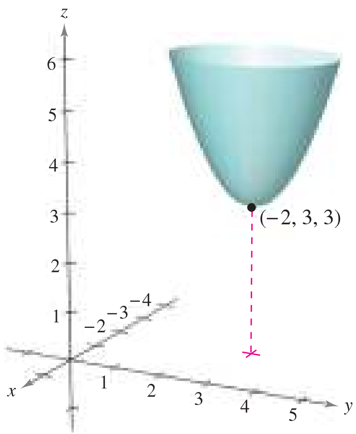
$$4x + 8 = 0 \quad \text{and} \quad 2y - 6 = 0$$

to obtain the critical point $(-2, 3)$. By completing the square, you can conclude that for all $(x, y) \neq (-2, 3)$

$$f(x, y) = 2(x + 2)^2 + (y - 3)^2 + 3 > 3.$$

So, a relative *minimum* of f occurs at $(-2, 3)$. The value of the relative minimum is $f(-2, 3) = 3$, as shown in Figure 13.67.

Surface:
$f(x, y) = 2x^2 + y^2 + 8x - 6y + 20$



The function $z = f(x, y)$ has a relative minimum at $(-2, 3)$.
Figure 13.67

Example 1 shows a relative minimum occurring at one type of critical point—the type for which both $f_x(x, y)$ and $f_y(x, y)$ are 0. The next example concerns a relative maximum that occurs at the other type of critical point—the type for which either $f_x(x, y)$ or $f_y(x, y)$ does not exist.

EXAMPLE 2 Finding a Relative Extremum

Determine the relative extrema of $f(x, y) = 1 - (x^2 + y^2)^{1/3}$.

Solution Because

$$f_x(x, y) = -\frac{2x}{3(x^2 + y^2)^{2/3}} \qquad \text{Partial with respect to } x$$

and

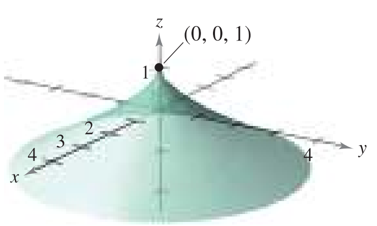
$$f_y(x, y) = -\frac{2y}{3(x^2 + y^2)^{2/3}} \qquad \text{Partial with respect to } y$$

it follows that both partial derivatives exist for all points in the xy-plane except for $(0, 0)$. Moreover, because the partial derivatives cannot both be 0 unless both x and y are 0, you can conclude that $(0, 0)$ is the only critical point. In Figure 13.68, note that $f(0, 0)$ is 1. For all other (x, y) it is clear that

$$f(x, y) = 1 - (x^2 + y^2)^{1/3} < 1.$$

So, f has a relative *maximum* at $(0, 0)$.

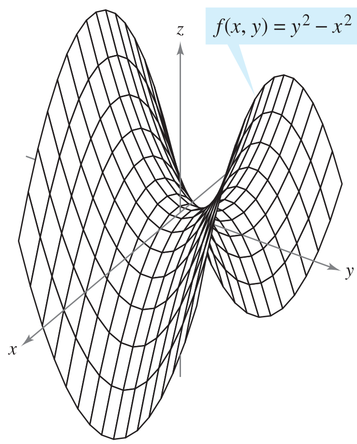
Surface:
$f(x, y) = 1 - (x^2 + y^2)^{1/3}$



$f_x(x, y)$ and $f_y(x, y)$ are undefined at $(0, 0)$.
Figure 13.68

NOTE In Example 2, $f_x(x, y) = 0$ for every point on the y-axis other than $(0, 0)$. However, because $f_y(x, y)$ is nonzero, these are not critical points. Remember that *one* of the partials must not exist or *both* must be 0 in order to yield a critical point.

The Second Partials Test



Saddle point at $(0, 0, 0)$:
$f_x(0, 0) = f_y(0, 0) = 0$
Figure 13.69

Theorem 13.16 tells you that to find relative extrema you need only examine values of $f(x, y)$ at critical points. However, as is true for a function of one variable, the critical points of a function of two variables do not always yield relative maxima or minima. Some critical points yield **saddle points,** which are neither relative maxima nor relative minima.

As an example of a critical point that does not yield a relative extremum, consider the surface given by

$$f(x, y) = y^2 - x^2 \qquad \text{Hyperbolic paraboloid}$$

as shown in Figure 13.69. At the point $(0, 0)$, both partial derivatives are 0. The function f does not, however, have a relative extremum at this point because in any open disk centered at $(0, 0)$ the function takes on both negative values (along the x-axis) *and* positive values (along the y-axis). So, the point $(0, 0, 0)$ is a saddle point of the surface. (The term "saddle point" comes from the fact that surfaces such as the one shown in Figure 13.69 resemble saddles.)

For the functions in Examples 1 and 2, it was relatively easy to determine the relative extrema, because each function was either given, or able to be written, in completed square form. For more complicated functions, algebraic arguments are less convenient and it is better to rely on the analytic means presented in the following Second Partials Test. This is the two-variable counterpart of the Second Derivative Test for functions of one variable. The proof of this theorem is best left to a course in advanced calculus.

THEOREM 13.17 SECOND PARTIALS TEST

Let f have continuous second partial derivatives on an open region containing a point (a, b) for which

$$f_x(a, b) = 0 \quad \text{and} \quad f_y(a, b) = 0.$$

To test for relative extrema of f, consider the quantity

$$d = f_{xx}(a, b)f_{yy}(a, b) - [f_{xy}(a, b)]^2.$$

1. If $d > 0$ and $f_{xx}(a, b) > 0$, then f has a **relative minimum** at (a, b).
2. If $d > 0$ and $f_{xx}(a, b) < 0$, then f has a **relative maximum** at (a, b).
3. If $d < 0$, then $(a, b, f(a, b))$ is a **saddle point.**
4. The test is inconclusive if $d = 0$.

NOTE If $d > 0$, then $f_{xx}(a, b)$ and $f_{yy}(a, b)$ must have the same sign. This means that $f_{xx}(a, b)$ can be replaced by $f_{yy}(a, b)$ in the first two parts of the test. ■

A convenient device for remembering the formula for d in the Second Partials Test is given by the 2×2 determinant

$$d = \begin{vmatrix} f_{xx}(a, b) & f_{xy}(a, b) \\ f_{yx}(a, b) & f_{yy}(a, b) \end{vmatrix}$$

where $f_{xy}(a, b) = f_{yx}(a, b)$ by Theorem 13.3.

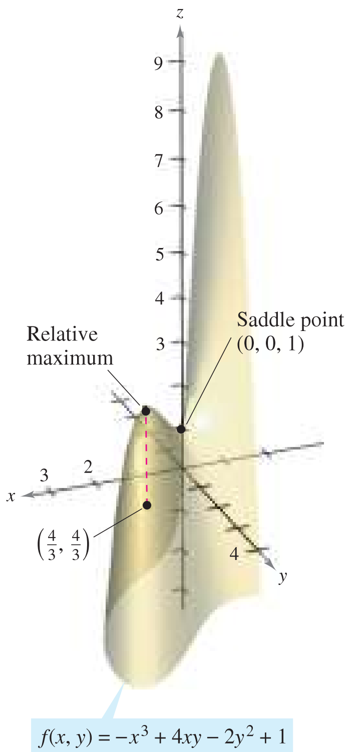


Figure 13.70

EXAMPLE 3 Using the Second Partials Test

Find the relative extrema of $f(x, y) = -x^3 + 4xy - 2y^2 + 1$.

Solution Begin by finding the critical points of f. Because

$$f_x(x, y) = -3x^2 + 4y \quad \text{and} \quad f_y(x, y) = 4x - 4y$$

exist for all x and y, the only critical points are those for which both first partial derivatives are 0. To locate these points, set $f_x(x, y)$ and $f_y(x, y)$ equal to 0 to obtain $-3x^2 + 4y = 0$ and $4x - 4y = 0$. From the second equation you know that $x = y$, and, by substitution into the first equation, you obtain two solutions: $y = x = 0$ and $y = x = \frac{4}{3}$. Because

$$f_{xx}(x, y) = -6x, \quad f_{yy}(x, y) = -4, \quad \text{and} \quad f_{xy}(x, y) = 4$$

it follows that, for the critical point $(0, 0)$,

$$d = f_{xx}(0, 0)f_{yy}(0, 0) - [f_{xy}(0, 0)]^2 = 0 - 16 < 0$$

and, by the Second Partials Test, you can conclude that $(0, 0, 1)$ is a saddle point of f. Furthermore, for the critical point $\left(\frac{4}{3}, \frac{4}{3}\right)$,

$$\begin{aligned} d &= f_{xx}\left(\tfrac{4}{3}, \tfrac{4}{3}\right) f_{yy}\left(\tfrac{4}{3}, \tfrac{4}{3}\right) - \left[f_{xy}\left(\tfrac{4}{3}, \tfrac{4}{3}\right)\right]^2 \\ &= -8(-4) - 16 \\ &= 16 \\ &> 0 \end{aligned}$$

and because $f_{xx}\left(\frac{4}{3}, \frac{4}{3}\right) = -8 < 0$ you can conclude that f has a relative maximum at $\left(\frac{4}{3}, \frac{4}{3}\right)$, as shown in Figure 13.70. ■

The Second Partials Test can fail to find relative extrema in two ways. If either of the first partial derivatives does not exist, you cannot use the test. Also, if

$$d = f_{xx}(a, b)f_{yy}(a, b) - [f_{xy}(a, b)]^2 = 0$$

the test fails. In such cases, you can try a sketch or some other approach, as demonstrated in the next example.

EXAMPLE 4 Failure of the Second Partials Test

Find the relative extrema of $f(x, y) = x^2y^2$.

Solution Because $f_x(x, y) = 2xy^2$ and $f_y(x, y) = 2x^2y$, you know that both partial derivatives are 0 if $x = 0$ or $y = 0$. That is, every point along the x- or y-axis is a critical point. Moreover, because

$$f_{xx}(x, y) = 2y^2, \qquad f_{yy}(x, y) = 2x^2, \qquad \text{and} \qquad f_{xy}(x, y) = 4xy$$

you know that if either $x = 0$ or $y = 0$, then

$$\begin{aligned} d &= f_{xx}(x, y)f_{yy}(x, y) - [f_{xy}(x, y)]^2 \\ &= 4x^2y^2 - 16x^2y^2 = -12x^2y^2 = 0. \end{aligned}$$

So, the Second Partials Test fails. However, because $f(x, y) = 0$ for every point along the x- or y-axis and $f(x, y) = x^2y^2 > 0$ for all other points, you can conclude that each of these critical points yields an absolute minimum, as shown in Figure 13.71. ■

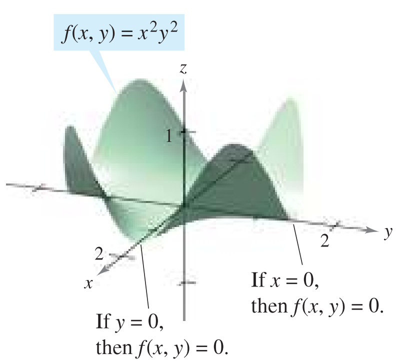


Figure 13.71

Absolute extrema of a function can occur in two ways. First, some relative extrema also happen to be absolute extrema. For instance, in Example 1, $f(-2, 3)$ is an absolute minimum of the function. (On the other hand, the relative maximum found in Example 3 is not an absolute maximum of the function.) Second, absolute extrema can occur at a boundary point of the domain. This is illustrated in Example 5.

EXAMPLE 5 Finding Absolute Extrema

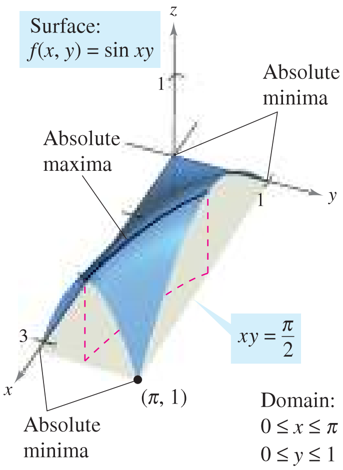


Figure 13.72

Find the absolute extrema of the function

$$f(x, y) = \sin xy$$

on the closed region given by $0 \leq x \leq \pi$ and $0 \leq y \leq 1$.

Solution From the partial derivatives

$$f_x(x, y) = y \cos xy \quad \text{and} \quad f_y(x, y) = x \cos xy$$

you can see that each point lying on the hyperbola given by $xy = \pi/2$ is a critical point. These points each yield the value

$$f(x, y) = \sin \frac{\pi}{2} = 1$$

which you know is the absolute maximum, as shown in Figure 13.72. The only other critical point of f *lying in the given region* is $(0, 0)$. It yields an absolute minimum of 0, because

$$0 \leq xy \leq \pi$$

implies that

$$0 \leq \sin xy \leq 1.$$

To locate other absolute extrema, you should consider the four boundaries of the region formed by taking traces with the vertical planes $x = 0$, $x = \pi$, $y = 0$, and $y = 1$. In doing this, you will find that $\sin xy = 0$ at all points on the x-axis, at all points on the y-axis, and at the point $(\pi, 1)$. Each of these points yields an absolute minimum for the surface, as shown in Figure 13.72. ■

The concepts of relative extrema and critical points can be extended to functions of three or more variables. If all first partial derivatives of

$$w = f(x_1, x_2, x_3, \ldots, x_n)$$

exist, it can be shown that a relative maximum or minimum can occur at $(x_1, x_2, x_3, \ldots, x_n)$ only if every first partial derivative is 0 at that point. This means that the critical points are obtained by solving the following system of equations.

$$\begin{aligned}
f_{x_1}(x_1, x_2, x_3, \ldots, x_n) &= 0 \\
f_{x_2}(x_1, x_2, x_3, \ldots, x_n) &= 0 \\
&\vdots \\
f_{x_n}(x_1, x_2, x_3, \ldots, x_n) &= 0
\end{aligned}$$

The extension of Theorem 13.17 to three or more variables is also possible, although you will not consider such an extension in this text.

13.8 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–6, identify any extrema of the function by recognizing its given form or its form after completing the square. Verify your results by using the partial derivatives to locate any critical points and test for relative extrema.

1. $g(x, y) = (x - 1)^2 + (y - 3)^2$
2. $g(x, y) = 5 - (x - 3)^2 - (y + 2)^2$
3. $f(x, y) = \sqrt{x^2 + y^2 + 1}$
4. $f(x, y) = \sqrt{25 - (x - 2)^2 - y^2}$
5. $f(x, y) = x^2 + y^2 + 2x - 6y + 6$
6. $f(x, y) = -x^2 - y^2 + 10x + 12y - 64$

In Exercises 7–16, examine the function for relative extrema.

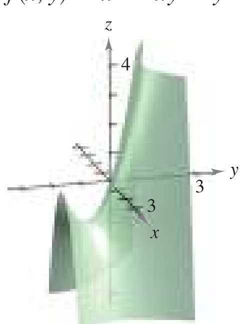
7. $f(x, y) = 3x^2 + 2y^2 - 6x - 4y + 16$
8. $f(x, y) = -3x^2 - 2y^2 + 3x - 4y + 5$
9. $f(x, y) = -x^2 - 5y^2 + 10x - 10y - 28$
10. $f(x, y) = 2x^2 + 2xy + y^2 + 2x - 3$
11. $z = x^2 + xy + \frac{1}{2}y^2 - 2x + y$
12. $z = -5x^2 + 4xy - y^2 + 16x + 10$
13. $f(x, y) = \sqrt{x^2 + y^2}$
14. $h(x, y) = (x^2 + y^2)^{1/3} + 2$
15. $g(x, y) = 4 - |x| - |y|$
16. $f(x, y) = |x + y| - 2$

CAS **In Exercises 17–20, use a computer algebra system to graph the surface and locate any relative extrema and saddle points.**

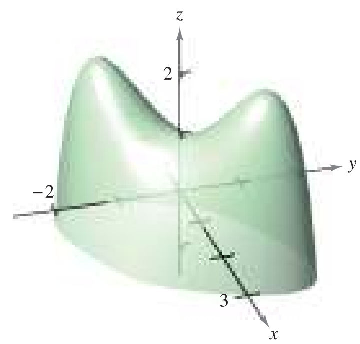
17. $z = \dfrac{-4x}{x^2 + y^2 + 1}$
18. $f(x, y) = y^3 - 3yx^2 - 3y^2 - 3x^2 + 1$
19. $z = (x^2 + 4y^2)e^{1 - x^2 - y^2}$
20. $z = e^{xy}$

In Exercises 21–28, examine the function for relative extrema and saddle points.

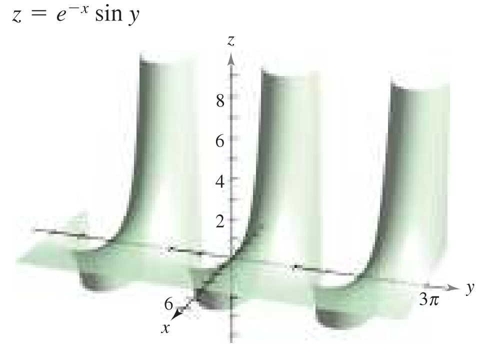
21. $h(x, y) = 80x + 80y - x^2 - y^2$
22. $g(x, y) = x^2 - y^2 - x - y$
23. $g(x, y) = xy$
24. $h(x, y) = x^2 - 3xy - y^2$
25. $f(x, y) = x^2 - xy - y^2 - 3x - y$



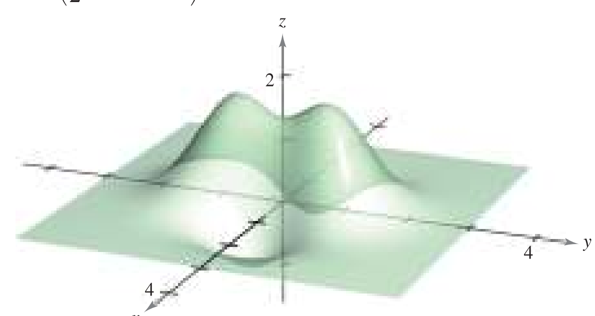
26. $f(x, y) = 2xy - \frac{1}{2}(x^4 + y^4) + 1$



27. $z = e^{-x} \sin y$



28. $z = \left(\dfrac{1}{2} - x^2 + y^2\right)e^{1 - x^2 - y^2}$



CAS **In Exercises 29 and 30, examine the function for extrema without using the derivative tests, and use a computer algebra system to graph the surface. (*Hint:* By observation, determine if it is possible for z to be negative. When is z equal to 0?)**

29. $z = \dfrac{(x - y)^4}{x^2 + y^2}$
30. $z = \dfrac{(x^2 - y^2)^2}{x^2 + y^2}$

Think About It **In Exercises 31–34, determine whether there is a relative maximum, a relative minimum, a saddle point, or insufficient information to determine the nature of the function $f(x, y)$ at the critical point (x_0, y_0).**

31. $f_{xx}(x_0, y_0) = 9, \quad f_{yy}(x_0, y_0) = 4, \quad f_{xy}(x_0, y_0) = 6$
32. $f_{xx}(x_0, y_0) = -3, \quad f_{yy}(x_0, y_0) = -8, \quad f_{xy}(x_0, y_0) = 2$
33. $f_{xx}(x_0, y_0) = -9, \quad f_{yy}(x_0, y_0) = 6, \quad f_{xy}(x_0, y_0) = 10$
34. $f_{xx}(x_0, y_0) = 25, \quad f_{yy}(x_0, y_0) = 8, \quad f_{xy}(x_0, y_0) = 10$

35. A function f has continuous second partial derivatives on an open region containing the critical point $(3, 7)$. The function has a minimum at $(3, 7)$, and $d > 0$ for the Second Partials Test. Determine the interval for $f_{xy}(3, 7)$ if $f_{xx}(3, 7) = 2$ and $f_{yy}(3, 7) = 8$.

36. A function f has continuous second partial derivatives on an open region containing the critical point (a, b). If $f_{xx}(a, b)$ and $f_{yy}(a, b)$ have opposite signs, what is implied? Explain.

CAS **In Exercises 37–42, (a) find the critical points, (b) test for relative extrema, (c) list the critical points for which the Second Partials Test fails, and (d) use a computer algebra system to graph the function, labeling any extrema and saddle points.**

37. $f(x, y) = x^3 + y^3$

38. $f(x, y) = x^3 + y^3 - 6x^2 + 9y^2 + 12x + 27y + 19$

39. $f(x, y) = (x - 1)^2(y + 4)^2$

40. $f(x, y) = \sqrt{(x - 1)^2 + (y + 2)^2}$

41. $f(x, y) = x^{2/3} + y^{2/3}$

42. $f(x, y) = (x^2 + y^2)^{2/3}$

In Exercises 43 and 44, find the critical points of the function and, from the form of the function, determine whether a relative maximum or a relative minimum occurs at each point.

43. $f(x, y, z) = x^2 + (y - 3)^2 + (z + 1)^2$

44. $f(x, y, z) = 9 - [x(y - 1)(z + 2)]^2$

In Exercises 45–54, find the absolute extrema of the function over the region R. (In each case, R contains the boundaries.) Use a computer algebra system to confirm your results.

45. $f(x, y) = x^2 - 4xy + 5$

$R = \{(x, y): 1 \le x \le 4,\ 0 \le y \le 2\}$

46. $f(x, y) = x^2 + xy, \quad R = \{(x, y): |x| \le 2,\ |y| \le 1\}$

47. $f(x, y) = 12 - 3x - 2y$

R: The triangular region in the xy-plane with vertices $(2, 0)$, $(0, 1)$, and $(1, 2)$

48. $f(x, y) = (2x - y)^2$

R: The triangular region in the xy-plane with vertices $(2, 0)$, $(0, 1)$, and $(1, 2)$

49. $f(x, y) = 3x^2 + 2y^2 - 4y$

R: The region in the xy-plane bounded by the graphs of $y = x^2$ and $y = 4$

50. $f(x, y) = 2x - 2xy + y^2$

R: The region in the xy-plane bounded by the graphs of $y = x^2$ and $y = 1$

51. $f(x, y) = x^2 + 2xy + y^2, \quad R = \{(x, y): |x| \le 2,\ |y| \le 1\}$

52. $f(x, y) = x^2 + 2xy + y^2, \quad R = \{(x, y): x^2 + y^2 \le 8\}$

53. $f(x, y) = \dfrac{4xy}{(x^2 + 1)(y^2 + 1)}$

$R = \{(x, y): 0 \le x \le 1,\ 0 \le y \le 1\}$

54. $f(x, y) = \dfrac{4xy}{(x^2 + 1)(y^2 + 1)}$

$R = \{(x, y): x \ge 0,\ y \ge 0,\ x^2 + y^2 \le 1\}$

WRITING ABOUT CONCEPTS

55. The figure shows the level curves for an unknown function $f(x, y)$. What, if any, information can be given about f at the point A? Explain your reasoning.

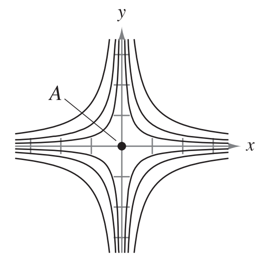


Figure for 55

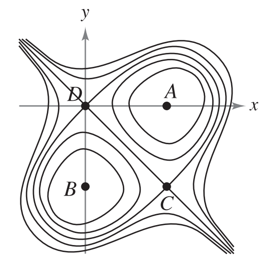


Figure for 56

56. The figure shows the level curves for an unknown function $f(x, y)$. What, if any, information can be given about f at the points A, B, C, and D? Explain your reasoning.

In Exercises 57–59, sketch the graph of an arbitrary function f satisfying the given conditions. State whether the function has any extrema or saddle points. (There are many correct answers.)

57. $f_x(x, y) > 0$ and $f_y(x, y) < 0$ for all (x, y).

58. All of the first and second partial derivatives of f are 0.

59. $f_x(0, 0) = 0,\ f_y(0, 0) = 0$

$$f_x(x, y)\begin{cases} < 0, & x < 0 \\ > 0, & x > 0 \end{cases}, \qquad f_y(x, y)\begin{cases} > 0, & y < 0 \\ < 0, & y > 0 \end{cases}$$

$f_{xx}(x, y) > 0$, $f_{yy}(x, y) < 0$, and $f_{xy}(x, y) = 0$ for all (x, y).

CAPSTONE

60. Consider the functions

$f(x, y) = x^2 - y^2$ and $g(x, y) = x^2 + y^2$.

(a) Show that both functions have a critical point at $(0, 0)$.

(b) Explain how f and g behave differently at this critical point.

True or False? **In Exercises 61–64, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

61. If f has a relative maximum at (x_0, y_0, z_0), then $f_x(x_0, y_0) = f_y(x_0, y_0) = 0$.

62. If $f_x(x_0, y_0) = f_y(x_0, y_0) = 0$, then f has a relative maximum at (x_0, y_0, z_0).

63. Between any two relative minima of f, there must be at least one relative maximum of f.

64. If f is continuous for all x and y and has two relative minima, then f must have at least one relative maximum.

13.9 Applications of Extrema of Functions of Two Variables

- **Solve optimization problems involving functions of several variables.**
- **Use the method of least squares.**

Applied Optimization Problems

In this section, you will survey a few of the many applications of extrema of functions of two (or more) variables.

EXAMPLE 1 Finding Maximum Volume

A rectangular box is resting on the xy-plane with one vertex at the origin. The opposite vertex lies in the plane

$$6x + 4y + 3z = 24$$

as shown in Figure 13.73. Find the maximum volume of such a box.

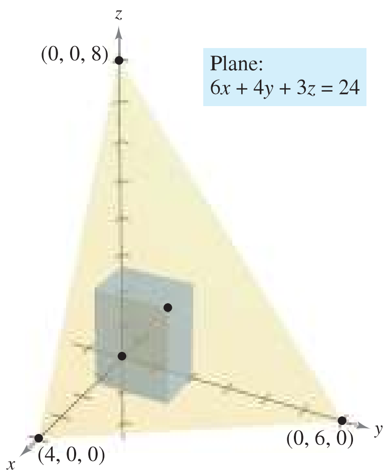


Figure 13.73

Solution Let x, y, and z represent the length, width, and height of the box. Because one vertex of the box lies in the plane $6x + 4y + 3z = 24$, you know that $z = \frac{1}{3}(24 - 6x - 4y)$, and you can write the volume xyz of the box as a function of two variables.

$$\begin{aligned} V(x, y) &= (x)(y)\left[\tfrac{1}{3}(24 - 6x - 4y)\right] \\ &= \tfrac{1}{3}(24xy - 6x^2y - 4xy^2) \end{aligned}$$

By setting the first partial derivatives equal to 0

$$V_x(x, y) = \frac{1}{3}(24y - 12xy - 4y^2) = \frac{y}{3}(24 - 12x - 4y) = 0$$

$$V_y(x, y) = \frac{1}{3}(24x - 6x^2 - 8xy) = \frac{x}{3}(24 - 6x - 8y) = 0$$

you obtain the critical points $(0, 0)$ and $\left(\frac{4}{3}, 2\right)$. At $(0, 0)$ the volume is 0, so that point does not yield a maximum volume. At the point $\left(\frac{4}{3}, 2\right)$, you can apply the Second Partials Test.

$$V_{xx}(x, y) = -4y$$

$$V_{yy}(x, y) = \frac{-8x}{3}$$

$$V_{xy}(x, y) = \frac{1}{3}(24 - 12x - 8y)$$

NOTE In many applied problems, the domain of the function to be optimized is a closed bounded region. To find minimum or maximum points, you must not only test critical points, but also consider the values of the function at points on the boundary.

Because

$$V_{xx}\left(\tfrac{4}{3}, 2\right)V_{yy}\left(\tfrac{4}{3}, 2\right) - \left[V_{xy}\left(\tfrac{4}{3}, 2\right)\right]^2 = (-8)\left(-\tfrac{32}{9}\right) - \left(-\tfrac{8}{3}\right)^2 = \tfrac{64}{3} > 0$$

and

$$V_{xx}\left(\tfrac{4}{3}, 2\right) = -8 < 0$$

you can conclude from the Second Partials Test that the maximum volume is

$$\begin{aligned} V\left(\tfrac{4}{3}, 2\right) &= \tfrac{1}{3}\left[24\left(\tfrac{4}{3}\right)(2) - 6\left(\tfrac{4}{3}\right)^2(2) - 4\left(\tfrac{4}{3}\right)(2^2)\right] \\ &= \tfrac{64}{9} \text{ cubic units.} \end{aligned}$$

Note that the volume is 0 at the boundary points of the triangular domain of V. ■

Applications of extrema in economics and business often involve more than one independent variable. For instance, a company may produce several models of one type of product. The price per unit and profit per unit are usually different for each model. Moreover, the demand for each model is often a function of the prices of the other models (as well as its own price). The next example illustrates an application involving two products.

■ **FOR FURTHER INFORMATION** For more information on the use of mathematics in economics, see the article "Mathematical Methods of Economics" by Joel Franklin in *The American Mathematical Monthly*. To view this article, go to the website *www.matharticles.com*.

EXAMPLE 2 Finding the Maximum Profit

An electronics manufacturer determines that the profit P (in dollars) obtained by producing and selling x units of a DVD player and y units of a DVD recorder is approximated by the model

$$P(x, y) = 8x + 10y - (0.001)(x^2 + xy + y^2) - 10{,}000.$$

Find the production level that produces a maximum profit. What is the maximum profit?

Solution The partial derivatives of the profit function are

$$P_x(x, y) = 8 - (0.001)(2x + y) \quad \text{and} \quad P_y(x, y) = 10 - (0.001)(x + 2y).$$

By setting these partial derivatives equal to 0, you obtain the following system of equations.

$$\begin{aligned} 8 - (0.001)(2x + y) &= 0 \\ 10 - (0.001)(x + 2y) &= 0 \end{aligned}$$

After simplifying, this system of linear equations can be written as

$$\begin{aligned} 2x + y &= 8000 \\ x + 2y &= 10{,}000. \end{aligned}$$

Solving this system produces $x = 2000$ and $y = 4000$. The second partial derivatives of P are

$$\begin{aligned} P_{xx}(2000, 4000) &= -0.002 \\ P_{yy}(2000, 4000) &= -0.002 \\ P_{xy}(2000, 4000) &= -0.001. \end{aligned}$$

Because $P_{xx} < 0$ and

$$P_{xx}(2000, 4000)P_{yy}(2000, 4000) - [P_{xy}(2000, 4000)]^2 = (-0.002)^2 - (-0.001)^2 > 0$$

you can conclude that the production level of $x = 2000$ units and $y = 4000$ units yields a *maximum* profit. The maximum profit is

$$\begin{aligned} P(2000, 4000) &= 8(2000) + 10(4000) - (0.001)[2000^2 + 2000(4000) + 4000^2)] - 10{,}000 \\ &= \$18{,}000. \end{aligned}$$

NOTE In Example 2, it was assumed that the manufacturing plant is able to produce the required number of units to yield a maximum profit. In actual practice, the production would be bounded by physical constraints. You will study such constrained optimization problems in the next section.

The Method of Least Squares

Many of the examples in this text have involved **mathematical models.** For instance, Example 2 involves a quadratic model for profit. There are several ways to develop such models; one is called the **method of least squares.**

In constructing a model to represent a particular phenomenon, the goals are simplicity and accuracy. Of course, these goals often conflict. For instance, a simple linear model for the points in Figure 13.74 is

$$y = 1.8566x - 5.0246.$$

However, Figure 13.75 shows that by choosing the slightly more complicated quadratic model*

$$y = 0.1996x^2 - 0.7281x + 1.3749$$

you can achieve greater accuracy.

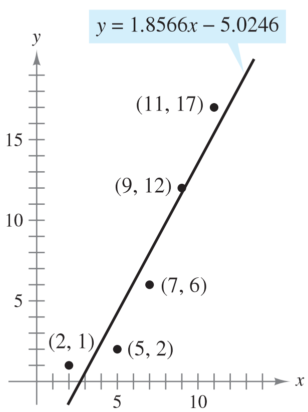


Figure 13.74

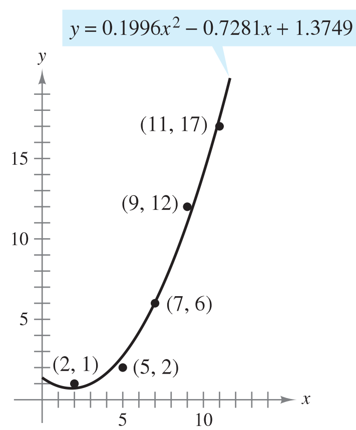


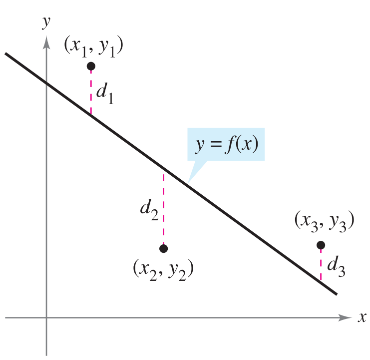
Figure 13.75

As a measure of how well the model $y = f(x)$ fits the collection of points

$$\{(x_1, y_1), (x_2, y_2), (x_3, y_3), \ldots, (x_n, y_n)\}$$

you can add the squares of the differences between the actual y-values and the values given by the model to obtain the **sum of the squared errors**

$$S = \sum_{i=1}^{n} [f(x_i) - y_i]^2. \qquad \text{Sum of the squared errors}$$



Sum of the squared errors:
$S = d_1^2 + d_2^2 + d_3^2$
Figure 13.76

Graphically, S can be interpreted as the sum of the squares of the vertical distances between the graph of f and the given points in the plane, as shown in Figure 13.76. If the model is perfect, then $S = 0$. However, when perfection is not feasible, you can settle for a model that minimizes S. For instance, the sum of the squared errors for the linear model in Figure 13.74 is $S \approx 17$. Statisticians call the *linear model* that minimizes S the **least squares regression line.** The proof that this line actually minimizes S involves the minimizing of a function of two variables.

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* *A method for finding the least squares quadratic model for a collection of data is described in Exercise 37.*

The Granger Collection

ADRIEN-MARIE LEGENDRE (1752–1833)

The method of least squares was introduced by the French mathematician Adrien-Marie Legendre. Legendre is best known for his work in geometry. In fact, his text Elements of Geometry was so popular in the United States that it continued to be used for 33 editions, spanning a period of more than 100 years.

THEOREM 13.18 LEAST SQUARES REGRESSION LINE

The **least squares regression line** for $\{(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)\}$ is given by $f(x) = ax + b$, where

$$a = \frac{n\sum_{i=1}^{n} x_i y_i - \sum_{i=1}^{n} x_i \sum_{i=1}^{n} y_i}{n\sum_{i=1}^{n} x_i^2 - \left(\sum_{i=1}^{n} x_i\right)^2} \quad \text{and} \quad b = \frac{1}{n}\left(\sum_{i=1}^{n} y_i - a\sum_{i=1}^{n} x_i\right).$$

PROOF Let $S(a, b)$ represent the sum of the squared errors for the model $f(x) = ax + b$ and the given set of points. That is,

$$\begin{aligned} S(a, b) &= \sum_{i=1}^{n} [f(x_i) - y_i]^2 \\ &= \sum_{i=1}^{n} (ax_i + b - y_i)^2 \end{aligned}$$

where the points (x_i, y_i) represent constants. Because S is a function of a and b, you can use the methods discussed in the preceding section to find the minimum value of S. Specifically, the first partial derivatives of S are

$$\begin{aligned} S_a(a, b) &= \sum_{i=1}^{n} 2x_i(ax_i + b - y_i) \\ &= 2a\sum_{i=1}^{n} x_i^2 + 2b\sum_{i=1}^{n} x_i - 2\sum_{i=1}^{n} x_i y_i \\ S_b(a, b) &= \sum_{i=1}^{n} 2(ax_i + b - y_i) \\ &= 2a\sum_{i=1}^{n} x_i + 2nb - 2\sum_{i=1}^{n} y_i. \end{aligned}$$

By setting these two partial derivatives equal to 0, you obtain the values of a and b that are listed in the theorem. It is left to you to apply the Second Partials Test (see Exercise 47) to verify that these values of a and b yield a minimum. ■

If the x-values are symmetrically spaced about the y-axis, then $\Sigma x_i = 0$ and the formulas for a and b simplify to

$$a = \frac{\sum_{i=1}^{n} x_i y_i}{\sum_{i=1}^{n} x_i^2}$$

and

$$b = \frac{1}{n}\sum_{i=1}^{n} y_i.$$

This simplification is often possible with a translation of the x-values. For instance, if the x-values in a data collection consist of the years 2005, 2006, 2007, 2008, and 2009, you could let 2007 be represented by 0.

EXAMPLE 3 Finding the Least Squares Regression Line

Find the least squares regression line for the points $(-3, 0)$, $(-1, 1)$, $(0, 2)$, and $(2, 3)$.

Solution The table shows the calculations involved in finding the least squares regression line using $n = 4$.

x	y	xy	x^2
-3	0	0	9
-1	1	-1	1
0	2	0	0
2	3	6	4
$\sum_{i=1}^{n} x_i = -2$	$\sum_{i=1}^{n} y_i = 6$	$\sum_{i=1}^{n} x_i y_i = 5$	$\sum_{i=1}^{n} x_i^2 = 14$

TECHNOLOGY Many calculators have "built-in" least squares regression programs. If your calculator has such a program, use it to duplicate the results of Example 3.

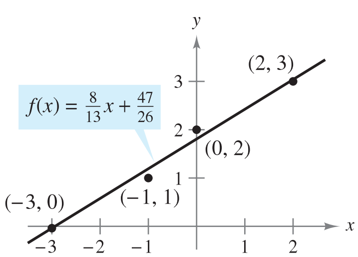
Applying Theorem 13.18 produces

$$a = \frac{n\sum_{i=1}^{n} x_i y_i - \sum_{i=1}^{n} x_i \sum_{i=1}^{n} y_i}{n\sum_{i=1}^{n} x_i^2 - \left(\sum_{i=1}^{n} x_i\right)^2} = \frac{4(5) - (-2)(6)}{4(14) - (-2)^2} = \frac{8}{13}$$

and

$$b = \frac{1}{n}\left(\sum_{i=1}^{n} y_i - a\sum_{i=1}^{n} x_i\right) = \frac{1}{4}\left[6 - \frac{8}{13}(-2)\right] = \frac{47}{26}.$$

The least squares regression line is $f(x) = \frac{8}{13}x + \frac{47}{26}$, as shown in Figure 13.77.



Least squares regression line
Figure 13.77

13.9 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1 and 2, find the minimum distance from the point to the plane $x - y + z = 3$. (*Hint:* To simplify the computations, minimize the square of the distance.)

1. $(0, 0, 0)$ **2.** $(1, 2, 3)$

In Exercises 3 and 4, find the minimum distance from the point to the surface $z = \sqrt{1 - 2x - 2y}$. (*Hint:* In Exercise 4, use the *root* feature of a graphing utility.)

3. $(-2, -2, 0)$

4. $(0, 0, 2)$

In Exercises 5–8, find three positive integers x, y, and z that satisfy the given conditions.

5. The product is 27 and the sum is a minimum.

6. The sum is 32 and $P = xy^2z$ is a maximum.

7. The sum is 30 and the sum of the squares is a minimum.

8. The product is 1 and the sum of the squares is a minimum.

9. ***Cost*** A home improvement contractor is painting the walls and ceiling of a rectangular room. The volume of the room is 668.25 cubic feet. The cost of wall paint is \$0.06 per square foot and the cost of ceiling paint is \$0.11 per square foot. Find the room dimensions that result in a minimum cost for the paint. What is the minimum cost for the paint?

10. ***Maximum Volume*** The material for constructing the base of an open box costs 1.5 times as much per unit area as the material for constructing the sides. For a fixed amount of money C, find the dimensions of the box of largest volume that can be made.

11. ***Maximum Volume*** The volume of an ellipsoid

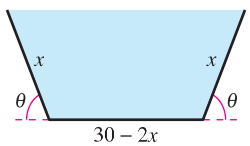
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

is $4\pi abc/3$. For a fixed sum $a + b + c$, show that the ellipsoid of maximum volume is a sphere.

12. ***Maximum Volume*** Show that the rectangular box of maximum volume inscribed in a sphere of radius r is a cube.

13. ***Volume and Surface Area*** Show that a rectangular box of given volume and minimum surface area is a cube.

14. ***Area*** A trough with trapezoidal cross sections is formed by turning up the edges of a 30-inch-wide sheet of aluminum (see figure). Find the cross section of maximum area.



15. ***Maximum Revenue*** A company manufactures two types of sneakers, running shoes and basketball shoes. The total revenue from x_1 units of running shoes and x_2 units of basketball shoes is $R = -5x_1^2 - 8x_2^2 - 2x_1x_2 + 42x_1 + 102x_2$, where x_1 and x_2 are in thousands of units. Find x_1 and x_2 so as to maximize the revenue.

16. ***Maximum Profit*** A corporation manufactures candles at two locations. The cost of producing x_1 units at location 1 is

$$C_1 = 0.02x_1^2 + 4x_1 + 500$$

and the cost of producing x_2 units at location 2 is

$$C_2 = 0.05x_2^2 + 4x_2 + 275.$$

The candles sell for \$15 per unit. Find the quantity that should be produced at each location to maximize the profit $P = 15(x_1 + x_2) - C_1 - C_2$.

17. ***Hardy-Weinberg Law*** Common blood types are determined genetically by three alleles A, B, and O. (An allele is any of a group of possible mutational forms of a gene.) A person whose blood type is AA, BB, or OO is homozygous. A person whose blood type is AB, AO, or BO is heterozygous. The Hardy-Weinberg Law states that the proportion P of heterozygous individuals in any given population is

$$P(p, q, r) = 2pq + 2pr + 2qr$$

where p represents the percent of allele A in the population, q represents the percent of allele B in the population, and r represents the percent of allele O in the population. Use the fact that $p + q + r = 1$ to show that the maximum proportion of heterozygous individuals in any population is $\frac{2}{3}$.

18. ***Shannon Diversity Index*** One way to measure species diversity is to use the Shannon diversity index H. If a habitat consists of three species, A, B, and C, its Shannon diversity index is

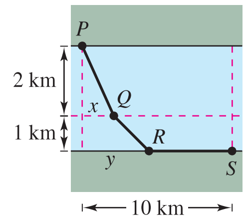
$$H = -x \ln x - y \ln y - z \ln z$$

where x is the percent of species A in the habitat, y is the percent of species B in the habitat, and z is the percent of species C in the habitat.

(a) Use the fact that $x + y + z = 1$ to show that the maximum value of H occurs when $x = y = z = \frac{1}{3}$.

(b) Use the results of part (a) to show that the maximum value of H in this habitat is $\ln 3$.

19. ***Minimum Cost*** A water line is to be built from point P to point S and must pass through regions where construction costs differ (see figure). The cost per kilometer in dollars is $3k$ from P to Q, $2k$ from Q to R, and k from R to S. Find x and y such that the total cost C will be minimized.



20. ***Distance*** A company has retail outlets located at the points $(0, 0)$, $(2, 2)$, and $(-2, 2)$ (see figure). Management plans to build a distribution center located such that the sum of the distances S from the center to the outlets is minimum. From the symmetry of the problem it is clear that the distribution center will be located on the y-axis, and therefore S is a function of the single variable y. Using techniques presented in Chapter 3, find the required value of y.

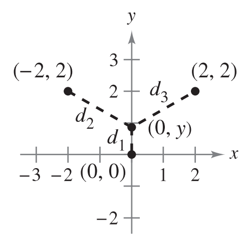


Figure for 20

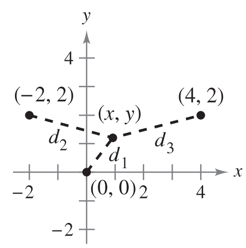


Figure for 21

CAS **21.** ***Investigation*** The retail outlets described in Exercise 20 are located at $(0, 0)$, $(4, 2)$, and $(-2, 2)$ (see figure). The location of the distribution center is (x, y), and therefore the sum of the distances S is a function of x and y.

(a) Write the expression giving the sum of the distances S. Use a computer algebra system to graph S. Does the surface have a minimum?

(b) Use a computer algebra system to obtain S_x and S_y. Observe that solving the system $S_x = 0$ and $S_y = 0$ is very difficult. So, approximate the location of the distribution center.

(c) An initial estimate of the critical point is $(x_1, y_1) = (1, 1)$. Calculate $-\nabla S(1, 1)$ with components $-S_x(1, 1)$ and $-S_y(1, 1)$. What direction is given by the vector $-\nabla S(1, 1)$?

(d) The second estimate of the critical point is

$$(x_2, y_2) = (x_1 - S_x(x_1, y_1)t, y_1 - S_y(x_1, y_1)t).$$

If these coordinates are substituted into $S(x, y)$, then S becomes a function of the single variable t. Find the value of t that minimizes S. Use this value of t to estimate (x_2, y_2).

(e) Complete two more iterations of the process in part (d) to obtain (x_4, y_4). For this location of the distribution center, what is the sum of the distances to the retail outlets?

(f) Explain why $-\nabla S(x, y)$ was used to approximate the minimum value of S. In what types of problems would you use $\nabla S(x, y)$?

22. ***Investigation*** Repeat Exercise 21 for retail outlets located at the points $(-4, 0)$, $(1, 6)$, and $(12, 2)$.

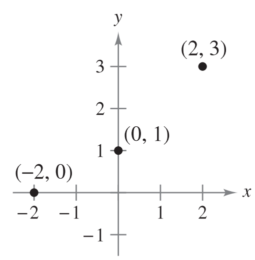
WRITING ABOUT CONCEPTS

23. In your own words, state the problem-solving strategy for applied minimum and maximum problems.

24. In your own words, describe the method of least squares for finding mathematical models.

In Exercises 25–28, (a) find the least squares regression line and (b) calculate S, the sum of the squared errors. Use the regression capabilities of a graphing utility to verify your results.

25.



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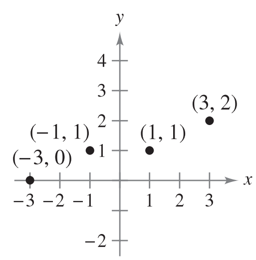
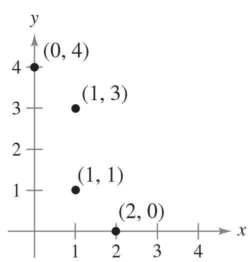
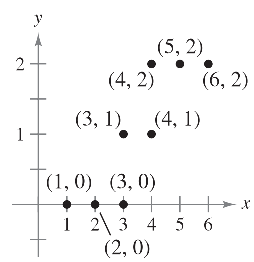


27.



28.



In Exercises 29–32, find the least squares regression line for the points. Use the regression capabilities of a graphing utility to verify your results. Use the graphing utility to plot the points and graph the regression line.

29. $(0, 0)$, $(1, 1)$, $(3, 4)$, $(4, 2)$, $(5, 5)$

30. $(1, 0)$, $(3, 3)$, $(5, 6)$

31. $(0, 6)$, $(4, 3)$, $(5, 0)$, $(8, -4)$, $(10, -5)$

32. $(6, 4)$, $(1, 2)$, $(3, 3)$, $(8, 6)$, $(11, 8)$, $(13, 8)$

33. ***Modeling Data*** The ages x (in years) and systolic blood pressures y of seven men are shown in the table.

Age, x	16	25	39	45	49	64	70
Systolic Blood Pressure, y	109	122	150	165	159	183	199

(a) Use the regression capabilities of a graphing utility to find the least squares regression line for the data.

(b) Use a graphing utility to plot the data and graph the model.

(c) Use the model to approximate the change in systolic blood pressure for each one-year increase in age.

34. ***Modeling Data*** A store manager wants to know the demand y for an energy bar as a function of price x. The daily sales for three different prices of the energy bar are shown in the table.

Price, x	\$1.29	\$1.49	\$1.69
Demand, y	450	375	330

(a) Use the regression capabilities of a graphing utility to find the least squares regression line for the data.

(b) Use the model to estimate the demand when the price is \$1.59.

35. ***Modeling Data*** An agronomist used four test plots to determine the relationship between the wheat yield y (in bushels per acre) and the amount of fertilizer x (in hundreds of pounds per acre). The results are shown in the table.

Fertilizer, x	1.0	1.5	2.0	2.5
Yield, y	32	41	48	53

Use the regression capabilities of a graphing utility to find the least squares regression line for the data, and estimate the yield for a fertilizer application of 160 pounds per acre.

36. ***Modeling Data*** The table shows the percents x and numbers y (in millions) of women in the work force for selected years. *(Source: U.S. Bureau of Labor Statistics)*

Year	1970	1975	1980	1985
Percent, x	43.3	46.3	51.5	54.5
Number, y	31.5	37.5	45.5	51.1

Year	1990	1995	2000	2005
Percent, x	57.5	58.9	59.9	59.3
Number, y	56.8	60.9	66.3	69.3

(a) Use the regression capabilities of a graphing utility to find the least squares regression line for the data.

(b) According to this model, approximately how many women enter the labor force for each one-point increase in the percent of women in the labor force?

37. Find a system of equations whose solution yields the coefficients a, b, and c for the least squares regression quadratic $y = ax^2 + bx + c$ for the points $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ by minimizing the sum

$$S(a, b, c) = \sum_{i=1}^{n} (y_i - ax_i^2 - bx_i - c)^2.$$

CAPSTONE

38. The sum of the length and the girth (perimeter of a cross section) of a package carried by a delivery service cannot exceed 108 inches. Find the dimensions of the rectangular package of largest volume that may be sent.

In Exercises 39–42, use the result of Exercise 37 to find the least squares regression quadratic for the given points. Use the regression capabilities of a graphing utility to confirm your results. Use the graphing utility to plot the points and graph the least squares regression quadratic.

39. $(-2, 0), (-1, 0), (0, 1), (1, 2), (2, 5)$

40. $(-4, 5), (-2, 6), (2, 6), (4, 2)$

41. $(0, 0), (2, 2), (3, 6), (4, 12)$

42. $(0, 10), (1, 9), (2, 6), (3, 0)$

43. ***Modeling Data*** After a new turbocharger for an automobile engine was developed, the following experimental data were obtained for speed y in miles per hour at two-second time intervals x.

Time, x	0	2	4	6	8	10
Speed, y	0	15	30	50	65	70

(a) Find a least squares regression quadratic for the data. Use a graphing utility to confirm your results.

(b) Use a graphing utility to plot the points and graph the model.

44. ***Modeling Data*** The table shows the world populations y (in billions) for five different years. Let $x = 8$ represent the year 1998. *(Source: U.S. Census Bureau, International Data Base)*

Year, x	1998	2000	2002	2004	2006
Population, y	5.9	6.1	6.2	6.4	6.5

(a) Use the regression capabilities of a graphing utility to find the least squares regression line for the data.

(b) Use the regression capabilities of a graphing utility to find the least squares regression quadratic for the data.

(c) Use a graphing utility to plot the data and graph the models.

(d) Use both models to forecast the world population for the year 2014. How do the two models differ as you extrapolate into the future?

45. ***Modeling Data*** A meteorologist measures the atmospheric pressure P (in kilograms per square meter) at altitude h (in kilometers). The data are shown below.

Altitude, h	0	5	10	15	20
Pressure, P	10,332	5583	2376	1240	517

(a) Use the regression capabilities of a graphing utility to find a least squares regression line for the points $(h, \ln P)$.

(b) The result in part (a) is an equation of the form $\ln P = ah + b$. Write this logarithmic form in exponential form.

(c) Use a graphing utility to plot the original data and graph the exponential model in part (b).

(d) If your graphing utility can fit logarithmic models to data, use it to verify the result in part (b).

46. ***Modeling Data*** The endpoints of the interval over which distinct vision is possible are called the near point and far point of the eye. With increasing age, these points normally change. The table shows the approximate near points y (in inches) for various ages x (in years). *(Source: Ophthalmology & Physiological Optics)*

Age, x	16	32	44	50	60
Near Point, y	3.0	4.7	9.8	19.7	39.4

(a) Find a rational model for the data by taking the reciprocals of the near points to generate the points $(x, 1/y)$. Use the regression capabilities of a graphing utility to find a least squares regression line for the revised data. The resulting line has the form $1/y = ax + b$. Solve for y.

(b) Use a graphing utility to plot the data and graph the model.

(c) Do you think the model can be used to predict the near point for a person who is 70 years old? Explain.

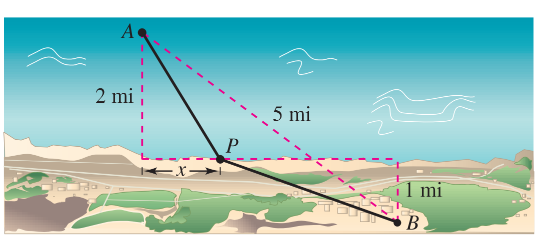
47. Use the Second Partials Test to verify that the formulas for a and b given in Theorem 13.18 yield a minimum.

$$\left[\textit{Hint: } \text{Use the fact that } n\sum_{i=1}^{n} x_i^2 \ge \left(\sum_{i=1}^{n} x_i\right)^2. \right]$$

SECTION PROJECT

Building a Pipeline

An oil company wishes to construct a pipeline from its offshore facility A to its refinery B. The offshore facility is 2 miles from shore, and the refinery is 1 mile inland. Furthermore, A and B are 5 miles apart, as shown in the figure.



The cost of building the pipeline is \$3 million per mile in the water and \$4 million per mile on land. So, the cost of the pipeline depends on the location of point P, where it meets the shore. What would be the most economical route of the pipeline?

Imagine that you are to write a report to the oil company about this problem. Let x be the distance shown in the figure. Determine the cost of building the pipeline from A to P, and the cost from P to B. Analyze some sample pipeline routes and their corresponding costs. For instance, what is the cost of the most direct route? Then use calculus to determine the route of the pipeline that minimizes the cost. Explain all steps of your development and include any relevant graphs.

13.10 Lagrange Multipliers

- Understand the Method of Lagrange Multipliers.
- Use Lagrange multipliers to solve constrained optimization problems.
- Use the Method of Lagrange Multipliers with two constraints.

Lagrange Multipliers

JOSEPH-LOUIS LAGRANGE (1736–1813)

The Method of Lagrange Multipliers is named after the French mathematician Joseph-Louis Lagrange. Lagrange first introduced the method in his famous paper on mechanics, written when he was just 19 years old.

Many optimization problems have restrictions, or **constraints,** on the values that can be used to produce the optimal solution. Such constraints tend to complicate optimization problems because the optimal solution can occur at a boundary point of the domain. In this section, you will study an ingenious technique for solving such problems. It is called the **Method of Lagrange Multipliers.**

To see how this technique works, suppose you want to find the rectangle of maximum area that can be inscribed in the ellipse given by

$$\frac{x^2}{3^2} + \frac{y^2}{4^2} = 1.$$

Let (x, y) be the vertex of the rectangle in the first quadrant, as shown in Figure 13.78. Because the rectangle has sides of lengths $2x$ and $2y$, its area is given by

$$f(x, y) = 4xy. \qquad \text{Objective function}$$

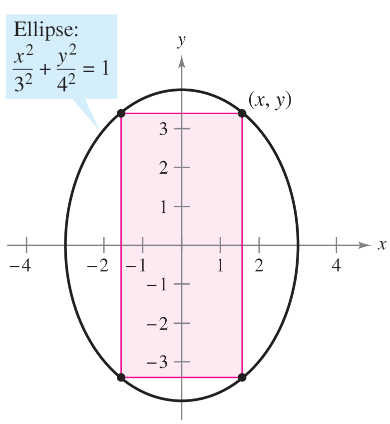
You want to find x and y such that $f(x, y)$ is a maximum. Your choice of (x, y) is restricted to first-quadrant points that lie on the ellipse

$$\frac{x^2}{3^2} + \frac{y^2}{4^2} = 1. \qquad \text{Constraint}$$

Now, consider the constraint equation to be a fixed level curve of

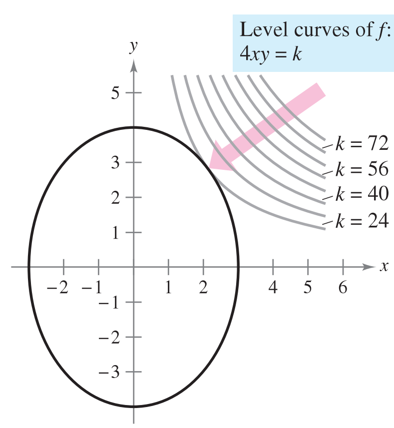
$$g(x, y) = \frac{x^2}{3^2} + \frac{y^2}{4^2}.$$

The level curves of f represent a family of hyperbolas $f(x, y) = 4xy = k$. In this family, the level curves that meet the given constraint correspond to the hyperbolas that intersect the ellipse. Moreover, to maximize $f(x, y)$, you want to find the hyperbola that just barely satisfies the constraint. The level curve that does this is the one that is *tangent* to the ellipse, as shown in Figure 13.79.



Objective function: $f(x, y) = 4xy$

Figure 13.78



Constraint: $g(x, y) = \frac{x^2}{3^2} + \frac{y^2}{4^2} = 1$

Figure 13.79

To find the appropriate hyperbola, use the fact that two curves are tangent at a point if and only if their gradient vectors are parallel. This means that $\nabla f(x, y)$ must be a scalar multiple of $\nabla g(x, y)$ at the point of tangency. In the context of constrained optimization problems, this scalar is denoted by λ (the lowercase Greek letter lambda).

$$\nabla f(x, y) = \lambda \nabla g(x, y)$$

The scalar λ is called a **Lagrange multiplier.** Theorem 13.19 gives the necessary conditions for the existence of such multipliers.

THEOREM 13.19 LAGRANGE'S THEOREM

Let f and g have continuous first partial derivatives such that f has an extremum at a point (x_0, y_0) on the smooth constraint curve $g(x, y) = c$. If $\nabla g(x_0, y_0) \neq \mathbf{0}$, then there is a real number λ such that

$$\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0).$$

PROOF To begin, represent the smooth curve given by $g(x, y) = c$ by the vector-valued function

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}, \quad \mathbf{r}'(t) \neq \mathbf{0}$$

where x' and y' are continuous on an open interval I. Define the function h as $h(t) = f(x(t), y(t))$. Then, because $f(x_0, y_0)$ is an extreme value of f, you know that

$$h(t_0) = f(x(t_0), y(t_0)) = f(x_0, y_0)$$

is an extreme value of h. This implies that $h'(t_0) = 0$, and, by the Chain Rule,

$$h'(t_0) = f_x(x_0, y_0)x'(t_0) + f_y(x_0, y_0)y'(t_0) = \nabla f(x_0, y_0) \cdot \mathbf{r}'(t_0) = 0.$$

So, $\nabla f(x_0, y_0)$ is orthogonal to $\mathbf{r}'(t_0)$. Moreover, by Theorem 13.12, $\nabla g(x_0, y_0)$ is also orthogonal to $\mathbf{r}'(t_0)$. Consequently, the gradients $\nabla f(x_0, y_0)$ and $\nabla g(x_0, y_0)$ are parallel, and there must exist a scalar λ such that

$$\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0).$$ ■

NOTE Lagrange's Theorem can be shown to be true for functions of three variables, using a similar argument with level surfaces and Theorem 13.14.

The Method of Lagrange Multipliers uses Theorem 13.19 to find the extreme values of a function f subject to a constraint.

METHOD OF LAGRANGE MULTIPLIERS

Let f and g satisfy the hypothesis of Lagrange's Theorem, and let f have a minimum or maximum subject to the constraint $g(x, y) = c$. To find the minimum or maximum of f, use the following steps.

1. Simultaneously solve the equations $\nabla f(x, y) = \lambda \nabla g(x, y)$ and $g(x, y) = c$ by solving the following system of equations.

$$f_x(x, y) = \lambda g_x(x, y)$$
$$f_y(x, y) = \lambda g_y(x, y)$$
$$g(x, y) = c$$

2. Evaluate f at each solution point obtained in the first step. The largest value yields the maximum of f subject to the constraint $g(x, y) = c$, and the smallest value yields the minimum of f subject to the constraint $g(x, y) = c$.

NOTE As you will see in Examples 1 and 2, the Method of Lagrange Multipliers requires solving systems of nonlinear equations. This often can require some tricky algebraic manipulation.

Constrained Optimization Problems

In the problem at the beginning of this section, you wanted to maximize the area of a rectangle that is inscribed in an ellipse. Example 1 shows how to use Lagrange multipliers to solve this problem.

EXAMPLE 1 Using a Lagrange Multiplier with One Constraint

Find the maximum value of $f(x, y) = 4xy$ where $x > 0$ and $y > 0$, subject to the constraint $(x^2/3^2) + (y^2/4^2) = 1$.

NOTE Example 1 can also be solved using the techniques you learned in Chapter 3. To see how, try to find the maximum value of $A = 4xy$ given that

$$\frac{x^2}{3^2} + \frac{y^2}{4^2} = 1.$$

To begin, solve the second equation for y to obtain

$$y = \tfrac{4}{3}\sqrt{9 - x^2}.$$

Then substitute into the first equation to obtain

$$A = 4x\left(\tfrac{4}{3}\sqrt{9 - x^2}\right).$$

Finally, use the techniques of Chapter 3 to maximize A.

Solution To begin, let

$$g(x, y) = \frac{x^2}{3^2} + \frac{y^2}{4^2} = 1.$$

By equating $\nabla f(x, y) = 4y\mathbf{i} + 4x\mathbf{j}$ and $\lambda\nabla g(x, y) = (2\lambda x/9)\mathbf{i} + (\lambda y/8)\mathbf{j}$, you can obtain the following system of equations.

$$4y = \frac{2}{9}\lambda x \qquad f_x(x, y) = \lambda g_x(x, y)$$

$$4x = \frac{1}{8}\lambda y \qquad f_y(x, y) = \lambda g_y(x, y)$$

$$\frac{x^2}{3^2} + \frac{y^2}{4^2} = 1 \qquad \text{Constraint}$$

From the first equation, you obtain $\lambda = 18y/x$, and substitution into the second equation produces

$$4x = \frac{1}{8}\left(\frac{18y}{x}\right)y \quad \Rightarrow \quad x^2 = \frac{9}{16}y^2.$$

Substituting this value for x^2 into the third equation produces

$$\frac{1}{9}\left(\frac{9}{16}y^2\right) + \frac{1}{16}y^2 = 1 \quad \Rightarrow \quad y^2 = 8.$$

So, $y = \pm 2\sqrt{2}$. Because it is required that $y > 0$, choose the positive value and find that

$$\begin{aligned} x^2 &= \frac{9}{16}y^2 \\ &= \frac{9}{16}(8) = \frac{9}{2} \\ x &= \frac{3}{\sqrt{2}}. \end{aligned}$$

So, the maximum value of f is

$$f\left(\frac{3}{\sqrt{2}}, 2\sqrt{2}\right) = 4xy = 4\left(\frac{3}{\sqrt{2}}\right)\left(2\sqrt{2}\right) = 24. \qquad \blacksquare$$

Note that writing the constraint as

$$g(x, y) = \frac{x^2}{3^2} + \frac{y^2}{4^2} = 1 \qquad \text{or} \qquad g(x, y) = \frac{x^2}{3^2} + \frac{y^2}{4^2} - 1 = 0$$

does not affect the solution—the constant is eliminated when you form ∇g.

EXAMPLE 2 A Business Application

The Cobb-Douglas production function (see Example 5, Section 13.1) for a software manufacturer is given by

$$f(x, y) = 100x^{3/4}y^{1/4} \qquad \text{Objective function}$$

where x represents the units of labor (at \$150 per unit) and y represents the units of capital (at \$250 per unit). The total cost of labor and capital is limited to \$50,000. Find the maximum production level for this manufacturer.

■ **FOR FURTHER INFORMATION** For more information on the use of Lagrange multipliers in economics, see the article "Lagrange Multiplier Problems in Economics" by John V. Baxley and John C. Moorhouse in *The American Mathematical Monthly.* To view this article, go to the website *www.matharticles.com*.

Solution From the given function, you have

$$\nabla f(x, y) = 75x^{-1/4}y^{1/4}\mathbf{i} + 25x^{3/4}y^{-3/4}\mathbf{j}.$$

The limit on the cost of labor and capital produces the constraint

$$g(x, y) = 150x + 250y = 50{,}000. \qquad \text{Constraint}$$

So, $\lambda\nabla g(x, y) = 150\lambda\mathbf{i} + 250\lambda\mathbf{j}$. This gives rise to the following system of equations.

$$\begin{aligned} 75x^{-1/4}y^{1/4} &= 150\lambda && f_x(x, y) = \lambda g_x(x, y) \\ 25x^{3/4}y^{-3/4} &= 250\lambda && f_y(x, y) = \lambda g_y(x, y) \\ 150x + 250y &= 50{,}000 && \text{Constraint} \end{aligned}$$

By solving for λ in the first equation

$$\lambda = \frac{75x^{-1/4}y^{1/4}}{150} = \frac{x^{-1/4}y^{1/4}}{2}$$

and substituting into the second equation, you obtain

$$\begin{aligned} 25x^{3/4}y^{-3/4} &= 250\left(\frac{x^{-1/4}y^{1/4}}{2}\right) \\ 25x &= 125y. && \text{Multiply by } x^{1/4}y^{3/4}. \end{aligned}$$

So, $x = 5y$. By substituting into the third equation, you have

$$\begin{aligned} 150(5y) + 250y &= 50{,}000 \\ 1000y &= 50{,}000 \\ y &= 50 \text{ units of capital} \\ x &= 250 \text{ units of labor.} \end{aligned}$$

So, the maximum production level is

$$\begin{aligned} f(250, 50) &= 100(250)^{3/4}(50)^{1/4} \\ &\approx 16{,}719 \text{ product units.} \end{aligned}$$

■

Economists call the Lagrange multiplier obtained in a production function the **marginal productivity of money.** For instance, in Example 2 the marginal productivity of money at $x = 250$ and $y = 50$ is

$$\lambda = \frac{x^{-1/4}y^{1/4}}{2} = \frac{(250)^{-1/4}(50)^{1/4}}{2} \approx 0.334$$

which means that for each additional dollar spent on production, an additional 0.334 unit of the product can be produced.

EXAMPLE 3 Lagrange Multipliers and Three Variables

Find the minimum value of

$$f(x, y, z) = 2x^2 + y^2 + 3z^2 \qquad \text{Objective function}$$

subject to the constraint $2x - 3y - 4z = 49$.

Solution Let $g(x, y, z) = 2x - 3y - 4z = 49$. Then, because

$$\nabla f(x, y, z) = 4x\mathbf{i} + 2y\mathbf{j} + 6z\mathbf{k} \quad \text{and} \quad \lambda\nabla g(x, y, z) = 2\lambda\mathbf{i} - 3\lambda\mathbf{j} - 4\lambda\mathbf{k}$$

you obtain the following system of equations.

$$\begin{aligned} 4x &= 2\lambda && f_x(x, y, z) = \lambda g_x(x, y, z) \\ 2y &= -3\lambda && f_y(x, y, z) = \lambda g_y(x, y, z) \\ 6z &= -4\lambda && f_z(x, y, z) = \lambda g_z(x, y, z) \\ 2x - 3y - 4z &= 49 && \text{Constraint} \end{aligned}$$

The solution of this system is $x = 3$, $y = -9$, and $z = -4$. So, the optimum value of f is

$$\begin{aligned} f(3, -9, -4) &= 2(3)^2 + (-9)^2 + 3(-4)^2 \\ &= 147. \end{aligned}$$

From the original function and constraint, it is clear that $f(x, y, z)$ has no maximum. So, the optimum value of f determined above is a minimum. ■

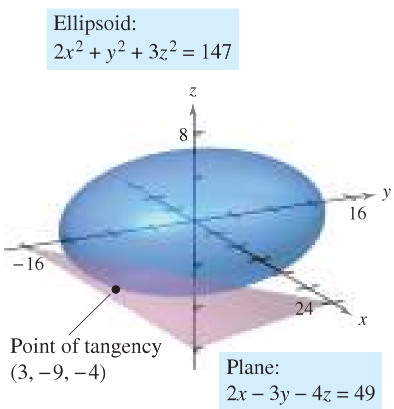


Figure 13.80

A graphical interpretation of constrained optimization problems in two variables was given at the beginning of this section. In three variables, the interpretation is similar, except that level surfaces are used instead of level curves. For instance, in Example 3, the level surfaces of f are ellipsoids centered at the origin, and the constraint

$$2x - 3y - 4z = 49$$

is a plane. The minimum value of f is represented by the ellipsoid that is tangent to the constraint plane, as shown in Figure 13.80.

EXAMPLE 4 Optimization Inside a Region

Find the extreme values of

$$f(x, y) = x^2 + 2y^2 - 2x + 3 \qquad \text{Objective function}$$

subject to the constraint $x^2 + y^2 \le 10$.

Solution To solve this problem, you can break the constraint into two cases.

a. For points *on the circle* $x^2 + y^2 = 10$, you can use Lagrange multipliers to find that the maximum value of $f(x, y)$ is 24—this value occurs at $(-1, 3)$ and at $(-1, -3)$. In a similar way, you can determine that the minimum value of $f(x, y)$ is approximately 6.675—this value occurs at $(\sqrt{10}, 0)$.

b. For points *inside the circle*, you can use the techniques discussed in Section 13.8 to conclude that the function has a relative minimum of 2 at the point $(1, 0)$.

By combining these two results, you can conclude that f has a maximum of 24 at $(-1, \pm 3)$ and a minimum of 2 at $(1, 0)$, as shown in Figure 13.81. ■

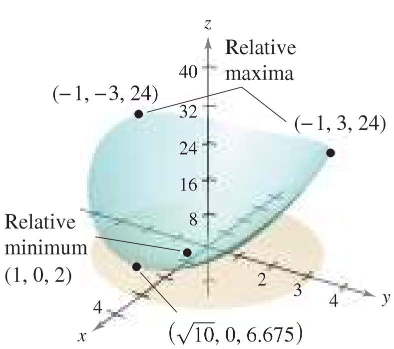


Figure 13.81

The Method of Lagrange Multipliers with Two Constraints

For optimization problems involving *two* constraint functions g and h, you can introduce a second Lagrange multiplier, μ (the lowercase Greek letter mu), and then solve the equation

$$\nabla f = \lambda \nabla g + \mu \nabla h$$

where the gradient vectors are not parallel, as illustrated in Example 5.

EXAMPLE 5 Optimization with Two Constraints

Let $T(x, y, z) = 20 + 2x + 2y + z^2$ represent the temperature at each point on the sphere $x^2 + y^2 + z^2 = 11$. Find the extreme temperatures on the curve formed by the intersection of the plane $x + y + z = 3$ and the sphere.

Solution The two constraints are

$$g(x, y, z) = x^2 + y^2 + z^2 = 11 \quad \text{and} \quad h(x, y, z) = x + y + z = 3.$$

Using

$$\begin{aligned} \nabla T(x, y, z) &= 2\mathbf{i} + 2\mathbf{j} + 2z\mathbf{k} \\ \lambda \nabla g(x, y, z) &= 2\lambda x\mathbf{i} + 2\lambda y\mathbf{j} + 2\lambda z\mathbf{k} \end{aligned}$$

and

$$\mu \nabla h(x, y, z) = \mu\mathbf{i} + \mu\mathbf{j} + \mu\mathbf{k}$$

you can write the following system of equations.

$$\begin{aligned} 2 &= 2\lambda x + \mu && T_x(x, y, z) = \lambda g_x(x, y, z) + \mu h_x(x, y, z) \\ 2 &= 2\lambda y + \mu && T_y(x, y, z) = \lambda g_y(x, y, z) + \mu h_y(x, y, z) \\ 2z &= 2\lambda z + \mu && T_z(x, y, z) = \lambda g_z(x, y, z) + \mu h_z(x, y, z) \\ x^2 + y^2 + z^2 &= 11 && \text{Constraint 1} \\ x + y + z &= 3 && \text{Constraint 2} \end{aligned}$$

By subtracting the second equation from the first, you can obtain the following system.

$$\begin{aligned} \lambda(x - y) &= 0 \\ 2z(1 - \lambda) - \mu &= 0 \\ x^2 + y^2 + z^2 &= 11 \\ x + y + z &= 3 \end{aligned}$$

STUDY TIP The systems of equations that arise when the Method of Lagrange Multipliers is used are not, in general, linear systems, and finding the solutions often requires ingenuity.

From the first equation, you can conclude that $\lambda = 0$ or $x = y$. If $\lambda = 0$, you can show that the critical points are $(3, -1, 1)$ and $(-1, 3, 1)$. (Try doing this—it takes a little work.) If $\lambda \neq 0$, then $x = y$ and you can show that the critical points occur when $x = y = \left(3 \pm 2\sqrt{3}\right)/3$ and $z = \left(3 \mp 4\sqrt{3}\right)/3$. Finally, to find the optimal solutions, compare the temperatures at the four critical points.

$$T(3, -1, 1) = T(-1, 3, 1) = 25$$

$$T\left(\frac{3 - 2\sqrt{3}}{3}, \frac{3 - 2\sqrt{3}}{3}, \frac{3 + 4\sqrt{3}}{3}\right) = \frac{91}{3} \approx 30.33$$

$$T\left(\frac{3 + 2\sqrt{3}}{3}, \frac{3 + 2\sqrt{3}}{3}, \frac{3 - 4\sqrt{3}}{3}\right) = \frac{91}{3} \approx 30.33$$

So, $T = 25$ is the minimum temperature and $T = \frac{91}{3}$ is the maximum temperature on the curve. ■

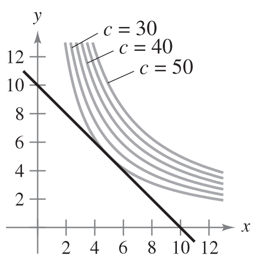
13.10 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–4, identify the constraint and level curves of the objective function shown in the figure. Use the figure to approximate the indicated extrema, assuming that x and y are positive. Use Lagrange multipliers to verify your result.

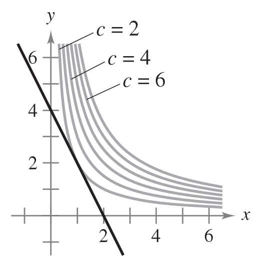
1. Maximize $z = xy$

Constraint: $x + y = 10$



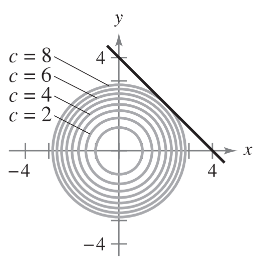
2. Maximize $z = xy$

Constraint: $2x + y = 4$



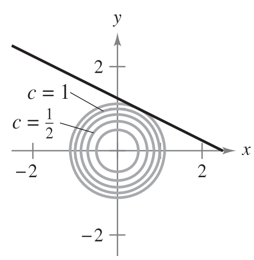
3. Minimize $z = x^2 + y^2$

Constraint: $x + y - 4 = 0$



4. Minimize $z = x^2 + y^2$

Constraint: $2x + 4y = 5$



In Exercises 5–10, use Lagrange multipliers to find the indicated extrema, assuming that x and y are positive.

5. Minimize $f(x, y) = x^2 + y^2$

Constraint: $x + 2y - 5 = 0$

6. Maximize $f(x, y) = x^2 - y^2$

Constraint: $2y - x^2 = 0$

7. Maximize $f(x, y) = 2x + 2xy + y$

Constraint: $2x + y = 100$

8. Minimize $f(x, y) = 3x + y + 10$

Constraint: $x^2y = 6$

9. Maximize $f(x, y) = \sqrt{6 - x^2 - y^2}$

Constraint: $x + y - 2 = 0$

10. Minimize $f(x, y) = \sqrt{x^2 + y^2}$

Constraint: $2x + 4y - 15 = 0$

In Exercises 11–14, use Lagrange multipliers to find the indicated extrema, assuming that x, y, and z are positive.

11. Minimize $f(x, y, z) = x^2 + y^2 + z^2$

Constraint: $x + y + z - 9 = 0$

12. Maximize $f(x, y, z) = xyz$

Constraint: $x + y + z - 3 = 0$

13. Minimize $f(x, y, z) = x^2 + y^2 + z^2$

Constraint: $x + y + z = 1$

14. Minimize $f(x, y) = x^2 - 10x + y^2 - 14y + 28$

Constraint: $x + y = 10$

In Exercises 15 and 16, use Lagrange multipliers to find any extrema of the function subject to the constraint $x^2 + y^2 \le 1$.

15. $f(x, y) = x^2 + 3xy + y^2$

16. $f(x, y) = e^{-xy/4}$

In Exercises 17 and 18, use Lagrange multipliers to find the indicated extrema of f subject to two constraints. In each case, assume that x, y, and z are nonnegative.

17. Maximize $f(x, y, z) = xyz$

Constraints: $x + y + z = 32, \quad x - y + z = 0$

18. Minimize $f(x, y, z) = x^2 + y^2 + z^2$

Constraints: $x + 2z = 6, \quad x + y = 12$

In Exercises 19–28, use Lagrange multipliers to find the minimum distance from the curve or surface to the indicated point. [*Hints:* In Exercise 19, minimize $f(x, y) = x^2 + y^2$ subject to the constraint $x + y = 1$. In Exercise 25, use the *root* feature of a graphing utility.]

	Curve	*Point*
19.	Line: $x + y = 1$	$(0, 0)$
20.	Line: $2x + 3y = -1$	$(0, 0)$
21.	Line: $x - y = 4$	$(0, 2)$
22.	Line: $x + 4y = 3$	$(1, 0)$
23.	Parabola: $y = x^2$	$(0, 3)$
24.	Parabola: $y = x^2$	$(-3, 0)$
25.	Parabola: $y = x^2 + 1$	$\left(\frac{1}{2}, 1\right)$
26.	Circle: $(x - 4)^2 + y^2 = 4$	$(0, 10)$
	Surface	*Point*
27.	Plane: $x + y + z = 1$	$(2, 1, 1)$
28.	Cone: $z = \sqrt{x^2 + y^2}$	$(4, 0, 0)$

In Exercises 29 and 30, find the highest point on the curve of intersection of the surfaces.

29. Cone: $x^2 + y^2 - z^2 = 0$, Plane: $x + 2z = 4$

30. Sphere: $x^2 + y^2 + z^2 = 36$, Plane: $2x + y - z = 2$

WRITING ABOUT CONCEPTS

31. Explain what is meant by constrained optimization problems.

32. Explain the Method of Lagrange Multipliers for solving constrained optimization problems.

In Exercises 33–42, use Lagrange multipliers to solve the indicated exercise in Section 13.9.

33. Exercise 1 **34.** Exercise 2
35. Exercise 5 **36.** Exercise 6
37. Exercise 9 **38.** Exercise 10
39. Exercise 11 **40.** Exercise 12
41. Exercise 17 **42.** Exercise 18

43. ***Maximum Volume*** Use Lagrange multipliers to find the dimensions of a rectangular box of maximum volume that can be inscribed (with edges parallel to the coordinate axes) in the ellipsoid $(x^2/a^2) + (y^2/b^2) + (z^2/c^2) = 1$.

CAPSTONE

44. The sum of the length and the girth (perimeter of a cross section) of a package carried by a delivery service cannot exceed 108 inches.

(a) Determine whether Lagrange multipliers can be used to find the dimensions of the rectangular package of largest volume that may be sent. Explain your reasoning.

(b) If Lagrange multipliers can be used, find the dimensions. Compare your answer with that obtained in Exercise 38, Section 13.9.

45. ***Minimum Cost*** A cargo container (in the shape of a rectangular solid) must have a volume of 480 cubic feet. The bottom will cost \$5 per square foot to construct and the sides and the top will cost \$3 per square foot to construct. Use Lagrange multipliers to find the dimensions of the container of this size that has minimum cost.

46. ***Geometric and Arithmetic Means***

(a) Use Lagrange multipliers to prove that the product of three positive numbers x, y, and z, whose sum has the constant value S, is a maximum when the three numbers are equal. Use this result to prove that $\sqrt[3]{xyz} \le (x + y + z)/3$.

(b) Generalize the result of part (a) to prove that the product $x_1 x_2 x_3 \cdot \cdot \cdot x_n$ is a maximum when $x_1 = x_2 = x_3 = \cdot \cdot \cdot = x_n$, $\sum_{i=1}^{n} x_i = S$, and all $x_i \ge 0$. Then prove that

$$\sqrt[n]{x_1 x_2 x_3 \cdot \cdot \cdot x_n} \le \frac{x_1 + x_2 + x_3 + \cdot \cdot \cdot + x_n}{n}.$$

This shows that the geometric mean is never greater than the arithmetic mean.

47. ***Minimum Surface Area*** Use Lagrange multipliers to find the dimensions of a right circular cylinder with volume V_0 cubic units and minimum surface area.

48. ***Temperature Distribution*** Let $T(x, y, z) = 100 + x^2 + y^2$ represent the temperature at each point on the sphere $x^2 + y^2 + z^2 = 50$. Find the maximum temperature on the curve formed by the intersection of the sphere and the plane $x - z = 0$.

49. ***Refraction of Light*** When light waves traveling in a transparent medium strike the surface of a second transparent medium, they tend to "bend" in order to follow the path of minimum time. This tendency is called *refraction* and is described by **Snell's Law of Refraction,**

$$\frac{\sin \theta_1}{v_1} = \frac{\sin \theta_2}{v_2}$$

where θ_1 and θ_2 are the magnitudes of the angles shown in the figure, and v_1 and v_2 are the velocities of light in the two media. Use Lagrange multipliers to derive this law using $x + y = a$.

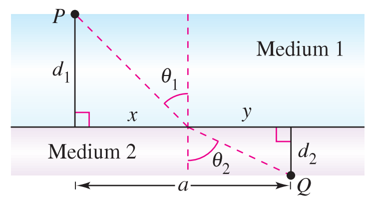


Figure for 49

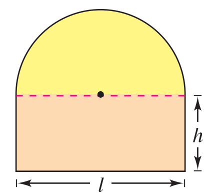


Figure for 50

50. ***Area and Perimeter*** A semicircle is on top of a rectangle (see figure). If the area is fixed and the perimeter is a minimum, or if the perimeter is fixed and the area is a maximum, use Lagrange multipliers to verify that the length of the rectangle is twice its height.

Production Level **In Exercises 51 and 52, find the maximum production level *P* if the total cost of labor (at \$72 per unit) and capital (at \$60 per unit) is limited to \$250,000, where *x* is the number of units of labor and *y* is the number of units of capital.**

51. $P(x, y) = 100x^{0.25}y^{0.75}$ **52.** $P(x, y) = 100x^{0.4}y^{0.6}$

Cost **In Exercises 53 and 54, find the minimum cost of producing 50,000 units of a product, where *x* is the number of units of labor (at \$72 per unit) and *y* is the number of units of capital (at \$60 per unit).**

53. $P(x, y) = 100x^{0.25}y^{0.75}$ **54.** $P(x, y) = 100x^{0.6}y^{0.4}$

55. ***Investigation*** Consider the objective function $g(\alpha, \beta, \gamma) = \cos \alpha \cos \beta \cos \gamma$ subject to the constraint that α, β, and γ are the angles of a triangle.

(a) Use Lagrange multipliers to maximize g.

CAS (b) Use the constraint to reduce the function g to a function of two independent variables. Use a computer algebra system to graph the surface represented by g. Identify the maximum values on the graph.

PUTNAM EXAM CHALLENGE

56. A can buoy is to be made of three pieces, namely, a cylinder and two equal cones, the altitude of each cone being equal to the altitude of the cylinder. For a given area of surface, what shape will have the greatest volume?

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13 REVIEW EXERCISES

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1 and 2, sketch the graph of the level surface $f(x, y, z) = c$ at the given value of c.

1. $f(x, y, z) = x^2 - y + z^2, \quad c = 2$
2. $f(x, y, z) = 4x^2 - y^2 + 4z^2, \quad c = 0$

3. ***Conjecture*** Consider the function $f(x, y) = x^2 + y^2$.
   (a) Sketch the graph of the surface given by f.
   (b) Make a conjecture about the relationship between the graphs of f and $g(x, y) = f(x, y) + 2$. Explain your reasoning.
   (c) Make a conjecture about the relationship between the graphs of f and $g(x, y) = f(x, y - 2)$. Explain your reasoning.
   (d) On the surface in part (a), sketch the graphs of $z = f(1, y)$ and $z = f(x, 1)$.

4. ***Conjecture*** Consider the function
   $f(x, y) = \sqrt{1 - x^2 - y^2}$.
   (a) Sketch the graph of the surface given by f.
   (b) Make a conjecture about the relationship between the graphs of f and $g(x, y) = f(x + 2, y)$. Explain your reasoning.
   (c) Make a conjecture about the relationship between the graphs of f and $g(x, y) = 4 - f(x, y)$. Explain your reasoning.
   (d) On the surface in part (a), sketch the graphs of $z = f(0, y)$ and $z = f(x, 0)$.

CAS **In Exercises 5–8, use a computer algebra system to graph several level curves of the function.**

5. $f(x, y) = e^{x^2 + y^2}$
6. $f(x, y) = \ln xy$
7. $f(x, y) = x^2 - y^2$
8. $f(x, y) = \dfrac{x}{x + y}$

CAS **In Exercises 9 and 10, use a computer algebra system to graph the function.**

9. $f(x, y) = e^{-(x^2 + y^2)}$
10. $g(x, y) = |y|^{1 + |x|}$

In Exercises 11–14, find the limit and discuss the continuity of the function (if it exists).

11. $\displaystyle\lim_{(x, y)\to(1, 1)} \frac{xy}{x^2 + y^2}$
12. $\displaystyle\lim_{(x, y)\to(1, 1)} \frac{xy}{x^2 - y^2}$
13. $\displaystyle\lim_{(x, y)\to(0, 0)} \frac{y + xe^{-y^2}}{1 + x^2}$
14. $\displaystyle\lim_{(x, y)\to(0, 0)} \frac{x^2 y}{x^4 + y^2}$

In Exercises 15–24, find all first partial derivatives.

15. $f(x, y) = e^x \cos y$
16. $f(x, y) = \dfrac{xy}{x + y}$
17. $z = e^{-y} + e^{-x}$
18. $z = \ln(x^2 + y^2 + 1)$
19. $g(x, y) = \dfrac{xy}{x^2 + y^2}$
20. $w = \sqrt{x^2 - y^2 - z^2}$
21. $f(x, y, z) = z \arctan \dfrac{y}{x}$
22. $f(x, y, z) = \dfrac{1}{\sqrt{1 + x^2 + y^2 + z^2}}$
23. $u(x, t) = ce^{-n^2 t} \sin nx$
24. $u(x, t) = c \sin(akx) \cos kt$

25. ***Think About It*** Sketch a graph of a function $z = f(x, y)$ whose derivative f_x is always negative and whose derivative f_y is always negative.

26. Find the slopes of the surface $z = x^2 \ln(y + 1)$ in the x- and y-directions at the point $(2, 0, 0)$.

In Exercises 27–30, find all second partial derivatives and verify that the second mixed partials are equal.

27. $f(x, y) = 3x^2 - xy + 2y^3$
28. $h(x, y) = \dfrac{x}{x + y}$
29. $h(x, y) = x \sin y + y \cos x$
30. $g(x, y) = \cos(x - 2y)$

Laplace's Equation **In Exercises 31–34, show that the function satisfies Laplace's equation**

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0.$$

31. $z = x^2 - y^2$
32. $z = x^3 - 3xy^2$
33. $z = \dfrac{y}{x^2 + y^2}$
34. $z = e^y \sin x$

In Exercises 35 and 36, find the total differential.

35. $z = x \sin xy$
36. $z = \dfrac{xy}{\sqrt{x^2 + y^2}}$

37. ***Error Analysis*** The legs of a right triangle are measured to be 5 centimeters and 12 centimeters, with a possible error of $\frac{1}{2}$ centimeter. Approximate the maximum possible error in computing the length of the hypotenuse. Approximate the maximum percent error.

38. ***Error Analysis*** To determine the height of a tower, the angle of elevation to the top of the tower is measured from a point 100 feet $\pm \frac{1}{2}$ foot from the base. The angle is measured at 33°, with a possible error of 1°. Assuming that the ground is horizontal, approximate the maximum error in determining the height of the tower.

39. ***Volume*** A right circular cone is measured, and the radius and height are found to be 2 inches and 5 inches, respectively. The possible error in measurement is $\frac{1}{8}$ inch. Approximate the maximum possible error in the computation of the volume.

40. ***Lateral Surface Area*** Approximate the error in the computation of the lateral surface area of the cone in Exercise 39. (The lateral surface area is given by $A = \pi r\sqrt{r^2 + h^2}$.)

In Exercises 41–44, find the indicated derivatives (a) using the appropriate Chain Rule and (b) using substitution before differentiating.

41. $w = \ln(x^2 + y), \quad \dfrac{dw}{dt}$

$x = 2t, \quad y = 4 - t$

42. $u = y^2 - x, \quad \dfrac{du}{dt}$

$x = \cos t, \quad y = \sin t$

43. $w = \dfrac{xy}{z}, \quad \dfrac{\partial w}{\partial r}, \quad \dfrac{\partial w}{\partial t}$

$x = 2r + t, \quad y = rt, \quad z = 2r - t$

44. $u = x^2 + y^2 + z^2, \quad \dfrac{\partial u}{\partial r}, \quad \dfrac{\partial u}{\partial t}$

$x = r\cos t, \quad y = r\sin t, \quad z = t$

In Exercises 45 and 46, differentiate implicitly to find the first partial derivatives of *z*.

45. $x^2 + xy + y^2 + yz + z^2 = 0$

46. $xz^2 - y\sin z = 0$

In Exercises 47–50, find the directional derivative of the function at *P* in the direction of v.

47. $f(x, y) = x^2y, \quad (-5, 5), \quad \mathbf{v} = 3\mathbf{i} - 4\mathbf{j}$

48. $f(x, y) = \frac{1}{4}y^2 - x^2, \quad (1, 4), \quad \mathbf{v} = 2\mathbf{i} + \mathbf{j}$

49. $w = y^2 + xz, \quad (1, 2, 2), \quad \mathbf{v} = 2\mathbf{i} - \mathbf{j} + 2\mathbf{k}$

50. $w = 5x^2 + 2xy - 3y^2z, \quad (1, 0, 1), \quad \mathbf{v} = \mathbf{i} + \mathbf{j} - \mathbf{k}$

In Exercises 51–54, find the gradient of the function and the maximum value of the directional derivative at the given point.

51. $z = x^2y, \quad (2, 1)$

52. $z = e^{-x}\cos y, \quad \left(0, \dfrac{\pi}{4}\right)$

53. $z = \dfrac{y}{x^2 + y^2}, \quad (1, 1)$

54. $z = \dfrac{x^2}{x - y}, \quad (2, 1)$

In Exercises 55 and 56, (a) find the gradient of the function at *P*, (b) find a unit normal vector to the level curve $f(x, y) = c$ at *P*, (c) find the tangent line to the level curve $f(x, y) = c$ at *P*, and (d) sketch the level curve, the unit normal vector, and the tangent line in the *xy*-plane.

55. $f(x, y) = 9x^2 - 4y^2$

$c = 65, \quad P(3, 2)$

56. $f(x, y) = 4y\sin x - y$

$c = 3, \quad P\left(\dfrac{\pi}{2}, 1\right)$

In Exercises 57–60, find an equation of the tangent plane and parametric equations of the normal line to the surface at the given point.

	Surface	*Point*
57.	$f(x, y) = x^2y$	$(2, 1, 4)$
58.	$f(x, y) = \sqrt{25 - y^2}$	$(2, 3, 4)$
59.	$z = -9 + 4x - 6y - x^2 - y^2$	$(2, -3, 4)$
60.	$z = \sqrt{9 - x^2 - y^2}$	$(1, 2, 2)$

In Exercises 61 and 62, find symmetric equations of the tangent line to the curve of intersection of the surfaces at the given point.

	Surfaces	*Point*
61.	$z = 9 - y^2, \quad y = x$	$(2, 2, 5)$
62.	$z = x^2 - y^2, \quad z = 3$	$(2, 1, 3)$

63. Find the angle of inclination θ of the tangent plane to the surface $x^2 + y^2 + z^2 = 14$ at the point $(2, 1, 3)$.

64. ***Approximation*** Consider the following approximations for a function $f(x, y)$ centered at $(0, 0)$.

Linear approximation:

$P_1(x, y) = f(0, 0) + f_x(0, 0)x + f_y(0, 0)y$

Quadratic approximation:

$$P_2(x, y) = f(0, 0) + f_x(0, 0)x + f_y(0, 0)y + \tfrac{1}{2}f_{xx}(0, 0)x^2 + f_{xy}(0, 0)xy + \tfrac{1}{2}f_{yy}(0, 0)y^2$$

[Note that the linear approximation is the tangent plane to the surface at $(0, 0, f(0, 0))$.]

(a) Find the linear approximation of $f(x, y) = \cos x + \sin y$ centered at $(0, 0)$.

(b) Find the quadratic approximation of $f(x, y) = \cos x + \sin y$ centered at $(0, 0)$.

(c) If $y = 0$ in the quadratic approximation, you obtain the second-degree Taylor polynomial for what function?

(d) Complete the table.

x	y	$f(x, y)$	$P_1(x, y)$	$P_2(x, y)$
0	0			
0	0.1			
0.2	0.1			
0.5	0.3			
1	0.5			

CAS (e) Use a computer algebra system to graph the surfaces $z = f(x, y)$, $z = P_1(x, y)$, and $z = P_2(x, y)$. How does the accuracy of the approximations change as the distance from $(0, 0)$ increases?

CAS **In Exercises 65–68, examine the function for relative extrema and saddle points. Use a computer algebra system to graph the function and confirm your results.**

65. $f(x, y) = 2x^2 + 6xy + 9y^2 + 8x + 14$

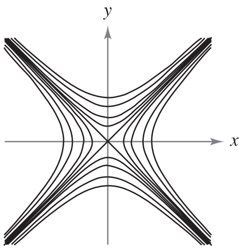
66. $f(x, y) = x^2 + 3xy + y^2 - 5x$

67. $f(x, y) = xy + \dfrac{1}{x} + \dfrac{1}{y}$

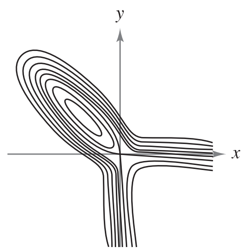
68. $z = 50(x + y) - (0.1x^3 + 20x + 150) - (0.05y^3 + 20.6y + 125)$

Writing **In Exercises 69 and 70, write a short paragraph about the surface whose level curves (c-values evenly spaced) are shown. Comment on possible extrema, saddle points, the magnitude of the gradient, etc.**

69.



70.



71. ***Maximum Profit*** A corporation manufactures digital cameras at two locations. The cost functions for producing x_1 units at location 1 and x_2 units at location 2 are

$$C_1 = 0.05x_1^2 + 15x_1 + 5400$$
$$C_2 = 0.03x_2^2 + 15x_2 + 6100$$

and the total revenue function is

$$R = [225 - 0.4(x_1 + x_2)](x_1 + x_2).$$

Find the production levels at the two locations that will maximize the profit $P(x_1, x_2) = R - C_1 - C_2$.

72. ***Minimum Cost*** A manufacturer has an order for 1000 units of wooden benches that can be produced at two locations. Let x_1 and x_2 be the numbers of units produced at the two locations. The cost function is

$$C = 0.25x_1^2 + 10x_1 + 0.15x_2^2 + 12x_2.$$

Find the number that should be produced at each location to meet the order and minimize cost.

73. ***Production Level*** The production function for a candy manufacturer is

$$f(x, y) = 4x + xy + 2y$$

where x is the number of units of labor and y is the number of units of capital. Assume that the total amount available for labor and capital is \$2000, and that units of labor and capital cost \$20 and \$4, respectively. Find the maximum production level for this manufacturer.

74. Find the minimum distance from the point $(2, 2, 0)$ to the surface $z = x^2 + y^2$.

75. ***Modeling Data*** The table shows the drag force y in kilograms for a motor vehicle at indicated speeds x in kilometers per hour.

Speed, x	25	50	75	100	125
Drag, y	24	34	50	71	98

(a) Use the regression capabilities of a graphing utility to find the least squares regression quadratic for the data.

(b) Use the model to estimate the total drag when the vehicle is moving at 80 kilometers per hour.

76. ***Modeling Data*** The data in the table show the yield y (in milligrams) of a chemical reaction after t minutes.

Minutes, t	1	2	3	4	5	6	7	8
Yield, y	1.2	7.1	9.9	13.1	15.5	16.0	17.9	18.0

(a) Use the regression capabilities of a graphing utility to find the least squares regression line for the data. Then use the graphing utility to plot the data and graph the model.

(b) Use a graphing utility to plot the points $(\ln t, y)$. Do these points appear to follow a linear pattern more closely than the plot of the given data in part (a)?

(c) Use the regression capabilities of a graphing utility to find the least squares regression line for the points $(\ln t, y)$ and obtain the logarithmic model $y = a + b \ln t$.

(d) Use a graphing utility to plot the data and graph the linear and logarithmic models. Which is a better model? Explain.

In Exercises 77 and 78, use Lagrange multipliers to locate and classify any extrema of the function.

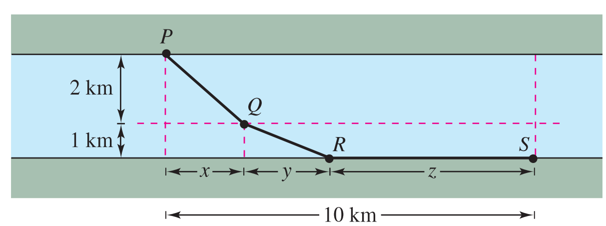
77. $w = xy + yz + xz$

Constraint: $x + y + z = 1$

78. $z = x^2y$

Constraint: $x + 2y = 2$

79. ***Minimum Cost*** A water line is to be built from point P to point S and must pass through regions where construction costs differ (see figure). The cost per kilometer in dollars is $3k$ from P to Q, $2k$ from Q to R, and k from R to S. For simplicity, let $k = 1$. Use Lagrange multipliers to find x, y, and z such that the total cost C will be minimized.



CAS **80.** ***Investigation*** Consider the objective function $f(x, y) = ax + by$ subject to the constraint $x^2/64 + y^2/36 = 1$. Assume that x and y are positive.

(a) Use a computer algebra system to graph the constraint. For $a = 4$ and $b = 3$, use the computer algebra system to graph the level curves of the objective function. By trial and error, find the level curve that appears to be tangent to the ellipse. Use the result to approximate the maximum of f subject to the constraint.

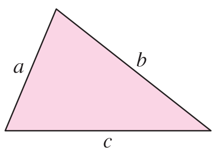
(b) Repeat part (a) for $a = 4$ and $b = 9$.

P.S. PROBLEM SOLVING

1. **Heron's Formula** states that the area of a triangle with sides of lengths a, b, and c is given by

$$A = \sqrt{s(s-a)(s-b)(s-c)}$$

where $s = \dfrac{a+b+c}{2}$, as shown in the figure.

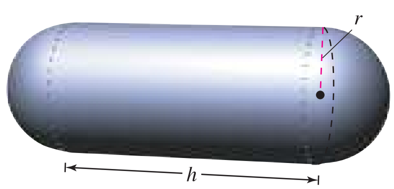


(a) Use Heron's Formula to find the area of the triangle with vertices $(0, 0)$, $(3, 4)$, and $(6, 0)$.

(b) Show that among all triangles having a fixed perimeter, the triangle with the largest area is an equilateral triangle.

(c) Show that among all triangles having a fixed area, the triangle with the smallest perimeter is an equilateral triangle.

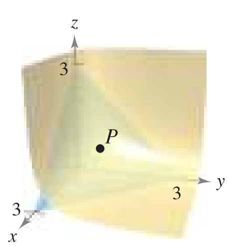
2. An industrial container is in the shape of a cylinder with hemispherical ends, as shown in the figure. The container must hold 1000 liters of fluid. Determine the radius r and length h that minimize the amount of material used in the construction of the tank.



3. Let $P(x_0, y_0, z_0)$ be a point in the first octant on the surface $xyz = 1$.

(a) Find the equation of the tangent plane to the surface at the point P.

(b) Show that the volume of the tetrahedron formed by the three coordinate planes and the tangent plane is constant, independent of the point of tangency (see figure).



4. Use a graphing utility to graph the functions $f(x) = \sqrt[3]{x^3 - 1}$ and $g(x) = x$ in the same viewing window.

(a) Show that

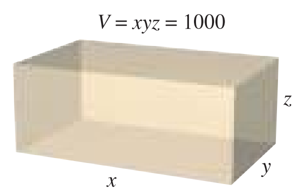
$$\lim_{x\to\infty} [f(x) - g(x)] = 0 \text{ and } \lim_{x\to-\infty} [f(x) - g(x)] = 0.$$

(b) Find the point on the graph of f that is farthest from the graph of g.

5. (a) Let $f(x, y) = x - y$ and $g(x, y) = x^2 + y^2 = 4$. Graph various level curves of f and the constraint g in the xy-plane. Use the graph to determine the maximum value of f subject to the constraint $g = 4$. Then verify your answer using Lagrange multipliers.

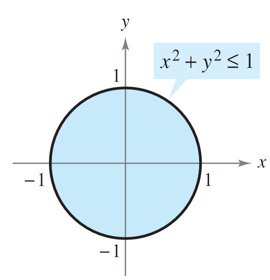
(b) Let $f(x, y) = x - y$ and $g(x, y) = x^2 + y^2 = 0$. Find the maximum and minimum values of f subject to the constraint $g = 0$. Does the method of Lagrange Multipliers work in this case? Explain.

6. A heated storage room has the shape of a rectangular box and has a volume of 1000 cubic feet, as shown in the figure. Because warm air rises, the heat loss per unit of area through the ceiling is five times as great as the heat loss through the floor. The heat loss through the four walls is three times as great as the heat loss through the floor. Determine the room dimensions that will minimize heat loss and therefore minimize heating costs.



7. Repeat Exercise 6 assuming that the heat loss through the walls and ceiling remain the same, but the floor is insulated so that there is no heat loss through the floor.

8. Consider a circular plate of radius 1 given by $x^2 + y^2 \le 1$, as shown in the figure. The temperature at any point $P(x, y)$ on the plate is $T(x, y) = 2x^2 + y^2 - y + 10$.



(a) Sketch the isotherm $T(x, y) = 10$. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.

(b) Find the hottest and coldest points on the plate.

9. Consider the Cobb-Douglas production function

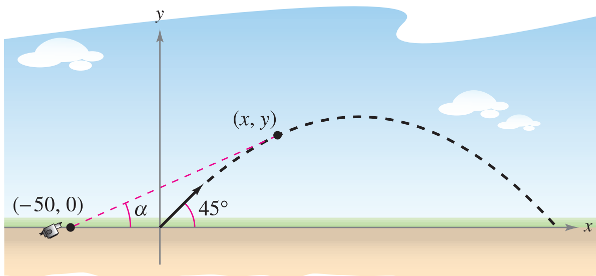
$$f(x, y) = Cx^a y^{1-a}, \quad 0 < a < 1.$$

(a) Show that f satisfies the equation $x\dfrac{\partial f}{\partial x} + y\dfrac{\partial f}{\partial y} = f$.

(b) Show that $f(tx, ty) = tf(x, y)$.

10. Rewrite Laplace's equation $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2} = 0$ in cylindrical coordinates.

11. A projectile is launched at an angle of 45° with the horizontal and with an initial velocity of 64 feet per second. A television camera is located in the plane of the path of the projectile 50 feet behind the launch site (see figure).



(a) Find parametric equations for the path of the projectile in terms of the parameter t representing time.

(b) Write the angle α that the camera makes with the horizontal in terms of x and y and in terms of t.

(c) Use the results of part (b) to find $d\alpha/dt$.

(d) Use a graphing utility to graph α in terms of t. Is the graph symmetric to the axis of the parabolic arch of the projectile? At what time is the rate of change of α greatest?

(e) At what time is the angle α maximum? Does this occur when the projectile is at its greatest height?

12. Consider the distance d between the launch site and the projectile in Exercise 11.

(a) Write the distance d in terms of x and y and in terms of the parameter t.

(b) Use the results of part (a) to find the rate of change of d.

(c) Find the rate of change of the distance when $t = 2$.

(d) When is the rate of change of d minimum during the flight of the projectile? Does this occur at the time when the projectile reaches its maximum height?

CAS **13.** Consider the function

$$f(x, y) = (\alpha x^2 + \beta y^2)e^{-(x^2+y^2)}, \quad 0 < |\alpha| < \beta.$$

(a) Use a computer algebra system to graph the function for $\alpha = 1$ and $\beta = 2$, and identify any extrema or saddle points.

(b) Use a computer algebra system to graph the function for $\alpha = -1$ and $\beta = 2$, and identify any extrema or saddle points.

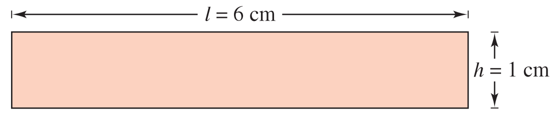
(c) Generalize the results in parts (a) and (b) for the function f.

14. Prove that if f is a differentiable function such that

$$\nabla f(x_0, y_0) = \mathbf{0}$$

then the tangent plane at (x_0, y_0) is horizontal.

15. The figure shows a rectangle that is approximately $l = 6$ centimeters long and $h = 1$ centimeter high.



(a) Draw a rectangular strip along the rectangular region showing a small increase in length.

(b) Draw a rectangular strip along the rectangular region showing a small increase in height.

(c) Use the results in parts (a) and (b) to identify the measurement that has more effect on the area A of the rectangle.

(d) Verify your answer in part (c) analytically by comparing the value of dA when $dl = 0.01$ and when $dh = 0.01$.

16. Consider converting a point $(5 \pm 0.05, \pi/18 \pm 0.05)$ in polar coordinates to rectangular coordinates (x, y).

(a) Use a geometric argument to determine whether the accuracy in x is more dependent on the accuracy in r or on the accuracy in θ. Explain. Verify your answer analytically.

(b) Use a geometric argument to determine whether the accuracy in y is more dependent on the accuracy in r or on the accuracy in θ. Explain. Verify your answer analytically.

17. Let f be a differentiable function of one variable. Show that all tangent planes to the surface $z = y f(x/y)$ intersect in a common point.

18. Consider the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

that encloses the circle $x^2 + y^2 = 2x$. Find values of a and b that minimize the area of the ellipse.

19. Show that

$$u(x, t) = \frac{1}{2}[\sin(x - t) + \sin(x + t)]$$

is a solution to the one-dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}.$$

20. Show that

$$u(x, t) = \frac{1}{2}[f(x - ct) + f(x + ct)]$$

is a solution to the one-dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}.$$

(This equation describes the small transverse vibration of an elastic string such as those on certain musical instruments.)

14 Multiple Integration

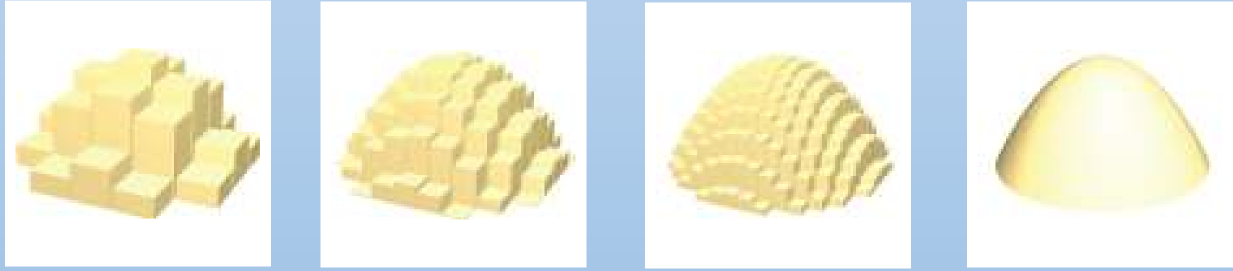
This chapter introduces the concepts of double integrals over regions in the plane and triple integrals over regions in space.

In this chapter, you should learn the following.

- How to evaluate an iterated integral and find the area of a plane region. (14.1)
- How to use a double integral to find the volume of a solid region. (14.2)
- How to write and evaluate double integrals in polar coordinates. (14.3)
- How to find the mass of a planar lamina, the center of mass of a planar lamina, and moments of inertia using double integrals. (14.4)
- How to use a double integral to find the area of a surface. (14.5)
- How to use a triple integral to find the volume, center of mass, and moments of inertia of a solid region. (14.6)
- How to write and evaluate triple integrals in cylindrical and spherical coordinates. (14.7)
- How to use a Jacobian to change variables in a double integral. (14.8)

Langley Photography/Getty Images

The center of pressure on a sail is that point at which the total aerodynamic force may be assumed to act. Letting the sail be represented by a plane region, how can you use double integrals to find the center of pressure on a sail? (See Section 14.4, Section Project.)



You can approximate the volume of a solid region by finding the sum of the volumes of representative rectangular prisms. As you increase the number of rectangular prisms, the approximation tends to become more and more accurate. In Chapter 14, you will learn how to use multiple integrals to find the volume of a solid region.

14.1 Iterated Integrals and Area in the Plane

- **Evaluate an iterated integral.**
- **Use an iterated integral to find the area of a plane region.**

Iterated Integrals

NOTE In Chapters 14 and 15, you will study several applications of integration involving functions of several variables. Chapter 14 is much like Chapter 7 in that it surveys the use of integration to find plane areas, volumes, surface areas, moments, and centers of mass.

In Chapter 13, you saw that it is meaningful to differentiate functions of several variables with respect to one variable while holding the other variables constant. You can *integrate* functions of several variables by a similar procedure. For example, if you are given the partial derivative

$$f_x(x, y) = 2xy$$

then, by considering y constant, you can integrate with respect to x to obtain

$$\begin{aligned}
f(x, y) &= \int f_x(x, y)\, dx && \text{Integrate with respect to } x. \\
&= \int 2xy\, dx && \text{Hold } y \text{ constant.} \\
&= y \int 2x\, dx && \text{Factor out constant } y. \\
&= y(x^2) + C(y) && \text{Antiderivative of } 2x \text{ is } x^2. \\
&= x^2 y + C(y). && C(y) \text{ is a function of } y.
\end{aligned}$$

The "constant" of integration, $C(y)$, is a function of y. In other words, by integrating with respect to x, you are able to recover $f(x, y)$ only partially. The total recovery of a function of x and y from its partial derivatives is a topic you will study in Chapter 15. For now, we are more concerned with extending definite integrals to functions of several variables. For instance, by considering y constant, you can apply the Fundamental Theorem of Calculus to evaluate

$$\int_1^{2y} 2xy\, dx = x^2 y \Big]_1^{2y} = (2y)^2 y - (1)^2 y = 4y^3 - y.$$

x is the variable of integration and y is fixed.

Replace x by the limits of integration.

The result is a function of y.

Similarly, you can integrate with respect to y by holding x fixed. Both procedures are summarized as follows.

$$\int_{h_1(y)}^{h_2(y)} f_x(x, y)\, dx = f(x, y)\Big]_{h_1(y)}^{h_2(y)} = f(h_2(y), y) - f(h_1(y), y) \qquad \text{With respect to } x$$

$$\int_{g_1(x)}^{g_2(x)} f_y(x, y)\, dy = f(x, y)\Big]_{g_1(x)}^{g_2(x)} = f(x, g_2(x)) - f(x, g_1(x)) \qquad \text{With respect to } y$$

Note that the variable of integration cannot appear in either limit of integration. For instance, it makes no sense to write

$$\int_0^x y\, dx.$$

EXAMPLE 1 Integrating with Respect to y

Evaluate $\displaystyle\int_1^x (2x^2y^{-2} + 2y)\,dy$.

Solution Considering x to be constant and integrating with respect to y produces

$$\begin{aligned}\int_1^x (2x^2y^{-2} + 2y)\,dy &= \left[\frac{-2x^2}{y} + y^2\right]_1^x && \text{Integrate with respect to } y.\\ &= \left(\frac{-2x^2}{x} + x^2\right) - \left(\frac{-2x^2}{1} + 1\right)\\ &= 3x^2 - 2x - 1.\end{aligned}$$

Notice in Example 1 that the integral defines a function of x and can *itself* be integrated, as shown in the next example.

EXAMPLE 2 The Integral of an Integral

Evaluate $\displaystyle\int_1^2 \left[\int_1^x (2x^2y^{-2} + 2y)\,dy\right] dx$.

Solution Using the result of Example 1, you have

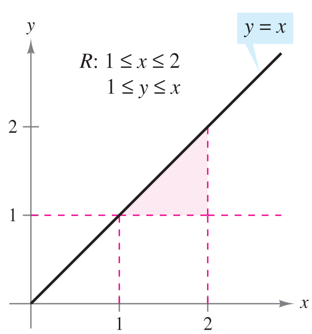
$$\begin{aligned}\int_1^2 \left[\int_1^x (2x^2y^{-2} + 2y)\,dy\right] dx &= \int_1^2 (3x^2 - 2x - 1)\,dx\\ &= \Big[x^3 - x^2 - x\Big]_1^2 && \text{Integrate with respect to } x.\\ &= 2 - (-1)\\ &= 3.\end{aligned}$$

The integral in Example 2 is an **iterated integral.** The brackets used in Example 2 are normally not written. Instead, iterated integrals are usually written simply as

$$\int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\,dy\,dx \quad \text{and} \quad \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\,dx\,dy.$$

The **inside limits of integration** can be variable with respect to the outer variable of integration. However, the **outside limits of integration** *must be* constant with respect to both variables of integration. After performing the inside integration, you obtain a "standard" definite integral, and the second integration produces a real number. The limits of integration for an iterated integral identify two sets of boundary intervals for the variables. For instance, in Example 2, the outside limits indicate that x lies in the interval $1 \le x \le 2$ and the inside limits indicate that y lies in the interval $1 \le y \le x$. Together, these two intervals determine the **region of integration R** of the iterated integral, as shown in Figure 14.1.

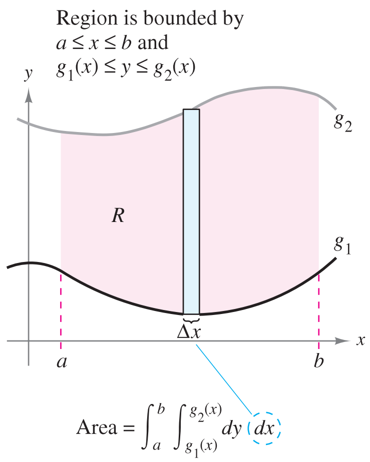
Because an iterated integral is just a special type of definite integral—one in which the integrand is also an integral—you can use the properties of definite integrals to evaluate iterated integrals.



The region of integration for $\displaystyle\int_1^2 \int_1^x f(x, y)\,dy\,dx$

Figure 14.1

Area of a Plane Region



Vertically simple region
Figure 14.2

In the remainder of this section, you will take a new look at an old problem—that of finding the area of a plane region. Consider the plane region R bounded by $a \le x \le b$ and $g_1(x) \le y \le g_2(x)$, as shown in Figure 14.2. The area of R is given by the definite integral

$$\int_a^b [g_2(x) - g_1(x)]\, dx. \qquad \text{Area of } R$$

Using the Fundamental Theorem of Calculus, you can rewrite the integrand $g_2(x) - g_1(x)$ as a definite integral. Specifically, if you consider x to be fixed and let y vary from $g_1(x)$ to $g_2(x)$, you can write

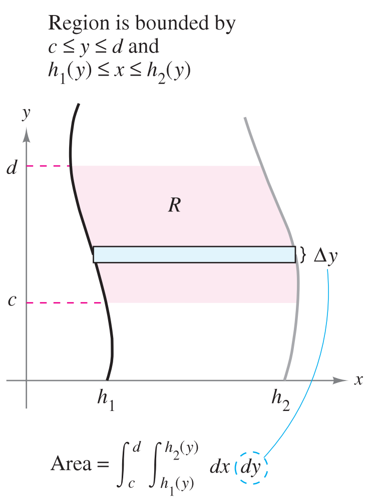
$$\int_{g_1(x)}^{g_2(x)} dy = y\Big]_{g_1(x)}^{g_2(x)} = g_2(x) - g_1(x).$$

Combining these two integrals, you can write the area of the region R as an iterated integral

$$\int_a^b \int_{g_1(x)}^{g_2(x)} dy\, dx = \int_a^b y\Big]_{g_1(x)}^{g_2(x)} dx \qquad \text{Area of } R$$
$$= \int_a^b [g_2(x) - g_1(x)]\, dx.$$

Placing a representative rectangle in the region R helps determine both the order and the limits of integration. A vertical rectangle implies the order $dy\, dx$, with the inside limits corresponding to the upper and lower bounds of the rectangle, as shown in Figure 14.2. This type of region is called **vertically simple,** because the outside limits of integration represent the vertical lines $x = a$ and $x = b$.

Similarly, a horizontal rectangle implies the order $dx\, dy$, with the inside limits determined by the left and right bounds of the rectangle, as shown in Figure 14.3. This type of region is called **horizontally simple,** because the outside limits represent the horizontal lines $y = c$ and $y = d$. The iterated integrals used for these two types of simple regions are summarized as follows.



Horizontally simple region
Figure 14.3

AREA OF A REGION IN THE PLANE

1. If R is defined by $a \le x \le b$ and $g_1(x) \le y \le g_2(x)$, where g_1 and g_2 are continuous on $[a, b]$, then the area of R is given by

$$A = \int_a^b \int_{g_1(x)}^{g_2(x)} dy\, dx. \qquad \text{Figure 14.2 (vertically simple)}$$

2. If R is defined by $c \le y \le d$ and $h_1(y) \le x \le h_2(y)$, where h_1 and h_2 are continuous on $[c, d]$, then the area of R is given by

$$A = \int_c^d \int_{h_1(y)}^{h_2(y)} dx\, dy. \qquad \text{Figure 14.3 (horizontally simple)}$$

NOTE Be sure you see that the orders of integration of these two integrals are different—the order $dy\, dx$ corresponds to a vertically simple region, and the order $dx\, dy$ corresponds to a horizontally simple region. ■

If all four limits of integration happen to be constants, the region of integration is rectangular, as shown in Example 3.

EXAMPLE 3 The Area of a Rectangular Region

Use an iterated integral to represent the area of the rectangle shown in Figure 14.4.

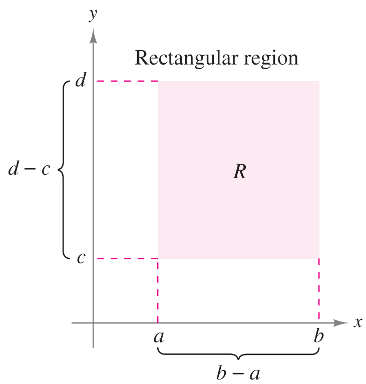


Figure 14.4

Solution The region shown in Figure 14.4 is both vertically simple and horizontally simple, so you can use either order of integration. By choosing the order $dy\,dx$, you obtain the following.

$$\begin{aligned}
\int_a^b \int_c^d dy\,dx &= \int_a^b y\Big]_c^d dx && \text{Integrate with respect to } y.\\
&= \int_a^b (d-c)\,dx\\
&= \Big[(d-c)x\Big]_a^b && \text{Integrate with respect to } x.\\
&= (d-c)(b-a)
\end{aligned}$$

Notice that this answer is consistent with what you know from geometry.

EXAMPLE 4 Finding Area by an Iterated Integral

Use an iterated integral to find the area of the region bounded by the graphs of

$$f(x) = \sin x \qquad \text{Sine curve forms upper boundary.}$$
$$g(x) = \cos x \qquad \text{Cosine curve forms lower boundary.}$$

between $x = \pi/4$ and $x = 5\pi/4$.

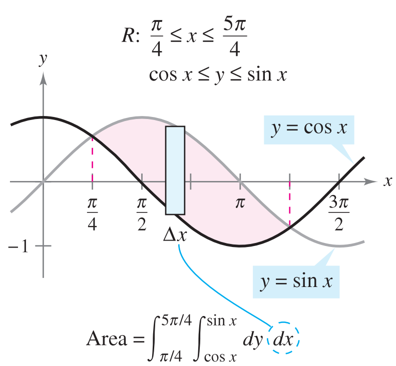


Figure 14.5

Solution Because f and g are given as functions of x, a vertical representative rectangle is convenient, and you can choose $dy\,dx$ as the order of integration, as shown in Figure 14.5. The outside limits of integration are $\pi/4 \le x \le 5\pi/4$. Moreover, because the rectangle is bounded above by $f(x) = \sin x$ and below by $g(x) = \cos x$, you have

$$\begin{aligned}
\text{Area of } R &= \int_{\pi/4}^{5\pi/4} \int_{\cos x}^{\sin x} dy\,dx\\
&= \int_{\pi/4}^{5\pi/4} y\Big]_{\cos x}^{\sin x} dx && \text{Integrate with respect to } y.\\
&= \int_{\pi/4}^{5\pi/4} (\sin x - \cos x)\,dx\\
&= \Big[-\cos x - \sin x\Big]_{\pi/4}^{5\pi/4} && \text{Integrate with respect to } x.\\
&= 2\sqrt{2}.
\end{aligned}$$

NOTE The region of integration of an iterated integral need not have any straight lines as boundaries. For instance, the region of integration shown in Figure 14.5 is *vertically simple* even though it has no vertical lines as left and right boundaries. The quality that makes the region vertically simple is that it is bounded above and below by the graphs of *functions of x*.

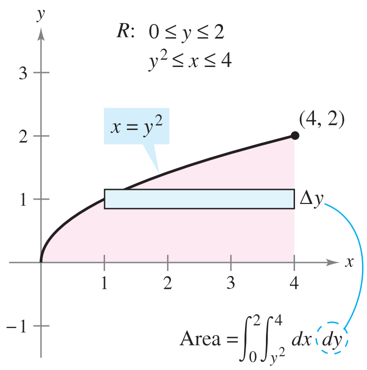
One order of integration will often produce a simpler integration problem than the other order. For instance, try reworking Example 4 with the order $dx\,dy$—you may be surprised to see that the task is formidable. However, if you succeed, you will see that the answer is the same. In other words, the order of integration affects the ease of integration, but not the value of the integral.

EXAMPLE 5 Comparing Different Orders of Integration

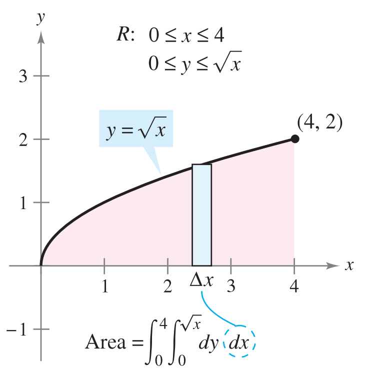
Sketch the region whose area is represented by the integral

$$\int_0^2 \int_{y^2}^4 dx\,dy.$$

Then find another iterated integral using the order $dy\,dx$ to represent the same area and show that both integrals yield the same value.



(a)



(b)

Figure 14.6

Solution From the given limits of integration, you know that

$$y^2 \le x \le 4 \qquad \text{Inner limits of integration}$$

which means that the region R is bounded on the left by the parabola $x = y^2$ and on the right by the line $x = 4$. Furthermore, because

$$0 \le y \le 2 \qquad \text{Outer limits of integration}$$

you know that R is bounded below by the x-axis, as shown in Figure 14.6(a). The value of this integral is

$$\begin{aligned}\int_0^2 \int_{y^2}^4 dx\,dy &= \int_0^2 x\Big]_{y^2}^4 dy && \text{Integrate with respect to } x.\\ &= \int_0^2 (4 - y^2)\,dy \\ &= \left[4y - \frac{y^3}{3}\right]_0^2 = \frac{16}{3}. && \text{Integrate with respect to } y.\end{aligned}$$

To change the order of integration to $dy\,dx$, place a vertical rectangle in the region, as shown in Figure 14.6(b). From this you can see that the constant bounds $0 \le x \le 4$ serve as the outer limits of integration. By solving for y in the equation $x = y^2$, you can conclude that the inner bounds are $0 \le y \le \sqrt{x}$. So, the area of the region can also be represented by

$$\int_0^4 \int_0^{\sqrt{x}} dy\,dx.$$

By evaluating this integral, you can see that it has the same value as the original integral.

$$\begin{aligned}\int_0^4 \int_0^{\sqrt{x}} dy\,dx &= \int_0^4 y\Big]_0^{\sqrt{x}} dx && \text{Integrate with respect to } y.\\ &= \int_0^4 \sqrt{x}\,dx \\ &= \frac{2}{3}x^{3/2}\Big]_0^4 = \frac{16}{3} && \text{Integrate with respect to } x.\end{aligned}$$

■

The icon indicates that you will find a CAS Investigation on the book's website. The CAS Investigation is a collaborative exploration of this example using the computer algebra systems Maple *and* Mathematica.

Sometimes it is not possible to calculate the area of a region with a single iterated integral. In these cases you can divide the region into subregions such that the area of each subregion can be calculated by an iterated integral. The total area is then the sum of the iterated integrals.

TECHNOLOGY Some computer software can perform symbolic integration for integrals such as those in Example 6. If you have access to such software, use it to evaluate the integrals in the exercises and examples given in this section.

EXAMPLE 6 An Area Represented by Two Iterated Integrals

Find the area of the region R that lies below the parabola

$$y = 4x - x^2 \qquad \text{Parabola forms upper boundary.}$$

above the x-axis, and above the line

$$y = -3x + 6. \qquad \text{Line and } x\text{-axis form lower boundary.}$$

Solution Begin by dividing R into the two subregions R_1 and R_2 shown in Figure 14.7.

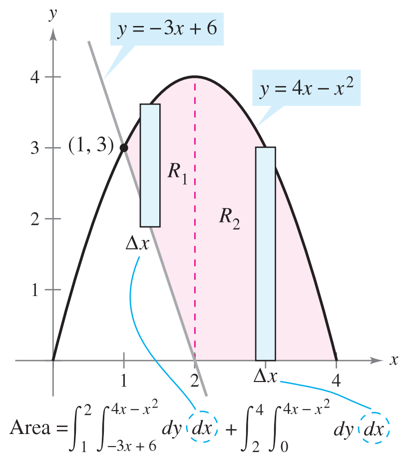


Figure 14.7

In both regions, it is convenient to use vertical rectangles, and you have

$$\begin{aligned}
\text{Area} &= \int_1^2 \int_{-3x+6}^{4x-x^2} dy\,dx + \int_2^4 \int_0^{4x-x^2} dy\,dx \\
&= \int_1^2 (4x - x^2 + 3x - 6)\,dx + \int_2^4 (4x - x^2)\,dx \\
&= \left[\frac{7x^2}{2} - \frac{x^3}{3} - 6x\right]_1^2 + \left[2x^2 - \frac{x^3}{3}\right]_2^4 \\
&= \left(14 - \frac{8}{3} - 12 - \frac{7}{2} + \frac{1}{3} + 6\right) + \left(32 - \frac{64}{3} - 8 + \frac{8}{3}\right) = \frac{15}{2}.
\end{aligned}$$

The area of the region is $15/2$ square units. Try checking this using the procedure for finding the area between two curves, as presented in Section 7.1. ■

NOTE In Examples 3 to 6, be sure you see the benefit of sketching the region of integration. You should develop the habit of making sketches to help you determine the limits of integration for all iterated integrals in this chapter.

At this point you may be wondering why you would need iterated integrals. After all, you already know how to use conventional integration to find the area of a region in the plane. (For instance, compare the solution of Example 4 in this section with that given in Example 3 in Section 7.1.) The need for iterated integrals will become clear in the next section. In this section, primary attention is given to procedures for finding the limits of integration of the region of an iterated integral, and the following exercise set is designed to develop skill in this important procedure.

14.1 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–10, evaluate the integral.

1. $\displaystyle\int_0^x (x + 2y)\, dy$

2. $\displaystyle\int_x^{x^2} \frac{y}{x}\, dy$

3. $\displaystyle\int_1^{2y} \frac{y}{x}\, dx, \quad y > 0$

4. $\displaystyle\int_0^{\cos y} y\, dx$

5. $\displaystyle\int_0^{\sqrt{4-x^2}} x^2 y\, dy$

6. $\displaystyle\int_{x^3}^{\sqrt{x}} (x^2 + 3y^2)\, dy$

7. $\displaystyle\int_{e^y}^{y} \frac{y \ln x}{x}\, dx, \quad y > 0$

8. $\displaystyle\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} (x^2 + y^2)\, dx$

9. $\displaystyle\int_0^{x^3} ye^{-y/x}\, dy$

10. $\displaystyle\int_y^{\pi/2} \sin^3 x \cos y\, dx$

In Exercises 11–30, evaluate the iterated integral.

11. $\displaystyle\int_0^1\int_0^2 (x + y)\, dy\, dx$

12. $\displaystyle\int_{-1}^1\int_{-2}^2 (x^2 - y^2)\, dy\, dx$

13. $\displaystyle\int_1^2\int_0^4 (x^2 - 2y^2)\, dx\, dy$

14. $\displaystyle\int_{-1}^2\int_1^3 (x + y^2)\, dx\, dy$

15. $\displaystyle\int_0^{\pi/2}\int_0^1 y \cos x\, dy\, dx$

16. $\displaystyle\int_0^{\ln 4}\int_0^{\ln 3} e^{x+y}\, dy\, dx$

17. $\displaystyle\int_0^{\pi}\int_0^{\sin x} (1 + \cos x)\, dy\, dx$

18. $\displaystyle\int_1^4\int_1^{\sqrt{x}} 2ye^{-x}\, dy\, dx$

19. $\displaystyle\int_0^1\int_0^x \sqrt{1 - x^2}\, dy\, dx$

20. $\displaystyle\int_{-4}^4\int_0^{x^2} \sqrt{64 - x^3}\, dy\, dx$

21. $\displaystyle\int_{-1}^5\int_0^{3y} \left(3 + x^2 + \frac{1}{4}y^2\right) dx\, dy$

22. $\displaystyle\int_0^2\int_y^{2y} (10 + 2x^2 + 2y^2)\, dx\, dy$

23. $\displaystyle\int_0^1\int_0^{\sqrt{1-y^2}} (x + y)\, dx\, dy$

24. $\displaystyle\int_0^2\int_{3y^2-6y}^{2y-y^2} 3y\, dx\, dy$

25. $\displaystyle\int_0^2\int_0^{\sqrt{4-y^2}} \frac{2}{\sqrt{4 - y^2}}\, dx\, dy$

26. $\displaystyle\int_1^3\int_0^y \frac{4}{x^2 + y^2}\, dx\, dy$

27. $\displaystyle\int_0^{\pi/2}\int_0^{2\cos\theta} r\, dr\, d\theta$

28. $\displaystyle\int_0^{\pi/4}\int_{\sqrt{3}}^{\sqrt{3}\cos\theta} r\, dr\, d\theta$

29. $\displaystyle\int_0^{\pi/2}\int_0^{\sin\theta} \theta r\, dr\, d\theta$

30. $\displaystyle\int_0^{\pi/4}\int_0^{\cos\theta} 3r^2 \sin\theta\, dr\, d\theta$

In Exercises 31–34, evaluate the improper iterated integral.

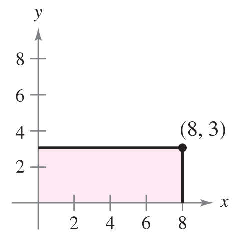
31. $\displaystyle\int_1^{\infty}\int_0^{1/x} y\, dy\, dx$

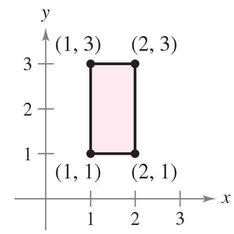
32. $\displaystyle\int_0^3\int_0^{\infty} \frac{x^2}{1 + y^2}\, dy\, dx$

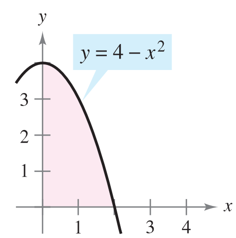
33. $\displaystyle\int_1^{\infty}\int_1^{\infty} \frac{1}{xy}\, dx\, dy$

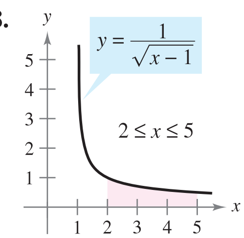
34. $\displaystyle\int_0^{\infty}\int_0^{\infty} xye^{-(x^2+y^2)}\, dx\, dy$

In Exercises 35–38, use an iterated integral to find the area of the region.

35.


36.


37.


38.


In Exercises 39–46, use an iterated integral to find the area of the region bounded by the graphs of the equations.

39. $\sqrt{x} + \sqrt{y} = 2, \quad x = 0, \quad y = 0$

40. $y = x^{3/2}, \quad y = 2x$

41. $2x - 3y = 0, \quad x + y = 5, \quad y = 0$

42. $xy = 9, \quad y = x, \quad y = 0, \quad x = 9$

43. $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

44. $y = x, \quad y = 2x, \quad x = 2$

45. $y = 4 - x^2, \quad y = x + 2$

46. $x^2 + y^2 = 4, \quad x = 0, \quad y = 0$

In Exercises 47–54, sketch the region R of integration and switch the order of integration.

47. $\displaystyle\int_0^4\int_0^y f(x, y)\, dx\, dy$

48. $\displaystyle\int_0^4\int_{\sqrt{y}}^2 f(x, y)\, dx\, dy$

49. $\displaystyle\int_{-2}^2\int_0^{\sqrt{4-x^2}} f(x, y)\, dy\, dx$

50. $\displaystyle\int_0^2\int_0^{4-x^2} f(x, y)\, dy\, dx$

51. $\displaystyle\int_1^{10}\int_0^{\ln y} f(x, y)\, dx\, dy$

52. $\displaystyle\int_{-1}^2\int_0^{e^{-x}} f(x, y)\, dy\, dx$

53. $\displaystyle\int_{-1}^1\int_{x^2}^1 f(x, y)\, dy\, dx$

54. $\displaystyle\int_{-\pi/2}^{\pi/2}\int_0^{\cos x} f(x, y)\, dy\, dx$

In Exercises 55–64, sketch the region R whose area is given by the iterated integral. Then switch the order of integration and show that both orders yield the same area.

55. $\displaystyle\int_0^1\int_0^2 dy\, dx$

56. $\displaystyle\int_1^2\int_2^4 dx\, dy$

57. $\displaystyle\int_0^1\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} dx\, dy$

58. $\displaystyle\int_{-2}^2\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dy\, dx$

59. $\int_0^2 \int_0^x dy\,dx + \int_2^4 \int_0^{4-x} dy\,dx$

60. $\int_0^4 \int_0^{x/2} dy\,dx + \int_4^6 \int_0^{6-x} dy\,dx$

61. $\int_0^2 \int_{x/2}^1 dy\,dx$

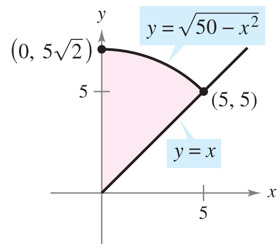
62. $\int_0^9 \int_{\sqrt{x}}^3 dy\,dx$

63. $\int_0^1 \int_{y^2}^{\sqrt[3]{y}} dx\,dy$

64. $\int_{-2}^2 \int_0^{4-y^2} dx\,dy$

65. ***Think About It*** Give a geometric argument for the equality. Verify the equality analytically.

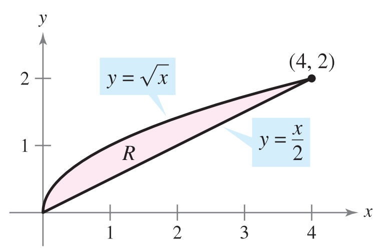
$$\int_0^5 \int_x^{\sqrt{50-x^2}} x^2y^2\,dy\,dx =$$
$$\int_0^5 \int_0^y x^2y^2\,dx\,dy + \int_5^{5\sqrt{2}} \int_0^{\sqrt{50-y^2}} x^2y^2\,dx\,dy$$



CAPSTONE

66. ***Think About It*** Complete the iterated integrals so that each one represents the area of the region R (see figure). Then show that both integrals yield the same area.

(a) Area $= \int\int dx\,dy$ (b) Area $= \int\int dy\,dx$



In Exercises 67–72, sketch the region of integration. Then evaluate the iterated integral. (Note that it is necessary to switch the order of integration.)

67. $\int_0^2 \int_x^2 x\sqrt{1+y^3}\,dy\,dx$

68. $\int_0^4 \int_{\sqrt{x}}^2 \frac{3}{2+y^3}\,dy\,dx$

69. $\int_0^1 \int_{2x}^2 4e^{y^2}\,dy\,dx$

70. $\int_0^2 \int_x^2 e^{-y^2}\,dy\,dx$

71. $\int_0^1 \int_y^1 \sin x^2\,dx\,dy$

72. $\int_0^2 \int_{y^2}^4 \sqrt{x}\sin x\,dx\,dy$

CAS **In Exercises 73–76, use a computer algebra system to evaluate the iterated integral.**

73. $\int_0^2 \int_{x^2}^{2x} (x^3 + 3y^2)\,dy\,dx$

74. $\int_0^1 \int_y^{2y} \sin(x+y)\,dx\,dy$

75. $\int_0^4 \int_0^y \frac{2}{(x+1)(y+1)}\,dx\,dy$

76. $\int_0^a \int_0^{a-x} (x^2+y^2)\,dy\,dx$

CAS **In Exercises 77 and 78, (a) sketch the region of integration, (b) switch the order of integration, and (c) use a computer algebra system to show that both orders yield the same value.**

77. $\int_0^2 \int_{y^3}^{4\sqrt{2y}} (x^2y - xy^2)\,dx\,dy$

78. $\int_0^2 \int_{\sqrt{4-x^2}}^{4-x^2/4} \frac{xy}{x^2+y^2+1}\,dy\,dx$

CAS **In Exercises 79–82, use a computer algebra system to approximate the iterated integral.**

79. $\int_0^2 \int_0^{4-x^2} e^{xy}\,dy\,dx$

80. $\int_0^2 \int_x^2 \sqrt{16 - x^3 - y^3}\,dy\,dx$

81. $\int_0^{2\pi} \int_0^{1+\cos\theta} 6r^2\cos\theta\,dr\,d\theta$

82. $\int_0^{\pi/2} \int_0^{1+\sin\theta} 15\theta r\,dr\,d\theta$

WRITING ABOUT CONCEPTS

83. Explain what is meant by an iterated integral. How is it evaluated?

84. Describe regions that are vertically simple and regions that are horizontally simple.

85. Give a geometric description of the region of integration if the inside and outside limits of integration are constants.

86. Explain why it is sometimes an advantage to change the order of integration.

True or False? **In Exercises 87 and 88, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

87. $\int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_a^b f(x,y)\,dx\,dy$

88. $\int_0^1 \int_0^x f(x,y)\,dy\,dx = \int_0^1 \int_0^y f(x,y)\,dx\,dy$

14.2 Double Integrals and Volume

- **Use a double integral to represent the volume of a solid region.**
- **Use properties of double integrals.**
- **Evaluate a double integral as an iterated integral.**
- **Find the average value of a function over a region.**

Double Integrals and Volume of a Solid Region

You already know that a definite integral over an *interval* uses a limit process to assign measures to quantities such as area, volume, arc length, and mass. In this section, you will use a similar process to define the **double integral** of a function of two variables over a *region in the plane*.

Consider a continuous function f such that $f(x, y) \geq 0$ for all (x, y) in a region R in the xy-plane. The goal is to find the volume of the solid region lying between the surface given by

$$z = f(x, y) \qquad \text{Surface lying above the } xy\text{-plane}$$

and the xy-plane, as shown in Figure 14.8. You can begin by superimposing a rectangular grid over the region, as shown in Figure 14.9. The rectangles lying entirely within R form an **inner partition** Δ, whose **norm** $\|\Delta\|$ is defined as the length of the longest diagonal of the n rectangles. Next, choose a point (x_i, y_i) in each rectangle and form the rectangular prism whose height is $f(x_i, y_i)$, as shown in Figure 14.10. Because the area of the ith rectangle is

$$\Delta A_i \qquad \text{Area of } i\text{th rectangle}$$

it follows that the volume of the ith prism is

$$f(x_i, y_i)\,\Delta A_i \qquad \text{Volume of } i\text{th prism}$$

and you can approximate the volume of the solid region by the Riemann sum of the volumes of all n prisms,

$$\sum_{i=1}^{n} f(x_i, y_i)\,\Delta A_i \qquad \text{Riemann sum}$$

as shown in Figure 14.11. This approximation can be improved by tightening the mesh of the grid to form smaller and smaller rectangles, as shown in Example 1.

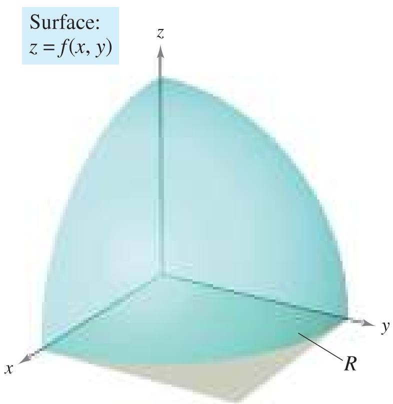
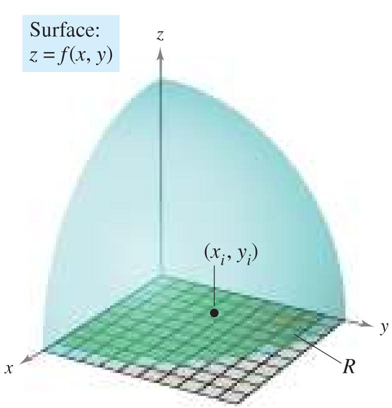
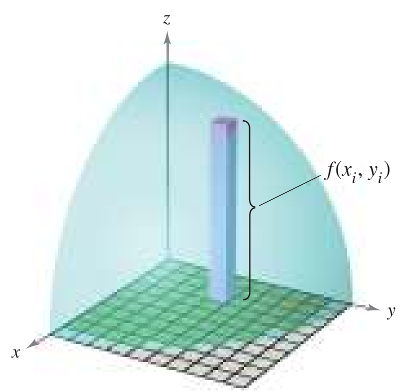


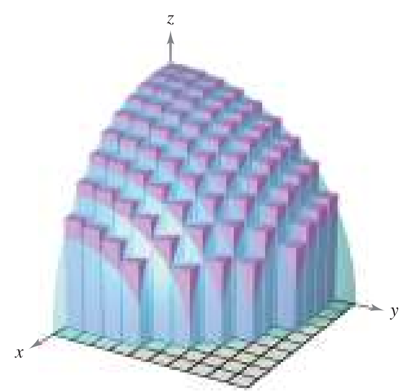
Figure 14.8



The rectangles lying within R form an inner partition of R.
Figure 14.9



Rectangular prism whose base has an area of ΔA_i and whose height is $f(x_i, y_i)$
Figure 14.10



Volume approximated by rectangular prisms
Figure 14.11

EXAMPLE 1 Approximating the Volume of a Solid

Approximate the volume of the solid lying between the paraboloid

$$f(x, y) = 1 - \frac{1}{2}x^2 - \frac{1}{2}y^2$$

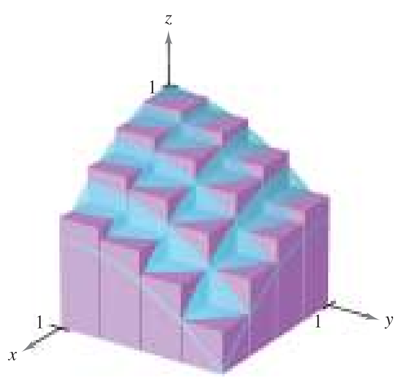
and the square region R given by $0 \le x \le 1$, $0 \le y \le 1$. Use a partition made up of squares whose sides have a length of $\frac{1}{4}$.

Solution Begin by forming the specified partition of R. For this partition, it is convenient to choose the centers of the subregions as the points at which to evaluate $f(x, y)$.

$$\begin{array}{cccc} \left(\frac{1}{8}, \frac{1}{8}\right) & \left(\frac{1}{8}, \frac{3}{8}\right) & \left(\frac{1}{8}, \frac{5}{8}\right) & \left(\frac{1}{8}, \frac{7}{8}\right) \\ \left(\frac{3}{8}, \frac{1}{8}\right) & \left(\frac{3}{8}, \frac{3}{8}\right) & \left(\frac{3}{8}, \frac{5}{8}\right) & \left(\frac{3}{8}, \frac{7}{8}\right) \\ \left(\frac{5}{8}, \frac{1}{8}\right) & \left(\frac{5}{8}, \frac{3}{8}\right) & \left(\frac{5}{8}, \frac{5}{8}\right) & \left(\frac{5}{8}, \frac{7}{8}\right) \\ \left(\frac{7}{8}, \frac{1}{8}\right) & \left(\frac{7}{8}, \frac{3}{8}\right) & \left(\frac{7}{8}, \frac{5}{8}\right) & \left(\frac{7}{8}, \frac{7}{8}\right) \end{array}$$

Because the area of each square is $\Delta A_i = \frac{1}{16}$, you can approximate the volume by the sum

$$\sum_{i=1}^{16} f(x_i\, y_i)\, \Delta A_i = \sum_{i=1}^{16} \left(1 - \frac{1}{2}x_i^2 - \frac{1}{2}y_i^2\right)\left(\frac{1}{16}\right)$$
$$\approx 0.672.$$



Surface:
$f(x, y) = 1 - \frac{1}{2}x^2 - \frac{1}{2}y^2$

Figure 14.12

This approximation is shown graphically in Figure 14.12. The exact volume of the solid is $\frac{2}{3}$ (see Example 2). You can obtain a better approximation by using a finer partition. For example, with a partition of squares with sides of length $\frac{1}{10}$, the approximation is 0.668. ■

TECHNOLOGY Some three-dimensional graphing utilities are capable of graphing figures such as that shown in Figure 14.12. For instance, the graph shown in Figure 14.13 was drawn with a computer program. In this graph, note that each of the rectangular prisms lies within the solid region.

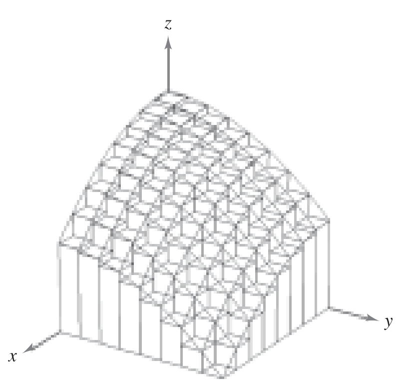


Figure 14.13

In Example 1, note that by using finer partitions, you obtain better approximations of the volume. This observation suggests that you could obtain the exact volume by taking a limit. That is,

$$\text{Volume} = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} f(x_i, y_i)\, \Delta A_i.$$

The precise meaning of this limit is that the limit is equal to L if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\left| L - \sum_{i=1}^{n} f(x_i, y_i)\, \Delta A_i \right| < \varepsilon$$

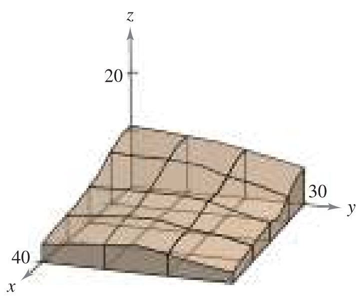
for all partitions Δ of the plane region R (that satisfy $\|\Delta\| < \delta$) and for all possible choices of x_i and y_i in the ith region.

Using the limit of a Riemann sum to define volume is a special case of using the limit to define a **double integral.** The general case, however, does not require that the function be positive or continuous.

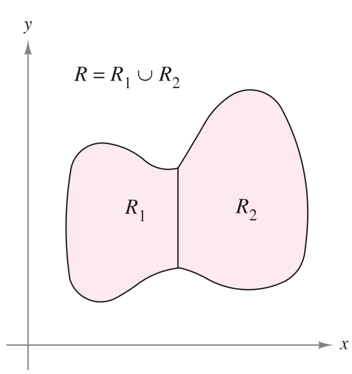
EXPLORATION

The entries in the table represent the depths (in 10-yard units) of earth at the centers of the squares in the figure below.

x \ y	1	2	3
1	10	9	7
2	7	7	4
3	5	5	4
4	4	5	3



Approximate the number of cubic yards of earth in the first octant. (This exploration was submitted by Robert Vojack, Ridgewood High School, Ridgewood, NJ.)



Two regions are nonoverlapping if their intersection is a set that has an area of 0. In this figure, the area of the line segment that is common to R_1 and R_2 is 0.
Figure 14.14

DEFINITION OF DOUBLE INTEGRAL

If f is defined on a closed, bounded region R in the xy-plane, then the **double integral of f over R** is given by

$$\int_R\int f(x, y)\,dA = \lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} f(x_i, y_i)\,\Delta A_i$$

provided the limit exists. If the limit exists, then f is **integrable** over R.

NOTE Having defined a double integral, you will see that a definite integral is occasionally referred to as a **single integral.**

Sufficient conditions for the double integral of f on the region R to exist are that R can be written as a union of a finite number of nonoverlapping subregions (see Figure 14.14) that are vertically or horizontally simple *and* that f is continuous on the region R.

A double integral can be used to find the volume of a solid region that lies between the xy-plane and the surface given by $z = f(x, y)$.

VOLUME OF A SOLID REGION

If f is integrable over a plane region R and $f(x, y) \geq 0$ for all (x, y) in R, then the volume of the solid region that lies above R and below the graph of f is defined as

$$V = \int_R\int f(x, y)\,dA.$$

Properties of Double Integrals

Double integrals share many properties of single integrals.

THEOREM 14.1 PROPERTIES OF DOUBLE INTEGRALS

Let f and g be continuous over a closed, bounded plane region R, and let c be a constant.

1. $\displaystyle\int_R\int cf(x, y)\,dA = c\int_R\int f(x, y)\,dA$
2. $\displaystyle\int_R\int [f(x, y) \pm g(x, y)]\,dA = \int_R\int f(x, y)\,dA \pm \int_R\int g(x, y)\,dA$
3. $\displaystyle\int_R\int f(x, y)\,dA \geq 0, \qquad \text{if } f(x, y) \geq 0$
4. $\displaystyle\int_R\int f(x, y)\,dA \geq \int_R\int g(x, y)\,dA, \qquad \text{if } f(x, y) \geq g(x, y)$
5. $\displaystyle\int_R\int f(x, y)\,dA = \int_{R_1}\int f(x, y)\,dA + \int_{R_2}\int f(x, y)\,dA$, where R is the union of two nonoverlapping subregions R_1 and R_2.

Evaluation of Double Integrals

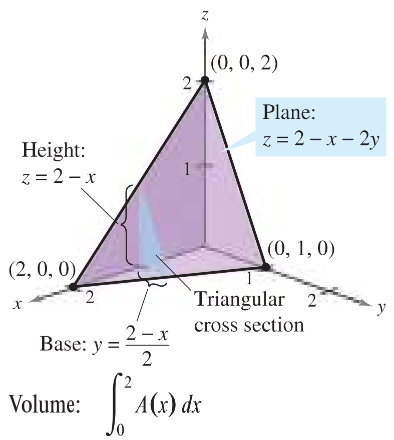


Figure 14.15

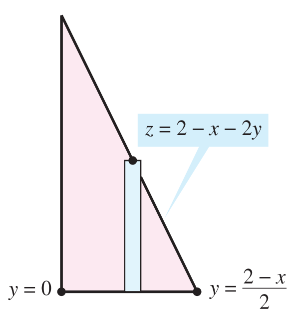
Normally, the first step in evaluating a double integral is to rewrite it as an iterated integral. To show how this is done, a geometric model of a double integral is used as the volume of a solid.

Consider the solid region bounded by the plane $z = f(x, y) = 2 - x - 2y$ and the three coordinate planes, as shown in Figure 14.15. Each vertical cross section taken parallel to the yz-plane is a triangular region whose base has a length of $y = (2 - x)/2$ and whose height is $z = 2 - x$. This implies that for a fixed value of x, the area of the triangular cross section is

$$A(x) = \frac{1}{2}(\text{base})(\text{height}) = \frac{1}{2}\left(\frac{2-x}{2}\right)(2-x) = \frac{(2-x)^2}{4}.$$

By the formula for the volume of a solid with known cross sections (Section 7.2), the volume of the solid is

$$\begin{aligned}\text{Volume} &= \int_a^b A(x)\,dx\\ &= \int_0^2 \frac{(2-x)^2}{4}\,dx\\ &= -\frac{(2-x)^3}{12}\Bigg]_0^2 = \frac{2}{3}.\end{aligned}$$



Triangular cross section
Figure 14.16

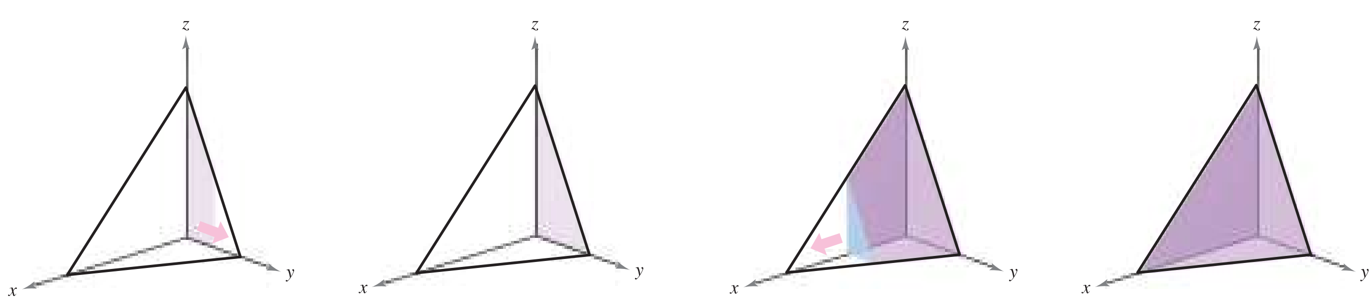
This procedure works no matter how $A(x)$ is obtained. In particular, you can find $A(x)$ by integration, as shown in Figure 14.16. That is, you consider x to be constant, and integrate $z = 2 - x - 2y$ from 0 to $(2 - x)/2$ to obtain

$$\begin{aligned}A(x) &= \int_0^{(2-x)/2} (2 - x - 2y)\,dy\\ &= \Big[(2-x)y - y^2\Big]_0^{(2-x)/2}\\ &= \frac{(2-x)^2}{4}.\end{aligned}$$

Combining these results, you have the *iterated integral*

$$\text{Volume} = \int_R\int f(x, y)\,dA = \int_0^2\int_0^{(2-x)/2} (2 - x - 2y)\,dy\,dx.$$

To understand this procedure better, it helps to imagine the integration as two sweeping motions. For the inner integration, a vertical line sweeps out the area of a cross section. For the outer integration, the triangular cross section sweeps out the volume, as shown in Figure 14.17.



Integrate with respect to y to obtain the area of the cross section.
Integrate with respect to x to obtain the volume of the solid.
Figure 14.17

The following theorem was proved by the Italian mathematician Guido Fubini (1879–1943). The theorem states that if R is a vertically or horizontally simple region and f is continuous on R, the double integral of f on R is equal to an iterated integral.

THEOREM 14.2 FUBINI'S THEOREM

Let f be continuous on a plane region R.

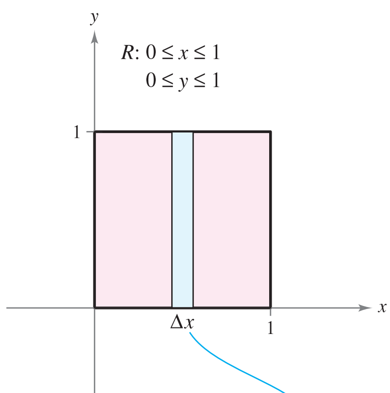
1. If R is defined by $a \le x \le b$ and $g_1(x) \le y \le g_2(x)$, where g_1 and g_2 are continuous on $[a, b]$, then
$$\int_R\int f(x, y)\,dA = \int_a^b\int_{g_1(x)}^{g_2(x)} f(x, y)\,dy\,dx.$$
2. If R is defined by $c \le y \le d$ and $h_1(y) \le x \le h_2(y)$, where h_1 and h_2 are continuous on $[c, d]$, then
$$\int_R\int f(x, y)\,dA = \int_c^d\int_{h_1(y)}^{h_2(y)} f(x, y)\,dx\,dy.$$

EXAMPLE 2 Evaluating a Double Integral as an Iterated Integral

Evaluate
$$\int_R\int\left(1 - \frac{1}{2}x^2 - \frac{1}{2}y^2\right)dA$$
where R is the region given by $0 \le x \le 1$, $0 \le y \le 1$.

Solution Because the region R is a square, it is both vertically and horizontally simple, and you can use either order of integration. Choose $dy\,dx$ by placing a vertical representative rectangle in the region, as shown in Figure 14.18. This produces the following.

$$\begin{aligned}\int_R\int\left(1 - \frac{1}{2}x^2 - \frac{1}{2}y^2\right)dA &= \int_0^1\int_0^1\left(1 - \frac{1}{2}x^2 - \frac{1}{2}y^2\right)dy\,dx\\ &= \int_0^1\left[\left(1 - \frac{1}{2}x^2\right)y - \frac{y^3}{6}\right]_0^1 dx\\ &= \int_0^1\left(\frac{5}{6} - \frac{1}{2}x^2\right)dx\\ &= \left[\frac{5}{6}x - \frac{x^3}{6}\right]_0^1\\ &= \frac{2}{3}\end{aligned}$$



$\int_R\int f(x, y)\,dA = \int_0^1\int_0^1 f(x, y)\,dy\,dx$

The volume of the solid region is $\frac{2}{3}$.
Figure 14.18

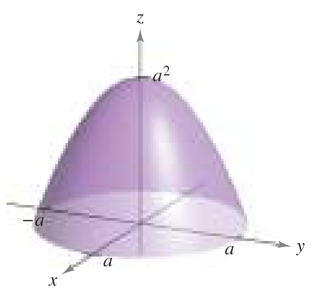
The double integral evaluated in Example 2 represents the volume of the solid region approximated in Example 1. Note that the approximation obtained in Example 1 is quite good $\left(0.672 \text{ vs. } \frac{2}{3}\right)$, even though you used a partition consisting of only 16 squares. The error resulted because the centers of the square subregions were used as the points in the approximation. This is comparable to the Midpoint Rule approximation of a single integral.

EXPLORATION

Volume of a Paraboloid Sector
The solid in Example 3 has an elliptical (not a circular) base. Consider the region bounded by the circular paraboloid

$$z = a^2 - x^2 - y^2, \quad a > 0$$

and the xy-plane. How many ways of finding the volume of this solid do you now know? For instance, you could use the disk method to find the volume as a solid of revolution. Does each method involve integration?



The difficulty of evaluating a single integral $\int_a^b f(x)\,dx$ usually depends on the function f, and not on the interval $[a, b]$. This is a major difference between single and double integrals. In the next example, you will integrate a function similar to the one in Examples 1 and 2. Notice that a change in the region R produces a much more difficult integration problem.

EXAMPLE 3 Finding Volume by a Double Integral

Find the volume of the solid region bounded by the paraboloid $z = 4 - x^2 - 2y^2$ and the xy-plane.

Solution By letting $z = 0$, you can see that the base of the region in the xy-plane is the ellipse $x^2 + 2y^2 = 4$, as shown in Figure 14.19(a). This plane region is both vertically and horizontally simple, so the order $dy\,dx$ is appropriate.

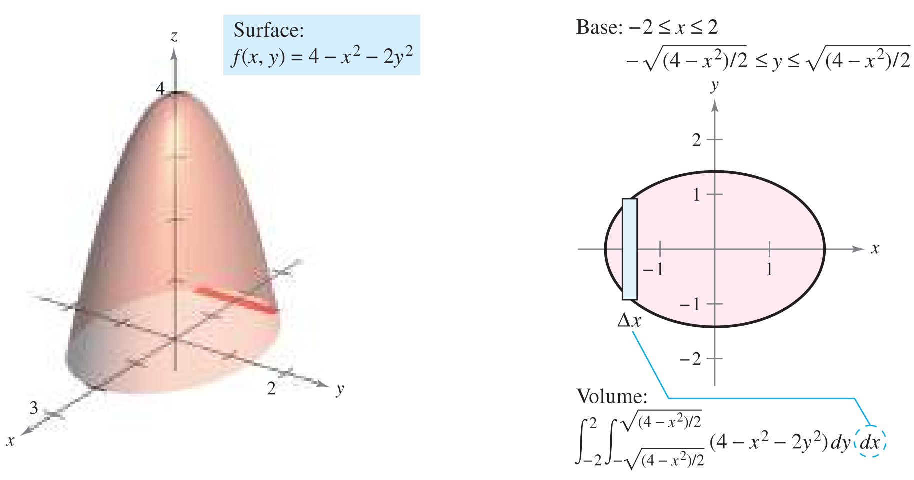
Variable bounds for y: $-\sqrt{\dfrac{(4 - x^2)}{2}} \le y \le \sqrt{\dfrac{(4 - x^2)}{2}}$

Constant bounds for x: $-2 \le x \le 2$

The volume is given by

$$
\begin{aligned}
V &= \int_{-2}^{2}\int_{-\sqrt{(4-x^2)/2}}^{\sqrt{(4-x^2)/2}} (4 - x^2 - 2y^2)\,dy\,dx && \text{See Figure 14.19(b).}\\
&= \int_{-2}^{2}\left[(4 - x^2)y - \frac{2y^3}{3}\right]_{-\sqrt{(4-x^2)/2}}^{\sqrt{(4-x^2)/2}} dx\\
&= \frac{4}{3\sqrt{2}}\int_{-2}^{2}(4 - x^2)^{3/2}\,dx\\
&= \frac{4}{3\sqrt{2}}\int_{-\pi/2}^{\pi/2} 16\cos^4\theta\,d\theta && x = 2\sin\theta\\
&= \frac{64}{3\sqrt{2}}(2)\int_{0}^{\pi/2}\cos^4\theta\,d\theta\\
&= \frac{128}{3\sqrt{2}}\left(\frac{3\pi}{16}\right) && \text{Wallis's Formula}\\
&= 4\sqrt{2}\pi.
\end{aligned}
$$

NOTE In Example 3, note the usefulness of Wallis's Formula to evaluate $\int_0^{\pi/2}\cos^n\theta\,d\theta$. You may want to review this formula in Section 8.3.



(a) **(b)**

Figure 14.19

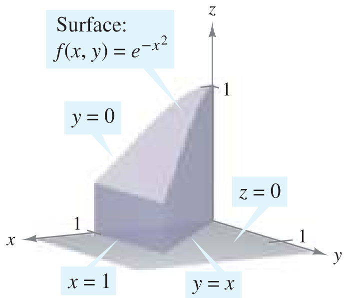
In Examples 2 and 3, the problems could be solved with either order of integration because the regions were both vertically and horizontally simple. Moreover, had you used the order $dx\,dy$, you would have obtained integrals of comparable difficulty. There are, however, some occasions in which one order of integration is much more convenient than the other. Example 4 shows such a case.

EXAMPLE 4 Comparing Different Orders of Integration

Find the volume of the solid region R bounded by the surface

$$f(x, y) = e^{-x^2} \qquad \text{Surface}$$

and the planes $z = 0$, $y = 0$, $y = x$, and $x = 1$, as shown in Figure 14.20.



Base is bounded by $y = 0$, $y = x$, and $x = 1$.
Figure 14.20

Solution The base of R in the xy-plane is bounded by the lines $y = 0$, $x = 1$, and $y = x$. The two possible orders of integration are shown in Figure 14.21.

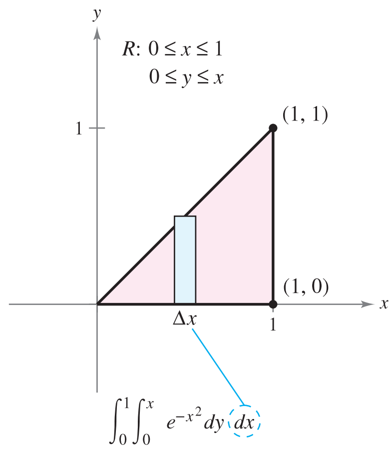
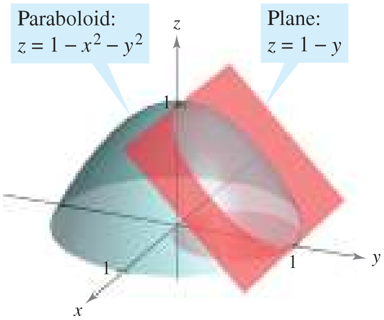


Figure 14.21

By setting up the corresponding iterated integrals, you can see that the order $dx\,dy$ requires the antiderivative $\int e^{-x^2}\,dx$, which is not an elementary function. On the other hand, the order $dy\,dx$ produces the integral

$$\begin{aligned}
\int_0^1 \int_0^x e^{-x^2}\,dy\,dx &= \int_0^1 e^{-x^2} y \Big]_0^x \,dx \\
&= \int_0^1 xe^{-x^2}\,dx \\
&= -\frac{1}{2}e^{-x^2}\Big]_0^1 \\
&= -\frac{1}{2}\left(\frac{1}{e} - 1\right) \\
&= \frac{e-1}{2e} \\
&\approx 0.316.
\end{aligned}$$

NOTE Try using a symbolic integration utility to evaluate the integral in Example 4.



R: $0 \le y \le 1$
$-\sqrt{y - y^2} \le x \le \sqrt{y - y^2}$

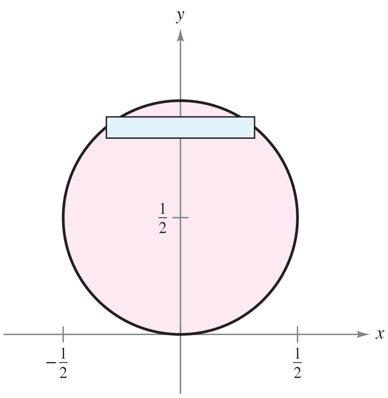


Figure 14.22

EXAMPLE 5 Volume of a Region Bounded by Two Surfaces

Find the volume of the solid region R bounded above by the paraboloid $z = 1 - x^2 - y^2$ and below by the plane $z = 1 - y$, as shown in Figure 14.22.

Solution Equating z-values, you can determine that the intersection of the two surfaces occurs on the right circular cylinder given by

$$1 - y = 1 - x^2 - y^2 \quad \Longrightarrow \quad x^2 = y - y^2.$$

Because the volume of R is the difference between the volume under the paraboloid and the volume under the plane, you have

$$\begin{aligned}
\text{Volume} &= \int_0^1 \int_{-\sqrt{y-y^2}}^{\sqrt{y-y^2}} (1 - x^2 - y^2)\,dx\,dy - \int_0^1 \int_{-\sqrt{y-y^2}}^{\sqrt{y-y^2}} (1 - y)\,dx\,dy \\
&= \int_0^1 \int_{-\sqrt{y-y^2}}^{\sqrt{y-y^2}} (y - y^2 - x^2)\,dx\,dy \\
&= \int_0^1 \left[(y - y^2)x - \frac{x^3}{3} \right]_{-\sqrt{y-y^2}}^{\sqrt{y-y^2}} dy \\
&= \frac{4}{3}\int_0^1 (y - y^2)^{3/2}\,dy \\
&= \left(\frac{4}{3}\right)\left(\frac{1}{8}\right)\int_0^1 [1 - (2y - 1)^2]^{3/2}\,dy \\
&= \frac{1}{6}\int_{-\pi/2}^{\pi/2} \frac{\cos^4\theta}{2}\,d\theta && 2y - 1 = \sin\theta \\
&= \frac{1}{6}\int_0^{\pi/2} \cos^4 d\theta \\
&= \left(\frac{1}{6}\right)\left(\frac{3\pi}{16}\right) && \text{Wallis's Formula} \\
&= \frac{\pi}{32}.
\end{aligned}$$

Average Value of a Function

Recall from Section 4.4 that for a function f in one variable, the average value of f on $[a, b]$ is

$$\frac{1}{b - a}\int_a^b f(x)\,dx.$$

Given a function f in two variables, you can find the average value of f over the region R as shown in the following definition.

DEFINITION OF THE AVERAGE VALUE OF A FUNCTION OVER A REGION

If f is integrable over the plane region R, then the **average value** of f over R is

$$\frac{1}{A}\int_R\int f(x, y)\,dA$$

where A is the area of R.

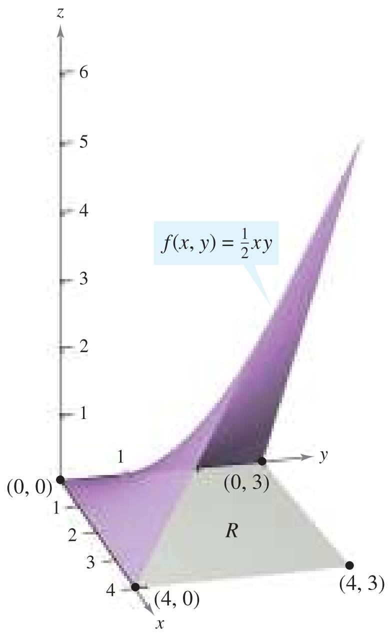


Figure 14.23

EXAMPLE 6 Finding the Average Value of a Function

Find the average value of $f(x, y) = \frac{1}{2}xy$ over the region R, where R is a rectangle with vertices $(0, 0)$, $(4, 0)$, $(4, 3)$, and $(0, 3)$.

Solution The area of the rectangular region R is $A = 12$ (see Figure 14.23). The average value is given by

$$\begin{aligned}
\frac{1}{A}\int_R\int f(x, y)\, dA &= \frac{1}{12}\int_0^4\int_0^3 \frac{1}{2}xy\, dy\, dx \\
&= \frac{1}{12}\int_0^4 \frac{1}{4}xy^2\Big]_0^3\, dx \\
&= \left(\frac{1}{12}\right)\left(\frac{9}{4}\right)\int_0^4 x\, dx \\
&= \frac{3}{16}\left[\frac{1}{2}x^2\right]_0^4 \\
&= \left(\frac{3}{16}\right)(8) \\
&= \frac{3}{2}.
\end{aligned}$$

14.2 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Approximation **In Exercises 1–4, approximate the integral $\int_R\int f(x, y)\, dA$ by dividing the rectangle R with vertices $(0, 0)$, $(4, 0)$, $(4, 2)$, and $(0, 2)$ into eight equal squares and finding the sum $\sum_{i=1}^{8} f(x_i, y_i)\,\Delta A_i$ where (x_i, y_i) is the center of the ith square. Evaluate the iterated integral and compare it with the approximation.**

1. $\displaystyle\int_0^4\int_0^2 (x + y)\, dy\, dx$

2. $\displaystyle\frac{1}{2}\int_0^4\int_0^2 x^2y\, dy\, dx$

3. $\displaystyle\int_0^4\int_0^2 (x^2 + y^2)\, dy\, dx$

4. $\displaystyle\int_0^4\int_0^2 \frac{1}{(x + 1)(y + 1)}\, dy\, dx$

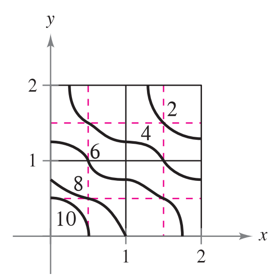
5. ***Approximation*** The table shows values of a function f over a square region R. Divide the region into 16 equal squares and select (x_i, y_i) to be the point in the ith square closest to the origin. Compare this approximation with that obtained by using the point in the ith square farthest from the origin.

$\int_0^4\int_0^4 f(x, y)\, dy\, dx$

x \ y	0	1	2	3	4
0	32	31	28	23	16
1	31	30	27	22	15
2	28	27	24	19	12
3	23	22	19	14	7
4	16	15	12	7	0

6. ***Approximation*** The figure shows the level curves for a function f over a square region R. Approximate the integral using four squares, selecting the midpoint of each square as (x_i, y_i).

$\displaystyle\int_0^2\int_0^2 f(x, y)\, dy\, dx$



In Exercises 7–12, sketch the region R and evaluate the iterated integral $\int_R\int f(x, y)\, dA$.

7. $\displaystyle\int_0^2\int_0^1 (1 + 2x + 2y)\, dy\, dx$

8. $\displaystyle\int_0^{\pi}\int_0^{\pi/2} \sin^2 x \cos^2 y\, dy\, dx$

9. $\displaystyle\int_0^6\int_{y/2}^3 (x + y)\, dx\, dy$

10. $\displaystyle\int_0^4\int_{\frac{1}{2}y}^{\sqrt{y}} x^2y^2\, dx\, dy$

11. $\displaystyle\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} (x + y)\, dy\, dx$

12. $\displaystyle\int_0^1\int_{y-1}^0 e^{x+y}\, dx\, dy + \int_0^1\int_0^{1-y} e^{x+y}\, dx\, dy$

In Exercises 13–20, set up integrals for both orders of integration, and use the more convenient order to evaluate the integral over the region R.

13. $\displaystyle\int_R\int xy\,dA$

R: rectangle with vertices $(0, 0), (0, 5), (3, 5), (3, 0)$

14. $\displaystyle\int_R\int \sin x \sin y\,dA$

R: rectangle with vertices $(-\pi, 0), (\pi, 0), (\pi, \pi/2), (-\pi, \pi/2)$

15. $\displaystyle\int_R\int \frac{y}{x^2 + y^2}\,dA$

R: trapezoid bounded by $y = x,\ y = 2x,\ x = 1,\ x = 2$

16. $\displaystyle\int_R\int xe^y\,dA$

R: triangle bounded by $y = 4 - x,\ y = 0, x = 0$

17. $\displaystyle\int_R\int -2y\,dA$

R: region bounded by $y = 4 - x^2,\ y = 4 - x$

18. $\displaystyle\int_R\int \frac{y}{1 + x^2}\,dA$

R: region bounded by $y = 0, y = \sqrt{x}, x = 4$

19. $\displaystyle\int_R\int x\,dA$

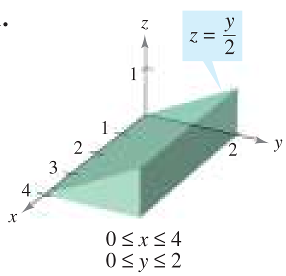
R: sector of a circle in the first quadrant bounded by $y = \sqrt{25 - x^2}, 3x - 4y = 0, y = 0$

20. $\displaystyle\int_R\int (x^2 + y^2)\,dA$

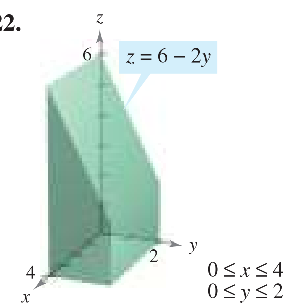
R: semicircle bounded by $y = \sqrt{4 - x^2}, y = 0$

In Exercises 21–30, use a double integral to find the volume of the indicated solid.

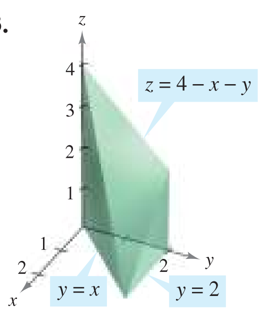
21.



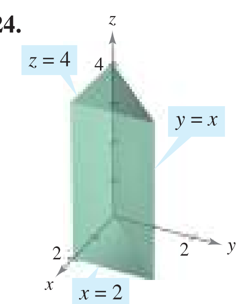
22.



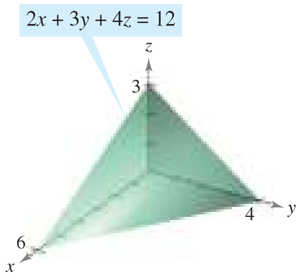
23.



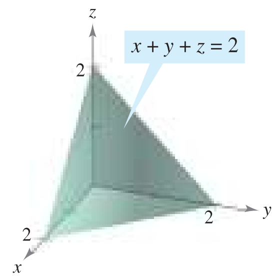
24.



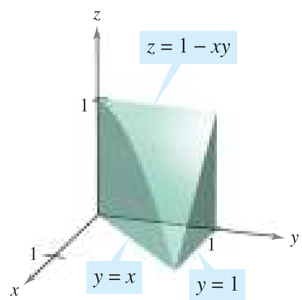
25.



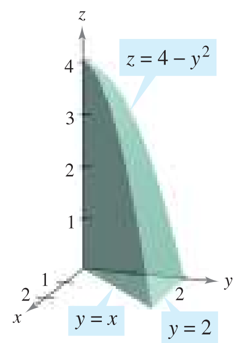
26.



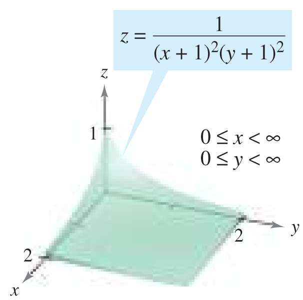
27.



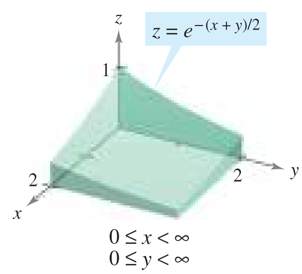
28.



29. Improper integral

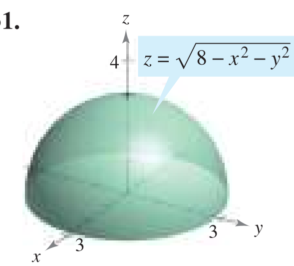


30. Improper integral

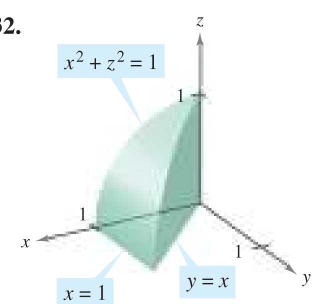


CAS **In Exercises 31 and 32, use a computer algebra system to find the volume of the solid.**

31.



32.



In Exercises 33–40, set up and evaluate a double integral to find the volume of the solid bounded by the graphs of the equations.

33. $z = xy,\ z = 0,\ y = x,\ x = 1,$ first octant

34. $y = 0,\ z = 0,\ y = x,\ z = x,\ x = 0,\ x = 5$

35. $z = 0,\ z = x^2,\ x = 0,\ x = 2,\ y = 0,\ y = 4$

36. $x^2 + y^2 + z^2 = r^2$

37. $x^2 + z^2 = 1,\ y^2 + z^2 = 1,$ first octant

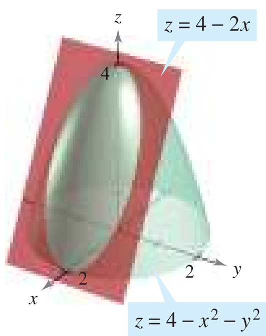
38. $y = 4 - x^2,\ z = 4 - x^2,$ first octant

39. $z = x + y,\ x^2 + y^2 = 4,$ first octant

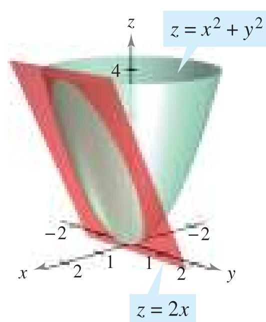
40. $z = \dfrac{1}{1 + y^2},\ x = 0,\ x = 2,\ y \geq 0$

In Exercises 41–46, set up a double integral to find the volume of the solid region bounded by the graphs of the equations. Do not evaluate the integral.

41.



42.



43. $z = x^2 + y^2,\ x^2 + y^2 = 4,\ z = 0$

44. $z = \sin^2 x,\ z = 0,\ 0 \leq x \leq \pi,\ 0 \leq y \leq 5$

45. $z = x^2 + 2y^2,\ z = 4y$

46. $z = x^2 + y^2,\ z = 18 - x^2 - y^2$

CAS **In Exercises 47–50, use a computer algebra system to find the volume of the solid bounded by the graphs of the equations.**

47. $z = 9 - x^2 - y^2,\ z = 0$

48. $x^2 = 9 - y,\ z^2 = 9 - y,$ first octant

49. $z = \dfrac{2}{1 + x^2 + y^2},\ z = 0,\ y = 0,\ x = 0,\ y = -0.5x + 1$

50. $z = \ln(1 + x + y),\ z = 0,\ y = 0,\ x = 0,\ x = 4 - \sqrt{y}$

51. If f is a continuous function such that $0 \leq f(x, y) \leq 1$ over a region R of area 1, prove that $0 \leq \int_R\int f(x, y)\, dA \leq 1$.

52. Find the volume of the solid in the first octant bounded by the coordinate planes and the plane $(x/a) + (y/b) + (z/c) = 1$, where $a > 0$, $b > 0$, and $c > 0$.

In Exercises 53–58, sketch the region of integration. Then evaluate the iterated integral, switching the order of integration if necessary.

53. $\displaystyle\int_0^1\int_{y/2}^{1/2} e^{-x^2}\, dx\, dy$

54. $\displaystyle\int_0^{\ln 10}\int_{e^x}^{10} \frac{1}{\ln y}\, dy\, dx$

55. $\displaystyle\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \sqrt{4 - y^2}\, dy\, dx$

56. $\displaystyle\int_0^3\int_{y/3}^{1} \frac{1}{1 + x^4}\, dx\, dy$

57. $\displaystyle\int_0^1\int_0^{\arccos y} \sin x\sqrt{1 + \sin^2 x}\, dx\, dy$

58. $\displaystyle\int_0^2\int_{(1/2)x^2}^{2} \sqrt{y}\cos y\, dy\, dx$

Average Value **In Exercises 59–64, find the average value of $f(x, y)$ over the region R.**

59. $f(x, y) = x$

R: rectangle with vertices $(0, 0), (4, 0), (4, 2), (0, 2)$

60. $f(x, y) = 2xy$

R: rectangle with vertices $(0, 0), (5, 0), (5, 3), (0, 3)$

61. $f(x, y) = x^2 + y^2$

R: square with vertices $(0, 0), (2, 0), (2, 2), (0, 2)$

62. $f(x, y) = \dfrac{1}{x + y}$

R: triangle with vertices $(0, 0), (1, 0), (1, 1)$

63. $f(x, y) = e^{x+y}$

R: triangle with vertices $(0, 0), (0, 1), (1, 1)$

64. $f(x, y) = \sin(x + y)$

R: rectangle with vertices $(0, 0), (\pi, 0), (\pi, \pi), (0, \pi)$

65. ***Average Production*** The Cobb-Douglas production function for an automobile manufacturer is $f(x, y) = 100x^{0.6}y^{0.4}$, where x is the number of units of labor and y is the number of units of capital. Estimate the average production level if the number of units of labor x varies between 200 and 250 and the number of units of capital y varies between 300 and 325.

66. ***Average Temperature*** The temperature in degrees Celsius on the surface of a metal plate is $T(x, y) = 20 - 4x^2 - y^2$, where x and y are measured in centimeters. Estimate the average temperature if x varies between 0 and 2 centimeters and y varies between 0 and 4 centimeters.

WRITING ABOUT CONCEPTS

67. State the definition of a double integral. If the integrand is a nonnegative function over the region of integration, give the geometric interpretation of a double integral.

68. Let R be a region in the xy-plane whose area is B. If $f(x, y) = k$ for every point (x, y) in R, what is the value of $\int_R\int f(x, y)\, dA$? Explain.

69. Let R represent a county in the northern part of the United States, and let $f(x, y)$ represent the total annual snowfall at the point (x, y) in R. Interpret each of the following.

(a) $\displaystyle\int_R\int f(x, y)\, dA$ (b) $\dfrac{\displaystyle\int_R\int f(x, y)\, dA}{\displaystyle\int_R\int dA}$

70. Identify the expression that is invalid. Explain your reasoning.

(a) $\displaystyle\int_0^2\int_0^3 f(x, y)\, dy\, dx$ (b) $\displaystyle\int_0^2\int_0^y f(x, y)\, dy\, dx$

(c) $\displaystyle\int_0^2\int_x^3 f(x, y)\, dy\, dx$ (d) $\displaystyle\int_0^2\int_0^x f(x, y)\, dy\, dx$

71. Let the plane region R be a unit circle and let the maximum value of f on R be 6. Is the greatest possible value of $\int_R\int f(x, y)\, dy\, dx$ equal to 6? Why or why not? If not, what is the greatest possible value?

CAPSTONE

72. The following iterated integrals represent the solution to the same problem. Which iterated integral is easier to evaluate? Explain your reasoning.

$$\int_0^4 \int_{x/2}^2 \sin y^2 \, dy \, dx = \int_0^2 \int_0^{2y} \sin y^2 \, dx \, dy$$

Probability **A joint density function of the continuous random variables x and y is a function $f(x, y)$ satisfying the following properties.**

(a) $f(x, y) \geq 0$ **for all** (x, y) **(b)** $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) \, dA = 1$

(c) $P[(x, y) \in R] = \int_R \int f(x, y) \, dA$

In Exercises 73–76, show that the function is a joint density function and find the required probability.

73. $f(x, y) = \begin{cases} \frac{1}{10}, & 0 \leq x \leq 5, \ 0 \leq y \leq 2 \\ 0, & \text{elsewhere} \end{cases}$

$P(0 \leq x \leq 2, \ 1 \leq y \leq 2)$

74. $f(x, y) = \begin{cases} \frac{1}{4}xy, & 0 \leq x \leq 2, \ 0 \leq y \leq 2 \\ 0, & \text{elsewhere} \end{cases}$

$P(0 \leq x \leq 1, \ 1 \leq y \leq 2)$

75. $f(x, y) = \begin{cases} \frac{1}{27}(9 - x - y), & 0 \leq x \leq 3, \ 3 \leq y \leq 6 \\ 0, & \text{elsewhere} \end{cases}$

$P(0 \leq x \leq 1, \ 4 \leq y \leq 6)$

76. $f(x, y) = \begin{cases} e^{-x-y}, & x \geq 0, \ y \geq 0 \\ 0, & \text{elsewhere} \end{cases}$

$P(0 \leq x \leq 1, \ x \leq y \leq 1)$

77. ***Approximation*** The base of a pile of sand at a cement plant is rectangular with approximate dimensions of 20 meters by 30 meters. If the base is placed on the xy-plane with one vertex at the origin, the coordinates on the surface of the pile are $(5, 5, 3)$, $(15, 5, 6)$, $(25, 5, 4)$, $(5, 15, 2)$, $(15, 15, 7)$, and $(25, 15, 3)$. Approximate the volume of sand in the pile.

78. ***Programming*** Consider a continuous function $f(x, y)$ over the rectangular region R with vertices (a, c), (b, c), (a, d), and (b, d), where $a < b$ and $c < d$. Partition the intervals $[a, b]$ and $[c, d]$ into m and n subintervals, so that the subintervals in a given direction are of equal length. Write a program for a graphing utility to compute the sum

$$\sum_{i=1}^{n} \sum_{j=1}^{m} f(x_i, y_j) \, \Delta A_i \approx \int_a^b \int_c^d f(x, y) \, dA$$

where (x_i, y_j) is the center of a representative rectangle in R.

CAS ***Approximation*** **In Exercises 79–82, (a) use a computer algebra system to approximate the iterated integral, and (b) use the program in Exercise 78 to approximate the iterated integral for the given values of m and n.**

79. $\int_0^1 \int_0^2 \sin \sqrt{x + y} \, dy \, dx$

$m = 4, \ n = 8$

80. $\int_0^2 \int_0^4 20e^{-x^3/8} dy \, dx$

$m = 10, \ n = 20$

81. $\int_4^6 \int_0^2 y \cos \sqrt{x} \, dx \, dy$

$m = 4, \ n = 8$

82. $\int_1^4 \int_1^2 \sqrt{x^3 + y^3} \, dx \, dy$

$m = 6, \ n = 4$

Approximation **In Exercises 83 and 84, determine which value best approximates the volume of the solid between the xy-plane and the function over the region. (Make your selection on the basis of a sketch of the solid and *not* by performing any calculations.)**

83. $f(x, y) = 4x$

R: square with vertices $(0, 0)$, $(4, 0)$, $(4, 4)$, $(0, 4)$

(a) -200 (b) 600 (c) 50 (d) 125 (e) 1000

84. $f(x, y) = \sqrt{x^2 + y^2}$

R: circle bounded by $x^2 + y^2 = 9$

(a) 50 (b) 500 (c) -500 (d) 5 (e) 5000

True or False? **In Exercises 85 and 86, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

85. The volume of the sphere $x^2 + y^2 + z^2 = 1$ is given by the integral

$$V = 8 \int_0^1 \int_0^1 \sqrt{1 - x^2 - y^2} \, dx \, dy.$$

86. If $f(x, y) \leq g(x, y)$ for all (x, y) in R, and both f and g are continuous over R, then $\int_R \int f(x, y) \, dA \leq \int_R \int g(x, y) \, dA$.

87. Let $f(x) = \int_1^x e^{t^2} \, dt$. Find the average value of f on the interval $[0, 1]$.

88. Find $\displaystyle\int_0^\infty \frac{e^{-x} - e^{-2x}}{x} \, dx$. $\left(\textit{Hint:} \text{ Evaluate } \displaystyle\int_1^2 e^{-xy} \, dy. \right)$

89. Determine the region R in the xy-plane that maximizes the value of $\int_R \int (9 - x^2 - y^2) \, dA$.

90. Determine the region R in the xy-plane that minimizes the value of $\int_R \int (x^2 + y^2 - 4) \, dA$.

91. Find $\int_0^2 [\arctan(\pi x) - \arctan x] \, dx$. (*Hint:* Convert the integral to a double integral.)

92. Use a geometric argument to show that

$$\int_0^3 \int_0^{\sqrt{9 - y^2}} \sqrt{9 - x^2 - y^2} \, dx \, dy = \frac{9\pi}{2}.$$

PUTNAM EXAM CHALLENGE

93. Evaluate $\int_0^a \int_0^b e^{\max\{b^2x^2, a^2y^2\}} \, dy \, dx$, where a and b are positive.

94. Show that if $\lambda > \frac{1}{2}$ there does not exist a real-valued function u such that for all x in the closed interval $0 \leq x \leq 1$, $u(x) = 1 + \lambda \int_x^1 u(y) u(y - x) \, dy$.

14.3 Change of Variables: Polar Coordinates

- **Write and evaluate double integrals in polar coordinates.**

Double Integrals in Polar Coordinates

Some double integrals are *much* easier to evaluate in polar form than in rectangular form. This is especially true for regions such as circles, cardioids, and rose curves, and for integrands that involve $x^2 + y^2$.

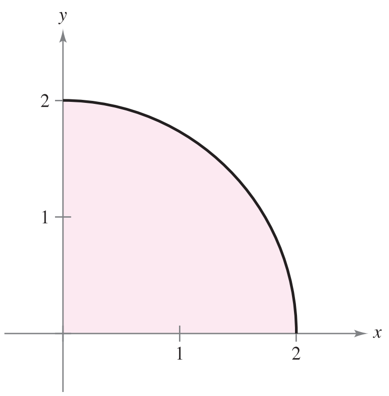
In Section 10.4, you learned that the polar coordinates (r, θ) of a point are related to the rectangular coordinates (x, y) of the point as follows.

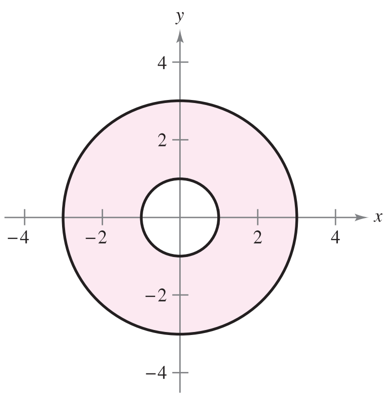
$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta$$

$$r^2 = x^2 + y^2 \quad \text{and} \quad \tan\theta = \frac{y}{x}$$

EXAMPLE 1 Using Polar Coordinates to Describe a Region

Use polar coordinates to describe each region shown in Figure 14.24.





(a) **(b)**
Figure 14.24

Solution

a. The region R is a quarter circle of radius 2. It can be described in polar coordinates as

$$R = \{(r, \theta): 0 \le r \le 2, \quad 0 \le \theta \le \pi/2\}.$$

b. The region R consists of all points between concentric circles of radii 1 and 3. It can be described in polar coordinates as

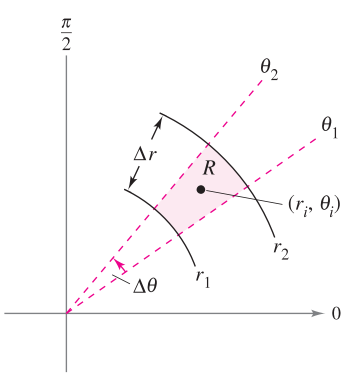
$$R = \{(r, \theta): 1 \le r \le 3, \quad 0 \le \theta \le 2\pi\}.$$

■

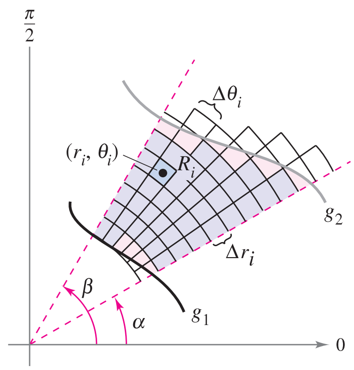
The regions in Example 1 are special cases of **polar sectors**

$$R = \{(r, \theta): r_1 \le r \le r_2, \quad \theta_1 \le \theta \le \theta_2\} \qquad \text{Polar sector}$$

as shown in Figure 14.25.



Polar sector
Figure 14.25



Polar grid superimposed over region R
Figure 14.26

To define a double integral of a continuous function $z = f(x, y)$ in polar coordinates, consider a region R bounded by the graphs of $r = g_1(\theta)$ and $r = g_2(\theta)$ and the lines $\theta = \alpha$ and $\theta = \beta$. Instead of partitioning R into small rectangles, use a partition of small polar sectors. On R, superimpose a polar grid made of rays and circular arcs, as shown in Figure 14.26. The polar sectors R_i lying entirely within R form an **inner polar partition** Δ, whose **norm** $\|\Delta\|$ is the length of the longest diagonal of the n polar sectors.

Consider a specific polar sector R_i, as shown in Figure 14.27. It can be shown (see Exercise 75) that the area of R_i is

$$\Delta A_i = r_i \Delta r_i \Delta\theta_i \qquad \text{Area of } R_i$$

where $\Delta r_i = r_2 - r_1$ and $\Delta\theta_i = \theta_2 - \theta_1$. This implies that the volume of the solid of height $f(r_i \cos\theta_i, r_i \sin\theta_i)$ above R_i is approximately

$$f(r_i \cos\theta_i, r_i \sin\theta_i) r_i \Delta r_i \Delta\theta_i$$

and you have

$$\int\!\!\int_R f(x, y)\, dA \approx \sum_{i=1}^{n} f(r_i \cos\theta_i, r_i \sin\theta_i) r_i \Delta r_i \Delta\theta_i.$$

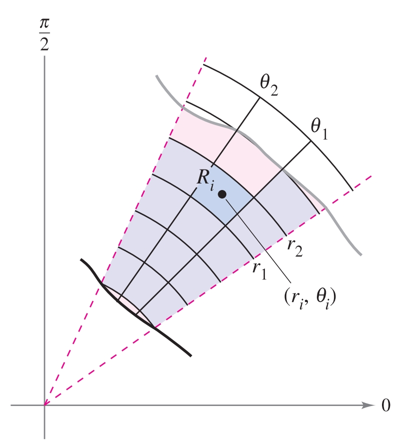
The sum on the right can be interpreted as a Riemann sum for $f(r\cos\theta, r\sin\theta)r$. The region R corresponds to a *horizontally simple* region S in the $r\theta$-plane, as shown in Figure 14.28. The polar sectors R_i correspond to rectangles S_i, and the area ΔA_i of S_i is $\Delta r_i \Delta\theta_i$. So, the right-hand side of the equation corresponds to the double integral

$$\int\!\!\int_S f(r\cos\theta, r\sin\theta) r\, dA.$$

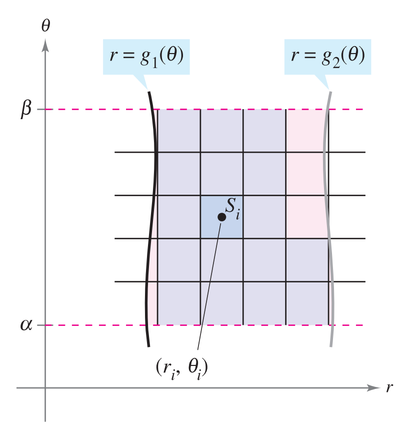
From this, you can apply Theorem 14.2 to write

$$\begin{aligned}\int\!\!\int_R f(x, y)\, dA &= \int\!\!\int_S f(r\cos\theta, r\sin\theta) r\, dA \\ &= \int_\alpha^\beta \int_{g_1(\theta)}^{g_2(\theta)} f(r\cos\theta, r\sin\theta) r\, dr\, d\theta.\end{aligned}$$

This suggests the following theorem, the proof of which is discussed in Section 14.8.



The polar sector R_i is the set of all points (r, θ) such that $r_1 \le r \le r_2$ and $\theta_1 \le \theta \le \theta_2$.
Figure 14.27



Horizontally simple region S
Figure 14.28

THEOREM 14.3 CHANGE OF VARIABLES TO POLAR FORM

Let R be a plane region consisting of all points $(x, y) = (r \cos \theta, r \sin \theta)$ satisfying the conditions $0 \le g_1(\theta) \le r \le g_2(\theta)$, $\alpha \le \theta \le \beta$, where $0 \le (\beta - \alpha) \le 2\pi$. If g_1 and g_2 are continuous on $[\alpha, \beta]$ and f is continuous on R, then

$$\int_R\int f(x, y)\, dA = \int_\alpha^\beta \int_{g_1(\theta)}^{g_2(\theta)} f(r \cos \theta, r \sin \theta) r\, dr\, d\theta.$$

EXPLORATION

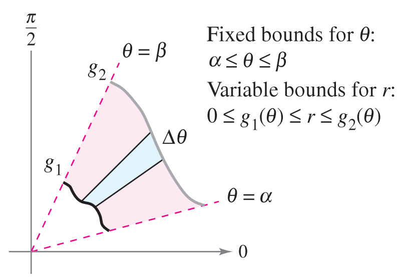
Volume of a Paraboloid Sector
In the Exploration feature on page 997, you were asked to summarize the different ways you know of finding the volume of the solid bounded by the paraboloid

$$z = a^2 - x^2 - y^2, \quad a > 0$$

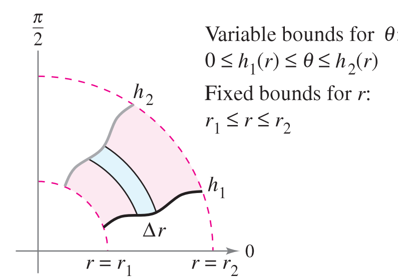
and the xy-plane. You now know another way. Use it to find the volume of the solid.

NOTE If $z = f(x, y)$ is nonnegative on R, then the integral in Theorem 14.3 can be interpreted as the volume of the solid region between the graph of f and the region R. When using the integral in Theorem 14.3, be certain not to omit the extra factor of r in the integrand. ■

The region R is restricted to two basic types, ***r*-simple** regions and ***θ*-simple** regions, as shown in Figure 14.29.



r-Simple region



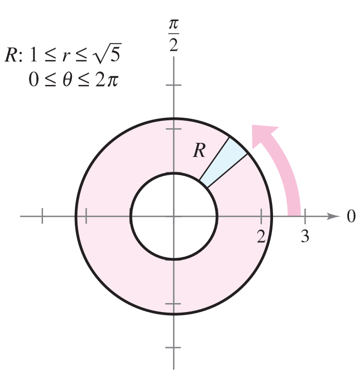
θ-Simple region

Figure 14.29

EXAMPLE 2 Evaluating a Double Polar Integral

Let R be the annular region lying between the two circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 5$. Evaluate the integral $\int_R\int (x^2 + y)\, dA$.

Solution The polar boundaries are $1 \le r \le \sqrt{5}$ and $0 \le \theta \le 2\pi$, as shown in Figure 14.30. Furthermore, $x^2 = (r \cos \theta)^2$ and $y = r \sin \theta$. So, you have



r-Simple region
Figure 14.30

$$\begin{aligned}
\int_R\int (x^2 + y)\, dA &= \int_0^{2\pi}\int_1^{\sqrt{5}} (r^2 \cos^2 \theta + r \sin \theta) r\, dr\, d\theta \\
&= \int_0^{2\pi}\int_1^{\sqrt{5}} (r^3 \cos^2 \theta + r^2 \sin \theta)\, dr\, d\theta \\
&= \int_0^{2\pi} \left(\frac{r^4}{4} \cos^2 \theta + \frac{r^3}{3} \sin \theta\right)\Big]_1^{\sqrt{5}} d\theta \\
&= \int_0^{2\pi} \left(6 \cos^2 \theta + \frac{5\sqrt{5} - 1}{3} \sin \theta\right) d\theta \\
&= \int_0^{2\pi} \left(3 + 3 \cos 2\theta + \frac{5\sqrt{5} - 1}{3} \sin \theta\right) d\theta \\
&= \left(3\theta + \frac{3 \sin 2\theta}{2} - \frac{5\sqrt{5} - 1}{3} \cos \theta\right)\Big]_0^{2\pi} \\
&= 6\pi.
\end{aligned}$$ ■

In Example 2, be sure to notice the extra factor of r in the integrand. This comes from the formula for the area of a polar sector. In differential notation, you can write

$$dA = r\,dr\,d\theta$$

which indicates that the area of a polar sector increases as you move away from the origin.

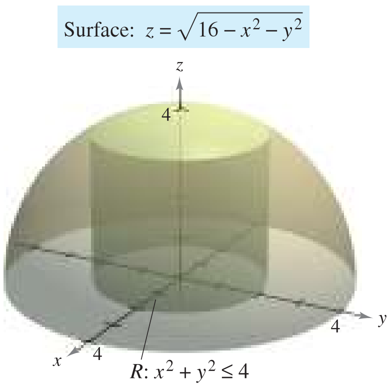


Figure 14.31

EXAMPLE 3 Change of Variables to Polar Coordinates

Use polar coordinates to find the volume of the solid region bounded above by the hemisphere

$$z = \sqrt{16 - x^2 - y^2} \qquad \text{Hemisphere forms upper surface.}$$

and below by the circular region R given by

$$x^2 + y^2 \le 4 \qquad \text{Circular region forms lower surface.}$$

as shown in Figure 14.31.

Solution In Figure 14.31, you can see that R has the bounds

$$-\sqrt{4 - y^2} \le x \le \sqrt{4 - y^2}, \quad -2 \le y \le 2$$

and that $0 \le z \le \sqrt{16 - x^2 - y^2}$. In polar coordinates, the bounds are

$$0 \le r \le 2 \quad \text{and} \quad 0 \le \theta \le 2\pi$$

with height $z = \sqrt{16 - x^2 - y^2} = \sqrt{16 - r^2}$. Consequently, the volume V is given by

$$\begin{aligned} V = \int_R\!\int f(x, y)\,dA &= \int_0^{2\pi}\!\int_0^2 \sqrt{16 - r^2}\,r\,dr\,d\theta \\ &= -\frac{1}{3}\int_0^{2\pi} (16 - r^2)^{3/2}\Big]_0^2\,d\theta \\ &= -\frac{1}{3}\int_0^{2\pi} \left(24\sqrt{3} - 64\right)d\theta \\ &= -\frac{8}{3}\left(3\sqrt{3} - 8\right)\theta\Big]_0^{2\pi} \\ &= \frac{16\pi}{3}\left(8 - 3\sqrt{3}\right) \approx 46.979. \end{aligned}$$

NOTE To see the benefit of polar coordinates in Example 3, you should try to evaluate the corresponding rectangular iterated integral

$$\int_{-2}^{2}\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} \sqrt{16 - x^2 - y^2}\,dx\,dy.$$

TECHNOLOGY Any computer algebra system that can handle double integrals in rectangular coordinates can also handle double integrals in polar coordinates. The reason this is true is that once you have formed the iterated integral, its value is not changed by using different variables. In other words, if you use a computer algebra system to evaluate

$$\int_0^{2\pi}\!\int_0^2 \sqrt{16 - x^2}\,x\,dx\,dy$$

you should obtain the same value as that obtained in Example 3.

Just as with rectangular coordinates, the double integral

$$\int_R\!\int dA$$

can be used to find the area of a region in the plane.

EXAMPLE 4 Finding Areas of Polar Regions

Use a double integral to find the area enclosed by the graph of $r = 3\cos 3\theta$.

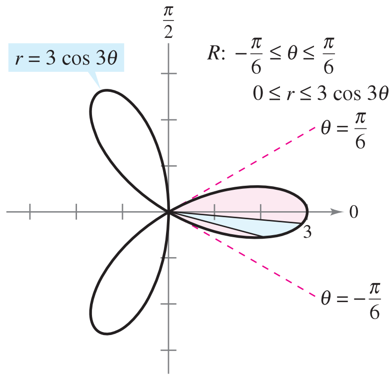


Figure 14.32

Solution Let R be one petal of the curve shown in Figure 14.32. This region is r-simple, and the boundaries are as follows.

$$-\frac{\pi}{6} \le \theta \le \frac{\pi}{6} \qquad \text{Fixed bounds on } \theta$$

$$0 \le r \le 3\cos 3\theta \qquad \text{Variable bounds on } r$$

So, the area of one petal is

$$\begin{aligned}
\frac{1}{3}A = \int_R\int dA &= \int_{-\pi/6}^{\pi/6}\int_0^{3\cos 3\theta} r\,dr\,d\theta \\
&= \int_{-\pi/6}^{\pi/6} \frac{r^2}{2}\Big]_0^{3\cos 3\theta} d\theta \\
&= \frac{9}{2}\int_{-\pi/6}^{\pi/6} \cos^2 3\theta\,d\theta \\
&= \frac{9}{4}\int_{-\pi/6}^{\pi/6} (1 + \cos 6\theta)\,d\theta = \frac{9}{4}\Big[\theta + \frac{1}{6}\sin 6\theta\Big]_{-\pi/6}^{\pi/6} = \frac{3\pi}{4}.
\end{aligned}$$

So, the total area is $A = 9\pi/4$. ■

As illustrated in Example 4, the area of a region in the plane can be represented by

$$A = \int_\alpha^\beta\int_{g_1(\theta)}^{g_2(\theta)} r\,dr\,d\theta.$$

If $g_1(\theta) = 0$, you obtain

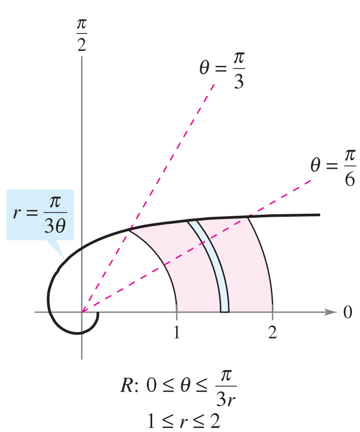
$$A = \int_\alpha^\beta\int_0^{g_2(\theta)} r\,dr\,d\theta = \int_\alpha^\beta \frac{r^2}{2}\Big]_0^{g_2(\theta)} d\theta = \int_\alpha^\beta \frac{1}{2}(g_2(\theta))^2\,d\theta$$

which agrees with Theorem 10.13.

So far in this section, all of the examples of iterated integrals in polar form have been of the form

$$\int_\alpha^\beta\int_{g_1(\theta)}^{g_2(\theta)} f(r\cos\theta, r\sin\theta)r\,dr\,d\theta$$

in which the order of integration is with respect to r first. Sometimes you can obtain a simpler integration problem by switching the order of integration, as illustrated in the next example.



θ-Simple region
Figure 14.33

EXAMPLE 5 Changing the Order of Integration

Find the area of the region bounded above by the spiral $r = \pi/(3\theta)$ and below by the polar axis, between $r = 1$ and $r = 2$.

Solution The region is shown in Figure 14.33. The polar boundaries for the region are

$$1 \le r \le 2 \quad \text{and} \quad 0 \le \theta \le \frac{\pi}{3r}.$$

So, the area of the region can be evaluated as follows.

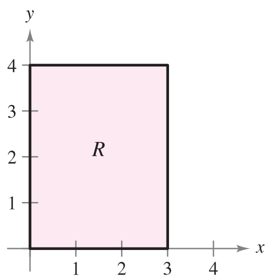
$$A = \int_1^2\int_0^{\pi/(3r)} r\,d\theta\,dr = \int_1^2 r\theta\Big]_0^{\pi/(3r)} dr = \int_1^2 \frac{\pi}{3}\,dr = \frac{\pi r}{3}\Big]_1^2 = \frac{\pi}{3}$$ ■

14.3 Exercises

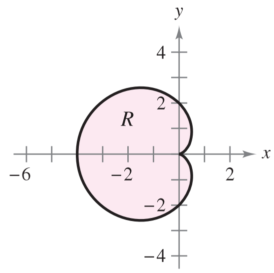
See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–4, the region R for the integral $\int_R\int f(x, y)\, dA$ is shown. State whether you would use rectangular or polar coordinates to evaluate the integral.

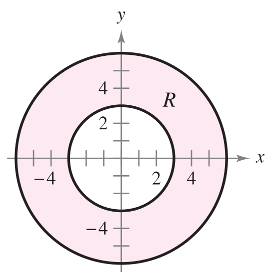
1.



2.



3.



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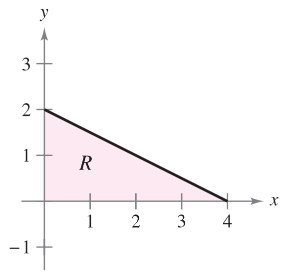
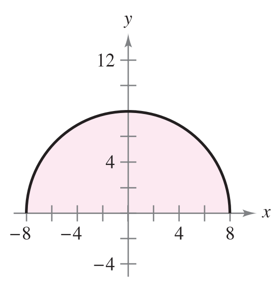
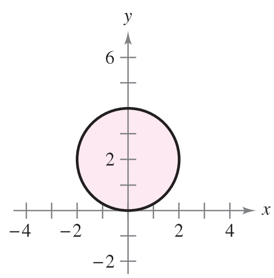


In Exercises 5–8, use polar coordinates to describe the region shown.

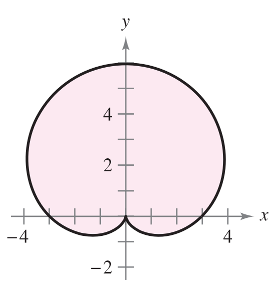
5.



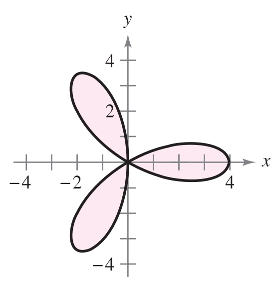
6.



7.



8.



In Exercises 9–16, evaluate the double integral $\int_R\int f(r, \theta)\, dA$, and sketch the region R.

9. $\displaystyle\int_0^{\pi}\int_0^{\cos\theta} r\, dr\, d\theta$

10. $\displaystyle\int_0^{\pi}\int_0^{\sin\theta} r^2\, dr\, d\theta$

11. $\displaystyle\int_0^{2\pi}\int_0^{6} 3r^2 \sin\theta\, dr\, d\theta$

12. $\displaystyle\int_0^{\pi/4}\int_0^{4} r^2 \sin\theta \cos\theta\, dr\, d\theta$

13. $\displaystyle\int_0^{\pi/2}\int_2^{3} \sqrt{9 - r^2}\, r\, dr\, d\theta$

14. $\displaystyle\int_0^{\pi/2}\int_0^{3} re^{-r^2}\, dr\, d\theta$

15. $\displaystyle\int_0^{\pi/2}\int_0^{1+\sin\theta} \theta r\, dr\, d\theta$

16. $\displaystyle\int_0^{\pi/2}\int_0^{1-\cos\theta} (\sin\theta) r\, dr\, d\theta$

In Exercises 17–26, evaluate the iterated integral by converting to polar coordinates.

17. $\displaystyle\int_0^{a}\int_0^{\sqrt{a^2-y^2}} y\, dx\, dy$

18. $\displaystyle\int_0^{a}\int_0^{\sqrt{a^2-x^2}} x\, dy\, dx$

19. $\displaystyle\int_{-2}^{2}\int_0^{\sqrt{4-x^2}} (x^2 + y^2)\, dy\, dx$

20. $\displaystyle\int_0^{1}\int_{-\sqrt{x-x^2}}^{\sqrt{x-x^2}} (x^2 + y^2)\, dy\, dx$

21. $\displaystyle\int_0^{3}\int_0^{\sqrt{9-x^2}} (x^2 + y^2)^{3/2}\, dy\, dx$

22. $\displaystyle\int_0^{2}\int_y^{\sqrt{8-y^2}} \sqrt{x^2 + y^2}\, dx\, dy$

23. $\displaystyle\int_0^{2}\int_0^{\sqrt{2x-x^2}} xy\, dy\, dx$

24. $\displaystyle\int_0^{4}\int_0^{\sqrt{4y-y^2}} x^2\, dx\, dy$

25. $\displaystyle\int_{-1}^{1}\int_0^{\sqrt{1-x^2}} \cos(x^2 + y^2)\, dy\, dx$

26. $\displaystyle\int_0^{2}\int_0^{\sqrt{4-x^2}} \sin\sqrt{x^2 + y^2}\, dy\, dx$

In Exercises 27 and 28, combine the sum of the two iterated integrals into a single iterated integral by converting to polar coordinates. Evaluate the resulting iterated integral.

27. $\displaystyle\int_0^{2}\int_0^{x} \sqrt{x^2 + y^2}\, dy\, dx + \int_2^{2\sqrt{2}}\int_0^{\sqrt{8-x^2}} \sqrt{x^2 + y^2}\, dy\, dx$

28. $\displaystyle\int_0^{5\sqrt{2}/2}\int_0^{x} xy\, dy\, dx + \int_{5\sqrt{2}/2}^{5}\int_0^{\sqrt{25-x^2}} xy\, dy\, dx$

In Exercises 29–32, use polar coordinates to set up and evaluate the double integral $\int_R\int f(x, y)\, dA$.

29. $f(x, y) = x + y,\ R: x^2 + y^2 \le 4,\ x \ge 0,\ y \ge 0$

30. $f(x, y) = e^{-(x^2+y^2)/2},\ R: x^2 + y^2 \le 25,\ x \ge 0$

31. $f(x, y) = \arctan\dfrac{y}{x},\ R: x^2 + y^2 \ge 1,\ x^2 + y^2 \le 4,\ 0 \le y \le x$

32. $f(x, y) = 9 - x^2 - y^2,\ R: x^2 + y^2 \le 9,\ x \ge 0,\ y \ge 0$

Volume **In Exercises 33–38, use a double integral in polar coordinates to find the volume of the solid bounded by the graphs of the equations.**

33. $z = xy,\ x^2 + y^2 = 1$, first octant

34. $z = x^2 + y^2 + 3,\ z = 0,\ x^2 + y^2 = 1$

35. $z = \sqrt{x^2 + y^2},\ z = 0,\ x^2 + y^2 = 25$

36. $z = \ln(x^2 + y^2),\ z = 0,\ x^2 + y^2 \ge 1,\ x^2 + y^2 \le 4$

37. Inside the hemisphere $z = \sqrt{16 - x^2 - y^2}$ and inside the cylinder $x^2 + y^2 - 4x = 0$

38. Inside the hemisphere $z = \sqrt{16 - x^2 - y^2}$ and outside the cylinder $x^2 + y^2 = 1$

39. ***Volume*** Find a such that the volume inside the hemisphere $z = \sqrt{16 - x^2 - y^2}$ and outside the cylinder $x^2 + y^2 = a^2$ is one-half the volume of the hemisphere.

40. ***Volume*** Use a double integral in polar coordinates to find the volume of a sphere of radius a.

41. ***Volume*** Determine the diameter of a hole that is drilled vertically through the center of the solid bounded by the graphs of the equations $z = 25e^{-(x^2+y^2)/4}$, $z = 0$, and $x^2 + y^2 = 16$ if one-tenth of the volume of the solid is removed.

CAS **42.** ***Machine Design*** The surfaces of a double-lobed cam are modeled by the inequalities $\frac{1}{4} \le r \le \frac{1}{2}(1 + \cos^2 \theta)$ and

$$\frac{-9}{4(x^2 + y^2 + 9)} \le z \le \frac{9}{4(x^2 + y^2 + 9)}$$

where all measurements are in inches.

(a) Use a computer algebra system to graph the cam.

(b) Use a computer algebra system to approximate the perimeter of the polar curve

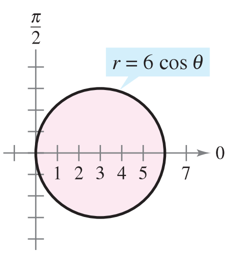
$r = \frac{1}{2}(1 + \cos^2 \theta)$.

This is the distance a roller must travel as it runs against the cam through one revolution of the cam.

(c) Use a computer algebra system to find the volume of steel in the cam.

Area **In Exercises 43–48, use a double integral to find the area of the shaded region.**

43.



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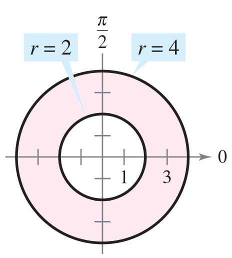

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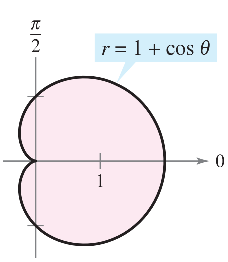

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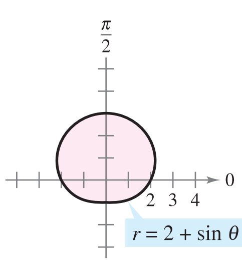

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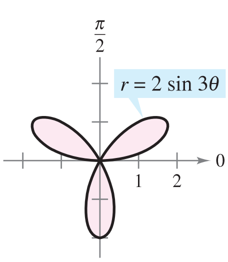

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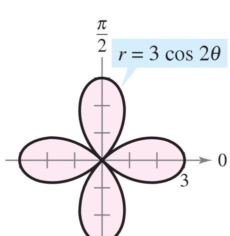


Area **In Exercises 49–54, sketch a graph of the region bounded by the graphs of the equations. Then use a double integral to find the area of the region.**

49. Inside the circle $r = 2 \cos \theta$ and outside the circle $r = 1$

50. Inside the cardioid $r = 2 + 2 \cos \theta$ and outside the circle $r = 1$

51. Inside the circle $r = 3 \cos \theta$ and outside the cardioid $r = 1 + \cos \theta$

52. Inside the cardioid $r = 1 + \cos \theta$ and outside the circle $r = 3 \cos \theta$

53. Inside the rose curve $r = 4 \sin 3\theta$ and outside the circle $r = 2$

54. Inside the circle $r = 2$ and outside the cardioid $r = 2 - 2 \cos \theta$

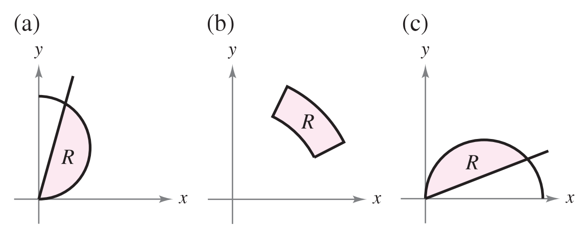
WRITING ABOUT CONCEPTS

55. Describe the partition of the region R of integration in the xy-plane when polar coordinates are used to evaluate a double integral.

56. Explain how to change from rectangular coordinates to polar coordinates in a double integral.

57. In your own words, describe r-simple regions and θ-simple regions.

58. Each figure shows a region of integration for the double integral $\int_R\int f(x, y)\, dA$. For each region, state whether horizontal representative elements, vertical representative elements, or polar sectors would yield the easiest method for obtaining the limits of integration. Explain your reasoning.



59. Let R be the region bounded by the circle $x^2 + y^2 = 9$.

(a) Set up the integral $\int_R\int f(x, y)\, dA$.

(b) Convert the integral in part (a) to polar coordinates.

(c) Which integral would you choose to evaluate? Why?

CAPSTONE

60. ***Think About It*** Without performing any calculations, identify the double integral that represents the integral of $f(x) = x^2 + y^2$ over a circle of radius 4. Explain your reasoning.

(a) $\displaystyle\int_0^{2\pi}\int_0^4 r^2\, dr\, d\theta$ (b) $\displaystyle\int_0^4\int_0^{2\pi} r^3\, dr\, d\theta$

(c) $\displaystyle\int_0^{2\pi}\int_0^4 r^3\, dr\, d\theta$ (d) $\displaystyle\int_0^{2\pi}\int_{-4}^4 r^3\, dr\, d\theta$

61. ***Think About It*** Consider the program you wrote to approximate double integrals in rectangular coordinates in Exercise 78, in Section 14.2. If the program is used to approximate the double integral

$$\int_R\int f(r, \theta)\, dA$$

in polar coordinates, how will you modify f when it is entered into the program? Because the limits of integration are constants, describe the plane region of integration.

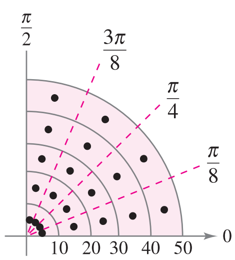
62. ***Approximation*** Horizontal cross sections of a piece of ice that broke from a glacier are in the shape of a quarter of a circle with a radius of approximately 50 feet. The base is divided into 20 subregions, as shown in the figure. At the center of each subregion, the height of the ice is measured, yielding the following points in cylindrical coordinates.

$\left(5, \frac{\pi}{16}, 7\right), \left(15, \frac{\pi}{16}, 8\right), \left(25, \frac{\pi}{16}, 10\right), \left(35, \frac{\pi}{16}, 12\right), \left(45, \frac{\pi}{16}, 9\right),$
$\left(5, \frac{3\pi}{16}, 9\right), \left(15, \frac{3\pi}{16}, 10\right), \left(25, \frac{3\pi}{16}, 14\right), \left(35, \frac{3\pi}{16}, 15\right), \left(45, \frac{3\pi}{16}, 10\right),$
$\left(5, \frac{5\pi}{16}, 9\right), \left(15, \frac{5\pi}{16}, 11\right), \left(25, \frac{5\pi}{16}, 15\right), \left(35, \frac{5\pi}{16}, 18\right), \left(45, \frac{5\pi}{16}, 14\right),$
$\left(5, \frac{7\pi}{16}, 5\right), \left(15, \frac{7\pi}{16}, 8\right), \left(25, \frac{7\pi}{16}, 11\right), \left(35, \frac{7\pi}{16}, 16\right), \left(45, \frac{7\pi}{16}, 12\right)$

(a) Approximate the volume of the solid.

(b) Ice weighs approximately 57 pounds per cubic foot. Approximate the weight of the solid.

(c) There are 7.48 gallons of water per cubic foot. Approximate the number of gallons of water in the solid.



CAS ***Approximation*** **In Exercises 63 and 64, use a computer algebra system to approximate the iterated integral.**

63. $\displaystyle\int_{\pi/4}^{\pi/2}\int_0^5 r\sqrt{1 + r^3}\sin\sqrt{\theta}\, dr\, d\theta$

64. $\displaystyle\int_0^{\pi/4}\int_0^4 5re^{\sqrt{r\theta}}\, dr\, d\theta$

Approximation **In Exercises 65 and 66, determine which value best approximates the volume of the solid between the xy-plane and the function over the region. (Make your selection on the basis of a sketch of the solid and *not* by performing any calculations.)**

65. $f(x, y) = 15 - 2y$; R: semicircle: $x^2 + y^2 = 16, y \geq 0$

(a) 100 (b) 200 (c) 300 (d) -200 (e) 800

66. $f(x, y) = xy + 2$; R: quarter circle: $x^2 + y^2 = 9, x \geq 0, y \geq 0$

(a) 25 (b) 8 (c) 100 (d) 50 (e) -30

True or False? **In Exercises 67 and 68, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

67. If $\int_R\int f(r, \theta)\, dA > 0$, then $f(r, \theta) > 0$ for all (r, θ) in R.

68. If $f(r, \theta)$ is a constant function and the area of the region S is twice that of the region R, then $2\int_R\int f(r, \theta)\, dA = \int_S\int f(r, \theta)\, dA$.

69. ***Probability*** The value of the integral $I = \int_{-\infty}^{\infty} e^{-x^2/2}\, dx$ is required in the development of the normal probability density function.

(a) Use polar coordinates to evaluate the improper integral.

$$I^2 = \left(\int_{-\infty}^{\infty} e^{-x^2/2}\, dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2/2}\, dy\right)$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)/2}\, dA$$

(b) Use the result of part (a) to determine I.

FOR FURTHER INFORMATION For more information on this problem, see the article "Integrating e^{-x^2} Without Polar Coordinates" by William Dunham in *Mathematics Teacher*. To view this article, go to the website *www.matharticles.com*.

70. Use the result of Exercise 69 and a change of variables to evaluate each integral. No integration is required.

(a) $\displaystyle\int_{-\infty}^{\infty} e^{-x^2}\, dx$ (b) $\displaystyle\int_{-\infty}^{\infty} e^{-4x^2}\, dx$

71. ***Population*** The population density of a city is approximated by the model $f(x, y) = 4000e^{-0.01(x^2+y^2)}$, $x^2 + y^2 \leq 49$, where x and y are measured in miles. Integrate the density function over the indicated circular region to approximate the population of the city.

72. ***Probability*** Find k such that the function

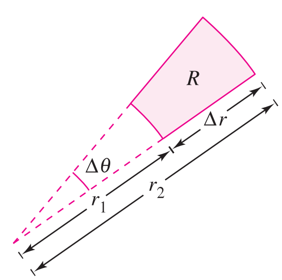
$$f(x, y) = \begin{cases} ke^{-(x^2+y^2)}, & x \geq 0, y \geq 0 \\ 0, & \text{elsewhere} \end{cases}$$

is a probability density function.

73. ***Think About It*** Consider the region bounded by the graphs of $y = 2$, $y = 4$, $y = x$, and $y = \sqrt{3}x$ and the double integral $\int_R\int f\, dA$. Determine the limits of integration if the region R is divided into (a) horizontal representative elements, (b) vertical representative elements, and (c) polar sectors.

74. Repeat Exercise 73 for a region R bounded by the graph of the equation $(x - 2)^2 + y^2 = 4$.

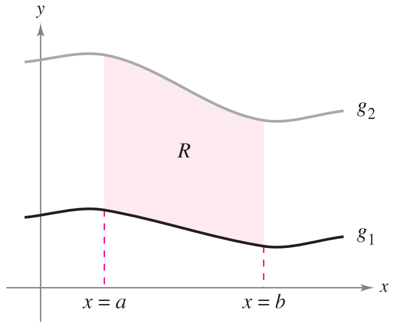
75. Show that the area A of the polar sector R (see figure) is $A = r\Delta r\Delta\theta$, where $r = (r_1 + r_2)/2$ is the average radius of R.



14.4 Center of Mass and Moments of Inertia

- **Find the mass of a planar lamina using a double integral.**
- **Find the center of mass of a planar lamina using double integrals.**
- **Find moments of inertia using double integrals.**

Mass



Lamina of constant density ρ
Figure 14.34

Section 7.6 discussed several applications of integration involving a lamina of *constant* density ρ. For example, if the lamina corresponding to the region R, as shown in Figure 14.34, has a constant density ρ, then the mass of the lamina is given by

$$\text{Mass} = \rho A = \rho \int_R\int dA = \int_R\int \rho\, dA. \quad \text{Constant density}$$

If not otherwise stated, a lamina is assumed to have a constant density. In this section, however, you will extend the definition of the term *lamina* to include thin plates of *variable* density. Double integrals can be used to find the mass of a lamina of variable density, where the density at (x, y) is given by the **density function ρ.**

DEFINITION OF MASS OF A PLANAR LAMINA OF VARIABLE DENSITY

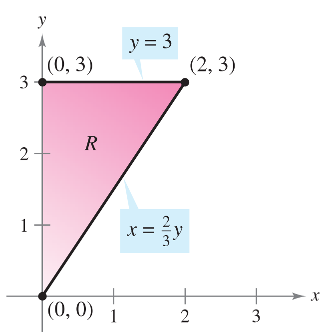
If ρ is a continuous density function on the lamina corresponding to a plane region R, then the mass m of the lamina is given by

$$m = \int_R\int \rho(x, y)\, dA. \quad \text{Variable density}$$

NOTE Density is normally expressed as mass per unit volume. For a planar lamina, however, density is mass per unit surface area.

EXAMPLE 1 Finding the Mass of a Planar Lamina

Find the mass of the triangular lamina with vertices $(0, 0)$, $(0, 3)$, and $(2, 3)$, given that the density at (x, y) is $\rho(x, y) = 2x + y$.



Lamina of variable density $\rho(x, y) = 2x + y$
Figure 14.35

Solution As shown in Figure 14.35, region R has the boundaries $x = 0$, $y = 3$, and $y = 3x/2$ (or $x = 2y/3$). Therefore, the mass of the lamina is

$$\begin{aligned} m = \int_R\int (2x + y)\, dA &= \int_0^3 \int_0^{2y/3} (2x + y)\, dx\, dy \\ &= \int_0^3 \Big[x^2 + xy\Big]_0^{2y/3} dy \\ &= \frac{10}{9}\int_0^3 y^2\, dy \\ &= \frac{10}{9}\left[\frac{y^3}{3}\right]_0^3 \\ &= 10. \end{aligned}$$

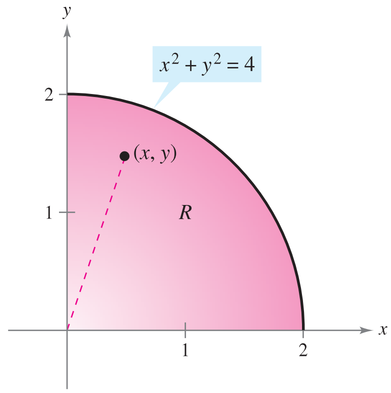
NOTE In Figure 14.35, note that the planar lamina is shaded so that the darkest shading corresponds to the densest part.

EXAMPLE 2 Finding Mass by Polar Coordinates

Find the mass of the lamina corresponding to the first-quadrant portion of the circle

$$x^2 + y^2 = 4$$

where the density at the point (x, y) is proportional to the distance between the point and the origin, as shown in Figure 14.36.



Density at (x, y): $\rho(x, y) = k\sqrt{x^2 + y^2}$
Figure 14.36

Solution At any point (x, y), the density of the lamina is

$$\begin{aligned}\rho(x, y) &= k\sqrt{(x-0)^2 + (y-0)^2}\\ &= k\sqrt{x^2 + y^2}.\end{aligned}$$

Because $0 \le x \le 2$ and $0 \le y \le \sqrt{4 - x^2}$, the mass is given by

$$\begin{aligned}m &= \iint_R k\sqrt{x^2 + y^2}\, dA\\ &= \int_0^2 \int_0^{\sqrt{4-x^2}} k\sqrt{x^2 + y^2}\, dy\, dx.\end{aligned}$$

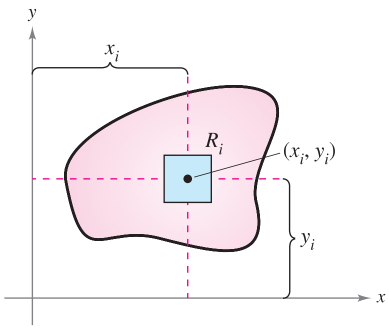
To simplify the integration, you can convert to polar coordinates, using the bounds $0 \le \theta \le \pi/2$ and $0 \le r \le 2$. So, the mass is

$$\begin{aligned}m = \iint_R k\sqrt{x^2 + y^2}\, dA &= \int_0^{\pi/2} \int_0^2 k\sqrt{r^2}\, r\, dr\, d\theta\\ &= \int_0^{\pi/2} \int_0^2 kr^2\, dr\, d\theta\\ &= \int_0^{\pi/2} \frac{kr^3}{3}\Big]_0^2\, d\theta\\ &= \frac{8k}{3}\int_0^{\pi/2} d\theta\\ &= \frac{8k}{3}\Big[\theta\Big]_0^{\pi/2}\\ &= \frac{4\pi k}{3}.\end{aligned}$$

TECHNOLOGY On many occasions, this text has mentioned the benefits of computer programs that perform symbolic integration. Even if you use such a program regularly, you should remember that its greatest benefit comes only in the hands of a knowledgeable user. For instance, notice how much simpler the integral in Example 2 becomes when it is converted to polar form.

Rectangular Form	*Polar Form*
$\int_0^2 \int_0^{\sqrt{4-x^2}} k\sqrt{x^2 + y^2}\, dy\, dx$	$\int_0^{\pi/2} \int_0^2 kr^2\, dr\, d\theta$

If you have access to software that performs symbolic integration, use it to evaluate both integrals. Some software programs cannot handle the first integral, but any program that can handle double integrals can evaluate the second integral.



$M_x = (\text{mass})(y_i)$
$M_y = (\text{mass})(x_i)$
Figure 14.37

Moments and Center of Mass

For a lamina of variable density, moments of mass are defined in a manner similar to that used for the uniform density case. For a partition Δ of a lamina corresponding to a plane region R, consider the ith rectangle R_i of one area ΔA_i, as shown in Figure 14.37. Assume that the mass of R_i is concentrated at one of its interior points (x_i, y_i). The moment of mass of R_i with respect to the x-axis can be approximated by

$$(\text{Mass})(y_i) \approx [\rho(x_i, y_i)\,\Delta A_i](y_i).$$

Similarly, the moment of mass with respect to the y-axis can be approximated by

$$(\text{Mass})(x_i) \approx [\rho(x_i, y_i)\,\Delta A_i](x_i).$$

By forming the Riemann sum of all such products and taking the limits as the norm of Δ approaches 0, you obtain the following definitions of moments of mass with respect to the x- and y-axes.

MOMENTS AND CENTER OF MASS OF A VARIABLE DENSITY PLANAR LAMINA

Let ρ be a continuous density function on the planar lamina R. The **moments of mass** with respect to the x- and y-axes are

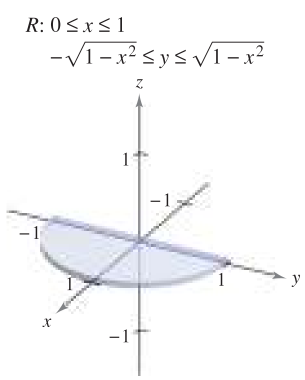
$$M_x = \int_R\int y\rho(x, y)\,dA \quad \text{and} \quad M_y = \int_R\int x\rho(x, y)\,dA.$$

If m is the mass of the lamina, then the **center of mass** is

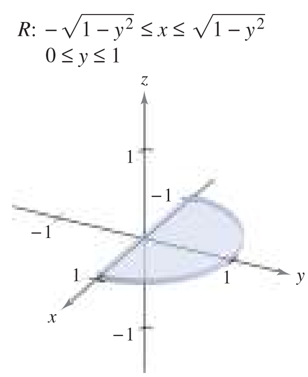
$$(\bar{x}, \bar{y}) = \left(\frac{M_y}{m}, \frac{M_x}{m}\right).$$

If R represents a simple plane region rather than a lamina, the point $(\bar{x}, \bar{y})$ is called the **centroid** of the region.

For some planar laminas with a constant density ρ, you can determine the center of mass (or one of its coordinates) using symmetry rather than using integration. For instance, consider the laminas of constant density shown in Figure 14.38. Using symmetry, you can see that $\bar{y} = 0$ for the first lamina and $\bar{x} = 0$ for the second lamina.

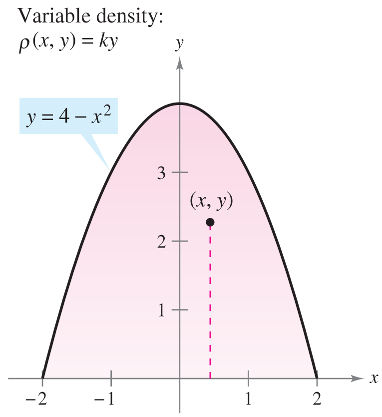


Lamina of constant density that is symmetric with respect to the x-axis



Lamina of constant density that is symmetric with respect to the y-axis

Figure 14.38



Parabolic region of variable density

Figure 14.39

EXAMPLE 3 Finding the Center of Mass

Find the center of mass of the lamina corresponding to the parabolic region

$$0 \le y \le 4 - x^2 \qquad \text{Parabolic region}$$

where the density at the point (x, y) is proportional to the distance between (x, y) and the x-axis, as shown in Figure 14.39.

Solution Because the lamina is symmetric with respect to the y-axis and

$$\rho(x, y) = ky$$

the center of mass lies on the y-axis. So, $\bar{x} = 0$. To find $\bar{y}$, first find the mass of the lamina.

$$\begin{aligned}
\text{Mass} = \int_{-2}^{2}\int_{0}^{4-x^2} ky\,dy\,dx &= \frac{k}{2}\int_{-2}^{2} y^2\Big]_0^{4-x^2}\,dx \\
&= \frac{k}{2}\int_{-2}^{2} (16 - 8x^2 + x^4)\,dx \\
&= \frac{k}{2}\left[16x - \frac{8x^3}{3} + \frac{x^5}{5}\right]_{-2}^{2} \\
&= k\left(32 - \frac{64}{3} + \frac{32}{5}\right) \\
&= \frac{256k}{15}
\end{aligned}$$

Next, find the moment about the x-axis.

$$\begin{aligned}
M_x = \int_{-2}^{2}\int_{0}^{4-x^2} (y)(ky)\,dy\,dx &= \frac{k}{3}\int_{-2}^{2} y^3\Big]_0^{4-x^2}\,dx \\
&= \frac{k}{3}\int_{-2}^{2} (64 - 48x^2 + 12x^4 - x^6)\,dx \\
&= \frac{k}{3}\left[64x - 16x^3 + \frac{12x^5}{5} - \frac{x^7}{7}\right]_{-2}^{2} \\
&= \frac{4096k}{105}
\end{aligned}$$

So,

$$\bar{y} = \frac{M_x}{m} = \frac{4096k/105}{256k/15} = \frac{16}{7}$$

and the center of mass is $\left(0, \frac{16}{7}\right)$. ■

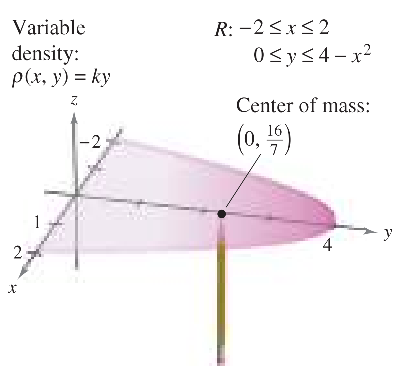


Figure 14.40

Although you can think of the moments M_x and M_y as measuring the tendency to rotate about the x- or y-axis, the calculation of moments is usually an intermediate step toward a more tangible goal. The use of the moments M_x and M_y is typical—to find the center of mass. Determination of the center of mass is useful in a variety of applications that allow you to treat a lamina as if its mass were concentrated at just one point. Intuitively, you can think of the center of mass as the balancing point of the lamina. For instance, the lamina in Example 3 should balance on the point of a pencil placed at $\left(0, \frac{16}{7}\right)$, as shown in Figure 14.40.

Moments of Inertia

The moments of M_x and M_y used in determining the center of mass of a lamina are sometimes called the **first moments** about the x- and y-axes. In each case, the moment is the product of a mass times a distance.

$$M_x = \int_R\int \underbrace{(y)}_{\substack{\text{Distance}\\ \text{to } x\text{-axis}}} \underbrace{\rho(x, y)\, dA}_{\text{Mass}} \qquad M_y = \int_R\int \underbrace{(x)}_{\substack{\text{Distance}\\ \text{to } y\text{-axis}}} \underbrace{\rho(x, y)\, dA}_{\text{Mass}}$$

You will now look at another type of moment—the **second moment,** or the **moment of inertia** of a lamina about a line. In the same way that mass is a measure of the tendency of matter to resist a change in straight-line motion, the moment of inertia about a line is a *measure of the tendency of matter to resist a change in rotational motion*. For example, if a particle of mass m is a distance d from a fixed line, its moment of inertia about the line is defined as

$$I = md^2 = (\text{mass})(\text{distance})^2.$$

As with moments of mass, you can generalize this concept to obtain the moments of inertia about the x- and y-axes of a lamina of variable density. These second moments are denoted by I_x and I_y, and in each case the moment is the product of a mass times the square of a distance.

$$I_x = \int_R\int \underbrace{(y^2)}_{\substack{\text{Square of distance}\\ \text{to } x\text{-axis}}} \underbrace{\rho(x, y)\, dA}_{\text{Mass}} \qquad I_y = \int_R\int \underbrace{(x^2)}_{\substack{\text{Square of distance}\\ \text{to } y\text{-axis}}} \underbrace{\rho(x, y)\, dA}_{\text{Mass}}$$

The sum of the moments I_x and I_y is called the **polar moment of inertia** and is denoted by I_0.

NOTE For a lamina in the xy-plane, I_0 represents the moment of inertia of the lamina about the z-axis. The term "polar moment of inertia" stems from the fact that the square of the polar distance r is used in the calculation.

$$\begin{aligned} I_0 &= \int_R\int (x^2 + y^2)\rho(x, y)\, dA \\ &= \int_R\int r^2\rho(x, y)\, dA \end{aligned}$$

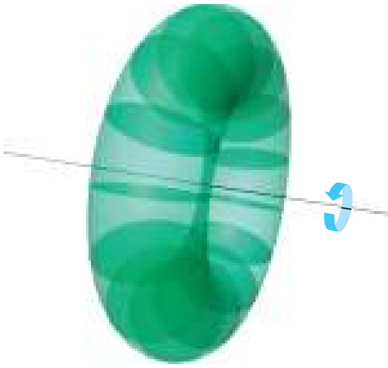
EXAMPLE 4 Finding the Moment of Inertia

Find the moment of inertia about the x-axis of the lamina in Example 3.

Solution From the definition of moment of inertia, you have

$$\begin{aligned} I_x &= \int_{-2}^{2}\int_{0}^{4-x^2} y^2(ky)\, dy\, dx \\ &= \frac{k}{4}\int_{-2}^{2} y^4\Big]_0^{4-x^2} dx \\ &= \frac{k}{4}\int_{-2}^{2} (256 - 256x^2 + 96x^4 - 16x^6 + x^8)\, dx \\ &= \frac{k}{4}\left[256x - \frac{256x^3}{3} + \frac{96x^5}{5} - \frac{16x^7}{7} + \frac{x^9}{9}\right]_{-2}^{2} \\ &= \frac{32{,}768k}{315}. \end{aligned}$$

■



Planar lamina revolving at ω radians per second
Figure 14.41

The moment of inertia I of a revolving lamina can be used to measure its kinetic energy. For example, suppose a planar lamina is revolving about a line with an **angular speed** of ω radians per second, as shown in Figure 14.41. The kinetic energy E of the revolving lamina is

$$E = \frac{1}{2} I\omega^2. \qquad \text{Kinetic energy for rotational motion}$$

On the other hand, the kinetic energy E of a mass m moving in a straight line at a velocity v is

$$E = \frac{1}{2} mv^2. \qquad \text{Kinetic energy for linear motion}$$

So, the kinetic energy of a mass moving in a straight line is proportional to its mass, but the kinetic energy of a mass revolving about an axis is proportional to its moment of inertia.

The **radius of gyration** $\bar{\bar{r}}$ of a revolving mass m with moment of inertia I is defined as

$$\bar{\bar{r}} = \sqrt{\frac{I}{m}}. \qquad \text{Radius of gyration}$$

If the entire mass were located at a distance $\bar{\bar{r}}$ from its axis of revolution, it would have the same moment of inertia and, consequently, the same kinetic energy. For instance, the radius of gyration of the lamina in Example 4 about the x-axis is given by

$$\bar{\bar{y}} = \sqrt{\frac{I_x}{m}} = \sqrt{\frac{32{,}768k/315}{256k/15}} = \sqrt{\frac{128}{21}} \approx 2.469.$$

EXAMPLE 5 Finding the Radius of Gyration

Find the radius of gyration about the y-axis for the lamina corresponding to the region R: $0 \le y \le \sin x$, $0 \le x \le \pi$, where the density at (x, y) is given by $\rho(x, y) = x$.

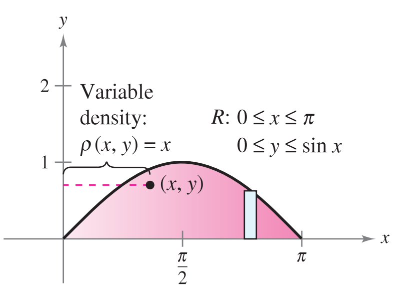


Figure 14.42

Solution The region R is shown in Figure 14.42. By integrating $\rho(x, y) = x$ over the region R, you can determine that the mass of the region is π. The moment of inertia about the y-axis is

$$\begin{aligned} I_y &= \int_0^{\pi} \int_0^{\sin x} x^3 \, dy \, dx \\ &= \int_0^{\pi} x^3 y \Big]_0^{\sin x} dx \\ &= \int_0^{\pi} x^3 \sin x \, dx \\ &= \Big[(3x^2 - 6)(\sin x) - (x^3 - 6x)(\cos x) \Big]_0^{\pi} \\ &= \pi^3 - 6\pi. \end{aligned}$$

So, the radius of gyration about the y-axis is

$$\begin{aligned} \bar{\bar{x}} &= \sqrt{\frac{I_y}{m}} \\ &= \sqrt{\frac{\pi^3 - 6\pi}{\pi}} \\ &= \sqrt{\pi^2 - 6} \approx 1.967. \end{aligned}$$

14.4 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–4, find the mass of the lamina described by the inequalities, given that its density is $\rho(x, y) = xy$. (*Hint:* Some of the integrals are simpler in polar coordinates.)

1. $0 \le x \le 2,\ 0 \le y \le 2$
2. $0 \le x \le 3,\ 0 \le y \le 9 - x^2$
3. $0 \le x \le 1,\ 0 \le y \le \sqrt{1 - x^2}$
4. $x \ge 0,\ 3 \le y \le 3 + \sqrt{9 - x^2}$

In Exercises 5–8, find the mass and center of mass of the lamina for each density.

5. R: square with vertices $(0, 0), (a, 0), (0, a), (a, a)$
   (a) $\rho = k$ (b) $\rho = ky$ (c) $\rho = kx$
6. R: rectangle with vertices $(0, 0), (a, 0), (0, b), (a, b)$
   (a) $\rho = kxy$ (b) $\rho = k(x^2 + y^2)$
7. R: triangle with vertices $(0, 0), (0, a), (a, a)$
   (a) $\rho = k$ (b) $\rho = ky$ (c) $\rho = kx$
8. R: triangle with vertices $(0, 0), (a/2, a), (a, 0)$
   (a) $\rho = k$ (b) $\rho = kxy$

9. ***Translations in the Plane*** Translate the lamina in Exercise 5 to the right five units and determine the resulting center of mass.
10. ***Conjecture*** Use the result of Exercise 9 to make a conjecture about the change in the center of mass when a lamina of constant density is translated c units horizontally or d units vertically. Is the conjecture true if the density is not constant? Explain.

In Exercises 11–22, find the mass and center of mass of the lamina bounded by the graphs of the equations for the given density or densities. (*Hint:* Some of the integrals are simpler in polar coordinates.)

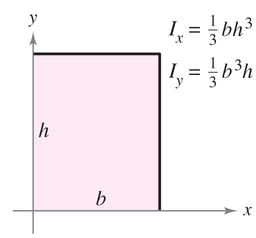
11. $y = \sqrt{x},\ y = 0,\ x = 1,\ \rho = ky$
12. $y = x^2,\ y = 0,\ x = 2,\ \rho = kxy$
13. $y = 4/x,\ y = 0,\ x = 1,\ x = 4,\ \rho = kx^2$
14. $y = \dfrac{1}{1 + x^2},\ y = 0,\ x = -1,\ x = 1,\ \rho = k$
15. $y = e^x,\ y = 0,\ x = 0,\ x = 1$
    (a) $\rho = k$ (b) $\rho = ky$
16. $y = e^{-x},\ y = 0,\ x = 0,\ x = 1$
    (a) $\rho = ky$ (b) $\rho = ky^2$
17. $y = 4 - x^2,\ y = 0,\ \rho = ky$
18. $x = 9 - y^2,\ x = 0,\ \rho = kx$
19. $y = \sin \dfrac{\pi x}{L},\ y = 0,\ x = 0,\ x = L,\ \rho = k$
20. $y = \cos \dfrac{\pi x}{L},\ y = 0,\ x = 0,\ x = \dfrac{L}{2},\ \rho = ky$
21. $y = \sqrt{a^2 - x^2},\ 0 \le y \le x,\ \rho = k$
22. $x^2 + y^2 = a^2,\ 0 \le x,\ 0 \le y,\ \rho = k(x^2 + y^2)$

CAS **In Exercises 23–26, use a computer algebra system to find the mass and center of mass of the lamina bounded by the graphs of the equations for the given density.**

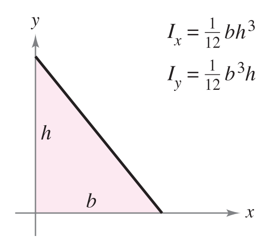
23. $y = e^{-x},\ y = 0,\ x = 0,\ x = 2,\ \rho = kxy$
24. $y = \ln x,\ y = 0,\ x = 1,\ x = e,\ \rho = k/x$
25. $r = 2\cos 3\theta,\ -\pi/6 \le \theta \le \pi/6,\ \rho = k$
26. $r = 1 + \cos\theta,\ \rho = k$

In Exercises 27–32, verify the given moment(s) of inertia and find $\bar{\bar{x}}$ and $\bar{\bar{y}}$. Assume that each lamina has a density of $\rho = 1$ gram per square centimeter. (These regions are common shapes used in engineering.)

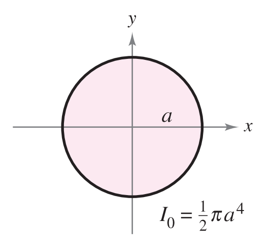
27. Rectangle



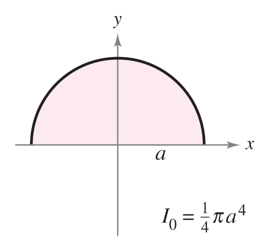
28. Right triangle



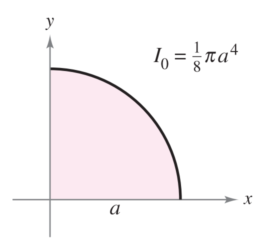
29. Circle



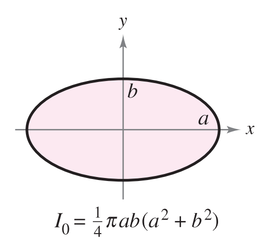
30. Semicircle



31. Quarter circle



32. Ellipse



CAS **In Exercises 33–40, find $I_x, I_y, I_0, \bar{\bar{x}}$, and $\bar{\bar{y}}$ for the lamina bounded by the graphs of the equations. Use a computer algebra system to evaluate the double integrals.**

33. $y = 0,\ y = b,\ x = 0,\ x = a,\ \rho = ky$
34. $y = \sqrt{a^2 - x^2},\ y = 0,\ \rho = ky$
35. $y = 4 - x^2,\ y = 0,\ x > 0,\ \rho = kx$
36. $y = x,\ y = x^2,\ \rho = kxy$
37. $y = \sqrt{x},\ y = 0,\ x = 4,\ \rho = kxy$
38. $y = x^2,\ y^2 = x,\ \rho = x^2 + y^2$
39. $y = x^2,\ y^2 = x,\ \rho = kx$
40. $y = x^3,\ y = 4x,\ \rho = k|y|$

CAS **In Exercises 41–46, set up the double integral required to find the moment of inertia I, about the given line, of the lamina bounded by the graphs of the equations. Use a computer algebra system to evaluate the double integral.**

41. $x^2 + y^2 = b^2,\ \rho = k,$ line: $x = a\ (a > b)$

42. $y = 0,\ y = 2,\ x = 0,\ x = 4,\ \rho = k,$ line: $x = 6$

43. $y = \sqrt{x},\ y = 0,\ x = 4,\ \rho = kx,$ line: $x = 6$

44. $y = \sqrt{a^2 - x^2},\ y = 0,\ \rho = ky,$ line: $y = a$

45. $y = \sqrt{a^2 - x^2},\ y = 0,\ x \geq 0,\ \rho = k(a - y),$ line: $y = a$

46. $y = 4 - x^2,\ y = 0,\ \rho = k,$ line: $y = 2$

WRITING ABOUT CONCEPTS

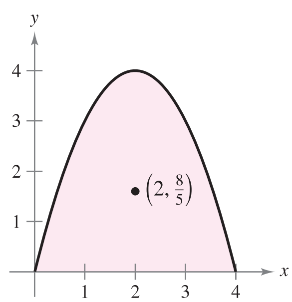
47. Give the formulas for finding the moments and center of mass of a variable density planar lamina.

48. Give the formulas for finding the moments of inertia about the x- and y-axes for a variable density planar lamina.

49. In your own words, describe what the radius of gyration measures.

CAPSTONE

50. The center of mass of the lamina of constant density shown in the figure is $\left(2, \frac{8}{5}\right)$. Make a conjecture about how the center of mass $(\bar{x}, \bar{y})$ changes for each given nonconstant density $\rho(x, y)$. Explain. (Make your conjecture *without* performing any calculations.)



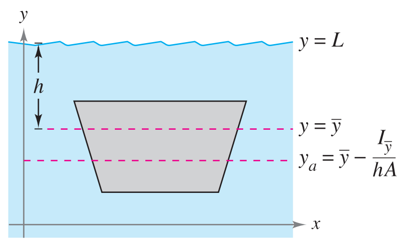
(a) $\rho(x, y) = ky$ (b) $\rho(x, y) = k|2 - x|$

(c) $\rho(x, y) = kxy$ (d) $\rho(x, y) = k(4 - x)(4 - y)$

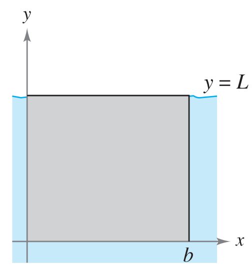
Hydraulics **In Exercises 51–54, determine the location of the horizontal axis y_a at which a vertical gate in a dam is to be hinged so that there is no moment causing rotation under the indicated loading (see figure). The model for y_a is**

$$y_a = \bar{y} - \frac{I_{\bar{y}}}{hA}$$

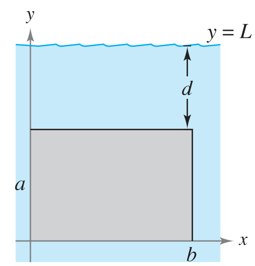
where $\bar{y}$ is the y-coordinate of the centroid of the gate, $I_{\bar{y}}$ is the moment of inertia of the gate about the line $y = \bar{y}$, h is the depth of the centroid below the surface, and A is the area of the gate.



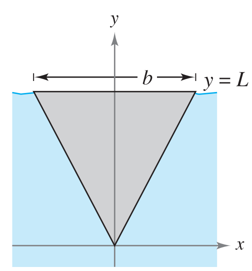
51.



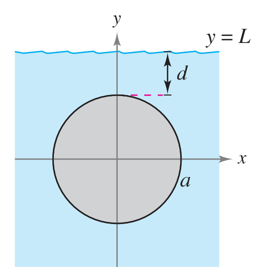
52.



53.



54.



55. Prove the following Theorem of Pappus: Let R be a region in a plane and let L be a line in the same plane such that L does not intersect the interior of R. If r is the distance between the centroid of R and the line, then the volume V of the solid of revolution formed by revolving R about the line is given by $V = 2\pi rA$, where A is the area of R.

SECTION PROJECT

Center of Pressure on a Sail

The center of pressure on a sail is that point (x_p, y_p) at which the total aerodynamic force may be assumed to act. If the sail is represented by a plane region R, the center of pressure is

$$x_p = \frac{\int_R\int xy\,dA}{\int_R\int y\,dA} \quad \text{and} \quad y_p = \frac{\int_R\int y^2\,dA}{\int_R\int y\,dA}.$$

Consider a triangular sail with vertices at $(0, 0)$, $(2, 1)$, and $(0, 5)$. Verify the value of each integral.

(a) $\displaystyle\int_R\int y\,dA = 10$ (b) $\displaystyle\int_R\int xy\,dA = \frac{35}{6}$ (c) $\displaystyle\int_R\int y^2\,dA = \frac{155}{6}$

Calculate the coordinates (x_p, y_p) of the center of pressure. Sketch a graph of the sail and indicate the location of the center of pressure.

14.5 Surface Area

- **Use a double integral to find the area of a surface.**

Surface Area

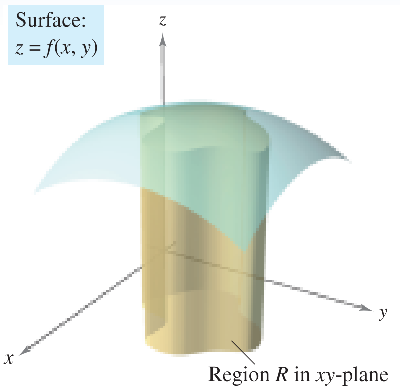


Figure 14.43

At this point you know a great deal about the solid region lying between a surface and a closed and bounded region R in the xy-plane, as shown in Figure 14.43. For example, you know how to find the extrema of f on R (Section 13.8), the area of the base R of the solid (Section 14.1), the volume of the solid (Section 14.2), and the centroid of the base R (Section 14.4).

In this section, you will learn how to find the upper **surface area** of the solid. Later, you will learn how to find the centroid of the solid (Section 14.6) and the lateral surface area (Section 15.2).

To begin, consider a surface S given by

$$z = f(x, y) \qquad \text{Surface defined over a region } R$$

defined over a region R. Assume that R is closed and bounded and that f has continuous first partial derivatives. To find the surface area, construct an inner partition of R consisting of n rectangles, where the area of the ith rectangle R_i is $\Delta A_i = \Delta x_i \, \Delta y_i$, as shown in Figure 14.44. In each R_i let (x_i, y_i) be the point that is closest to the origin. At the point $(x_i, y_i, z_i) = (x_i, y_i, f(x_i, y_i))$ on the surface S, construct a tangent plane T_i. The area of the portion of the tangent plane that lies directly above R_i is approximately equal to the area of the surface lying directly above R_i. That is, $\Delta T_i \approx \Delta S_i$. So, the surface area of S is given by

$$\sum_{i=1}^{n} \Delta S_i \approx \sum_{i=1}^{n} \Delta T_i.$$

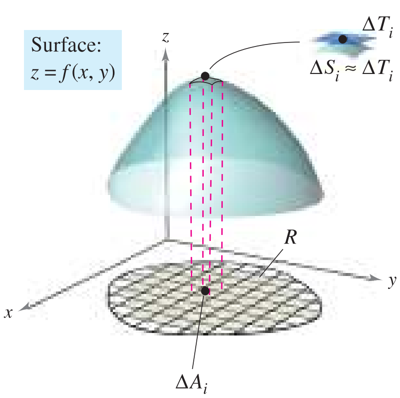


Figure 14.44

To find the area of the parallelogram ΔT_i, note that its sides are given by the vectors

$$\mathbf{u} = \Delta x_i \mathbf{i} + f_x(x_i, y_i)\, \Delta x_i \mathbf{k}$$

and

$$\mathbf{v} = \Delta y_i \mathbf{j} + f_y(x_i, y_i)\, \Delta y_i \mathbf{k}.$$

From Theorem 11.8, the area of ΔT_i is given by $\|\mathbf{u} \times \mathbf{v}\|$, where

$$\begin{aligned}
\mathbf{u} \times \mathbf{v} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \Delta x_i & 0 & f_x(x_i, y_i)\, \Delta x_i \\ 0 & \Delta y_i & f_y(x_i, y_i)\, \Delta y_i \end{vmatrix} \\
&= -f_x(x_i, y_i)\, \Delta x_i \Delta y_i \mathbf{i} - f_y(x_i, y_i)\, \Delta x_i \Delta y_i \mathbf{j} + \Delta x_i \Delta y_i \mathbf{k} \\
&= (-f_x(x_i, y_i)\mathbf{i} - f_y(x_i, y_i)\mathbf{j} + \mathbf{k})\, \Delta A_i.
\end{aligned}$$

So, the area of ΔT_i is $\|\mathbf{u} \times \mathbf{v}\| = \sqrt{[f_x(x_i, y_i)]^2 + [f_y(x_i, y_i)]^2 + 1}\, \Delta A_i$, and

$$\begin{aligned}
\text{Surface area of } S &\approx \sum_{i=1}^{n} \Delta S_i \\
&\approx \sum_{i=1}^{n} \sqrt{1 + [f_x(x_i, y_i)]^2 + [f_y(x_i, y_i)]^2}\, \Delta A_i.
\end{aligned}$$

This suggests the following definition of surface area.

DEFINITION OF SURFACE AREA

If f and its first partial derivatives are continuous on the closed region R in the xy-plane, then the **area of the surface S** given by $z = f(x, y)$ over R is defined as

$$\text{Surface area} = \int_R\int dS$$
$$= \int_R\int \sqrt{1 + [f_x(x, y)]^2 + [f_y(x, y)]^2}\, dA.$$

As an aid to remembering the double integral for surface area, it is helpful to note its similarity to the integral for arc length.

Length on x-axis: $\displaystyle\int_a^b dx$

Arc length in xy-plane: $\displaystyle\int_a^b ds = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$

Area in xy-plane: $\displaystyle\int_R\int dA$

Surface area in space: $\displaystyle\int_R\int dS = \int_R\int \sqrt{1 + [f_x(x, y)]^2 + [f_y(x, y)]^2}\, dA$

Like integrals for arc length, integrals for surface area are often very difficult to evaluate. However, one type that is easily evaluated is demonstrated in the next example.

EXAMPLE 1 The Surface Area of a Plane Region

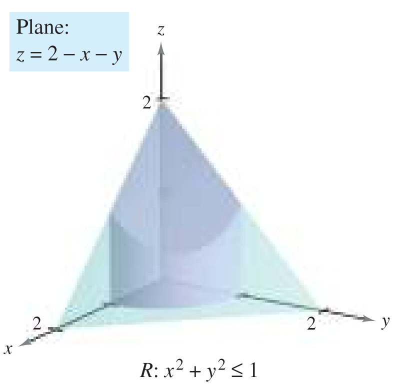


Figure 14.45

Find the surface area of the portion of the plane

$$z = 2 - x - y$$

that lies above the circle $x^2 + y^2 \le 1$ in the first quadrant, as shown in Figure 14.45.

Solution Because $f_x(x, y) = -1$ and $f_y(x, y) = -1$, the surface area is given by

$$S = \int_R\int \sqrt{1 + [f_x(x, y)]^2 + [f_y(x, y)]^2}\, dA \qquad \text{Formula for surface area}$$
$$= \int_R\int \sqrt{1 + (-1)^2 + (-1)^2}\, dA \qquad \text{Substitute.}$$
$$= \int_R\int \sqrt{3}\, dA$$
$$= \sqrt{3}\int_R\int dA.$$

Note that the last integral is simply $\sqrt{3}$ times the area of the region R. R is a quarter circle of radius 1, with an area of $\frac{1}{4}\pi(1^2)$ or $\pi/4$. So, the area of S is

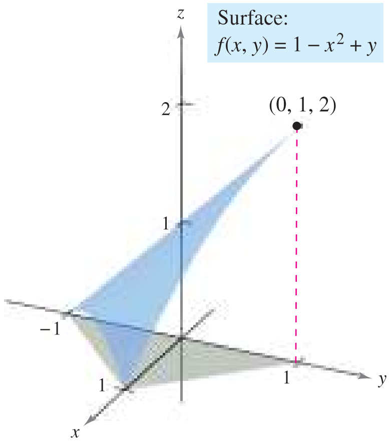
$$S = \sqrt{3}\,(\text{area of } R)$$
$$= \sqrt{3}\left(\frac{\pi}{4}\right)$$
$$= \frac{\sqrt{3}\,\pi}{4}.$$

EXAMPLE 2 Finding Surface Area

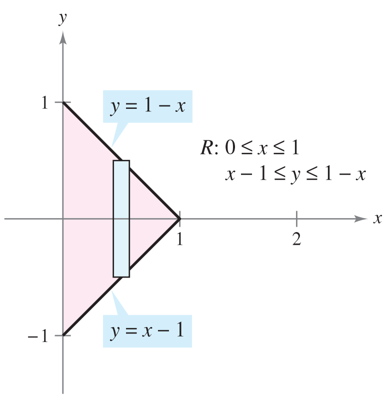
Find the area of the portion of the surface

$$f(x, y) = 1 - x^2 + y$$

that lies above the triangular region with vertices $(1, 0, 0)$, $(0, -1, 0)$, and $(0, 1, 0)$, as shown in Figure 14.46(a).



(a)



(b)

Figure 14.46

Solution Because $f_x(x, y) = -2x$ and $f_y(x, y) = 1$, you have

$$S = \iint_R \sqrt{1 + [f_x(x, y)]^2 + [f_y(x, y)]^2}\, dA = \iint_R \sqrt{1 + 4x^2 + 1}\, dA.$$

In Figure 14.46(b), you can see that the bounds for R are $0 \le x \le 1$ and $x - 1 \le y \le 1 - x$. So, the integral becomes

$$\begin{aligned}
S &= \int_0^1 \int_{x-1}^{1-x} \sqrt{2 + 4x^2}\, dy\, dx \\
&= \int_0^1 y\sqrt{2 + 4x^2}\Big]_{x-1}^{1-x}\, dx \\
&= \int_0^1 \Big[(1 - x)\sqrt{2 + 4x^2} - (x - 1)\sqrt{2 + 4x^2}\Big]\, dx \\
&= \int_0^1 \left(2\sqrt{2 + 4x^2} - 2x\sqrt{2 + 4x^2}\right) dx \qquad \text{Integration tables (Appendix B), Formula 26 and Power Rule} \\
&= \left[x\sqrt{2 + 4x^2} + \ln\left(2x + \sqrt{2 + 4x^2}\right) - \frac{(2 + 4x^2)^{3/2}}{6}\right]_0^1 \\
&= \sqrt{6} + \ln\left(2 + \sqrt{6}\right) - \sqrt{6} - \ln\sqrt{2} + \frac{1}{3}\sqrt{2} \approx 1.618.
\end{aligned}$$

EXAMPLE 3 Change of Variables to Polar Coordinates

Find the surface area of the paraboloid $z = 1 + x^2 + y^2$ that lies above the unit circle, as shown in Figure 14.47.

Solution Because $f_x(x, y) = 2x$ and $f_y(x, y) = 2y$, you have

$$S = \iint_R \sqrt{1 + [f_x(x, y)]^2 + [f_y(x, y)]^2}\, dA = \iint_R \sqrt{1 + 4x^2 + 4y^2}\, dA.$$

You can convert to polar coordinates by letting $x = r\cos\theta$ and $y = r\sin\theta$. Then, because the region R is bounded by $0 \le r \le 1$ and $0 \le \theta \le 2\pi$, you have

$$\begin{aligned}
S &= \int_0^{2\pi} \int_0^1 \sqrt{1 + 4r^2}\, r\, dr\, d\theta \\
&= \int_0^{2\pi} \frac{1}{12}(1 + 4r^2)^{3/2}\Big]_0^1\, d\theta \\
&= \int_0^{2\pi} \frac{5\sqrt{5} - 1}{12}\, d\theta \\
&= \frac{5\sqrt{5} - 1}{12}\,\theta\Big]_0^{2\pi} \\
&= \frac{\pi\left(5\sqrt{5} - 1\right)}{6} \\
&\approx 5.33.
\end{aligned}$$

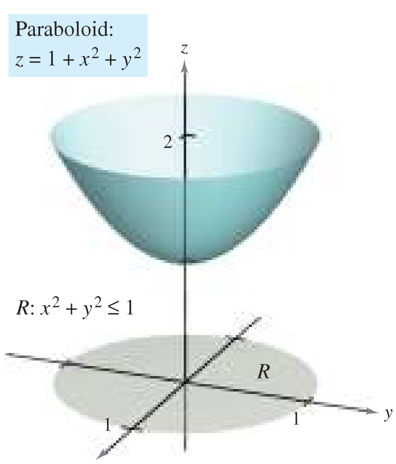


Figure 14.47

EXAMPLE 4 Finding Surface Area

Find the surface area S of the portion of the hemisphere

$$f(x, y) = \sqrt{25 - x^2 - y^2} \qquad \text{Hemisphere}$$

that lies above the region R bounded by the circle $x^2 + y^2 \leq 9$, as shown in Figure 14.48.

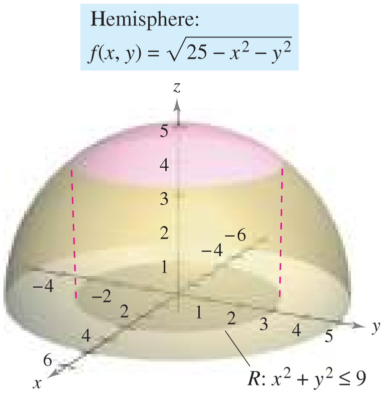


Figure 14.48

Solution The first partial derivatives of f are

$$f_x(x, y) = \frac{-x}{\sqrt{25 - x^2 - y^2}} \quad \text{and} \quad f_y(x, y) = \frac{-y}{\sqrt{25 - x^2 - y^2}}$$

and, from the formula for surface area, you have

$$\begin{aligned} dS &= \sqrt{1 + [f_x(x, y)]^2 + [f_y(x, y)]^2}\, dA \\ &= \sqrt{1 + \left(\frac{-x}{\sqrt{25 - x^2 - y^2}}\right)^2 + \left(\frac{-y}{\sqrt{25 - x^2 - y^2}}\right)^2}\, dA \\ &= \frac{5}{\sqrt{25 - x^2 - y^2}}\, dA. \end{aligned}$$

So, the surface area is

$$S = \iint_R \frac{5}{\sqrt{25 - x^2 - y^2}}\, dA.$$

You can convert to polar coordinates by letting $x = r\cos\theta$ and $y = r\sin\theta$. Then, because the region R is bounded by $0 \leq r \leq 3$ and $0 \leq \theta \leq 2\pi$, you obtain

$$\begin{aligned} S &= \int_0^{2\pi}\int_0^3 \frac{5}{\sqrt{25 - r^2}}\, r\, dr\, d\theta \\ &= 5\int_0^{2\pi} -\sqrt{25 - r^2}\,\Big]_0^3\, d\theta \\ &= 5\int_0^{2\pi} d\theta \\ &= 10\pi. \end{aligned}$$

The procedure used in Example 4 can be extended to find the surface area of a sphere by using the region R bounded by the circle $x^2 + y^2 \leq a^2$, where $0 < a < 5$, as shown in Figure 14.49. The surface area of the portion of the hemisphere

$$f(x, y) = \sqrt{25 - x^2 - y^2}$$

lying above the circular region can be shown to be

$$\begin{aligned} S &= \iint_R \frac{5}{\sqrt{25 - x^2 - y^2}}\, dA \\ &= \int_0^{2\pi}\int_0^a \frac{5}{\sqrt{25 - r^2}}\, r\, dr\, d\theta \\ &= 10\pi\left(5 - \sqrt{25 - a^2}\right). \end{aligned}$$

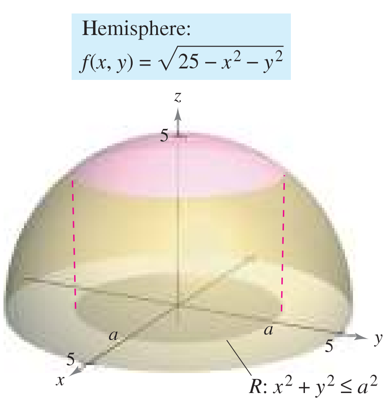


Figure 14.49

By taking the limit as a approaches 5 and doubling the result, you obtain a total area of 100π. (The surface area of a sphere of radius r is $S = 4\pi r^2$.)

You can use Simpson's Rule or the Trapezoidal Rule to approximate the value of a double integral, *provided* you can get through the first integration. This is demonstrated in the next example.

EXAMPLE 5 Approximating Surface Area by Simpson's Rule

Find the area of the surface of the paraboloid

$$f(x, y) = 2 - x^2 - y^2 \qquad \text{Paraboloid}$$

that lies above the square region bounded by $-1 \le x \le 1$ and $-1 \le y \le 1$, as shown in Figure 14.50.

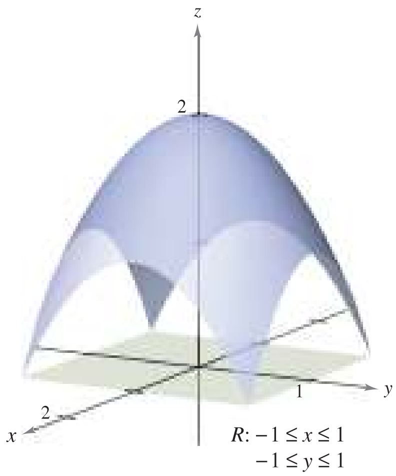


Figure 14.50

Solution Using the partial derivatives

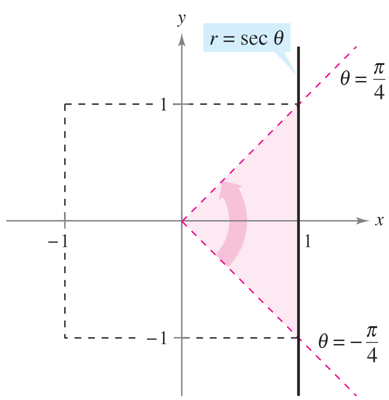
$$f_x(x, y) = -2x \quad \text{and} \quad f_y(x, y) = -2y$$

you have a surface area of

$$\begin{aligned} S &= \int_R\int \sqrt{1 + [f_x(x, y)]^2 + [f_y(x, y)]^2}\, dA \\ &= \int_R\int \sqrt{1 + (-2x)^2 + (-2y)^2}\, dA \\ &= \int_R\int \sqrt{1 + 4x^2 + 4y^2}\, dA. \end{aligned}$$

In polar coordinates, the line $x = 1$ is given by $r \cos \theta = 1$ or $r = \sec \theta$, and you can determine from Figure 14.51 that one-fourth of the region R is bounded by

$$0 \le r \le \sec \theta \quad \text{and} \quad -\frac{\pi}{4} \le \theta \le \frac{\pi}{4}.$$



One-fourth of the region R is bounded by $0 \le r \le \sec \theta$ and $-\frac{\pi}{4} \le \theta \le \frac{\pi}{4}$.

Figure 14.51

Letting $x = r \cos \theta$ and $y = r \sin \theta$ produces

$$\begin{aligned} \frac{1}{4}S &= \frac{1}{4}\int_R\int \sqrt{1 + 4x^2 + 4y^2}\, dA \\ &= \int_{-\pi/4}^{\pi/4}\int_0^{\sec\theta} \sqrt{1 + 4r^2}\, r\, dr\, d\theta \\ &= \int_{-\pi/4}^{\pi/4} \frac{1}{12}(1 + 4r^2)^{3/2}\Big]_0^{\sec\theta} d\theta \\ &= \frac{1}{12}\int_{-\pi/4}^{\pi/4} [(1 + 4\sec^2\theta)^{3/2} - 1]\, d\theta. \end{aligned}$$

Finally, using Simpson's Rule with $n = 10$, you can approximate this single integral to be

$$\begin{aligned} S &= \frac{1}{3}\int_{-\pi/4}^{\pi/4} [(1 + 4\sec^2\theta)^{3/2} - 1]\, d\theta \\ &\approx 7.450. \end{aligned}$$

■

TECHNOLOGY Most computer programs that are capable of performing symbolic integration for multiple integrals are also capable of performing numerical approximation techniques. If you have access to such software, use it to approximate the value of the integral in Example 5.

14.5 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–14, find the area of the surface given by $z = f(x, y)$ over the region R. (*Hint:* Some of the integrals are simpler in polar coordinates.)

1. $f(x, y) = 2x + 2y$

R: triangle with vertices $(0, 0)$, $(4, 0)$, $(0, 4)$

2. $f(x, y) = 15 + 2x - 3y$

R: square with vertices $(0, 0)$, $(3, 0)$, $(0, 3)$, $(3, 3)$

3. $f(x, y) = 7 + 2x + 2y$

$R = \{(x, y): x^2 + y^2 \le 4\}$

4. $f(x, y) = 12 + 2x - 3y$

$R = \{(x, y): x^2 + y^2 \le 9\}$

5. $f(x, y) = 9 - x^2$

R: square with vertices $(0, 0)$, $(2, 0)$, $(0, 2)$, $(2, 2)$

6. $f(x, y) = y^2$

R: square with vertices $(0, 0)$, $(3, 0)$, $(0, 3)$, $(3, 3)$

7. $f(x, y) = 3 + x^{3/2}$

R: rectangle with vertices $(0, 0)$, $(0, 4)$, $(3, 4)$, $(3, 0)$

8. $f(x, y) = 2 + \frac{2}{3}y^{3/2}$

$R = \{(x, y): 0 \le x \le 2, 0 \le y \le 2 - x\}$

9. $f(x, y) = \ln|\sec x|$

$R = \left\{(x, y): 0 \le x \le \dfrac{\pi}{4}, 0 \le y \le \tan x\right\}$

10. $f(x, y) = 13 + x^2 - y^2$

$R = \{(x, y): x^2 + y^2 \le 4\}$

11. $f(x, y) = \sqrt{x^2 + y^2}$, $R = \{(x, y): 0 \le f(x, y) \le 1\}$

12. $f(x, y) = xy$, $R = \{(x, y): x^2 + y^2 \le 16\}$

13. $f(x, y) = \sqrt{a^2 - x^2 - y^2}$

$R = \{(x, y): x^2 + y^2 \le b^2, 0 < b < a\}$

14. $f(x, y) = \sqrt{a^2 - x^2 - y^2}$

$R = \{(x, y): x^2 + y^2 \le a^2\}$

In Exercises 15–18, find the area of the surface.

15. The portion of the plane $z = 24 - 3x - 2y$ in the first octant

16. The portion of the paraboloid $z = 16 - x^2 - y^2$ in the first octant

17. The portion of the sphere $x^2 + y^2 + z^2 = 25$ inside the cylinder $x^2 + y^2 = 9$

18. The portion of the cone $z = 2\sqrt{x^2 + y^2}$ inside the cylinder $x^2 + y^2 = 4$

CAS **In Exercises 19–24, write a double integral that represents the surface area of $z = f(x, y)$ over the region R. Use a computer algebra system to evaluate the double integral.**

19. $f(x, y) = 2y + x^2$

R: triangle with vertices $(0, 0)$, $(1, 0)$, $(1, 1)$

20. $f(x, y) = 2x + y^2$

R: triangle with vertices $(0, 0)$, $(2, 0)$, $(2, 2)$

21. $f(x, y) = 9 - x^2 - y^2$

$R = \{(x, y): 0 \le f(x, y)\}$

22. $f(x, y) = x^2 + y^2$

$R = \{(x, y): 0 \le f(x, y) \le 16\}$

23. $f(x, y) = 4 - x^2 - y^2$

$R = \{(x, y): 0 \le x \le 1, 0 \le y \le 1\}$

24. $f(x, y) = \frac{2}{3}x^{3/2} + \cos x$

$R = \{(x, y): 0 \le x \le 1, 0 \le y \le 1\}$

Approximation **In Exercises 25 and 26, determine which value best approximates the surface area of $z = f(x, y)$ over the region R. (Make your selection on the basis of a sketch of the surface and *not* by performing any calculations.)**

25. $f(x, y) = 10 - \frac{1}{2}y^2$

R: square with vertices $(0, 0)$, $(4, 0)$, $(4, 4)$, $(0, 4)$

(a) 16 (b) 200 (c) -100 (d) 72 (e) 36

26. $f(x, y) = \frac{1}{4}\sqrt{x^2 + y^2}$

R: circle bounded by $x^2 + y^2 = 9$

(a) -100 (b) 150 (c) 9π (d) 55 (e) 500

CAS **In Exercises 27 and 28, use a computer algebra system to approximate the double integral that gives the surface area of the graph of f over the region $R = \{(x, y): 0 \le x \le 1, 0 \le y \le 1\}$.**

27. $f(x, y) = e^x$

28. $f(x, y) = \frac{2}{5}y^{5/2}$

In Exercises 29–34, set up a double integral that gives the area of the surface on the graph of f over the region R.

29. $f(x, y) = x^3 - 3xy + y^3$

R: square with vertices $(1, 1)$, $(-1, 1)$, $(-1, -1)$, $(1, -1)$

30. $f(x, y) = x^2 - 3xy - y^2$

$R = \{(x, y): 0 \le x \le 4, 0 \le y \le x\}$

31. $f(x, y) = e^{-x} \sin y$

$R = \{(x, y): x^2 + y^2 \le 4\}$

32. $f(x, y) = \cos(x^2 + y^2)$

$R = \left\{(x, y): x^2 + y^2 \le \dfrac{\pi}{2}\right\}$

33. $f(x, y) = e^{xy}$

$R = \{(x, y): 0 \le x \le 4, 0 \le y \le 10\}$

34. $f(x, y) = e^{-x} \sin y$

$R = \{(x, y): 0 \le x \le 4, 0 \le y \le x\}$

WRITING ABOUT CONCEPTS

35. State the double integral definition of the area of a surface S given by $z = f(x, y)$ over the region R in the xy-plane.

36. Consider the surface $f(x, y) = x^2 + y^2$ and the surface area of f over each region R. Without integrating, order the surface areas from least to greatest. Explain your reasoning.

(a) R: rectangle with vertices $(0, 0)$, $(2, 0)$, $(2, 2)$, $(0, 2)$

(b) R: triangle with vertices $(0, 0)$, $(2, 0)$, $(0, 2)$

(c) $R = \{(x, y): x^2 + y^2 \le 4$, first quadrant only$\}$

37. Will the surface area of the graph of a function $z = f(x, y)$ over a region R increase if the graph is shifted k units vertically? Why or why not?

CAPSTONE

38. Answer the following questions about the surface area S on a surface given by a positive function $z = f(x, y)$ over a region R in the xy-plane. Explain each answer.

(a) Is it possible for S to equal the area of R?

(b) Can S be greater than the area of R?

(c) Can S be less than the area of R?

39. Find the surface area of the solid of intersection of the cylinders $x^2 + z^2 = 1$ and $y^2 + z^2 = 1$ (see figure).

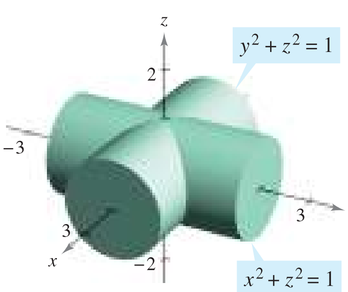


Figure for 39

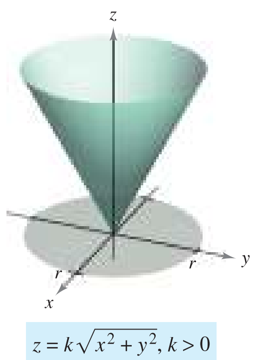
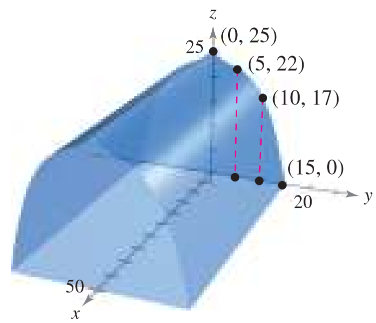


Figure for 40

40. Show that the surface area of the cone $z = k\sqrt{x^2 + y^2}$, $k > 0$, over the circular region $x^2 + y^2 \le r^2$ in the xy-plane is $\pi r^2\sqrt{k^2 + 1}$ (see figure).

41. ***Product Design*** A company produces a spherical object of radius 25 centimeters. A hole of radius 4 centimeters is drilled through the center of the object. Find (a) the volume of the object and (b) the outer surface area of the object.

42. ***Modeling Data*** A rancher builds a barn with dimensions 30 feet by 50 feet. The symmetrical shape and selected heights of the roof are shown in the figure.



(a) Use the regression capabilities of a graphing utility to find a model of the form $z = ay^3 + by^2 + cy + d$ for the roof line.

(b) Use the numerical integration capabilities of a graphing utility and the model in part (a) to approximate the volume of storage space in the barn.

(c) Use the numerical integration capabilities of a graphing utility and the model in part (a) to approximate the surface area of the roof.

(d) Approximate the arc length of the roof line and find the surface area of the roof by multiplying the arc length by the length of the barn. Compare the results and the integrations with those found in part (c).

SECTION PROJECT

Capillary Action

A well-known property of liquids is that they will rise in narrow vertical channels—this property is called "capillary action." The figure shows two plates, which form a narrow wedge, in a container of liquid. The upper surface of the liquid follows a hyperbolic shape given by

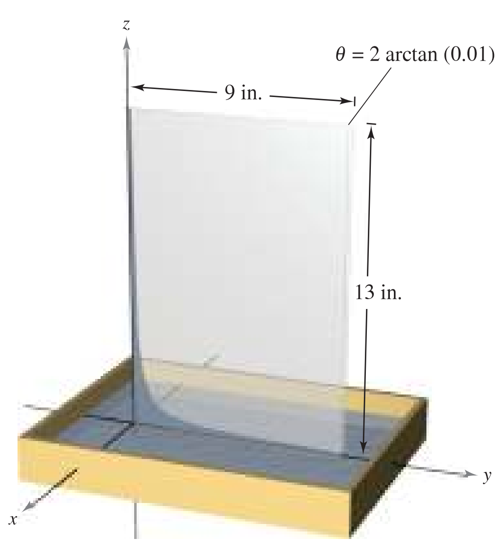
$$z = \frac{k}{\sqrt{x^2 + y^2}}$$

where x, y, and z are measured in inches. The constant k depends on the angle of the wedge, the type of liquid, and the material that comprises the flat plates.

(a) Find the volume of the liquid that has risen in the wedge. (Assume $k = 1$.)

(b) Find the horizontal surface area of the liquid that has risen in the wedge.

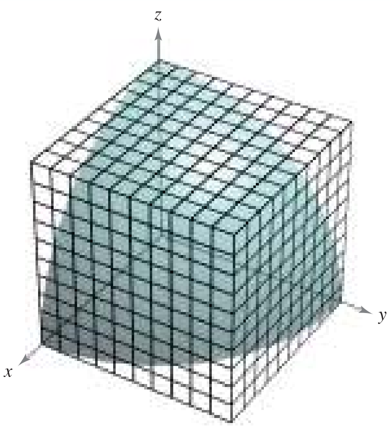
Adaptation of Capillary Action problem from "Capillary Phenomena" by Thomas B. Greenslade, Jr., *Physics Teacher*, May 1992. By permission of the author.



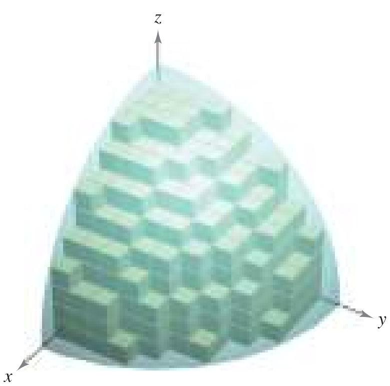
14.6 Triple Integrals and Applications

- **Use a triple integral to find the volume of a solid region.**
- **Find the center of mass and moments of inertia of a solid region.**

Triple Integrals



Solid region Q



Volume of $Q \approx \sum_{i=1}^{n} \Delta V_i$

Figure 14.52

The procedure used to define a **triple integral** follows that used for double integrals. Consider a function f of three variables that is continuous over a bounded solid region Q. Then, encompass Q with a network of boxes and form the **inner partition** consisting of all boxes lying entirely within Q, as shown in Figure 14.52. The volume of the ith box is

$$\Delta V_i = \Delta x_i \Delta y_i \Delta z_i. \qquad \text{Volume of } i\text{th box}$$

The **norm** $\|\Delta\|$ of the partition is the length of the longest diagonal of the n boxes in the partition. Choose a point (x_i, y_i, z_i) in each box and form the Riemann sum

$$\sum_{i=1}^{n} f(x_i, y_i, z_i)\,\Delta V_i.$$

Taking the limit as $\|\Delta\| \to 0$ leads to the following definition.

DEFINITION OF TRIPLE INTEGRAL

If f is continuous over a bounded solid region Q, then the **triple integral of f over Q** is defined as

$$\iiint_Q f(x, y, z)\,dV = \lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} f(x_i, y_i, z_i)\,\Delta V_i$$

provided the limit exists. The **volume** of the solid region Q is given by

$$\text{Volume of } Q = \iiint_Q dV.$$

Some of the properties of double integrals in Theorem 14.1 can be restated in terms of triple integrals.

1. $\displaystyle\iiint_Q cf(x, y, z)\,dV = c\iiint_Q f(x, y, z)\,dV$

2. $\displaystyle\iiint_Q [f(x, y, z) \pm g(x, y, z)]\,dV = \iiint_Q f(x, y, z)\,dV \pm \iiint_Q g(x, y, z)\,dV$

3. $\displaystyle\iiint_Q f(x, y, z)\,dV = \iiint_{Q_1} f(x, y, z)\,dV + \iiint_{Q_2} f(x, y, z)\,dV$

In the properties above, Q is the union of two nonoverlapping solid subregions Q_1 and Q_2. If the solid region Q is simple, the triple integral $\iiint f(x, y, z)\,dV$ can be evaluated with an iterated integral using one of the six possible orders of integration:

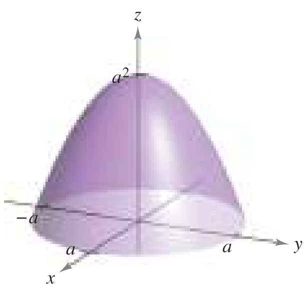
$$dx\,dy\,dz \quad dy\,dx\,dz \quad dz\,dx\,dy \quad dx\,dz\,dy \quad dy\,dz\,dx \quad dz\,dy\,dx.$$

EXPLORATION

Volume of a Paraboloid Sector
On pages 997 and 1006, you were asked to summarize the different ways you know of finding the volume of the solid bounded by the paraboloid

$$z = a^2 - x^2 - y^2, \quad a > 0$$

and the xy-plane. You now know one more way. Use it to find the volume of the solid.



The following version of Fubini's Theorem describes a region that is considered simple with respect to the order $dz\,dy\,dx$. Similar descriptions can be given for the other five orders.

THEOREM 14.4 EVALUATION BY ITERATED INTEGRALS

Let f be continuous on a solid region Q defined by

$$a \le x \le b, \quad h_1(x) \le y \le h_2(x), \quad g_1(x, y) \le z \le g_2(x, y)$$

where h_1, h_2, g_1, and g_2 are continuous functions. Then,

$$\iiint_Q f(x, y, z)\,dV = \int_a^b \int_{h_1(x)}^{h_2(x)} \int_{g_1(x, y)}^{g_2(x, y)} f(x, y, z)\,dz\,dy\,dx.$$

To evaluate a triple iterated integral in the order $dz\,dy\,dx$, hold *both* x and y constant for the innermost integration. Then, hold x constant for the second integration.

EXAMPLE 1 Evaluating a Triple Iterated Integral

Evaluate the triple iterated integral

$$\int_0^2 \int_0^x \int_0^{x+y} e^x(y + 2z)\,dz\,dy\,dx.$$

Solution For the first integration, hold x and y constant and integrate with respect to z.

$$\begin{aligned}\int_0^2 \int_0^x \int_0^{x+y} e^x(y + 2z)\,dz\,dy\,dx &= \int_0^2 \int_0^x e^x(yz + z^2)\Big]_0^{x+y} dy\,dx\\ &= \int_0^2 \int_0^x e^x(x^2 + 3xy + 2y^2)\,dy\,dx\end{aligned}$$

For the second integration, hold x constant and integrate with respect to y.

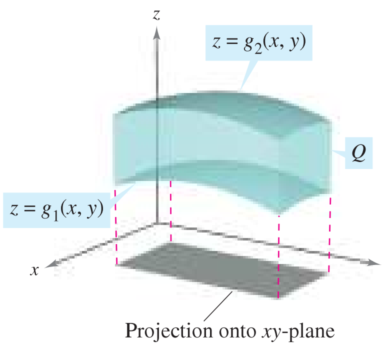
$$\begin{aligned}\int_0^2 \int_0^x e^x(x^2 + 3xy + 2y^2)\,dy\,dx &= \int_0^2 \left[e^x\left(x^2y + \frac{3xy^2}{2} + \frac{2y^3}{3}\right)\right]_0^x dx\\ &= \frac{19}{6}\int_0^2 x^3e^x\,dx\end{aligned}$$

Finally, integrate with respect to x.

$$\begin{aligned}\frac{19}{6}\int_0^2 x^3e^x\,dx &= \frac{19}{6}\Big[e^x(x^3 - 3x^2 + 6x - 6)\Big]_0^2\\ &= 19\left(\frac{e^2}{3} + 1\right)\\ &\approx 65.797\end{aligned}$$

■

Example 1 demonstrates the integration order $dz\,dy\,dx$. For other orders, you can follow a similar procedure. For instance, to evaluate a triple iterated integral in the order $dx\,dy\,dz$, hold both y and z constant for the innermost integration and integrate with respect to x. Then, for the second integration, hold z constant and integrate with respect to y. Finally, for the third integration, integrate with respect to z.



Solid region Q lies between two surfaces.
Figure 14.53

To find the limits for a particular order of integration, it is generally advisable first to determine the innermost limits, which may be functions of the outer two variables. Then, by projecting the solid Q onto the coordinate plane of the outer two variables, you can determine their limits of integration by the methods used for double integrals. For instance, to evaluate

$$\iiint_Q f(x, y, z)\, dz\, dy\, dx$$

first determine the limits for z, and then the integral has the form

$$\iint \left[\int_{g_1(x, y)}^{g_2(x, y)} f(x, y, z)\, dz\right] dy\, dx.$$

By projecting the solid Q onto the xy-plane, you can determine the limits for x and y as you did for double integrals, as shown in Figure 14.53.

EXAMPLE 2 Using a Triple Integral to Find Volume

Find the volume of the ellipsoid given by $4x^2 + 4y^2 + z^2 = 16$.

Solution Because x, y, and z play similar roles in the equation, the order of integration is probably immaterial, and you can arbitrarily choose $dz\, dy\, dx$. Moreover, you can simplify the calculation by considering only the portion of the ellipsoid lying in the first octant, as shown in Figure 14.54. From the order $dz\, dy\, dx$, you first determine the bounds for z.

$$0 \le z \le 2\sqrt{4 - x^2 - y^2}$$

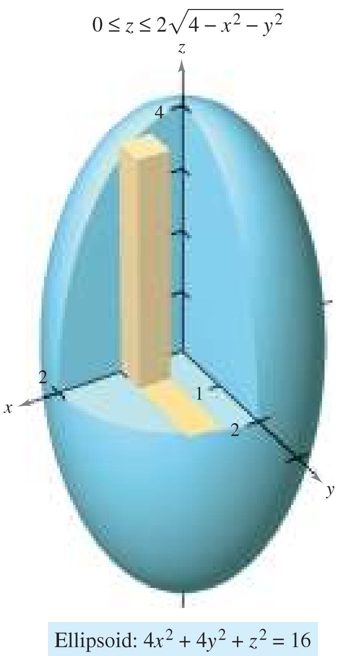


Figure 14.54

In Figure 14.55, you can see that the boundaries for x and y are $0 \le x \le 2$ and $0 \le y \le \sqrt{4 - x^2}$, so the volume of the ellipsoid is

$$\begin{aligned}
V &= \iiint_Q dV \\
&= 8\int_0^2 \int_0^{\sqrt{4-x^2}} \int_0^{2\sqrt{4-x^2-y^2}} dz\, dy\, dx \\
&= 8\int_0^2 \int_0^{\sqrt{4-x^2}} z\Big]_0^{2\sqrt{4-x^2-y^2}} dy\, dx \\
&= 16\int_0^2 \int_0^{\sqrt{4-x^2}} \sqrt{(4 - x^2) - y^2}\, dy\, dx \qquad \text{Integration tables (Appendix B), Formula 37} \\
&= 8\int_0^2 \left[y\sqrt{4 - x^2 - y^2} + (4 - x^2)\arcsin\left(\frac{y}{\sqrt{4 - x^2}}\right)\right]_0^{\sqrt{4-x^2}} dx \\
&= 8\int_0^2 [0 + (4 - x^2)\arcsin(1) - 0 - 0]\, dx \\
&= 8\int_0^2 (4 - x^2)\left(\frac{\pi}{2}\right) dx \\
&= 4\pi\left[4x - \frac{x^3}{3}\right]_0^2 \\
&= \frac{64\pi}{3}.
\end{aligned}$$

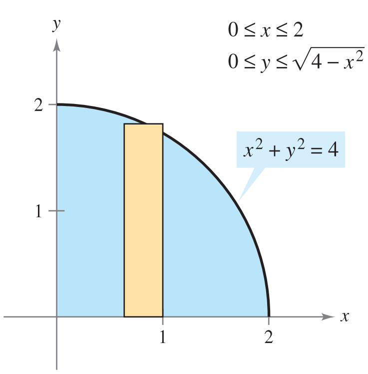


Figure 14.55

Example 2 is unusual in that all six possible orders of integration produce integrals of comparable difficulty. Try setting up some other possible orders of integration to find the volume of the ellipsoid. For instance, the order $dx\,dy\,dz$ yields the integral

$$V = 8\int_0^4 \int_0^{\sqrt{16-z^2}/2} \int_0^{\sqrt{16-4y^2-z^2}/2} dx\,dy\,dz.$$

If you solve this integral, you will obtain the same volume obtained in Example 2. This is always the case—the order of integration does not affect the value of the integral. However, the order of integration often does affect the complexity of the integral. In Example 3, the given order of integration is not convenient, so you can change the order to simplify the problem.

EXAMPLE 3 Changing the Order of Integration

Evaluate $\displaystyle\int_0^{\sqrt{\pi/2}} \int_x^{\sqrt{\pi/2}} \int_1^3 \sin(y^2)\,dz\,dy\,dx.$

Solution Note that after one integration in the given order, you would encounter the integral $2\int \sin(y^2)\,dy$, which is not an elementary function. To avoid this problem, change the order of integration to $dz\,dx\,dy$, so that y is the outer variable. From Figure 14.56, you can see that the solid region Q is given by

$$0 \le x \le \sqrt{\frac{\pi}{2}}, \quad x \le y \le \sqrt{\frac{\pi}{2}}, \quad 1 \le z \le 3$$

and the projection of Q in the xy-plane yields the bounds

$$0 \le y \le \sqrt{\frac{\pi}{2}} \quad \text{and} \quad 0 \le x \le y.$$

So, evaluating the triple integral using the order $dz\,dx\,dy$ produces

$$\begin{aligned}
\int_0^{\sqrt{\pi/2}} \int_0^y \int_1^3 \sin(y^2)\,dz\,dx\,dy &= \int_0^{\sqrt{\pi/2}} \int_0^y z\sin(y^2)\Big]_1^3\,dx\,dy \\
&= 2\int_0^{\sqrt{\pi/2}} \int_0^y \sin(y^2)\,dx\,dy \\
&= 2\int_0^{\sqrt{\pi/2}} x\sin(y^2)\Big]_0^y\,dy \\
&= 2\int_0^{\sqrt{\pi/2}} y\sin(y^2)\,dy \\
&= -\cos(y^2)\Big]_0^{\sqrt{\pi/2}} \\
&= 1.
\end{aligned}$$

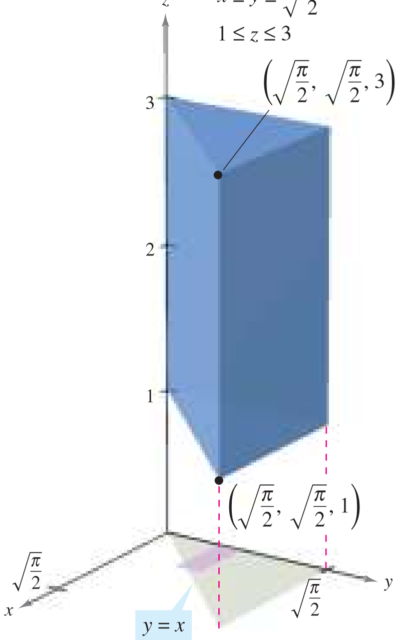
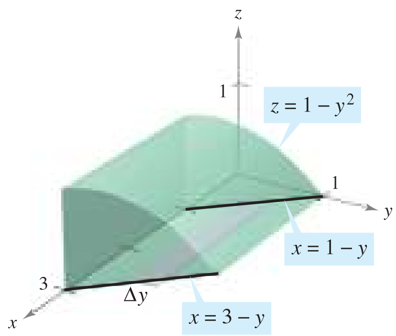
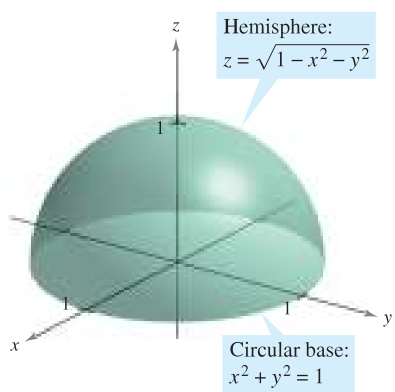


Figure 14.56



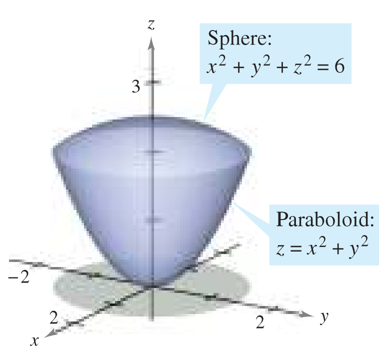
Q: $0 \le z \le 1 - y^2$
$1 - y \le x \le 3 - y$
$0 \le y \le 1$

Figure 14.57



Q: $0 \le z \le \sqrt{1 - x^2 - y^2}$
$-\sqrt{1 - y^2} \le x \le \sqrt{1 - y^2}$
$-1 \le y \le 1$

Figure 14.58



Q: $x^2 + y^2 \le z \le \sqrt{6 - x^2 - y^2}$
$-\sqrt{2 - x^2} \le y \le \sqrt{2 - x^2}$
$-\sqrt{2} \le x \le \sqrt{2}$

Figure 14.59

EXAMPLE 4 Determining the Limits of Integration

Set up a triple integral for the volume of each solid region.

a. The region in the first octant bounded above by the cylinder $z = 1 - y^2$ and lying between the vertical planes $x + y = 1$ and $x + y = 3$

b. The upper hemisphere given by $z = \sqrt{1 - x^2 - y^2}$

c. The region bounded below by the paraboloid $z = x^2 + y^2$ and above by the sphere $x^2 + y^2 + z^2 = 6$

Solution

a. In Figure 14.57, note that the solid is bounded below by the xy-plane $(z = 0)$ and above by the cylinder $z = 1 - y^2$. So,

$$0 \le z \le 1 - y^2. \qquad \text{Bounds for } z$$

Projecting the region onto the xy-plane produces a parallelogram. Because two sides of the parallelogram are parallel to the x-axis, you have the following bounds:

$$1 - y \le x \le 3 - y \quad \text{and} \quad 0 \le y \le 1.$$

So, the volume of the region is given by

$$V = \iiint_Q dV = \int_0^1 \int_{1-y}^{3-y} \int_0^{1-y^2} dz\,dx\,dy.$$

b. For the upper hemisphere given by $z = \sqrt{1 - x^2 - y^2}$, you have

$$0 \le z \le \sqrt{1 - x^2 - y^2}. \qquad \text{Bounds for } z$$

In Figure 14.58, note that the projection of the hemisphere onto the xy-plane is the circle given by $x^2 + y^2 = 1$, and you can use either order $dx\,dy$ or $dy\,dx$. Choosing the first produces

$$-\sqrt{1 - y^2} \le x \le \sqrt{1 - y^2} \quad \text{and} \quad -1 \le y \le 1$$

which implies that the volume of the region is given by

$$V = \iiint_Q dV = \int_{-1}^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \int_0^{\sqrt{1-x^2-y^2}} dz\,dx\,dy.$$

c. For the region bounded below by the paraboloid $z = x^2 + y^2$ and above by the sphere $x^2 + y^2 + z^2 = 6$, you have

$$x^2 + y^2 \le z \le \sqrt{6 - x^2 - y^2}. \qquad \text{Bounds for } z$$

The sphere and the paraboloid intersect at $z = 2$. Moreover, you can see in Figure 14.59 that the projection of the solid region onto the xy-plane is the circle given by $x^2 + y^2 = 2$. Using the order $dy\,dx$ produces

$$-\sqrt{2 - x^2} \le y \le \sqrt{2 - x^2} \quad \text{and} \quad -\sqrt{2} \le x \le \sqrt{2}$$

which implies that the volume of the region is given by

$$V = \iiint_Q dV = \int_{-\sqrt{2}}^{\sqrt{2}} \int_{-\sqrt{2-x^2}}^{\sqrt{2-x^2}} \int_{x^2+y^2}^{\sqrt{6-x^2-y^2}} dz\,dy\,dx.$$ ■

Center of Mass and Moments of Inertia

EXPLORATION

Sketch the solid (of uniform density) bounded by $z = 0$ and

$$z = \frac{1}{1 + x^2 + y^2}$$

where $x^2 + y^2 \leq 1$. From your sketch, estimate the coordinates of the center of mass of the solid. Now use a computer algebra system to verify your estimate. What do you observe?

In the remainder of this section, two important engineering applications of triple integrals are discussed. Consider a solid region Q whose density is given by the **density function ρ.** The **center of mass** of a solid region Q of mass m is given by $(\bar{x}, \bar{y}, \bar{z})$, where

$$m = \iiint_Q \rho(x, y, z)\, dV \qquad \text{Mass of the solid}$$

$$M_{yz} = \iiint_Q x\rho(x, y, z)\, dV \qquad \text{First moment about } yz\text{-plane}$$

$$M_{xz} = \iiint_Q y\rho(x, y, z)\, dV \qquad \text{First moment about } xz\text{-plane}$$

$$M_{xy} = \iiint_Q z\rho(x, y, z)\, dV \qquad \text{First moment about } xy\text{-plane}$$

and

$$\bar{x} = \frac{M_{yz}}{m}, \quad \bar{y} = \frac{M_{xz}}{m}, \quad \bar{z} = \frac{M_{xy}}{m}.$$

The quantities M_{yz}, M_{xz}, and M_{xy} are called the **first moments** of the region Q about the yz-, xz-, and xy-planes, respectively.

The first moments for solid regions are taken about a plane, whereas the second moments for solids are taken about a line. The **second moments** (or **moments of inertia**) about the x-, y-, and z-axes are as follows.

$$I_x = \iiint_Q (y^2 + z^2)\rho(x, y, z)\, dV \qquad \text{Moment of inertia about } x\text{-axis}$$

$$I_y = \iiint_Q (x^2 + z^2)\rho(x, y, z)\, dV \qquad \text{Moment of inertia about } y\text{-axis}$$

$$I_z = \iiint_Q (x^2 + y^2)\rho(x, y, z)\, dV \qquad \text{Moment of inertia about } z\text{-axis}$$

For problems requiring the calculation of all three moments, considerable effort can be saved by applying the additive property of triple integrals and writing

$$I_x = I_{xz} + I_{xy}, \quad I_y = I_{yz} + I_{xy}, \quad \text{and} \quad I_z = I_{yz} + I_{xz}$$

where I_{xy}, I_{xz}, and I_{yz} are as follows.

$$I_{xy} = \iiint_Q z^2\rho(x, y, z)\, dV$$

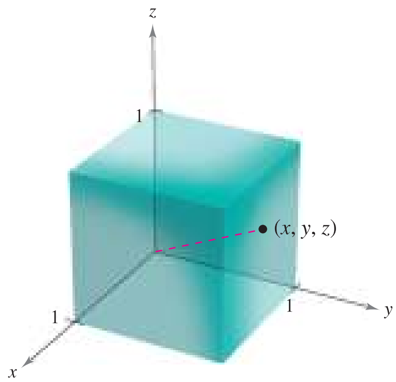
$$I_{xz} = \iiint_Q y^2\rho(x, y, z)\, dV$$

$$I_{yz} = \iiint_Q x^2\rho(x, y, z)\, dV$$

NOTE In engineering and physics, the moment of inertia of a mass is used to find the time required for the mass to reach a given speed of rotation about an axis, as shown in Figure 14.60. The greater the moment of inertia, the longer a force must be applied for the mass to reach the given speed.

Figure 14.60

EXAMPLE 5 Finding the Center of Mass of a Solid Region



Variable density:
$\rho(x, y, z) = k(x^2 + y^2 + z^2)$
Figure 14.61

Find the center of mass of the unit cube shown in Figure 14.61, given that the density at the point (x, y, z) is proportional to the square of its distance from the origin.

Solution Because the density at (x, y, z) is proportional to the square of the distance between $(0, 0, 0)$ and (x, y, z), you have

$$\rho(x, y, z) = k(x^2 + y^2 + z^2).$$

You can use this density function to find the mass of the cube. Because of the symmetry of the region, any order of integration will produce an integral of comparable difficulty.

$$\begin{aligned}
m &= \int_0^1\int_0^1\int_0^1 k(x^2 + y^2 + z^2)\,dz\,dy\,dx \\
&= k\int_0^1\int_0^1 \left[(x^2 + y^2)z + \frac{z^3}{3}\right]_0^1 dy\,dx \\
&= k\int_0^1\int_0^1 \left(x^2 + y^2 + \frac{1}{3}\right) dy\,dx \\
&= k\int_0^1 \left[\left(x^2 + \frac{1}{3}\right)y + \frac{y^3}{3}\right]_0^1 dx \\
&= k\int_0^1 \left(x^2 + \frac{2}{3}\right) dx \\
&= k\left[\frac{x^3}{3} + \frac{2x}{3}\right]_0^1 = k
\end{aligned}$$

The first moment about the yz-plane is

$$\begin{aligned}
M_{yz} &= k\int_0^1\int_0^1\int_0^1 x(x^2 + y^2 + z^2)\,dz\,dy\,dx \\
&= k\int_0^1 x\left[\int_0^1\int_0^1 (x^2 + y^2 + z^2)\,dz\,dy\right] dx.
\end{aligned}$$

Note that x can be factored out of the two inner integrals, because it is constant with respect to y and z. After factoring, the two inner integrals are the same as for the mass m. Therefore, you have

$$\begin{aligned}
M_{yz} &= k\int_0^1 x\left(x^2 + \frac{2}{3}\right) dx \\
&= k\left[\frac{x^4}{4} + \frac{x^2}{3}\right]_0^1 \\
&= \frac{7k}{12}.
\end{aligned}$$

So,

$$\bar{x} = \frac{M_{yz}}{m} = \frac{7k/12}{k} = \frac{7}{12}.$$

Finally, from the nature of ρ and the symmetry of x, y, and z in this solid region, you have $\bar{x} = \bar{y} = \bar{z}$, and the center of mass is $\left(\frac{7}{12}, \frac{7}{12}, \frac{7}{12}\right)$. ■

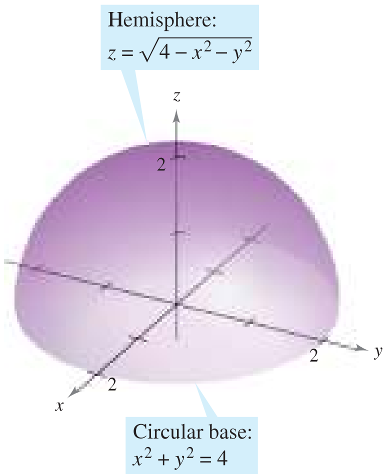
EXAMPLE 6 Moments of Inertia for a Solid Region

Find the moments of inertia about the x- and y-axes for the solid region lying between the hemisphere

$$z = \sqrt{4 - x^2 - y^2}$$

and the xy-plane, given that the density at (x, y, z) is proportional to the distance between (x, y, z) and the xy-plane.

$0 \le z \le \sqrt{4-x^2-y^2}$
$-\sqrt{4-x^2} \le y \le \sqrt{4-x^2}$
$-2 \le x \le 2$



Variable density: $\rho(x, y, z) = kz$
Figure 14.62

Solution The density of the region is given by $\rho(x, y, z) = kz$. Considering the symmetry of this problem, you know that $I_x = I_y$, and you need to compute only one moment, say I_x. From Figure 14.62, choose the order $dz\,dy\,dx$ and write

$$\begin{aligned}
I_x &= \iiint\limits_Q (y^2 + z^2)\rho(x, y, z)\, dV \\
&= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{0}^{\sqrt{4-x^2-y^2}} (y^2 + z^2)(kz)\, dz\, dy\, dx \\
&= k\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \left[\frac{y^2z^2}{2} + \frac{z^4}{4}\right]_0^{\sqrt{4-x^2-y^2}} dy\, dx \\
&= k\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \left[\frac{y^2(4 - x^2 - y^2)}{2} + \frac{(4 - x^2 - y^2)^2}{4}\right] dy\, dx \\
&= \frac{k}{4}\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \left[(4 - x^2)^2 - y^4\right] dy\, dx \\
&= \frac{k}{4}\int_{-2}^{2} \left[(4 - x^2)^2 y - \frac{y^5}{5}\right]_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dx \\
&= \frac{k}{4}\int_{-2}^{2} \frac{8}{5}(4 - x^2)^{5/2}\, dx \\
&= \frac{4k}{5}\int_{0}^{2} (4 - x^2)^{5/2}\, dx && x = 2\sin\theta \\
&= \frac{4k}{5}\int_{0}^{\pi/2} 64\cos^6\theta\, d\theta \\
&= \left(\frac{256k}{5}\right)\left(\frac{5\pi}{32}\right) && \text{Wallis's Formula} \\
&= 8k\pi.
\end{aligned}$$

So, $I_x = 8k\pi = I_y$. ■

In Example 6, notice that the moments of inertia about the x- and y-axes are equal to each other. The moment about the z-axis, however, is different. Does it seem that the moment of inertia about the z-axis should be less than or greater than the moments calculated in Example 6? By performing the calculations, you can determine that

$$I_z = \frac{16}{3}k\pi.$$

This tells you that the solid shown in Figure 14.62 has a greater resistance to rotation about the x- or y-axis than about the z-axis.

14.6 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–8, evaluate the iterated integral.

1. $\displaystyle\int_0^3\int_0^2\int_0^1 (x+y+z)\,dx\,dz\,dy$

2. $\displaystyle\int_{-1}^1\int_{-1}^1\int_{-1}^1 x^2y^2z^2\,dx\,dy\,dz$

3. $\displaystyle\int_0^1\int_0^x\int_0^{xy} x\,dz\,dy\,dx$

4. $\displaystyle\int_0^9\int_0^{y/3}\int_0^{\sqrt{y^2-9x^2}} z\,dz\,dx\,dy$

5. $\displaystyle\int_1^4\int_0^1\int_0^x 2ze^{-x^2}\,dy\,dx\,dz$

6. $\displaystyle\int_1^4\int_1^{e^2}\int_0^{1/xz} \ln z\,dy\,dz\,dx$

7. $\displaystyle\int_0^4\int_0^{\pi/2}\int_0^{1-x} x\cos y\,dz\,dy\,dx$

8. $\displaystyle\int_0^{\pi/2}\int_0^{y/2}\int_0^{1/y} \sin y\,dz\,dx\,dy$

CAS **In Exercises 9 and 10, use a computer algebra system to evaluate the iterated integral.**

9. $\displaystyle\int_0^3\int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}}\int_0^{y^2} y\,dz\,dx\,dy$

10. $\displaystyle\int_0^{\sqrt{2}}\int_0^{\sqrt{2-x^2}}\int_{2x^2+y^2}^{4-y^2} y\,dz\,dy\,dx$

CAS **In Exercises 11 and 12, use a computer algebra system to approximate the iterated integral.**

11. $\displaystyle\int_0^2\int_0^{\sqrt{4-x^2}}\int_1^4 \frac{x^2\sin y}{z}\,dz\,dy\,dx$

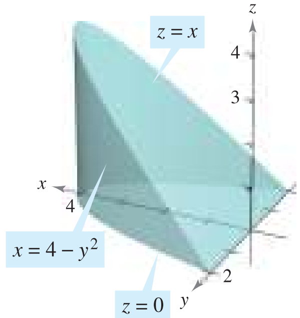
12. $\displaystyle\int_0^3\int_0^{2-(2y/3)}\int_0^{6-2y-3z} ze^{-x^2y^2}\,dx\,dz\,dy$

In Exercises 13–18, set up a triple integral for the volume of the solid.

13. The solid in the first octant bounded by the coordinate planes and the plane $z = 5 - x - y$
14. The solid bounded by $z = 9 - x^2$, $z = 0$, $y = 0$, and $y = 2x$
15. The solid bounded by $z = 6 - x^2 - y^2$ and $z = 0$
16. The solid bounded by $z = \sqrt{16 - x^2 - y^2}$ and $z = 0$
17. The solid that is the common interior below the sphere $x^2 + y^2 + z^2 = 80$ and above the paraboloid $z = \frac{1}{2}(x^2 + y^2)$
18. The solid bounded above by the cylinder $z = 4 - x^2$ and below by the paraboloid $z = x^2 + 3y^2$

Volume **In Exercises 19–22, use a triple integral to find the volume of the solid shown in the figure.**

19.



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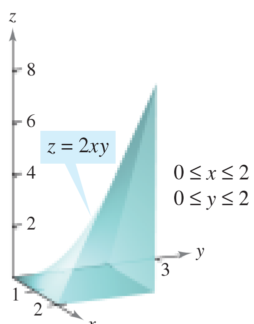

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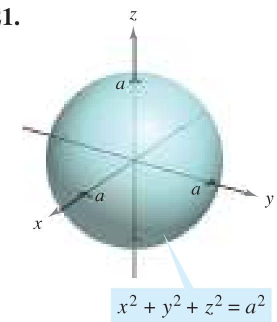
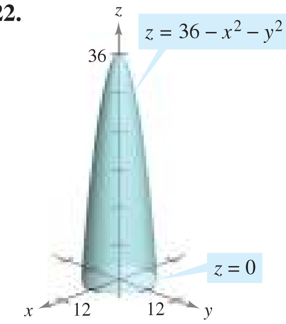


22.



Volume **In Exercises 23–26, use a triple integral to find the volume of the solid bounded by the graphs of the equations.**

23. $z = 4 - x^2$, $y = 4 - x^2$, first octant
24. $z = 9 - x^3$, $y = -x^2 + 2$, $y = 0$, $z = 0$, $x \ge 0$
25. $z = 2 - y$, $z = 4 - y^2$, $x = 0$, $x = 3$, $y = 0$
26. $z = x$, $y = x + 2$, $y = x^2$, first octant

In Exercises 27–32, sketch the solid whose volume is given by the iterated integral and rewrite the integral using the indicated order of integration.

27. $\displaystyle\int_0^1\int_{-1}^0\int_0^{y^2} dz\,dy\,dx$

Rewrite using the order $dy\,dz\,dx$.

28. $\displaystyle\int_{-1}^1\int_{y^2}^1\int_0^{1-x} dz\,dx\,dy$

Rewrite using the order $dx\,dz\,dy$.

29. $\displaystyle\int_0^4\int_0^{(4-x)/2}\int_0^{(12-3x-6y)/4} dz\,dy\,dx$

Rewrite using the order $dy\,dx\,dz$.

30. $\displaystyle\int_0^3\int_0^{\sqrt{9-x^2}}\int_0^{6-x-y} dz\,dy\,dx$

Rewrite using the order $dz\,dx\,dy$.

31. $\displaystyle\int_0^1\int_y^1\int_0^{\sqrt{1-y^2}} dz\,dx\,dy$

Rewrite using the order $dz\,dy\,dx$.

32. $\displaystyle\int_0^2\int_{2x}^4\int_0^{\sqrt{y^2-4x^2}} dz\,dy\,dx$

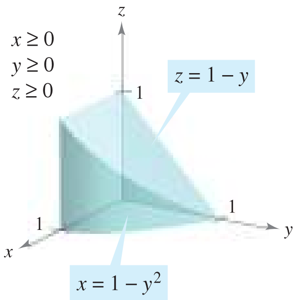
Rewrite using the order $dx\,dy\,dz$.

In Exercises 33–36, list the six possible orders of integration for the triple integral over the solid region Q, $\displaystyle\iiint_Q xyz\,dV$.

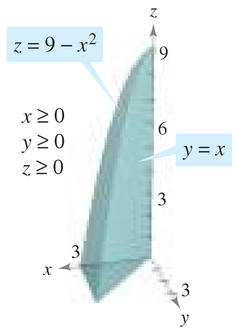
33. $Q = \{(x, y, z): 0 \le x \le 1, 0 \le y \le x, 0 \le z \le 3\}$
34. $Q = \{(x, y, z): 0 \le x \le 2, x^2 \le y \le 4, 0 \le z \le 2 - x\}$
35. $Q = \{(x, y, z): x^2 + y^2 \le 9, 0 \le z \le 4\}$
36. $Q = \{(x, y, z): 0 \le x \le 1, y \le 1 - x^2, 0 \le z \le 6\}$

In Exercises 37 and 38, the figure shows the region of integration for the given integral. Rewrite the integral as an equivalent iterated integral in the five other orders.

37. $\displaystyle\int_0^1\int_0^{1-y^2}\int_0^{1-y} dz\,dx\,dy$



38. $\displaystyle\int_0^3\int_0^{x}\int_0^{9-x^2} dz\,dy\,dx$



Mass and Center of Mass **In Exercises 39–42, find the mass and the indicated coordinates of the center of mass of the solid of given density bounded by the graphs of the equations.**

39. Find $\bar{x}$ using $\rho(x, y, z) = k$.

Q: $2x + 3y + 6z = 12, x = 0, y = 0, z = 0$

40. Find $\bar{y}$ using $\rho(x, y, z) = ky$.

Q: $3x + 3y + 5z = 15, x = 0, y = 0, z = 0$

41. Find $\bar{z}$ using $\rho(x, y, z) = kx$.

Q: $z = 4 - x, z = 0, y = 0, y = 4, x = 0$

42. Find $\bar{y}$ using $\rho(x, y, z) = k$.

Q: $\dfrac{x}{a} + \dfrac{y}{b} + \dfrac{z}{c} = 1\ (a, b, c > 0), x = 0, y = 0, z = 0$

Mass and Center of Mass **In Exercises 43 and 44, set up the triple integrals for finding the mass and the center of mass of the solid bounded by the graphs of the equations.**

43. $x = 0, x = b, y = 0, y = b, z = 0, z = b$

$\rho(x, y, z) = kxy$

44. $x = 0, x = a, y = 0, y = b, z = 0, z = c$

$\rho(x, y, z) = kz$

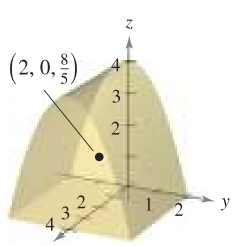
Think About It **The center of mass of a solid of constant density is shown in the figure. In Exercises 45–48, make a conjecture about how the center of mass $(\bar{x}, \bar{y}, \bar{z})$ will change for the nonconstant density $\rho(x, y, z)$. Explain.**

45. $\rho(x, y, z) = kx$

46. $\rho(x, y, z) = kz$

47. $\rho(x, y, z) = k(y + 2)$

48. $\rho(x, y, z) = kxz^2(y + 2)^2$



CAS ***Centroid*** **In Exercises 49–54, find the centroid of the solid region bounded by the graphs of the equations or described by the figure. Use a computer algebra system to evaluate the triple integrals. (Assume uniform density and find the center of mass.)**

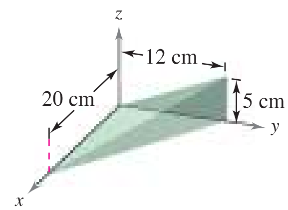
49. $z = \dfrac{h}{r}\sqrt{x^2 + y^2}, z = h$

50. $y = \sqrt{9 - x^2}, z = y, z = 0$

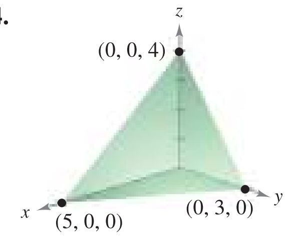
51. $z = \sqrt{16 - x^2 - y^2}, z = 0$

52. $z = \dfrac{1}{y^2 + 1}, z = 0, x = -2, x = 2, y = 0, y = 1$

53.



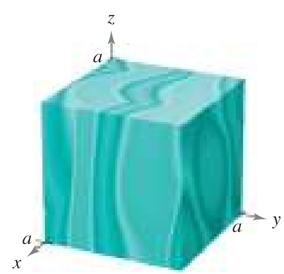
54.



CAS ***Moments of Inertia*** **In Exercises 55–58, find I_x, I_y, and I_z for the solid of given density. Use a computer algebra system to evaluate the triple integrals.**

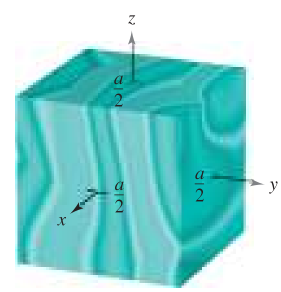
55. (a) $\rho = k$

(b) $\rho = kxyz$



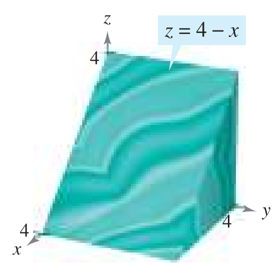
56. (a) $\rho(x, y, z) = k$

(b) $\rho(x, y, z) = k(x^2 + y^2)$



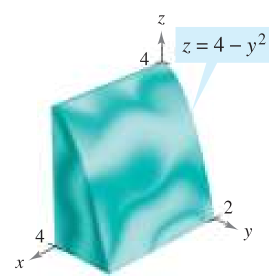
57. (a) $\rho(x, y, z) = k$

(b) $\rho = ky$



58. (a) $\rho = kz$

(b) $\rho = k(4 - z)$

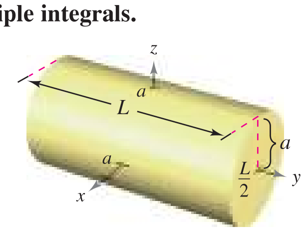


CAS ***Moments of Inertia*** **In Exercises 59 and 60, verify the moments of inertia for the solid of uniform density. Use a computer algebra system to evaluate the triple integrals.**

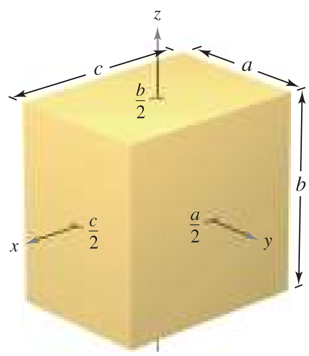
59. $I_x = \frac{1}{12}m(3a^2 + L^2)$

$I_y = \frac{1}{2}ma^2$

$I_z = \frac{1}{12}m(3a^2 + L^2)$



60. $I_x = \frac{1}{12}m(a^2 + b^2)$
$I_y = \frac{1}{12}m(b^2 + c^2)$
$I_z = \frac{1}{12}m(a^2 + c^2)$



Moments of Inertia **In Exercises 61 and 62, set up a triple integral that gives the moment of inertia about the z-axis of the solid region Q of density ρ.**

61. $Q = \{(x, y, z): -1 \le x \le 1, -1 \le y \le 1, 0 \le z \le 1 - x\}$
$\rho = \sqrt{x^2 + y^2 + z^2}$

62. $Q = \{(x, y, z): x^2 + y^2 \le 1, 0 \le z \le 4 - x^2 - y^2\}$
$\rho = kx^2$

In Exercises 63 and 64, using the description of the solid region, set up the integral for (a) the mass, (b) the center of mass, and (c) the moment of inertia about the z-axis.

63. The solid bounded by $z = 4 - x^2 - y^2$ and $z = 0$ with density function $\rho = kz$

64. The solid in the first octant bounded by the coordinate planes and $x^2 + y^2 + z^2 = 25$ with density function $\rho = kxy$

WRITING ABOUT CONCEPTS

65. Define a triple integral and describe a method of evaluating a triple integral.

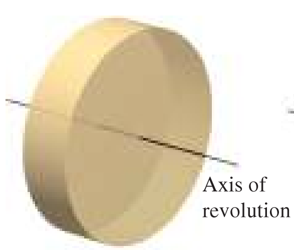
66. Determine whether the moment of inertia about the y-axis of the cylinder in Exercise 59 will increase or decrease for the nonconstant density $\rho(x, y, z) = \sqrt{x^2 + z^2}$ and $a = 4$.

67. Consider two solids, solid A and solid B, of equal weight as shown below.

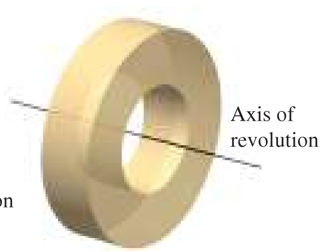
(a) Because the solids have the same weight, which has the greater density?

(b) Which solid has the greater moment of inertia? Explain.

(c) The solids are rolled down an inclined plane. They are started at the same time and at the same height. Which will reach the bottom first? Explain.



Solid A



Solid B

CAPSTONE

68. ***Think About It*** Of the integrals (a)–(c), which one is equal to $\int_1^3 \int_0^2 \int_{-1}^1 f(x, y, z)\, dz\, dy\, dx$? Explain.

(a) $\int_1^3 \int_0^2 \int_{-1}^1 f(x, y, z)\, dz\, dx\, dy$

(b) $\int_{-1}^1 \int_0^2 \int_1^3 f(x, y, z)\, dx\, dy\, dz$

(c) $\int_0^2 \int_1^3 \int_{-1}^1 f(x, y, z)\, dy\, dx\, dz$

Average Value **In Exercises 69–72, find the average value of the function over the given solid. The average value of a continuous function $f(x, y, z)$ over a solid region Q is**

$$\frac{1}{V}\iiint_Q f(x, y, z)\, dV$$

where V is the volume of the solid region Q.

69. $f(x, y, z) = z^2 + 4$ over the cube in the first octant bounded by the coordinate planes and the planes $x = 1$, $y = 1$, and $z = 1$

70. $f(x, y, z) = xyz$ over the cube in the first octant bounded by the coordinate planes and the planes $x = 4$, $y = 4$, and $z = 4$

71. $f(x, y, z) = x + y + z$ over the tetrahedron in the first octant with vertices $(0, 0, 0)$, $(2, 0, 0)$, $(0, 2, 0)$ and $(0, 0, 2)$

72. $f(x, y, z) = x + y$ over the solid bounded by the sphere $x^2 + y^2 + z^2 = 3$

CAS **73.** Find the solid region Q where the triple integral

$$\iiint_Q (1 - 2x^2 - y^2 - 3z^2)\, dV$$

is a maximum. Use a computer algebra system to approximate the maximum value. What is the exact maximum value?

CAS **74.** Find the solid region Q where the triple integral

$$\iiint_Q (1 - x^2 - y^2 - z^2)\, dV$$

is a maximum. Use a computer algebra system to approximate the maximum value. What is the exact maximum value?

75. Solve for a in the triple integral.

$$\int_0^1 \int_0^{3-a-y^2} \int_a^{4-x-y^2} dz\, dx\, dy = \frac{14}{15}$$

76. Determine the value of b such that the volume of the ellipsoid $x^2 + (y^2/b^2) + (z^2/9) = 1$ is 16π.

PUTNAM EXAM CHALLENGE

77. Evaluate

$$\lim_{n\to\infty} \int_0^1 \int_0^1 \cdots \int_0^1 \cos^2\left\{\frac{\pi}{2n}(x_1 + x_2 + \cdots + x_n)\right\} dx_1\, dx_2 \cdots dx_n.$$

14.7 Triple Integrals in Cylindrical and Spherical Coordinates

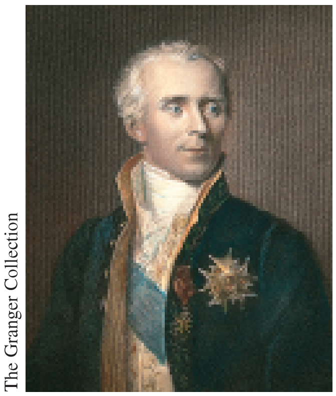
- **Write and evaluate a triple integral in cylindrical coordinates.**
- **Write and evaluate a triple integral in spherical coordinates.**

Triple Integrals in Cylindrical Coordinates

Many common solid regions such as spheres, ellipsoids, cones, and paraboloids can yield difficult triple integrals in rectangular coordinates. In fact, it is precisely this difficulty that led to the introduction of nonrectangular coordinate systems. In this section, you will learn how to use *cylindrical* and *spherical* coordinates to evaluate triple integrals.

Recall from Section 11.7 that the rectangular conversion equations for cylindrical coordinates are

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$z = z.$$



PIERRE SIMON DE LAPLACE (1749–1827)

One of the first to use a cylindrical coordinate system was the French mathematician Pierre Simon de Laplace. Laplace has been called the "Newton of France," and he published many important works in mechanics, differential equations, and probability.

STUDY TIP An easy way to remember these conversions is to note that the equations for x and y are the same as in polar coordinates and z is unchanged.

In this coordinate system, the simplest solid region is a cylindrical block determined by

$$r_1 \le r \le r_2, \quad \theta_1 \le \theta \le \theta_2, \quad z_1 \le z \le z_2$$

as shown in Figure 14.63. To obtain the cylindrical coordinate form of a triple integral, suppose that Q is a solid region whose projection R onto the xy-plane can be described in polar coordinates. That is,

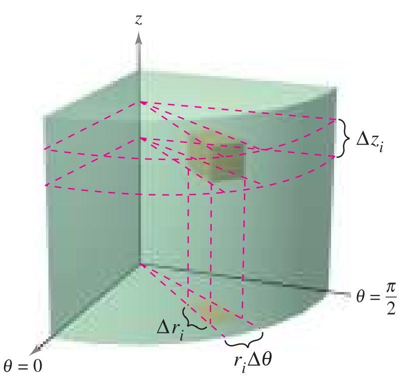
$$Q = \{(x, y, z): (x, y) \text{ is in } R, \quad h_1(x, y) \le z \le h_2(x, y)\}$$

and

$$R = \{(r, \theta): \theta_1 \le \theta \le \theta_2, \quad g_1(\theta) \le r \le g_2(\theta)\}.$$

If f is a continuous function on the solid Q, you can write the triple integral of f over Q as

$$\iiint_Q f(x, y, z)\, dV = \iint_R \left[\int_{h_1(x, y)}^{h_2(x, y)} f(x, y, z)\, dz\right] dA$$

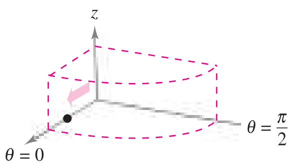


Volume of cylindrical block:
$\Delta V_i = r_i \Delta r_i \Delta\theta_i \Delta z_i$
Figure 14.63

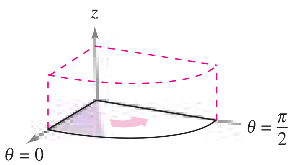
where the double integral over R is evaluated in polar coordinates. That is, R is a plane region that is either r-simple or θ-simple. If R is r-simple, the iterated form of the triple integral in cylindrical form is

$$\iiint_Q f(x, y, z)\, dV = \int_{\theta_1}^{\theta_2}\int_{g_1(\theta)}^{g_2(\theta)}\int_{h_1(r\cos\theta,\, r\sin\theta)}^{h_2(r\cos\theta,\, r\sin\theta)} f(r\cos\theta, r\sin\theta, z) r\, dz\, dr\, d\theta.$$

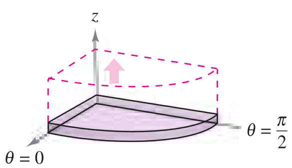
NOTE This is only one of six possible orders of integration. The other five are $dz\,d\theta\,dr$, $dr\,dz\,d\theta$, $dr\,d\theta\,dz$, $d\theta\,dz\,dr$, and $d\theta\,dr\,dz$.



Integrate with respect to r.



Integrate with respect to θ.



Integrate with respect to z.
Figure 14.64

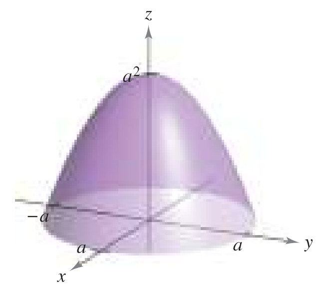
To visualize a particular order of integration, it helps to view the iterated integral in terms of three sweeping motions—each adding another dimension to the solid. For instance, in the order $dr\,d\theta\,dz$, the first integration occurs in the r-direction as a point sweeps out a ray. Then, as θ increases, the line sweeps out a sector. Finally, as z increases, the sector sweeps out a solid wedge, as shown in Figure 14.64.

EXPLORATION

Volume of a Paraboloid Sector On pages 997, 1006, and 1028, you were asked to summarize the different ways you know of finding the volume of the solid bounded by the paraboloid

$$z = a^2 - x^2 - y^2, \quad a > 0$$

and the xy-plane. You now know one more way. Use it to find the volume of the solid. Compare the different methods. What are the advantages and disadvantages of each?



EXAMPLE 1 Finding Volume in Cylindrical Coordinates

Find the volume of the solid region Q cut from the sphere $x^2 + y^2 + z^2 = 4$ by the cylinder $r = 2 \sin \theta$, as shown in Figure 14.65.

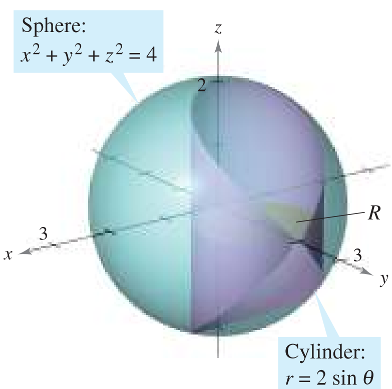


Figure 14.65

Solution Because $x^2 + y^2 + z^2 = r^2 + z^2 = 4$, the bounds on z are

$$-\sqrt{4 - r^2} \leq z \leq \sqrt{4 - r^2}.$$

Let R be the circular projection of the solid onto the $r\theta$-plane. Then the bounds on R are $0 \leq r \leq 2 \sin \theta$ and $0 \leq \theta \leq \pi$. So, the volume of Q is

$$\begin{aligned}
V &= \int_0^{\pi} \int_0^{2 \sin \theta} \int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} r\,dz\,dr\,d\theta \\
&= 2 \int_0^{\pi/2} \int_0^{2 \sin \theta} \int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} r\,dz\,dr\,d\theta \\
&= 2 \int_0^{\pi/2} \int_0^{2 \sin \theta} 2r\sqrt{4 - r^2}\,dr\,d\theta \\
&= 2 \int_0^{\pi/2} -\frac{2}{3}(4 - r^2)^{3/2} \Big]_0^{2 \sin \theta} d\theta \\
&= \frac{4}{3} \int_0^{\pi/2} (8 - 8 \cos^3 \theta)\,d\theta \\
&= \frac{32}{3} \int_0^{\pi/2} [1 - (\cos \theta)(1 - \sin^2 \theta)]\,d\theta \\
&= \frac{32}{3} \left[\theta - \sin \theta + \frac{\sin^3 \theta}{3}\right]_0^{\pi/2} \\
&= \frac{16}{9}(3\pi - 4) \\
&\approx 9.644.
\end{aligned}$$

■

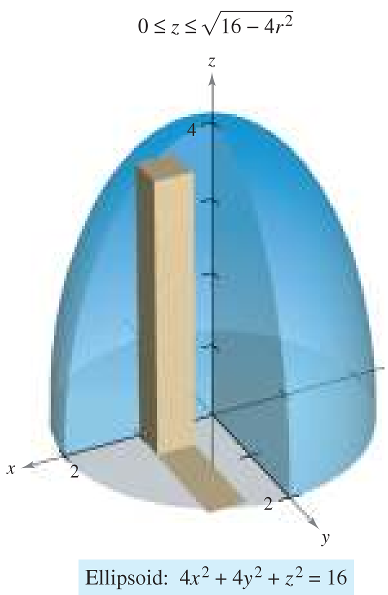


Figure 14.66

EXAMPLE 2 Finding Mass in Cylindrical Coordinates

Find the mass of the ellipsoid Q given by $4x^2 + 4y^2 + z^2 = 16$, lying above the xy-plane. The density at a point in the solid is proportional to the distance between the point and the xy-plane.

Solution The density function is $\rho(r, \theta, z) = kz$. The bounds on z are

$$0 \le z \le \sqrt{16 - 4x^2 - 4y^2} = \sqrt{16 - 4r^2}$$

where $0 \le r \le 2$ and $0 \le \theta \le 2\pi$, as shown in Figure 14.66. The mass of the solid is

$$\begin{aligned} m &= \int_0^{2\pi}\int_0^2\int_0^{\sqrt{16-4r^2}} kzr \, dz \, dr \, d\theta \\ &= \frac{k}{2}\int_0^{2\pi}\int_0^2 z^2 r \Big]_0^{\sqrt{16-4r^2}} dr \, d\theta \\ &= \frac{k}{2}\int_0^{2\pi}\int_0^2 (16r - 4r^3) \, dr \, d\theta \\ &= \frac{k}{2}\int_0^{2\pi}\Big[8r^2 - r^4\Big]_0^2 d\theta \\ &= 8k\int_0^{2\pi} d\theta = 16\pi k. \end{aligned}$$

Integration in cylindrical coordinates is useful when factors involving $x^2 + y^2$ appear in the integrand, as illustrated in Example 3.

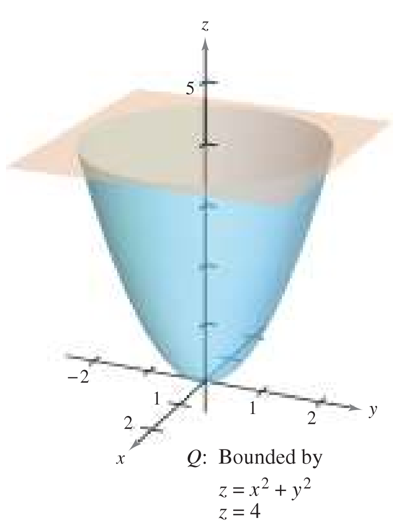


Figure 14.67

EXAMPLE 3 Finding a Moment of Inertia

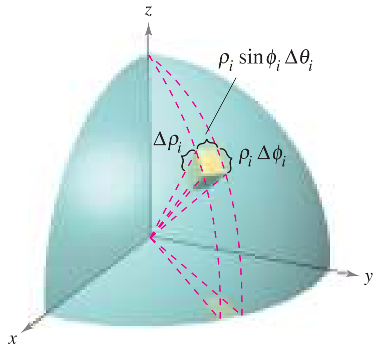
Find the moment of inertia about the axis of symmetry of the solid Q bounded by the paraboloid $z = x^2 + y^2$ and the plane $z = 4$, as shown in Figure 14.67. The density at each point is proportional to the distance between the point and the z-axis.

Solution Because the z-axis is the axis of symmetry, and $\rho(x, y, z) = k\sqrt{x^2 + y^2}$, it follows that

$$I_z = \iiint_Q k(x^2 + y^2)\sqrt{x^2 + y^2} \, dV.$$

In cylindrical coordinates, $0 \le r \le \sqrt{x^2 + y^2} = \sqrt{z}$. So, you have

$$\begin{aligned} I_z &= k\int_0^4\int_0^{2\pi}\int_0^{\sqrt{z}} r^2(r)r \, dr \, d\theta \, dz \\ &= k\int_0^4\int_0^{2\pi} \frac{r^5}{5}\Big]_0^{\sqrt{z}} d\theta \, dz \\ &= k\int_0^4\int_0^{2\pi} \frac{z^{5/2}}{5} \, d\theta \, dz \\ &= \frac{k}{5}\int_0^4 z^{5/2}(2\pi) \, dz \\ &= \frac{2\pi k}{5}\left[\frac{2}{7}z^{7/2}\right]_0^4 = \frac{512k\pi}{35}. \end{aligned}$$



Spherical block:
$\Delta V_i \approx \rho_i^2 \sin\phi_i \Delta\rho_i \Delta\phi_i \Delta\theta_i$
Figure 14.68

Triple Integrals in Spherical Coordinates

Triple integrals involving spheres or cones are often easier to evaluate by converting to spherical coordinates. Recall from Section 11.7 that the rectangular conversion equations for spherical coordinates are

$$x = \rho \sin\phi \cos\theta$$
$$y = \rho \sin\phi \sin\theta$$
$$z = \rho \cos\phi.$$

In this coordinate system, the simplest region is a spherical block determined by

$$\{(\rho, \theta, \phi): \rho_1 \le \rho \le \rho_2, \quad \theta_1 \le \theta \le \theta_2, \quad \phi_1 \le \phi \le \phi_2\}$$

where $\rho_1 \ge 0$, $\theta_2 - \theta_1 \le 2\pi$, and $0 \le \phi_1 \le \phi_2 \le \pi$, as shown in Figure 14.68. If (ρ, θ, ϕ) is a point in the interior of such a block, then the volume of the block can be approximated by $\Delta V \approx \rho^2 \sin\phi\, \Delta\rho\, \Delta\phi\, \Delta\theta$ (see Exercise 18 in the Problem Solving exercises at the end of this chapter).

Using the usual process involving an inner partition, summation, and a limit, you can develop the following version of a triple integral in spherical coordinates for a continuous function f defined on the solid region Q.

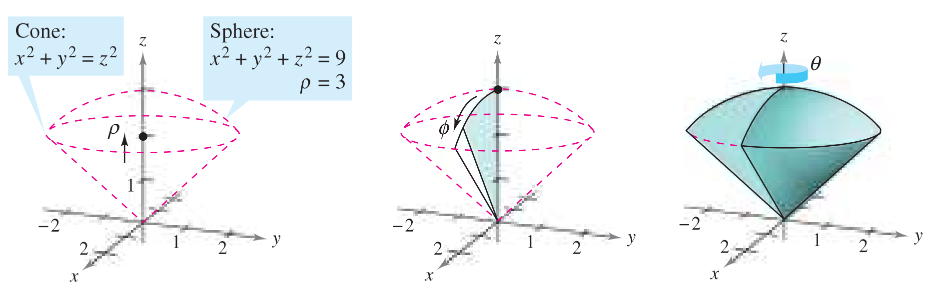
$$\iiint_Q f(x, y, z)\, dV = \int_{\theta_1}^{\theta_2}\int_{\phi_1}^{\phi_2}\int_{\rho_1}^{\rho_2} f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\rho^2\sin\phi\, d\rho\, d\phi\, d\theta$$

This formula can be modified for different orders of integration and generalized to include regions with variable boundaries.

Like triple integrals in cylindrical coordinates, triple integrals in spherical coordinates are evaluated with iterated integrals. As with cylindrical coordinates, you can visualize a particular order of integration by viewing the iterated integral in terms of three sweeping motions—each adding another dimension to the solid. For instance, the iterated integral

$$\int_0^{2\pi}\int_0^{\pi/4}\int_0^3 \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$$

(which is used in Example 4) is illustrated in Figure 14.69.



ρ varies from 0 to 3 with ϕ and θ held constant.

ϕ varies from 0 to $\pi/4$ with θ held constant.

θ varies from 0 to 2π.

Figure 14.69

NOTE The Greek letter ρ used in spherical coordinates is not related to density. Rather, it is the three-dimensional analog of the r used in polar coordinates. For problems involving spherical coordinates and a density function, this text uses a different symbol to denote density.

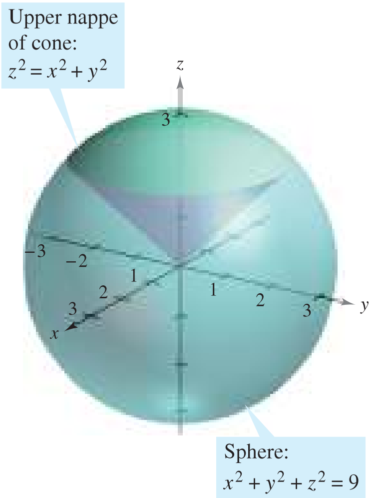


Figure 14.70

EXAMPLE 4 Finding Volume in Spherical Coordinates

Find the volume of the solid region Q bounded below by the upper nappe of the cone $z^2 = x^2 + y^2$ and above by the sphere $x^2 + y^2 + z^2 = 9$, as shown in Figure 14.70.

Solution In spherical coordinates, the equation of the sphere is

$$\rho^2 = x^2 + y^2 + z^2 = 9 \quad \Rightarrow \quad \rho = 3.$$

Furthermore, the sphere and cone intersect when

$$(x^2 + y^2) + z^2 = (z^2) + z^2 = 9 \quad \Rightarrow \quad z = \frac{3}{\sqrt{2}}$$

and, because $z = \rho \cos \phi$, it follows that

$$\left(\frac{3}{\sqrt{2}}\right)\left(\frac{1}{3}\right) = \cos \phi \quad \Rightarrow \quad \phi = \frac{\pi}{4}.$$

Consequently, you can use the integration order $d\rho\, d\phi\, d\theta$, where $0 \le \rho \le 3$, $0 \le \phi \le \pi/4$, and $0 \le \theta \le 2\pi$. The volume is

$$\begin{aligned}
V = \iiint_Q dV &= \int_0^{2\pi}\int_0^{\pi/4}\int_0^3 \rho^2 \sin \phi \, d\rho\, d\phi\, d\theta \\
&= \int_0^{2\pi}\int_0^{\pi/4} 9 \sin \phi \, d\phi\, d\theta \\
&= 9\int_0^{2\pi} -\cos \phi \Big]_0^{\pi/4} d\theta \\
&= 9\int_0^{2\pi} \left(1 - \frac{\sqrt{2}}{2}\right) d\theta = 9\pi\left(2 - \sqrt{2}\right) \approx 16.563.
\end{aligned}$$

EXAMPLE 5 Finding the Center of Mass of a Solid Region

Find the center of mass of the solid region Q of uniform density, bounded below by the upper nappe of the cone $z^2 = x^2 + y^2$ and above by the sphere $x^2 + y^2 + z^2 = 9$.

Solution Because the density is uniform, you can consider the density at the point (x, y, z) to be k. By symmetry, the center of mass lies on the z-axis, and you need only calculate $\bar{z} = M_{xy}/m$, where $m = kV = 9k\pi\left(2 - \sqrt{2}\right)$ from Example 4. Because $z = \rho \cos \phi$, it follows that

$$\begin{aligned}
M_{xy} = \iiint_Q kz\, dV &= k\int_0^3\int_0^{2\pi}\int_0^{\pi/4} (\rho \cos \phi)\rho^2 \sin \phi \, d\phi\, d\theta\, d\rho \\
&= k\int_0^3\int_0^{2\pi} \rho^3 \frac{\sin^2 \phi}{2}\Big]_0^{\pi/4} d\theta\, d\rho \\
&= \frac{k}{4}\int_0^3\int_0^{2\pi} \rho^3\, d\theta\, d\rho = \frac{k\pi}{2}\int_0^3 \rho^3\, d\rho = \frac{81k\pi}{8}.
\end{aligned}$$

So,

$$\bar{z} = \frac{M_{xy}}{m} = \frac{81k\pi/8}{9k\pi\left(2 - \sqrt{2}\right)} = \frac{9\left(2 + \sqrt{2}\right)}{16} \approx 1.920$$

and the center of mass is approximately $(0, 0, 1.92)$. ■

14.7 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–6, evaluate the iterated integral.

1. $\displaystyle\int_{-1}^{5}\int_{0}^{\pi/2}\int_{0}^{3} r\cos\theta\,dr\,d\theta\,dz$

2. $\displaystyle\int_{0}^{\pi/4}\int_{0}^{6}\int_{0}^{6-r} rz\,dz\,dr\,d\theta$

3. $\displaystyle\int_{0}^{\pi/2}\int_{0}^{2\cos^2\theta}\int_{0}^{4-r^2} r\sin\theta\,dz\,dr\,d\theta$

4. $\displaystyle\int_{0}^{\pi/2}\int_{0}^{\pi}\int_{0}^{2} e^{-\rho^3}\rho^2\,d\rho\,d\theta\,d\phi$

5. $\displaystyle\int_{0}^{2\pi}\int_{0}^{\pi/4}\int_{0}^{\cos\phi} \rho^2\sin\phi\,d\rho\,d\phi\,d\theta$

6. $\displaystyle\int_{0}^{\pi/4}\int_{0}^{\pi/4}\int_{0}^{\cos\theta} \rho^2\sin\phi\cos\phi\,d\rho\,d\theta\,d\phi$

CAS **In Exercises 7 and 8, use a computer algebra system to evaluate the iterated integral.**

7. $\displaystyle\int_{0}^{4}\int_{0}^{z}\int_{0}^{\pi/2} re^r\,d\theta\,dr\,dz$

8. $\displaystyle\int_{0}^{\pi/2}\int_{0}^{\pi}\int_{0}^{\sin\theta} (2\cos\phi)\rho^2\,d\rho\,d\theta\,d\phi$

In Exercises 9–12, sketch the solid region whose volume is given by the iterated integral, and evaluate the iterated integral.

9. $\displaystyle\int_{0}^{\pi/2}\int_{0}^{3}\int_{0}^{e^{-r^2}} r\,dz\,dr\,d\theta$

10. $\displaystyle\int_{0}^{2\pi}\int_{0}^{\sqrt{5}}\int_{0}^{5-r^2} r\,dz\,dr\,d\theta$

11. $\displaystyle\int_{0}^{2\pi}\int_{\pi/6}^{\pi/2}\int_{0}^{4} \rho^2\sin\phi\,d\rho\,d\phi\,d\theta$

12. $\displaystyle\int_{0}^{2\pi}\int_{0}^{\pi}\int_{2}^{5} \rho^2\sin\phi\,d\rho\,d\phi\,d\theta$

In Exercises 13–16, convert the integral from rectangular coordinates to both cylindrical and spherical coordinates, and evaluate the simplest iterated integral.

13. $\displaystyle\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{x^2+y^2}^{4} x\,dz\,dy\,dx$

14. $\displaystyle\int_{0}^{2}\int_{0}^{\sqrt{4-x^2}}\int_{0}^{\sqrt{16-x^2-y^2}} \sqrt{x^2+y^2}\,dz\,dy\,dx$

15. $\displaystyle\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}\int_{a}^{a+\sqrt{a^2-x^2-y^2}} x\,dz\,dy\,dx$

16. $\displaystyle\int_{0}^{3}\int_{0}^{\sqrt{9-x^2}}\int_{0}^{\sqrt{9-x^2-y^2}} \sqrt{x^2+y^2+z^2}\,dz\,dy\,dx$

Volume **In Exercises 17–22, use cylindrical coordinates to find the volume of the solid.**

17. Solid inside both $x^2 + y^2 + z^2 = a^2$ and $(x - a/2)^2 + y^2 = (a/2)^2$

18. Solid inside $x^2 + y^2 + z^2 = 16$ and outside $z = \sqrt{x^2 + y^2}$

19. Solid bounded above by $z = 2x$ and below by $z = 2x^2 + 2y^2$

20. Solid bounded above by $z = 2 - x^2 - y^2$ and below by $z = x^2 + y^2$

21. Solid bounded by the graphs of the sphere $r^2 + z^2 = a^2$ and the cylinder $r = a\cos\theta$

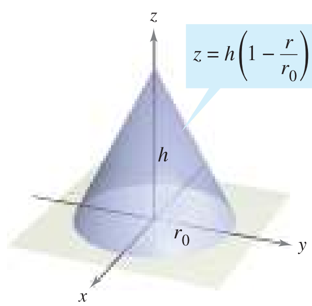
22. Solid inside the sphere $x^2 + y^2 + z^2 = 4$ and above the upper nappe of the cone $z^2 = x^2 + y^2$

Mass **In Exercises 23 and 24, use cylindrical coordinates to find the mass of the solid Q.**

23. $Q = \{(x, y, z)\colon 0 \le z \le 9 - x - 2y,\ x^2 + y^2 \le 4\}$
$\rho(x, y, z) = k\sqrt{x^2 + y^2}$

24. $Q = \{(x, y, z)\colon 0 \le z \le 12e^{-(x^2+y^2)},\ x^2 + y^2 \le 4,\ x \ge 0,\ y \ge 0\}$
$\rho(x, y, z) = k$

In Exercises 25–30, use cylindrical coordinates to find the indicated characteristic of the cone shown in the figure.



25. ***Volume*** Find the volume of the cone.

26. ***Centroid*** Find the centroid of the cone.

CAS 27. ***Center of Mass*** Find the center of mass of the cone assuming that its density at any point is proportional to the distance between the point and the axis of the cone. Use a computer algebra system to evaluate the triple integral.

CAS 28. ***Center of Mass*** Find the center of mass of the cone assuming that its density at any point is proportional to the distance between the point and the base. Use a computer algebra system to evaluate the triple integral.

29. ***Moment of Inertia*** Assume that the cone has uniform density and show that the moment of inertia about the z-axis is

$$I_z = \tfrac{3}{10}mr_0^2.$$

30. ***Moment of Inertia*** Assume that the density of the cone is $\rho(x, y, z) = k\sqrt{x^2 + y^2}$ and find the moment of inertia about the z-axis.

Moment of Inertia **In Exercises 31 and 32, use cylindrical coordinates to verify the given formula for the moment of inertia of the solid of uniform density.**

31. Cylindrical shell: $I_z = \frac{1}{2}m(a^2 + b^2)$
$0 < a \le r \le b,\quad 0 \le z \le h$

CAS **32.** Right circular cylinder: $I_z = \frac{3}{2}ma^2$

$r = 2a \sin \theta, \quad 0 \le z \le h$

Use a computer algebra system to evaluate the triple integral.

Volume **In Exercises 33–36, use spherical coordinates to find the volume of the solid.**

33. Solid inside $x^2 + y^2 + z^2 = 9$, outside $z = \sqrt{x^2 + y^2}$, and above the xy-plane

34. Solid bounded above by $x^2 + y^2 + z^2 = z$ and below by $z = \sqrt{x^2 + y^2}$

CAS **35.** The torus given by $\rho = 4 \sin \phi$ (Use a computer algebra system to evaluate the triple integral.)

36. The solid between the spheres $x^2 + y^2 + z^2 = a^2$ and $x^2 + y^2 + z^2 = b^2$, $b > a$, and inside the cone $z^2 = x^2 + y^2$

Mass **In Exercises 37 and 38, use spherical coordinates to find the mass of the sphere $x^2 + y^2 + z^2 = a^2$ with the given density.**

37. The density at any point is proportional to the distance between the point and the origin.

38. The density at any point is proportional to the distance of the point from the z-axis.

Center of Mass **In Exercises 39 and 40, use spherical coordinates to find the center of mass of the solid of uniform density.**

39. Hemispherical solid of radius r

40. Solid lying between two concentric hemispheres of radii r and R, where $r < R$

Moment of Inertia **In Exercises 41 and 42, use spherical coordinates to find the moment of inertia about the z-axis of the solid of uniform density.**

41. Solid bounded by the hemisphere $\rho = \cos \phi$, $\pi/4 \le \phi \le \pi/2$, and the cone $\phi = \pi/4$

42. Solid lying between two concentric hemispheres of radii r and R, where $r < R$

WRITING ABOUT CONCEPTS

43. Give the equations for conversion from rectangular to cylindrical coordinates and vice versa.

44. Give the equations for conversion from rectangular to spherical coordinates and vice versa.

45. Give the iterated form of the triple integral $\iiint\limits_Q f(x, y, z)\, dV$ in cylindrical form.

46. Give the iterated form of the triple integral $\iiint\limits_Q f(x, y, z)\, dV$ in spherical form.

47. Describe the surface whose equation is a coordinate equal to a constant for each of the coordinates in (a) the cylindrical coordinate system and (b) the spherical coordinate system.

CAPSTONE

48. Convert the integral from rectangular coordinates to both (a) cylindrical and (b) spherical coordinates. Without calculating, which integral appears to be the simplest to evaluate? Why?

$$\int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} \sqrt{x^2 + y^2 + z^2}\, dz\, dy\, dx$$

49. Find the "volume" of the "four-dimensional sphere"

$x^2 + y^2 + z^2 + w^2 = a^2$

by evaluating

$$16 \int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} \int_0^{\sqrt{a^2-x^2-y^2-z^2}} dw\, dz\, dy\, dx.$$

50. Use spherical coordinates to show that

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sqrt{x^2 + y^2 + z^2}\, e^{-(x^2+y^2+z^2)}\, dx\, dy\, dz = 2\pi.$$

PUTNAM EXAM CHALLENGE

51. Find the volume of the region of points (x, y, z) such that $(x^2 + y^2 + z^2 + 8)^2 \le 36(x^2 + y^2)$.

SECTION PROJECT

Wrinkled and Bumpy Spheres

In parts (a) and (b), find the volume of the wrinkled sphere or bumpy sphere. These solids are used as models for tumors.

(a) Wrinkled sphere

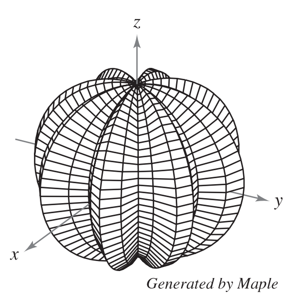
$\rho = 1 + 0.2 \sin 8\theta \sin \phi$

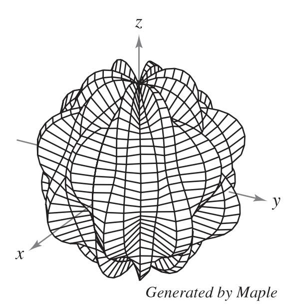
$0 \le \theta \le 2\pi, 0 \le \phi \le \pi$

(b) Bumpy sphere

$\rho = 1 + 0.2 \sin 8\theta \sin 4\phi$

$0 \le \theta \le 2\pi, 0 \le \phi \le \pi$





■ **FOR FURTHER INFORMATION** For more information on these types of spheres, see the article "Heat Therapy for Tumors" by Leah Edelstein-Keshet in *The UMAP Journal*.

14.8 Change of Variables: Jacobians

- **Understand the concept of a Jacobian.**
- **Use a Jacobian to change variables in a double integral.**

CARL GUSTAV JACOBI (1804–1851)

The Jacobian is named after the German mathematician Carl Gustav Jacobi. Jacobi is known for his work in many areas of mathematics, but his interest in integration stemmed from the problem of finding the circumference of an ellipse.

Jacobians

For the single integral

$$\int_a^b f(x)\,dx$$

you can change variables by letting $x = g(u)$, so that $dx = g'(u)\,du$, and obtain

$$\int_a^b f(x)\,dx = \int_c^d f(g(u))g'(u)\,du$$

where $a = g(c)$ and $b = g(d)$. Note that the change of variables process introduces an additional factor $g'(u)$ into the integrand. This also occurs in the case of double integrals

$$\int_R\!\int f(x, y)\,dA = \int_S\!\int f(g(u, v), h(u, v))\underbrace{\left|\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}\right|}_{\text{Jacobian}}\,du\,dv$$

where the change of variables $x = g(u, v)$ and $y = h(u, v)$ introduces a factor called the **Jacobian** of x and y with respect to u and v. In defining the Jacobian, it is convenient to use the following determinant notation.

DEFINITION OF THE JACOBIAN

If $x = g(u, v)$ and $y = h(u, v)$, then the **Jacobian** of x and y with respect to u and v, denoted by $\partial(x, y)/\partial(u, v)$, is

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\[2mm] \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}.$$

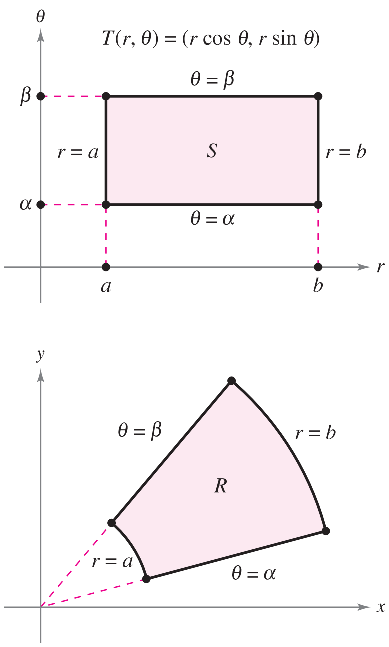
EXAMPLE 1 The Jacobian for Rectangular-to-Polar Conversion

Find the Jacobian for the change of variables defined by

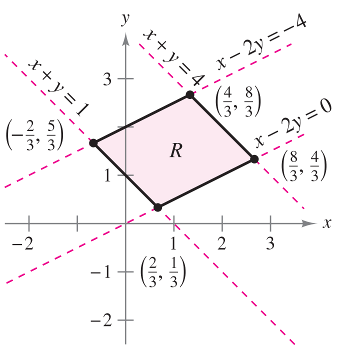
$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta.$$

Solution From the definition of the Jacobian, you obtain

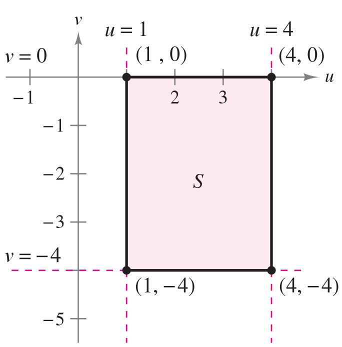
$$\begin{aligned}
\frac{\partial(x, y)}{\partial(r, \theta)} &= \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} \\[2mm] \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} \end{vmatrix} \\
&= \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} \\
&= r\cos^2\theta + r\sin^2\theta \\
&= r.
\end{aligned}$$



S is the region in the $r\theta$-plane that corresponds to R in the xy-plane.
Figure 14.71



Region R in the xy-plane
Figure 14.72



Region S in the uv-plane
Figure 14.73

Example 1 points out that the change of variables from rectangular to polar coordinates for a double integral can be written as

$$\begin{aligned}\int_R\!\!\int f(x, y)\, dA &= \int_S\!\!\int f(r\cos\theta, r\sin\theta) r\, dr\, d\theta, \quad r > 0\\ &= \int_S\!\!\int f(r\cos\theta, r\sin\theta)\left|\frac{\partial(x, y)}{\partial(r, \theta)}\right| dr\, d\theta\end{aligned}$$

where S is the region in the $r\theta$-plane that corresponds to the region R in the xy-plane, as shown in Figure 14.71. This formula is similar to that found in Theorem 14.3 on page 1006.

In general, a change of variables is given by a one-to-one **transformation** T from a region S in the uv-plane to a region R in the xy-plane, to be given by

$$T(u, v) = (x, y) = (g(u, v), h(u, v))$$

where g and h have continuous first partial derivatives in the region S. Note that the point (u, v) lies in S and the point (x, y) lies in R. In most cases, you are hunting for a transformation in which the region S is simpler than the region R.

EXAMPLE 2 Finding a Change of Variables to Simplify a Region

Let R be the region bounded by the lines

$$x - 2y = 0, \quad x - 2y = -4, \quad x + y = 4, \quad \text{and} \quad x + y = 1$$

as shown in Figure 14.72. Find a transformation T from a region S to R such that S is a rectangular region (with sides parallel to the u- or v-axis).

Solution To begin, let $u = x + y$ and $v = x - 2y$. Solving this system of equations for x and y produces $T(u, v) = (x, y)$, where

$$x = \frac{1}{3}(2u + v) \quad \text{and} \quad y = \frac{1}{3}(u - v).$$

The four boundaries for R in the xy-plane give rise to the following bounds for S in the uv-plane.

Bounds in the xy-Plane		*Bounds in the uv-Plane*
$x + y = 1$	⇨	$u = 1$
$x + y = 4$	⇨	$u = 4$
$x - 2y = 0$	⇨	$v = 0$
$x - 2y = -4$	⇨	$v = -4$

The region S is shown in Figure 14.73. Note that the transformation

$$T(u, v) = (x, y) = \left(\frac{1}{3}[2u + v], \frac{1}{3}[u - v]\right)$$

maps the vertices of the region S onto the vertices of the region R. For instance,

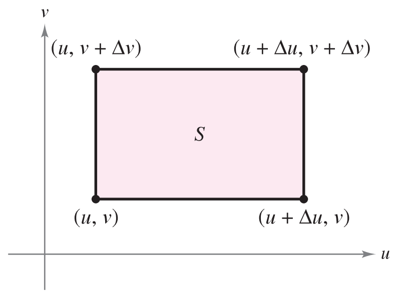
$$\begin{aligned}T(1, 0) &= \left(\tfrac{1}{3}[2(1) + 0], \tfrac{1}{3}[1 - 0]\right) = \left(\tfrac{2}{3}, \tfrac{1}{3}\right)\\ T(4, 0) &= \left(\tfrac{1}{3}[2(4) + 0], \tfrac{1}{3}[4 - 0]\right) = \left(\tfrac{8}{3}, \tfrac{4}{3}\right)\\ T(4, -4) &= \left(\tfrac{1}{3}[2(4) - 4], \tfrac{1}{3}[4 - (-4)]\right) = \left(\tfrac{4}{3}, \tfrac{8}{3}\right)\\ T(1, -4) &= \left(\tfrac{1}{3}[2(1) - 4], \tfrac{1}{3}[1 - (-4)]\right) = \left(-\tfrac{2}{3}, \tfrac{5}{3}\right).\end{aligned}$$

Change of Variables for Double Integrals

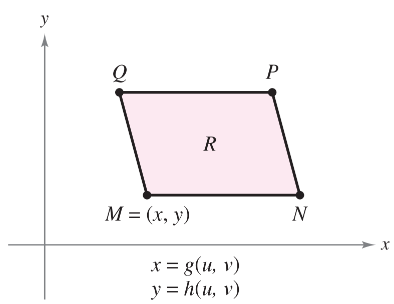
THEOREM 14.5 CHANGE OF VARIABLES FOR DOUBLE INTEGRALS

Let R be a vertically or horizontally simple region in the xy-plane, and let S be a vertically or horizontally simple region in the uv-plane. Let T from S to R be given by $T(u, v) = (x, y) = (g(u, v), h(u, v))$, where g and h have continuous first partial derivatives. Assume that T is one-to-one except possibly on the boundary of S. If f is continuous on R, and $\partial(x, y)/\partial(u, v)$ is nonzero on S, then

$$\int_R\int f(x, y)\,dx\,dy = \int_S\int f(g(u, v), h(u, v))\left|\frac{\partial(x, y)}{\partial(u, v)}\right| du\,dv.$$



Area of $S = \Delta u\,\Delta v$
$\Delta u > 0,\ \Delta v > 0$
Figure 14.74



The vertices in the xy-plane are $M(g(u, v), h(u, v))$, $N(g(u + \Delta u, v), h(u + \Delta u, v))$, $P(g(u + \Delta u, v + \Delta v), h(u + \Delta u, v + \Delta v))$, and $Q(g(u, v + \Delta v), h(u, v + \Delta v))$.
Figure 14.75

PROOF Consider the case in which S is a rectangular region in the uv-plane with vertices (u, v), $(u + \Delta u, v)$, $(u + \Delta u, v + \Delta v)$, and $(u, v + \Delta v)$, as shown in Figure 14.74. The images of these vertices in the xy-plane are shown in Figure 14.75. If Δu and Δv are small, the continuity of g and h implies that R is approximately a parallelogram determined by the vectors $\overrightarrow{MN}$ and $\overrightarrow{MQ}$. So, the area of R is

$$\Delta A \approx \|\overrightarrow{MN} \times \overrightarrow{MQ}\|.$$

Moreover, for small Δu and Δv, the partial derivatives of g and h with respect to u can be approximated by

$$g_u(u, v) \approx \frac{g(u + \Delta u, v) - g(u, v)}{\Delta u} \quad \text{and} \quad h_u(u, v) \approx \frac{h(u + \Delta u, v) - h(u, v)}{\Delta u}.$$

Consequently,

$$\begin{aligned}\overrightarrow{MN} &= [g(u + \Delta u, v) - g(u, v)]\mathbf{i} + [h(u + \Delta u, v) - h(u, v)]\mathbf{j}\\ &\approx [g_u(u, v)\,\Delta u]\mathbf{i} + [h_u(u, v)\,\Delta u]\mathbf{j}\\ &= \frac{\partial x}{\partial u}\Delta u\,\mathbf{i} + \frac{\partial y}{\partial u}\Delta u\,\mathbf{j}.\end{aligned}$$

Similarly, you can approximate $\overrightarrow{MQ}$ by $\dfrac{\partial x}{\partial v}\Delta v\,\mathbf{i} + \dfrac{\partial y}{\partial v}\Delta v\,\mathbf{j}$, which implies that

$$\overrightarrow{MN} \times \overrightarrow{MQ} \approx \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial x}{\partial u}\Delta u & \dfrac{\partial y}{\partial u}\Delta u & 0 \\ \dfrac{\partial x}{\partial v}\Delta v & \dfrac{\partial y}{\partial v}\Delta v & 0 \end{vmatrix} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial y}{\partial u} \\ \dfrac{\partial x}{\partial v} & \dfrac{\partial y}{\partial v} \end{vmatrix} \Delta u\,\Delta v\,\mathbf{k}.$$

It follows that, in Jacobian notation,

$$\Delta A \approx \|\overrightarrow{MN} \times \overrightarrow{MQ}\| \approx \left|\frac{\partial(x, y)}{\partial(u, v)}\right| \Delta u\,\Delta v.$$

Because this approximation improves as Δu and Δv approach 0, the limiting case can be written as

$$dA \approx \|\overrightarrow{MN} \times \overrightarrow{MQ}\| \approx \left|\frac{\partial(x, y)}{\partial(u, v)}\right| du\,dv.$$

So,

$$\int_R\int f(x, y)\,dx\,dy = \int_S\int f(g(u, v), h(u, v))\left|\frac{\partial(x, y)}{\partial(u, v)}\right| du\,dv. \qquad \blacksquare$$

The next two examples show how a change of variables can simplify the integration process. The simplification can occur in various ways. You can make a change of variables to simplify either the *region* R or the *integrand* $f(x, y)$, or both.

EXAMPLE 3 Using a Change of Variables to Simplify a Region

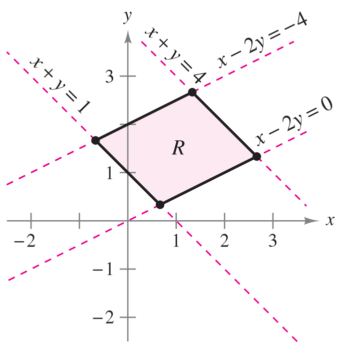


Figure 14.76

Let R be the region bounded by the lines

$$x - 2y = 0, \quad x - 2y = -4, \quad x + y = 4, \quad \text{and} \quad x + y = 1$$

as shown in Figure 14.76. Evaluate the double integral

$$\int_R\int 3xy\, dA.$$

Solution From Example 2, you can use the following change of variables.

$$x = \frac{1}{3}(2u + v) \quad \text{and} \quad y = \frac{1}{3}(u - v)$$

The partial derivatives of x and y are

$$\frac{\partial x}{\partial u} = \frac{2}{3}, \quad \frac{\partial x}{\partial v} = \frac{1}{3}, \quad \frac{\partial y}{\partial u} = \frac{1}{3}, \quad \text{and} \quad \frac{\partial y}{\partial v} = -\frac{1}{3}$$

which implies that the Jacobian is

$$\begin{aligned}
\frac{\partial(x, y)}{\partial(u, v)} &= \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \\
&= \begin{vmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} \end{vmatrix} \\
&= -\frac{2}{9} - \frac{1}{9} \\
&= -\frac{1}{3}.
\end{aligned}$$

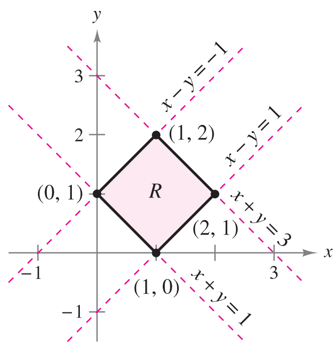
So, by Theorem 14.5, you obtain

$$\begin{aligned}
\int_R\int 3xy\, dA &= \int_S\int 3\left[\frac{1}{3}(2u + v)\frac{1}{3}(u - v)\right]\left|\frac{\partial(x, y)}{\partial(u, v)}\right| dv\, du \\
&= \int_1^4\int_{-4}^0 \frac{1}{9}(2u^2 - uv - v^2)\, dv\, du \\
&= \frac{1}{9}\int_1^4 \left[2u^2v - \frac{uv^2}{2} - \frac{v^3}{3}\right]_{-4}^0 du \\
&= \frac{1}{9}\int_1^4 \left(8u^2 + 8u - \frac{64}{3}\right) du \\
&= \frac{1}{9}\left[\frac{8u^3}{3} + 4u^2 - \frac{64}{3}u\right]_1^4 \\
&= \frac{164}{9}.
\end{aligned}$$

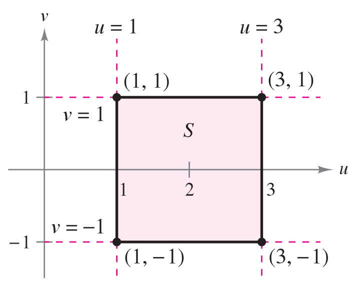
EXAMPLE 4 Using a Change of Variables to Simplify an Integrand

Let R be the region bounded by the square with vertices $(0, 1)$, $(1, 2)$, $(2, 1)$, and $(1, 0)$. Evaluate the integral

$$\int_R\int (x + y)^2 \sin^2(x - y)\, dA.$$



Region R in the xy-plane
Figure 14.77



Region S in the uv-plane
Figure 14.78

Solution Note that the sides of R lie on the lines $x + y = 1$, $x - y = 1$, $x + y = 3$, and $x - y = -1$, as shown in Figure 14.77. Letting $u = x + y$ and $v = x - y$, you can determine the bounds for region S in the uv-plane to be

$$1 \le u \le 3 \qquad \text{and} \qquad -1 \le v \le 1$$

as shown in Figure 14.78. Solving for x and y in terms of u and v produces

$$x = \frac{1}{2}(u + v) \qquad \text{and} \qquad y = \frac{1}{2}(u - v).$$

The partial derivatives of x and y are

$$\frac{\partial x}{\partial u} = \frac{1}{2}, \quad \frac{\partial x}{\partial v} = \frac{1}{2}, \quad \frac{\partial y}{\partial u} = \frac{1}{2}, \quad \text{and} \quad \frac{\partial y}{\partial v} = -\frac{1}{2}$$

which implies that the Jacobian is

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \dfrac{1}{2} & \dfrac{1}{2} \\ \dfrac{1}{2} & -\dfrac{1}{2} \end{vmatrix} = -\frac{1}{4} - \frac{1}{4} = -\frac{1}{2}.$$

By Theorem 14.5, it follows that

$$\begin{aligned}
\int_R\int (x + y)^2 \sin^2(x - y)\, dA &= \int_{-1}^{1}\int_{1}^{3} u^2 \sin^2 v\left(\frac{1}{2}\right) du\, dv \\
&= \frac{1}{2}\int_{-1}^{1} (\sin^2 v)\, \frac{u^3}{3}\Big]_1^3\, dv \\
&= \frac{13}{3}\int_{-1}^{1} \sin^2 v\, dv \\
&= \frac{13}{6}\int_{-1}^{1} (1 - \cos 2v)\, dv \\
&= \frac{13}{6}\left[v - \frac{1}{2}\sin 2v\right]_{-1}^{1} \\
&= \frac{13}{6}\left[2 - \frac{1}{2}\sin 2 + \frac{1}{2}\sin(-2)\right] \\
&= \frac{13}{6}(2 - \sin 2) \\
&\approx 2.363.
\end{aligned}$$

In each of the change of variables examples in this section, the region S has been a rectangle with sides parallel to the u- or v-axis. Occasionally, a change of variables can be used for other types of regions. For instance, letting $T(u, v) = \left(x, \frac{1}{2}y\right)$ changes the circular region $u^2 + v^2 = 1$ to the elliptical region $x^2 + (y^2/4) = 1$.

14.8 Exercises

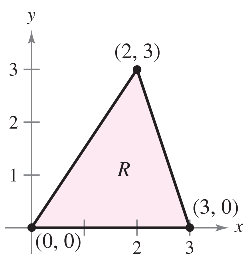
See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–8, find the Jacobian $\partial(x, y)/\partial(u, v)$ for the indicated change of variables.

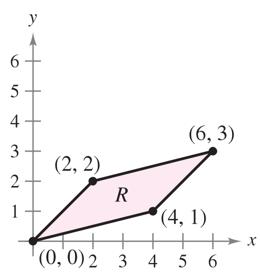
1. $x = -\frac{1}{2}(u - v),\ y = \frac{1}{2}(u + v)$
2. $x = au + bv,\ y = cu + dv$
3. $x = u - v^2,\ y = u + v$
4. $x = uv - 2u,\ y = uv$
5. $x = u\cos\theta - v\sin\theta,\ y = u\sin\theta + v\cos\theta$
6. $x = u + a,\ y = v + a$
7. $x = e^u \sin v,\ y = e^u \cos v$
8. $x = \dfrac{u}{v},\ y = u + v$

In Exercises 9–12, sketch the image S in the uv-plane of the region R in the xy-plane using the given transformations.

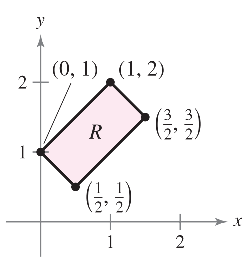
9. $x = 3u + 2v$
$y = 3v$



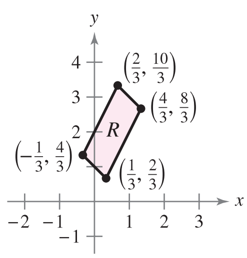
10. $x = \frac{1}{3}(4u - v)$
$y = \frac{1}{3}(u - v)$



11. $x = \frac{1}{2}(u + v)$
$y = \frac{1}{2}(u - v)$



12. $x = \frac{1}{3}(v - u)$
$y = \frac{1}{3}(2v + u)$



CAS **In Exercises 13 and 14, verify the result of the indicated example by setting up the integral using $dy\,dx$ or $dx\,dy$ for dA. Then use a computer algebra system to evaluate the integral.**

13. Example 3

14. Example 4

In Exercises 15–20, use the indicated change of variables to evaluate the double integral.

15. $\displaystyle\int_R\int 4(x^2 + y^2)\,dA$
$x = \frac{1}{2}(u + v)$
$y = \frac{1}{2}(u - v)$

16. $\displaystyle\int_R\int 60xy\,dA$
$x = \frac{1}{2}(u + v)$
$y = -\frac{1}{2}(u - v)$

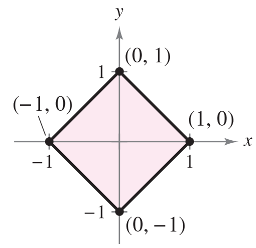


Figure for 15

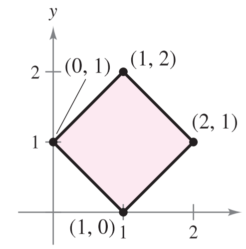
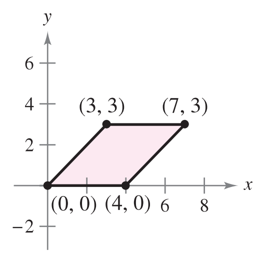
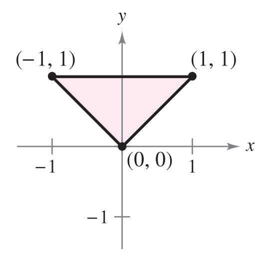


Figure for 16

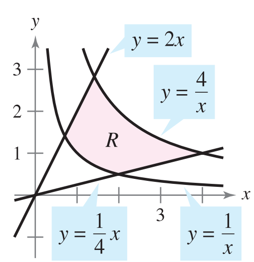
17. $\displaystyle\int_R\int y(x - y)\,dA$
$x = u + v$
$y = u$



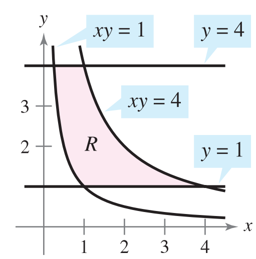
18. $\displaystyle\int_R\int 4(x + y)e^{x-y}\,dA$
$x = \frac{1}{2}(u + v)$
$y = \frac{1}{2}(u - v)$



19. $\displaystyle\int_R\int e^{-xy/2}\,dA$
$x = \sqrt{\dfrac{v}{u}},\ y = \sqrt{uv}$



20. $\displaystyle\int_R\int y\sin xy\,dA$
$x = \dfrac{u}{v},\quad y = v$



In Exercises 21–28, use a change of variables to find the volume of the solid region lying below the surface $z = f(x, y)$ and above the plane region R.

21. $f(x, y) = 48xy$
R: region bounded by the square with vertices $(1, 0)$, $(0, 1)$, $(1, 2)$, $(2, 1)$

22. $f(x, y) = (3x + 2y)^2\sqrt{2y - x}$
R: region bounded by the parallelogram with vertices $(0, 0)$, $(-2, 3)$, $(2, 5)$, $(4, 2)$

23. $f(x, y) = (x + y)e^{x-y}$
R: region bounded by the square with vertices $(4, 0)$, $(6, 2)$, $(4, 4)$, $(2, 2)$

24. $f(x, y) = (x + y)^2 \sin^2(x - y)$

R: region bounded by the square with vertices $(\pi, 0)$, $(3\pi/2, \pi/2)$, (π, π), $(\pi/2, \pi/2)$

25. $f(x, y) = \sqrt{(x - y)(x + 4y)}$

R: region bounded by the parallelogram with vertices $(0, 0)$, $(1, 1)$, $(5, 0)$, $(4, -1)$

26. $f(x, y) = (3x + 2y)(2y - x)^{3/2}$

R: region bounded by the parallelogram with vertices $(0, 0)$, $(-2, 3)$, $(2, 5)$, $(4, 2)$

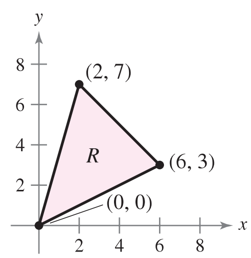
27. $f(x, y) = \sqrt{x + y}$

R: region bounded by the triangle with vertices $(0, 0)$, $(a, 0)$, $(0, a)$, where $a > 0$

28. $f(x, y) = \dfrac{xy}{1 + x^2y^2}$

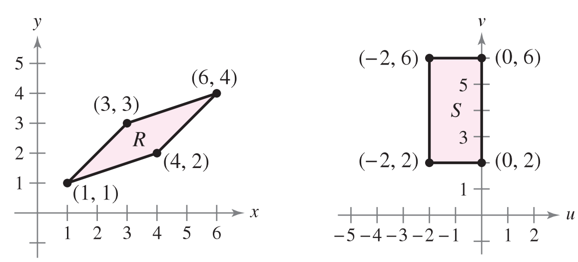
R: region bounded by the graphs of $xy = 1$, $xy = 4$, $x = 1$, $x = 4$ (*Hint:* Let $x = u$, $y = v/u$.)

29. The substitutions $u = 2x - y$ and $v = x + y$ make the region R (see figure) into a simpler region S in the uv-plane. Determine the total number of sides of S that are parallel to either the u-axis or the v-axis.



CAPSTONE

30. Find a transformation $T(u, v) = (x, y) = (g(u, v), h(u, v))$ that when applied to the region R will result in the image S (see figure). Explain your reasoning.



31. Consider the region R in the xy-plane bounded by the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

and the transformations $x = au$ and $y = bv$.

(a) Sketch the graph of the region R and its image S under the given transformation.

(b) Find $\dfrac{\partial(x, y)}{\partial(u, v)}$.

(c) Find the area of the ellipse.

32. Use the result of Exercise 31 to find the volume of each dome-shaped solid lying below the surface $z = f(x, y)$ and above the elliptical region R. (*Hint:* After making the change of variables given by the results in Exercise 31, make a second change of variables to polar coordinates.)

(a) $f(x, y) = 16 - x^2 - y^2$

R: $\dfrac{x^2}{16} + \dfrac{y^2}{9} \le 1$

(b) $f(x, y) = A \cos\left(\dfrac{\pi}{2}\sqrt{\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2}}\right)$

R: $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} \le 1$

WRITING ABOUT CONCEPTS

33. State the definition of the Jacobian.

34. Describe how to use the Jacobian to change variables in double integrals.

In Exercises 35–40, find the Jacobian $\partial(x, y, z)/\partial(u, v, w)$ for the indicated change of variables. If $x = f(u, v, w)$, $y = g(u, v, w)$, and $z = h(u, v, w)$, then the Jacobian of x, y, and z with respect to u, v, and w is

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} & \dfrac{\partial x}{\partial w} \\[2mm] \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} & \dfrac{\partial y}{\partial w} \\[2mm] \dfrac{\partial z}{\partial u} & \dfrac{\partial z}{\partial v} & \dfrac{\partial z}{\partial w} \end{vmatrix}.$$

35. $x = u(1 - v)$, $y = uv(1 - w)$, $z = uvw$

36. $x = 4u - v$, $y = 4v - w$, $z = u + w$

37. $x = \frac{1}{2}(u + v)$, $y = \frac{1}{2}(u - v)$, $z = 2uvw$

38. $x = u - v + w$, $y = 2uv$, $z = u + v + w$

39. ***Spherical Coordinates***

$x = \rho \sin\phi \cos\theta$, $y = \rho \sin\phi \sin\theta$, $z = \rho \cos\phi$

40. ***Cylindrical Coordinates***

$x = r\cos\theta$, $y = r\sin\theta$, $z = z$

PUTNAM EXAM CHALLENGE

41. Let A be the area of the region in the first quadrant bounded by the line $y = \frac{1}{2}x$, the x-axis, and the ellipse $\frac{1}{9}x^2 + y^2 = 1$. Find the positive number m such that A is equal to the area of the region in the first quadrant bounded by the line $y = mx$, the y-axis, and the ellipse $\frac{1}{9}x^2 + y^2 = 1$.

14 REVIEW EXERCISES

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1 and 2, evaluate the integral.

1. $\displaystyle\int_1^{x^2} x \ln y \, dy$

2. $\displaystyle\int_y^{2y} (x^2 + y^2)\, dx$

In Exercises 3–6, sketch the region of integration. Then evaluate the iterated integral. Change the coordinate system when convenient.

3. $\displaystyle\int_0^1 \int_0^{1+x} (3x + 2y)\, dy\, dx$

4. $\displaystyle\int_0^2 \int_{x^2}^{2x} (x^2 + 2y)\, dy\, dx$

5. $\displaystyle\int_0^3 \int_0^{\sqrt{9-x^2}} 4x\, dy\, dx$

6. $\displaystyle\int_0^{\sqrt{3}} \int_{2-\sqrt{4-y^2}}^{2+\sqrt{4-y^2}} dx\, dy$

Area **In Exercises 7–14, write the limits for the double integral**

$$\int_R\int f(x, y)\, dA$$

for both orders of integration. Compute the area of R by letting $f(x, y) = 1$ and integrating.

7. Triangle: vertices $(0, 0), (3, 0), (0, 1)$
8. Triangle: vertices $(0, 0), (3, 0), (2, 2)$
9. The larger area between the graphs of $x^2 + y^2 = 25$ and $x = 3$
10. Region bounded by the graphs of $y = 6x - x^2$ and $y = x^2 - 2x$
11. Region enclosed by the graph of $y^2 = x^2 - x^4$
12. Region bounded by the graphs of $x = y^2 + 1$, $x = 0$, $y = 0$, and $y = 2$
13. Region bounded by the graphs of $x = y + 3$ and $x = y^2 + 1$
14. Region bounded by the graphs of $x = -y$ and $x = 2y - y^2$

Think About It **In Exercises 15 and 16, give a geometric argument for the given equality. Verify the equality analytically.**

15. $\displaystyle\int_0^1 \int_{2y}^{2\sqrt{2-y^2}} (x + y)\, dx\, dy = \int_0^2 \int_0^{x/2} (x + y)\, dy\, dx + \int_2^{2\sqrt{2}} \int_0^{\sqrt{8-x^2}/2} (x + y)\, dy\, dx$

16. $\displaystyle\int_0^2 \int_{3y/2}^{5-y} e^{x+y}\, dx\, dy = \int_0^3 \int_0^{2x/3} e^{x+y}\, dy\, dx + \int_3^5 \int_0^{5-x} e^{x+y}\, dy\, dx$

Volume **In Exercises 17 and 18, use a multiple integral and a convenient coordinate system to find the volume of the solid.**

17. Solid bounded by the graphs of $z = x^2 - y + 4$, $z = 0$, $y = 0$, $x = 0$, and $x = 4$

18. Solid bounded by the graphs of $z = x + y$, $z = 0$, $y = 0$, $x = 3$, and $y = x$

Average Value **In Exercises 19 and 20, find the average of $f(x, y)$ over the region R.**

19. $f(x) = 16 - x^2 - y^2$

    R: rectangle with vertices $(2, 2), (-2, 2), (-2, -2), (2, -2)$

20. $f(x) = 2x^2 + y^2$

    R: square with vertices $(0, 0), (3, 0), (3, 3), (0, 3)$

21. ***Average Temperature*** The temperature in degrees Celsius on the surface of a metal plate is

    $T(x, y) = 40 - 6x^2 - y^2$

    where x and y are measured in centimeters. Estimate the average temperature if x varies between 0 and 3 centimeters and y varies between 0 and 5 centimeters.

CAS 22. ***Average Profit*** A firm's profit P from marketing two soft drinks is

    $P = 192x + 576y - x^2 - 5y^2 - 2xy - 5000$

    where x and y represent the numbers of units of the two soft drinks. Use a computer algebra system to evaluate the double integral yielding the average weekly profit if x varies between 40 and 50 units and y varies between 45 and 60 units.

Probability **In Exercises 23 and 24, find k such that the function is a joint density function and find the required probability, where**

$$P(a \le x \le b, c \le y \le d) = \int_c^d \int_a^b f(x, y)\, dx\, dy.$$

23. $f(x, y) = \begin{cases} kxye^{-(x+y)}, & x \ge 0, y \ge 0 \\ 0, & \text{elsewhere} \end{cases}$

    $P(0 \le x \le 1, 0 \le y \le 1)$

24. $f(x, y) = \begin{cases} kxy, & 0 \le x \le 1, 0 \le y \le x \\ 0, & \text{elsewhere} \end{cases}$

    $P(0 \le x \le 0.5, 0 \le y \le 0.25)$

Approximation **In Exercises 25 and 26, determine which value best approximates the volume of the solid between the xy-plane and the function over the region. (Make your selection on the basis of a sketch of the solid and *not* by performing any calculations.)**

25. $f(x, y) = x + y$

    R: triangle with vertices $(0, 0), (3, 0), (3, 3)$

    (a) $\frac{9}{2}$ (b) 5 (c) 13 (d) 100 (e) -100

26. $f(x, y) = 10x^2y^2$

    R: circle bounded by $x^2 + y^2 = 1$

    (a) π (b) -15 (c) $\frac{2}{3}$ (d) 3 (e) 15

True or False? **In Exercises 27–30, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

27. $\int_a^b \int_c^d f(x)g(y)\,dy\,dx = \left[\int_a^b f(x)\,dx\right]\left[\int_c^d g(y)\,dy\right]$

28. If f is continuous over R_1 and R_2, and

$$\int_{R_1}\int dA = \int_{R_2}\int dA$$

then

$$\int_{R_1}\int f(x, y)\,dA = \int_{R_2}\int f(x, y)\,dA.$$

29. $\int_{-1}^1 \int_{-1}^1 \cos(x^2 + y^2)\,dx\,dy = 4\int_0^1 \int_0^1 \cos(x^2 + y^2)\,dx\,dy$

30. $\int_0^1 \int_0^1 \frac{1}{1 + x^2 + y^2}\,dx\,dy < \frac{\pi}{4}$

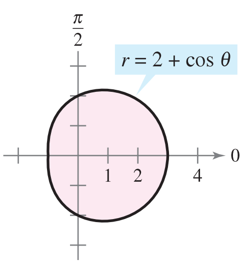
In Exercises 31 and 32, evaluate the iterated integral by converting to polar coordinates.

31. $\int_0^h \int_0^x \sqrt{x^2 + y^2}\,dy\,dx$

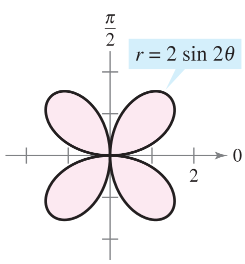
32. $\int_0^4 \int_0^{\sqrt{16-y^2}} (x^2 + y^2)\,dx\,dy$

Area **In Exercises 33 and 34, use a double integral to find the area of the shaded region.**

33.



34.



Volume **In Exercises 35 and 36, use a multiple integral and a convenient coordinate system to find the volume of the solid.**

35. Solid bounded by the graphs of $z = 0$ and $z = h$, outside the cylinder $x^2 + y^2 = 1$ and inside the hyperboloid $x^2 + y^2 - z^2 = 1$

36. Solid that remains after drilling a hole of radius b through the center of a sphere of radius R $(b < R)$

37. Consider the region R in the xy-plane bounded by the graph of the equation

$(x^2 + y^2)^2 = 9(x^2 - y^2)$.

(a) Convert the equation to polar coordinates. Use a graphing utility to graph the equation.

(b) Use a double integral to find the area of the region R.

CAS (c) Use a computer algebra system to determine the volume of the solid over the region R and beneath the hemisphere $z = \sqrt{9 - x^2 - y^2}$.

38. Combine the sum of the two iterated integrals into a single iterated integral by converting to polar coordinates. Evaluate the resulting iterated integral.

$$\int_0^{8/\sqrt{13}} \int_0^{3x/2} xy\,dy\,dx + \int_{8/\sqrt{13}}^4 \int_0^{\sqrt{16-x^2}} xy\,dy\,dx$$

CAS ***Mass and Center of Mass*** **In Exercises 39 and 40, find the mass and center of mass of the lamina bounded by the graphs of the equations for the given density or densities. Use a computer algebra system to evaluate the multiple integrals.**

39. $y = 2x$, $y = 2x^3$, first quadrant

(a) $\rho = kxy$

(b) $\rho = k(x^2 + y^2)$

40. $y = \frac{h}{2}\left(2 - \frac{x}{L} - \frac{x^2}{L^2}\right)$, $\rho = k$, first quadrant

CAS **In Exercises 41 and 42, find I_x, I_y, I_0, $\bar{\bar{x}}$, and $\bar{\bar{y}}$ for the lamina bounded by the graphs of the equations. Use a computer algebra system to evaluate the double integrals.**

41. $y = 0$, $y = b$, $x = 0$, $x = a$, $\rho = kx$

42. $y = 4 - x^2$, $y = 0$, $x > 0$, $\rho = ky$

Surface Area **In Exercises 43–46, find the area of the surface given by $z = f(x, y)$ over the region R.**

43. $f(x, y) = 25 - x^2 - y^2$

$R = \{(x, y): x^2 + y^2 \le 25\}$

CAS **44.** $f(x, y) = 16 - x - y^2$

$R = \{(x, y): 0 \le x \le 2, 0 \le y \le x\}$

Use a computer algebra system to evaluate the integral.

45. $f(x, y) = 9 - y^2$

R: triangle bounded by the graphs of the equations $y = x$, $y = -x$, and $y = 3$

46. $f(x, y) = 4 - x^2$

R: triangle bounded by the graphs of the equations $y = x$, $y = -x$, and $y = 2$

47. ***Building Design*** A new auditorium is built with a foundation in the shape of one-fourth of a circle of radius 50 feet. So, it forms a region R bounded by the graph of

$x^2 + y^2 = 50^2$

with $x \ge 0$ and $y \ge 0$. The following equations are models for the floor and ceiling.

Floor: $z = \frac{x + y}{5}$

Ceiling: $z = 20 + \frac{xy}{100}$

(a) Calculate the volume of the room, which is needed to determine the heating and cooling requirements.

(b) Find the surface area of the ceiling.

CAS **48.** ***Surface Area*** The roof over the stage of an open air theater at a theme park is modeled by

$$f(x, y) = 25\left[1 + e^{-(x^2+y^2)/1000}\cos^2\left(\frac{x^2+y^2}{1000}\right)\right]$$

where the stage is a semicircle bounded by the graphs of $y = \sqrt{50^2 - x^2}$ and $y = 0$.

(a) Use a computer algebra system to graph the surface.

(b) Use a computer algebra system to approximate the number of square feet of roofing required to cover the surface.

In Exercises 49–52, evaluate the iterated integral.

49. $\displaystyle\int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_{x^2+y^2}^{9}\sqrt{x^2+y^2}\,dz\,dy\,dx$

50. $\displaystyle\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{0}^{(x^2+y^2)/2}(x^2+y^2)\,dz\,dy\,dx$

51. $\displaystyle\int_{0}^{a}\int_{0}^{b}\int_{0}^{c}(x^2+y^2+z^2)\,dx\,dy\,dz$

52. $\displaystyle\int_{0}^{5}\int_{0}^{\sqrt{25-x^2}}\int_{0}^{\sqrt{25-x^2-y^2}}\frac{1}{1+x^2+y^2+z^2}\,dz\,dy\,dx$

CAS **In Exercises 53 and 54, use a computer algebra system to evaluate the iterated integral.**

53. $\displaystyle\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}}(x^2+y^2)\,dz\,dy\,dx$

54. $\displaystyle\int_{0}^{2}\int_{0}^{\sqrt{4-x^2}}\int_{0}^{\sqrt{4-x^2-y^2}}xyz\,dz\,dy\,dx$

Volume **In Exercises 55 and 56, use a multiple integral to find the volume of the solid.**

55. Solid inside the graphs of $r = 2\cos\theta$ and $r^2 + z^2 = 4$

56. Solid inside the graphs of $r^2 + z = 16$, $z = 0$, and $r = 2\sin\theta$

Center of Mass **In Exercises 57–60, find the center of mass of the solid of uniform density bounded by the graphs of the equations.**

57. Solid inside the hemisphere $\rho = \cos\phi$, $\pi/4 \le \phi \le \pi/2$, and outside the cone $\phi = \pi/4$

58. Wedge: $x^2 + y^2 = a^2$, $z = cy\ (c > 0)$, $y \ge 0$, $z \ge 0$

59. $x^2 + y^2 + z^2 = a^2$, first octant

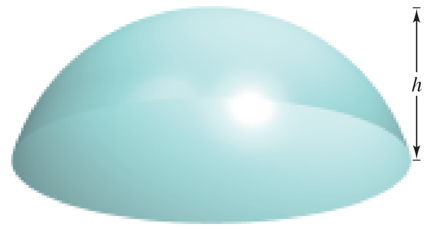
60. $x^2 + y^2 + z^2 = 25$, $z = 4$ (the larger solid)

Moment of Inertia **In Exercises 61 and 62, find the moment of inertia I_z of the solid of given density.**

61. The solid of uniform density inside the paraboloid $z = 16 - x^2 - y^2$, and outside the cylinder $x^2 + y^2 = 9$, $z \ge 0$.

62. $x^2 + y^2 + z^2 = a^2$, density proportional to the distance from the center

63. ***Investigation*** Consider a spherical segment of height h from a sphere of radius a, where $h \le a$, and constant density $\rho(x, y, z) = k$ (see figure).



(a) Find the volume of the solid.

(b) Find the centroid of the solid.

(c) Use the result of part (b) to find the centroid of a hemisphere of radius a.

(d) Find $\lim_{h\to 0}\bar{z}$.

(e) Find I_z.

(f) Use the result of part (e) to find I_z for a hemisphere.

64. ***Moment of Inertia*** Find the moment of inertia about the z-axis of the ellipsoid $x^2 + y^2 + \dfrac{z^2}{a^2} = 1$, where $a > 0$.

In Exercises 65 and 66, give a geometric interpretation of the iterated integral.

65. $\displaystyle\int_{0}^{2\pi}\int_{0}^{\pi}\int_{0}^{6\sin\phi}\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$

66. $\displaystyle\int_{0}^{\pi}\int_{0}^{2}\int_{0}^{1+r^2}r\,dz\,dr\,d\theta$

In Exercises 67 and 68, find the Jacobian $\partial(x, y)/\partial(u, v)$ for the indicated change of variables.

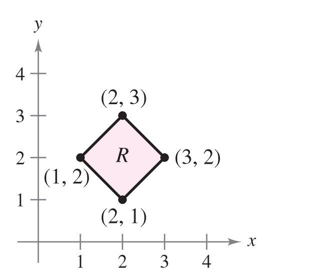
67. $x = u + 3v$, $y = 2u - 3v$

68. $x = u^2 + v^2$, $y = u^2 - v^2$

In Exercises 69 and 70, use the indicated change of variables to evaluate the double integral.

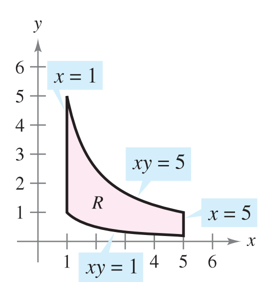
69. $\displaystyle\int\!\!\int_R \ln(x+y)\,dA$

$x = \frac{1}{2}(u + v)$, $y = \frac{1}{2}(u - v)$



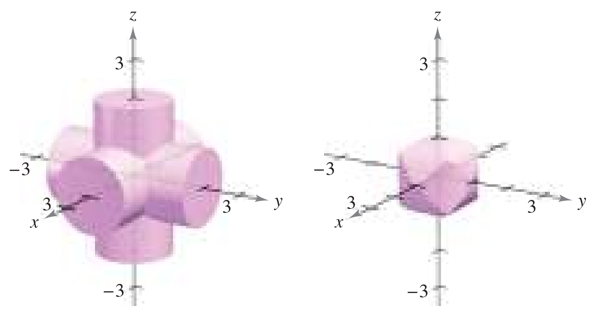
70. $\displaystyle\int\!\!\int_R \frac{x}{1+x^2y^2}\,dA$

$x = u$, $y = \dfrac{v}{u}$



P.S. PROBLEM SOLVING

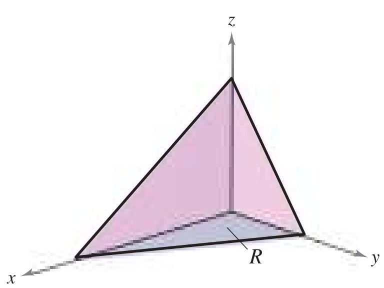
1. Find the volume of the solid of intersection of the three cylinders $x^2 + z^2 = 1$, $y^2 + z^2 = 1$, and $x^2 + y^2 = 1$ (see figure).



2. Let a, b, c, and d be positive real numbers. The first octant of the plane $ax + by + cz = d$ is shown in the figure. Show that the surface area of this portion of the plane is equal to

$$\frac{A(R)}{c}\sqrt{a^2 + b^2 + c^2}$$

where $A(R)$ is the area of the triangular region R in the xy-plane, as shown in the figure.



3. Derive Euler's famous result that was mentioned in Section 9.3, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$, by completing each step.

(a) Prove that $\displaystyle\int \frac{dv}{2 - u^2 + v^2} = \frac{1}{\sqrt{2 - u^2}} \arctan \frac{v}{\sqrt{2 - u^2}} + C.$

(b) Prove that $\displaystyle I_1 = \int_0^{\sqrt{2}/2} \int_{-u}^{u} \frac{2}{2 - u^2 + v^2}\, dv\, du = \frac{\pi^2}{18}$ by using the substitution $u = \sqrt{2} \sin \theta$.

(c) Prove that

$$I_2 = \int_{\sqrt{2}/2}^{\sqrt{2}} \int_{u-\sqrt{2}}^{-u+\sqrt{2}} \frac{2}{2 - u^2 + v^2}\, dv\, du = 4\int_{\pi/6}^{\pi/2} \arctan \frac{1 - \sin \theta}{\cos \theta}\, d\theta$$

by using the substitution $u = \sqrt{2} \sin \theta$.

(d) Prove the trigonometric identity $\displaystyle\frac{1 - \sin \theta}{\cos \theta} = \tan\left(\frac{(\pi/2) - \theta}{2}\right).$

(e) Prove that $\displaystyle I_2 = \int_{\sqrt{2}/2}^{\sqrt{2}} \int_{u-\sqrt{2}}^{-u+\sqrt{2}} \frac{2}{2 - u^2 + v^2}\, dv\, du = \frac{\pi^2}{9}.$

(f) Use the formula for the sum of an infinite geometric series to verify that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} = \int_0^1 \int_0^1 \frac{1}{1 - xy}\, dx\, dy.$

(g) Use the change of variables $\displaystyle u = \frac{x + y}{\sqrt{2}}$ and $\displaystyle v = \frac{y - x}{\sqrt{2}}$ to prove that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} = \int_0^1 \int_0^1 \frac{1}{1 - xy}\, dx\, dy = I_1 + I_2 = \frac{\pi^2}{6}.$

4. Consider a circular lawn with a radius of 10 feet, as shown in the figure. Assume that a sprinkler distributes water in a radial fashion according to the formula

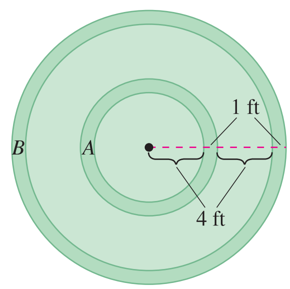
$$f(r) = \frac{r}{16} - \frac{r^2}{160}$$

(measured in cubic feet of water per hour per square foot of lawn), where r is the distance in feet from the sprinkler. Find the amount of water that is distributed in 1 hour in the following two annular regions.

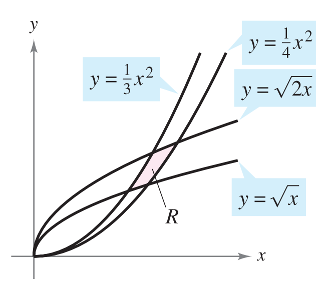
$A = \{(r, \theta): 4 \le r \le 5, 0 \le \theta \le 2\pi\}$

$B = \{(r, \theta): 9 \le r \le 10, 0 \le \theta \le 2\pi\}$

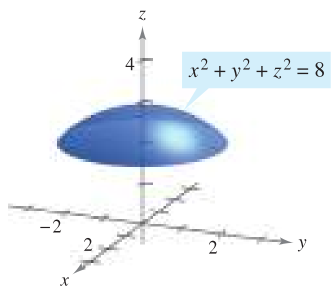
Is the distribution of water uniform? Determine the amount of water the entire lawn receives in 1 hour.



5. The figure shows the region R bounded by the curves $y = \sqrt{x}$, $y = \sqrt{2x}$, $\displaystyle y = \frac{x^2}{3}$, and $\displaystyle y = \frac{x^2}{4}$. Use the change of variables $x = u^{1/3}v^{2/3}$ and $y = u^{2/3}v^{1/3}$ to find the area of the region R.



6. The figure shows a solid bounded below by the plane $z = 2$ and above by the sphere $x^2 + y^2 + z^2 = 8$.



(a) Find the volume of the solid using cylindrical coordinates.

(b) Find the volume of the solid using spherical coordinates.

7. Sketch the solid whose volume is given by the sum of the iterated integrals

$$\int_0^6 \int_{z/2}^3 \int_{z/2}^y dx\,dy\,dz + \int_0^6 \int_3^{(12-z)/2} \int_{z/2}^{6-y} dx\,dy\,dz.$$

Then write the volume as a single iterated integral in the order $dy\,dz\,dx$.

8. Prove that $\displaystyle\lim_{n\to\infty} \int_0^1 \int_0^1 x^n y^n\,dx\,dy = 0$.

In Exercises 9 and 10, evaluate the integral. (*Hint:* See Exercise 69 in Section 14.3.)

9. $\displaystyle\int_0^\infty x^2 e^{-x^2}\,dx$

10. $\displaystyle\int_0^1 \sqrt{\ln \frac{1}{x}}\,dx$

11. Consider the function

$$f(x, y) = \begin{cases} ke^{-(x+y)/a}, & x \geq 0,\ y \geq 0 \\ 0, & \text{elsewhere.} \end{cases}$$

Find the relationship between the positive constants a and k such that f is a joint density function of the continuous random variables x and y.

12. Find the volume of the solid generated by revolving the region in the first quadrant bounded by $y = e^{-x^2}$ about the y-axis. Use this result to find

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx.$$

13. From 1963 to 1986, the volume of the Great Salt Lake approximately tripled while its top surface area approximately doubled. Read the article "Relations between Surface Area and Volume in Lakes" by Daniel Cass and Gerald Wildenberg in *The College Mathematics Journal*. Then give examples of solids that have "water levels" a and b such that $V(b) = 3V(a)$ and $A(b) = 2A(a)$ (see figure), where V is volume and A is area.

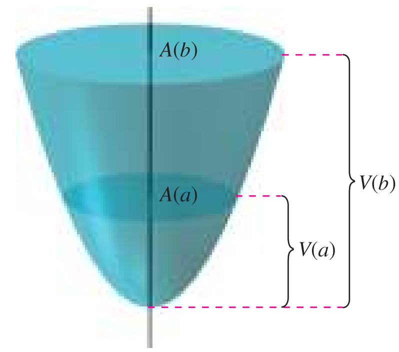
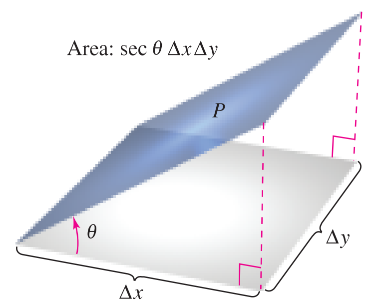


Figure for 13

14. The angle between a plane P and the xy-plane is θ, where $0 \leq \theta < \pi/2$. The projection of a rectangular region in P onto the xy-plane is a rectangle whose sides have lengths Δx and Δy, as shown in the figure. Prove that the area of the rectangular region in P is $\sec\theta\,\Delta x\,\Delta y$.



15. Use the result of Exercise 14 to order the planes in ascending order of their surface areas for a fixed region R in the xy-plane. Explain your ordering without doing any calculations.

(a) $z_1 = 2 + x$

(b) $z_2 = 5$

(c) $z_3 = 10 - 5x + 9y$

(d) $z_4 = 3 + x - 2y$

16. Evaluate the integral $\displaystyle\int_0^\infty \int_0^\infty \frac{1}{(1 + x^2 + y^2)^2}\,dx\,dy$.

17. Evaluate the integrals

$$\int_0^1 \int_0^1 \frac{x - y}{(x + y)^3}\,dx\,dy$$

and

$$\int_0^1 \int_0^1 \frac{x - y}{(x + y)^3}\,dy\,dx.$$

Are the results the same? Why or why not?

18. Show that the volume of a spherical block can be approximated by

$$\Delta V \approx \rho^2 \sin\phi\,\Delta\rho\,\Delta\phi\,\Delta\theta.$$

15 Vector Analysis

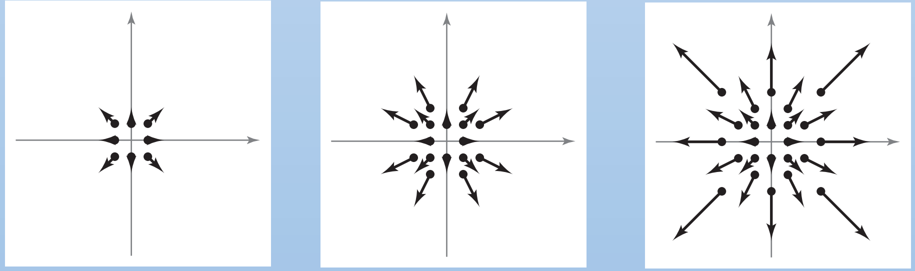
In this chapter, you will study vector fields, line integrals, and surface integrals. You will learn to use these to determine real-life quantities such as surface area, mass, flux, work, and energy.

In this chapter, you should learn the following.

- How to sketch a vector field, determine whether a vector field is conservative, find a potential function, find curl, and find divergence. (15.1)
- How to find a piecewise smooth parametrization, write and evaluate a line integral, and use Green's Theorem. (15.2, 15.4)
- How to use the Fundamental Theorem of Line Integrals, independence of path, and conservation of energy. (15.3)
- How to sketch a parametric surface, find a set of parametric equations to represent a surface, find a normal vector, find a tangent plane, and find the area of a parametric surface. (15.5)
- How to evaluate a surface integral, determine the orientation of a surface, evaluate a flux integral, and use the Divergence Theorem. (15.6, 15.7)
- How to use Stokes's Theorem to evaluate a line integral or a surface integral and how to use curl to analyze the motion of a rotating liquid. (15.8)

NASA

While on the ground awaiting liftoff, space shuttle astronauts have access to a basket and slide wire system that is designed to move them as far away from the shuttle as possible in an emergency situation. Does the amount of work done by the gravitational force field vary for different slide wire paths between two fixed points? (See Section 15.3, Exercise 39.)



In Chapter 15, you will combine your knowledge of vectors with your knowledge of integral calculus. Section 15.1 introduces *vector fields,* such as those shown above. Examples of vector fields include velocity fields, electromagnetic fields, and gravitational fields.

15.1 Vector Fields

- **Understand the concept of a vector field.**
- **Determine whether a vector field is conservative.**
- **Find the curl of a vector field.**
- **Find the divergence of a vector field.**

Vector Fields

In Chapter 12, you studied vector-valued functions—functions that assign a vector to a *real number*. There you saw that vector-valued functions of real numbers are useful in representing curves and motion along a curve. In this chapter, you will study two other types of vector-valued functions—functions that assign a vector to a *point in the plane* or a *point in space*. Such functions are called **vector fields,** and they are useful in representing various types of **force fields** and **velocity fields.**

DEFINITION OF VECTOR FIELD

A **vector field over a plane region R** is a function $\mathbf{F}$ that assigns a vector $\mathbf{F}(x, y)$ to each point in R.

A **vector field over a solid region Q in space** is a function $\mathbf{F}$ that assigns a vector $\mathbf{F}(x, y, z)$ to each point in Q.

NOTE Although a vector field consists of infinitely many vectors, you can get a good idea of what the vector field looks like by sketching several representative vectors $\mathbf{F}(x, y)$ whose initial points are (x, y). ■

The *gradient* is one example of a vector field. For example, if

$$f(x, y) = x^2y + 3xy^3$$

then the gradient of f

$$\begin{aligned}\nabla f(x, y) &= f_x(x, y)\mathbf{i} + f_y(x, y)\mathbf{j}\\ &= (2xy + 3y^3)\mathbf{i} + (x^2 + 9xy^2)\mathbf{j}\end{aligned}$$

Vector field in the plane

is a vector field in the plane. From Chapter 13, the graphical interpretation of this field is a family of vectors, each of which points in the direction of maximum increase along the surface given by $z = f(x, y)$.

Similarly, if

$$f(x, y, z) = x^2 + y^2 + z^2$$

then the gradient of f

$$\begin{aligned}\nabla f(x, y, z) &= f_x(x, y, z)\mathbf{i} + f_y(x, y, z)\mathbf{j} + f_z(x, y, z)\mathbf{k}\\ &= 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}\end{aligned}$$

Vector field in space

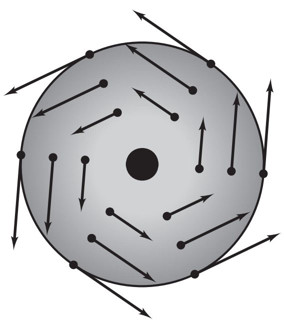
is a vector field in space. Note that the component functions for this particular vector field are $2x$, $2y$, and $2z$.

A vector field

$$\mathbf{F}(x, y, z) = M(x, y, z)\mathbf{i} + N(x, y, z)\mathbf{j} + P(x, y, z)\mathbf{k}$$

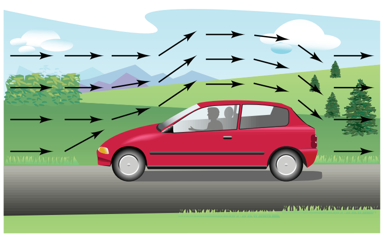
is **continuous** at a point if and only if each of its component functions M, N, and P is continuous at that point.

Some common *physical* examples of vector fields are **velocity fields, gravitational fields,** and **electric force fields.**



Rotating wheel
Figure 15.1

1. *Velocity fields* describe the motions of systems of particles in the plane or in space. For instance, Figure 15.1 shows the vector field determined by a wheel rotating on an axle. Notice that the velocity vectors are determined by the locations of their initial points—the farther a point is from the axle, the greater its velocity. Velocity fields are also determined by the flow of liquids through a container or by the flow of air currents around a moving object, as shown in Figure 15.2.



Air flow vector field
Figure 15.2

2. *Gravitational fields* are defined by **Newton's Law of Gravitation,** which states that the force of attraction exerted on a particle of mass m_1 located at (x, y, z) by a particle of mass m_2 located at $(0, 0, 0)$ is given by

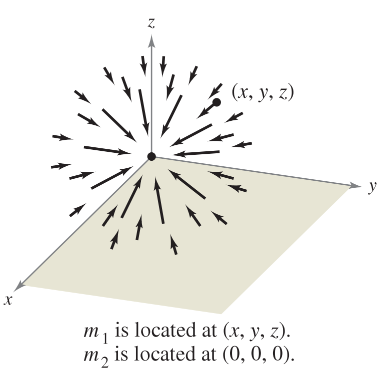
$$\mathbf{F}(x, y, z) = \frac{-Gm_1m_2}{x^2 + y^2 + z^2}\mathbf{u}$$

where G is the gravitational constant and $\mathbf{u}$ is the unit vector in the direction from the origin to (x, y, z). In Figure 15.3, you can see that the gravitational field $\mathbf{F}$ has the properties that $\mathbf{F}(x, y, z)$ always points toward the origin, and that the magnitude of $\mathbf{F}(x, y, z)$ is the same at all points equidistant from the origin. A vector field with these two properties is called a **central force field.** Using the position vector

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

for the point (x, y, z), you can write the gravitational field $\mathbf{F}$ as

$$\begin{aligned}\mathbf{F}(x, y, z) &= \frac{-Gm_1m_2}{\|\mathbf{r}\|^2}\left(\frac{\mathbf{r}}{\|\mathbf{r}\|}\right)\\ &= \frac{-Gm_1m_2}{\|\mathbf{r}\|^2}\mathbf{u}.\end{aligned}$$



Gravitational force field
Figure 15.3

3. *Electric force fields* are defined by **Coulomb's Law,** which states that the force exerted on a particle with electric charge q_1 located at (x, y, z) by a particle with electric charge q_2 located at $(0, 0, 0)$ is given by

$$\mathbf{F}(x, y, z) = \frac{cq_1q_2}{\|\mathbf{r}\|^2}\mathbf{u}$$

where $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, $\mathbf{u} = \mathbf{r}/\|\mathbf{r}\|$, and c is a constant that depends on the choice of units for $\|\mathbf{r}\|$, q_1, and q_2.

Note that an electric force field has the same form as a gravitational field. That is,

$$\mathbf{F}(x, y, z) = \frac{k}{\|\mathbf{r}\|^2}\mathbf{u}.$$

Such a force field is called an **inverse square field.**

DEFINITION OF INVERSE SQUARE FIELD

Let $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ be a position vector. The vector field $\mathbf{F}$ is an **inverse square field** if

$$\mathbf{F}(x, y, z) = \frac{k}{\|\mathbf{r}\|^2}\mathbf{u}$$

where k is a real number and $\mathbf{u} = \mathbf{r}/\|\mathbf{r}\|$ is a unit vector in the direction of $\mathbf{r}$.

Because vector fields consist of infinitely many vectors, it is not possible to create a sketch of the entire field. Instead, when you sketch a vector field, your goal is to sketch representative vectors that help you visualize the field.

EXAMPLE 1 Sketching a Vector Field

Sketch some vectors in the vector field given by

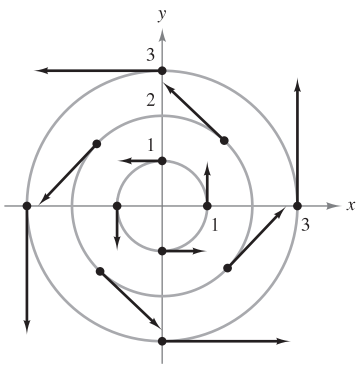
$$\mathbf{F}(x, y) = -y\mathbf{i} + x\mathbf{j}.$$

Solution You could plot vectors at several random points in the plane. However, it is more enlightening to plot vectors of equal magnitude. This corresponds to finding level curves in scalar fields. In this case, vectors of equal magnitude lie on circles.

$$\|\mathbf{F}\| = c \qquad \text{Vectors of length } c$$

$$\sqrt{x^2 + y^2} = c$$

$$x^2 + y^2 = c^2 \qquad \text{Equation of circle}$$



Vector field:
$\mathbf{F}(x, y) = -y\mathbf{i} + x\mathbf{j}$

Figure 15.4

To begin making the sketch, choose a value for c and plot several vectors on the resulting circle. For instance, the following vectors occur on the unit circle.

Point	*Vector*
$(1, 0)$	$\mathbf{F}(1, 0) = \mathbf{j}$
$(0, 1)$	$\mathbf{F}(0, 1) = -\mathbf{i}$
$(-1, 0)$	$\mathbf{F}(-1, 0) = -\mathbf{j}$
$(0, -1)$	$\mathbf{F}(0, -1) = \mathbf{i}$

These and several other vectors in the vector field are shown in Figure 15.4. Note in the figure that this vector field is similar to that given by the rotating wheel shown in Figure 15.1.

EXAMPLE 2 Sketching a Vector Field

Sketch some vectors in the vector field given by

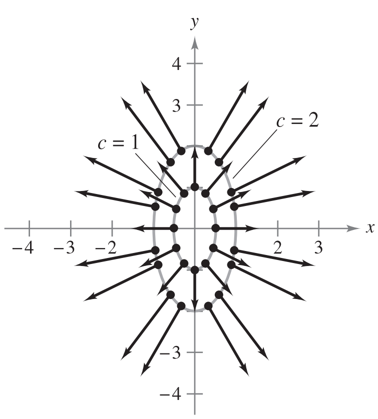
$$\mathbf{F}(x, y) = 2x\mathbf{i} + y\mathbf{j}.$$

Solution For this vector field, vectors of equal length lie on ellipses given by

$$\|\mathbf{F}\| = \sqrt{(2x)^2 + (y)^2} = c$$

which implies that

$$4x^2 + y^2 = c^2.$$



Vector field:
$\mathbf{F}(x, y) = 2x\mathbf{i} + y\mathbf{j}$

Figure 15.5

For $c = 1$, sketch several vectors $2x\mathbf{i} + y\mathbf{j}$ of magnitude 1 at points on the ellipse given by

$$4x^2 + y^2 = 1.$$

For $c = 2$, sketch several vectors $2x\mathbf{i} + y\mathbf{j}$ of magnitude 2 at points on the ellipse given by

$$4x^2 + y^2 = 4.$$

These vectors are shown in Figure 15.5. ■

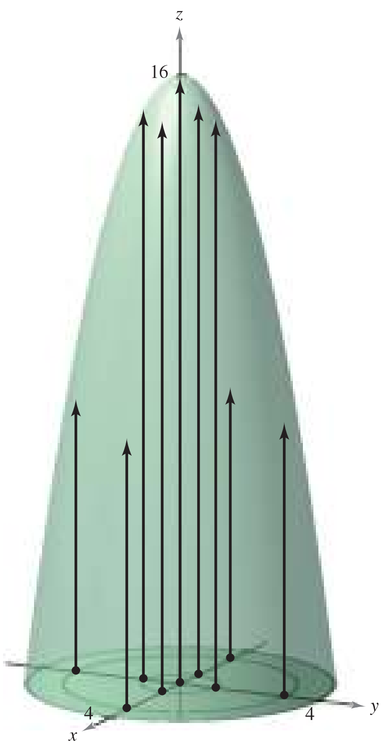
EXAMPLE 3 Sketching a Velocity Field

Sketch some vectors in the velocity field given by

$$\mathbf{v}(x, y, z) = (16 - x^2 - y^2)\mathbf{k}$$

where $x^2 + y^2 \leq 16$.

Solution You can imagine that $\mathbf{v}$ describes the velocity of a liquid flowing through a tube of radius 4. Vectors near the z-axis are longer than those near the edge of the tube. For instance, at the point $(0, 0, 0)$, the velocity vector is $\mathbf{v}(0, 0, 0) = 16\mathbf{k}$, whereas at the point $(0, 3, 0)$, the velocity vector is $\mathbf{v}(0, 3, 0) = 7\mathbf{k}$. Figure 15.6 shows these and several other vectors for the velocity field. From the figure, you can see that the speed of the liquid is greater near the center of the tube than near the edges of the tube. ■



Velocity field:
$\mathbf{v}(x, y, z) = (16 - x^2 - y^2)\mathbf{k}$

Figure 15.6

Conservative Vector Fields

Notice in Figure 15.5 that all the vectors appear to be normal to the level curve from which they emanate. Because this is a property of gradients, it is natural to ask whether the vector field given by $\mathbf{F}(x, y) = 2x\mathbf{i} + y\mathbf{j}$ is the *gradient* of some differentiable function f. The answer is that some vector fields can be represented as the gradients of differentiable functions and some cannot—those that can are called **conservative** vector fields.

DEFINITION OF CONSERVATIVE VECTOR FIELD

A vector field $\mathbf{F}$ is called **conservative** if there exists a differentiable function f such that $\mathbf{F} = \nabla f$. The function f is called the **potential function** for $\mathbf{F}$.

EXAMPLE 4 Conservative Vector Fields

a. The vector field given by $\mathbf{F}(x, y) = 2x\mathbf{i} + y\mathbf{j}$ is conservative. To see this, consider the potential function $f(x, y) = x^2 + \frac{1}{2}y^2$. Because

$$\nabla f = 2x\mathbf{i} + y\mathbf{j} = \mathbf{F}$$

it follows that $\mathbf{F}$ is conservative.

b. Every inverse square field is conservative. To see this, let

$$\mathbf{F}(x, y, z) = \frac{k}{\|\mathbf{r}\|^2}\mathbf{u} \quad \text{and} \quad f(x, y, z) = \frac{-k}{\sqrt{x^2 + y^2 + z^2}}$$

where $\mathbf{u} = \mathbf{r}/\|\mathbf{r}\|$. Because

$$\begin{aligned}\nabla f &= \frac{kx}{(x^2 + y^2 + z^2)^{3/2}}\mathbf{i} + \frac{ky}{(x^2 + y^2 + z^2)^{3/2}}\mathbf{j} + \frac{kz}{(x^2 + y^2 + z^2)^{3/2}}\mathbf{k} \\ &= \frac{k}{x^2 + y^2 + z^2}\left(\frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{\sqrt{x^2 + y^2 + z^2}}\right) \\ &= \frac{k}{\|\mathbf{r}\|^2}\frac{\mathbf{r}}{\|\mathbf{r}\|} \\ &= \frac{k}{\|\mathbf{r}\|^2}\mathbf{u}\end{aligned}$$

it follows that $\mathbf{F}$ is conservative. ■

As can be seen in Example 4(b), many important vector fields, including gravitational fields and electric force fields, are conservative. Most of the terminology in this chapter comes from physics. For example, the term "conservative" is derived from the classic physical law regarding the conservation of energy. This law states that the sum of the kinetic energy and the potential energy of a particle moving in a conservative force field is constant. (The kinetic energy of a particle is the energy due to its motion, and the potential energy is the energy due to its position in the force field.)

The following important theorem gives a necessary and sufficient condition for a vector field *in the plane* to be conservative.

THEOREM 15.1 TEST FOR CONSERVATIVE VECTOR FIELD IN THE PLANE

Let M and N have continuous first partial derivatives on an open disk R. The vector field given by $\mathbf{F}(x, y) = M\mathbf{i} + N\mathbf{j}$ is conservative if and only if

$$\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}.$$

PROOF To prove that the given condition is necessary for $\mathbf{F}$ to be conservative, suppose there exists a potential function f such that

$$\mathbf{F}(x, y) = \nabla f(x, y) = M\mathbf{i} + N\mathbf{j}.$$

Then you have

$$f_x(x, y) = M \quad \Rightarrow \quad f_{xy}(x, y) = \frac{\partial M}{\partial y}$$

$$f_y(x, y) = N \quad \Rightarrow \quad f_{yx}(x, y) = \frac{\partial N}{\partial x}$$

and, by the equivalence of the mixed partials f_{xy} and f_{yx}, you can conclude that $\partial N/\partial x = \partial M/\partial y$ for all (x, y) in R. The sufficiency of this condition is proved in Section 15.4. ■

NOTE Theorem 15.1 is valid on *simply connected* domains. A plane region R is simply connected if every simple closed curve in R encloses only points that are in R. See Figure 15.26 in Section 15.4. ■

EXAMPLE 5 Testing for Conservative Vector Fields in the Plane

Decide whether the vector field given by $\mathbf{F}$ is conservative.

a. $\mathbf{F}(x, y) = x^2y\mathbf{i} + xy\mathbf{j}$ **b.** $\mathbf{F}(x, y) = 2x\mathbf{i} + y\mathbf{j}$

Solution

a. The vector field given by $\mathbf{F}(x, y) = x^2y\mathbf{i} + xy\mathbf{j}$ is not conservative because

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}[x^2y] = x^2 \quad \text{and} \quad \frac{\partial N}{\partial x} = \frac{\partial}{\partial x}[xy] = y.$$

b. The vector field given by $\mathbf{F}(x, y) = 2x\mathbf{i} + y\mathbf{j}$ is conservative because

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}[2x] = 0 \quad \text{and} \quad \frac{\partial N}{\partial x} = \frac{\partial}{\partial x}[y] = 0.$$ ■

Theorem 15.1 tells you whether a vector field is conservative. It does not tell you how to find a potential function of **F**. The problem is comparable to antidifferentiation. Sometimes you will be able to find a potential function by simple inspection. For instance, in Example 4 you observed that

$$f(x, y) = x^2 + \frac{1}{2}y^2$$

has the property that $\nabla f(x, y) = 2x\mathbf{i} + y\mathbf{j}$.

EXAMPLE 6 Finding a Potential Function for F(*x*, *y*)

Find a potential function for

$$\mathbf{F}(x, y) = 2xy\mathbf{i} + (x^2 - y)\mathbf{j}.$$

Solution From Theorem 15.1 it follows that **F** is conservative because

$$\frac{\partial}{\partial y}[2xy] = 2x \quad \text{and} \quad \frac{\partial}{\partial x}[x^2 - y] = 2x.$$

If f is a function whose gradient is equal to $\mathbf{F}(x, y)$, then

$$\nabla f(x, y) = 2xy\mathbf{i} + (x^2 - y)\mathbf{j}$$

which implies that

$$f_x(x, y) = 2xy$$

and

$$f_y(x, y) = x^2 - y.$$

To reconstruct the function f from these two partial derivatives, integrate $f_x(x, y)$ with respect to x and integrate $f_y(x, y)$ with respect to y, as follows.

$$f(x, y) = \int f_x(x, y)\,dx = \int 2xy\,dx = x^2y + g(y)$$

$$f(x, y) = \int f_y(x, y)\,dy = \int (x^2 - y)\,dy = x^2y - \frac{y^2}{2} + h(x)$$

Notice that $g(y)$ is constant with respect to x and $h(x)$ is constant with respect to y. To find a single expression that represents $f(x, y)$, let

$$g(y) = -\frac{y^2}{2} \quad \text{and} \quad h(x) = K.$$

Then, you can write

$$\begin{aligned} f(x, y) &= x^2y + g(y) + K \\ &= x^2y - \frac{y^2}{2} + K. \end{aligned}$$

You can check this result by forming the gradient of f. You will see that it is equal to the original function **F**. ■

NOTE Notice that the solution in Example 6 is comparable to that given by an indefinite integral. That is, the solution represents a family of potential functions, any two of which differ by a constant. To find a unique solution, you would have to be given an initial condition satisfied by the potential function. ■

Curl of a Vector Field

Theorem 15.1 has a counterpart for vector fields in space. Before stating that result, the definition of the **curl of a vector field** in space is given.

DEFINITION OF CURL OF A VECTOR FIELD

The curl of $\mathbf{F}(x, y, z) = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ is

$$\begin{aligned}\text{curl } \mathbf{F}(x, y, z) &= \nabla \times \mathbf{F}(x, y, z)\\ &= \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right)\mathbf{i} - \left(\frac{\partial P}{\partial x} - \frac{\partial M}{\partial z}\right)\mathbf{j} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\mathbf{k}.\end{aligned}$$

NOTE If curl $\mathbf{F} = \mathbf{0}$, then $\mathbf{F}$ is said to be **irrotational.** ■

The cross product notation used for curl comes from viewing the gradient ∇f as the result of the **differential operator** ∇ acting on the function f. In this context, you can use the following determinant form as an aid in remembering the formula for curl.

$$\begin{aligned}\text{curl } \mathbf{F}(x, y, z) &= \nabla \times \mathbf{F}(x, y, z)\\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ M & N & P \end{vmatrix}\\ &= \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right)\mathbf{i} - \left(\frac{\partial P}{\partial x} - \frac{\partial M}{\partial z}\right)\mathbf{j} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\mathbf{k}\end{aligned}$$

EXAMPLE 7 Finding the Curl of a Vector Field

Find curl $\mathbf{F}$ of the vector field given by

$$\mathbf{F}(x, y, z) = 2xy\mathbf{i} + (x^2 + z^2)\mathbf{j} + 2yz\mathbf{k}.$$

Is $\mathbf{F}$ irrotational?

Solution The curl of $\mathbf{F}$ is given by

$$\begin{aligned}\text{curl } \mathbf{F}(x, y, z) &= \nabla \times \mathbf{F}(x, y, z)\\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2xy & x^2 + z^2 & 2yz \end{vmatrix}\\ &= \begin{vmatrix} \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2 + z^2 & 2yz \end{vmatrix}\mathbf{i} - \begin{vmatrix} \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial z} \\ 2xy & 2yz \end{vmatrix}\mathbf{j} + \begin{vmatrix} \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} \\ 2xy & x^2 + z^2 \end{vmatrix}\mathbf{k}\\ &= (2z - 2z)\mathbf{i} - (0 - 0)\mathbf{j} + (2x - 2x)\mathbf{k}\\ &= \mathbf{0}.\end{aligned}$$

Because curl $\mathbf{F} = \mathbf{0}$, $\mathbf{F}$ is irrotational. ■

The icon indicates that you will find a CAS Investigation on the book's website. The CAS Investigation is a collaborative exploration of this example using the computer algebra systems Maple *and* Mathematica.

Later in this chapter, you will assign a physical interpretation to the curl of a vector field. But for now, the primary use of curl is shown in the following test for conservative vector fields in space. The test states that for a vector field in space, the curl is $\mathbf{0}$ at every point in its domain if and only if $\mathbf{F}$ is conservative. The proof is similar to that given for Theorem 15.1.

THEOREM 15.2 TEST FOR CONSERVATIVE VECTOR FIELD IN SPACE

Suppose that M, N, and P have continuous first partial derivatives in an open sphere Q in space. The vector field given by $\mathbf{F}(x, y, z) = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ is conservative if and only if

$$\text{curl } \mathbf{F}(x, y, z) = \mathbf{0}.$$

That is, $\mathbf{F}$ is conservative if and only if

$$\frac{\partial P}{\partial y} = \frac{\partial N}{\partial z}, \quad \frac{\partial P}{\partial x} = \frac{\partial M}{\partial z}, \quad \text{and} \quad \frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}.$$

NOTE Theorem 15.2 is valid for *simply connected* domains in space. A simply connected domain in space is a domain D for which every simple closed curve in D (see Section 15.4) can be shrunk to a point in D without leaving D.

From Theorem 15.2, you can see that the vector field given in Example 7 is conservative because curl $\mathbf{F}(x, y, z) = \mathbf{0}$. Try showing that the vector field

$$\mathbf{F}(x, y, z) = x^3y^2z\mathbf{i} + x^2z\mathbf{j} + x^2y\mathbf{k}$$

is not conservative—you can do this by showing that its curl is

$$\text{curl } \mathbf{F}(x, y, z) = (x^3y^2 - 2xy)\mathbf{j} + (2xz - 2x^3yz)\mathbf{k} \neq \mathbf{0}.$$

For vector fields in space that pass the test for being conservative, you can find a potential function by following the same pattern used in the plane (as demonstrated in Example 6).

EXAMPLE 8 Finding a Potential Function for $\mathbf{F}(x, y, z)$

NOTE Examples 6 and 8 are illustrations of a type of problem called *recovering a function from its gradient*. If you go on to take a course in differential equations, you will study other methods for solving this type of problem. One popular method gives an interplay between successive "partial integrations" and partial differentiations.

Find a potential function for $\mathbf{F}(x, y, z) = 2xy\mathbf{i} + (x^2 + z^2)\mathbf{j} + 2yz\mathbf{k}$.

Solution From Example 7, you know that the vector field given by $\mathbf{F}$ is conservative. If f is a function such that $\mathbf{F}(x, y, z) = \nabla f(x, y, z)$, then

$$f_x(x, y, z) = 2xy, \quad f_y(x, y, z) = x^2 + z^2, \quad \text{and} \quad f_z(x, y, z) = 2yz$$

and integrating with respect to x, y, and z separately produces

$$f(x, y, z) = \int M\,dx = \int 2xy\,dx = x^2y + g(y, z)$$

$$f(x, y, z) = \int N\,dy = \int (x^2 + z^2)\,dy = x^2y + yz^2 + h(x, z)$$

$$f(x, y, z) = \int P\,dz = \int 2yz\,dz = yz^2 + k(x, y).$$

Comparing these three versions of $f(x, y, z)$, you can conclude that

$$g(y, z) = yz^2 + K, \quad h(x, z) = K, \quad \text{and} \quad k(x, y) = x^2y + K.$$

So, $f(x, y, z)$ is given by

$$f(x, y, z) = x^2y + yz^2 + K.$$

Divergence of a Vector Field

NOTE Divergence can be viewed as a type of derivative of **F** in that, for vector fields representing velocities of moving particles, the divergence measures the rate of particle flow per unit volume at a point. In hydrodynamics (the study of fluid motion), a velocity field that is divergence free is called **incompressible.** In the study of electricity and magnetism, a vector field that is divergence free is called **solenoidal.**

You have seen that the curl of a vector field **F** is itself a vector field. Another important function defined on a vector field is **divergence,** which is a scalar function.

DEFINITION OF DIVERGENCE OF A VECTOR FIELD

The **divergence** of $\mathbf{F}(x, y) = M\mathbf{i} + N\mathbf{j}$ is

$$\text{div } \mathbf{F}(x, y) = \nabla \cdot \mathbf{F}(x, y) = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}. \qquad \text{Plane}$$

The **divergence** of $\mathbf{F}(x, y, z) = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ is

$$\text{div } \mathbf{F}(x, y, z) = \nabla \cdot \mathbf{F}(x, y, z) = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}. \qquad \text{Space}$$

If $\text{div } \mathbf{F} = 0$, then **F** is said to be **divergence free.**

The dot product notation used for divergence comes from considering ∇ as a **differential operator,** as follows.

$$\begin{aligned}\nabla \cdot \mathbf{F}(x, y, z) &= \left[\left(\frac{\partial}{\partial x}\right)\mathbf{i} + \left(\frac{\partial}{\partial y}\right)\mathbf{j} + \left(\frac{\partial}{\partial z}\right)\mathbf{k}\right] \cdot (M\mathbf{i} + N\mathbf{j} + P\mathbf{k}) \\ &= \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}\end{aligned}$$

EXAMPLE 9 Finding the Divergence of a Vector Field

Find the divergence at $(2, 1, -1)$ for the vector field

$$\mathbf{F}(x, y, z) = x^3y^2z\mathbf{i} + x^2z\mathbf{j} + x^2y\mathbf{k}.$$

Solution The divergence of **F** is

$$\text{div } \mathbf{F}(x, y, z) = \frac{\partial}{\partial x}[x^3y^2z] + \frac{\partial}{\partial y}[x^2z] + \frac{\partial}{\partial z}[x^2y] = 3x^2y^2z.$$

At the point $(2, 1, -1)$, the divergence is

$$\text{div } \mathbf{F}(2, 1, -1) = 3(2^2)(1^2)(-1) = -12.$$

There are many important properties of the divergence and curl of a vector field **F** (see Exercises 83–89). One that is used often is described in Theorem 15.3. You are asked to prove this theorem in Exercise 90.

THEOREM 15.3 DIVERGENCE AND CURL

If $\mathbf{F}(x, y, z) = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ is a vector field and M, N, and P have continuous second partial derivatives, then

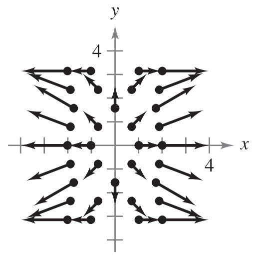
$$\text{div}(\text{curl } \mathbf{F}) = 0.$$

15.1 Exercises

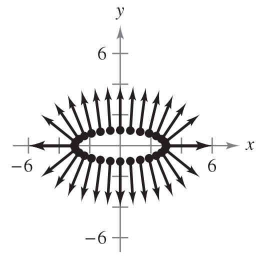
See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–6, match the vector field with its graph. [The graphs are labeled (a), (b), (c), (d), (e), and (f).]

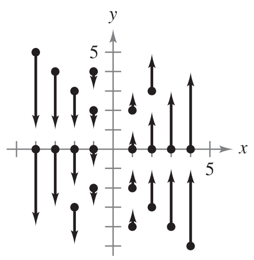
(a)



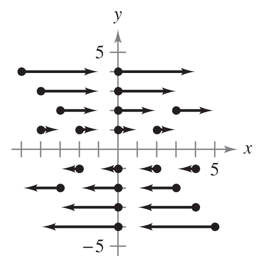
(b)



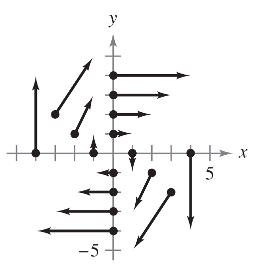
(c)



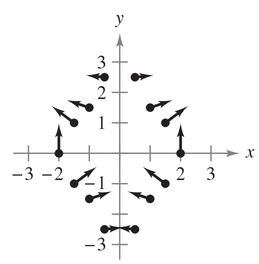
(d)



(e)



(f)



1. $\mathbf{F}(x, y) = y\mathbf{i}$
2. $\mathbf{F}(x, y) = x\mathbf{j}$
3. $\mathbf{F}(x, y) = y\mathbf{i} - x\mathbf{j}$
4. $\mathbf{F}(x, y) = x\mathbf{i} + 3y\mathbf{j}$
5. $\mathbf{F}(x, y) = \langle x, \sin y\rangle$
6. $\mathbf{F}(x, y) = \left\langle \frac{1}{2}xy, \frac{1}{4}x^2\right\rangle$

In Exercises 7–16, compute $\|\mathbf{F}\|$ and sketch several representative vectors in the vector field.

7. $\mathbf{F}(x, y) = \mathbf{i} + \mathbf{j}$
8. $\mathbf{F}(x, y) = 2\mathbf{i}$
9. $\mathbf{F}(x, y) = y\mathbf{i} + x\mathbf{j}$
10. $\mathbf{F}(x, y) = y\mathbf{i} - 2x\mathbf{j}$
11. $\mathbf{F}(x, y, z) = 3y\mathbf{j}$
12. $\mathbf{F}(x, y) = x\mathbf{i}$
13. $\mathbf{F}(x, y) = 4x\mathbf{i} + y\mathbf{j}$
14. $\mathbf{F}(x, y) = (x^2 + y^2)\mathbf{i} + \mathbf{j}$
15. $\mathbf{F}(x, y, z) = \mathbf{i} + \mathbf{j} + \mathbf{k}$
16. $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

CAS **In Exercises 17–20, use a computer algebra system to graph several representative vectors in the vector field.**

17. $\mathbf{F}(x, y) = \frac{1}{8}(2xy\mathbf{i} + y^2\mathbf{j})$
18. $\mathbf{F}(x, y) = (2y - 3x)\mathbf{i} + (2y + 3x)\mathbf{j}$
19. $\mathbf{F}(x, y, z) = \dfrac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{\sqrt{x^2 + y^2 + z^2}}$
20. $\mathbf{F}(x, y, z) = x\mathbf{i} - y\mathbf{j} + z\mathbf{k}$

In Exercises 21–30, find the conservative vector field for the potential function by finding its gradient.

21. $f(x, y) = x^2 + 2y^2$
22. $f(x, y) = x^2 - \frac{1}{4}y^2$
23. $g(x, y) = 5x^2 + 3xy + y^2$
24. $g(x, y) = \sin 3x \cos 4y$
25. $f(x, y, z) = 6xyz$
26. $f(x, y, z) = \sqrt{x^2 + 4y^2 + z^2}$
27. $g(x, y, z) = z + ye^{x^2}$
28. $g(x, y, z) = \dfrac{y}{z} + \dfrac{z}{x} - \dfrac{xz}{y}$
29. $h(x, y, z) = xy \ln(x + y)$
30. $h(x, y, z) = x \arcsin yz$

In Exercises 31–34, verify that the vector field is conservative.

31. $\mathbf{F}(x, y) = xy^2\mathbf{i} + x^2y\mathbf{j}$
32. $\mathbf{F}(x, y) = \dfrac{1}{x^2}(y\mathbf{i} - x\mathbf{j})$
33. $\mathbf{F}(x, y) = \sin y\mathbf{i} + x \cos y\mathbf{j}$
34. $\mathbf{F}(x, y) = \dfrac{1}{xy}(y\mathbf{i} - x\mathbf{j})$

In Exercises 35–38, determine whether the vector field is conservative. Justify your answer.

35. $\mathbf{F}(x, y) = 5y^2(y\mathbf{i} + 3x\mathbf{j})$
36. $\mathbf{F}(x, y) = \dfrac{2}{y^2}e^{2x/y}(y\mathbf{i} - x\mathbf{j})$
37. $\mathbf{F}(x, y) = \dfrac{1}{\sqrt{x^2 + y^2}}(\mathbf{i} + \mathbf{j})$
38. $\mathbf{F}(x, y) = \dfrac{1}{\sqrt{1 + xy}}(y\mathbf{i} + x\mathbf{j})$

In Exercises 39–48, determine whether the vector field is conservative. If it is, find a potential function for the vector field.

39. $\mathbf{F}(x, y) = y\mathbf{i} + x\mathbf{j}$
40. $\mathbf{F}(x, y) = 3x^2y^2\mathbf{i} + 2x^3y\mathbf{j}$
41. $\mathbf{F}(x, y) = 2xy\mathbf{i} + x^2\mathbf{j}$
42. $\mathbf{F}(x, y) = xe^{x^2y}(2y\mathbf{i} + x\mathbf{j})$
43. $\mathbf{F}(x, y) = 15y^3\mathbf{i} - 5xy^2\mathbf{j}$
44. $\mathbf{F}(x, y) = \dfrac{1}{y^2}(y\mathbf{i} - 2x\mathbf{j})$
45. $\mathbf{F}(x, y) = \dfrac{2y}{x}\mathbf{i} - \dfrac{x^2}{y^2}\mathbf{j}$
46. $\mathbf{F}(x, y) = \dfrac{x\mathbf{i} + y\mathbf{j}}{x^2 + y^2}$
47. $\mathbf{F}(x, y) = e^x(\cos y\mathbf{i} - \sin y\mathbf{j})$
48. $\mathbf{F}(x, y) = \dfrac{2x\mathbf{i} + 2y\mathbf{j}}{(x^2 + y^2)^2}$

In Exercises 49–52, find curl F for the vector field at the given point.

	Vector Field	*Point*
49.	$\mathbf{F}(x, y, z) = xyz\mathbf{i} + xyz\mathbf{j} + xyz\mathbf{k}$	$(2, 1, 3)$
50.	$\mathbf{F}(x, y, z) = x^2z\mathbf{i} - 2xz\mathbf{j} + yz\mathbf{k}$	$(2, -1, 3)$
51.	$\mathbf{F}(x, y, z) = e^x \sin y\mathbf{i} - e^x \cos y\mathbf{j}$	$(0, 0, 1)$
52.	$\mathbf{F}(x, y, z) = e^{-xyz}(\mathbf{i} + \mathbf{j} + \mathbf{k})$	$(3, 2, 0)$

CAS **In Exercises 53–56, use a computer algebra system to find the curl F for the vector field.**

53. $\mathbf{F}(x, y, z) = \arctan\left(\frac{x}{y}\right)\mathbf{i} + \ln\sqrt{x^2 + y^2}\,\mathbf{j} + \mathbf{k}$

54. $\mathbf{F}(x, y, z) = \frac{yz}{y - z}\mathbf{i} + \frac{xz}{x - z}\mathbf{j} + \frac{xy}{x - y}\mathbf{k}$

55. $\mathbf{F}(x, y, z) = \sin(x - y)\mathbf{i} + \sin(y - z)\mathbf{j} + \sin(z - x)\mathbf{k}$

56. $\mathbf{F}(x, y, z) = \sqrt{x^2 + y^2 + z^2}\,(\mathbf{i} + \mathbf{j} + \mathbf{k})$

In Exercises 57–62, determine whether the vector field F is conservative. If it is, find a potential function for the vector field.

57. $\mathbf{F}(x, y, z) = xy^2z^2\mathbf{i} + x^2yz^2\mathbf{j} + x^2y^2z\mathbf{k}$

58. $\mathbf{F}(x, y, z) = y^2z^3\mathbf{i} + 2xyz^3\mathbf{j} + 3xy^2z^2\mathbf{k}$

59. $\mathbf{F}(x, y, z) = \sin z\,\mathbf{i} + \sin x\,\mathbf{j} + \sin y\,\mathbf{k}$

60. $\mathbf{F}(x, y, z) = ye^z\mathbf{i} + ze^x\mathbf{j} + xe^y\mathbf{k}$

61. $\mathbf{F}(x, y, z) = \frac{z}{y}\mathbf{i} - \frac{xz}{y^2}\mathbf{j} + \frac{x}{y}\mathbf{k}$

62. $\mathbf{F}(x, y, z) = \frac{x}{x^2 + y^2}\mathbf{i} + \frac{y}{x^2 + y^2}\mathbf{j} + \mathbf{k}$

In Exercises 63–66, find the divergence of the vector field F.

63. $\mathbf{F}(x, y) = x^2\mathbf{i} + 2y^2\mathbf{j}$

64. $\mathbf{F}(x, y) = xe^x\mathbf{i} + ye^y\mathbf{j}$

65. $\mathbf{F}(x, y, z) = \sin x\,\mathbf{i} + \cos y\,\mathbf{j} + z^2\mathbf{k}$

66. $\mathbf{F}(x, y, z) = \ln(x^2 + y^2)\mathbf{i} + xy\mathbf{j} + \ln(y^2 + z^2)\mathbf{k}$

In Exercises 67–70, find the divergence of the vector field F at the given point.

	Vector Field	*Point*
67.	$\mathbf{F}(x, y, z) = xyz\mathbf{i} + xy\mathbf{j} + z\mathbf{k}$	$(2, 1, 1)$
68.	$\mathbf{F}(x, y, z) = x^2z\mathbf{i} - 2xz\mathbf{j} + yz\mathbf{k}$	$(2, -1, 3)$
69.	$\mathbf{F}(x, y, z) = e^x \sin y\mathbf{i} - e^x \cos y\mathbf{j} + z^2\mathbf{k}$	$(3, 0, 0)$
70.	$\mathbf{F}(x, y, z) = \ln(xyz)(\mathbf{i} + \mathbf{j} + \mathbf{k})$	$(3, 2, 1)$

WRITING ABOUT CONCEPTS

71. Define a vector field in the plane and in space. Give some physical examples of vector fields.

72. What is a conservative vector field, and how do you test for it in the plane and in space?

73. Define the curl of a vector field.

74. Define the divergence of a vector field in the plane and in space.

In Exercises 75 and 76, find curl(F × G) = ∇ × (F × G).

75. $\mathbf{F}(x, y, z) = \mathbf{i} + 3x\mathbf{j} + 2y\mathbf{k}$
$\mathbf{G}(x, y, z) = x\mathbf{i} - y\mathbf{j} + z\mathbf{k}$

76. $\mathbf{F}(x, y, z) = x\mathbf{i} - z\mathbf{k}$
$\mathbf{G}(x, y, z) = x^2\mathbf{i} + y\mathbf{j} + z^2\mathbf{k}$

In Exercises 77 and 78, find curl(curl F) = ∇ × (∇ × F).

77. $\mathbf{F}(x, y, z) = xyz\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

78. $\mathbf{F}(x, y, z) = x^2z\mathbf{i} - 2xz\mathbf{j} + yz\mathbf{k}$

In Exercises 79 and 80, find div(F × G) = ∇ · (F × G).

79. $\mathbf{F}(x, y, z) = \mathbf{i} + 3x\mathbf{j} + 2y\mathbf{k}$
$\mathbf{G}(x, y, z) = x\mathbf{i} - y\mathbf{j} + z\mathbf{k}$

80. $\mathbf{F}(x, y, z) = x\mathbf{i} - z\mathbf{k}$
$\mathbf{G}(x, y, z) = x^2\mathbf{i} + y\mathbf{j} + z^2\mathbf{k}$

In Exercises 81 and 82, find div(curl F) = ∇ · (∇ × F).

81. $\mathbf{F}(x, y, z) = xyz\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

82. $\mathbf{F}(x, y, z) = x^2z\mathbf{i} - 2xz\mathbf{j} + yz\mathbf{k}$

In Exercises 83–90, prove the property for vector fields F and G and scalar function *f*. (Assume that the required partial derivatives are continuous.)

83. $\text{curl}(\mathbf{F} + \mathbf{G}) = \text{curl}\,\mathbf{F} + \text{curl}\,\mathbf{G}$

84. $\text{curl}(\nabla f) = \nabla \times (\nabla f) = \mathbf{0}$

85. $\text{div}(\mathbf{F} + \mathbf{G}) = \text{div}\,\mathbf{F} + \text{div}\,\mathbf{G}$

86. $\text{div}(\mathbf{F} \times \mathbf{G}) = (\text{curl}\,\mathbf{F}) \cdot \mathbf{G} - \mathbf{F} \cdot (\text{curl}\,\mathbf{G})$

87. $\nabla \times [\nabla f + (\nabla \times \mathbf{F})] = \nabla \times (\nabla \times \mathbf{F})$

88. $\nabla \times (f\mathbf{F}) = f(\nabla \times \mathbf{F}) + (\nabla f) \times \mathbf{F}$

89. $\text{div}(f\mathbf{F}) = f\,\text{div}\,\mathbf{F} + \nabla f \cdot \mathbf{F}$

90. $\text{div}(\text{curl}\,\mathbf{F}) = 0$ (Theorem 15.3)

In Exercises 91–93, let $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, and let $f(x, y, z) = \|\mathbf{F}(x, y, z)\|$.

91. Show that $\nabla(\ln f) = \frac{\mathbf{F}}{f^2}$.

92. Show that $\nabla\left(\frac{1}{f}\right) = -\frac{\mathbf{F}}{f^3}$.

93. Show that $\nabla f^n = nf^{n-2}\mathbf{F}$.

CAPSTONE

94. (a) Sketch several representative vectors in the vector field given by

$$\mathbf{F}(x, y) = \frac{x\mathbf{i} + y\mathbf{j}}{\sqrt{x^2 + y^2}}.$$

(b) Sketch several representative vectors in the vector field given by

$$\mathbf{G}(x, y) = \frac{x\mathbf{i} - y\mathbf{j}}{\sqrt{x^2 + y^2}}.$$

(c) Explain any similarities or differences in the vector fields $\mathbf{F}(x, y)$ and $\mathbf{G}(x, y)$.

True or False? **In Exercises 95–98, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

95. If $\mathbf{F}(x, y) = 4x\mathbf{i} - y^2\mathbf{j}$, then $\|\mathbf{F}(x, y)\| \to 0$ as $(x, y) \to (0, 0)$.

96. If $\mathbf{F}(x, y) = 4x\mathbf{i} - y^2\mathbf{j}$ and (x, y) is on the positive y-axis, then the vector points in the negative y-direction.

97. If f is a scalar field, then curl f is a meaningful expression.

98. If $\mathbf{F}$ is a vector field and curl $\mathbf{F} = \mathbf{0}$, then $\mathbf{F}$ is irrotational but not conservative.

15.2 Line Integrals

- Understand and use the concept of a piecewise smooth curve.
- Write and evaluate a line integral.
- Write and evaluate a line integral of a vector field.
- Write and evaluate a line integral in differential form.

Piecewise Smooth Curves

A classic property of gravitational fields is that, subject to certain physical constraints, the work done by gravity on an object moving between two points in the field is independent of the path taken by the object. One of the constraints is that the **path** must be a piecewise smooth curve. Recall that a plane curve C given by

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}, \quad a \le t \le b$$

is **smooth** if

$$\frac{dx}{dt} \quad \text{and} \quad \frac{dy}{dt}$$

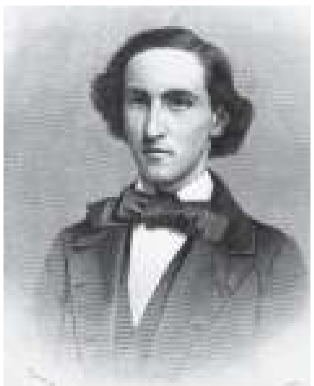
are continuous on $[a, b]$ and not simultaneously 0 on (a, b). Similarly, a space curve C given by

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}, \quad a \le t \le b$$

is **smooth** if

$$\frac{dx}{dt}, \quad \frac{dy}{dt}, \quad \text{and} \quad \frac{dz}{dt}$$

are continuous on $[a, b]$ and not simultaneously 0 on (a, b). A curve C is **piecewise smooth** if the interval $[a, b]$ can be partitioned into a finite number of subintervals, on each of which C is smooth.



The Granger Collection

JOSIAH WILLARD GIBBS (1839–1903)

Many physicists and mathematicians have contributed to the theory and applications described in this chapter—Newton, Gauss, Laplace, Hamilton, and Maxwell, among others. However, the use of vector analysis to describe these results is attributed primarily to the American mathematical physicist Josiah Willard Gibbs.

EXAMPLE 1 Finding a Piecewise Smooth Parametrization

Find a piecewise smooth parametrization of the graph of C shown in Figure 15.7.

Solution Because C consists of three line segments C_1, C_2, and C_3, you can construct a smooth parametrization for each segment and piece them together by making the last t-value in C_i correspond to the first t-value in C_{i+1}, as follows.

$$\begin{array}{llll} C_1\!: x(t) = 0, & y(t) = 2t, & z(t) = 0, & 0 \le t \le 1 \\ C_2\!: x(t) = t - 1, & y(t) = 2, & z(t) = 0, & 1 \le t \le 2 \\ C_3\!: x(t) = 1, & y(t) = 2, & z(t) = t - 2, & 2 \le t \le 3 \end{array}$$

So, C is given by

$$\mathbf{r}(t) = \begin{cases} 2t\mathbf{j}, & 0 \le t \le 1 \\ (t-1)\mathbf{i} + 2\mathbf{j}, & 1 \le t \le 2. \\ \mathbf{i} + 2\mathbf{j} + (t-2)\mathbf{k}, & 2 \le t \le 3 \end{cases}$$

Because C_1, C_2, and C_3 are smooth, it follows that C is piecewise smooth. ■

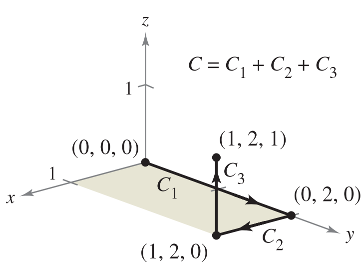


Figure 15.7

Recall that parametrization of a curve induces an **orientation** to the curve. For instance, in Example 1, the curve is oriented such that the positive direction is from (0, 0, 0), following the curve to (1, 2, 1). Try finding a parametrization that induces the opposite orientation.

Line Integrals

Up to this point in the text, you have studied various types of integrals. For a single integral

$$\int_a^b f(x)\,dx \qquad \text{Integrate over interval } [a, b].$$

you integrated over the interval $[a, b]$. Similarly, for a double integral

$$\int_R\int f(x, y)\,dA \qquad \text{Integrate over region } R.$$

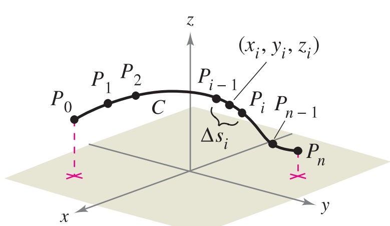
you integrated over the region R in the plane. In this section, you will study a new type of integral called a **line integral**

$$\int_C f(x, y)\,ds \qquad \text{Integrate over curve } C.$$

for which you integrate over a piecewise smooth curve C. (The terminology is somewhat unfortunate—this type of integral might be better described as a "curve integral.")

To introduce the concept of a line integral, consider the mass of a wire of finite length, given by a curve C in space. The density (mass per unit length) of the wire at the point (x, y, z) is given by $f(x, y, z)$. Partition the curve C by the points

$$P_0, P_1, \ldots, P_n$$



Partitioning of curve C
Figure 15.8

producing n subarcs, as shown in Figure 15.8. The length of the ith subarc is given by Δs_i. Next, choose a point (x_i, y_i, z_i) in each subarc. If the length of each subarc is small, the total mass of the wire can be approximated by the sum

$$\text{Mass of wire} \approx \sum_{i=1}^{n} f(x_i, y_i, z_i)\,\Delta s_i.$$

If you let $\|\Delta\|$ denote the length of the longest subarc and let $\|\Delta\|$ approach 0, it seems reasonable that the limit of this sum approaches the mass of the wire. This leads to the following definition.

DEFINITION OF LINE INTEGRAL

If f is defined in a region containing a smooth curve C of finite length, then the **line integral of f along C** is given by

$$\int_C f(x, y)\,ds = \lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} f(x_i, y_i)\,\Delta s_i \qquad \text{Plane}$$

or

$$\int_C f(x, y, z)\,ds = \lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} f(x_i, y_i, z_i)\,\Delta s_i \qquad \text{Space}$$

provided this limit exists.

As with the integrals discussed in Chapter 14, evaluation of a line integral is best accomplished by converting it to a definite integral. It can be shown that if f is *continuous*, the limit given above exists and is the same for all smooth parametrizations of C.

To evaluate a line integral over a plane curve C given by $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$, use the fact that

$$ds = \|\mathbf{r}'(t)\|\, dt = \sqrt{[x'(t)]^2 + [y'(t)]^2}\, dt.$$

A similar formula holds for a space curve, as indicated in Theorem 15.4.

THEOREM 15.4 EVALUATION OF A LINE INTEGRAL AS A DEFINITE INTEGRAL

Let f be continuous in a region containing a smooth curve C. If C is given by $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$, where $a \le t \le b$, then

$$\int_C f(x, y)\, ds = \int_a^b f(x(t), y(t))\sqrt{[x'(t)]^2 + [y'(t)]^2}\, dt.$$

If C is given by $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$, where $a \le t \le b$, then

$$\int_C f(x, y, z)\, ds = \int_a^b f(x(t), y(t), z(t))\sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}\, dt.$$

Note that if $f(x, y, z) = 1$, the line integral gives the arc length of the curve C, as defined in Section 12.5. That is,

$$\int_C 1\, ds = \int_a^b \|\mathbf{r}'(t)\|\, dt = \text{length of curve } C.$$

EXAMPLE 2 Evaluating a Line Integral

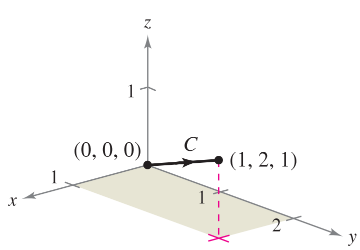


Figure 15.9

Evaluate

$$\int_C (x^2 - y + 3z)\, ds$$

where C is the line segment shown in Figure 15.9.

Solution Begin by writing a parametric form of the equation of the line segment:

$$x = t, \quad y = 2t, \quad \text{and} \quad z = t, \quad 0 \le t \le 1.$$

Therefore, $x'(t) = 1$, $y'(t) = 2$, and $z'(t) = 1$, which implies that

$$\sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2} = \sqrt{1^2 + 2^2 + 1^2} = \sqrt{6}.$$

So, the line integral takes the following form.

$$\begin{aligned}\int_C (x^2 - y + 3z)\, ds &= \int_0^1 (t^2 - 2t + 3t)\sqrt{6}\, dt \\ &= \sqrt{6}\int_0^1 (t^2 + t)\, dt \\ &= \sqrt{6}\left[\frac{t^3}{3} + \frac{t^2}{2}\right]_0^1 \\ &= \frac{5\sqrt{6}}{6}\end{aligned}$$

■

NOTE The value of the line integral in Example 2 does not depend on the parametrization of the line segment C (any smooth parametrization will produce the same value). To convince yourself of this, try some other parametrizations, such as $x = 1 + 2t$, $y = 2 + 4t$, $z = 1 + 2t$, $-\frac{1}{2} \le t \le 0$, or $x = -t$, $y = -2t$, $z = -t$, $-1 \le t \le 0$.

Suppose C is a path composed of smooth curves $C_1, C_2, \ldots, C_n$. If f is continuous on C, it can be shown that

$$\int_C f(x, y)\, ds = \int_{C_1} f(x, y)\, ds + \int_{C_2} f(x, y)\, ds + \cdots + \int_{C_n} f(x, y)\, ds.$$

This property is used in Example 3.

EXAMPLE 3 Evaluating a Line Integral Over a Path

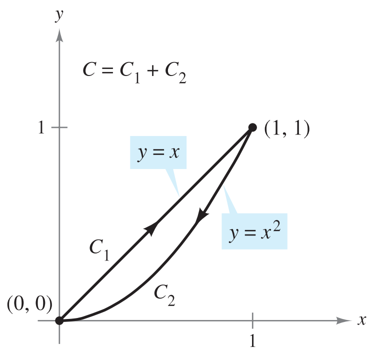


Figure 15.10

Evaluate $\int_C x\, ds$, where C is the piecewise smooth curve shown in Figure 15.10.

Solution Begin by integrating up the line $y = x$, using the following parametrization.

$$C_1\colon\ x = t,\ y = t, \quad 0 \le t \le 1$$

For this curve, $\mathbf{r}(t) = t\mathbf{i} + t\mathbf{j}$, which implies that $x'(t) = 1$ and $y'(t) = 1$. So,

$$\sqrt{[x'(t)]^2 + [y'(t)]^2} = \sqrt{2}$$

and you have

$$\int_{C_1} x\, ds = \int_0^1 t\sqrt{2}\, dt = \frac{\sqrt{2}}{2} t^2 \Big]_0^1 = \frac{\sqrt{2}}{2}.$$

Next, integrate down the parabola $y = x^2$, using the parametrization

$$C_2\colon\ x = 1 - t, \quad y = (1 - t)^2, \quad 0 \le t \le 1.$$

For this curve, $\mathbf{r}(t) = (1 - t)\mathbf{i} + (1 - t)^2\mathbf{j}$, which implies that $x'(t) = -1$ and $y'(t) = -2(1 - t)$. So,

$$\sqrt{[x'(t)]^2 + [y'(t)]^2} = \sqrt{1 + 4(1 - t)^2}$$

and you have

$$\begin{aligned}\int_{C_2} x\, ds &= \int_0^1 (1 - t)\sqrt{1 + 4(1 - t)^2}\, dt \\ &= -\frac{1}{8}\left[\frac{2}{3}[1 + 4(1 - t)^2]^{3/2}\right]_0^1 \\ &= \frac{1}{12}(5^{3/2} - 1).\end{aligned}$$

Consequently,

$$\int_C x\, ds = \int_{C_1} x\, ds + \int_{C_2} x\, ds = \frac{\sqrt{2}}{2} + \frac{1}{12}(5^{3/2} - 1) \approx 1.56.$$

For parametrizations given by $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$, it is helpful to remember the form of ds as

$$ds = \|\mathbf{r}'(t)\|\, dt = \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}\, dt.$$

This is demonstrated in Example 4.

EXAMPLE 4 Evaluating a Line Integral

Evaluate $\int_C (x + 2)\, ds$, where C is the curve represented by

$$\mathbf{r}(t) = t\mathbf{i} + \frac{4}{3}t^{3/2}\mathbf{j} + \frac{1}{2}t^2\mathbf{k}, \quad 0 \le t \le 2.$$

Solution Because $\mathbf{r}'(t) = \mathbf{i} + 2t^{1/2}\mathbf{j} + t\mathbf{k}$ and

$$\|\mathbf{r}'(t)\| = \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2} = \sqrt{1 + 4t + t^2}$$

it follows that

$$\begin{aligned}\int_C (x + 2)\, ds &= \int_0^2 (t + 2)\sqrt{1 + 4t + t^2}\, dt\\ &= \frac{1}{2}\int_0^2 2(t + 2)(1 + 4t + t^2)^{1/2}\, dt\\ &= \frac{1}{3}\Big[(1 + 4t + t^2)^{3/2}\Big]_0^2\\ &= \frac{1}{3}\left(13\sqrt{13} - 1\right)\\ &\approx 15.29.\end{aligned}$$

The next example shows how a line integral can be used to find the mass of a spring whose density varies. In Figure 15.11, note that the density of this spring increases as the spring spirals up the z-axis.

EXAMPLE 5 Finding the Mass of a Spring

Find the mass of a spring in the shape of the circular helix

$$\mathbf{r}(t) = \frac{1}{\sqrt{2}}(\cos t\,\mathbf{i} + \sin t\,\mathbf{j} + t\mathbf{k}), \quad 0 \le t \le 6\pi$$

where the density of the spring is $\rho(x, y, z) = 1 + z$, as shown in Figure 15.11.

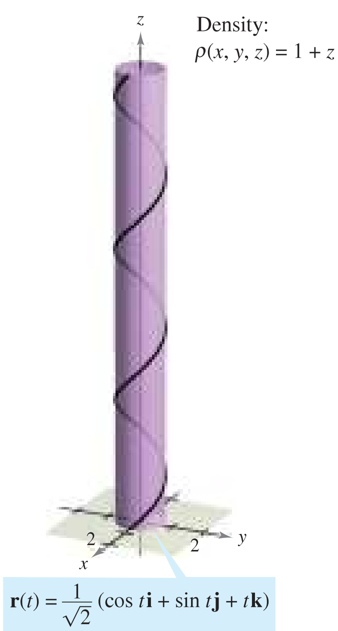


Figure 15.11

Solution Because

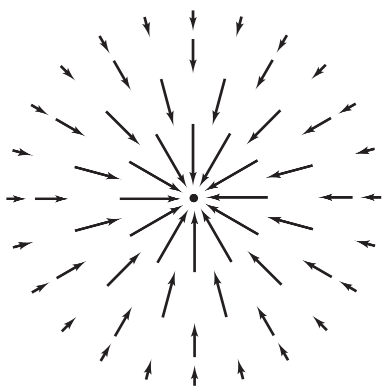
$$\|\mathbf{r}'(t)\| = \frac{1}{\sqrt{2}}\sqrt{(-\sin t)^2 + (\cos t)^2 + (1)^2} = 1$$

it follows that the mass of the spring is

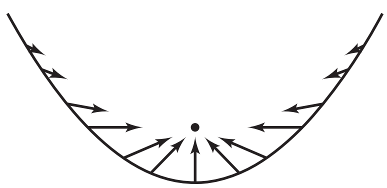
$$\begin{aligned}\text{Mass} = \int_C (1 + z)\, ds &= \int_0^{6\pi}\left(1 + \frac{t}{\sqrt{2}}\right) dt\\ &= \left[t + \frac{t^2}{2\sqrt{2}}\right]_0^{6\pi}\\ &= 6\pi\left(1 + \frac{3\pi}{\sqrt{2}}\right)\\ &\approx 144.47.\end{aligned}$$

The mass of the spring is approximately 144.47.

Line Integrals of Vector Fields



Inverse square force field $\mathbf{F}$



Vectors along a parabolic path in the force field $\mathbf{F}$

Figure 15.12

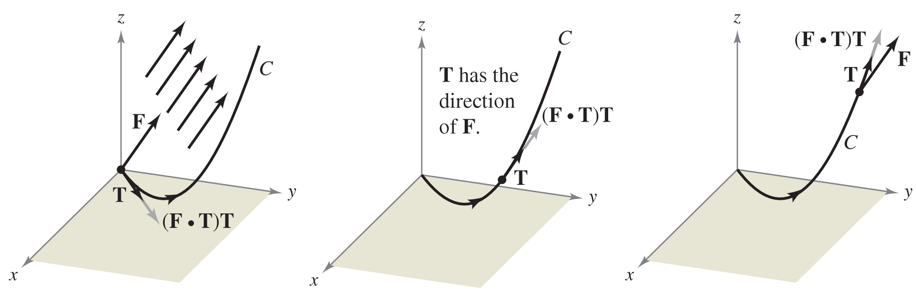
One of the most important physical applications of line integrals is that of finding the **work** done on an object moving in a force field. For example, Figure 15.12 shows an inverse square force field similar to the gravitational field of the sun. Note that the magnitude of the force along a circular path about the center is constant, whereas the magnitude of the force along a parabolic path varies from point to point.

To see how a line integral can be used to find work done in a force field $\mathbf{F}$, consider an object moving along a path C in the field, as shown in Figure 15.13. To determine the work done by the force, you need consider only that part of the force that is acting in the same direction as that in which the object is moving (or the opposite direction). This means that at each point on C, you can consider the projection $\mathbf{F} \cdot \mathbf{T}$ of the force vector $\mathbf{F}$ onto the unit tangent vector $\mathbf{T}$. On a small subarc of length Δs_i, the increment of work is

$$\begin{aligned}\Delta W_i &= (\text{force})(\text{distance})\\ &\approx [\mathbf{F}(x_i, y_i, z_i) \cdot \mathbf{T}(x_i, y_i, z_i)]\,\Delta s_i\end{aligned}$$

where (x_i, y_i, z_i) is a point in the ith subarc. Consequently, the total work done is given by the following integral.

$$W = \int_C \mathbf{F}(x, y, z) \cdot \mathbf{T}(x, y, z)\, ds$$



At each point on C, the force in the direction of motion is $(\mathbf{F} \cdot \mathbf{T})\mathbf{T}$.

Figure 15.13

This line integral appears in other contexts and is the basis of the following definition of the **line integral of a vector field.** Note in the definition that

$$\begin{aligned}\mathbf{F} \cdot \mathbf{T}\, ds &= \mathbf{F} \cdot \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}\|\mathbf{r}'(t)\|\, dt\\ &= \mathbf{F} \cdot \mathbf{r}'(t)\, dt\\ &= \mathbf{F} \cdot d\mathbf{r}.\end{aligned}$$

DEFINITION OF THE LINE INTEGRAL OF A VECTOR FIELD

Let $\mathbf{F}$ be a continuous vector field defined on a smooth curve C given by $\mathbf{r}(t)$, $a \le t \le b$. The **line integral** of $\mathbf{F}$ on C is given by

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot \mathbf{T}\, ds = \int_a^b \mathbf{F}(x(t),\; y(t), z(t)) \cdot \mathbf{r}'(t)\, dt.$$

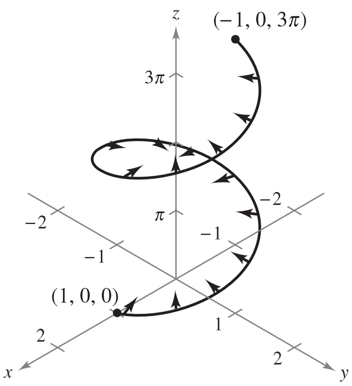


Figure 15.14

EXAMPLE 6 Work Done by a Force

Find the work done by the force field

$$\mathbf{F}(x, y, z) = -\frac{1}{2}x\mathbf{i} - \frac{1}{2}y\mathbf{j} + \frac{1}{4}\mathbf{k} \qquad \text{Force field } \mathbf{F}$$

on a particle as it moves along the helix given by

$$\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + t\mathbf{k} \qquad \text{Space curve } C$$

from the point $(1, 0, 0)$ to $(-1, 0, 3\pi)$, as shown in Figure 15.14.

Solution Because

$$\begin{aligned}\mathbf{r}(t) &= x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k} \\ &= \cos t\mathbf{i} + \sin t\mathbf{j} + t\mathbf{k}\end{aligned}$$

it follows that $x(t) = \cos t$, $y(t) = \sin t$, and $z(t) = t$. So, the force field can be written as

$$\mathbf{F}(x(t), y(t), z(t)) = -\frac{1}{2}\cos t\mathbf{i} - \frac{1}{2}\sin t\mathbf{j} + \frac{1}{4}\mathbf{k}.$$

To find the work done by the force field in moving a particle along the curve C, use the fact that

$$\mathbf{r}'(t) = -\sin t\mathbf{i} + \cos t\mathbf{j} + \mathbf{k}$$

and write the following.

$$\begin{aligned}W &= \int_C \mathbf{F} \cdot d\mathbf{r} \\ &= \int_a^b \mathbf{F}(x(t), y(t), z(t)) \cdot \mathbf{r}'(t)\, dt \\ &= \int_0^{3\pi} \left(-\frac{1}{2}\cos t\mathbf{i} - \frac{1}{2}\sin t\mathbf{j} + \frac{1}{4}\mathbf{k}\right) \cdot (-\sin t\mathbf{i} + \cos t\mathbf{j} + \mathbf{k})\, dt \\ &= \int_0^{3\pi} \left(\frac{1}{2}\sin t\cos t - \frac{1}{2}\sin t\cos t + \frac{1}{4}\right) dt \\ &= \int_0^{3\pi} \frac{1}{4}\, dt \\ &= \frac{1}{4}t\Big]_0^{3\pi} \\ &= \frac{3\pi}{4}\end{aligned}$$

NOTE In Example 6, note that the x- and y-components of the force field end up contributing nothing to the total work. This occurs because *in this particular example* the z-component of the force field is the only portion of the force that is acting in the same (or opposite) direction in which the particle is moving (see Figure 15.15).

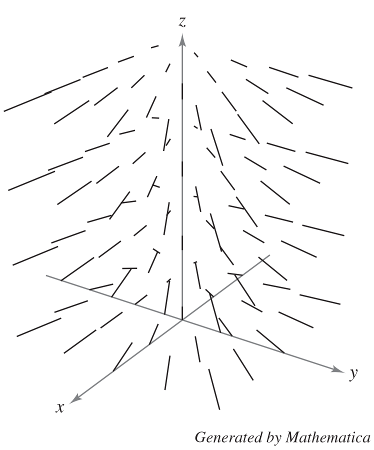


Figure 15.15

TECHNOLOGY The computer-generated view of the force field in Example 6 shown in Figure 15.15 indicates that each vector in the force field points toward the z-axis.

For line integrals of vector functions, the orientation of the curve C is important. If the orientation of the curve is reversed, the unit tangent vector $\mathbf{T}(t)$ is changed to $-\mathbf{T}(t)$, and you obtain

$$\int_{-C} \mathbf{F} \cdot d\mathbf{r} = -\int_C \mathbf{F} \cdot d\mathbf{r}.$$

EXAMPLE 7 Orientation and Parametrization of a Curve

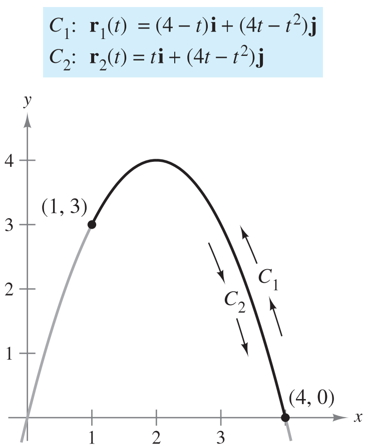


Figure 15.16

Let $\mathbf{F}(x, y) = y\mathbf{i} + x^2\mathbf{j}$ and evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ for each parabolic curve shown in Figure 15.16.

a. C_1: $\mathbf{r}_1(t) = (4-t)\mathbf{i} + (4t-t^2)\mathbf{j}, \quad 0 \le t \le 3$

b. C_2: $\mathbf{r}_2(t) = t\mathbf{i} + (4t-t^2)\mathbf{j}, \quad 1 \le t \le 4$

Solution

a. Because $\mathbf{r}_1'(t) = -\mathbf{i} + (4-2t)\mathbf{j}$ and

$$\mathbf{F}(x(t), y(t)) = (4t - t^2)\mathbf{i} + (4-t)^2\mathbf{j}$$

the line integral is

$$\begin{aligned}
\int_{C_1} \mathbf{F} \cdot d\mathbf{r} &= \int_0^3 [(4t-t^2)\mathbf{i} + (4-t)^2\mathbf{j}] \cdot [-\mathbf{i} + (4-2t)\mathbf{j}]\, dt \\
&= \int_0^3 (-4t + t^2 + 64 - 64t + 20t^2 - 2t^3)\, dt \\
&= \int_0^3 (-2t^3 + 21t^2 - 68t + 64)\, dt \\
&= \left[-\frac{t^4}{2} + 7t^3 - 34t^2 + 64t\right]_0^3 \\
&= \frac{69}{2}.
\end{aligned}$$

b. Because $\mathbf{r}_2'(t) = \mathbf{i} + (4-2t)\mathbf{j}$ and

$$\mathbf{F}(x(t), y(t)) = (4t - t^2)\mathbf{i} + t^2\mathbf{j}$$

the line integral is

$$\begin{aligned}
\int_{C_2} \mathbf{F} \cdot d\mathbf{r} &= \int_1^4 [(4t-t^2)\mathbf{i} + t^2\mathbf{j}] \cdot [\mathbf{i} + (4-2t)\mathbf{j}]\, dt \\
&= \int_1^4 (4t - t^2 + 4t^2 - 2t^3)\, dt \\
&= \int_1^4 (-2t^3 + 3t^2 + 4t)\, dt \\
&= \left[-\frac{t^4}{2} + t^3 + 2t^2\right]_1^4 \\
&= -\frac{69}{2}.
\end{aligned}$$

The answer in part (b) is the negative of that in part (a) because C_1 and C_2 represent opposite orientations of the same parabolic segment. ■

NOTE Although the value of the line integral in Example 7 depends on the orientation of C, it does not depend on the parametrization of C. To see this, let C_3 be represented by

$$\mathbf{r}_3 = (t+2)\mathbf{i} + (4-t^2)\mathbf{j}$$

where $-1 \le t \le 2$. The graph of this curve is the same parabolic segment shown in Figure 15.16. Does the value of the line integral over C_3 agree with the value over C_1 or C_2? Why or why not?

Line Integrals in Differential Form

A second commonly used form of line integrals is derived from the vector field notation used in the preceding section. If $\mathbf{F}$ is a vector field of the form $\mathbf{F}(x, y) = M\mathbf{i} + N\mathbf{j}$, and C is given by $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$, then $\mathbf{F} \cdot d\mathbf{r}$ is often written as $M\,dx + N\,dy$.

$$\begin{aligned}
\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_C \mathbf{F} \cdot \frac{d\mathbf{r}}{dt}\,dt \\
&= \int_a^b (M\mathbf{i} + N\mathbf{j}) \cdot (x'(t)\mathbf{i} + y'(t)\mathbf{j})\,dt \\
&= \int_a^b \left(M\frac{dx}{dt} + N\frac{dy}{dt}\right) dt \\
&= \int_C (M\,dx + N\,dy)
\end{aligned}$$

This **differential form** can be extended to three variables. The parentheses are often omitted, as follows.

$$\int_C M\,dx + N\,dy \qquad \text{and} \qquad \int_C M\,dx + N\,dy + P\,dz$$

Notice how this differential notation is used in Example 8.

EXAMPLE 8 Evaluating a Line Integral in Differential Form

Let C be the circle of radius 3 given by

$$\mathbf{r}(t) = 3\cos t\,\mathbf{i} + 3\sin t\,\mathbf{j}, \quad 0 \le t \le 2\pi$$

as shown in Figure 15.17. Evaluate the line integral

$$\int_C y^3\,dx + (x^3 + 3xy^2)\,dy.$$

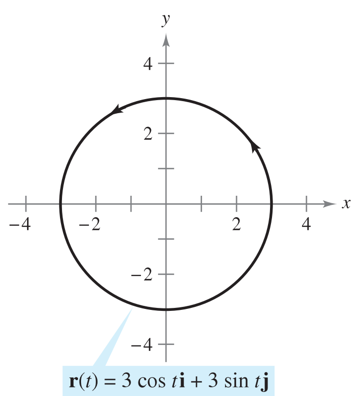


Figure 15.17

Solution Because $x = 3\cos t$ and $y = 3\sin t$, you have $dx = -3\sin t\,dt$ and $dy = 3\cos t\,dt$. So, the line integral is

$$\begin{aligned}
&\int_C M\,dx + N\,dy \\
&= \int_C y^3\,dx + (x^3 + 3xy^2)\,dy \\
&= \int_0^{2\pi} \left[(27\sin^3 t)(-3\sin t) + (27\cos^3 t + 81\cos t\sin^2 t)(3\cos t)\right] dt \\
&= 81\int_0^{2\pi} (\cos^4 t - \sin^4 t + 3\cos^2 t\sin^2 t)\,dt \\
&= 81\int_0^{2\pi} \left(\cos^2 t - \sin^2 t + \frac{3}{4}\sin^2 2t\right) dt \\
&= 81\int_0^{2\pi} \left[\cos 2t + \frac{3}{4}\left(\frac{1 - \cos 4t}{2}\right)\right] dt \\
&= 81\left[\frac{\sin 2t}{2} + \frac{3}{8}t - \frac{3\sin 4t}{32}\right]_0^{2\pi} \\
&= \frac{243\pi}{4}.
\end{aligned}$$

NOTE The orientation of C affects the value of the differential form of a line integral. Specifically, if $-C$ has the orientation opposite to that of C, then

$$\int_{-C} M\,dx + N\,dy = -\int_C M\,dx + N\,dy.$$

So, of the three line integral forms presented in this section, the orientation of C does not affect the form $\int_C f(x, y)\,ds$, but it does affect the vector form and the differential form.

For curves represented by $y = g(x)$, $a \le x \le b$, you can let $x = t$ and obtain the parametric form

$$x = t \quad \text{and} \quad y = g(t), \quad a \le t \le b.$$

Because $dx = dt$ for this form, you have the option of evaluating the line integral in the variable x or the variable t. This is demonstrated in Example 9.

EXAMPLE 9 Evaluating a Line Integral in Differential Form

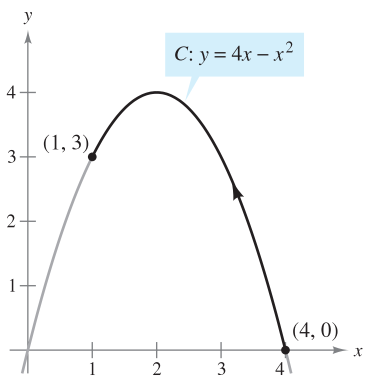


Figure 15.18

Evaluate

$$\int_C y\,dx + x^2\,dy$$

where C is the parabolic arc given by $y = 4x - x^2$ from $(4, 0)$ to $(1, 3)$, as shown in Figure 15.18.

Solution Rather than converting to the parameter t, you can simply retain the variable x and write

$$y = 4x - x^2 \quad \Rightarrow \quad dy = (4 - 2x)\,dx.$$

Then, in the direction from $(4, 0)$ to $(1, 3)$, the line integral is

$$\begin{aligned}\int_C y\,dx + x^2\,dy &= \int_4^1 [(4x - x^2)\,dx + x^2(4 - 2x)\,dx] \\ &= \int_4^1 (4x + 3x^2 - 2x^3)\,dx \\ &= \left[2x^2 + x^3 - \frac{x^4}{2}\right]_4^1 = \frac{69}{2}.\end{aligned}$$

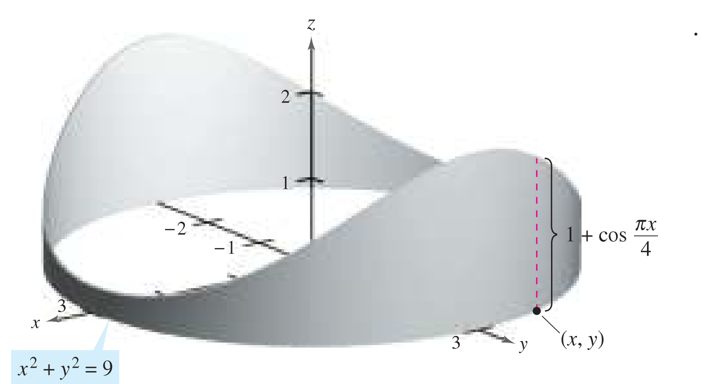
See Example 7.

EXPLORATION

Finding Lateral Surface Area The figure below shows a piece of tin that has been cut from a circular cylinder. The base of the circular cylinder is modeled by $x^2 + y^2 = 9$. At any point (x, y) on the base, the height of the object is given by

$$f(x, y) = 1 + \cos\frac{\pi x}{4}.$$

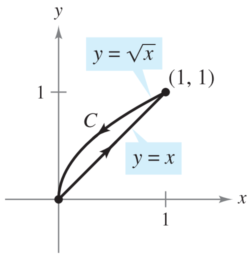
Explain how to use a line integral to find the surface area of the piece of tin.



15.2 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–6, find a piecewise smooth parametrization of the path C. (Note that there is more than one correct answer.)

1.


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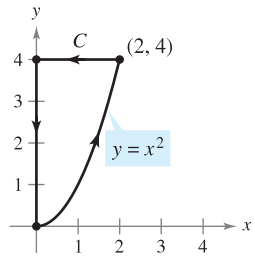

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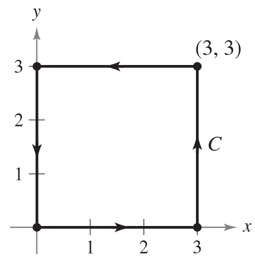

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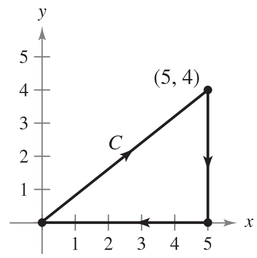
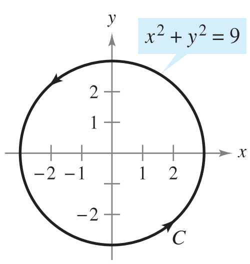
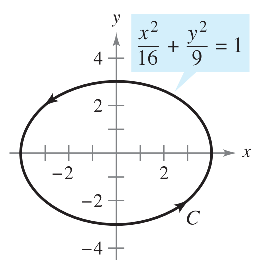


5.


6.


In Exercises 7–10, evaluate the line integral along the given path.

7. $\int_C xy\, ds$
C: $\mathbf{r}(t) = 4t\mathbf{i} + 3t\mathbf{j}$
$0 \le t \le 1$

8. $\int_C 3(x - y)\, ds$
C: $\mathbf{r}(t) = t\mathbf{i} + (2 - t)\mathbf{j}$
$0 \le t \le 2$

9. $\int_C (x^2 + y^2 + z^2)\, ds$
C: $\mathbf{r}(t) = \sin t\mathbf{i} + \cos t\mathbf{j} + 2\mathbf{k}$
$0 \le t \le \pi/2$

10. $\int_C 2xyz\, ds$
C: $\mathbf{r}(t) = 12t\mathbf{i} + 5t\mathbf{j} + 84t\mathbf{k}$
$0 \le t \le 1$

In Exercises 11–14, (a) find a parametrization of the path C, and (b) evaluate

$$\int_C (x^2 + y^2)\, ds$$

along C.

11. C: line segment from $(0, 0)$ to $(1, 1)$
12. C: line segment from $(0, 0)$ to $(2, 4)$
13. C: counterclockwise around the circle $x^2 + y^2 = 1$ from $(1, 0)$ to $(0, 1)$
14. C: counterclockwise around the circle $x^2 + y^2 = 4$ from $(2, 0)$ to $(0, 2)$

In Exercises 15–18, (a) find a parametrization of the path C, and (b) evaluate

$$\int_C (x + 4\sqrt{y})\, ds$$

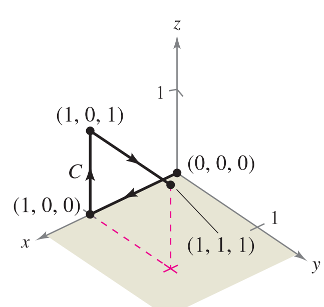
along C.

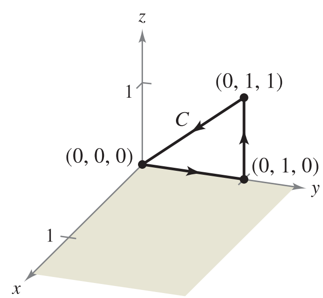
15. C: x-axis from $x = 0$ to $x = 1$
16. C: y-axis from $y = 1$ to $y = 9$
17. C: counterclockwise around the triangle with vertices $(0, 0)$, $(1, 0)$, and $(0, 1)$
18. C: counterclockwise around the square with vertices $(0, 0)$, $(2, 0)$, $(2, 2)$, and $(0, 2)$

In Exercises 19 and 20, (a) find a piecewise smooth parametrization of the path C shown in the figure, and (b) evaluate

$$\int_C (2x + y^2 - z)\, ds$$

along C.

19.


20.


Mass **In Exercises 21 and 22, find the total mass of two turns of a spring with density ρ in the shape of the circular helix**

$$\mathbf{r}(t) = 2\cos t\,\mathbf{i} + 2\sin t\,\mathbf{j} + t\mathbf{k},\quad 0 \le t \le 4\pi.$$

21. $\rho(x, y, z) = \frac{1}{2}(x^2 + y^2 + z^2)$
22. $\rho(x, y, z) = z$

Mass **In Exercises 23–26, find the total mass of the wire with density ρ.**

23. $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j}$, $\rho(x, y) = x + y$, $0 \le t \le \pi$
24. $\mathbf{r}(t) = t^2\mathbf{i} + 2t\mathbf{j}$, $\rho(x, y) = \frac{3}{4}y$, $0 \le t \le 1$
25. $\mathbf{r}(t) = t^2\mathbf{i} + 2t\mathbf{j} + t\mathbf{k}$, $\rho(x, y, z) = kz$ $(k > 0)$, $1 \le t \le 3$
26. $\mathbf{r}(t) = 2\cos t\mathbf{i} + 2\sin t\mathbf{j} + 3t\mathbf{k}$, $\rho(x, y, z) = k + z$ $(k > 0)$, $0 \le t \le 2\pi$

In Exercises 27–32, evaluate

$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

where C is represented by $\mathbf{r}(t)$.

27. $\mathbf{F}(x, y) = x\mathbf{i} + y\mathbf{j}$

C: $\mathbf{r}(t) = t\mathbf{i} + t\mathbf{j}, \quad 0 \le t \le 1$

28. $\mathbf{F}(x, y) = xy\mathbf{i} + y\mathbf{j}$

C: $\mathbf{r}(t) = 4\cos t\mathbf{i} + 4\sin t\mathbf{j}, \quad 0 \le t \le \pi/2$

29. $\mathbf{F}(x, y) = 3x\mathbf{i} + 4y\mathbf{j}$

C: $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j}, \quad 0 \le t \le \pi/2$

30. $\mathbf{F}(x, y) = 3x\mathbf{i} + 4y\mathbf{j}$

C: $\mathbf{r}(t) = t\mathbf{i} + \sqrt{4 - t^2}\mathbf{j}, \quad -2 \le t \le 2$

31. $\mathbf{F}(x, y, z) = xy\mathbf{i} + xz\mathbf{j} + yz\mathbf{k}$

C: $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + 2t\mathbf{k}, \quad 0 \le t \le 1$

32. $\mathbf{F}(x, y, z) = x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}$

C: $\mathbf{r}(t) = 2\sin t\mathbf{i} + 2\cos t\mathbf{j} + \frac{1}{2}t^2\mathbf{k}, \quad 0 \le t \le \pi$

CAS **In Exercises 33 and 34, use a computer algebra system to evaluate the integral**

$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

where C is represented by $\mathbf{r}(t)$.

33. $\mathbf{F}(x, y, z) = x^2z\mathbf{i} + 6y\mathbf{j} + yz^2\mathbf{k}$

C: $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + \ln t\mathbf{k}, \quad 1 \le t \le 3$

34. $\mathbf{F}(x, y, z) = \dfrac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{\sqrt{x^2 + y^2 + z^2}}$

C: $\mathbf{r}(t) = t\mathbf{i} + t\mathbf{j} + e^t\mathbf{k}, \quad 0 \le t \le 2$

Work **In Exercises 35–40, find the work done by the force field F on a particle moving along the given path.**

35. $\mathbf{F}(x, y) = x\mathbf{i} + 2y\mathbf{j}$

C: $x = t, y = t^3$ from $(0, 0)$ to $(2, 8)$

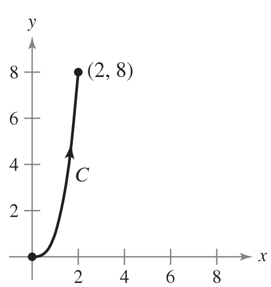


Figure for 35

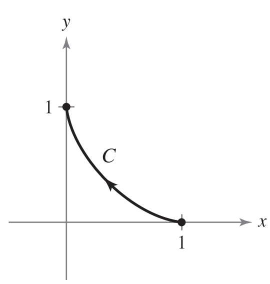


Figure for 36

36. $\mathbf{F}(x, y) = x^2\mathbf{i} - xy\mathbf{j}$

C: $x = \cos^3 t, y = \sin^3 t$ from $(1, 0)$ to $(0, 1)$

37. $\mathbf{F}(x, y) = x\mathbf{i} + y\mathbf{j}$

C: counterclockwise around the triangle with vertices $(0, 0)$, $(1, 0)$, and $(0, 1)$ (*Hint:* See Exercise 17a.)

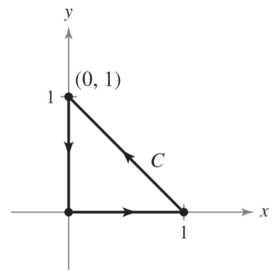


Figure for 37

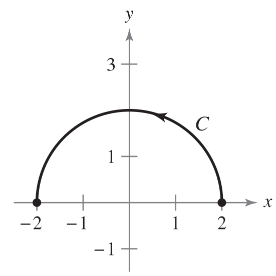


Figure for 38

38. $\mathbf{F}(x, y) = -y\mathbf{i} - x\mathbf{j}$

C: counterclockwise along the semicircle $y = \sqrt{4 - x^2}$ from $(2, 0)$ to $(-2, 0)$

39. $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} - 5z\mathbf{k}$

C: $\mathbf{r}(t) = 2\cos t\mathbf{i} + 2\sin t\mathbf{j} + t\mathbf{k}, \quad 0 \le t \le 2\pi$

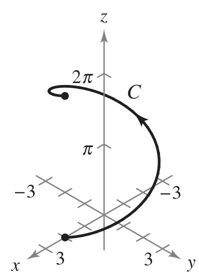


Figure for 39

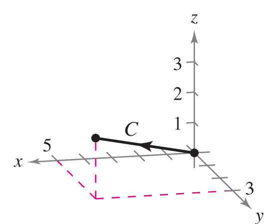


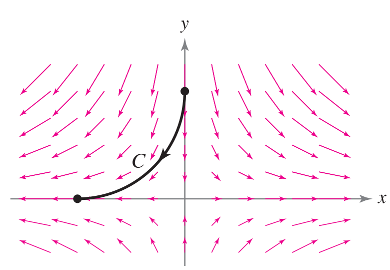
Figure for 40

40. $\mathbf{F}(x, y, z) = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$

C: line from $(0, 0, 0)$ to $(5, 3, 2)$

In Exercises 41–44, determine whether the work done along the path C is positive, negative, or zero. Explain.

41.



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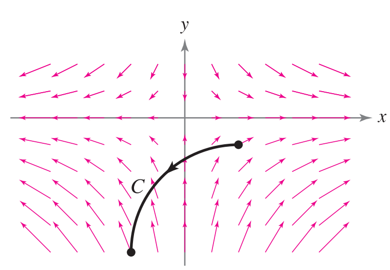

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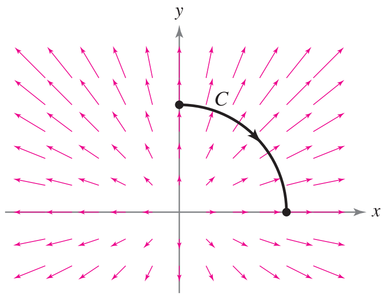
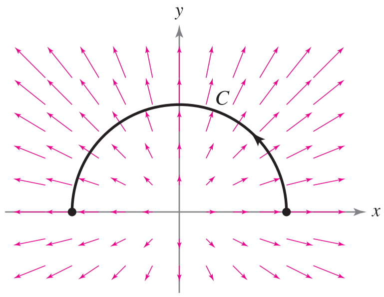


44.



In Exercises 45 and 46, evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ for each curve. Discuss the orientation of the curve and its effect on the value of the integral.

45. $\mathbf{F}(x, y) = x^2\mathbf{i} + xy\mathbf{j}$

(a) $\mathbf{r}_1(t) = 2t\mathbf{i} + (t - 1)\mathbf{j}, \quad 1 \le t \le 3$

(b) $\mathbf{r}_2(t) = 2(3 - t)\mathbf{i} + (2 - t)\mathbf{j}, \quad 0 \le t \le 2$

46. $\mathbf{F}(x, y) = x^2y\mathbf{i} + xy^{3/2}\mathbf{j}$

(a) $\mathbf{r}_1(t) = (t + 1)\mathbf{i} + t^2\mathbf{j}, \quad 0 \le t \le 2$

(b) $\mathbf{r}_2(t) = (1 + 2\cos t)\mathbf{i} + (4\cos^2 t)\mathbf{j}, \quad 0 \le t \le \pi/2$

In Exercises 47–50, demonstrate the property that

$$\int_C \mathbf{F} \cdot d\mathbf{r} = 0$$

regardless of the initial and terminal points of C, if the tangent vector $\mathbf{r}'(t)$ is orthogonal to the force field $\mathbf{F}$.

47. $\mathbf{F}(x, y) = y\mathbf{i} - x\mathbf{j}$

C: $\mathbf{r}(t) = t\mathbf{i} - 2t\mathbf{j}$

48. $\mathbf{F}(x, y) = -3y\mathbf{i} + x\mathbf{j}$

C: $\mathbf{r}(t) = t\mathbf{i} - t^3\mathbf{j}$

49. $\mathbf{F}(x, y) = (x^3 - 2x^2)\mathbf{i} + \left(x - \dfrac{y}{2}\right)\mathbf{j}$

C: $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j}$

50. $\mathbf{F}(x, y) = x\mathbf{i} + y\mathbf{j}$

C: $\mathbf{r}(t) = 3\sin t\mathbf{i} + 3\cos t\mathbf{j}$

In Exercises 51–54, evaluate the line integral along the path C given by $x = 2t$, $y = 10t$, where $0 \le t \le 1$.

51. $\displaystyle\int_C (x + 3y^2)\, dy$

52. $\displaystyle\int_C (x + 3y^2)\, dx$

53. $\displaystyle\int_C xy\, dx + y\, dy$

54. $\displaystyle\int_C (3y - x)\, dx + y^2\, dy$

In Exercises 55–62, evaluate the integral

$$\int_C (2x - y)\, dx + (x + 3y)\, dy$$

along the path C.

55. C: x-axis from $x = 0$ to $x = 5$

56. C: y-axis from $y = 0$ to $y = 2$

57. C: line segments from $(0, 0)$ to $(3, 0)$ and $(3, 0)$ to $(3, 3)$

58. C: line segments from $(0, 0)$ to $(0, -3)$ and $(0, -3)$ to $(2, -3)$

59. C: arc on $y = 1 - x^2$ from $(0, 1)$ to $(1, 0)$

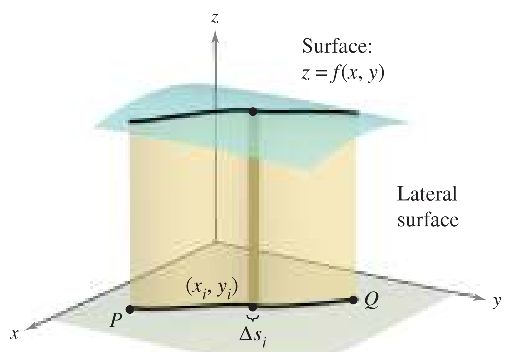
60. C: arc on $y = x^{3/2}$ from $(0, 0)$ to $(4, 8)$

61. C: parabolic path $x = t$, $y = 2t^2$, from $(0, 0)$ to $(2, 8)$

62. C: elliptic path $x = 4\sin t$, $y = 3\cos t$, from $(0, 3)$ to $(4, 0)$

Lateral Surface Area **In Exercises 63–70, find the area of the lateral surface (see figure) over the curve C in the xy-plane and under the surface $z = f(x, y)$, where**

$$\text{Lateral surface area} = \int_C f(x, y)\, ds.$$



C: Curve in xy-plane

63. $f(x, y) = h$, C: line from $(0, 0)$ to $(3, 4)$

64. $f(x, y) = y$, C: line from $(0, 0)$ to $(4, 4)$

65. $f(x, y) = xy$, C: $x^2 + y^2 = 1$ from $(1, 0)$ to $(0, 1)$

66. $f(x, y) = x + y$, C: $x^2 + y^2 = 1$ from $(1, 0)$ to $(0, 1)$

67. $f(x, y) = h$, C: $y = 1 - x^2$ from $(1, 0)$ to $(0, 1)$

68. $f(x, y) = y + 1$, C: $y = 1 - x^2$ from $(1, 0)$ to $(0, 1)$

69. $f(x, y) = xy$, C: $y = 1 - x^2$ from $(1, 0)$ to $(0, 1)$

70. $f(x, y) = x^2 - y^2 + 4$, C: $x^2 + y^2 = 4$

71. ***Engine Design*** A tractor engine has a steel component with a circular base modeled by the vector-valued function $\mathbf{r}(t) = 2\cos t\mathbf{i} + 2\sin t\mathbf{j}$. Its height is given by $z = 1 + y^2$. (All measurements of the component are in centimeters.)

(a) Find the lateral surface area of the component.

(b) The component is in the form of a shell of thickness 0.2 centimeter. Use the result of part (a) to approximate the amount of steel used in its manufacture.

(c) Draw a sketch of the component.

72. ***Building Design*** The ceiling of a building has a height above the floor given by $z = 20 + \frac{1}{4}x$, and one of the walls follows a path modeled by $y = x^{3/2}$. Find the surface area of the wall if $0 \le x \le 40$. (All measurements are in feet.)

Moments of Inertia **Consider a wire of density $\rho(x, y)$ given by the space curve**

$$C\text{: } \mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}, \quad 0 \le t \le b.$$

The moments of inertia about the x- and y-axes are given by

$$I_x = \int_C y^2\rho(x, y)\, ds$$

$$I_y = \int_C x^2\rho(x, y)\, ds.$$

In Exercises 73 and 74, find the moments of inertia for the wire of density ρ.

73. A wire lies along $\mathbf{r}(t) = a\cos t\mathbf{i} + a\sin t\mathbf{j}$, $0 \le t \le 2\pi$ and $a > 0$, with density $\rho(x, y) = 1$.

74. A wire lies along $\mathbf{r}(t) = a\cos t\mathbf{i} + a\sin t\mathbf{j}$, $0 \le t \le 2\pi$ and $a > 0$, with density $\rho(x, y) = y$.

CAS **75.** ***Investigation*** The top outer edge of a solid with vertical sides and resting on the xy-plane is modeled by $\mathbf{r}(t) = 3\cos t\mathbf{i} + 3\sin t\mathbf{j} + (1 + \sin^2 2t)\mathbf{k}$, where all measurements are in centimeters. The intersection of the plane $y = b\ (-3 < b < 3)$ with the top of the solid is a horizontal line.

(a) Use a computer algebra system to graph the solid.

(b) Use a computer algebra system to approximate the lateral surface area of the solid.

(c) Find (if possible) the volume of the solid.

76. ***Work*** A particle moves along the path $y = x^2$ from the point $(0, 0)$ to the point $(1, 1)$. The force field $\mathbf{F}$ is measured at five points along the path, and the results are shown in the table. Use Simpson's Rule or a graphing utility to approximate the work done by the force field.

(x, y)	$(0, 0)$	$\left(\frac{1}{4}, \frac{1}{16}\right)$	$\left(\frac{1}{2}, \frac{1}{4}\right)$	$\left(\frac{3}{4}, \frac{9}{16}\right)$	$(1, 1)$
$\mathbf{F}(x, y)$	$\langle 5, 0\rangle$	$\langle 3.5, 1\rangle$	$\langle 2, 2\rangle$	$\langle 1.5, 3\rangle$	$\langle 1, 5\rangle$

77. ***Work*** Find the work done by a person weighing 175 pounds walking exactly one revolution up a circular helical staircase of radius 3 feet if the person rises 10 feet.

78. ***Investigation*** Determine the value of c such that the work done by the force field

$$\mathbf{F}(x, y) = 15[(4 - x^2y)\mathbf{i} - xy\mathbf{j}]$$

on an object moving along the parabolic path $y = c(1 - x^2)$ between the points $(-1, 0)$ and $(1, 0)$ is a minimum. Compare the result with the work required to move the object along the straight-line path connecting the points.

WRITING ABOUT CONCEPTS

79. Define a line integral of a function f along a smooth curve C in the plane and in space. How do you evaluate the line integral as a definite integral?

80. Define a line integral of a continuous vector field $\mathbf{F}$ on a smooth curve C. How do you evaluate the line integral as a definite integral?

81. Order the surfaces in ascending order of the lateral surface area under the surface and over the curve $y = \sqrt{x}$ from $(0, 0)$ to $(4, 2)$ in the xy-plane. Explain your ordering without doing any calculations.

(a) $z_1 = 2 + x$ (b) $z_2 = 5 + x$

(c) $z_3 = 2$ (d) $z_4 = 10 + x + 2y$

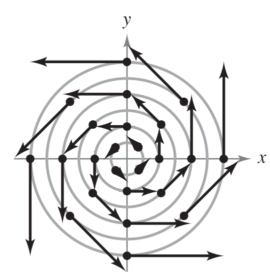
CAPSTONE

82. For each of the following, determine whether the work done in moving an object from the first to the second point through the force field shown in the figure is positive, negative, or zero. Explain your answer.

(a) From $(-3, -3)$ to $(3, 3)$

(b) From $(-3, 0)$ to $(0, 3)$

(c) From $(5, 0)$ to $(0, 3)$



True or False? **In Exercises 83–86, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

83. If C is given by $x(t) = t$, $y(t) = t$, $0 \le t \le 1$, then

$$\int_C xy\, ds = \int_0^1 t^2\, dt.$$

84. If $C_2 = -C_1$, then $\int_{C_1} f(x, y)\, ds + \int_{C_2} f(x, y)\, ds = 0$.

85. The vector functions $\mathbf{r}_1 = t\mathbf{i} + t^2\mathbf{j}$, $0 \le t \le 1$, and $\mathbf{r}_2 = (1 - t)\mathbf{i} + (1 - t)^2\mathbf{j}$, $0 \le t \le 1$, define the same curve.

86. If $\int_C \mathbf{F} \cdot \mathbf{T}\, ds = 0$, then $\mathbf{F}$ and $\mathbf{T}$ are orthogonal.

87. ***Work*** Consider a particle that moves through the force field $\mathbf{F}(x, y) = (y - x)\mathbf{i} + xy\mathbf{j}$ from the point $(0, 0)$ to the point $(0, 1)$ along the curve $x = kt(1 - t)$, $y = t$. Find the value of k such that the work done by the force field is 1.

15.3 Conservative Vector Fields and Independence of Path

- Understand and use the Fundamental Theorem of Line Integrals.
- Understand the concept of independence of path.
- Understand the concept of conservation of energy.

Fundamental Theorem of Line Integrals

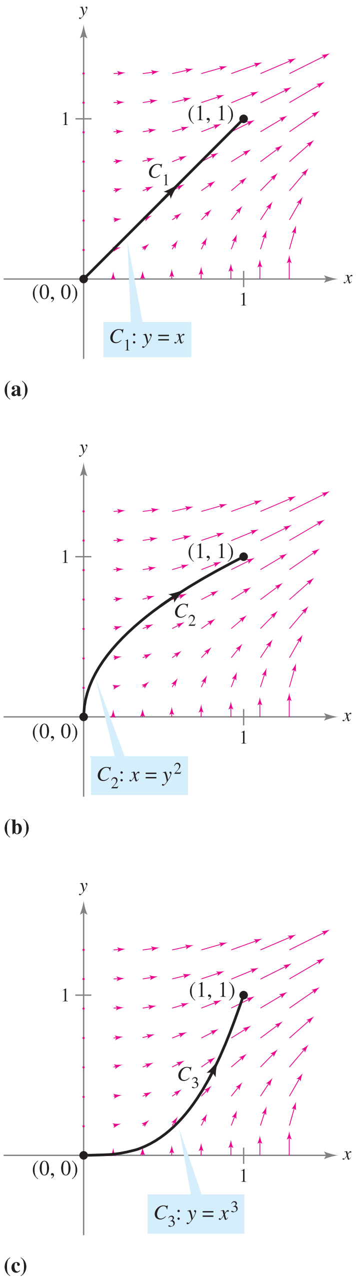


Figure 15.19

The discussion at the beginning of the preceding section pointed out that in a gravitational field the work done by gravity on an object moving between two points in the field is independent of the path taken by the object. In this section, you will study an important generalization of this result—it is called the **Fundamental Theorem of Line Integrals.**

To begin, an example is presented in which the line integral of a *conservative vector field* is evaluated over three different paths.

EXAMPLE 1 Line Integral of a Conservative Vector Field

Find the work done by the force field

$$\mathbf{F}(x, y) = \frac{1}{2}xy\mathbf{i} + \frac{1}{4}x^2\mathbf{j}$$

on a particle that moves from $(0, 0)$ to $(1, 1)$ along each path, as shown in Figure 15.19.

a. C_1: $y = x$ **b.** C_2: $x = y^2$ **c.** C_3: $y = x^3$

Solution

a. Let $\mathbf{r}(t) = t\mathbf{i} + t\mathbf{j}$ for $0 \leq t \leq 1$, so that

$$d\mathbf{r} = (\mathbf{i} + \mathbf{j})\,dt \quad \text{and} \quad \mathbf{F}(x, y) = \frac{1}{2}t^2\mathbf{i} + \frac{1}{4}t^2\mathbf{j}.$$

Then, the work done is

$$W = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_0^1 \frac{3}{4}t^2\,dt = \frac{1}{4}t^3\Big]_0^1 = \frac{1}{4}.$$

b. Let $\mathbf{r}(t) = t\mathbf{i} + \sqrt{t}\,\mathbf{j}$ for $0 \leq t \leq 1$, so that

$$d\mathbf{r} = \left(\mathbf{i} + \frac{1}{2\sqrt{t}}\mathbf{j}\right)dt \quad \text{and} \quad \mathbf{F}(x, y) = \frac{1}{2}t^{3/2}\mathbf{i} + \frac{1}{4}t^2\mathbf{j}.$$

Then, the work done is

$$W = \int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_0^1 \frac{5}{8}t^{3/2}\,dt = \frac{1}{4}t^{5/2}\Big]_0^1 = \frac{1}{4}.$$

c. Let $\mathbf{r}(t) = \frac{1}{2}t\mathbf{i} + \frac{1}{8}t^3\mathbf{j}$ for $0 \leq t \leq 2$, so that

$$d\mathbf{r} = \left(\frac{1}{2}\mathbf{i} + \frac{3}{8}t^2\mathbf{j}\right)dt \quad \text{and} \quad \mathbf{F}(x, y) = \frac{1}{32}t^4\mathbf{i} + \frac{1}{16}t^2\mathbf{j}.$$

Then, the work done is

$$W = \int_{C_3} \mathbf{F} \cdot d\mathbf{r} = \int_0^2 \frac{5}{128}t^4\,dt = \frac{1}{128}t^5\Big]_0^2 = \frac{1}{4}.$$

So, the work done by a conservative vector field is the same for all paths. ■

In Example 1, note that the vector field $\mathbf{F}(x, y) = \frac{1}{2}xy\mathbf{i} + \frac{1}{4}x^2\mathbf{j}$ is conservative because $\mathbf{F}(x, y) = \nabla f(x, y)$, where $f(x, y) = \frac{1}{4}x^2y$. In such cases, the following theorem states that the value of $\int_C \mathbf{F} \cdot d\mathbf{r}$ is given by

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} &= f(x(1), y(1)) - f(x(0), y(0)) \\ &= \frac{1}{4} - 0 \\ &= \frac{1}{4}.\end{aligned}$$

THEOREM 15.5 FUNDAMENTAL THEOREM OF LINE INTEGRALS

Let C be a piecewise smooth curve lying in an open region R and given by

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}, \quad a \le t \le b.$$

If $\mathbf{F}(x, y) = M\mathbf{i} + N\mathbf{j}$ is conservative in R, and M and N are continuous in R, then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \nabla f \cdot d\mathbf{r} = f(x(b), y(b)) - f(x(a), y(a))$$

where f is a potential function of $\mathbf{F}$. That is, $\mathbf{F}(x, y) = \nabla f(x, y)$.

NOTE Notice how the Fundamental Theorem of Line Integrals is similar to the Fundamental Theorem of Calculus (Section 4.4), which states that

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where $F'(x) = f(x)$.

PROOF A proof is provided only for a smooth curve. For piecewise smooth curves, the procedure is carried out separately on each smooth portion. Because $\mathbf{F}(x, y) = \nabla f(x, y) = f_x(x, y)\mathbf{i} + f_y(x, y)\mathbf{j}$, it follows that

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_a^b \mathbf{F} \cdot \frac{d\mathbf{r}}{dt}\,dt \\ &= \int_a^b \left[f_x(x, y)\frac{dx}{dt} + f_y(x, y)\frac{dy}{dt}\right] dt\end{aligned}$$

and, by the Chain Rule (Theorem 13.6), you have

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_a^b \frac{d}{dt}[f(x(t), y(t))]\,dt \\ &= f(x(b), y(b)) - f(x(a), y(a)).\end{aligned}$$

The last step is an application of the Fundamental Theorem of Calculus. ■

In space, the Fundamental Theorem of Line Integrals takes the following form. Let C be a piecewise smooth curve lying in an open region Q and given by

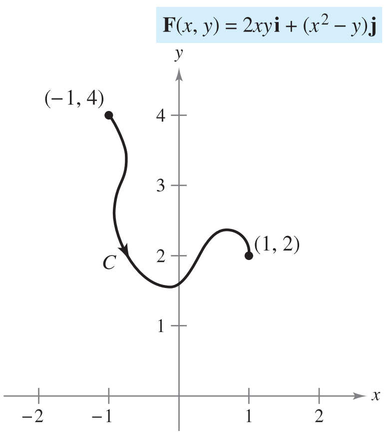
$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}, \quad a \le t \le b.$$

If $\mathbf{F}(x, y, z) = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ is conservative and M, N, and P are continuous, then

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_C \nabla f \cdot d\mathbf{r} \\ &= f(x(b), y(b), z(b)) - f(x(a), y(a), z(a))\end{aligned}$$

where $\mathbf{F}(x, y, z) = \nabla f(x, y, z)$.

The Fundamental Theorem of Line Integrals states that if the vector field $\mathbf{F}$ is conservative, then the line integral between any two points is simply the difference in the values of the *potential* function f at these points.



Using the Fundamental Theorem of Line Integrals, $\int_C \mathbf{F} \cdot d\mathbf{r}$.
Figure 15.20

EXAMPLE 2 Using the Fundamental Theorem of Line Integrals

Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is a piecewise smooth curve from $(-1, 4)$ to $(1, 2)$ and

$$\mathbf{F}(x, y) = 2xy\mathbf{i} + (x^2 - y)\mathbf{j}$$

as shown in Figure 15.20.

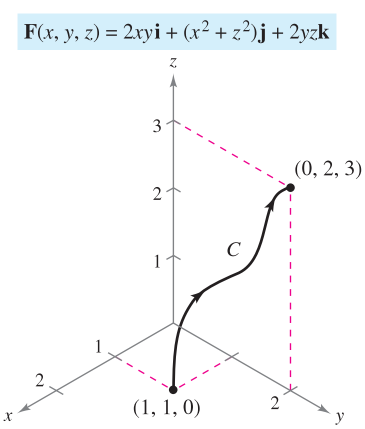
Solution From Example 6 in Section 15.1, you know that $\mathbf{F}$ is the gradient of f where

$$f(x, y) = x^2y - \frac{y^2}{2} + K.$$

Consequently, $\mathbf{F}$ is conservative, and by the Fundamental Theorem of Line Integrals, it follows that

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} &= f(1, 2) - f(-1, 4)\\ &= \left[1^2(2) - \frac{2^2}{2}\right] - \left[(-1)^2(4) - \frac{4^2}{2}\right]\\ &= 4.\end{aligned}$$

Note that it is unnecessary to include a constant K as part of f, because it is canceled by subtraction.



Using the Fundamental Theorem of Line Integrals, $\int_C \mathbf{F} \cdot d\mathbf{r}$.
Figure 15.21

EXAMPLE 3 Using the Fundamental Theorem of Line Integrals

Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is a piecewise smooth curve from $(1, 1, 0)$ to $(0, 2, 3)$ and

$$\mathbf{F}(x, y, z) = 2xy\mathbf{i} + (x^2 + z^2)\mathbf{j} + 2yz\mathbf{k}$$

as shown in Figure 15.21.

Solution From Example 8 in Section 15.1, you know that $\mathbf{F}$ is the gradient of f where $f(x, y, z) = x^2y + yz^2 + K$. Consequently, $\mathbf{F}$ is conservative, and by the Fundamental Theorem of Line Integrals, it follows that

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} &= f(0, 2, 3) - f(1, 1, 0)\\ &= [(0)^2(2) + (2)(3)^2] - [(1)^2(1) + (1)(0)^2]\\ &= 17.\end{aligned}$$

■

In Examples 2 and 3, be sure you see that the value of the line integral is the same for any smooth curve C that has the given initial and terminal points. For instance, in Example 3, try evaluating the line integral for the curve given by

$$\mathbf{r}(t) = (1 - t)\mathbf{i} + (1 + t)\mathbf{j} + 3t\mathbf{k}.$$

You should obtain

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^1 (30t^2 + 16t - 1)\, dt\\ &= 17.\end{aligned}$$

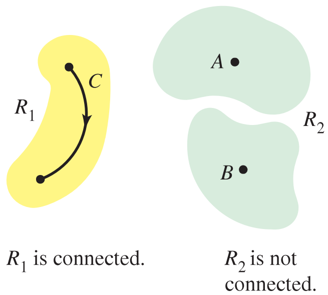


Figure 15.22

Independence of Path

From the Fundamental Theorem of Line Integrals it is clear that if $\mathbf{F}$ is continuous and conservative in an open region R, the value of $\int_C \mathbf{F} \cdot d\mathbf{r}$ is the same for every piecewise smooth curve C from one fixed point in R to another fixed point in R. This result is described by saying that the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ is **independent of path** in the region R.

A region in the plane (or in space) is **connected** if any two points in the region can be joined by a piecewise smooth curve lying entirely within the region, as shown in Figure 15.22. In open regions that are *connected*, the path independence of $\int_C \mathbf{F} \cdot d\mathbf{r}$ is equivalent to the condition that $\mathbf{F}$ is conservative.

THEOREM 15.6 INDEPENDENCE OF PATH AND CONSERVATIVE VECTOR FIELDS

If $\mathbf{F}$ is continuous on an open connected region, then the line integral

$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

is independent of path if and only if $\mathbf{F}$ is conservative.

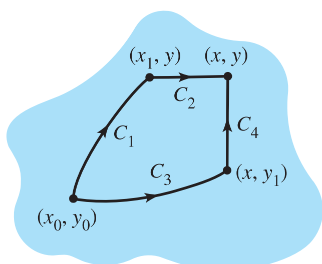


Figure 15.23

PROOF If $\mathbf{F}$ is conservative, then, by the Fundamental Theorem of Line Integrals, the line integral is independent of path. Now establish the converse for a plane region R. Let $\mathbf{F}(x, y) = M\mathbf{i} + N\mathbf{j}$, and let (x_0, y_0) be a fixed point in R. If (x, y) is any point in R, choose a piecewise smooth curve C running from (x_0, y_0) to (x, y), and define f by

$$f(x, y) = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C M\,dx + N\,dy.$$

The existence of C in R is guaranteed by the fact that R is connected. You can show that f is a potential function of $\mathbf{F}$ by considering two different paths between (x_0, y_0) and (x, y). For the *first* path, choose (x_1, y) in R such that $x \neq x_1$. This is possible because R is open. Then choose C_1 and C_2, as shown in Figure 15.23. Using the independence of path, it follows that

$$\begin{aligned} f(x, y) &= \int_C M\,dx + N\,dy \\ &= \int_{C_1} M\,dx + N\,dy + \int_{C_2} M\,dx + N\,dy. \end{aligned}$$

Because the first integral does not depend on x, and because $dy = 0$ in the second integral, you have

$$f(x, y) = g(y) + \int_{C_2} M\,dx$$

and it follows that the partial derivative of f with respect to x is $f_x(x, y) = M$. For the *second* path, choose a point (x, y_1). Using reasoning similar to that used for the first path, you can conclude that $f_y(x, y) = N$. Therefore,

$$\begin{aligned} \nabla f(x, y) &= f_x(x, y)\mathbf{i} + f_y(x, y)\mathbf{j} \\ &= M\mathbf{i} + N\mathbf{j} \\ &= \mathbf{F}(x, y) \end{aligned}$$

and it follows that $\mathbf{F}$ is conservative. ■

EXAMPLE 4 Finding Work in a Conservative Force Field

For the force field given by

$$\mathbf{F}(x, y, z) = e^x \cos y\mathbf{i} - e^x \sin y\mathbf{j} + 2\mathbf{k}$$

show that $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path, and calculate the work done by $\mathbf{F}$ on an object moving along a curve C from $(0, \pi/2, 1)$ to $(1, \pi, 3)$.

Solution Writing the force field in the form $\mathbf{F}(x, y, z) = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$, you have $M = e^x \cos y$, $N = -e^x \sin y$, and $P = 2$, and it follows that

$$\frac{\partial P}{\partial y} = 0 = \frac{\partial N}{\partial z}$$
$$\frac{\partial P}{\partial x} = 0 = \frac{\partial M}{\partial z}$$
$$\frac{\partial N}{\partial x} = -e^x \sin y = \frac{\partial M}{\partial y}.$$

So, $\mathbf{F}$ is conservative. If f is a potential function of $\mathbf{F}$, then

$$f_x(x, y, z) = e^x \cos y$$
$$f_y(x, y, z) = -e^x \sin y$$
$$f_z(x, y, z) = 2.$$

By integrating with respect to x, y, and z separately, you obtain

$$f(x, y, z) = \int f_x(x, y, z)\, dx = \int e^x \cos y\, dx = e^x \cos y + g(y, z)$$
$$f(x, y, z) = \int f_y(x, y, z)\, dy = \int -e^x \sin y\, dy = e^x \cos y + h(x, z)$$
$$f(x, y, z) = \int f_z(x, y, z)\, dz = \int 2\, dz = 2z + k(x, y).$$

By comparing these three versions of $f(x, y, z)$, you can conclude that

$$f(x, y, z) = e^x \cos y + 2z + K.$$

Therefore, the work done by $\mathbf{F}$ along *any* curve C from $(0, \pi/2, 1)$ to $(1, \pi, 3)$ is

$$\begin{aligned} W &= \int_C \mathbf{F} \cdot d\mathbf{r} \\ &= \Big[e^x \cos y + 2z \Big]_{(0,\, \pi/2,\, 1)}^{(1,\, \pi,\, 3)} \\ &= (-e + 6) - (0 + 2) \\ &= 4 - e. \end{aligned}$$

How much work would be done if the object in Example 4 moved from the point $(0, \pi/2, 1)$ to $(1, \pi, 3)$ and then back to the starting point $(0, \pi/2, 1)$? The Fundamental Theorem of Line Integrals states that there is zero work done. Remember that, by definition, work can be negative. So, by the time the object gets back to its starting point, the amount of work that registers positively is canceled out by the amount of work that registers negatively.

A curve C given by $\mathbf{r}(t)$ for $a \le t \le b$ is **closed** if $\mathbf{r}(a) = \mathbf{r}(b)$. By the Fundamental Theorem of Line Integrals, you can conclude that if $\mathbf{F}$ is continuous and conservative on an open region R, then the line integral over every closed curve C is 0.

NOTE Theorem 15.7 gives you options for evaluating a line integral involving a conservative vector field. You can use a potential function, or it might be more convenient to choose a particularly simple path, such as a straight line.

THEOREM 15.7 EQUIVALENT CONDITIONS

Let $\mathbf{F}(x, y, z) = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ have continuous first partial derivatives in an open connected region R, and let C be a piecewise smooth curve in R. The following conditions are equivalent.

1. $\mathbf{F}$ is conservative. That is, $\mathbf{F} = \nabla f$ for some function f.
2. $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path.
3. $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for every *closed* curve C in R.

EXAMPLE 5 Evaluating a Line Integral

Evaluate $\displaystyle\int_{C_1} \mathbf{F} \cdot d\mathbf{r}$, where

$$\mathbf{F}(x, y) = (y^3 + 1)\mathbf{i} + (3xy^2 + 1)\mathbf{j}$$

and C_1 is the semicircular path from $(0, 0)$ to $(2, 0)$, as shown in Figure 15.24.

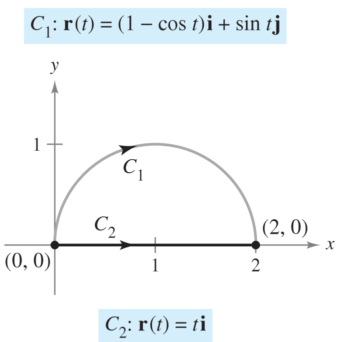


Figure 15.24

Solution You have the following three options.

a. You can use the method presented in the preceding section to evaluate the line integral along the *given curve*. To do this, you can use the parametrization $\mathbf{r}(t) = (1 - \cos t)\mathbf{i} + \sin t\mathbf{j}$, where $0 \le t \le \pi$. For this parametrization, it follows that $d\mathbf{r} = \mathbf{r}'(t)\,dt = (\sin t\mathbf{i} + \cos t\mathbf{j})\,dt$, and

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_0^{\pi} (\sin t + \sin^4 t + \cos t + 3\sin^2 t \cos t - 3\sin^2 t \cos^2 t)\,dt.$$

This integral should dampen your enthusiasm for this option.

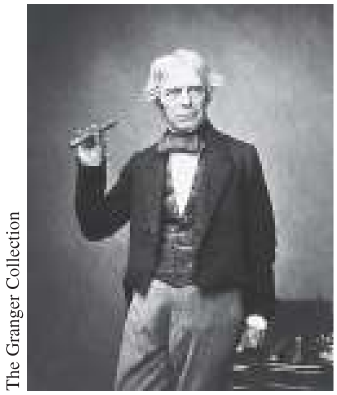
b. You can try to find a *potential function* and evaluate the line integral by the Fundamental Theorem of Line Integrals. Using the technique demonstrated in Example 4, you can find the potential function to be $f(x, y) = xy^3 + x + y + K$, and, by the Fundamental Theorem,

$$W = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} = f(2, 0) - f(0, 0) = 2.$$

c. Knowing that $\mathbf{F}$ is conservative, you have a third option. Because the value of the line integral is independent of path, you can replace the semicircular path with a *simpler path*. Suppose you choose the straight-line path C_2 from $(0, 0)$ to $(2, 0)$. Then, $\mathbf{r}(t) = t\mathbf{i}$, where $0 \le t \le 2$. So, $d\mathbf{r} = \mathbf{i}\,dt$ and $\mathbf{F}(x, y) = (y^3 + 1)\mathbf{i} + (3xy^2 + 1)\mathbf{j} = \mathbf{i} + \mathbf{j}$, so that

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_0^2 1\,dt = t\Big]_0^2 = 2.$$

Of the three options, obviously the third one is the easiest. ■



MICHAEL FARADAY (1791–1867)

Several philosophers of science have considered Faraday's Law of Conservation of Energy to be the greatest generalization ever conceived by humankind. Many physicists have contributed to our knowledge of this law. Two early and influential ones were James Prescott Joule (1818–1889) and Hermann Ludwig Helmholtz (1821–1894).

Conservation of Energy

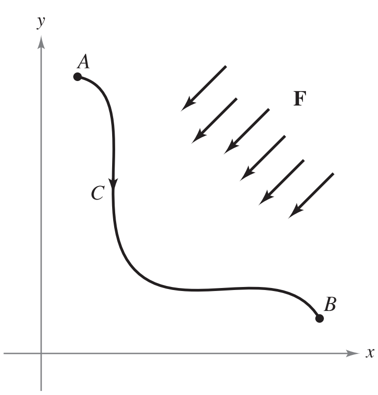
In 1840, the English physicist Michael Faraday wrote, "Nowhere is there a pure creation or production of power without a corresponding exhaustion of something to supply it." This statement represents the first formulation of one of the most important laws of physics—**the Law of Conservation of Energy.** In modern terminology, the law is stated as follows: *In a conservative force field, the sum of the potential and kinetic energies of an object remains constant from point to point.*

You can use the Fundamental Theorem of Line Integrals to derive this law. From physics, the **kinetic energy** of a particle of mass m and speed v is $k = \frac{1}{2}mv^2$. The **potential energy** p of a particle at point (x, y, z) in a conservative vector field $\mathbf{F}$ is defined as $p(x, y, z) = -f(x, y, z)$, where f is the potential function for $\mathbf{F}$. Consequently, the work done by $\mathbf{F}$ along a smooth curve C from A to B is

$$\begin{aligned}
W = \int_C \mathbf{F} \cdot d\mathbf{r} &= f(x, y, z)\Big]_A^B \\
&= -p(x, y, z)\Big]_A^B \\
&= p(A) - p(B)
\end{aligned}$$

as shown in Figure 15.25. In other words, work W is equal to the difference in the potential energies of A and B. Now, suppose that $\mathbf{r}(t)$ is the position vector for a particle moving along C from $A = \mathbf{r}(a)$ to $B = \mathbf{r}(b)$. At any time t, the particle's velocity, acceleration, and speed are $\mathbf{v}(t) = \mathbf{r}'(t)$, $\mathbf{a}(t) = \mathbf{r}''(t)$, and $v(t) = \|\mathbf{v}(t)\|$, respectively. So, by Newton's Second Law of Motion, $\mathbf{F} = m\mathbf{a}(t) = m(\mathbf{v}'(t))$, and the work done by $\mathbf{F}$ is

$$\begin{aligned}
W = \int_C \mathbf{F} \cdot d\mathbf{r} &= \int_a^b \mathbf{F} \cdot \mathbf{r}'(t)\, dt \\
&= \int_a^b \mathbf{F} \cdot \mathbf{v}(t)\, dt = \int_a^b [m\mathbf{v}'(t)] \cdot \mathbf{v}(t)\, dt \\
&= \int_a^b m[\mathbf{v}'(t) \cdot \mathbf{v}(t)]\, dt \\
&= \frac{m}{2}\int_a^b \frac{d}{dt}[\mathbf{v}(t) \cdot \mathbf{v}(t)]\, dt \\
&= \frac{m}{2}\int_a^b \frac{d}{dt}[\|\mathbf{v}(t)\|^2]\, dt \\
&= \frac{m}{2}\Big[\|\mathbf{v}(t)\|^2\Big]_a^b \\
&= \frac{m}{2}\Big[[v(t)]^2\Big]_a^b \\
&= \frac{1}{2}m[v(b)]^2 - \frac{1}{2}m[v(a)]^2 \\
&= k(B) - k(A).
\end{aligned}$$



The work done by $\mathbf{F}$ along C is
$W = \int_C \mathbf{F} \cdot d\mathbf{r} = p(A) - p(B).$
Figure 15.25

Equating these two results for W produces

$$\begin{aligned}
p(A) - p(B) &= k(B) - k(A) \\
p(A) + k(A) &= p(B) + k(B)
\end{aligned}$$

which implies that the sum of the potential and kinetic energies remains constant from point to point.

15.3 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–4, show that the value of $\int_C \mathbf{F} \cdot d\mathbf{r}$ is the same for each parametric representation of C.

1. $\mathbf{F}(x, y) = x^2\mathbf{i} + xy\mathbf{j}$

(a) $\mathbf{r}_1(t) = t\mathbf{i} + t^2\mathbf{j}, \quad 0 \le t \le 1$

(b) $\mathbf{r}_2(\theta) = \sin\theta\mathbf{i} + \sin^2\theta\mathbf{j}, \quad 0 \le \theta \le \dfrac{\pi}{2}$

2. $\mathbf{F}(x, y) = (x^2 + y^2)\mathbf{i} - x\mathbf{j}$

(a) $\mathbf{r}_1(t) = t\mathbf{i} + \sqrt{t}\,\mathbf{j}, \quad 0 \le t \le 4$

(b) $\mathbf{r}_2(w) = w^2\mathbf{i} + w\mathbf{j}, \quad 0 \le w \le 2$

3. $\mathbf{F}(x, y) = y\mathbf{i} - x\mathbf{j}$

(a) $\mathbf{r}_1(\theta) = \sec\theta\mathbf{i} + \tan\theta\mathbf{j}, \quad 0 \le \theta \le \dfrac{\pi}{3}$

(b) $\mathbf{r}_2(t) = \sqrt{t+1}\,\mathbf{i} + \sqrt{t}\,\mathbf{j}, \quad 0 \le t \le 3$

4. $\mathbf{F}(x, y) = y\mathbf{i} + x^2\mathbf{j}$

(a) $\mathbf{r}_1(t) = (2 + t)\mathbf{i} + (3 - t)\mathbf{j}, \quad 0 \le t \le 3$

(b) $\mathbf{r}_2(w) = (2 + \ln w)\mathbf{i} + (3 - \ln w)\mathbf{j}, \quad 1 \le w \le e^3$

In Exercises 5–10, determine whether or not the vector field is conservative.

5. $\mathbf{F}(x, y) = e^x(\sin y\mathbf{i} + \cos y\mathbf{j})$

6. $\mathbf{F}(x, y) = 15x^2y^2\mathbf{i} + 10x^3y\mathbf{j}$

7. $\mathbf{F}(x, y) = \dfrac{1}{y^2}(y\mathbf{i} + x\mathbf{j})$

8. $\mathbf{F}(x, y, z) = y\ln z\mathbf{i} - x\ln z\mathbf{j} + \dfrac{xy}{z}\mathbf{k}$

9. $\mathbf{F}(x, y, z) = y^2z\mathbf{i} + 2xyz\mathbf{j} + xy^2\mathbf{k}$

10. $\mathbf{F}(x, y, z) = \sin yz\mathbf{i} + xz\cos yz\mathbf{j} + xy\sin yz\mathbf{k}$

In Exercises 11–24, find the value of the line integral

$$\int_C \mathbf{F} \cdot d\mathbf{r}.$$

(*Hint:* If F is conservative, the integration may be easier on an alternative path.)

11. $\mathbf{F}(x, y) = 2xy\mathbf{i} + x^2\mathbf{j}$

(a) $\mathbf{r}_1(t) = t\mathbf{i} + t^2\mathbf{j}, \quad 0 \le t \le 1$

(b) $\mathbf{r}_2(t) = t\mathbf{i} + t^3\mathbf{j}, \quad 0 \le t \le 1$

12. $\mathbf{F}(x, y) = ye^{xy}\mathbf{i} + xe^{xy}\mathbf{j}$

(a) $\mathbf{r}_1(t) = t\mathbf{i} - (t - 3)\mathbf{j}, \quad 0 \le t \le 3$

(b) The closed path consisting of line segments from $(0, 3)$ to $(0, 0)$, from $(0, 0)$ to $(3, 0)$, and then from $(3, 0)$ to $(0, 3)$

13. $\mathbf{F}(x, y) = y\mathbf{i} - x\mathbf{j}$

(a) $\mathbf{r}_1(t) = t\mathbf{i} + t\mathbf{j}, \quad 0 \le t \le 1$

(b) $\mathbf{r}_2(t) = t\mathbf{i} + t^2\mathbf{j}, \quad 0 \le t \le 1$

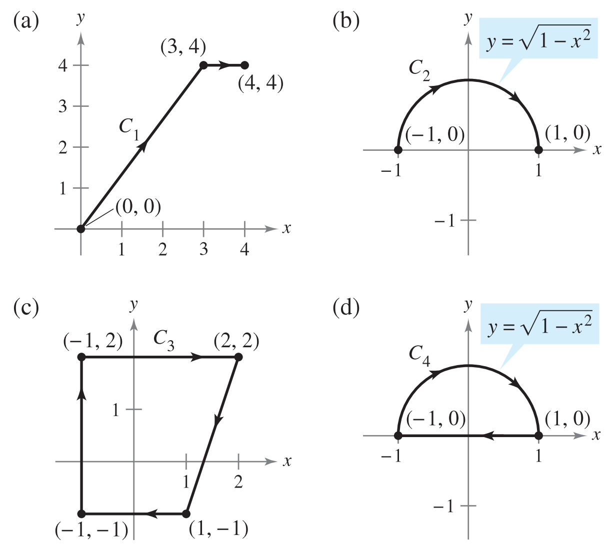
(c) $\mathbf{r}_3(t) = t\mathbf{i} + t^3\mathbf{j}, \quad 0 \le t \le 1$

14. $\mathbf{F}(x, y) = xy^2\mathbf{i} + 2x^2y\mathbf{j}$

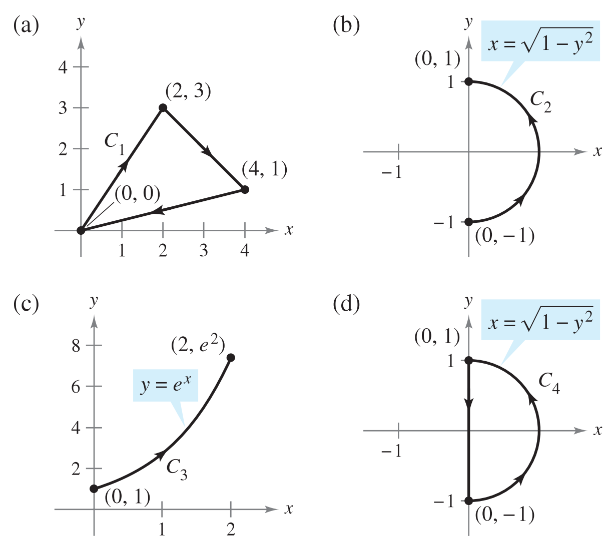
(a) $\mathbf{r}_1(t) = t\mathbf{i} + \dfrac{1}{t}\mathbf{j}, \quad 1 \le t \le 3$

(b) $\mathbf{r}_2(t) = (t + 1)\mathbf{i} - \frac{1}{3}(t - 3)\mathbf{j}, \quad 0 \le t \le 2$

15. $\displaystyle\int_C y^2\,dx + 2xy\,dy$



16. $\displaystyle\int_C (2x - 3y + 1)\,dx - (3x + y - 5)\,dy$



17. $\displaystyle\int_C 2xy\,dx + (x^2 + y^2)\,dy$

(a) C: ellipse $\dfrac{x^2}{25} + \dfrac{y^2}{16} = 1$ from $(5, 0)$ to $(0, 4)$

(b) C: parabola $y = 4 - x^2$ from $(2, 0)$ to $(0, 4)$

18. $\int_C (x^2 + y^2)\, dx + 2xy\, dy$

(a) $\mathbf{r}_1(t) = t^3\mathbf{i} + t^2\mathbf{j}, \quad 0 \le t \le 2$

(b) $\mathbf{r}_2(t) = 2\cos t\,\mathbf{i} + 2\sin t\,\mathbf{j}, \quad 0 \le t \le \dfrac{\pi}{2}$

19. $\mathbf{F}(x, y, z) = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$

(a) $\mathbf{r}_1(t) = t\mathbf{i} + 2\mathbf{j} + t\mathbf{k}, \quad 0 \le t \le 4$

(b) $\mathbf{r}_2(t) = t^2\mathbf{i} + t\mathbf{j} + t^2\mathbf{k}, \quad 0 \le t \le 2$

20. $\mathbf{F}(x, y, z) = \mathbf{i} + z\mathbf{j} + y\mathbf{k}$

(a) $\mathbf{r}_1(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + t^2\mathbf{k}, \quad 0 \le t \le \pi$

(b) $\mathbf{r}_2(t) = (1 - 2t)\mathbf{i} + \pi^2 t\mathbf{k}, \quad 0 \le t \le 1$

21. $\mathbf{F}(x, y, z) = (2y + x)\mathbf{i} + (x^2 - z)\mathbf{j} + (2y - 4z)\mathbf{k}$

(a) $\mathbf{r}_1(t) = t\mathbf{i} + t^2\mathbf{j} + \mathbf{k}, \quad 0 \le t \le 1$

(b) $\mathbf{r}_2(t) = t\mathbf{i} + t\mathbf{j} + (2t - 1)^2\mathbf{k}, \quad 0 \le t \le 1$

22. $\mathbf{F}(x, y, z) = -y\mathbf{i} + x\mathbf{j} + 3xz^2\mathbf{k}$

(a) $\mathbf{r}_1(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + t\mathbf{k}, \quad 0 \le t \le \pi$

(b) $\mathbf{r}_2(t) = (1 - 2t)\mathbf{i} + \pi t\mathbf{k}, \quad 0 \le t \le 1$

23. $\mathbf{F}(x, y, z) = e^z(y\mathbf{i} + x\mathbf{j} + xy\mathbf{k})$

(a) $\mathbf{r}_1(t) = 4\cos t\,\mathbf{i} + 4\sin t\,\mathbf{j} + 3\mathbf{k}, \quad 0 \le t \le \pi$

(b) $\mathbf{r}_2(t) = (4 - 8t)\mathbf{i} + 3\mathbf{k}, \quad 0 \le t \le 1$

24. $\mathbf{F}(x, y, z) = y\sin z\,\mathbf{i} + x\sin z\,\mathbf{j} + xy\cos x\mathbf{k}$

(a) $\mathbf{r}_1(t) = t^2\mathbf{i} + t^2\mathbf{j}, \quad 0 \le t \le 2$

(b) $\mathbf{r}_2(t) = 4t\mathbf{i} + 4t\mathbf{j}, \quad 0 \le t \le 1$

In Exercises 25–34, evaluate the line integral using the Fundamental Theorem of Line Integrals. Use a computer algebra system to verify your results.

25. $\int_C (3y\mathbf{i} + 3x\mathbf{j}) \cdot d\mathbf{r}$

C: smooth curve from $(0, 0)$ to $(3, 8)$

26. $\int_C [2(x + y)\mathbf{i} + 2(x + y)\mathbf{j}] \cdot d\mathbf{r}$

C: smooth curve from $(-1, 1)$ to $(3, 2)$

27. $\int_C \cos x \sin y\, dx + \sin x \cos y\, dy$

C: line segment from $(0, -\pi)$ to $\left(\dfrac{3\pi}{2}, \dfrac{\pi}{2}\right)$

28. $\int_C \dfrac{y\, dx - x\, dy}{x^2 + y^2}$

C: line segment from $(1, 1)$ to $\left(2\sqrt{3}, 2\right)$

29. $\int_C e^x \sin y\, dx + e^x \cos y\, dy$

C: cycloid $x = \theta - \sin\theta$, $y = 1 - \cos\theta$ from $(0, 0)$ to $(2\pi, 0)$

30. $\int_C \dfrac{2x}{(x^2 + y^2)^2}\, dx + \dfrac{2y}{(x^2 + y^2)^2}\, dy$

C: circle $(x - 4)^2 + (y - 5)^2 = 9$ clockwise from $(7, 5)$ to $(1, 5)$

31. $\int_C (z + 2y)\, dx + (2x - z)\, dy + (x - y)\, dz$

(a) C: line segment from $(0, 0, 0)$ to $(1, 1, 1)$

(b) C: line segments from $(0, 0, 0)$ to $(0, 0, 1)$ to $(1, 1, 1)$

(c) C: line segments from $(0, 0, 0)$ to $(1, 0, 0)$ to $(1, 1, 0)$ to $(1, 1, 1)$

32. Repeat Exercise 31 using the integral

$$\int_C zy\, dx + xz\, dy + xy\, dz.$$

33. $\int_C -\sin x\, dx + z\, dy + y\, dz$

C: smooth curve from $(0, 0, 0)$ to $\left(\dfrac{\pi}{2}, 3, 4\right)$

34. $\int_C 6x\, dx - 4z\, dy - (4y - 20z)\, dz$

C: smooth curve from $(0, 0, 0)$ to $(3, 4, 0)$

Work **In Exercises 35 and 36, find the work done by the force field F in moving an object from P to Q.**

35. $\mathbf{F}(x, y) = 9x^2y^2\mathbf{i} + (6x^3y - 1)\mathbf{j}$; $P(0, 0)$, $Q(5, 9)$

36. $\mathbf{F}(x, y) = \dfrac{2x}{y}\mathbf{i} - \dfrac{x^2}{y^2}\mathbf{j}$; $P(-1, 1)$, $Q(3, 2)$

37. ***Work*** A stone weighing 1 pound is attached to the end of a two-foot string and is whirled horizontally with one end held fixed. It makes 1 revolution per second. Find the work done by the force $\mathbf{F}$ that keeps the stone moving in a circular path. [*Hint:* Use Force = (mass)(centripetal acceleration).]

38. ***Work*** If $\mathbf{F}(x, y, z) = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ is a constant force vector field, show that the work done in moving a particle along any path from P to Q is $W = \mathbf{F} \cdot \overrightarrow{PQ}$.

39. ***Work*** To allow a means of escape for workers in a hazardous job 50 meters above ground level, a slide wire is installed. It runs from their position to a point on the ground 50 meters from the base of the installation where they are located. Show that the work done by the gravitational force field for a 175-pound worker moving the length of the slide wire is the same for each path.

(a) $\mathbf{r}(t) = t\mathbf{i} + (50 - t)\mathbf{j}$

(b) $\mathbf{r}(t) = t\mathbf{i} + \frac{1}{50}(50 - t)^2\mathbf{j}$

40. ***Work*** Can you find a path for the slide wire in Exercise 39 such that the work done by the gravitational force field would differ from the amounts of work done for the two paths given? Explain why or why not.

WRITING ABOUT CONCEPTS

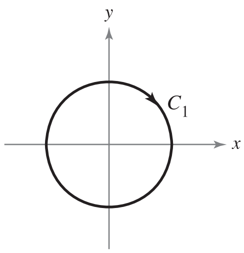
41. State the Fundamental Theorem of Line Integrals.

42. What does it mean that a line integral is independent of path? State the method for determining if a line integral is independent of path.

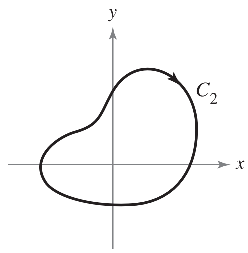
43. ***Think About It*** Let $\mathbf{F}(x, y) = \dfrac{y}{x^2 + y^2}\mathbf{i} - \dfrac{x}{x^2 + y^2}\mathbf{j}$. Find the value of the line integral

$$\int_C \mathbf{F} \cdot d\mathbf{r}.$$

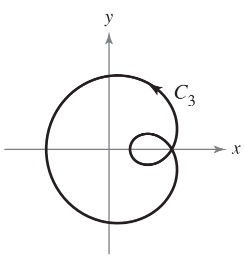
(a)



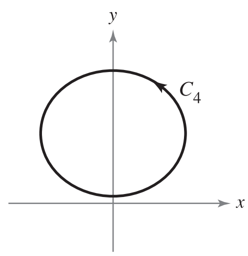
(b)



(c)

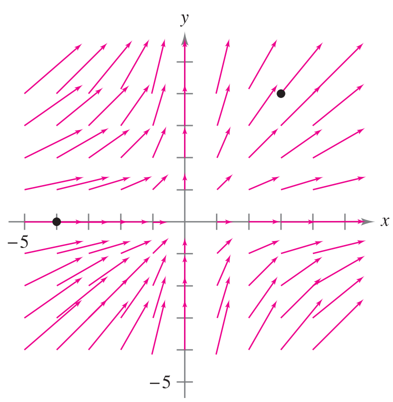


(d)



CAPSTONE

44. Consider the force field shown in the figure.



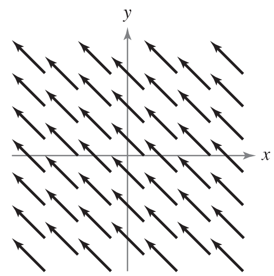
(a) Give a verbal argument that the force field is not conservative because you can identify two paths that require different amounts of work to move an object from $(-4, 0)$ to $(3, 4)$. Identify two paths and state which requires the greater amount of work. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.

(b) Give a verbal argument that the force field is not conservative because you can find a closed curve C such that

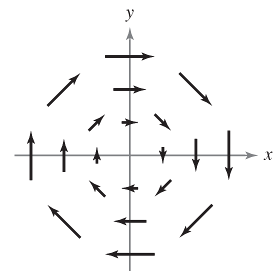
$$\int_C \mathbf{F} \cdot d\mathbf{r} \neq 0.$$

In Exercises 45 and 46, consider the force field shown in the figure. Is the force field conservative? Explain why or why not.

45.



46.



True or False? **In Exercises 47–50, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

47. If C_1, C_2, and C_3 have the same initial and terminal points and $\int_{C_1} \mathbf{F} \cdot d\mathbf{r}_1 = \int_{C_2} \mathbf{F} \cdot d\mathbf{r}_2$, then $\int_{C_1} \mathbf{F} \cdot d\mathbf{r}_1 = \int_{C_3} \mathbf{F} \cdot d\mathbf{r}_3$.

48. If $\mathbf{F} = y\mathbf{i} + x\mathbf{j}$ and C is given by $\mathbf{r}(t) = (4 \sin t)\mathbf{i} + (3 \cos t)\mathbf{j}$, $0 \leq t \leq \pi$, then $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$.

49. If $\mathbf{F}$ is conservative in a region R bounded by a simple closed path and C lies within R, then $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path.

50. If $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$ and $\partial M/\partial x = \partial N/\partial y$, then $\mathbf{F}$ is conservative.

51. A function f is called *harmonic* if $\dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2} = 0$. Prove that if f is harmonic, then

$$\int_C \left(\frac{\partial f}{\partial y}\,dx - \frac{\partial f}{\partial x}\,dy\right) = 0$$

where C is a smooth closed curve in the plane.

52. ***Kinetic and Potential Energy*** The kinetic energy of an object moving through a conservative force field is decreasing at a rate of 15 units per minute. At what rate is the potential energy changing?

53. Let $\mathbf{F}(x, y) = \dfrac{y}{x^2 + y^2}\mathbf{i} - \dfrac{x}{x^2 + y^2}\mathbf{j}$.

(a) Show that

$$\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$$

where

$$M = \frac{y}{x^2 + y^2} \text{ and } N = \frac{-x}{x^2 + y^2}.$$

(b) If $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j}$ for $0 \leq t \leq \pi$, find $\int_C \mathbf{F} \cdot d\mathbf{r}$.

(c) If $\mathbf{r}(t) = \cos t\,\mathbf{i} - \sin t\,\mathbf{j}$ for $0 \leq t \leq \pi$, find $\int_C \mathbf{F} \cdot d\mathbf{r}$.

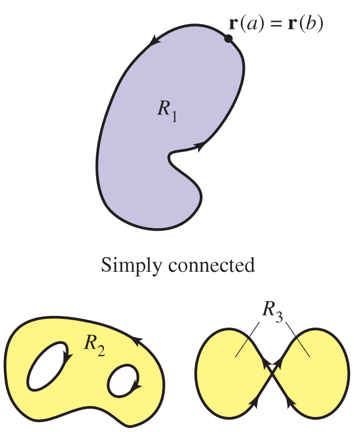
(d) If $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j}$ for $0 \leq t \leq 2\pi$, find $\int_C \mathbf{F} \cdot d\mathbf{r}$. Why doesn't this contradict Theorem 15.7?

(e) Show that $\nabla\left(\arctan \dfrac{x}{y}\right) = \mathbf{F}$.

15.4 Green's Theorem

- **Use Green's Theorem to evaluate a line integral.**
- **Use alternative forms of Green's Theorem.**

Green's Theorem



Simply connected

Not simply connected

Figure 15.26

In this section, you will study **Green's Theorem,** named after the English mathematician George Green (1793–1841). This theorem states that the value of a double integral over a *simply connected* plane region R is determined by the value of a line integral around the boundary of R.

A curve C given by $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$, where $a \le t \le b$, is **simple** if it does not cross itself—that is, $\mathbf{r}(c) \ne \mathbf{r}(d)$ for all c and d in the open interval (a, b). A plane region R is **simply connected** if every simple closed curve in R encloses only points that are in R (see Figure 15.26).

> **THEOREM 15.8 GREEN'S THEOREM**
>
> Let R be a simply connected region with a piecewise smooth boundary C, oriented counterclockwise (that is, C is traversed *once* so that the region R always lies to the *left*). If M and N have continuous first partial derivatives in an open region containing R, then
>
> $$\int_C M\,dx + N\,dy = \int_R\!\!\int \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA.$$

PROOF A proof is given only for a region that is both vertically simple and horizontally simple, as shown in Figure 15.27.

$$\begin{aligned}\int_C M\,dx &= \int_{C_1} M\,dx + \int_{C_2} M\,dx \\ &= \int_a^b M(x, f_1(x))\,dx + \int_b^a M(x, f_2(x))\,dx \\ &= \int_a^b [M(x, f_1(x)) - M(x, f_2(x))]\,dx\end{aligned}$$

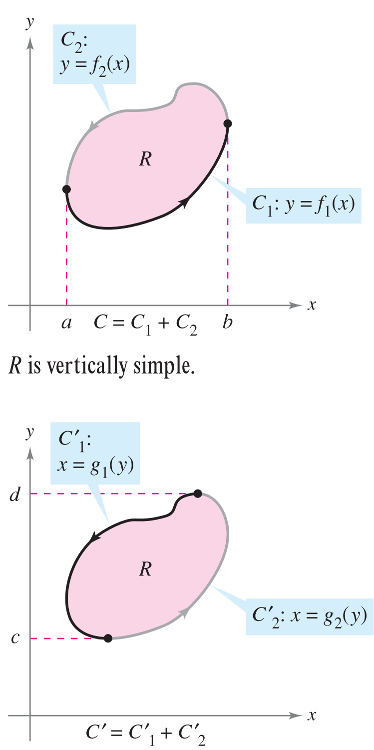
On the other hand,

$$\begin{aligned}\int_R\!\!\int \frac{\partial M}{\partial y}\,dA &= \int_a^b \int_{f_1(x)}^{f_2(x)} \frac{\partial M}{\partial y}\,dy\,dx \\ &= \int_a^b M(x, y)\Big]_{f_1(x)}^{f_2(x)}\,dx \\ &= \int_a^b [M(x, f_2(x)) - M(x, f_1(x))]\,dx.\end{aligned}$$

Consequently,

$$\int_C M\,dx = -\int_R\!\!\int \frac{\partial M}{\partial y}\,dA.$$

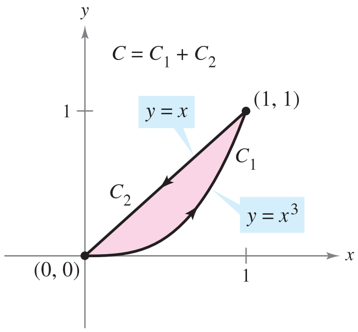
Similarly, you can use $g_1(y)$ and $g_2(y)$ to show that $\int_C N\,dy = \int_R\!\int \partial N/\partial x\,dA$. By adding the integrals $\int_C M\,dx$ and $\int_C N\,dy$, you obtain the conclusion stated in the theorem. ■



R is vertically simple.

R is horizontally simple.

Figure 15.27



C is simple and closed, and the region R always lies to the left of C.
Figure 15.28

EXAMPLE 1 Using Green's Theorem

Use Green's Theorem to evaluate the line integral

$$\int_C y^3\,dx + (x^3 + 3xy^2)\,dy$$

where C is the path from $(0, 0)$ to $(1, 1)$ along the graph of $y = x^3$ and from $(1, 1)$ to $(0, 0)$ along the graph of $y = x$, as shown in Figure 15.28.

Solution Because $M = y^3$ and $N = x^3 + 3xy^2$, it follows that

$$\frac{\partial N}{\partial x} = 3x^2 + 3y^2 \quad \text{and} \quad \frac{\partial M}{\partial y} = 3y^2.$$

Applying Green's Theorem, you then have

$$\begin{aligned}
\int_C y^3\,dx + (x^3 + 3xy^2)\,dy &= \int_R\int \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA \\
&= \int_0^1 \int_{x^3}^{x} [(3x^2 + 3y^2) - 3y^2]\,dy\,dx \\
&= \int_0^1 \int_{x^3}^{x} 3x^2\,dy\,dx \\
&= \int_0^1 3x^2y\Big]_{x^3}^{x}\,dx \\
&= \int_0^1 (3x^3 - 3x^5)\,dx \\
&= \left[\frac{3x^4}{4} - \frac{x^6}{2}\right]_0^1 \\
&= \frac{1}{4}.
\end{aligned}$$

GEORGE GREEN (1793–1841)

Green, a self-educated miller's son, first published the theorem that bears his name in 1828 in an essay on electricity and magnetism. At that time there was almost no mathematical theory to explain electrical phenomena. "Considering how desirable it was that a power of universal agency, like electricity, should, as far as possible, be submitted to calculation, . . . I was induced to try whether it would be possible to discover any general relations existing between this function and the quantities of electricity in the bodies producing it."

Green's Theorem cannot be applied to every line integral. Among other restrictions stated in Theorem 15.8, the curve C must be simple and closed. When Green's Theorem does apply, however, it can save time. To see this, try using the techniques described in Section 15.2 to evaluate the line integral in Example 1. To do this, you would need to write the line integral as

$$\int_C y^3\,dx + (x^3 + 3xy^2)\,dy =$$
$$\int_{C_1} y^3\,dx + (x^3 + 3xy^2)\,dy + \int_{C_2} y^3\,dx + (x^3 + 3xy^2)\,dy$$

where C_1 is the cubic path given by

$$\mathbf{r}(t) = t\mathbf{i} + t^3\mathbf{j}$$

from $t = 0$ to $t = 1$, and C_2 is the line segment given by

$$\mathbf{r}(t) = (1 - t)\mathbf{i} + (1 - t)\mathbf{j}$$

from $t = 0$ to $t = 1$.

EXAMPLE 2 Using Green's Theorem to Calculate Work

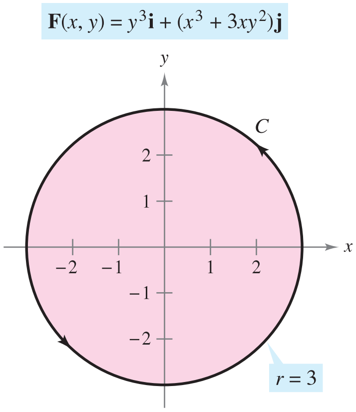


Figure 15.29

While subject to the force

$$\mathbf{F}(x, y) = y^3\mathbf{i} + (x^3 + 3xy^2)\mathbf{j}$$

a particle travels once around the circle of radius 3 shown in Figure 15.29. Use Green's Theorem to find the work done by $\mathbf{F}$.

Solution From Example 1, you know by Green's Theorem that

$$\int_C y^3\,dx + (x^3 + 3xy^2)\,dy = \int_R\int 3x^2\,dA.$$

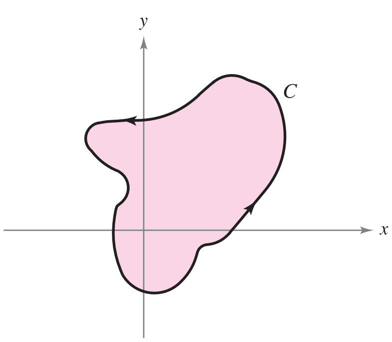
In polar coordinates, using $x = r\cos\theta$ and $dA = r\,dr\,d\theta$, the work done is

$$\begin{aligned} W = \int_R\int 3x^2\,dA &= \int_0^{2\pi}\int_0^3 3(r\cos\theta)^2\,r\,dr\,d\theta \\ &= 3\int_0^{2\pi}\int_0^3 r^3\cos^2\theta\,dr\,d\theta \\ &= 3\int_0^{2\pi} \frac{r^4}{4}\cos^2\theta\Big]_0^3\,d\theta \\ &= 3\int_0^{2\pi}\frac{81}{4}\cos^2\theta\,d\theta \\ &= \frac{243}{8}\int_0^{2\pi}(1 + \cos 2\theta)\,d\theta \\ &= \frac{243}{8}\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{2\pi} \\ &= \frac{243\pi}{4}. \end{aligned}$$

When evaluating line integrals over closed curves, remember that for conservative vector fields (those for which $\partial N/\partial x = \partial M/\partial y$), the value of the line integral is 0. This is easily seen from the statement of Green's Theorem:

$$\int_C M\,dx + N\,dy = \int_R\int\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dA = 0.$$

EXAMPLE 3 Green's Theorem and Conservative Vector Fields



C is closed.
Figure 15.30

Evaluate the line integral

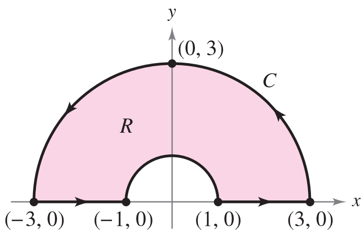
$$\int_C y^3\,dx + 3xy^2\,dy$$

where C is the path shown in Figure 15.30.

Solution From this line integral, $M = y^3$ and $N = 3xy^2$. So, $\partial N/\partial x = 3y^2$ and $\partial M/\partial y = 3y^2$. This implies that the vector field $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$ is conservative, and because C is closed, you can conclude that

$$\int_C y^3\,dx + 3xy^2\,dy = 0.$$

EXAMPLE 4 Using Green's Theorem for a Piecewise Smooth Curve



C is piecewise smooth.
Figure 15.31

Evaluate

$$\int_C (\arctan x + y^2)\,dx + (e^y - x^2)\,dy$$

where C is the path enclosing the annular region shown in Figure 15.31.

Solution In polar coordinates, R is given by $1 \le r \le 3$ for $0 \le \theta \le \pi$. Moreover,

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = -2x - 2y = -2(r\cos\theta + r\sin\theta).$$

So, by Green's Theorem,

$$\begin{aligned}
\int_C (\arctan x + y^2)\,dx + (e^y - x^2)\,dy &= \int_R\!\!\int -2(x+y)\,dA \\
&= \int_0^{\pi}\!\!\int_1^3 -2r(\cos\theta + \sin\theta)r\,dr\,d\theta \\
&= \int_0^{\pi} -2(\cos\theta + \sin\theta)\frac{r^3}{3}\bigg]_1^3\,d\theta \\
&= \int_0^{\pi}\left(-\frac{52}{3}\right)(\cos\theta + \sin\theta)\,d\theta \\
&= -\frac{52}{3}\Big[\sin\theta - \cos\theta\Big]_0^{\pi} \\
&= -\frac{104}{3}.
\end{aligned}$$

In Examples 1, 2, and 4, Green's Theorem was used to evaluate line integrals as double integrals. You can also use the theorem to evaluate double integrals as line integrals. One useful application occurs when $\partial N/\partial x - \partial M/\partial y = 1$.

$$\begin{aligned}
\int_C M\,dx + N\,dy &= \int_R\!\!\int \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dA \\
&= \int_R\!\!\int 1\,dA \qquad \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 1 \\
&= \text{area of region } R
\end{aligned}$$

Among the many choices for M and N satisfying the stated condition, the choice of $M = -y/2$ and $N = x/2$ produces the following line integral for the area of region R.

THEOREM 15.9 LINE INTEGRAL FOR AREA

If R is a plane region bounded by a piecewise smooth simple closed curve C, oriented counterclockwise, then the area of R is given by

$$A = \frac{1}{2}\int_C x\,dy - y\,dx.$$

EXAMPLE 5 Finding Area by a Line Integral

Use a line integral to find the area of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

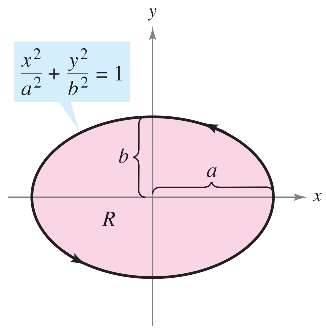


Figure 15.32

Solution Using Figure 15.32, you can induce a counterclockwise orientation to the elliptical path by letting

$$x = a\cos t \quad \text{and} \quad y = b\sin t, \quad 0 \le t \le 2\pi.$$

So, the area is

$$\begin{aligned} A = \frac{1}{2}\int_C x\,dy - y\,dx &= \frac{1}{2}\int_0^{2\pi} [(a\cos t)(b\cos t)\,dt - (b\sin t)(-a\sin t)\,dt] \\ &= \frac{ab}{2}\int_0^{2\pi} (\cos^2 t + \sin^2 t)\,dt \\ &= \frac{ab}{2}\Big[t\Big]_0^{2\pi} \\ &= \pi ab. \end{aligned}$$

Green's Theorem can be extended to cover some regions that are not simply connected. This is demonstrated in the next example.

EXAMPLE 6 Green's Theorem Extended to a Region with a Hole

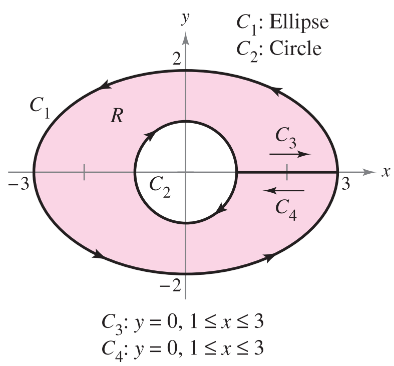


Figure 15.33

Let R be the region inside the ellipse $(x^2/9) + (y^2/4) = 1$ and outside the circle $x^2 + y^2 = 1$. Evaluate the line integral

$$\int_C 2xy\,dx + (x^2 + 2x)\,dy$$

where $C = C_1 + C_2$ is the boundary of R, as shown in Figure 15.33.

Solution To begin, you can introduce the line segments C_3 and C_4, as shown in Figure 15.33. Note that because the curves C_3 and C_4 have opposite orientations, the line integrals over them cancel. Furthermore, you can apply Green's Theorem to the region R using the boundary $C_1 + C_4 + C_2 + C_3$ to obtain

$$\begin{aligned} \int_C 2xy\,dx + (x^2 + 2x)\,dy &= \int_R\!\!\int \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA \\ &= \int_R\!\!\int (2x + 2 - 2x)\,dA \\ &= 2\int_R\!\!\int dA \\ &= 2(\text{area of } R) \\ &= 2(\pi ab - \pi r^2) \\ &= 2[\pi(3)(2) - \pi(1^2)] \\ &= 10\pi. \end{aligned}$$

In Section 15.1, a necessary and sufficient condition for conservative vector fields was listed. There, only one direction of the proof was shown. You can now outline the other direction, using Green's Theorem. Let $\mathbf{F}(x, y) = M\mathbf{i} + N\mathbf{j}$ be defined on an open disk R. You want to show that if M and N have continuous first partial derivatives and

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

then $\mathbf{F}$ is conservative. Suppose that C is a closed path forming the boundary of a connected region lying in R. Then, using the fact that $\partial M/\partial y = \partial N/\partial x$, you can apply Green's Theorem to conclude that

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_C M\,dx + N\,dy \\ &= \int_R\int \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA \\ &= 0.\end{aligned}$$

This, in turn, is equivalent to showing that $\mathbf{F}$ is conservative (see Theorem 15.7).

Alternative Forms of Green's Theorem

This section concludes with the derivation of two vector forms of Green's Theorem for regions in the plane. The extension of these vector forms to three dimensions is the basis for the discussion in the remaining sections of this chapter. If $\mathbf{F}$ is a vector field in the plane, you can write

$$\mathbf{F}(x, y, z) = M\mathbf{i} + N\mathbf{j} + 0\mathbf{k}$$

so that the curl of $\mathbf{F}$, as described in Section 15.1, is given by

$$\begin{aligned}\text{curl}\,\mathbf{F} = \nabla \times \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ M & N & 0 \end{vmatrix} \\ &= -\frac{\partial N}{\partial z}\mathbf{i} + \frac{\partial M}{\partial z}\mathbf{j} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\mathbf{k}.\end{aligned}$$

Consequently,

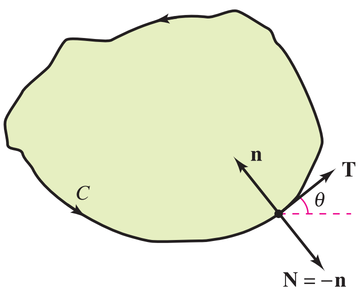
$$\begin{aligned}(\text{curl}\,\mathbf{F}) \cdot \mathbf{k} &= \left[-\frac{\partial N}{\partial z}\mathbf{i} + \frac{\partial M}{\partial z}\mathbf{j} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\mathbf{k}\right] \cdot \mathbf{k} \\ &= \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}.\end{aligned}$$

With appropriate conditions on $\mathbf{F}$, C, and R, you can write Green's Theorem in the vector form

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_R\int \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA \\ &= \int_R\int (\text{curl}\,\mathbf{F}) \cdot \mathbf{k}\,dA.\end{aligned}$$

First alternative form

The extension of this vector form of Green's Theorem to surfaces in space produces **Stokes's Theorem,** discussed in Section 15.8.



$\mathbf{T} = \cos\theta\mathbf{i} + \sin\theta\mathbf{j}$

$\mathbf{n} = \cos\left(\theta + \frac{\pi}{2}\right)\mathbf{i} + \sin\left(\theta + \frac{\pi}{2}\right)\mathbf{j}$

$\quad = -\sin\theta\mathbf{i} + \cos\theta\mathbf{j}$

$\mathbf{N} = \sin\theta\mathbf{i} - \cos\theta\mathbf{j}$

Figure 15.34

For the second vector form of Green's Theorem, assume the same conditions for $\mathbf{F}$, C, and R. Using the arc length parameter s for C, you have $\mathbf{r}(s) = x(s)\mathbf{i} + y(s)\mathbf{j}$. So, a unit tangent vector $\mathbf{T}$ to curve C is given by $\mathbf{r}'(s) = \mathbf{T} = x'(s)\mathbf{i} + y'(s)\mathbf{j}$. From Figure 15.34 you can see that the *outward* unit normal vector $\mathbf{N}$ can then be written as

$$\mathbf{N} = y'(s)\mathbf{i} - x'(s)\mathbf{j}.$$

Consequently, for $\mathbf{F}(x, y) = M\mathbf{i} + N\mathbf{j}$, you can apply Green's Theorem to obtain

$$\begin{aligned}
\int_C \mathbf{F} \cdot \mathbf{N}\, ds &= \int_a^b (M\mathbf{i} + N\mathbf{j}) \cdot (y'(s)\mathbf{i} - x'(s)\mathbf{j})\, ds \\
&= \int_a^b \left(M\frac{dy}{ds} - N\frac{dx}{ds}\right) ds \\
&= \int_C M\, dy - N\, dx \\
&= \int_C -N\, dx + M\, dy \\
&= \int_R\int \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dA \qquad \text{Green's Theorem} \\
&= \int_R\int \operatorname{div} \mathbf{F}\, dA.
\end{aligned}$$

Therefore,

$$\int_C \mathbf{F} \cdot \mathbf{N}\, ds = \int_R\int \operatorname{div} \mathbf{F}\, dA. \qquad \text{Second alternative form}$$

The extension of this form to three dimensions is called the **Divergence Theorem,** discussed in Section 15.7. The physical interpretations of divergence and curl will be discussed in Sections 15.7 and 15.8.

15.4 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–4, verify Green's Theorem by evaluating both integrals

$$\int_C y^2\, dx + x^2\, dy = \int_R\int \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA$$

for the given path.

1. C: boundary of the region lying between the graphs of $y = x$ and $y = x^2$
2. C: boundary of the region lying between the graphs of $y = x$ and $y = \sqrt{x}$
3. C: square with vertices $(0, 0)$, $(1, 0)$, $(1, 1)$, $(0, 1)$
4. C: rectangle with vertices $(0, 0)$, $(3, 0)$, $(3, 4)$, and $(0, 4)$

CAS **In Exercises 5 and 6, verify Green's Theorem by using a computer algebra system to evaluate both integrals**

$$\int_C xe^y\, dx + e^x\, dy = \int_R\int \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA$$

for the given path.

5. C: circle given by $x^2 + y^2 = 4$
6. C: boundary of the region lying between the graphs of $y = x$ and $y = x^3$ in the first quadrant

In Exercises 7–10, use Green's Theorem to evaluate the integral

$$\int_C (y - x)\, dx + (2x - y)\, dy$$

for the given path.

7. C: boundary of the region lying between the graphs of $y = x$ and $y = x^2 - 2x$
8. C: $x = 2\cos\theta$, $y = \sin\theta$
9. C: boundary of the region lying inside the rectangle bounded by $x = -5$, $x = 5$, $y = -3$, and $y = 3$, and outside the square bounded by $x = -1$, $x = 1$, $y = -1$, and $y = 1$
10. C: boundary of the region lying inside the semicircle $y = \sqrt{25 - x^2}$ and outside the semicircle $y = \sqrt{9 - x^2}$

In Exercises 11–20, use Green's Theorem to evaluate the line integral.

11. $\int_C 2xy\,dx + (x + y)\,dy$

C: boundary of the region lying between the graphs of $y = 0$ and $y = 1 - x^2$

12. $\int_C y^2\,dx + xy\,dy$

C: boundary of the region lying between the graphs of $y = 0$, $y = \sqrt{x}$, and $x = 9$

13. $\int_C (x^2 - y^2)\,dx + 2xy\,dy$

C: $x^2 + y^2 = 16$

14. $\int_C (x^2 - y^2)\,dx + 2xy\,dy$

C: $r = 1 + \cos\theta$

15. $\int_C e^x \cos 2y\,dx - 2e^x \sin 2y\,dy$

C: $x^2 + y^2 = a^2$

16. $\int_C 2\arctan\frac{y}{x}\,dx + \ln(x^2 + y^2)\,dy$

C: $x = 4 + 2\cos\theta$, $y = 4 + \sin\theta$

17. $\int_C \cos y\,dx + (xy - x\sin y)\,dy$

C: boundary of the region lying between the graphs of $y = x$ and $y = \sqrt{x}$

18. $\int_C (e^{-x^2/2} - y)\,dx + (e^{-y^2/2} + x)\,dy$

C: boundary of the region lying between the graphs of the circle $x = 6\cos\theta$, $y = 6\sin\theta$ and the ellipse $x = 3\cos\theta$, $y = 2\sin\theta$

19. $\int_C (x - 3y)\,dx + (x + y)\,dy$

C: boundary of the region lying between the graphs of $x^2 + y^2 = 1$ and $x^2 + y^2 = 9$

20. $\int_C 3x^2 e^y\,dx + e^y\,dy$

C: boundary of the region lying between the squares with vertices $(1, 1)$, $(-1, 1)$, $(-1, -1)$, and $(1, -1)$, and $(2, 2)$, $(-2, 2)$, $(-2, -2)$, and $(2, -2)$

Work **In Exercises 21–24, use Green's Theorem to calculate the work done by the force F on a particle that is moving counterclockwise around the closed path *C*.**

21. $\mathbf{F}(x, y) = xy\mathbf{i} + (x + y)\mathbf{j}$

C: $x^2 + y^2 = 1$

22. $\mathbf{F}(x, y) = (e^x - 3y)\mathbf{i} + (e^y + 6x)\mathbf{j}$

C: $r = 2\cos\theta$

23. $\mathbf{F}(x, y) = (x^{3/2} - 3y)\mathbf{i} + \left(6x + 5\sqrt{y}\right)\mathbf{j}$

C: boundary of the triangle with vertices $(0, 0)$, $(5, 0)$, and $(0, 5)$

24. $\mathbf{F}(x, y) = (3x^2 + y)\mathbf{i} + 4xy^2\,\mathbf{j}$

C: boundary of the region lying between the graphs of $y = \sqrt{x}$, $y = 0$, and $x = 9$

Area **In Exercises 25–28, use a line integral to find the area of the region *R*.**

25. R: region bounded by the graph of $x^2 + y^2 = a^2$

26. R: triangle bounded by the graphs of $x = 0$, $3x - 2y = 0$, and $x + 2y = 8$

27. R: region bounded by the graphs of $y = 5x - 3$ and $y = x^2 + 1$

28. R: region inside the loop of the folium of Descartes bounded by the graph of

$$x = \frac{3t}{t^3 + 1}, \quad y = \frac{3t^2}{t^3 + 1}$$

WRITING ABOUT CONCEPTS

29. State Green's Theorem.

30. Give the line integral for the area of a region R bounded by a piecewise smooth simple curve C.

In Exercises 31 and 32, use Green's Theorem to verify the line integral formulas.

31. The centroid of the region having area A bounded by the simple closed path C is

$$\bar{x} = \frac{1}{2A}\int_C x^2\,dy, \quad \bar{y} = -\frac{1}{2A}\int_C y^2\,dx.$$

32. The area of a plane region bounded by the simple closed path C given in polar coordinates is $A = \frac{1}{2}\int_C r^2\,d\theta$.

CAS ***Centroid*** **In Exercises 33–36, use a computer algebra system and the results of Exercise 31 to find the centroid of the region.**

33. R: region bounded by the graphs of $y = 0$ and $y = 4 - x^2$

34. R: region bounded by the graphs of $y = \sqrt{a^2 - x^2}$ and $y = 0$

35. R: region bounded by the graphs of $y = x^3$ and $y = x$, $0 \le x \le 1$

36. R: triangle with vertices $(-a, 0)$, $(a, 0)$, and (b, c), where $-a \le b \le a$

CAS ***Area*** **In Exercises 37–40, use a computer algebra system and the results of Exercise 32 to find the area of the region bounded by the graph of the polar equation.**

37. $r = a(1 - \cos\theta)$

38. $r = a\cos 3\theta$

39. $r = 1 + 2\cos\theta$ (inner loop)

40. $r = \dfrac{3}{2 - \cos\theta}$

41. (a) Evaluate $\int_{C_1} y^3\,dx + (27x - x^3)\,dy$, where C_1 is the unit circle given by $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j}$, $0 \le t \le 2\pi$.

(b) Find the maximum value of $\int_C y^3\,dx + (27x - x^3)\,dy$, where C is any closed curve in the xy-plane, oriented counterclockwise.

CAPSTONE

42. For each given path, verify Green's Theorem by showing that

$$\int_C y^2\,dx + x^2\,dy = \int_R\int \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA.$$

For each path, which integral is easier to evaluate? Explain.

(a) C: triangle with vertices $(0, 0)$, $(4, 0)$, $(4, 4)$

(b) C: circle given by $x^2 + y^2 = 1$

43. ***Think About It*** Let

$$I = \int_C \frac{y\,dx - x\,dy}{x^2 + y^2}$$

where C is a circle oriented counterclockwise. Show that $I = 0$ if C does not contain the origin. What is I if C does contain the origin?

44. (a) Let C be the line segment joining (x_1, y_1) and (x_2, y_2). Show that $\int_C -y\,dx + x\,dy = x_1 y_2 - x_2 y_1$.

(b) Let (x_1, y_1), $(x_2, y_2), \ldots, (x_n, y_n)$ be the vertices of a polygon. Prove that the area enclosed is

$$\tfrac{1}{2}[(x_1y_2 - x_2y_1) + (x_2 y_3 - x_3y_2) + \cdots + (x_{n-1} y_n - x_n y_{n-1}) + (x_n y_1 - x_1 y_n)].$$

Area **In Exercises 45 and 46, use the result of Exercise 44(b) to find the area enclosed by the polygon with the given vertices.**

45. Pentagon: $(0, 0)$, $(2, 0)$, $(3, 2)$, $(1, 4)$, $(-1, 1)$

46. Hexagon: $(0, 0)$, $(2, 0)$, $(3, 2)$, $(2, 4)$, $(0, 3)$, $(-1, 1)$

In Exercises 47 and 48, prove the identity where R is a simply connected region with boundary C. Assume that the required partial derivatives of the scalar functions f and g are continuous. The expressions $D_{\mathbf{N}}f$ and $D_{\mathbf{N}}g$ are the derivatives in the direction of the outward normal vector $\mathbf{N}$ of C, and are defined by $D_{\mathbf{N}}f = \nabla f \cdot \mathbf{N}$, and $D_{\mathbf{N}}g = \nabla g \cdot \mathbf{N}$.

47. Green's first identity:

$$\int_R\int (f\nabla^2 g + \nabla f \cdot \nabla g)\,dA = \int_C f D_{\mathbf{N}}g\,ds$$

[*Hint:* Use the second alternative form of Green's Theorem and the property $\operatorname{div}(f\mathbf{G}) = f \operatorname{div}\mathbf{G} + \nabla f \cdot \mathbf{G}$.]

48. Green's second identity:

$$\int_R\int (f\nabla^2 g - g\nabla^2 f)\,dA = \int_C (f D_{\mathbf{N}}g - g\,D_{\mathbf{N}}f)\,ds$$

(*Hint:* Use Green's first identity, given in Exercise 47, twice.)

49. Use Green's Theorem to prove that

$$\int_C f(x)\,dx + g(y)\,dy = 0$$

if f and g are differentiable functions and C is a piecewise smooth simple closed path.

50. Let $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$, where M and N have continuous first partial derivatives in a simply connected region R. Prove that if C is simple, smooth, and closed, and $N_x = M_y$, then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = 0.$$

SECTION PROJECT

Hyperbolic and Trigonometric Functions

(a) Sketch the plane curve represented by the vector-valued function $\mathbf{r}(t) = \cosh t\,\mathbf{i} + \sinh t\,\mathbf{j}$ on the interval $0 \le t \le 5$. Show that the rectangular equation corresponding to $\mathbf{r}(t)$ is the hyperbola $x^2 - y^2 = 1$. Verify your sketch by using a graphing utility to graph the hyperbola.

(b) Let $P = (\cosh \phi, \sinh \phi)$ be the point on the hyperbola corresponding to $\mathbf{r}(\phi)$ for $\phi > 0$. Use the formula for area

$$A = \frac{1}{2}\int_C x\,dy - y\,dx$$

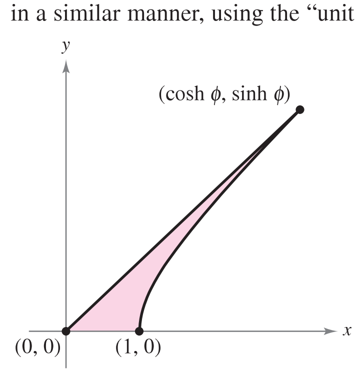
to verify that the area of the region shown in the figure is $\frac{1}{2}\phi$.

(c) Show that the area of the indicated region is also given by the integral

$$A = \int_0^{\sinh \phi} \left[\sqrt{1 + y^2} - (\coth \phi)y\right] dy.$$

Confirm your answer in part (b) by numerically approximating this integral for $\phi = 1, 2, 4,$ and 10.

(d) Consider the unit circle given by $x^2 + y^2 = 1$. Let θ be the angle formed by the x-axis and the radius to (x, y). The area of the corresponding sector is $\frac{1}{2}\theta$. That is, the trigonometric functions $f(\theta) = \cos\theta$ and $g(\theta) = \sin\theta$ could have been defined as the coordinates of that point $(\cos\theta, \sin\theta)$ on the unit circle that determines a sector of area $\frac{1}{2}\theta$. Write a short paragraph explaining how you could define the hyperbolic functions in a similar manner, using the "unit hyperbola" $x^2 - y^2 = 1$.



15.5 Parametric Surfaces

- Understand the definition of a parametric surface, and sketch the surface.
- Find a set of parametric equations to represent a surface.
- Find a normal vector and a tangent plane to a parametric surface.
- Find the area of a parametric surface.

Parametric Surfaces

You already know how to represent a curve in the plane or in space by a set of parametric equations—or, equivalently, by a vector-valued function.

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} \qquad \text{Plane curve}$$

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k} \qquad \text{Space curve}$$

In this section, you will learn how to represent a surface in space by a set of parametric equations—or by a vector-valued function. For curves, note that the vector-valued function $\mathbf{r}$ is a function of a *single* parameter t. For surfaces, the vector-valued function is a function of *two* parameters u and v.

DEFINITION OF PARAMETRIC SURFACE

Let x, y, and z be functions of u and v that are continuous on a domain D in the uv-plane. The set of points (x, y, z) given by

$$\mathbf{r}(u, v) = x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + z(u, v)\mathbf{k} \qquad \text{Parametric surface}$$

is called a **parametric surface.** The equations

$$x = x(u, v), \quad y = y(u, v), \quad \text{and} \quad z = z(u, v) \qquad \text{Parametric equations}$$

are the **parametric equations** for the surface.

If S is a parametric surface given by the vector-valued function $\mathbf{r}$, then S is traced out by the position vector $\mathbf{r}(u, v)$ as the point (u, v) moves throughout the domain D, as shown in Figure 15.35.

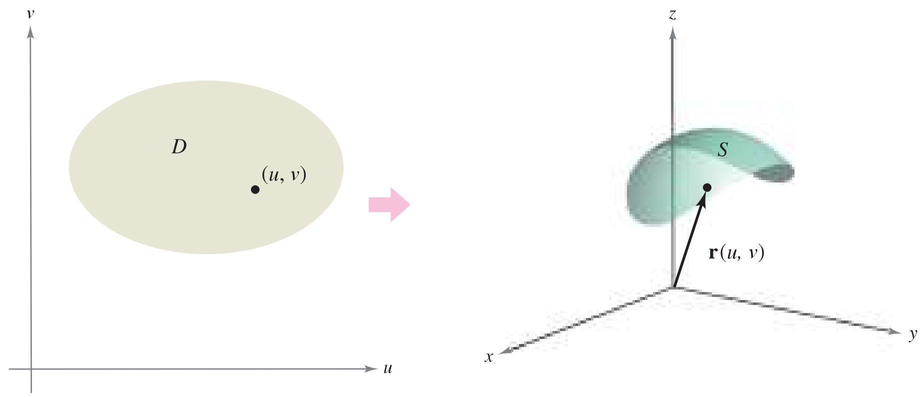


Figure 15.35

TECHNOLOGY Some computer algebra systems are capable of graphing surfaces that are represented parametrically. If you have access to such software, use it to graph some of the surfaces in the examples and exercises in this section.

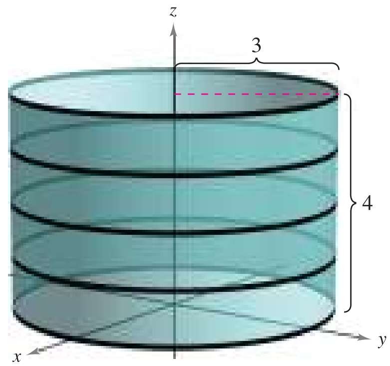


Figure 15.36

EXAMPLE 1 Sketching a Parametric Surface

Identify and sketch the parametric surface S given by

$$\mathbf{r}(u, v) = 3 \cos u\mathbf{i} + 3 \sin u\mathbf{j} + v\mathbf{k}$$

where $0 \le u \le 2\pi$ and $0 \le v \le 4$.

Solution Because $x = 3 \cos u$ and $y = 3 \sin u$, you know that for each point (x, y, z) on the surface, x and y are related by the equation $x^2 + y^2 = 3^2$. In other words, each cross section of S taken parallel to the xy-plane is a circle of radius 3, centered on the z-axis. Because $z = v$, where $0 \le v \le 4$, you can see that the surface is a right circular cylinder of height 4. The radius of the cylinder is 3, and the z-axis forms the axis of the cylinder, as shown in Figure 15.36. ■

As with parametric representations of curves, parametric representations of surfaces are not unique. That is, there are many other sets of parametric equations that could be used to represent the surface shown in Figure 15.36.

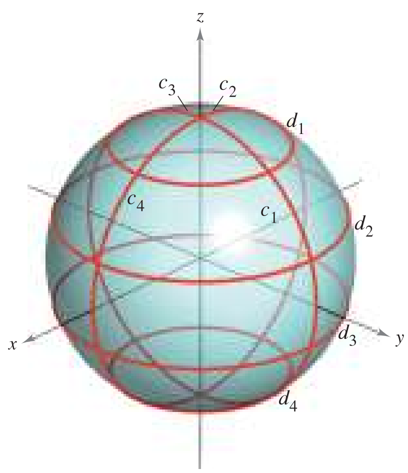


Figure 15.37

EXAMPLE 2 Sketching a Parametric Surface

Identify and sketch the parametric surface S given by

$$\mathbf{r}(u, v) = \sin u \cos v\mathbf{i} + \sin u \sin v\mathbf{j} + \cos u\mathbf{k}$$

where $0 \le u \le \pi$ and $0 \le v \le 2\pi$.

Solution To identify the surface, you can try to use trigonometric identities to eliminate the parameters. After some experimentation, you can discover that

$$\begin{aligned}
x^2 + y^2 + z^2 &= (\sin u \cos v)^2 + (\sin u \sin v)^2 + (\cos u)^2 \\
&= \sin^2 u \cos^2 v + \sin^2 u \sin^2 v + \cos^2 u \\
&= \sin^2 u(\cos^2 v + \sin^2 v) + \cos^2 u \\
&= \sin^2 u + \cos^2 u \\
&= 1.
\end{aligned}$$

So, each point on S lies on the unit sphere, centered at the origin, as shown in Figure 15.37. For fixed $u = d_i$, $\mathbf{r}(u, v)$ traces out latitude circles

$$x^2 + y^2 = \sin^2 d_i, \quad 0 \le d_i \le \pi$$

that are parallel to the xy-plane, and for fixed $v = c_i$, $\mathbf{r}(u, v)$ traces out longitude (or meridian) half-circles. ■

NOTE To convince yourself further that the vector-valued function in Example 2 traces out the entire unit sphere, recall that the parametric equations

$$x = \rho \sin \phi \cos \theta, \quad y = \rho \sin \phi \sin \theta, \quad \text{and} \quad z = \rho \cos \phi$$

where $0 \le \theta \le 2\pi$ and $0 \le \phi \le \pi$, describe the conversion from spherical to rectangular coordinates, as discussed in Section 11.7. ■

Finding Parametric Equations for Surfaces

In Examples 1 and 2, you were asked to identify the surface described by a given set of parametric equations. The reverse problem—that of writing a set of parametric equations for a given surface—is generally more difficult. One type of surface for which this problem is straightforward, however, is a surface that is given by $z = f(x, y)$. You can parametrize such a surface as

$$\mathbf{r}(x, y) = x\mathbf{i} + y\mathbf{j} + f(x, y)\mathbf{k}.$$

EXAMPLE 3 Representing a Surface Parametrically

Write a set of parametric equations for the cone given by

$$z = \sqrt{x^2 + y^2}$$

as shown in Figure 15.38.

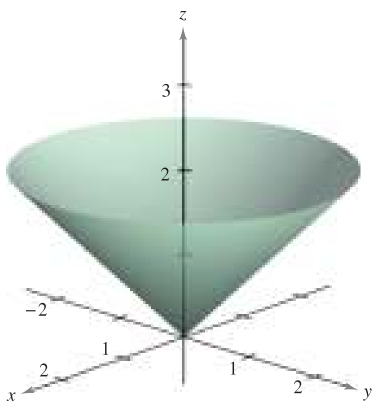


Figure 15.38

Solution Because this surface is given in the form $z = f(x, y)$, you can let x and y be the parameters. Then the cone is represented by the vector-valued function

$$\mathbf{r}(x, y) = x\mathbf{i} + y\mathbf{j} + \sqrt{x^2 + y^2}\,\mathbf{k}$$

where (x, y) varies over the entire xy-plane. ■

A second type of surface that is easily represented parametrically is a surface of revolution. For instance, to represent the surface formed by revolving the graph of $y = f(x)$, $a \le x \le b$, about the x-axis, use

$$x = u, \quad y = f(u)\cos v, \quad \text{and} \quad z = f(u)\sin v$$

where $a \le u \le b$ and $0 \le v \le 2\pi$.

EXAMPLE 4 Representing a Surface of Revolution Parametrically

Write a set of parametric equations for the surface of revolution obtained by revolving

$$f(x) = \frac{1}{x}, \quad 1 \le x \le 10$$

about the x-axis.

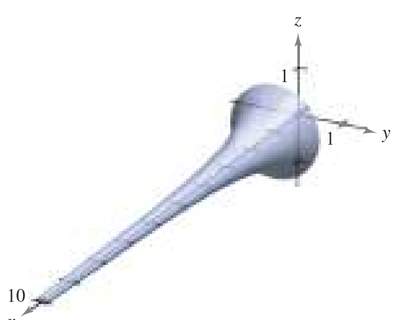


Figure 15.39

Solution Use the parameters u and v as described above to write

$$x = u, \quad y = f(u)\cos v = \frac{1}{u}\cos v, \quad \text{and} \quad z = f(u)\sin v = \frac{1}{u}\sin v$$

where $1 \le u \le 10$ and $0 \le v \le 2\pi$. The resulting surface is a portion of *Gabriel's Horn*, as shown in Figure 15.39. ■

The surface of revolution in Example 4 is formed by revolving the graph of $y = f(x)$ about the x-axis. For other types of surfaces of revolution, a similar parametrization can be used. For instance, to parametrize the surface formed by revolving the graph of $x = f(z)$ about the z-axis, you can use

$$z = u, \quad x = f(u)\cos v, \quad \text{and} \quad y = f(u)\sin v.$$

Normal Vectors and Tangent Planes

Let S be a parametric surface given by

$$\mathbf{r}(u, v) = x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + z(u, v)\mathbf{k}$$

over an open region D such that x, y, and z have continuous partial derivatives on D. The **partial derivatives of r** with respect to u and v are defined as

$$\mathbf{r}_u = \frac{\partial x}{\partial u}(u, v)\mathbf{i} + \frac{\partial y}{\partial u}(u, v)\mathbf{j} + \frac{\partial z}{\partial u}(u, v)\mathbf{k}$$

and

$$\mathbf{r}_v = \frac{\partial x}{\partial v}(u, v)\mathbf{i} + \frac{\partial y}{\partial v}(u, v)\mathbf{j} + \frac{\partial z}{\partial v}(u, v)\mathbf{k}.$$

Each of these partial derivatives is a vector-valued function that can be interpreted geometrically in terms of tangent vectors. For instance, if $v = v_0$ is held constant, then $\mathbf{r}(u, v_0)$ is a vector-valued function of a single parameter and defines a curve C_1 that lies on the surface S. The tangent vector to C_1 at the point $(x(u_0, v_0),\ y(u_0, v_0),\ z(u_0, v_0))$ is given by

$$\mathbf{r}_u(u_0, v_0) = \frac{\partial x}{\partial u}(u_0, v_0)\mathbf{i} + \frac{\partial y}{\partial u}(u_0, v_0)\mathbf{j} + \frac{\partial z}{\partial u}(u_0, v_0)\mathbf{k}$$

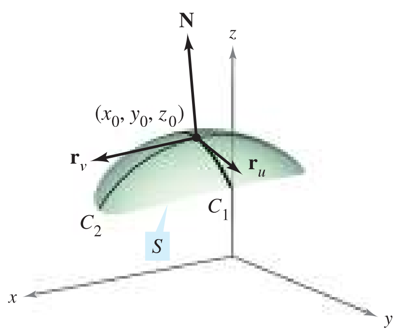


Figure 15.40

as shown in Figure 15.40. In a similar way, if $u = u_0$ is held constant, then $\mathbf{r}(u_0, v)$ is a vector-valued function of a single parameter and defines a curve C_2 that lies on the surface S. The tangent vector to C_2 at the point $(x(u_0, v_0),\ y(u_0, v_0),\ z(u_0, v_0))$ is given by

$$\mathbf{r}_v(u_0, v_0) = \frac{\partial x}{\partial v}(u_0, v_0)\mathbf{i} + \frac{\partial y}{\partial v}(u_0, v_0)\mathbf{j} + \frac{\partial z}{\partial v}(u_0, v_0)\mathbf{k}.$$

If the normal vector $\mathbf{r}_u \times \mathbf{r}_v$ is not $\mathbf{0}$ for any (u, v) in D, the surface S is called **smooth** and will have a tangent plane. Informally, a smooth surface is one that has no sharp points or cusps. For instance, spheres, ellipsoids, and paraboloids are smooth, whereas the cone given in Example 3 is not smooth.

NORMAL VECTOR TO A SMOOTH PARAMETRIC SURFACE

Let S be a smooth parametric surface

$$\mathbf{r}(u, v) = x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + z(u, v)\mathbf{k}$$

defined over an open region D in the uv-plane. Let (u_0, v_0) be a point in D. A normal vector at the point

$$(x_0, y_0, z_0) = (x(u_0, v_0), y(u_0, v_0), z(u_0, v_0))$$

is given by

$$\mathbf{N} = \mathbf{r}_u(u_0, v_0) \times \mathbf{r}_v(u_0, v_0) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial x}{\partial u} & \dfrac{\partial y}{\partial u} & \dfrac{\partial z}{\partial u} \\ \dfrac{\partial x}{\partial v} & \dfrac{\partial y}{\partial v} & \dfrac{\partial z}{\partial v} \end{vmatrix}.$$

NOTE Figure 15.40 shows the normal vector $\mathbf{r}_u \times \mathbf{r}_v$. The vector $\mathbf{r}_v \times \mathbf{r}_u$ is also normal to S and points in the opposite direction. ■

EXAMPLE 5 Finding a Tangent Plane to a Parametric Surface

Find an equation of the tangent plane to the paraboloid given by

$$\mathbf{r}(u, v) = u\mathbf{i} + v\mathbf{j} + (u^2 + v^2)\mathbf{k}$$

at the point $(1, 2, 5)$.

Solution The point in the uv-plane that is mapped to the point $(x, y, z) = (1, 2, 5)$ is $(u, v) = (1, 2)$. The partial derivatives of $\mathbf{r}$ are

$$\mathbf{r}_u = \mathbf{i} + 2u\mathbf{k} \quad \text{and} \quad \mathbf{r}_v = \mathbf{j} + 2v\mathbf{k}.$$

The normal vector is given by

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 2u \\ 0 & 1 & 2v \end{vmatrix} = -2u\mathbf{i} - 2v\mathbf{j} + \mathbf{k}$$

which implies that the normal vector at $(1, 2, 5)$ is $\mathbf{r}_u \times \mathbf{r}_v = -2\mathbf{i} - 4\mathbf{j} + \mathbf{k}$. So, an equation of the tangent plane at $(1, 2, 5)$ is

$$-2(x - 1) - 4(y - 2) + (z - 5) = 0$$
$$-2x - 4y + z = -5.$$

The tangent plane is shown in Figure 15.41. ■

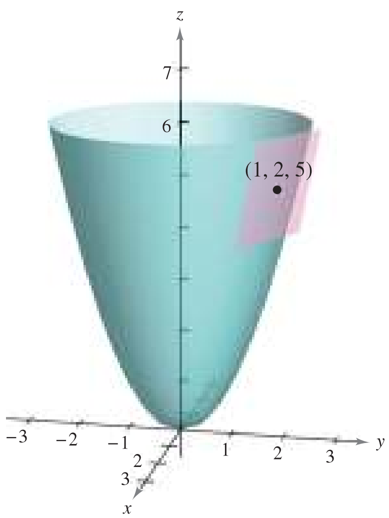


Figure 15.41

Area of a Parametric Surface

To define the area of a parametric surface, you can use a development that is similar to that given in Section 14.5. Begin by constructing an inner partition of D consisting of n rectangles, where the area of the ith rectangle D_i is $\Delta A_i = \Delta u_i \, \Delta v_i$, as shown in Figure 15.42. In each D_i let (u_i, v_i) be the point that is closest to the origin. At the point $(x_i, y_i, z_i) = (x(u_i, v_i), y(u_i, v_i), z(u_i, v_i))$ on the surface S, construct a tangent plane T_i. The area of the portion of S that corresponds to D_i, ΔT_i, can be approximated by a parallelogram in the tangent plane. That is, $\Delta T_i \approx \Delta S_i$. So, the surface of S is given by $\Sigma \, \Delta S_i \approx \Sigma \, \Delta T_i$. The area of the parallelogram in the tangent plane is

$$\|\Delta u_i \mathbf{r}_u \times \Delta v_i \mathbf{r}_v\| = \|\mathbf{r}_u \times \mathbf{r}_v\| \, \Delta u_i \, \Delta v_i$$

which leads to the following definition.

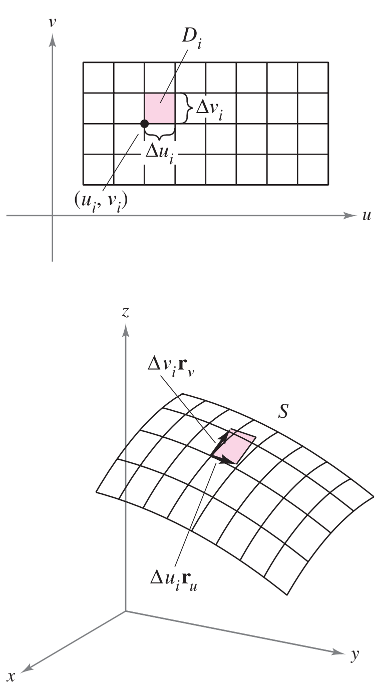


Figure 15.42

AREA OF A PARAMETRIC SURFACE

Let S be a smooth parametric surface

$$\mathbf{r}(u, v) = x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + z(u, v)\mathbf{k}$$

defined over an open region D in the uv-plane. If each point on the surface S corresponds to exactly one point in the domain D, then the **surface area** of S is given by

$$\text{Surface area} = \iint_S dS = \iint_D \|\mathbf{r}_u \times \mathbf{r}_v\| \, dA$$

where $\mathbf{r}_u = \dfrac{\partial x}{\partial u}\mathbf{i} + \dfrac{\partial y}{\partial u}\mathbf{j} + \dfrac{\partial z}{\partial u}\mathbf{k}$ and $\mathbf{r}_v = \dfrac{\partial x}{\partial v}\mathbf{i} + \dfrac{\partial y}{\partial v}\mathbf{j} + \dfrac{\partial z}{\partial v}\mathbf{k}$.

For a surface S given by $z = f(x, y)$, this formula for surface area corresponds to that given in Section 14.5. To see this, you can parametrize the surface using the vector-valued function

$$\mathbf{r}(x, y) = x\mathbf{i} + y\mathbf{j} + f(x, y)\mathbf{k}$$

defined over the region R in the xy-plane. Using

$$\mathbf{r}_x = \mathbf{i} + f_x(x, y)\mathbf{k} \quad \text{and} \quad \mathbf{r}_y = \mathbf{j} + f_y(x, y)\mathbf{k}$$

you have

$$\mathbf{r}_x \times \mathbf{r}_y = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & f_x(x, y) \\ 0 & 1 & f_y(x, y) \end{vmatrix} = -f_x(x, y)\mathbf{i} - f_y(x, y)\mathbf{j} + \mathbf{k}$$

and $\|\mathbf{r}_x \times \mathbf{r}_y\| = \sqrt{[f_x(x, y)]^2 + [f_y(x, y)]^2 + 1}$. This implies that the surface area of S is

$$\begin{aligned} \text{Surface area} &= \int\!\!\int_R \|\mathbf{r}_x \times \mathbf{r}_y\|\, dA \\ &= \int\!\!\int_R \sqrt{1 + [f_x(x, y)]^2 + [f_y(x, y)]^2}\, dA. \end{aligned}$$

EXAMPLE 6 Finding Surface Area

NOTE The surface in Example 6 does not quite fulfill the hypothesis that each point on the surface corresponds to exactly one point in D. For this surface, $\mathbf{r}(u, 0) = \mathbf{r}(u, 2\pi)$ for any fixed value of u. However, because the overlap consists of only a semicircle (which has no area), you can still apply the formula for the area of a parametric surface.

Find the surface area of the unit sphere given by

$$\mathbf{r}(u, v) = \sin u \cos v\mathbf{i} + \sin u \sin v\mathbf{j} + \cos u\mathbf{k}$$

where the domain D is given by $0 \le u \le \pi$ and $0 \le v \le 2\pi$.

Solution Begin by calculating $\mathbf{r}_u$ and $\mathbf{r}_v$.

$$\begin{aligned} \mathbf{r}_u &= \cos u \cos v\mathbf{i} + \cos u \sin v\mathbf{j} - \sin u\mathbf{k} \\ \mathbf{r}_v &= -\sin u \sin v\mathbf{i} + \sin u \cos v\mathbf{j} \end{aligned}$$

The cross product of these two vectors is

$$\begin{aligned} \mathbf{r}_u \times \mathbf{r}_v &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos u \cos v & \cos u \sin v & -\sin u \\ -\sin u \sin v & \sin u \cos v & 0 \end{vmatrix} \\ &= \sin^2 u \cos v\mathbf{i} + \sin^2 u \sin v\mathbf{j} + \sin u \cos u\mathbf{k} \end{aligned}$$

which implies that

$$\begin{aligned} \|\mathbf{r}_u \times \mathbf{r}_v\| &= \sqrt{(\sin^2 u \cos v)^2 + (\sin^2 u \sin v)^2 + (\sin u \cos u)^2} \\ &= \sqrt{\sin^4 u + \sin^2 u \cos^2 u} \\ &= \sqrt{\sin^2 u} \\ &= \sin u. \qquad \sin u > 0 \text{ for } 0 \le u \le \pi \end{aligned}$$

Finally, the surface area of the sphere is

$$\begin{aligned} A = \int\!\!\int_D \|\mathbf{r}_u \times \mathbf{r}_v\|\, dA &= \int_0^{2\pi} \int_0^{\pi} \sin u\, du\, dv \\ &= \int_0^{2\pi} 2\, dv \\ &= 4\pi. \end{aligned}$$

■

EXPLORATION

For the torus in Example 7, describe the function $\mathbf{r}(u, v)$ for fixed u. Then describe the function $\mathbf{r}(u, v)$ for fixed v.

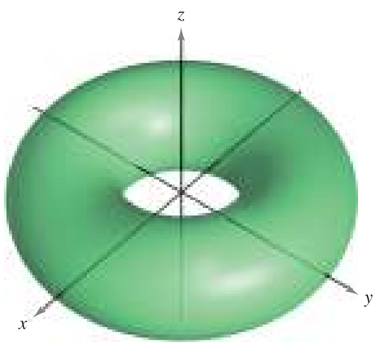


Figure 15.43

EXAMPLE 7 Finding Surface Area

Find the surface area of the torus given by

$$\mathbf{r}(u, v) = (2 + \cos u)\cos v\mathbf{i} + (2 + \cos u)\sin v\mathbf{j} + \sin u\mathbf{k}$$

where the domain D is given by $0 \le u \le 2\pi$ and $0 \le v \le 2\pi$. (See Figure 15.43.)

Solution Begin by calculating $\mathbf{r}_u$ and $\mathbf{r}_v$.

$$\mathbf{r}_u = -\sin u \cos v\mathbf{i} - \sin u \sin v\mathbf{j} + \cos u\mathbf{k}$$
$$\mathbf{r}_v = -(2 + \cos u)\sin v\mathbf{i} + (2 + \cos u)\cos v\mathbf{j}$$

The cross product of these two vectors is

$$\begin{aligned}\mathbf{r}_u \times \mathbf{r}_v &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sin u \cos v & -\sin u \sin v & \cos u \\ -(2 + \cos u)\sin v & (2 + \cos u)\cos v & 0 \end{vmatrix} \\ &= -(2 + \cos u)(\cos v \cos u\mathbf{i} + \sin v \cos u\mathbf{j} + \sin u\mathbf{k})\end{aligned}$$

which implies that

$$\begin{aligned}\|\mathbf{r}_u \times \mathbf{r}_v\| &= (2 + \cos u)\sqrt{(\cos v \cos u)^2 + (\sin v \cos u)^2 + \sin^2 u} \\ &= (2 + \cos u)\sqrt{\cos^2 u(\cos^2 v + \sin^2 v) + \sin^2 u} \\ &= (2 + \cos u)\sqrt{\cos^2 u + \sin^2 u} \\ &= 2 + \cos u.\end{aligned}$$

Finally, the surface area of the torus is

$$\begin{aligned}A = \int\!\!\int_D \|\mathbf{r}_u \times \mathbf{r}_v\|\, dA &= \int_0^{2\pi}\!\!\int_0^{2\pi} (2 + \cos u)\, du\, dv \\ &= \int_0^{2\pi} 4\pi\, dv \\ &= 8\pi^2.\end{aligned}$$

If the surface S is a surface of revolution, you can show that the formula for surface area given in Section 7.4 is equivalent to the formula given in this section. For instance, suppose f is a nonnegative function such that f' is continuous over the interval $[a, b]$. Let S be the surface of revolution formed by revolving the graph of f, where $a \le x \le b$, about the x-axis. From Section 7.4, you know that the surface area is given by

$$\text{Surface area} = 2\pi\int_a^b f(x)\sqrt{1 + [f'(x)]^2}\, dx.$$

To represent S parametrically, let $x = u$, $y = f(u)\cos v$, and $z = f(u)\sin v$, where $a \le u \le b$ and $0 \le v \le 2\pi$. Then,

$$\mathbf{r}(u, v) = u\mathbf{i} + f(u)\cos v\mathbf{j} + f(u)\sin v\mathbf{k}.$$

Try showing that the formula

$$\text{Surface area} = \int\!\!\int_D \|\mathbf{r}_u \times \mathbf{r}_v\|\, dA$$

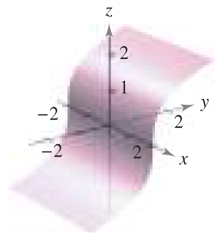
is equivalent to the formula given above (see Exercise 58).

15.5 Exercises

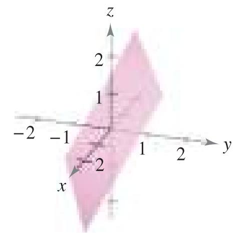
See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–6, match the vector-valued function with its graph. [The graphs are labeled (a), (b), (c), (d), (e), and (f).]

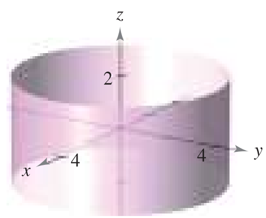
(a)



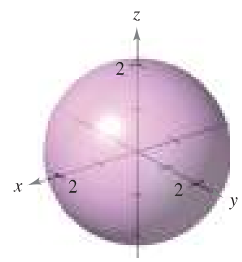
(b)



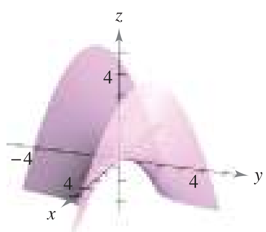
(c)



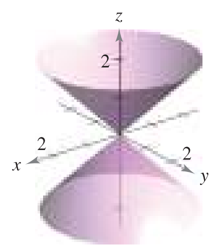
(d)



(e)



(f)



1. $\mathbf{r}(u, v) = u\mathbf{i} + v\mathbf{j} + uv\mathbf{k}$
2. $\mathbf{r}(u, v) = u \cos v\mathbf{i} + u \sin v\mathbf{j} + u\mathbf{k}$
3. $\mathbf{r}(u, v) = u\mathbf{i} + \frac{1}{2}(u + v)\mathbf{j} + v\mathbf{k}$
4. $\mathbf{r}(u, v) = u\mathbf{i} + \frac{1}{4}v^3\mathbf{j} + v\mathbf{k}$
5. $\mathbf{r}(u, v) = 2 \cos v \cos u\mathbf{i} + 2 \cos v \sin u\mathbf{j} + 2 \sin v\mathbf{k}$
6. $\mathbf{r}(u, v) = 4 \cos u\mathbf{i} + 4 \sin u\mathbf{j} + v\mathbf{k}$

In Exercises 7–10, find the rectangular equation for the surface by eliminating the parameters from the vector-valued function. Identify the surface and sketch its graph.

7. $\mathbf{r}(u, v) = u\mathbf{i} + v\mathbf{j} + \dfrac{v}{2}\mathbf{k}$
8. $\mathbf{r}(u, v) = 2u \cos v\mathbf{i} + 2u \sin v\mathbf{j} + \frac{1}{2}u^2\mathbf{k}$
9. $\mathbf{r}(u, v) = 2 \cos u\mathbf{i} + v\mathbf{j} + 2 \sin u\mathbf{k}$
10. $\mathbf{r}(u, v) = 3 \cos v \cos u\mathbf{i} + 3 \cos v \sin u\mathbf{j} + 5 \sin v\mathbf{k}$

CAS **In Exercises 11–16, use a computer algebra system to graph the surface represented by the vector-valued function.**

11. $\mathbf{r}(u, v) = 2u \cos v\mathbf{i} + 2u \sin v\mathbf{j} + u^4\mathbf{k}$

    $0 \le u \le 1, \quad 0 \le v \le 2\pi$
12. $\mathbf{r}(u, v) = 2 \cos v \cos u\mathbf{i} + 4 \cos v \sin u\mathbf{j} + \sin v\mathbf{k}$

    $0 \le u \le 2\pi, \quad 0 \le v \le 2\pi$
13. $\mathbf{r}(u, v) = 2 \sinh u \cos v\mathbf{i} + \sinh u \sin v\mathbf{j} + \cosh u\mathbf{k}$

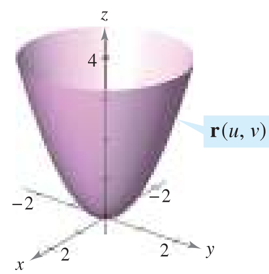
    $0 \le u \le 2, \quad 0 \le v \le 2\pi$
14. $\mathbf{r}(u, v) = 2u \cos v\mathbf{i} + 2u \sin v\mathbf{j} + v\mathbf{k}$

    $0 \le u \le 1, \quad 0 \le v \le 3\pi$
15. $\mathbf{r}(u, v) = (u - \sin u)\cos v\mathbf{i} + (1 - \cos u)\sin v\mathbf{j} + u\mathbf{k}$

    $0 \le u \le \pi, \quad 0 \le v \le 2\pi$
16. $\mathbf{r}(u, v) = \cos^3 u \cos v\mathbf{i} + \sin^3 u \sin v\mathbf{j} + u\mathbf{k}$

    $0 \le u \le \dfrac{\pi}{2}, \quad 0 \le v \le 2\pi$

Think About It **In Exercises 17–20, determine how the graph of the surface $\mathbf{s}(u, v)$ differs from the graph of $\mathbf{r}(u, v) = u \cos v\mathbf{i} + u \sin v\mathbf{j} + u^2\mathbf{k}$ (see figure), where $0 \le u \le 2$ and $0 \le v \le 2\pi$. (It is not necessary to graph s.)**



17. $\mathbf{s}(u, v) = u \cos v\mathbf{i} + u \sin v\mathbf{j} - u^2\mathbf{k}$

    $0 \le u \le 2, \quad 0 \le v \le 2\pi$
18. $\mathbf{s}(u, v) = u \cos v\mathbf{i} + u^2\mathbf{j} + u \sin v\mathbf{k}$

    $0 \le u \le 2, \quad 0 \le v \le 2\pi$
19. $\mathbf{s}(u, v) = u \cos v\mathbf{i} + u \sin v\mathbf{j} + u^2\mathbf{k}$

    $0 \le u \le 3, \quad 0 \le v \le 2\pi$
20. $\mathbf{s}(u, v) = 4u \cos v\mathbf{i} + 4u \sin v\mathbf{j} + u^2\mathbf{k}$

    $0 \le u \le 2, \quad 0 \le v \le 2\pi$

In Exercises 21–30, find a vector-valued function whose graph is the indicated surface.

21. The plane $z = y$
22. The plane $x + y + z = 6$
23. The cone $y = \sqrt{4x^2 + 9z^2}$
24. The cone $x = \sqrt{16y^2 + z^2}$
25. The cylinder $x^2 + y^2 = 25$
26. The cylinder $4x^2 + y^2 = 16$
27. The cylinder $z = x^2$
28. The ellipsoid $\dfrac{x^2}{9} + \dfrac{y^2}{4} + \dfrac{z^2}{1} = 1$
29. The part of the plane $z = 4$ that lies inside the cylinder $x^2 + y^2 = 9$
30. The part of the paraboloid $z = x^2 + y^2$ that lies inside the cylinder $x^2 + y^2 = 9$

Surface of Revolution **In Exercises 31–34, write a set of parametric equations for the surface of revolution obtained by revolving the graph of the function about the given axis.**

Function	*Axis of Revolution*
31. $y = \dfrac{x}{2}, \quad 0 \le x \le 6$	x-axis
32. $y = \sqrt{x}, \quad 0 \le x \le 4$	x-axis
33. $x = \sin z, \quad 0 \le z \le \pi$	z-axis
34. $z = y^2 + 1, \quad 0 \le y \le 2$	y-axis

Tangent Plane **In Exercises 35–38, find an equation of the tangent plane to the surface represented by the vector-valued function at the given point.**

35. $\mathbf{r}(u, v) = (u + v)\mathbf{i} + (u - v)\mathbf{j} + v\mathbf{k}, \quad (1, -1, 1)$

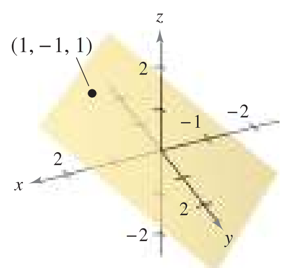


Figure for 35

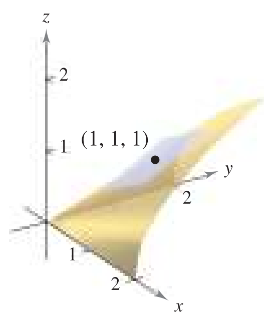
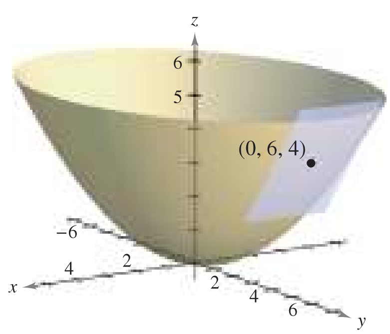


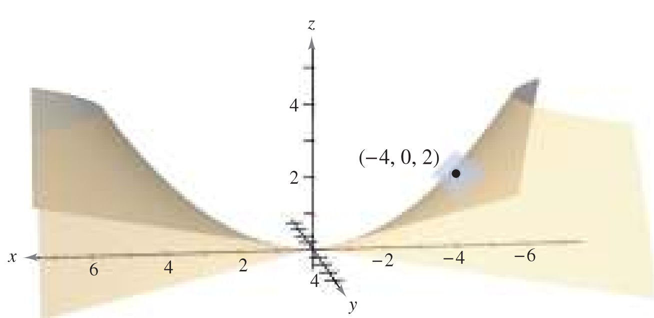
Figure for 36

36. $\mathbf{r}(u, v) = u\mathbf{i} + v\mathbf{j} + \sqrt{uv}\,\mathbf{k}, \quad (1, 1, 1)$

37. $\mathbf{r}(u, v) = 2u \cos v\mathbf{i} + 3u \sin v\mathbf{j} + u^2\mathbf{k}, \quad (0, 6, 4)$



38. $\mathbf{r}(u, v) = 2u \cosh v\mathbf{i} + 2u \sinh v\mathbf{j} + \frac{1}{2}u^2\mathbf{k}, \quad (-4, 0, 2)$



Area **In Exercises 39–46, find the area of the surface over the given region. Use a computer algebra system to verify your results.**

39. The part of the plane $\mathbf{r}(u, v) = 4u\mathbf{i} - v\mathbf{j} + v\mathbf{k}$, where $0 \le u \le 2$ and $0 \le v \le 1$

40. The part of the paraboloid $\mathbf{r}(u, v) = 2u \cos v\mathbf{i} + 2u \sin v\mathbf{j} + u^2\mathbf{k}$, where $0 \le u \le 2$ and $0 \le v \le 2\pi$

41. The part of the cylinder $\mathbf{r}(u, v) = a \cos u\mathbf{i} + a \sin u\mathbf{j} + v\mathbf{k}$, where $0 \le u \le 2\pi$ and $0 \le v \le b$

42. The sphere $\mathbf{r}(u, v) = a \sin u \cos v\mathbf{i} + a \sin u \sin v\mathbf{j} + a \cos u\mathbf{k}$, where $0 \le u \le \pi$ and $0 \le v \le 2\pi$

43. The part of the cone $\mathbf{r}(u, v) = au \cos v\mathbf{i} + au \sin v\mathbf{j} + u\mathbf{k}$, where $0 \le u \le b$ and $0 \le v \le 2\pi$

44. The torus $\mathbf{r}(u, v) = (a + b \cos v)\cos u\mathbf{i} + (a + b \cos v)\sin u\mathbf{j} + b \sin v\mathbf{k}$, where $a > b$, $0 \le u \le 2\pi$, and $0 \le v \le 2\pi$

45. The surface of revolution $\mathbf{r}(u, v) = \sqrt{u} \cos v\mathbf{i} + \sqrt{u} \sin v\mathbf{j} + u\mathbf{k}$, where $0 \le u \le 4$ and $0 \le v \le 2\pi$

46. The surface of revolution $\mathbf{r}(u, v) = \sin u \cos v\mathbf{i} + u\mathbf{j} + \sin u \sin v\mathbf{k}$, where $0 \le u \le \pi$ and $0 \le v \le 2\pi$

WRITING ABOUT CONCEPTS

47. Define a parametric surface.

48. Give the double integral that yields the surface area of a parametric surface over an open region D.

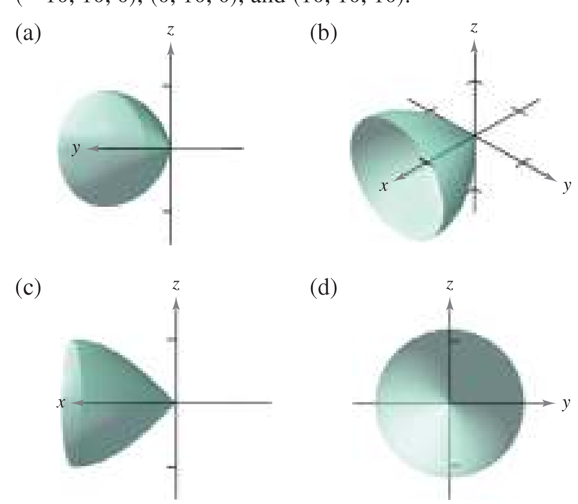
49. Show that the cone in Example 3 can be represented parametrically by $\mathbf{r}(u, v) = u \cos v\mathbf{i} + u \sin v\mathbf{j} + u\mathbf{k}$, where $0 \le u$ and $0 \le v \le 2\pi$.

CAPSTONE

50. The four figures below are graphs of the surface $\mathbf{r}(u, v) = u\mathbf{i} + \sin u \cos v\mathbf{j} + \sin u \sin v\mathbf{k}$,

$$0 \le u \le \pi/2, \quad 0 \le v \le 2\pi.$$

Match each of the four graphs with the point in space from which the surface is viewed. The four points are $(10, 0, 0)$, $(-10, 10, 0)$, $(0, 10, 0)$, and $(10, 10, 10)$.



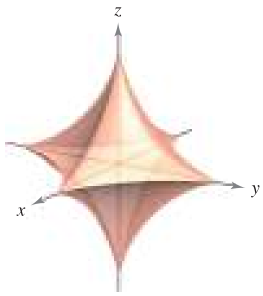
51. ***Astroidal Sphere*** An equation of an **astroidal sphere** in x, y, and z is

$$x^{2/3} + y^{2/3} + z^{2/3} = a^{2/3}.$$

A graph of an astroidal sphere is shown below. Show that this surface can be represented parametrically by

$$\mathbf{r}(u, v) = a \sin^3 u \cos^3 v\mathbf{i} + a \sin^3 u \sin^3 v\mathbf{j} + a \cos^3 u\mathbf{k}$$

where $0 \le u \le \pi$ and $0 \le v \le 2\pi$.



CAS **52.** Use a computer algebra system to graph three views of the graph of the vector-valued function

$$\mathbf{r}(u, v) = u \cos v\mathbf{i} + u \sin v\mathbf{j} + v\mathbf{k}, \quad 0 \le u \le \pi, \quad 0 \le v \le \pi$$

from the points $(10, 0, 0)$, $(0, 0, 10)$, and $(10, 10, 10)$.

CAS **53.** ***Investigation*** Use a computer algebra system to graph the torus

$$\mathbf{r}(u, v) = (a + b \cos v) \cos u\mathbf{i} + (a + b \cos v) \sin u\mathbf{j} + b \sin v\mathbf{k}$$

for each set of values of a and b, where $0 \le u \le 2\pi$ and $0 \le v \le 2\pi$. Use the results to describe the effects of a and b on the shape of the torus.

(a) $a = 4, \quad b = 1$ (b) $a = 4, \quad b = 2$

(c) $a = 8, \quad b = 1$ (d) $a = 8, \quad b = 3$

54. ***Investigation*** Consider the function in Exercise 14.

(a) Sketch a graph of the function where u is held constant at $u = 1$. Identify the graph.

(b) Sketch a graph of the function where v is held constant at $v = 2\pi/3$. Identify the graph.

(c) Assume that a surface is represented by the vector-valued function $\mathbf{r} = \mathbf{r}(u, v)$. What generalization can you make about the graph of the function if one of the parameters is held constant?

55. ***Surface Area*** The surface of the dome on a new museum is given by

$$\mathbf{r}(u, v) = 20 \sin u \cos v\mathbf{i} + 20 \sin u \sin v\mathbf{j} + 20 \cos u\mathbf{k}$$

where $0 \le u \le \pi/3$, $0 \le v \le 2\pi$, and $\mathbf{r}$ is in meters. Find the surface area of the dome.

56. Find a vector-valued function for the hyperboloid

$$x^2 + y^2 - z^2 = 1$$

and determine the tangent plane at $(1, 0, 0)$.

57. Graph and find the area of one turn of the spiral ramp

$$\mathbf{r}(u, v) = u \cos v\mathbf{i} + u \sin v\mathbf{j} + 2v\mathbf{k}$$

where $0 \le u \le 3$ and $0 \le v \le 2\pi$.

58. Let f be a nonnegative function such that f' is continuous over the interval $[a, b]$. Let S be the surface of revolution formed by revolving the graph of f, where $a \le x \le b$, about the x-axis. Let $x = u$, $y = f(u) \cos v$, and $z = f(u)\sin v$, where $a \le u \le b$ and $0 \le v \le 2\pi$. Then, S is represented parametrically by $\mathbf{r}(u, v) = u\mathbf{i} + f(u) \cos v\mathbf{j} + f(u) \sin v\mathbf{k}$. Show that the following formulas are equivalent.

$$\text{Surface area } = 2\pi \int_a^b f(x)\sqrt{1 + [f'(x)]^2}\, dx$$

$$\text{Surface area } = \int\!\!\int_D \|\mathbf{r}_u \times \mathbf{r}_v\|\, dA$$

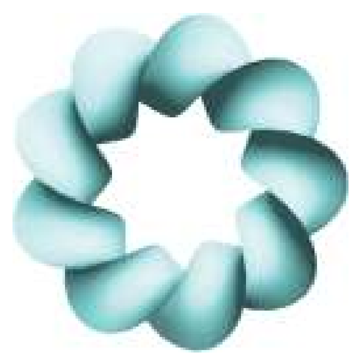
CAS **59.** ***Open-Ended Project*** The parametric equations

$$x = 3 + \sin u[7 - \cos(3u - 2v) - 2 \cos(3u + v)]$$

$$y = 3 + \cos u[7 - \cos(3u - 2v) - 2 \cos(3u + v)]$$

$$z = \sin(3u - 2v) + 2 \sin(3u + v)$$

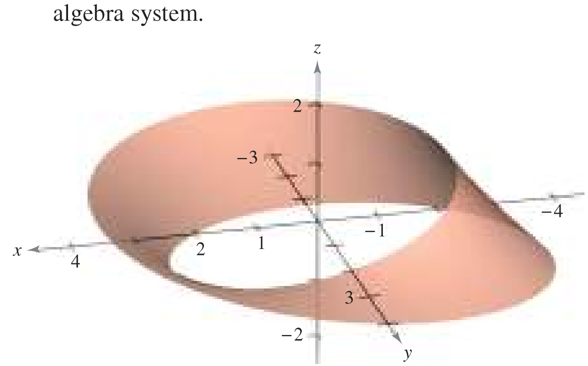
where $-\pi \le u \le \pi$ and $-\pi \le v \le \pi$, represent the surface shown below. Try to create your own parametric surface using a computer algebra system.



CAS **60.** ***Möbius Strip*** The surface shown in the figure is called a **Möbius strip** and can be represented by the parametric equations

$$x = \left(a + u \cos\frac{v}{2}\right) \cos v, \; y = \left(a + u \cos\frac{v}{2}\right) \sin v, \; z = u \sin\frac{v}{2}$$

where $-1 \le u \le 1$, $0 \le v \le 2\pi$, and $a = 3$. Try to graph other Möbius strips for different values of a using a computer algebra system.



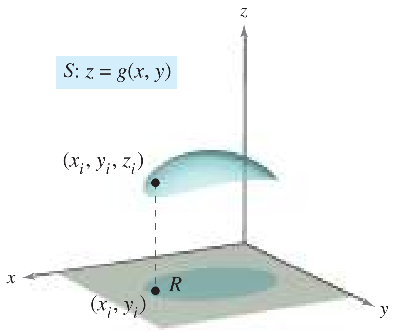
15.6 Surface Integrals

- **Evaluate a surface integral as a double integral.**
- **Evaluate a surface integral for a parametric surface.**
- **Determine the orientation of a surface.**
- **Understand the concept of a flux integral.**

Surface Integrals

The remainder of this chapter deals primarily with **surface integrals.** You will first consider surfaces given by $z = g(x, y)$. Later in this section you will consider more general surfaces given in parametric form.

Let S be a surface given by $z = g(x, y)$ and let R be its projection onto the xy-plane, as shown in Figure 15.44. Suppose that g, g_x, and g_y are continuous at all points in R and that f is defined on S. Employing the procedure used to find surface area in Section 14.5, evaluate f at (x_i, y_i, z_i) and form the sum



Scalar function f assigns a number to each point of S.
Figure 15.44

$$\sum_{i=1}^{n} f(x_i, y_i, z_i)\, \Delta S_i$$

where $\Delta S_i \approx \sqrt{1 + [g_x(x_i, y_i)]^2 + [g_y(x_i, y_i)]^2}\, \Delta A_i$. Provided the limit of this sum as $\|\Delta\|$ approaches 0 exists, the **surface integral of f over S** is defined as

$$\int_S\int f(x, y, z)\, dS = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} f(x_i, y_i, z_i)\, \Delta S_i.$$

This integral can be evaluated by a double integral.

THEOREM 15.10 EVALUATING A SURFACE INTEGRAL

Let S be a surface with equation $z = g(x, y)$ and let R be its projection onto the xy-plane. If g, g_x, and g_y are continuous on R and f is continuous on S, then the surface integral of f over S is

$$\int_S\int f(x, y, z)\, dS = \int_R\int f(x, y, g(x, y))\sqrt{1 + [g_x(x, y)]^2 + [g_y(x, y)]^2}\, dA.$$

For surfaces described by functions of x and z (or y and z), you can make the following adjustments to Theorem 15.10. If S is the graph of $y = g(x, z)$ and R is its projection onto the xz-plane, then

$$\int_S\int f(x, y, z)\, dS = \int_R\int f(x, g(x, z), z)\sqrt{1 + [g_x(x, z)]^2 + [g_z(x, z)]^2}\, dA.$$

If S is the graph of $x = g(y, z)$ and R is its projection onto the yz-plane, then

$$\int_S\int f(x, y, z)\, dS = \int_R\int f(g(y, z), y, z)\sqrt{1 + [g_y(y, z)]^2 + [g_z(y, z)]^2}\, dA.$$

If $f(x, y, z) = 1$, the surface integral over S yields the surface area of S. For instance, suppose the surface S is the plane given by $z = x$, where $0 \le x \le 1$ and $0 \le y \le 1$. The surface area of S is $\sqrt{2}$ square units. Try verifying that $\int_S\int f(x, y, z)\, dS = \sqrt{2}$.

EXAMPLE 1 Evaluating a Surface Integral

Evaluate the surface integral

$$\iint_S (y^2 + 2yz)\, dS$$

where S is the first-octant portion of the plane $2x + y + 2z = 6$.

Solution Begin by writing S as

$$z = \frac{1}{2}(6 - 2x - y)$$

$$g(x, y) = \frac{1}{2}(6 - 2x - y).$$

Using the partial derivatives $g_x(x, y) = -1$ and $g_y(x, y) = -\frac{1}{2}$, you can write

$$\sqrt{1 + [g_x(x, y)]^2 + [g_y(x, y)]^2} = \sqrt{1 + 1 + \frac{1}{4}} = \frac{3}{2}.$$

Using Figure 15.45 and Theorem 15.10, you obtain

$$\begin{aligned}
\iint_S (y^2 + 2yz)\, dS &= \iint_R f(x, y, g(x, y))\sqrt{1 + [g_x(x, y)]^2 + [g_y(x, y)]^2}\, dA \\
&= \iint_R \left[y^2 + 2y\left(\frac{1}{2}\right)(6 - 2x - y)\right]\left(\frac{3}{2}\right) dA \\
&= 3\int_0^3 \int_0^{2(3-x)} y(3 - x)\, dy\, dx \\
&= 6\int_0^3 (3 - x)^3\, dx \\
&= -\frac{3}{2}(3 - x)^4\Big]_0^3 \\
&= \frac{243}{2}.
\end{aligned}$$

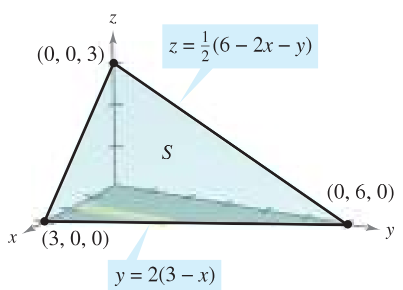


Figure 15.45

An alternative solution to Example 1 would be to project S onto the yz-plane, as shown in Figure 15.46. Then, $x = \frac{1}{2}(6 - y - 2z)$, and

$$\sqrt{1 + [g_y(y, z)]^2 + [g_z(y, z)]^2} = \sqrt{1 + \frac{1}{4} + 1} = \frac{3}{2}.$$

So, the surface integral is

$$\begin{aligned}
\iint_S (y^2 + 2yz)\, dS &= \iint_R f(g(y, z), y, z)\sqrt{1 + [g_y(y, z)]^2 + [g_z(y, z)]^2}\, dA \\
&= \int_0^6 \int_0^{(6-y)/2} (y^2 + 2yz)\left(\frac{3}{2}\right) dz\, dy \\
&= \frac{3}{8}\int_0^6 (36y - y^3)\, dy \\
&= \frac{243}{2}.
\end{aligned}$$

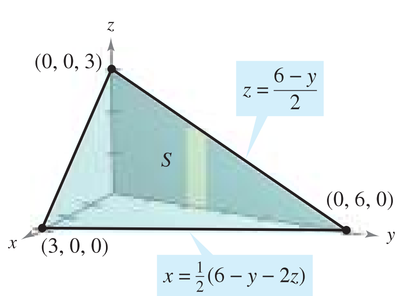


Figure 15.46

Try reworking Example 1 by projecting S onto the xz-plane.

In Example 1, you could have projected the surface S onto any one of the three coordinate planes. In Example 2, S is a portion of a cylinder centered about the x-axis, and you can project it onto either the xz-plane or the xy-plane.

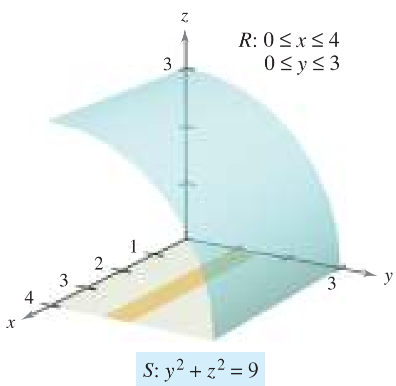


Figure 15.47

EXAMPLE 2 Evaluating a Surface Integral

Evaluate the surface integral

$$\int_S\int (x + z)\, dS$$

where S is the first-octant portion of the cylinder $y^2 + z^2 = 9$ between $x = 0$ and $x = 4$, as shown in Figure 15.47.

Solution Project S onto the xy-plane, so that $z = g(x, y) = \sqrt{9 - y^2}$, and obtain

$$\sqrt{1 + [g_x(x, y)]^2 + [g_y(x, y)]^2} = \sqrt{1 + \left(\frac{-y}{\sqrt{9 - y^2}}\right)^2}$$

$$= \frac{3}{\sqrt{9 - y^2}}.$$

Theorem 15.10 does not apply directly, because g_y is not continuous when $y = 3$. However, you can apply Theorem 15.10 for $0 \le b < 3$ and then take the limit as b approaches 3, as follows.

$$\begin{aligned}
\int_S\int (x + z)\, dS &= \lim_{b\to 3^-} \int_0^b\int_0^4 \left(x + \sqrt{9 - y^2}\right)\frac{3}{\sqrt{9 - y^2}}\, dx\, dy \\
&= \lim_{b\to 3^-} 3\int_0^b\int_0^4 \left(\frac{x}{\sqrt{9 - y^2}} + 1\right) dx\, dy \\
&= \lim_{b\to 3^-} 3\int_0^b \left[\frac{x^2}{2\sqrt{9 - y^2}} + x\right]_0^4 dy \\
&= \lim_{b\to 3^-} 3\int_0^b \left(\frac{8}{\sqrt{9 - y^2}} + 4\right) dy \\
&= \lim_{b\to 3^-} 3\left[4y + 8 \arcsin\frac{y}{3}\right]_0^b \\
&= \lim_{b\to 3^-} 3\left(4b + 8 \arcsin\frac{b}{3}\right) \\
&= 36 + 24\left(\frac{\pi}{2}\right) \\
&= 36 + 12\pi
\end{aligned}$$

TECHNOLOGY Some computer algebra systems are capable of evaluating improper integrals. If you have access to such computer software, use it to evaluate the improper integral

$$\int_0^3\int_0^4 \left(x + \sqrt{9 - y^2}\right)\frac{3}{\sqrt{9 - y^2}}\, dx\, dy.$$

Do you obtain the same result as in Example 2?

You have already seen that if the function f defined on the surface S is simply $f(x, y, z) = 1$, the surface integral yields the *surface area* of S.

$$\text{Area of surface} = \int\int_S 1\, dS$$

On the other hand, if S is a lamina of variable density and $\rho(x, y, z)$ is the density at the point (x, y, z), then the *mass* of the lamina is given by

$$\text{Mass of lamina} = \int\int_S \rho(x, y, z)\, dS.$$

EXAMPLE 3 Finding the Mass of a Surface Lamina

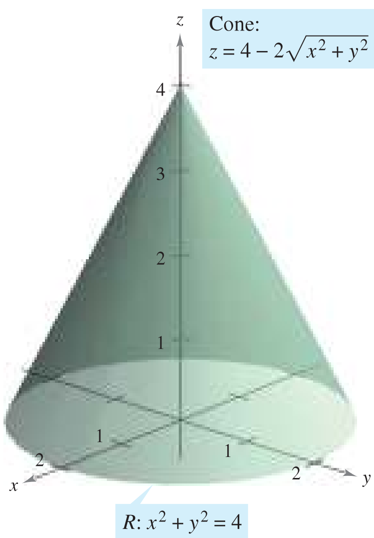


Figure 15.48

A cone-shaped surface lamina S is given by

$$z = 4 - 2\sqrt{x^2 + y^2}, \quad 0 \le z \le 4$$

as shown in Figure 15.48. At each point on S, the density is proportional to the distance between the point and the z-axis. Find the mass m of the lamina.

Solution Projecting S onto the xy-plane produces

$$S: z = 4 - 2\sqrt{x^2 + y^2} = g(x, y), \quad 0 \le z \le 4$$
$$R: x^2 + y^2 \le 4$$

with a density of $\rho(x, y, z) = k\sqrt{x^2 + y^2}$. Using a surface integral, you can find the mass to be

$$\begin{aligned}
m &= \int\int_S \rho(x, y, z)\, dS \\
&= \int\int_R k\sqrt{x^2 + y^2}\sqrt{1 + [g_x(x, y)]^2 + [g_y(x, y)]^2}\, dA \\
&= k\int\int_R \sqrt{x^2 + y^2}\sqrt{1 + \frac{4x^2}{x^2 + y^2} + \frac{4y^2}{x^2 + y^2}}\, dA \\
&= k\int\int_R \sqrt{5}\sqrt{x^2 + y^2}\, dA \\
&= k\int_0^{2\pi}\int_0^2 \left(\sqrt{5}r\right) r\, dr\, d\theta \qquad \text{Polar coordinates} \\
&= \frac{\sqrt{5}k}{3}\int_0^{2\pi} r^3\Big]_0^2\, d\theta \\
&= \frac{8\sqrt{5}k}{3}\int_0^{2\pi} d\theta \\
&= \frac{8\sqrt{5}k}{3}\Big[\theta\Big]_0^{2\pi} = \frac{16\sqrt{5}k\pi}{3}.
\end{aligned}$$

TECHNOLOGY Use a computer algebra system to confirm the result shown in Example 3. The computer algebra system *Maple* evaluated the integral as follows.

$$k\int_{-2}^{2}\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} \sqrt{5}\sqrt{x^2 + y^2}\, dx\, dy = k\int_0^{2\pi}\int_0^2 \left(\sqrt{5}r\right) r\, dr\, d\theta = \frac{16\sqrt{5}k\pi}{3}$$

Parametric Surfaces and Surface Integrals

For a surface S given by the vector-valued function

$$\mathbf{r}(u, v) = x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + z(u, v)\mathbf{k} \qquad \text{Parametric surface}$$

defined over a region D in the uv-plane, you can show that the surface integral of $f(x, y, z)$ over S is given by

$$\int_S\int f(x, y, z)\, dS = \int_D\int f(x(u, v), y(u, v), z(u, v))\, \|\mathbf{r}_u(u, v) \times \mathbf{r}_v(u, v)\|\, dA.$$

Note the similarity to a line integral over a space curve C.

$$\int_C f(x, y, z)\, ds = \int_a^b f(x(t), y(t), z(t))\, \|\mathbf{r}'(t)\|\, dt \qquad \text{Line integral}$$

NOTE Notice that ds and dS can be written as $ds = \|\mathbf{r}'(t)\|\, dt$ and $dS = \|\mathbf{r}_u(u, v) \times \mathbf{r}_v(u, v)\|\, dA$. ■

EXAMPLE 4 Evaluating a Surface Integral

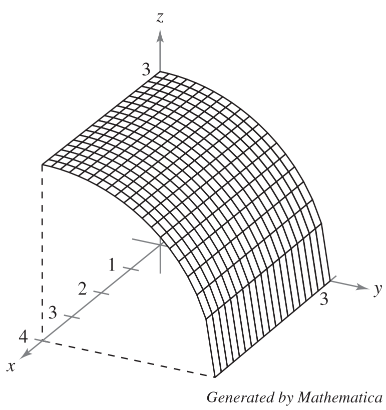


Figure 15.49

Example 2 demonstrated an evaluation of the surface integral

$$\int_S\int (x + z)\, dS$$

where S is the first-octant portion of the cylinder $y^2 + z^2 = 9$ between $x = 0$ and $x = 4$ (see Figure 15.49). Reevaluate this integral in parametric form.

Solution In parametric form, the surface is given by

$$\mathbf{r}(x, \theta) = x\mathbf{i} + 3\cos\theta\mathbf{j} + 3\sin\theta\mathbf{k}$$

where $0 \le x \le 4$ and $0 \le \theta \le \pi/2$. To evaluate the surface integral in parametric form, begin by calculating the following.

$$\mathbf{r}_x = \mathbf{i}$$

$$\mathbf{r}_\theta = -3\sin\theta\mathbf{j} + 3\cos\theta\mathbf{k}$$

$$\mathbf{r}_x \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 0 \\ 0 & -3\sin\theta & 3\cos\theta \end{vmatrix} = -3\cos\theta\mathbf{j} - 3\sin\theta\mathbf{k}$$

$$\|\mathbf{r}_x \times \mathbf{r}_\theta\| = \sqrt{9\cos^2\theta + 9\sin^2\theta} = 3$$

So, the surface integral can be evaluated as follows.

$$\begin{aligned}\int_D\int (x + 3\sin\theta)3\, dA &= \int_0^4\int_0^{\pi/2} (3x + 9\sin\theta)\, d\theta\, dx \\ &= \int_0^4 \Big[3x\theta - 9\cos\theta\Big]_0^{\pi/2} dx \\ &= \int_0^4 \left(\frac{3\pi}{2}x + 9\right) dx \\ &= \left[\frac{3\pi}{4}x^2 + 9x\right]_0^4 \\ &= 12\pi + 36\end{aligned}$$

■

Orientation of a Surface

Unit normal vectors are used to induce an orientation to a surface S in space. A surface is called **orientable** if a unit normal vector $\mathbf{N}$ can be defined at every nonboundary point of S in such a way that the normal vectors vary continuously over the surface S. If this is possible, S is called an **oriented surface.**

An orientable surface S has two distinct sides. So, when you orient a surface, you are selecting one of the two possible unit normal vectors. If S is a closed surface such as a sphere, it is customary to choose the unit normal vector $\mathbf{N}$ to be the one that points outward from the sphere.

Most common surfaces, such as spheres, paraboloids, ellipses, and planes, are orientable. (See Exercise 43 for an example of a surface that is *not* orientable.) Moreover, for an orientable surface, the gradient vector provides a convenient way to find a unit normal vector. That is, for an orientable surface S given by

$$z = g(x, y) \qquad \text{Orientable surface}$$

let

$$G(x, y, z) = z - g(x, y).$$

Then, S can be oriented by either the unit normal vector

$$\begin{aligned}\mathbf{N} &= \frac{\nabla G(x, y, z)}{\|\nabla G(x, y, z)\|}\\ &= \frac{-g_x(x, y)\mathbf{i} - g_y(x, y)\mathbf{j} + \mathbf{k}}{\sqrt{1 + [g_x(x, y)]^2 + [g_y(x, y)]^2}} \qquad \text{Upward unit normal vector}\end{aligned}$$

or the unit normal vector

$$\begin{aligned}\mathbf{N} &= \frac{-\nabla G(x, y, z)}{\|\nabla G(x, y, z)\|}\\ &= \frac{g_x(x, y)\mathbf{i} + g_y(x, y)\mathbf{j} - \mathbf{k}}{\sqrt{1 + [g_x(x, y)]^2 + [g_y(x, y)]^2}} \qquad \text{Downward unit normal vector}\end{aligned}$$

as shown in Figure 15.50. If the smooth orientable surface S is given in parametric form by

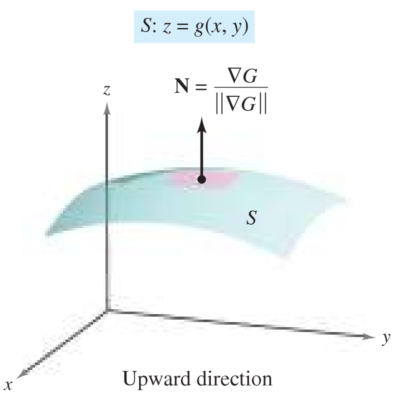
$$\mathbf{r}(u, v) = x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + z(u, v)\mathbf{k} \qquad \text{Parametric surface}$$

the unit normal vectors are given by

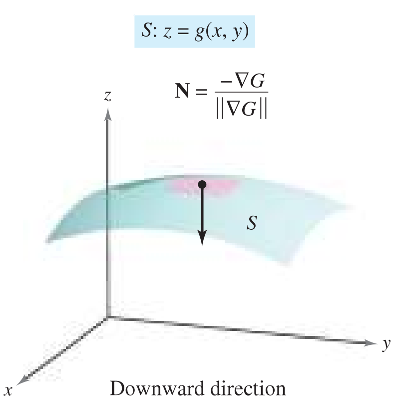
$$\mathbf{N} = \frac{\mathbf{r}_u \times \mathbf{r}_v}{\|\mathbf{r}_u \times \mathbf{r}_v\|}$$

and

$$\mathbf{N} = \frac{\mathbf{r}_v \times \mathbf{r}_u}{\|\mathbf{r}_v \times \mathbf{r}_u\|}.$$



S is oriented in an upward direction.



S is oriented in a downward direction.
Figure 15.50

NOTE Suppose that the orientable surface is given by $y = g(x, z)$ or $x = g(y, z)$. Then you can use the gradient vector

$$\nabla G(x, y, z) = -g_x(x, z)\mathbf{i} + \mathbf{j} - g_z(x, z)\mathbf{k} \qquad G(x, y, z) = y - g(x, z)$$

or

$$\nabla G(x, y, z) = \mathbf{i} - g_y(y, z)\mathbf{j} - g_z(y, z)\mathbf{k} \qquad G(x, y, z) = x - g(y, z)$$

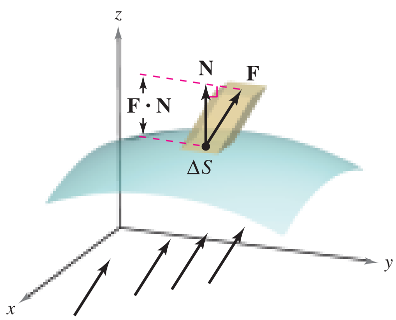
to orient the surface. ■

Flux Integrals

One of the principal applications involving the vector form of a surface integral relates to the flow of a fluid through a surface S. Suppose an oriented surface S is submerged in a fluid having a continuous velocity field $\mathbf{F}$. Let ΔS be the area of a small patch of the surface S over which $\mathbf{F}$ is nearly constant. Then the amount of fluid crossing this region per unit of time is approximated by the volume of the column of height $\mathbf{F} \cdot \mathbf{N}$, as shown in Figure 15.51. That is,

$$\Delta V = (\text{height})(\text{area of base}) = (\mathbf{F} \cdot \mathbf{N})\Delta S.$$

Consequently, the volume of fluid crossing the surface S per unit of time (called the **flux of F across S**) is given by the surface integral in the following definition.



The velocity field $\mathbf{F}$ indicates the direction of the fluid flow.
Figure 15.51

DEFINITION OF FLUX INTEGRAL

Let $\mathbf{F}(x, y, z) = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$, where M, N, and P have continuous first partial derivatives on the surface S oriented by a unit normal vector $\mathbf{N}$. The **flux integral of F across S** is given by

$$\int_S\int \mathbf{F} \cdot \mathbf{N}\, dS.$$

Geometrically, a flux integral is the surface integral over S of the *normal component* of $\mathbf{F}$. If $\rho(x, y, z)$ is the density of the fluid at (x, y, z), the flux integral

$$\int_S\int \rho\, \mathbf{F} \cdot \mathbf{N}\, dS$$

represents the *mass* of the fluid flowing across S per unit of time.

To evaluate a flux integral for a surface given by $z = g(x, y)$, let

$$G(x, y, z) = z - g(x, y).$$

Then, $\mathbf{N}\, dS$ can be written as follows.

$$\begin{aligned}\mathbf{N}\, dS &= \frac{\nabla G(x, y, z)}{\|\nabla G(x, y, z)\|}\, dS \\ &= \frac{\nabla G(x, y, z)}{\sqrt{(g_x)^2 + (g_y)^2 + 1}}\sqrt{(g_x)^2 + (g_y)^2 + 1}\, dA \\ &= \nabla G(x, y, z)\, dA\end{aligned}$$

THEOREM 15.11 EVALUATING A FLUX INTEGRAL

Let S be an oriented surface given by $z = g(x, y)$ and let R be its projection onto the xy-plane.

$$\int_S\int \mathbf{F} \cdot \mathbf{N}\, dS = \int_R\int \mathbf{F} \cdot [-g_x(x, y)\mathbf{i} - g_y(x, y)\mathbf{j} + \mathbf{k}]\, dA \quad \text{Oriented upward}$$

$$\int_S\int \mathbf{F} \cdot \mathbf{N}\, dS = \int_R\int \mathbf{F} \cdot [g_x(x, y)\mathbf{i} + g_y(x, y)\mathbf{j} - \mathbf{k}]\, dA \quad \text{Oriented downward}$$

For the first integral, the surface is oriented upward, and for the second integral, the surface is oriented downward.

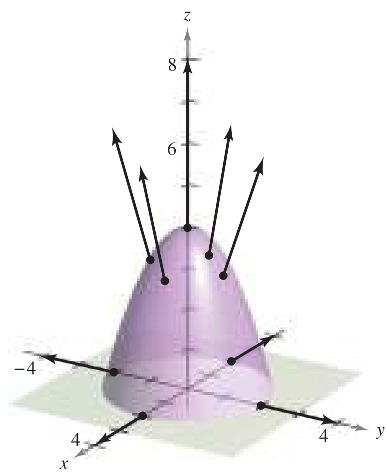


Figure 15.52

EXAMPLE 5 Using a Flux Integral to Find the Rate of Mass Flow

Let S be the portion of the paraboloid

$$z = g(x, y) = 4 - x^2 - y^2$$

lying above the xy-plane, oriented by an upward unit normal vector, as shown in Figure 15.52. A fluid of constant density ρ is flowing through the surface S according to the vector field

$$\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}.$$

Find the rate of mass flow through S.

Solution Begin by computing the partial derivatives of g.

$$g_x(x, y) = -2x$$

and

$$g_y(x, y) = -2y$$

The rate of mass flow through the surface S is

$$\begin{aligned}
\int_S\!\!\int \rho\mathbf{F}\cdot\mathbf{N}\,dS &= \rho\int_R\!\!\int \mathbf{F}\cdot[-g_x(x, y)\mathbf{i} - g_y(x, y)\mathbf{j} + \mathbf{k}]\,dA\\
&= \rho\int_R\!\!\int [x\mathbf{i} + y\mathbf{j} + (4 - x^2 - y^2)\mathbf{k}]\cdot(2x\mathbf{i} + 2y\mathbf{j} + \mathbf{k})\,dA\\
&= \rho\int_R\!\!\int [2x^2 + 2y^2 + (4 - x^2 - y^2)]\,dA\\
&= \rho\int_R\!\!\int (4 + x^2 + y^2)\,dA\\
&= \rho\int_0^{2\pi}\!\!\int_0^2 (4 + r^2)r\,dr\,d\theta \qquad \text{Polar coordinates}\\
&= \rho\int_0^{2\pi} 12\,d\theta\\
&= 24\pi\rho.
\end{aligned}$$

For an oriented surface S given by the vector-valued function

$$\mathbf{r}(u, v) = x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + z(u, v)\mathbf{k} \qquad \text{Parametric surface}$$

defined over a region D in the uv-plane, you can define the flux integral of $\mathbf{F}$ across S as

$$\begin{aligned}
\int_S\!\!\int \mathbf{F}\cdot\mathbf{N}\,dS &= \int_D\!\!\int \mathbf{F}\cdot\left(\frac{\mathbf{r}_u\times\mathbf{r}_v}{\|\mathbf{r}_u\times\mathbf{r}_v\|}\right)\|\mathbf{r}_u\times\mathbf{r}_v\|\,dA\\
&= \int_D\!\!\int \mathbf{F}\cdot(\mathbf{r}_u\times\mathbf{r}_v)\,dA.
\end{aligned}$$

Note the similarity of this integral to the line integral

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C \mathbf{F}\cdot\mathbf{T}\,ds.$$

A summary of formulas for line and surface integrals is presented on page 1121.

EXAMPLE 6 Finding the Flux of an Inverse Square Field

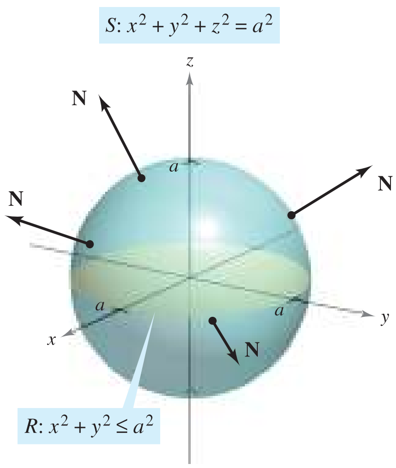


Figure 15.53

Find the flux over the sphere S given by

$$x^2 + y^2 + z^2 = a^2 \qquad \text{Sphere } S$$

where $\mathbf{F}$ is an inverse square field given by

$$\mathbf{F}(x, y, z) = \frac{kq}{\|\mathbf{r}\|^2}\frac{\mathbf{r}}{\|\mathbf{r}\|} = \frac{kq\mathbf{r}}{\|\mathbf{r}\|^3} \qquad \text{Inverse square field } \mathbf{F}$$

and $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. Assume S is oriented outward, as shown in Figure 15.53.

Solution The sphere is given by

$$\begin{aligned}\mathbf{r}(u, v) &= x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + z(u, v)\mathbf{k}\\ &= a \sin u \cos v\mathbf{i} + a \sin u \sin v\mathbf{j} + a \cos u\mathbf{k}\end{aligned}$$

where $0 \le u \le \pi$ and $0 \le v \le 2\pi$. The partial derivatives of $\mathbf{r}$ are

$$\mathbf{r}_u(u, v) = a \cos u \cos v\,\mathbf{i} + a \cos u \sin v\mathbf{j} - a \sin u\mathbf{k}$$

and

$$\mathbf{r}_v(u, v) = -a \sin u \sin v\mathbf{i} + a \sin u \cos v\mathbf{j}$$

which implies that the normal vector $\mathbf{r}_u \times \mathbf{r}_v$ is

$$\begin{aligned}\mathbf{r}_u \times \mathbf{r}_v &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a \cos u \cos v & a \cos u \sin v & -a \sin u \\ -a \sin u \sin v & a \sin u \cos v & 0 \end{vmatrix}\\ &= a^2(\sin^2 u \cos v\mathbf{i} + \sin^2 u \sin v\mathbf{j} + \sin u \cos u\mathbf{k}).\end{aligned}$$

Now, using

$$\begin{aligned}\mathbf{F}(x, y, z) &= \frac{kq\mathbf{r}}{\|\mathbf{r}\|^3}\\ &= kq\frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{\|x\mathbf{i} + y\mathbf{j} + z\mathbf{k}\|^3}\\ &= \frac{kq}{a^3}(a \sin u \cos v\mathbf{i} + a \sin u \sin v\mathbf{j} + a \cos u\mathbf{k})\end{aligned}$$

it follows that

$$\begin{aligned}\mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v) &= \frac{kq}{a^3}[(a \sin u \cos v\mathbf{i} + a \sin u \sin v\mathbf{j} + a \cos u\mathbf{k}) \cdot \\ &\qquad a^2(\sin^2 u \cos v\mathbf{i} + \sin^2 u \sin v\mathbf{j} + \sin u \cos u\mathbf{k})]\\ &= kq(\sin^3 u \cos^2 v + \sin^3 u \sin^2 v + \sin u \cos^2 u)\\ &= kq \sin u.\end{aligned}$$

Finally, the flux over the sphere S is given by

$$\begin{aligned}\int_S\!\!\int \mathbf{F} \cdot \mathbf{N}\, dS &= \int_D\!\!\int (kq \sin u)\, dA\\ &= \int_0^{2\pi}\!\!\int_0^{\pi} kq \sin u\, du\, dv\\ &= 4\pi kq.\end{aligned}$$

■

The result in Example 6 shows that the flux across a sphere S in an inverse square field is independent of the radius of S. In particular, if $\mathbf{E}$ is an electric field, the result in Example 6, along with Coulomb's Law, yields one of the basic laws of electrostatics, known as **Gauss's Law:**

$$\int_S\!\!\int \mathbf{E} \cdot \mathbf{N}\, dS = 4\pi kq \qquad \text{Gauss's Law}$$

where q is a point charge located at the center of the sphere and k is the Coulomb constant. Gauss's Law is valid for more general closed surfaces that enclose the origin, and relates the flux out of the surface to the total charge q inside the surface.

This section concludes with a summary of different forms of line integrals and surface integrals.

SUMMARY OF LINE AND SURFACE INTEGRALS

Line Integrals

$$ds = \|\mathbf{r}'(t)\|\, dt$$
$$= \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}\, dt$$

$$\int_C f(x, y, z)\, ds = \int_a^b f(x(t), y(t), z(t))\, ds \qquad \text{Scalar form}$$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot \mathbf{T}\, ds$$
$$= \int_a^b \mathbf{F}(x(t), y(t), z(t)) \cdot \mathbf{r}'(t)\, dt \qquad \text{Vector form}$$

Surface Integrals $[z = g(x, y)]$

$$dS = \sqrt{1 + [g_x(x, y)]^2 + [g_y(x, y)]^2}\, dA$$

$$\int_S\!\!\int f(x, y, z)\, dS = \int_R\!\!\int f(x, y, g(x, y))\sqrt{1 + [g_x(x, y)]^2 + [g_y(x, y)]^2}\, dA \qquad \text{Scalar form}$$

$$\int_S\!\!\int \mathbf{F} \cdot \mathbf{N}\, dS = \int_R\!\!\int \mathbf{F} \cdot [-g_x(x, y)\mathbf{i} - g_y(x, y)\mathbf{j} + \mathbf{k}]\, dA \qquad \text{Vector form (upward normal)}$$

Surface Integrals (*parametric form*)

$$dS = \|\mathbf{r}_u(u, v) \times \mathbf{r}_v(u, v)\|\, dA$$

$$\int_S\!\!\int f(x, y, z)\, dS = \int_D\!\!\int f(x(u, v), y(u, v), z(u, v))\, dS \qquad \text{Scalar form}$$

$$\int_S\!\!\int \mathbf{F} \cdot \mathbf{N}\, dS = \int_D\!\!\int \mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v)\, dA \qquad \text{Vector form}$$

15.6 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–4, evaluate $\displaystyle\iint_S (x - 2y + z)\, dS$.

1. S: $z = 4 - x, \quad 0 \le x \le 4, \quad 0 \le y \le 3$
2. S: $z = 15 - 2x + 3y, \quad 0 \le x \le 2, \quad 0 \le y \le 4$
3. S: $z = 2, \quad x^2 + y^2 \le 1$
4. S: $z = \frac{2}{3}x^{3/2}, \quad 0 \le x \le 1, \quad 0 \le y \le x$

In Exercises 5 and 6, evaluate $\displaystyle\iint_S xy\, dS$.

5. S: $z = 3 - x - y$, first octant
6. S: $z = h, \quad 0 \le x \le 2, \quad 0 \le y \le \sqrt{4 - x^2}$

CAS **In Exercises 7 and 8, use a computer algebra system to evaluate**

$$\iint_S xy\, dS.$$

7. S: $z = 9 - x^2, \quad 0 \le x \le 2, \quad 0 \le y \le x$
8. S: $z = \frac{1}{2}xy, \quad 0 \le x \le 4, \quad 0 \le y \le 4$

CAS **In Exercises 9 and 10, use a computer algebra system to evaluate**

$$\iint_S (x^2 - 2xy)\, dS.$$

9. S: $z = 10 - x^2 - y^2, \quad 0 \le x \le 2, \quad 0 \le y \le 2$
10. S: $z = \cos x, \quad 0 \le x \le \dfrac{\pi}{2}, \quad 0 \le y \le \dfrac{1}{2}x$

Mass **In Exercises 11 and 12, find the mass of the surface lamina S of density ρ.**

11. S: $2x + 3y + 6z = 12$, first octant, $\rho(x, y, z) = x^2 + y^2$
12. S: $z = \sqrt{a^2 - x^2 - y^2}, \quad \rho(x, y, z) = kz$

In Exercises 13–16, evaluate $\displaystyle\iint_S f(x, y)\, dS$.

13. $f(x, y) = y + 5$
    S: $\mathbf{r}(u, v) = u\mathbf{i} + v\mathbf{j} + 2v\mathbf{k}, \quad 0 \le u \le 1, \quad 0 \le v \le 2$
14. $f(x, y) = xy$
    S: $\mathbf{r}(u, v) = 2\cos u\,\mathbf{i} + 2\sin u\,\mathbf{j} + v\mathbf{k}$
    $0 \le u \le \dfrac{\pi}{2}, \quad 0 \le v \le 1$
15. $f(x, y) = x + y$
    S: $\mathbf{r}(u, v) = 2\cos u\,\mathbf{i} + 2\sin u\,\mathbf{j} + v\mathbf{k}$
    $0 \le u \le \dfrac{\pi}{2}, \quad 0 \le v \le 1$
16. $f(x, y) = x + y$
    S: $\mathbf{r}(u, v) = 4u\cos v\,\mathbf{i} + 4u\sin v\,\mathbf{j} + 3u\mathbf{k}$
    $0 \le u \le 4, \quad 0 \le v \le \pi$

In Exercises 17–22, evaluate $\displaystyle\iint_S f(x, y, z)\, dS$.

17. $f(x, y, z) = x^2 + y^2 + z^2$
    S: $z = x + y, \quad x^2 + y^2 \le 1$
18. $f(x, y, z) = \dfrac{xy}{z}$
    S: $z = x^2 + y^2, \quad 4 \le x^2 + y^2 \le 16$
19. $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$
    S: $z = \sqrt{x^2 + y^2}, \quad x^2 + y^2 \le 4$
20. $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$
    S: $z = \sqrt{x^2 + y^2}, \quad (x - 1)^2 + y^2 \le 1$
21. $f(x, y, z) = x^2 + y^2 + z^2$
    S: $x^2 + y^2 = 9, \quad 0 \le x \le 3, \quad 0 \le y \le 3, \quad 0 \le z \le 9$
22. $f(x, y, z) = x^2 + y^2 + z^2$
    S: $x^2 + y^2 = 9, \quad 0 \le x \le 3, \quad 0 \le z \le x$

In Exercises 23–28, find the flux of F through S,

$$\iint_S \mathbf{F} \cdot \mathbf{N}\, dS$$

where N is the upward unit normal vector to S.

23. $\mathbf{F}(x, y, z) = 3z\mathbf{i} - 4\mathbf{j} + y\mathbf{k}$
    S: $z = 1 - x - y$, first octant
24. $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j}$
    S: $z = 6 - 3x - 2y$, first octant
25. $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$
    S: $z = 1 - x^2 - y^2, \quad z \ge 0$
26. $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$
    S: $x^2 + y^2 + z^2 = 36$, first octant
27. $\mathbf{F}(x, y, z) = 4\mathbf{i} - 3\mathbf{j} + 5\mathbf{k}$
    S: $z = x^2 + y^2, \quad x^2 + y^2 \le 4$
28. $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} - 2z\mathbf{k}$
    S: $z = \sqrt{a^2 - x^2 - y^2}$

In Exercises 29 and 30, find the flux of F over the closed surface. (Let N be the outward unit normal vector of the surface.)

29. $\mathbf{F}(x, y, z) = (x + y)\mathbf{i} + y\mathbf{j} + z\mathbf{k}$
    S: $z = 16 - x^2 - y^2, \quad z = 0$
30. $\mathbf{F}(x, y, z) = 4xy\mathbf{i} + z^2\mathbf{j} + yz\mathbf{k}$
    S: unit cube bounded by $x = 0, x = 1, y = 0, y = 1, z = 0, z = 1$

31. ***Electrical Charge*** Let $\mathbf{E} = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$ be an electrostatic field. Use Gauss's Law to find the total charge enclosed by the closed surface consisting of the hemisphere $z = \sqrt{1 - x^2 - y^2}$ and its circular base in the xy-plane.

32. *Electrical Charge* Let $\mathbf{E} = x\mathbf{i} + y\mathbf{j} + 2z\mathbf{k}$ be an electrostatic field. Use Gauss's Law to find the total charge enclosed by the closed surface consisting of the hemisphere $z = \sqrt{1 - x^2 - y^2}$ and its circular base in the xy-plane.

Moment of Inertia **In Exercises 33 and 34, use the following formulas for the moments of inertia about the coordinate axes of a surface lamina of density ρ.**

$$I_x = \int\int_S (y^2 + z^2)\rho(x, y, z)\, dS$$

$$I_y = \int\int_S (x^2 + z^2)\rho(x, y, z)\, dS$$

$$I_z = \int\int_S (x^2 + y^2)\rho(x, y, z)\, dS$$

33. Verify that the moment of inertia of a conical shell of uniform density about its axis is $\frac{1}{2}ma^2$, where m is the mass and a is the radius and height.

34. Verify that the moment of inertia of a spherical shell of uniform density about its diameter is $\frac{2}{3}ma^2$, where m is the mass and a is the radius.

Moment of Inertia **In Exercises 35 and 36, find I_z for the given lamina with uniform density of 1. Use a computer algebra system to verify your results.**

35. $x^2 + y^2 = a^2, \quad 0 \le z \le h$

36. $z = x^2 + y^2, \quad 0 \le z \le h$

CAS ***Flow Rate*** **In Exercises 37 and 38, use a computer algebra system to find the rate of mass flow of a fluid of density ρ through the surface S oriented upward if the velocity field is given by $\mathbf{F}(x, y, z) = 0.5z\mathbf{k}$.**

37. $S\colon\ z = 16 - x^2 - y^2, \quad z \ge 0$

38. $S\colon\ z = \sqrt{16 - x^2 - y^2}$

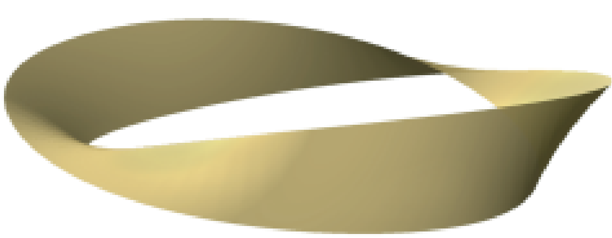
WRITING ABOUT CONCEPTS

39. Define a surface integral of the scalar function f over a surface $z = g(x, y)$. Explain how to evaluate the surface integral.

40. Describe an orientable surface.

41. Define a flux integral and explain how it is evaluated.

42. Is the surface shown in the figure orientable? Explain.



Double twist

CAS **43.** *Investigation*

(a) Use a computer algebra system to graph the vector-valued function

$\mathbf{r}(u, v) = (4 - v\sin u)\cos(2u)\mathbf{i} + (4 - v\sin u)\sin(2u)\mathbf{j} + v\cos u\mathbf{k}, \quad 0 \le u \le \pi, \quad -1 \le v \le 1.$

This surface is called a Möbius strip.

(b) Explain why this surface is not orientable.

(c) Use a computer algebra system to graph the space curve represented by $\mathbf{r}(u, 0)$. Identify the curve.

(d) Construct a Möbius strip by cutting a strip of paper, making a single twist, and pasting the ends together.

(e) Cut the Möbius strip along the space curve graphed in part (c), and describe the result.

CAPSTONE

44. Consider the vector field

$\mathbf{F}(x, y, z) = z\mathbf{i} + x\mathbf{j} + y\mathbf{k}$

and the orientable surface S given in parametric form by

$\mathbf{r}(u, v) = (u + v^2)\mathbf{i} + (u - v)\mathbf{j} + u^2\mathbf{k},$

$0 \le u \le 2, -1 \le v \le 1.$

(a) Find and interpret $\mathbf{r}_u \times \mathbf{r}_v$.

(b) Find $\mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v)$ as a function of u and v.

(c) Find u and v at the point $P(3, 1, 4)$.

(d) Explain how to find the normal component of $\mathbf{F}$ to the surface at P. Then find this value.

(e) Evaluate the flux integral $\int\int_S \mathbf{F} \cdot \mathbf{N}\, dS$.

SECTION PROJECT

Hyperboloid of One Sheet

Consider the parametric surface given by the function

$\mathbf{r}(u, v) = a\cosh u\cos v\mathbf{i} + a\cosh u\sin v\mathbf{j} + b\sinh u\mathbf{k}.$

(a) Use a graphing utility to graph $\mathbf{r}$ for various values of the constants a and b. Describe the effect of the constants on the shape of the surface.

(b) Show that the surface is a hyperboloid of one sheet given by

$$\frac{x^2}{a^2} + \frac{y^2}{a^2} - \frac{z^2}{b^2} = 1.$$

(c) For fixed values $u = u_0$, describe the curves given by

$\mathbf{r}(u_0, v) = a\cosh u_0\cos v\mathbf{i} + a\cosh u_0\sin v\mathbf{j} + b\sinh u_0\mathbf{k}.$

(d) For fixed values $v = v_0$, describe the curves given by

$\mathbf{r}(u, v_0) = a\cosh u\cos v_0\mathbf{i} + a\cosh u\sin v_0\mathbf{j} + b\sinh u\mathbf{k}.$

(e) Find a normal vector to the surface at $(u, v) = (0, 0)$.

15.7 Divergence Theorem

- **Understand and use the Divergence Theorem.**
- **Use the Divergence Theorem to calculate flux.**

CARL FRIEDRICH GAUSS (1777–1855)

The **Divergence Theorem** is also called **Gauss's Theorem,** after the famous German mathematician Carl Friedrich Gauss. Gauss is recognized, with Newton and Archimedes, as one of the three greatest mathematicians in history. One of his many contributions to mathematics was made at the age of 22, when, as part of his doctoral dissertation, he proved the *Fundamental Theorem of Algebra*.

Divergence Theorem

Recall from Section 15.4 that an alternative form of Green's Theorem is

$$\int_C \mathbf{F} \cdot \mathbf{N}\, ds = \int\!\!\int_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dA$$
$$= \int\!\!\int_R \operatorname{div} \mathbf{F}\, dA.$$

In an analogous way, the **Divergence Theorem** gives the relationship between a triple integral over a solid region Q and a surface integral over the surface of Q. In the statement of the theorem, the surface S is **closed** in the sense that it forms the complete boundary of the solid Q. Regions bounded by spheres, ellipsoids, cubes, tetrahedrons, or combinations of these surfaces are typical examples of closed surfaces. Assume that Q is a solid region on which a triple integral can be evaluated, and that the closed surface S is oriented by *outward* unit normal vectors, as shown in Figure 15.54. With these restrictions on S and Q, the Divergence Theorem is as follows.

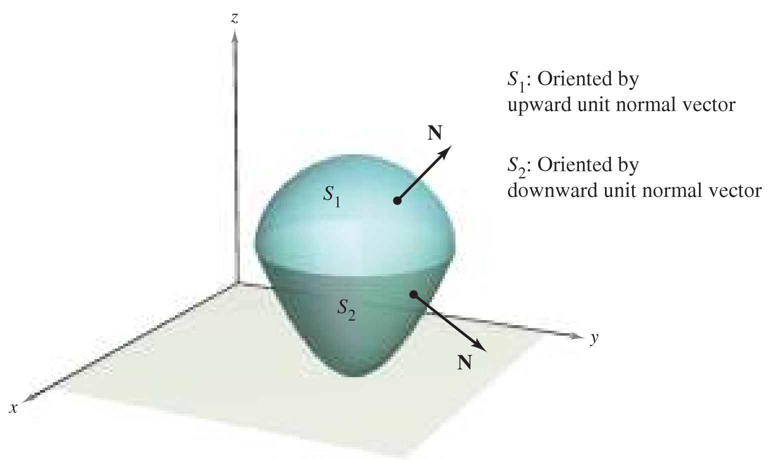


Figure 15.54

THEOREM 15.12 THE DIVERGENCE THEOREM

Let Q be a solid region bounded by a closed surface S oriented by a unit normal vector directed outward from Q. If $\mathbf{F}$ is a vector field whose component functions have continuous first partial derivatives in Q, then

$$\int\!\!\int_S \mathbf{F} \cdot \mathbf{N}\, dS = \int\!\!\int\!\!\int_Q \operatorname{div} \mathbf{F}\, dV.$$

NOTE As noted at the left above, the Divergence Theorem is sometimes called Gauss's Theorem. It is also sometimes called Ostrogradsky's Theorem, after the Russian mathematician Michel Ostrogradsky (1801–1861). ■

NOTE This proof is restricted to a *simple* solid region. The general proof is best left to a course in advanced calculus.

PROOF If you let $\mathbf{F}(x, y, z) = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$, the theorem takes the form

$$\int_S\!\!\int \mathbf{F} \cdot \mathbf{N}\, dS = \int_S\!\!\int (M\mathbf{i} \cdot \mathbf{N} + N\mathbf{j} \cdot \mathbf{N} + P\mathbf{k} \cdot \mathbf{N})\, dS$$
$$= \iiint_Q \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}\right) dV.$$

You can prove this by verifying that the following three equations are valid.

$$\int_S\!\!\int M\mathbf{i} \cdot \mathbf{N}\, dS = \iiint_Q \frac{\partial M}{\partial x}\, dV$$

$$\int_S\!\!\int N\mathbf{j} \cdot \mathbf{N}\, dS = \iiint_Q \frac{\partial N}{\partial y}\, dV$$

$$\int_S\!\!\int P\mathbf{k} \cdot \mathbf{N}\, dS = \iiint_Q \frac{\partial P}{\partial z}\, dV$$

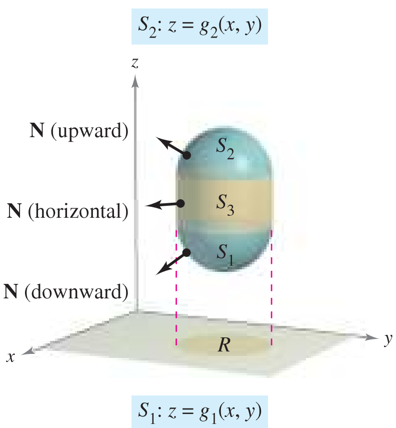


Figure 15.55

Because the verifications of the three equations are similar, only the third is discussed. Restrict the proof to a **simple solid** region with upper surface

$z = g_2(x, y)$ Upper surface

and lower surface

$z = g_1(x, y)$ Lower surface

whose projections onto the xy-plane coincide and form region R. If Q has a lateral surface like S_3 in Figure 15.55, then a normal vector is horizontal, which implies that $P\mathbf{k} \cdot \mathbf{N} = 0$. Consequently, you have

$$\int_S\!\!\int P\mathbf{k} \cdot \mathbf{N}\, dS = \int_{S_1}\!\!\int P\mathbf{k} \cdot \mathbf{N}\, dS + \int_{S_2}\!\!\int P\mathbf{k} \cdot \mathbf{N}\, dS + 0.$$

On the upper surface S_2, the outward normal vector is upward, whereas on the lower surface S_1, the outward normal vector is downward. So, by Theorem 15.11, you have the following.

$$\int_{S_1}\!\!\int P\mathbf{k} \cdot \mathbf{N}\, dS = \int_R\!\!\int P(x, y, g_1(x, y))\mathbf{k} \cdot \left(\frac{\partial g_1}{\partial x}\mathbf{i} + \frac{\partial g_1}{\partial y}\mathbf{j} - \mathbf{k}\right) dA$$
$$= -\int_R\!\!\int P(x, y, g_1(x, y))\, dA$$

$$\int_{S_2}\!\!\int P\mathbf{k} \cdot \mathbf{N}\, dS = \int_R\!\!\int P(x, y, g_2(x, y))\mathbf{k} \cdot \left(-\frac{\partial g_2}{\partial x}\mathbf{i} - \frac{\partial g_2}{\partial y}\mathbf{j} + \mathbf{k}\right) dA$$
$$= \int_R\!\!\int P(x, y, g_2(x, y))\, dA$$

Adding these results, you obtain

$$\int_S\!\!\int P\mathbf{k} \cdot \mathbf{N}\, dS = \int_R\!\!\int [P(x, y, g_2(x, y)) - P(x, y, g_1(x, y))]\, dA$$
$$= \int_R\!\!\int \left[\int_{g_1(x, y)}^{g_2(x, y)} \frac{\partial P}{\partial z}\, dz\right] dA$$
$$= \iiint_Q \frac{\partial P}{\partial z}\, dV.$$

EXAMPLE 1 Using the Divergence Theorem

Let Q be the solid region bounded by the coordinate planes and the plane $2x + 2y + z = 6$, and let $\mathbf{F} = x\mathbf{i} + y^2\mathbf{j} + z\mathbf{k}$. Find

$$\int_S\int \mathbf{F} \cdot \mathbf{N}\, dS$$

where S is the surface of Q.

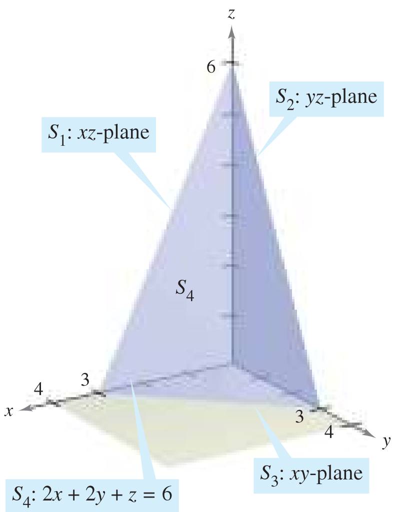


Figure 15.56

Solution From Figure 15.56, you can see that Q is bounded by four subsurfaces. So, you would need four *surface integrals* to evaluate

$$\int_S\int \mathbf{F} \cdot \mathbf{N}\, dS.$$

However, by the Divergence Theorem, you need only one triple integral. Because

$$\begin{aligned} \operatorname{div} \mathbf{F} &= \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z} \\ &= 1 + 2y + 1 \\ &= 2 + 2y \end{aligned}$$

you have

$$\begin{aligned} \int_S\int \mathbf{F} \cdot \mathbf{N}\, dS &= \iiint_Q \operatorname{div} \mathbf{F}\, dV \\ &= \int_0^3 \int_0^{3-y} \int_0^{6-2x-2y} (2 + 2y)\, dz\, dx\, dy \\ &= \int_0^3 \int_0^{3-y} (2z + 2yz)\Big]_0^{6-2x-2y} dx\, dy \\ &= \int_0^3 \int_0^{3-y} (12 - 4x + 8y - 4xy - 4y^2)\, dx\, dy \\ &= \int_0^3 \left[12x - 2x^2 + 8xy - 2x^2y - 4xy^2\right]_0^{3-y} dy \\ &= \int_0^3 (18 + 6y - 10y^2 + 2y^3)\, dy \\ &= \left[18y + 3y^2 - \frac{10y^3}{3} + \frac{y^4}{2}\right]_0^3 \\ &= \frac{63}{2}. \end{aligned}$$

TECHNOLOGY If you have access to a computer algebra system that can evaluate triple-iterated integrals, use it to verify the result in Example 1. When you are using such a utility, note that the first step is to convert the triple integral to an iterated integral—this step must be done by hand. To give yourself some practice with this important step, find the limits of integration for the following iterated integrals. Then use a computer to verify that the value is the same as that obtained in Example 1.

$$\int_?^? \int_?^? \int_?^? (2 + 2y)\, dy\, dz\, dx, \qquad \int_?^? \int_?^? \int_?^? (2 + 2y)\, dx\, dy\, dz$$

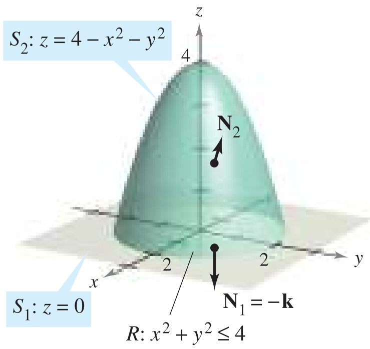


Figure 15.57

EXAMPLE 2 Verifying the Divergence Theorem

Let Q be the solid region between the paraboloid

$$z = 4 - x^2 - y^2$$

and the xy-plane. Verify the Divergence Theorem for

$$\mathbf{F}(x, y, z) = 2z\mathbf{i} + x\mathbf{j} + y^2\mathbf{k}.$$

Solution From Figure 15.57 you can see that the outward normal vector for the surface S_1 is $\mathbf{N}_1 = -\mathbf{k}$, whereas the outward normal vector for the surface S_2 is

$$\mathbf{N}_2 = \frac{2x\mathbf{i} + 2y\mathbf{j} + \mathbf{k}}{\sqrt{4x^2 + 4y^2 + 1}}.$$

So, by Theorem 15.11, you have

$$\begin{aligned}
\int_S\!\!\int \mathbf{F}\cdot\mathbf{N}\,dS
&= \int_{S_1}\!\!\int \mathbf{F}\cdot\mathbf{N}_1\,dS + \int_{S_2}\!\!\int \mathbf{F}\cdot\mathbf{N}_2\,dS\\
&= \int_{S_1}\!\!\int \mathbf{F}\cdot(-\mathbf{k})\,dS + \int_{S_2}\!\!\int \mathbf{F}\cdot\frac{(2x\mathbf{i} + 2y\mathbf{j} + \mathbf{k})}{\sqrt{4x^2 + 4y^2 + 1}}\,dS\\
&= \int_R\!\!\int -y^2\,dA + \int_R\!\!\int (4xz + 2xy + y^2)\,dA\\
&= -\int_{-2}^{2}\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} y^2\,dx\,dy + \int_{-2}^{2}\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} (4xz + 2xy + y^2)\,dx\,dy\\
&= \int_{-2}^{2}\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} (4xz + 2xy)\,dx\,dy\\
&= \int_{-2}^{2}\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} [4x(4 - x^2 - y^2) + 2xy]\,dx\,dy\\
&= \int_{-2}^{2}\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} (16x - 4x^3 - 4xy^2 + 2xy)\,dx\,dy\\
&= \int_{-2}^{2}\Big[8x^2 - x^4 - 2x^2y^2 + x^2y\Big]_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}}\,dy\\
&= \int_{-2}^{2} 0\,dy\\
&= 0.
\end{aligned}$$

On the other hand, because

$$\text{div}\,\mathbf{F} = \frac{\partial}{\partial x}[2z] + \frac{\partial}{\partial y}[x] + \frac{\partial}{\partial z}[y^2] = 0 + 0 + 0 = 0$$

you can apply the Divergence Theorem to obtain the equivalent result

$$\begin{aligned}
\int_S\!\!\int \mathbf{F}\cdot\mathbf{N}\,dS &= \iiint_Q \text{div}\,\mathbf{F}\,dV\\
&= \iiint_Q 0\,dV = 0.
\end{aligned}$$

■

EXAMPLE 3 Using the Divergence Theorem

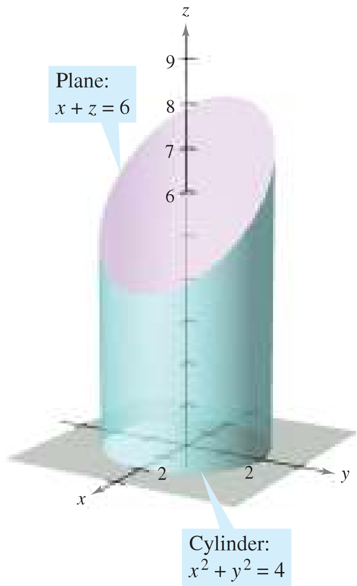


Figure 15.58

Let Q be the solid bounded by the cylinder $x^2 + y^2 = 4$, the plane $x + z = 6$, and the xy-plane, as shown in Figure 15.58. Find

$$\int_S\int \mathbf{F} \cdot \mathbf{N}\, dS$$

where S is the surface of Q and

$$\mathbf{F}(x, y, z) = (x^2 + \sin z)\mathbf{i} + (xy + \cos z)\mathbf{j} + e^y\mathbf{k}.$$

Solution Direct evaluation of this surface integral would be difficult. However, by the Divergence Theorem, you can evaluate the integral as follows.

$$\begin{aligned}
\int_S\int \mathbf{F} \cdot \mathbf{N}\, dS &= \iiint_Q \operatorname{div} \mathbf{F}\, dV \\
&= \iiint_Q (2x + x + 0)\, dV \\
&= \iiint_Q 3x\, dV \\
&= \int_0^{2\pi}\int_0^2\int_0^{6 - r\cos\theta} (3r\cos\theta)r\, dz\, dr\, d\theta \\
&= \int_0^{2\pi}\int_0^2 (18r^2\cos\theta - 3r^3\cos^2\theta)\, dr\, d\theta \\
&= \int_0^{2\pi} (48\cos\theta - 12\cos^2\theta)d\theta \\
&= \left[48\sin\theta - 6\left(\theta + \frac{1}{2}\sin 2\theta\right)\right]_0^{2\pi} \\
&= -12\pi
\end{aligned}$$

Notice that cylindrical coordinates with $x = r\cos\theta$ and $dV = r\, dz\, dr\, d\theta$ were used to evaluate the triple integral. ■

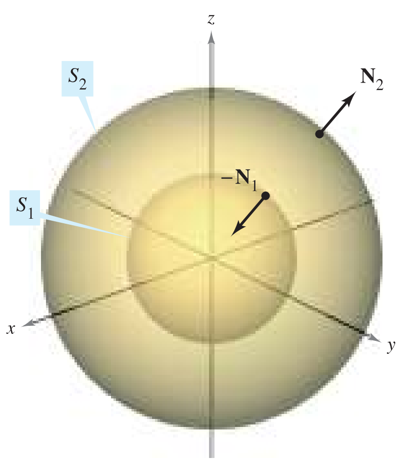


Figure 15.59

Even though the Divergence Theorem was stated for a simple solid region Q bounded by a closed surface, the theorem is also valid for regions that are the finite unions of simple solid regions. For example, let Q be the solid bounded by the closed surfaces S_1 and S_2, as shown in Figure 15.59. To apply the Divergence Theorem to this solid, let $S = S_1 \cup S_2$. The normal vector $\mathbf{N}$ to S is given by $-\mathbf{N}_1$ on S_1 and by $\mathbf{N}_2$ on S_2. So, you can write

$$\begin{aligned}
\iiint_Q \operatorname{div} \mathbf{F}\, dV &= \int_S\int \mathbf{F} \cdot \mathbf{N}\, dS \\
&= \int_{S_1}\int \mathbf{F} \cdot (-\mathbf{N}_1)\, dS + \int_{S_2}\int \mathbf{F} \cdot \mathbf{N}_2\, dS \\
&= -\int_{S_1}\int \mathbf{F} \cdot \mathbf{N}_1\, dS + \int_{S_2}\int \mathbf{F} \cdot \mathbf{N}_2\, dS.
\end{aligned}$$

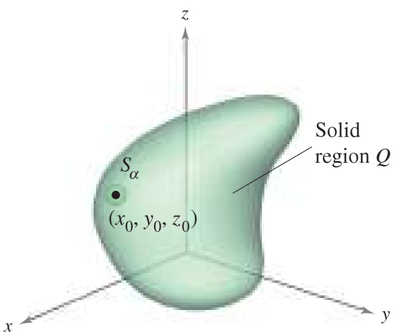


Figure 15.60

Flux and the Divergence Theorem

To help understand the Divergence Theorem, consider the two sides of the equation

$$\int_S\!\!\int \mathbf{F} \cdot \mathbf{N}\, dS = \iiint_Q \operatorname{div} \mathbf{F}\, dV.$$

You know from Section 15.6 that the flux integral on the left determines the total fluid flow across the surface S per unit of time. This can be approximated by summing the fluid flow across small patches of the surface. The triple integral on the right measures this same fluid flow across S, but from a very different perspective—namely, by calculating the flow of fluid into (or out of) small *cubes* of volume ΔV_i. The flux of the ith cube is approximately

$$\text{Flux of } i\text{th cube} \approx \operatorname{div} \mathbf{F}(x_i, y_i, z_i)\, \Delta V_i$$

for some point (x_i, y_i, z_i) in the ith cube. Note that for a cube in the interior of Q, the gain (or loss) of fluid through any one of its six sides is offset by a corresponding loss (or gain) through one of the sides of an adjacent cube. After summing over all the cubes in Q, the only fluid flow that is not canceled by adjoining cubes is that on the outside edges of the cubes on the boundary. So, the sum

$$\sum_{i=1}^{n} \operatorname{div} \mathbf{F}(x_i, y_i, z_i)\, \Delta V_i$$

approximates the total flux into (or out of) Q, and therefore through the surface S.

To see what is meant by the divergence of $\mathbf{F}$ at a point, consider ΔV_α to be the volume of a small sphere S_α of radius α and center (x_0, y_0, z_0), contained in region Q, as shown in Figure 15.60. Applying the Divergence Theorem to S_α produces

$$\begin{aligned}\text{Flux of } \mathbf{F} \text{ across } S_\alpha &= \iiint_{Q_\alpha} \operatorname{div} \mathbf{F}\, dV \\ &\approx \operatorname{div} \mathbf{F}(x_0, y_0, z_0,)\, \Delta V_\alpha\end{aligned}$$

where Q_α is the interior of S_α. Consequently, you have

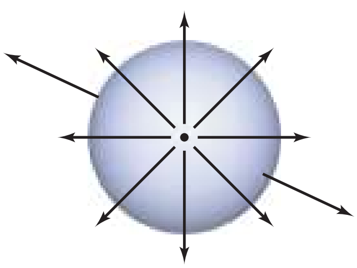
$$\operatorname{div} \mathbf{F}(x_0, y_0, z_0) \approx \frac{\text{flux of } \mathbf{F} \text{ across } S_\alpha}{\Delta V_\alpha}$$

and, by taking the limit as $\alpha \to 0$, you obtain the divergence of $\mathbf{F}$ at the point (x_0, y_0, z_0).

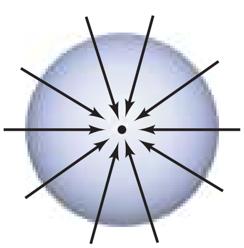
$$\begin{aligned}\operatorname{div} \mathbf{F}(x_0, y_0, z_0) &= \lim_{\alpha \to 0} \frac{\text{flux of } \mathbf{F} \text{ across } S_\alpha}{\Delta V_\alpha} \\ &= \text{flux per unit volume at } (x_0, y_0, z_0)\end{aligned}$$

The point (x_0, y_0, z_0) in a vector field is classified as a source, a sink, or incompressible, as follows.

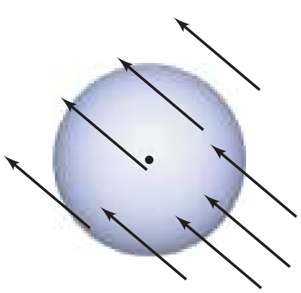
1. **Source,** if $\operatorname{div} \mathbf{F} > 0$ See Figure 15.61(a).
2. **Sink,** if $\operatorname{div} \mathbf{F} < 0$ See Figure 15.61(b).
3. **Incompressible,** if $\operatorname{div} \mathbf{F} = 0$ See Figure 15.61(c).



(a) Source: $\operatorname{div} \mathbf{F} > 0$



(b) Sink: $\operatorname{div} \mathbf{F} < 0$



(c) Incompressible: $\operatorname{div} \mathbf{F} = 0$

Figure 15.61

NOTE In hydrodynamics, a *source* is a point at which additional fluid is considered as being introduced to the region occupied by the fluid. A *sink* is a point at which fluid is considered as being removed. ■

EXAMPLE 4 Calculating Flux by the Divergence Theorem

Let Q be the region bounded by the sphere $x^2 + y^2 + z^2 = 4$. Find the outward flux of the vector field $\mathbf{F}(x, y, z) = 2x^3\mathbf{i} + 2y^3\mathbf{j} + 2z^3\mathbf{k}$ through the sphere.

Solution By the Divergence Theorem, you have

$$\begin{aligned}
\text{Flux across } S &= \int_S\!\!\int \mathbf{F}\cdot\mathbf{N}\,dS = \iiint_Q \operatorname{div}\mathbf{F}\,dV \\
&= \iiint_Q 6(x^2+y^2+z^2)\,dV \\
&= 6\int_0^2\int_0^\pi\int_0^{2\pi} \rho^4 \sin\phi\,d\theta\,d\phi\,d\rho \qquad \text{Spherical coordinates} \\
&= 6\int_0^2\int_0^\pi 2\pi\rho^4\sin\phi\,d\phi\,d\rho \\
&= 12\pi\int_0^2 2\rho^4\,d\rho \\
&= 24\pi\left(\frac{32}{5}\right) \\
&= \frac{768\pi}{5}.
\end{aligned}$$

15.7 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–6, verify the Divergence Theorem by evaluating

$$\int_S\!\!\int \mathbf{F}\cdot\mathbf{N}\,dS$$

as a surface integral and as a triple integral.

1. $\mathbf{F}(x, y, z) = 2x\mathbf{i} - 2y\mathbf{j} + z^2\mathbf{k}$

S: cube bounded by the planes $x = 0$, $x = a$, $y = 0$, $y = a$, $z = 0$, $z = a$

2. $\mathbf{F}(x, y, z) = 2x\mathbf{i} - 2y\mathbf{j} + z^2\mathbf{k}$

S: cylinder $x^2 + y^2 = 4$, $0 \le z \le h$

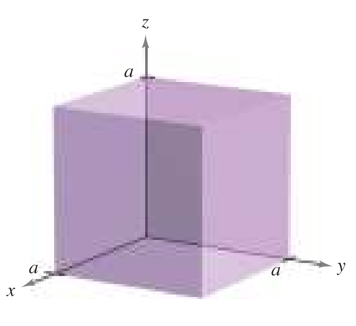


Figure for 1

Figure for 2

3. $\mathbf{F}(x, y, z) = (2x - y)\mathbf{i} - (2y - z)\mathbf{j} + z\mathbf{k}$

S: surface bounded by the plane $2x + 4y + 2z = 12$ and the coordinate planes

4. $\mathbf{F}(x, y, z) = xy\mathbf{i} + z\mathbf{j} + (x + y)\mathbf{k}$

S: surface bounded by the planes $y = 4$ and $z = 4 - x$ and the coordinate planes

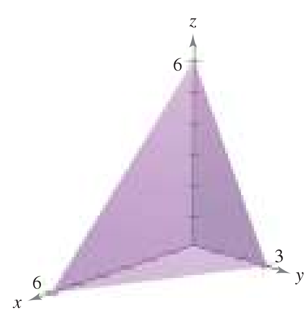


Figure for 3

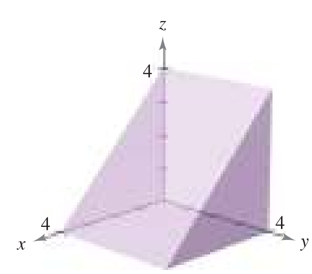


Figure for 4

5. $\mathbf{F}(x, y, z) = xz\mathbf{i} + zy\mathbf{j} + 2z^2\mathbf{k}$

S: surface bounded by $z = 1 - x^2 - y^2$ and $z = 0$

6. $\mathbf{F}(x, y, z) = xy^2\mathbf{i} + yx^2\mathbf{j} + e\mathbf{k}$

S: surface bounded by $z = \sqrt{x^2 + y^2}$ and $z = 4$

In Exercises 7–18, use the Divergence Theorem to evaluate

$$\int_S\!\!\int \mathbf{F}\cdot\mathbf{N}\,dS$$

and find the outward flux of F through the surface of the solid bounded by the graphs of the equations. Use a computer algebra system to verify your results.

7. $\mathbf{F}(x, y, z) = x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}$

S: $x = 0, x = a, y = 0, y = a, z = 0, z = a$

8. $\mathbf{F}(x, y, z) = x^2z^2\mathbf{i} - 2y\mathbf{j} + 3xyz\mathbf{k}$

S: $x = 0, x = a, y = 0, y = a, z = 0, z = a$

9. $\mathbf{F}(x, y, z) = x^2\mathbf{i} - 2xy\mathbf{j} + xyz^2\mathbf{k}$

S: $z = \sqrt{a^2 - x^2 - y^2}, z = 0$

10. $\mathbf{F}(x, y, z) = xy\mathbf{i} + yz\mathbf{j} - yz\mathbf{k}$

S: $z = \sqrt{a^2 - x^2 - y^2}, z = 0$

11. $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

S: $x^2 + y^2 + z^2 = 9$

12. $\mathbf{F}(x, y, z) = xyz\mathbf{j}$

S: $x^2 + y^2 = 4, z = 0, z = 5$

13. $\mathbf{F}(x, y, z) = x\mathbf{i} + y^2\mathbf{j} - z\mathbf{k}$

S: $x^2 + y^2 = 25, z = 0, z = 7$

14. $\mathbf{F}(x, y, z) = (xy^2 + \cos z)\mathbf{i} + (x^2y + \sin z)\mathbf{j} + e^z\mathbf{k}$

S: $z = \frac{1}{2}\sqrt{x^2 + y^2}, z = 8$

15. $\mathbf{F}(x, y, z) = x^3\mathbf{i} + x^2y\mathbf{j} + x^2e^y\mathbf{k}$

S: $z = 4 - y, z = 0, x = 0, x = 6, y = 0$

16. $\mathbf{F}(x, y, z) = xe^z\mathbf{i} + ye^z\mathbf{j} + e^z\mathbf{k}$

S: $z = 4 - y, z = 0, x = 0, x = 6, y = 0$

17. $\mathbf{F}(x, y, z) = xy\mathbf{i} + 4y\mathbf{j} + xz\mathbf{k}$

S: $x^2 + y^2 + z^2 = 16$

18. $\mathbf{F}(x, y, z) = 2(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$

S: $z = \sqrt{4 - x^2 - y^2}, z = 0$

In Exercises 19 and 20, evaluate

$$\int_S\!\!\int \operatorname{curl}\mathbf{F}\cdot\mathbf{N}\,dS$$

where S is the closed surface of the solid bounded by the graphs of $x = 4$ and $z = 9 - y^2$, and the coordinate planes.

19. $\mathbf{F}(x, y, z) = (4xy + z^2)\mathbf{i} + (2x^2 + 6yz)\mathbf{j} + 2xz\mathbf{k}$

20. $\mathbf{F}(x, y, z) = xy\cos z\,\mathbf{i} + yz\sin x\,\mathbf{j} + xyz\mathbf{k}$

WRITING ABOUT CONCEPTS

21. State the Divergence Theorem.

22. How do you determine if a point (x_0, y_0, z_0) in a vector field is a source, a sink, or incompressible?

23. (a) Use the Divergence Theorem to verify that the volume of the solid bounded by a surface S is

$$\int_S\!\!\int x\,dy\,dz = \int_S\!\!\int y\,dz\,dx = \int_S\!\!\int z\,dx\,dy.$$

(b) Verify the result of part (a) for the cube bounded by $x = 0$, $x = a$, $y = 0$, $y = a$, $z = 0$, and $z = a$.

CAPSTONE

24. Let $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and let S be the cube bounded by the planes $x = 0$, $x = 1$, $y = 0$, $y = 1$, $z = 0$, and $z = 1$. Verify the Divergence Theorem by evaluating

$$\int_S\!\!\int \mathbf{F}\cdot\mathbf{N}\,dS$$

as a surface integral and as a triple integral.

25. Verify that

$$\int_S\!\!\int \operatorname{curl}\mathbf{F}\cdot\mathbf{N}\,dS = 0$$

for any closed surface S.

26. For the constant vector field $\mathbf{F}(x, y, z) = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$, verify that

$$\int_S\!\!\int \mathbf{F}\cdot\mathbf{N}\,dS = 0$$

where V is the volume of the solid bounded by the closed surface S.

27. Given the vector field $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, verify that

$$\int_S\!\!\int \mathbf{F}\cdot\mathbf{N}\,dS = 3V$$

where V is the volume of the solid bounded by the closed surface S.

28. Given the vector field $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, verify that

$$\frac{1}{\|\mathbf{F}\|}\int_S\!\!\int \mathbf{F}\cdot\mathbf{N}\,dS = \frac{3}{\|\mathbf{F}\|}\iiint_Q dV.$$

In Exercises 29 and 30, prove the identity, assuming that Q, S, and N meet the conditions of the Divergence Theorem and that the required partial derivatives of the scalar functions f and g are continuous. The expressions $D_{\mathbf{N}}f$ and $D_{\mathbf{N}}g$ are the derivatives in the direction of the vector N and are defined by

$$D_{\mathbf{N}}f = \nabla f\cdot\mathbf{N},\quad D_{\mathbf{N}}g = \nabla g\cdot\mathbf{N}.$$

29. $\displaystyle\iiint_Q (f\nabla^2 g + \nabla f\cdot\nabla g)\,dV = \int_S\!\!\int f D_{\mathbf{N}}g\,dS$

[*Hint:* Use $\operatorname{div}(f\mathbf{G}) = f\operatorname{div}\mathbf{G} + \nabla f\cdot\mathbf{G}$.]

30. $\displaystyle\iiint_Q (f\nabla^2 g - g\nabla^2 f)\,dV = \int_S\!\!\int (f D_{\mathbf{N}}g - g D_{\mathbf{N}}f)\,dS$

(*Hint:* Use Exercise 29 twice.)

15.8 Stokes's Theorem

- **Understand and use Stokes's Theorem.**
- **Use curl to analyze the motion of a rotating liquid.**

Bettmann/Corbis

GEORGE GABRIEL STOKES (1819–1903)

Stokes became a Lucasian professor of mathematics at Cambridge in 1849. Five years later, he published the theorem that bears his name as a prize examination question there.

Stokes's Theorem

A second higher-dimension analog of Green's Theorem is called **Stokes's Theorem,** after the English mathematical physicist George Gabriel Stokes. Stokes was part of a group of English mathematical physicists referred to as the Cambridge School, which included William Thomson (Lord Kelvin) and James Clerk Maxwell. In addition to making contributions to physics, Stokes worked with infinite series and differential equations, as well as with the integration results presented in this section.

Stokes's Theorem gives the relationship between a surface integral over an oriented surface S and a line integral along a closed space curve C forming the boundary of S, as shown in Figure 15.62. The positive direction along C is counterclockwise relative to the normal vector $\mathbf{N}$. That is, if you imagine grasping the normal vector $\mathbf{N}$ with your right hand, with your thumb pointing in the direction of $\mathbf{N}$, your fingers will point in the positive direction C, as shown in Figure 15.63.

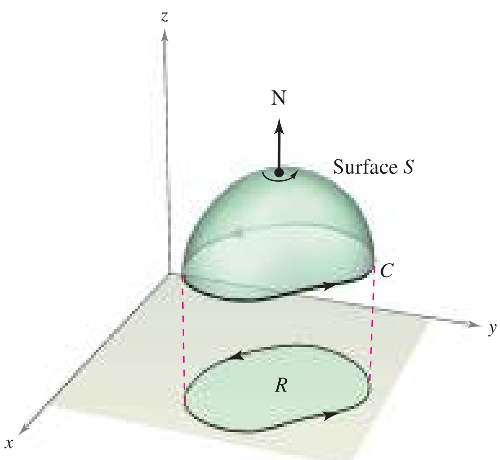
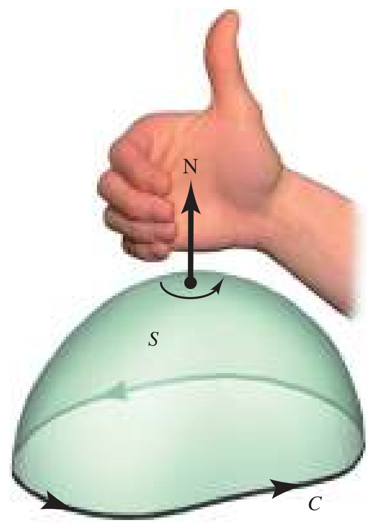


Figure 15.62



Direction along C is counterclockwise relative to $\mathbf{N}$.
Figure 15.63

THEOREM 15.13 STOKES'S THEOREM

Let S be an oriented surface with unit normal vector $\mathbf{N}$, bounded by a piecewise smooth simple closed curve C with a positive orientation. If $\mathbf{F}$ is a vector field whose component functions have continuous first partial derivatives on an open region containing S and C, then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_S\int (\text{curl } \mathbf{F}) \cdot \mathbf{N}\, dS.$$

NOTE The line integral may be written in the differential form $\int_C M\,dx + N\,dy + P\,dz$ or in the vector form $\int_C \mathbf{F} \cdot \mathbf{T}\,ds$.

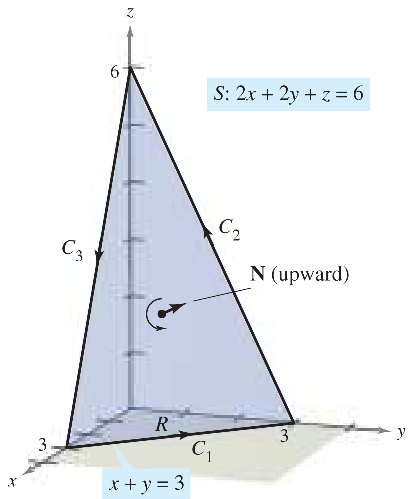


Figure 15.64

EXAMPLE 1 Using Stokes's Theorem

Let C be the oriented triangle lying in the plane $2x + 2y + z = 6$, as shown in Figure 15.64. Evaluate

$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

where $\mathbf{F}(x, y, z) = -y^2\mathbf{i} + z\mathbf{j} + x\mathbf{k}$.

Solution Using Stokes's Theorem, begin by finding the curl of $\mathbf{F}$.

$$\text{curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ -y^2 & z & x \end{vmatrix} = -\mathbf{i} - \mathbf{j} + 2y\mathbf{k}$$

Considering $z = 6 - 2x - 2y = g(x, y)$, you can use Theorem 15.11 for an upward normal vector to obtain

$$\begin{aligned}
\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_S\int (\text{curl } \mathbf{F}) \cdot \mathbf{N}\, dS \\
&= \int_R\int (-\mathbf{i} - \mathbf{j} + 2y\mathbf{k}) \cdot [-g_x(x, y)\mathbf{i} - g_y(x, y)\mathbf{j} + \mathbf{k}]\, dA \\
&= \int_R\int (-\mathbf{i} - \mathbf{j} + 2y\mathbf{k}) \cdot (2\mathbf{i} + 2\mathbf{j} + \mathbf{k})\, dA \\
&= \int_0^3\int_0^{3-y} (2y - 4)\, dx\, dy \\
&= \int_0^3 (-2y^2 + 10y - 12)\, dy \\
&= \left[-\frac{2y^3}{3} + 5y^2 - 12y\right]_0^3 \\
&= -9.
\end{aligned}$$

Try evaluating the line integral in Example 1 directly, *without* using Stokes's Theorem. One way to do this would be to consider C as the union of C_1, C_2, and C_3, as follows.

$$\begin{aligned}
&C_1: \mathbf{r}_1(t) = (3 - t)\mathbf{i} + t\mathbf{j}, \quad 0 \le t \le 3 \\
&C_2: \mathbf{r}_2(t) = (6 - t)\mathbf{j} + (2t - 6)\mathbf{k}, \quad 3 \le t \le 6 \\
&C_3: \mathbf{r}_3(t) = (t - 6)\mathbf{i} + (18 - 2t)\mathbf{k}, \quad 6 \le t \le 9
\end{aligned}$$

The value of the line integral is

$$\begin{aligned}
\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_{C_1} \mathbf{F} \cdot \mathbf{r}_1'(t)\, dt + \int_{C_2} \mathbf{F} \cdot \mathbf{r}_2'(t)\, dt + \int_{C_3} \mathbf{F} \cdot \mathbf{r}_3'(t)\, dt \\
&= \int_0^3 t^2\, dt + \int_3^6 (-2t + 6)\, dt + \int_6^9 (-2t + 12)\, dt \\
&= 9 - 9 - 9 \\
&= -9.
\end{aligned}$$

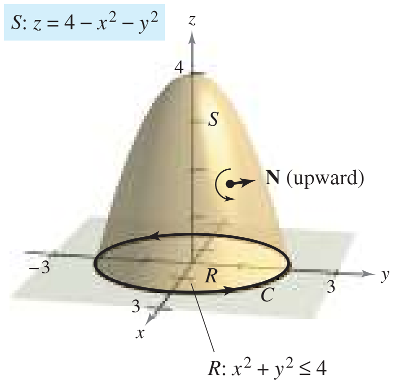


Figure 15.65

EXAMPLE 2 Verifying Stokes's Theorem

Let S be the portion of the paraboloid $z = 4 - x^2 - y^2$ lying above the xy-plane, oriented upward (see Figure 15.65). Let C be its boundary curve in the xy-plane, oriented counterclockwise. Verify Stokes's Theorem for

$$\mathbf{F}(x, y, z) = 2z\mathbf{i} + x\mathbf{j} + y^2\mathbf{k}$$

by evaluating the surface integral and the equivalent line integral.

Solution As a *surface integral*, you have $z = g(x, y) = 4 - x^2 - y^2$, $g_x = -2x$, $g_y = -2y$, and

$$\text{curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2z & x & y^2 \end{vmatrix} = 2y\mathbf{i} + 2\mathbf{j} + \mathbf{k}.$$

By Theorem 15.11, you obtain

$$\begin{aligned}
\int_S\!\!\int (\text{curl } \mathbf{F}) \cdot \mathbf{N}\, dS &= \int_R\!\!\int (2y\mathbf{i} + 2\mathbf{j} + \mathbf{k}) \cdot (2x\mathbf{i} + 2y\mathbf{j} + \mathbf{k})\, dA \\
&= \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (4xy + 4y + 1)\, dy\, dx \\
&= \int_{-2}^{2} \Big[2xy^2 + 2y^2 + y \Big]_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dx \\
&= \int_{-2}^{2} 2\sqrt{4 - x^2}\, dx \\
&= \text{Area of circle of radius } 2 = 4\pi.
\end{aligned}$$

As a *line integral*, you can parametrize C as

$$\mathbf{r}(t) = 2\cos t\,\mathbf{i} + 2\sin t\,\mathbf{j} + 0\mathbf{k}, \quad 0 \le t \le 2\pi.$$

For $\mathbf{F}(x, y, z) = 2z\mathbf{i} + x\mathbf{j} + y^2\mathbf{k}$, you obtain

$$\begin{aligned}
\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_C M\, dx + N\, dy + P\, dz \\
&= \int_C 2z\, dx + x\, dy + y^2\, dz \\
&= \int_0^{2\pi} [0 + 2\cos t(2\cos t) + 0]\, dt \\
&= \int_0^{2\pi} 4\cos^2 t\, dt \\
&= 2\int_0^{2\pi} (1 + \cos 2t)\, dt \\
&= 2\left[t + \frac{1}{2}\sin 2t\right]_0^{2\pi} \\
&= 4\pi.
\end{aligned}$$

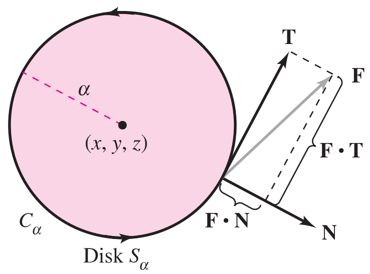


Figure 15.66

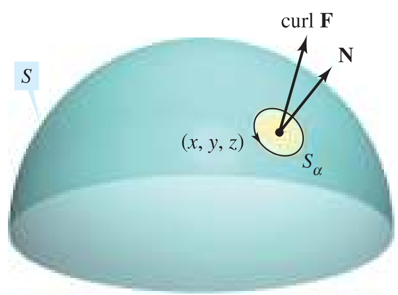


Figure 15.67

Physical Interpretation of Curl

Stokes's Theorem provides insight into a physical interpretation of curl. In a vector field $\mathbf{F}$, let S_α be a *small* circular disk of radius α, centered at (x, y, z) and with boundary C_α, as shown in Figure 15.66. At each point on the circle C_α, $\mathbf{F}$ has a normal component $\mathbf{F} \cdot \mathbf{N}$ and a tangential component $\mathbf{F} \cdot \mathbf{T}$. The more closely $\mathbf{F}$ and $\mathbf{T}$ are aligned, the greater the value of $\mathbf{F} \cdot \mathbf{T}$. So, a fluid tends to move along the circle rather than across it. Consequently, you say that the line integral around C_α measures the **circulation of F around C_α.** That is,

$$\int_{C_\alpha} \mathbf{F} \cdot \mathbf{T}\, ds = \text{circulation of } \mathbf{F} \text{ around } C_\alpha.$$

Now consider a small disk S_α to be centered at some point (x, y, z) on the surface S, as shown in Figure 15.67. On such a small disk, curl $\mathbf{F}$ is nearly constant, because it varies little from its value at (x, y, z). Moreover, curl $\mathbf{F} \cdot \mathbf{N}$ is also nearly constant on S_α, because all unit normals to S_α are about the same. Consequently, Stokes's Theorem yields

$$\begin{aligned}\int_{C_\alpha} \mathbf{F} \cdot \mathbf{T}\, ds &= \int_{S_\alpha}\!\!\int (\text{curl } \mathbf{F}) \cdot \mathbf{N}\, dS \\ &\approx (\text{curl } \mathbf{F}) \cdot \mathbf{N} \int_{S_\alpha}\!\!\int dS \\ &\approx (\text{curl } \mathbf{F}) \cdot \mathbf{N}(\pi\alpha^2).\end{aligned}$$

So,

$$\begin{aligned}(\text{curl } \mathbf{F}) \cdot \mathbf{N} &\approx \frac{\int_{C_\alpha} \mathbf{F} \cdot \mathbf{T}\, ds}{\pi\alpha^2} \\ &= \frac{\text{circulation of } \mathbf{F} \text{ around } C_\alpha}{\text{area of disk } S_\alpha} \\ &= \text{rate of circulation.}\end{aligned}$$

Assuming conditions are such that the approximation improves for smaller and smaller disks $(\alpha \to 0)$, it follows that

$$(\text{curl } \mathbf{F}) \cdot \mathbf{N} = \lim_{\alpha \to 0} \frac{1}{\pi\alpha^2} \int_{C_\alpha} \mathbf{F} \cdot \mathbf{T}\, ds$$

which is referred to as the **rotation of F about N.** That is,

$$\text{curl } \mathbf{F}(x, y, z) \cdot \mathbf{N} = \text{rotation of } \mathbf{F} \text{ about } \mathbf{N} \text{ at } (x, y, z).$$

In this case, the rotation of $\mathbf{F}$ is maximum when curl $\mathbf{F}$ and $\mathbf{N}$ have the same direction. Normally, this tendency to rotate will vary from point to point on the surface S, and Stokes's Theorem

$$\underbrace{\int_S\!\!\int (\text{curl } \mathbf{F}) \cdot \mathbf{N}\, dS}_{\text{Surface integral}} = \underbrace{\int_C \mathbf{F} \cdot d\mathbf{r}}_{\text{Line integral}}$$

says that the collective measure of this *rotational* tendency taken over the entire surface S (surface integral) is equal to the tendency of a fluid to *circulate* around the boundary C (line integral).

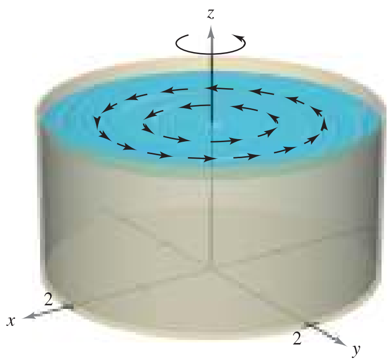


Figure 15.68

EXAMPLE 3 An Application of Curl

A liquid is swirling around in a cylindrical container of radius 2, so that its motion is described by the velocity field

$$\mathbf{F}(x, y, z) = -y\sqrt{x^2 + y^2}\,\mathbf{i} + x\sqrt{x^2 + y^2}\,\mathbf{j}$$

as shown in Figure 15.68. Find

$$\int_S\!\!\int (\text{curl } \mathbf{F}) \cdot \mathbf{N}\, dS$$

where S is the upper surface of the cylindrical container.

Solution The curl of $\mathbf{F}$ is given by

$$\text{curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ -y\sqrt{x^2+y^2} & x\sqrt{x^2+y^2} & 0 \end{vmatrix} = 3\sqrt{x^2+y^2}\,\mathbf{k}.$$

Letting $\mathbf{N} = \mathbf{k}$, you have

$$\begin{aligned} \int_S\!\!\int (\text{curl } \mathbf{F}) \cdot \mathbf{N}\, dS &= \int_R\!\!\int 3\sqrt{x^2+y^2}\, dA \\ &= \int_0^{2\pi}\!\!\int_0^2 (3r)r\, dr\, d\theta \\ &= \int_0^{2\pi} r^3\Big]_0^2\, d\theta \\ &= \int_0^{2\pi} 8\, d\theta \\ &= 16\pi. \end{aligned}$$

NOTE If curl $\mathbf{F} = \mathbf{0}$ throughout region Q, the rotation of $\mathbf{F}$ about each unit normal $\mathbf{N}$ is 0. That is, $\mathbf{F}$ is irrotational. From earlier work, you know that this is a characteristic of conservative vector fields.

SUMMARY OF INTEGRATION FORMULAS

Fundamental Theorem of Calculus:

$$\int_a^b F'(x)\, dx = F(b) - F(a)$$

Fundamental Theorem of Line Integrals:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \nabla f \cdot d\mathbf{r} = f(x(b), y(b)) - f(x(a), y(a))$$

Green's Theorem:

$$\int_C M\, dx + N\, dy = \int_R\!\!\int \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA = \int_C \mathbf{F} \cdot \mathbf{T}\, ds = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_R\!\!\int (\text{curl } \mathbf{F}) \cdot \mathbf{k}\, dA$$

$$\int_C \mathbf{F} \cdot \mathbf{N}\, ds = \int_R\!\!\int \text{div } \mathbf{F}\, dA$$

Divergence Theorem:

$$\int_S\!\!\int \mathbf{F} \cdot \mathbf{N}\, dS = \int\!\!\int_Q\!\!\int \text{div } \mathbf{F}\, dV$$

Stokes's Theorem:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_S\!\!\int (\text{curl } \mathbf{F}) \cdot \mathbf{N}\, dS$$

15.8 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–6, find the curl of the vector field F.

1. $\mathbf{F}(x, y, z) = (2y - z)\mathbf{i} + e^z\mathbf{j} + xyz\mathbf{k}$
2. $\mathbf{F}(x, y, z) = x^2\mathbf{i} + y^2\mathbf{j} + x^2\mathbf{k}$
3. $\mathbf{F}(x, y, z) = 2z\mathbf{i} - 4x^2\mathbf{j} + \arctan x\mathbf{k}$
4. $\mathbf{F}(x, y, z) = x \sin y\mathbf{i} - y \cos x\mathbf{j} + yz^2\mathbf{k}$
5. $\mathbf{F}(x, y, z) = e^{x^2+y^2}\mathbf{i} + e^{y^2+z^2}\mathbf{j} + xyz\mathbf{k}$
6. $\mathbf{F}(x, y, z) = \arcsin y\mathbf{i} + \sqrt{1 - x^2}\,\mathbf{j} + y^2\mathbf{k}$

In Exercises 7–10, verify Stokes's Theorem by evaluating $\int_C \mathbf{F} \cdot \mathbf{T}\, ds = \int_C \mathbf{F} \cdot d\mathbf{r}$ as a line integral and as a double integral.

7. $\mathbf{F}(x, y, z) = (-y + z)\mathbf{i} + (x - z)\mathbf{j} + (x - y)\mathbf{k}$
   S: $z = 9 - x^2 - y^2, \quad z \geq 0$
8. $\mathbf{F}(x, y, z) = (-y + z)\mathbf{i} + (x - z)\mathbf{j} + (x - y)\mathbf{k}$
   S: $z = \sqrt{1 - x^2 - y^2}$
9. $\mathbf{F}(x, y, z) = xyz\mathbf{i} + y\mathbf{j} + z\mathbf{k}$
   S: $6x + 6y + z = 12$, first octant
10. $\mathbf{F}(x, y, z) = z^2\mathbf{i} + x^2\mathbf{j} + y^2\mathbf{k}$
    S: $z = y^2, \quad 0 \leq x \leq a, \quad 0 \leq y \leq a$

In Exercises 11–20, use Stokes's Theorem to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$. Use a computer algebra system to verify your results. In each case, C is oriented counterclockwise as viewed from above.

11. $\mathbf{F}(x, y, z) = 2y\mathbf{i} + 3z\mathbf{j} + x\mathbf{k}$
    C: triangle with vertices $(2, 0, 0)$, $(0, 2, 0)$, $(0, 0, 2)$
12. $\mathbf{F}(x, y, z) = \arctan \dfrac{x}{y}\mathbf{i} + \ln\sqrt{x^2 + y^2}\,\mathbf{j} + \mathbf{k}$
    C: triangle with vertices $(0, 0, 0)$, $(1, 1, 1)$, $(0, 0, 2)$
13. $\mathbf{F}(x, y, z) = z^2\mathbf{i} + 2x\mathbf{j} + y^2\mathbf{k}$
    S: $z = 1 - x^2 - y^2, \quad z \geq 0$
14. $\mathbf{F}(x, y, z) = 4xz\mathbf{i} + y\mathbf{j} + 4xy\mathbf{k}$
    S: $z = 9 - x^2 - y^2, \quad z \geq 0$
15. $\mathbf{F}(x, y, z) = z^2\mathbf{i} + y\mathbf{j} + z\mathbf{k}$
    S: $z = \sqrt{4 - x^2 - y^2}$
16. $\mathbf{F}(x, y, z) = x^2\mathbf{i} + z^2\mathbf{j} - xyz\mathbf{k}$
    S: $z = \sqrt{4 - x^2 - y^2}$
17. $\mathbf{F}(x, y, z) = -\ln\sqrt{x^2 + y^2}\,\mathbf{i} + \arctan \dfrac{x}{y}\mathbf{j} + \mathbf{k}$
    S: $z = 9 - 2x - 3y$ over $r = 2 \sin 2\theta$ in the first octant
18. $\mathbf{F}(x, y, z) = yz\mathbf{i} + (2 - 3y)\mathbf{j} + (x^2 + y^2)\mathbf{k}, \quad x^2 + y^2 \leq 16$
    S: the first-octant portion of $x^2 + z^2 = 16$ over $x^2 + y^2 = 16$
19. $\mathbf{F}(x, y, z) = xyz\mathbf{i} + y\mathbf{j} + z\mathbf{k}$
    S: $z = x^2, \quad 0 \leq x \leq a, \quad 0 \leq y \leq a$
    $\mathbf{N}$ is the downward unit normal to the surface.
20. $\mathbf{F}(x, y, z) = xyz\mathbf{i} + y\mathbf{j} + z\mathbf{k}, \quad x^2 + y^2 \leq a^2$
    S: the first-octant portion of $z = x^2$ over $x^2 + y^2 = a^2$

Motion of a Liquid **In Exercises 21 and 22, the motion of a liquid in a cylindrical container of radius 1 is described by the velocity field $\mathbf{F}(x, y, z)$. Find $\int_S\int (\text{curl } \mathbf{F}) \cdot \mathbf{N}\, dS$, where S is the upper surface of the cylindrical container.**

21. $\mathbf{F}(x, y, z) = \mathbf{i} + \mathbf{j} - 2\mathbf{k}$
22. $\mathbf{F}(x, y, z) = -z\mathbf{i} + y\mathbf{k}$

WRITING ABOUT CONCEPTS

23. State Stokes's Theorem.
24. Give a physical interpretation of curl.

25. Let f and g be scalar functions with continuous partial derivatives, and let C and S satisfy the conditions of Stokes's Theorem. Verify each identity.
    (a) $\int_C (f\nabla g) \cdot d\mathbf{r} = \int_S\int (\nabla f \times \nabla g) \cdot \mathbf{N}\, dS$
    (b) $\int_C (f\nabla f) \cdot d\mathbf{r} = 0$
    (c) $\int_C (f\nabla g + g\nabla f) \cdot d\mathbf{r} = 0$
26. Demonstrate the results of Exercise 25 for the functions $f(x, y, z) = xyz$ and $g(x, y, z) = z$. Let S be the hemisphere $z = \sqrt{4 - x^2 - y^2}$.
27. Let $\mathbf{C}$ be a constant vector. Let S be an oriented surface with a unit normal vector $\mathbf{N}$, bounded by a smooth curve C. Prove that

$$\int_S\int \mathbf{C} \cdot \mathbf{N}\, dS = \frac{1}{2}\int_C (\mathbf{C} \times \mathbf{r}) \cdot d\mathbf{r}.$$

CAPSTONE

28. Verify Stokes's Theorem for each given vector field and upward oriented surface. Is the line integral or the double integral easier to set up? to evaluate? Explain.
    (a) $\mathbf{F}(x, y, z) = e^{y+z}\mathbf{i}$
    C: square with vertices $(0, 0, 0)$, $(1, 0, 0)$, $(1, 1, 0)$, $(0, 1, 0)$
    (b) $\mathbf{F}(x, y, z) = z^2\mathbf{i} + x^2\mathbf{j} + y^2\mathbf{k}$
    S: the portion of the paraboloid $z = x^2 + y^2$ that lies below the plane $z = 4$

PUTNAM EXAM CHALLENGE

29. Let $\mathbf{G}(x, y) = \left(\dfrac{-y}{x^2 + 4y^2}, \dfrac{x}{x^2 + 4y^2}, 0\right)$.
    Prove or disprove that there is a vector-valued function $\mathbf{F}(x, y, z) = (M(x, y, z),\ N(x, y, z),\ P(x, y, z))$ with the following properties.
    (i) M, N, P have continuous partial derivatives for all $(x, y, z) \neq (0, 0, 0)$;
    (ii) Curl $\mathbf{F} = \mathbf{0}$ for all $(x, y, z) \neq (0, 0, 0)$;
    (iii) $\mathbf{F}(x, y, 0) = \mathbf{G}(x, y)$.

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15 REVIEW EXERCISES

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1 and 2, compute $\|\mathbf{F}\|$ and sketch several representative vectors in the vector field. Use a computer algebra system to verify your results.

1. $\mathbf{F}(x, y, z) = x\mathbf{i} + \mathbf{j} + 2\mathbf{k}$
2. $\mathbf{F}(x, y) = \mathbf{i} - 2y\mathbf{j}$

In Exercises 3 and 4, find the gradient vector field for the scalar function.

3. $f(x, y, z) = 2x^2 + xy + z^2$
4. $f(x, y, z) = x^2e^{yz}$

In Exercises 5–12, determine whether the vector field is conservative. If it is, find a potential function for the vector field.

5. $\mathbf{F}(x, y) = -\dfrac{y}{x^2}\mathbf{i} + \dfrac{1}{x}\mathbf{j}$
6. $\mathbf{F}(x, y) = \dfrac{1}{y}\mathbf{i} - \dfrac{y}{x^2}\mathbf{j}$
7. $\mathbf{F}(x, y) = (xy^2 - x^2)\mathbf{i} + (x^2y + y^2)\mathbf{j}$
8. $\mathbf{F}(x, y) = (-2y^3 \sin 2x)\mathbf{i} + 3y^2(1 + \cos 2x)\mathbf{j}$
9. $\mathbf{F}(x, y, z) = 4xy^2\mathbf{i} + 2x^2\mathbf{j} + 2z\mathbf{k}$
10. $\mathbf{F}(x, y, z) = (4xy + z^2)\mathbf{i} + (2x^2 + 6yz)\mathbf{j} + 2xz\mathbf{k}$
11. $\mathbf{F}(x, y, z) = \dfrac{yz\mathbf{i} - xz\mathbf{j} - xy\mathbf{k}}{y^2z^2}$
12. $\mathbf{F}(x, y, z) = \sin z(y\mathbf{i} + x\mathbf{j} + \mathbf{k})$

In Exercises 13–20, find (a) the divergence of the vector field F and (b) the curl of the vector field F.

13. $\mathbf{F}(x, y, z) = x^2\mathbf{i} + xy^2\mathbf{j} + x^2z\mathbf{k}$
14. $\mathbf{F}(x, y, z) = y^2\mathbf{j} - z^2\mathbf{k}$
15. $\mathbf{F}(x, y, z) = (\cos y + y \cos x)\mathbf{i} + (\sin x - x \sin y)\mathbf{j} + xyz\mathbf{k}$
16. $\mathbf{F}(x, y, z) = (3x - y)\mathbf{i} + (y - 2z)\mathbf{j} + (z - 3x)\mathbf{k}$
17. $\mathbf{F}(x, y, z) = \arcsin x\,\mathbf{i} + xy^2\mathbf{j} + yz^2\mathbf{k}$
18. $\mathbf{F}(x, y, z) = (x^2 - y)\mathbf{i} - (x + \sin^2 y)\mathbf{j}$
19. $\mathbf{F}(x, y, z) = \ln(x^2 + y^2)\mathbf{i} + \ln(x^2 + y^2)\mathbf{j} + z\mathbf{k}$
20. $\mathbf{F}(x, y, z) = \dfrac{z}{x}\mathbf{i} + \dfrac{z}{y}\mathbf{j} + z^2\mathbf{k}$

In Exercises 21–26, evaluate the line integral along the given path(s).

21. $\displaystyle\int_C (x^2 + y^2)\, ds$

 (a) C: line segment from $(0, 0)$ to $(3, 4)$

 (b) C: $x^2 + y^2 = 1$, one revolution counterclockwise, starting at $(1, 0)$

22. $\displaystyle\int_C xy\, ds$

 (a) C: line segment from $(0, 0)$ to $(5, 4)$

 (b) C: counterclockwise around the triangle with vertices $(0, 0), (4, 0), (0, 2)$

23. $\displaystyle\int_C (x^2 + y^2)\, ds$

 C: $\mathbf{r}(t) = (1 - \sin t)\mathbf{i} + (1 - \cos t)\mathbf{j}, \quad 0 \le t \le 2\pi$

24. $\displaystyle\int_C (x^2 + y^2)\, ds$

 C: $\mathbf{r}(t) = (\cos t + t \sin t)\mathbf{i} + (\sin t - t \cos t)\mathbf{j}, \quad 0 \le t \le 2\pi$

25. $\displaystyle\int_C (2x - y)\, dx + (x + 2y)\, dy$

 (a) C: line segment from $(0, 0)$ to $(3, -3)$

 (b) C: one revolution counterclockwise around the circle $x = 3 \cos t, y = 3 \sin t$

26. $\displaystyle\int_C (2x - y)\, dx + (x + 3y)\, dy$

 C: $\mathbf{r}(t) = (\cos t + t \sin t)\mathbf{i} + (\sin t - t \sin t)\mathbf{j}, \quad 0 \le t \le \pi/2$

CAS **In Exercises 27 and 28, use a computer algebra system to evaluate the line integral over the given path.**

27. $\displaystyle\int_C (2x + y)\, ds$

 $\mathbf{r}(t) = a \cos^3 t\,\mathbf{i} + a \sin^3 t\,\mathbf{j},$

 $0 \le t \le \pi/2$

28. $\displaystyle\int_C (x^2 + y^2 + z^2)\, ds$

 $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + t^{3/2}\mathbf{k},$

 $0 \le t \le 4$

Lateral Surface Area **In Exercises 29 and 30, find the lateral surface area over the curve C in the xy-plane and under the surface $z = f(x, y)$.**

29. $f(x, y) = 3 + \sin(x + y)$

 C: $y = 2x$ from $(0, 0)$ to $(2, 4)$

30. $f(x, y) = 12 - x - y$

 C: $y = x^2$ from $(0, 0)$ to $(2, 4)$

In Exercises 31–36, evaluate $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{r}$.

31. $\mathbf{F}(x, y) = xy\mathbf{i} + 2xy\mathbf{j}$

 C: $\mathbf{r}(t) = t^2\mathbf{i} + t^2\mathbf{j}, \quad 0 \le t \le 1$

32. $\mathbf{F}(x, y) = (x - y)\mathbf{i} + (x + y)\mathbf{j}$

 C: $\mathbf{r}(t) = 4 \cos t\,\mathbf{i} + 3 \sin t\,\mathbf{j}, \quad 0 \le t \le 2\pi$

33. $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

 C: $\mathbf{r}(t) = 2 \cos t\,\mathbf{i} + 2 \sin t\,\mathbf{j} + t\mathbf{k}, \quad 0 \le t \le 2\pi$

34. $\mathbf{F}(x, y, z) = (2y - z)\mathbf{i} + (z - x)\mathbf{j} + (x - y)\mathbf{k}$

 C: curve of intersection of $x^2 + z^2 = 4$ and $y^2 + z^2 = 4$ from $(2, 2, 0)$ to $(0, 0, 2)$

35. $\mathbf{F}(x, y, z) = (y + z)\mathbf{i} + (x + z)\mathbf{j} + (x + y)\mathbf{k}$

 C: curve of intersection of $z = x^2 + y^2$ and $y = x$ from $(0, 0, 0)$ to $(2, 2, 8)$

36. $\mathbf{F}(x, y, z) = (x^2 - z)\mathbf{i} + (y^2 + z)\mathbf{j} + x\mathbf{k}$

 C: curve of intersection of $z = x^2$ and $x^2 + y^2 = 4$ from $(0, -2, 0)$ to $(0, 2, 0)$

CAS **In Exercises 37 and 38, use a computer algebra system to evaluate the line integral.**

37. $\int_C xy\,dx + (x^2 + y^2)\,dy$

C: $y = x^2$ from $(0, 0)$ to $(2, 4)$ and $y = 2x$ from $(2, 4)$ to $(0, 0)$

38. $\int_C \mathbf{F} \cdot d\mathbf{r}$

$\mathbf{F}(x, y) = (2x - y)\mathbf{i} + (2y - x)\mathbf{j}$

C: $\mathbf{r}(t) = (2\cos t + 2t\sin t)\mathbf{i} + (2\sin t - 2t\cos t)\mathbf{j}$, $0 \le t \le \pi$

39. ***Work*** Find the work done by the force field $\mathbf{F} = x\mathbf{i} - \sqrt{y}\,\mathbf{j}$ along the path $y = x^{3/2}$ from $(0, 0)$ to $(4, 8)$.

40. ***Work*** A 20-ton aircraft climbs 2000 feet while making a 90° turn in a circular arc of radius 10 miles. Find the work done by the engines.

In Exercises 41 and 42, evaluate the integral using the Fundamental Theorem of Line Integrals.

41. $\int_C 2xyz\,dx + x^2z\,dy + x^2y\,dz$

C: smooth curve from $(0, 0, 0)$ to $(1, 3, 2)$

42. $\int_C y\,dx + x\,dy + \frac{1}{z}\,dz$

C: smooth curve from $(0, 0, 1)$ to $(4, 4, 4)$

43. Evaluate the line integral $\int_C y^2\,dx + 2xy\,dy$.

(a) C: $\mathbf{r}(t) = (1 + 3t)\mathbf{i} + (1 + t)\mathbf{j}$, $0 \le t \le 1$

(b) C: $\mathbf{r}(t) = t\mathbf{i} + \sqrt{t}\,\mathbf{j}$, $1 \le t \le 4$

(c) Use the Fundamental Theorem of Line Integrals, where C is a smooth curve from $(1, 1)$ to $(4, 2)$.

44. ***Area and Centroid*** Consider the region bounded by the x-axis and one arch of the cycloid with parametric equations $x = a(\theta - \sin\theta)$ and $y = a(1 - \cos\theta)$. Use line integrals to find (a) the area of the region and (b) the centroid of the region.

In Exercises 45–50, use Green's Theorem to evaluate the line integral.

45. $\int_C y\,dx + 2x\,dy$

C: boundary of the square with vertices $(0, 0)$, $(0, 1)$, $(1, 0)$, $(1, 1)$

46. $\int_C xy\,dx + (x^2 + y^2)\,dy$

C: boundary of the square with vertices $(0, 0)$, $(0, 2)$, $(2, 0)$, $(2, 2)$

47. $\int_C xy^2\,dx + x^2y\,dy$

C: $x = 4\cos t$, $y = 4\sin t$

48. $\int_C (x^2 - y^2)\,dx + 2xy\,dy$

C: $x^2 + y^2 = a^2$

49. $\int_C xy\,dx + x^2\,dy$

C: boundary of the region between the graphs of $y = x^2$ and $y = 1$

50. $\int_C y^2\,dx + x^{4/3}\,dy$

C: $x^{2/3} + y^{2/3} = 1$

CAS **In Exercises 51 and 52, use a computer algebra system to graph the surface represented by the vector-valued function.**

51. $\mathbf{r}(u, v) = \sec u \cos v\,\mathbf{i} + (1 + 2\tan u)\sin v\,\mathbf{j} + 2u\,\mathbf{k}$

$0 \le u \le \frac{\pi}{3}$, $0 \le v \le 2\pi$

52. $\mathbf{r}(u, v) = e^{-u/4}\cos v\,\mathbf{i} + e^{-u/4}\sin v\,\mathbf{j} + \frac{u}{6}\mathbf{k}$

$0 \le u \le 4$, $0 \le v \le 2\pi$

CAS **53.** ***Investigation*** Consider the surface represented by the vector-valued function

$\mathbf{r}(u, v) = 3\cos v\cos u\,\mathbf{i} + 3\cos v\sin u\,\mathbf{j} + \sin v\,\mathbf{k}$.

Use a computer algebra system to do the following.

(a) Graph the surface for $0 \le u \le 2\pi$ and $-\frac{\pi}{2} \le v \le \frac{\pi}{2}$.

(b) Graph the surface for $0 \le u \le 2\pi$ and $\frac{\pi}{4} \le v \le \frac{\pi}{2}$.

(c) Graph the surface for $0 \le u \le \frac{\pi}{4}$ and $0 \le v \le \frac{\pi}{2}$.

(d) Graph and identify the space curve for $0 \le u \le 2\pi$ and $v = \frac{\pi}{4}$.

(e) Approximate the area of the surface graphed in part (b).

(f) Approximate the area of the surface graphed in part (c).

54. Evaluate the surface integral $\int_S\int z\,dS$ over the surface S:

$\mathbf{r}(u, v) = (u + v)\mathbf{i} + (u - v)\mathbf{j} + \sin v\,\mathbf{k}$

where $0 \le u \le 2$ and $0 \le v \le \pi$.

CAS **55.** Use a computer algebra system to graph the surface S and approximate the surface integral

$$\int_S\int (x + y)\,dS$$

where S is the surface

S: $\mathbf{r}(u, v) = u\cos v\,\mathbf{i} + u\sin v\,\mathbf{j} + (u - 1)(2 - u)\mathbf{k}$

over $0 \le u \le 2$ and $0 \le v \le 2\pi$.

56. ***Mass*** A cone-shaped surface lamina S is given by

$$z = a\left(a - \sqrt{x^2 + y^2}\right), \quad 0 \le z \le a^2.$$

At each point on S, the density is proportional to the distance between the point and the z-axis.

(a) Sketch the cone-shaped surface.

(b) Find the mass m of the lamina.

In Exercises 57 and 58, verify the Divergence Theorem by evaluating

$$\int_S\!\!\int \mathbf{F} \cdot \mathbf{N}\, dS$$

as a surface integral and as a triple integral.

57. $\mathbf{F}(x, y, z) = x^2\mathbf{i} + xy\mathbf{j} + z\mathbf{k}$

Q: solid region bounded by the coordinate planes and the plane $2x + 3y + 4z = 12$

58. $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

Q: solid region bounded by the coordinate planes and the plane $2x + 3y + 4z = 12$

In Exercises 59 and 60, verify Stokes's Theorem by evaluating

$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

as a line integral and as a double integral.

59. $\mathbf{F}(x, y, z) = (\cos y + y\cos x)\mathbf{i} + (\sin x - x\sin y)\mathbf{j} + xyz\mathbf{k}$

S: portion of $z = y^2$ over the square in the xy-plane with vertices $(0, 0)$, $(a, 0)$, (a, a), $(0, a)$

$\mathbf{N}$ is the upward unit normal vector to the surface.

60. $\mathbf{F}(x, y, z) = (x - z)\mathbf{i} + (y - z)\mathbf{j} + x^2\mathbf{k}$

S: first-octant portion of the plane $3x + y + 2z = 12$

61. Prove that it is not possible for a vector field with twice-differentiable components to have a curl of $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

SECTION PROJECT

The Planimeter

You have learned many calculus techniques for finding the area of a planar region. Engineers use a mechanical device called a *planimeter* for measuring planar areas, which is based on the area formula given in Theorem 15.9 (page 1096). As you can see in the figure, the planimeter is fixed at point O (but free to pivot) and has a hinge at A. The end of the tracer arm AB moves counterclockwise around the region R. A small wheel at B is perpendicular to $\overline{AB}$ and is marked with a scale to measure how much it rolls as B traces out the boundary of region R. In this project you will show that the area of R is given by the length L of the tracer arm $\overline{AB}$ multiplied by the distance D that the wheel rolls.

Assume that point B traces out the boundary of R for $a \le t \le b$. Point A will move back and forth along a circular arc around the origin O. Let $\theta(t)$ denote the angle in the figure and let $(x(t), y(t))$ denote the coordinates of A.

(a) Show that the vector $\overrightarrow{OB}$ is given by the vector-valued function

$$\mathbf{r}(t) = [x(t) + L\cos\theta(t)]\mathbf{i} + [y(t) + L\sin\theta(t)]\mathbf{j}.$$

(b) Show that the following two integrals are equal to zero.

$$I_1 = \int_a^b \frac{1}{2}L^2\frac{d\theta}{dt}\,dt$$

$$I_2 = \int_a^b \frac{1}{2}\left(x\frac{dy}{dt} - y\frac{dx}{dt}\right)dt$$

(c) Use the integral $\int_a^b [x(t)\sin\theta(t) - y(t)\cos\theta(t)]'\,dt$ to show that the following two integrals are equal.

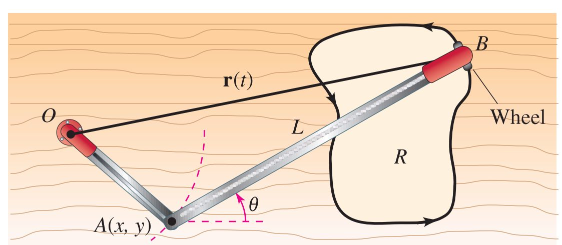
$$I_3 = \int_a^b \frac{1}{2}L\left(y\sin\theta\frac{d\theta}{dt} + x\cos\theta\frac{d\theta}{dt}\right)dt$$

$$I_4 = \int_a^b \frac{1}{2}L\left(-\sin\theta\frac{dx}{dt} + \cos\theta\frac{dy}{dt}\right)dt$$

(d) Let $\mathbf{N} = -\sin\theta\mathbf{i} + \cos\theta\mathbf{j}$. Explain why the distance D that the wheel rolls is given by

$$D = \int_C \mathbf{N} \cdot \mathbf{T}\,ds.$$

(e) Show that the area of region R is given by $I_1 + I_2 + I_3 + I_4 = DL$.



FOR FURTHER INFORMATION For more information about using calculus to find irregular areas, see "The Amateur Scientist" by C. L. Strong in the August 1958 issue of *Scientific American*.

P.S. PROBLEM SOLVING

1. Heat flows from areas of higher temperature to areas of lower temperature in the direction of greatest change. As a result, measuring heat flux involves the gradient of the temperature. The flux depends on the area of the surface. It is the normal direction to the surface that is important, because heat that flows in directions tangential to the surface will produce no heat loss. So, assume that the heat flux across a portion of the surface of area ΔS is given by $\Delta H \approx -k\nabla T \cdot \mathbf{N}\, dS$, where T is the temperature, $\mathbf{N}$ is the unit normal vector to the surface in the direction of the heat flow, and k is the thermal diffusivity of the material. The heat flux across the surface S is given by

$$H = \iint_S -k\nabla T \cdot \mathbf{N}\, dS.$$

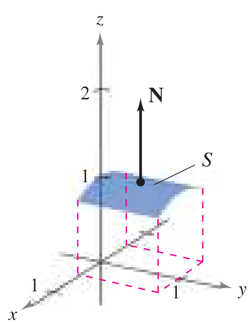
Consider a single heat source located at the origin with temperature

$$T(x, y, z) = \frac{25}{\sqrt{x^2 + y^2 + z^2}}.$$

(a) Calculate the heat flux across the surface

$$S = \left\{(x, y, z): z = \sqrt{1 - x^2},\ -\frac{1}{2} \le x \le \frac{1}{2},\ 0 \le y \le 1\right\}$$

as shown in the figure.



(b) Repeat the calculation in part (a) using the parametrization

$$x = \cos u,\quad y = v,\quad z = \sin u,\quad \frac{\pi}{3} \le u \le \frac{2\pi}{3},\quad 0 \le v \le 1.$$

2. Consider a single heat source located at the origin with temperature

$$T(x, y, z) = \frac{25}{\sqrt{x^2 + y^2 + z^2}}.$$

(a) Calculate the heat flux across the surface

$$S = \left\{(x, y, z): z = \sqrt{1 - x^2 - y^2},\ x^2 + y^2 \le 1\right\}$$

as shown in the figure.

(b) Repeat the calculation in part (a) using the parametrization

$x = \sin u \cos v,\quad y = \sin u \sin v,\quad z = \cos u,\quad 0 \le u \le \dfrac{\pi}{2},$
$0 \le v \le 2\pi.$

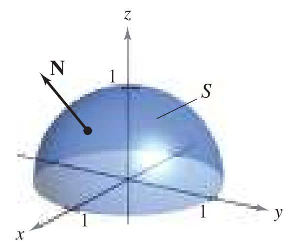


Figure for 2

3. Consider a wire of density $\rho(x, y, z)$ given by the space curve

$$C: \mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k},\quad a \le t \le b.$$

The **moments of inertia** about the x-, y-, and z-axes are given by

$$I_x = \int_C (y^2 + z^2)\rho(x, y, z)\, ds$$

$$I_y = \int_C (x^2 + z^2)\rho(x, y, z)\, ds$$

$$I_z = \int_C (x^2 + y^2)\rho(x, y, z)\, ds.$$

Find the moments of inertia for a wire of uniform density $\rho = 1$ in the shape of the helix

$\mathbf{r}(t) = 3 \cos t\mathbf{i} + 3 \sin t\mathbf{j} + 2t\mathbf{k},\quad 0 \le t \le 2\pi$ (see figure).

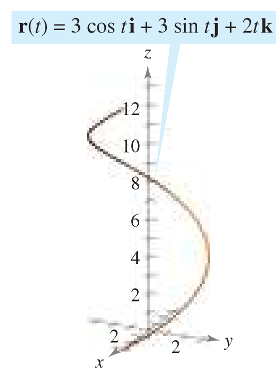


Figure for 3

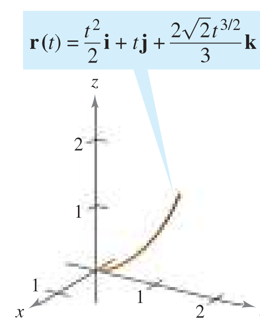


Figure for 4

4. Find the moments of inertia for the wire of density $\rho = \dfrac{1}{1 + t}$ given by the curve

$C: \mathbf{r}(t) = \dfrac{t^2}{2}\mathbf{i} + t\mathbf{j} + \dfrac{2\sqrt{2}\,t^{3/2}}{3}\mathbf{k},\quad 0 \le t \le 1$ (see figure).

5. The **Laplacian** is the differential operator

$$\nabla^2 = \nabla \cdot \nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

and **Laplace's equation** is

$$\nabla^2 w = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} = 0.$$

Any function that satisfies this equation is called **harmonic.** Show that the function $w = 1/f$ is harmonic.

CAS **6.** Consider the line integral

$$\int_C y^n\,dx + x^n\,dy$$

where C is the boundary of the region lying between the graphs of $y = \sqrt{a^2 - x^2}$ $(a > 0)$ and $y = 0$.

(a) Use a computer algebra system to verify Green's Theorem for n, an odd integer from 1 through 7.

(b) Use a computer algebra system to verify Green's Theorem for n, an even integer from 2 through 8.

(c) For n an odd integer, make a conjecture about the value of the integral.

7. Use a line integral to find the area bounded by one arch of the cycloid $x(\theta) = a(\theta - \sin\theta)$, $y(\theta) = a(1 - \cos\theta)$, $0 \le \theta \le 2\pi$, as shown in the figure.

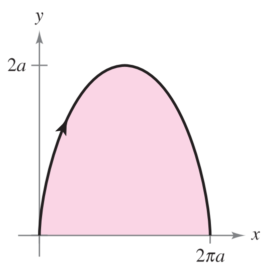


Figure for 7

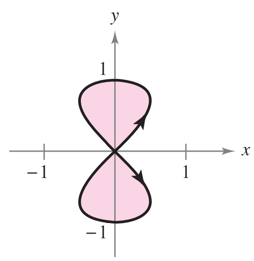


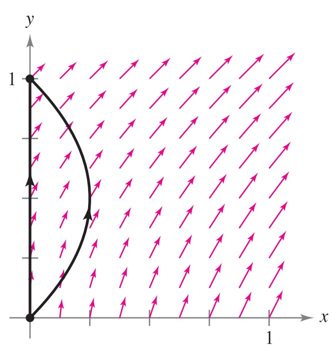
Figure for 8

8. Use a line integral to find the area bounded by the two loops of the eight curve

$$x(t) = \frac{1}{2}\sin 2t, \quad y(t) = \sin t, \quad 0 \le t \le 2\pi$$

as shown in the figure.

9. The force field $\mathbf{F}(x, y) = (x + y)\mathbf{i} + (x^2 + 1)\mathbf{j}$ acts on an object moving from the point $(0, 0)$ to the point $(0, 1)$, as shown in the figure.

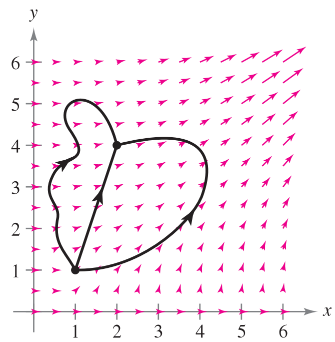


(a) Find the work done if the object moves along the path $x = 0$, $0 \le y \le 1$.

(b) Find the work done if the object moves along the path $x = y - y^2$, $0 \le y \le 1$.

(c) Suppose the object moves along the path $x = c(y - y^2)$, $0 \le y \le 1$, $c > 0$. Find the value of the constant c that minimizes the work.

10. The force field $\mathbf{F}(x, y) = (3x^2y^2)\mathbf{i} + (2x^3y)\mathbf{j}$ is shown in the figure below. Three particles move from the point $(1, 1)$ to the point $(2, 4)$ along different paths. Explain why the work done is the same for each particle, and find the value of the work.



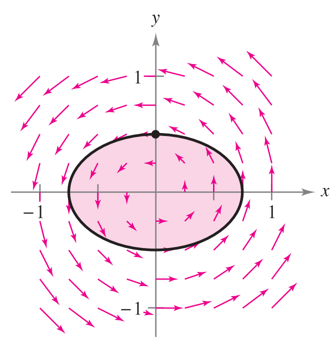
11. Let S be a smooth oriented surface with normal vector $\mathbf{N}$, bounded by a smooth simple closed curve C. Let $\mathbf{v}$ be a constant vector, and prove that

$$\int_S\int (2\mathbf{v} \cdot \mathbf{N})\,dS = \int_C (\mathbf{v} \times \mathbf{r}) \cdot d\mathbf{r}.$$

12. How does the area of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ compare with the magnitude of the work done by the force field

$$\mathbf{F}(x, y) = -\frac{1}{2}y\mathbf{i} + \frac{1}{2}x\mathbf{j}$$

on a particle that moves once around the ellipse (see figure)?



13. A cross section of Earth's magnetic field can be represented as a vector field in which the center of Earth is located at the origin and the positive y-axis points in the direction of the magnetic north pole. The equation for this field is

$$\begin{aligned}\mathbf{F}(x, y) &= M(x, y)\mathbf{i} + N(x, y)\mathbf{j}\\ &= \frac{m}{(x^2 + y^2)^{5/2}}[3xy\mathbf{i} + (2y^2 - x^2)\mathbf{j}]\end{aligned}$$

where m is the magnetic moment of Earth. Show that this vector field is conservative.

Appendices

A Proofs of Selected Theorems

THEOREM 1.2 PROPERTIES OF LIMITS (PROPERTIES 2, 3, 4, AND 5) (PAGE 59)

Let b and c be real numbers, let n be a positive integer, and let f and g be functions with the following limits.

$$\lim_{x \to c} f(x) = L \qquad \text{and} \qquad \lim_{x \to c} g(x) = K$$

2. Sum or difference: $\lim_{x \to c} [f(x) \pm g(x)] = L \pm K$

3. Product: $\lim_{x \to c} [f(x)g(x)] = LK$

4. Quotient: $\lim_{x \to c} \dfrac{f(x)}{g(x)} = \dfrac{L}{K}$, provided $K \neq 0$

5. Power: $\lim_{x \to c} [f(x)]^n = L^n$

PROOF To prove Property 2, choose $\varepsilon > 0$. Because $\varepsilon/2 > 0$, you know that there exists $\delta_1 > 0$ such that $0 < |x - c| < \delta_1$ implies $|f(x) - L| < \varepsilon/2$. You also know that there exists $\delta_2 > 0$ such that $0 < |x - c| < \delta_2$ implies $|g(x) - K| < \varepsilon/2$. Let δ be the smaller of δ_1 and δ_2; then $0 < |x - c| < \delta$ implies that

$$|f(x) - L| < \frac{\varepsilon}{2} \quad \text{and} \quad |g(x) - K| < \frac{\varepsilon}{2}.$$

So, you can apply the triangle inequality to conclude that

$$|[f(x) + g(x)] - (L + K)| \leq |f(x) - L| + |g(x) - K| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

which implies that

$$\lim_{x \to c} [f(x) + g(x)] = L + K = \lim_{x \to c} f(x) + \lim_{x \to c} g(x).$$

The proof that

$$\lim_{x \to c} [f(x) - g(x)] = L - K$$

is similar.

To prove Property 3, given that

$$\lim_{x \to c} f(x) = L \quad \text{and} \quad \lim_{x \to c} g(x) = K$$

you can write

$$f(x)g(x) = [f(x) - L][g(x) - K] + [Lg(x) + Kf(x)] - LK.$$

Because the limit of $f(x)$ is L, and the limit of $g(x)$ is K, you have

$$\lim_{x \to c} [f(x) - L] = 0 \quad \text{and} \quad \lim_{x \to c} [g(x) - K] = 0.$$

Let $0 < \varepsilon < 1$. Then there exists $\delta > 0$ such that if $0 < |x - c| < \delta$, then

$$|f(x) - L - 0| < \varepsilon \quad \text{and} \quad |g(x) - K - 0| < \varepsilon$$

which implies that

$$|[f(x) - L][g(x) - K] - 0| = |f(x) - L|\,|g(x) - K| < \varepsilon\varepsilon < \varepsilon.$$

So,

$$\lim_{x \to c} [f(x) - L][g(x) - K] = 0.$$

Furthermore, by Property 1, you have

$$\lim_{x \to c} Lg(x) = LK \quad \text{and} \quad \lim_{x \to c} Kf(x) = KL.$$

Finally, by Property 2, you obtain

$$\begin{aligned}\lim_{x \to c} f(x)g(x) &= \lim_{x \to c} [f(x) - L][g(x) - K] + \lim_{x \to c} Lg(x) + \lim_{x \to c} Kf(x) - \lim_{x \to c} LK \\ &= 0 + LK + KL - LK \\ &= LK.\end{aligned}$$

To prove Property 4, note that it is sufficient to prove that

$$\lim_{x \to c} \frac{1}{g(x)} = \frac{1}{K}.$$

Then you can use Property 3 to write

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} f(x) \frac{1}{g(x)} = \lim_{x \to c} f(x) \cdot \lim_{x \to c} \frac{1}{g(x)} = \frac{L}{K}.$$

Let $\varepsilon > 0$. Because $\lim_{x \to c} g(x) = K$, there exists $\delta_1 > 0$ such that if

$$0 < |x - c| < \delta_1, \text{ then } |g(x) - K| < \frac{|K|}{2}$$

which implies that

$$|K| = |g(x) + [|K| - g(x)]| \leq |g(x)| + ||K| - g(x)| < |g(x)| + \frac{|K|}{2}.$$

That is, for $0 < |x - c| < \delta_1$,

$$\frac{|K|}{2} < |g(x)| \quad \text{or} \quad \frac{1}{|g(x)|} < \frac{2}{|K|}.$$

Similarly, there exists a $\delta_2 > 0$ such that if $0 < |x - c| < \delta_2$, then

$$|g(x) - K| < \frac{|K|^2}{2} \varepsilon.$$

Let δ be the smaller of δ_1 and δ_2. For $0 < |x - c| < \delta$, you have

$$\left|\frac{1}{g(x)} - \frac{1}{K}\right| = \left|\frac{K - g(x)}{g(x)K}\right| = \frac{1}{|K|} \cdot \frac{1}{|g(x)|} |K - g(x)| < \frac{1}{|K|} \cdot \frac{2}{|K|} \frac{|K|^2}{2} \varepsilon = \varepsilon.$$

So, $\lim_{x \to c} \frac{1}{g(x)} = \frac{1}{K}$.

Finally, the proof of Property 5 can be obtained by a straightforward application of mathematical induction coupled with Property 3. ■

THEOREM 1.4 THE LIMIT OF A FUNCTION INVOLVING A RADICAL (PAGE 60)

Let n be a positive integer. The following limit is valid for all c if n is odd, and is valid for $c > 0$ if n is even.

$$\lim_{x\to c} \sqrt[n]{x} = \sqrt[n]{c}.$$

PROOF Consider the case for which $c > 0$ and n is any positive integer. For a given $\varepsilon > 0$, you need to find $\delta > 0$ such that

$$\left|\sqrt[n]{x} - \sqrt[n]{c}\right| < \varepsilon \quad \text{whenever} \quad 0 < |x - c| < \delta$$

which is the same as saying

$$-\varepsilon < \sqrt[n]{x} - \sqrt[n]{c} < \varepsilon \quad \text{whenever} \quad -\delta < x - c < \delta.$$

Assume $\varepsilon < \sqrt[n]{c}$, which implies that $0 < \sqrt[n]{c} - \varepsilon < \sqrt[n]{c}$. Now, let δ be the smaller of the two numbers.

$$c - \left(\sqrt[n]{c} - \varepsilon\right)^n \quad \text{and} \quad \left(\sqrt[n]{c} + \varepsilon\right)^n - c$$

Then you have

$$\begin{aligned}
-\delta < x - c &< \delta \\
-\left[c - \left(\sqrt[n]{c} - \varepsilon\right)^n\right] < x - c &< \left(\sqrt[n]{c} + \varepsilon\right)^n - c \\
\left(\sqrt[n]{c} - \varepsilon\right)^n - c < x - c &< \left(\sqrt[n]{c} + \varepsilon\right)^n - c \\
\left(\sqrt[n]{c} - \varepsilon\right)^n < x &< \left(\sqrt[n]{c} + \varepsilon\right)^n \\
\sqrt[n]{c} - \varepsilon < \sqrt[n]{x} &< \sqrt[n]{c} + \varepsilon \\
-\varepsilon < \sqrt[n]{x} - \sqrt[n]{c} &< \varepsilon.
\end{aligned}$$

■

THEOREM 1.5 THE LIMIT OF A COMPOSITE FUNCTION (PAGE 61)

If f and g are functions such that $\lim_{x\to c} g(x) = L$ and $\lim_{x\to L} f(x) = f(L)$, then

$$\lim_{x\to c} f(g(x)) = f\left(\lim_{x\to c} g(x)\right) = f(L).$$

PROOF For a given $\varepsilon > 0$, you must find $\delta > 0$ such that

$$|f(g(x)) - f(L)| < \varepsilon \quad \text{whenever} \quad 0 < |x - c| < \delta.$$

Because the limit of $f(x)$ as $x \to L$ is $f(L)$, you know there exists $\delta_1 > 0$ such that

$$|f(u) - f(L)| < \varepsilon \quad \text{whenever} \quad |u - L| < \delta_1.$$

Moreover, because the limit of $g(x)$ as $x \to c$ is L, you know there exists $\delta > 0$ such that

$$|g(x) - L| < \delta_1 \quad \text{whenever} \quad 0 < |x - c| < \delta.$$

Finally, letting $u = g(x)$, you have

$$|f(g(x)) - f(L)| < \varepsilon \quad \text{whenever} \quad 0 < |x - c| < \delta.$$

■

THEOREM 1.7 FUNCTIONS THAT AGREE AT ALL BUT ONE POINT (PAGE 62)

Let c be a real number and let $f(x) = g(x)$ for all $x \neq c$ in an open interval containing c. If the limit of $g(x)$ as x approaches c exists, then the limit of $f(x)$ also exists and

$$\lim_{x \to c} f(x) = \lim_{x \to c} g(x).$$

PROOF Let L be the limit of $g(x)$ as $x \to c$. Then, for each $\varepsilon > 0$ there exists a $\delta > 0$ such that $f(x) = g(x)$ in the open intervals $(c - \delta, c)$ and $(c, c + \delta)$, and

$$|g(x) - L| < \varepsilon \quad \text{whenever} \quad 0 < |x - c| < \delta.$$

Because $f(x) = g(x)$ for all x in the open interval other than $x = c$, it follows that

$$|f(x) - L| < \varepsilon \quad \text{whenever} \quad 0 < |x - c| < \delta.$$

So, the limit of $f(x)$ as $x \to c$ is also L. ■

THEOREM 1.8 THE SQUEEZE THEOREM (PAGE 65)

If $h(x) \le f(x) \le g(x)$ for all x in an open interval containing c, except possibly at c itself, and if

$$\lim_{x \to c} h(x) = L = \lim_{x \to c} g(x)$$

then $\lim_{x \to c} f(x)$ exists and is equal to L.

PROOF For $\varepsilon > 0$ there exist $\delta_1 > 0$ and $\delta_2 > 0$ such that

$$|h(x) - L| < \varepsilon \quad \text{whenever} \quad 0 < |x - c| < \delta_1$$

and

$$|g(x) - L| < \varepsilon \quad \text{whenever} \quad 0 < |x - c| < \delta_2.$$

Because $h(x) \le f(x) \le g(x)$ for all x in an open interval containing c, except possibly at c itself, there exists $\delta_3 > 0$ such that $h(x) \le f(x) \le g(x)$ for $0 < |x - c| < \delta_3$. Let δ be the smallest of δ_1, δ_2, and δ_3. Then, if $0 < |x - c| < \delta$, it follows that $|h(x) - L| < \varepsilon$ and $|g(x) - L| < \varepsilon$, which implies that

$$-\varepsilon < h(x) - L < \varepsilon \quad \text{and} \quad -\varepsilon < g(x) - L < \varepsilon$$
$$L - \varepsilon < h(x) \quad \text{and} \quad g(x) < L + \varepsilon.$$

Now, because $h(x) \le f(x) \le g(x)$, it follows that $L - \varepsilon < f(x) < L + \varepsilon$, which implies that $|f(x) - L| < \varepsilon$. Therefore,

$$\lim_{x \to c} f(x) = L.$$ ■

THEOREM 1.11 PROPERTIES OF CONTINUITY (PAGE 75)

If b is a real number and f and g are continuous at $x = c$, then the following functions are also continuous at c.

1. Scalar multiple: bf
2. Sum or difference: $f \pm g$
3. Product: fg
4. Quotient: $\dfrac{f}{g}$, if $g(c) \neq 0$

PROOF Because f and g are continuous at $x = c$, you can write

$$\lim_{x \to c} f(x) = f(c) \quad \text{and} \quad \lim_{x \to c} g(x) = g(c).$$

For Property 1, when b is a real number, it follows from Theorem 1.2 that

$$\lim_{x \to c} [(bf)(x)] = \lim_{x \to c} [bf(x)] = b \lim_{x \to c} [f(x)] = b\, f(c) = (bf)(c).$$

Thus, bf is continuous at $x = c$.

For Property 2, it follows from Theorem 1.2 that

$$\begin{aligned}
\lim_{x \to c} (f \pm g)(x) &= \lim_{x \to c} [f(x) \pm g(x)] \\
&= \lim_{x \to c} [f(x)] \pm \lim_{x \to c} [g(x)] \\
&= f(c) \pm g(c) \\
&= (f \pm g)(c).
\end{aligned}$$

Thus, $f \pm g$ is continuous at $x = c$.

For Property 3, it follows from Theorem 1.2 that

$$\begin{aligned}
\lim_{x \to c} (fg)(x) &= \lim_{x \to c} [f(x)g(x)] \\
&= \lim_{x \to c} [f(x)] \lim_{x \to c} [g(x)] \\
&= f(c)g(c) \\
&= (fg)(c).
\end{aligned}$$

Thus, fg is continuous at $x = c$.

For Property 4, when $g(c) \neq 0$, it follows from Theorem 1.2 that

$$\begin{aligned}
\lim_{x \to c} \frac{f}{g}(x) &= \lim_{x \to c} \frac{f(x)}{g(x)} \\
&= \frac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)} \\
&= \frac{f(c)}{g(c)} \\
&= \frac{f}{g}(c).
\end{aligned}$$

Thus, $\dfrac{f}{g}$ is continuous at $x = c$. ■

THEOREM 1.14 VERTICAL ASYMPTOTES (PAGE 85)

Let f and g be continuous on an open interval containing c. If $f(c) \neq 0$, $g(c) = 0$, and there exists an open interval containing c such that $g(x) \neq 0$ for all $x \neq c$ in the interval, then the graph of the function given by

$$h(x) = \frac{f(x)}{g(x)}$$

has a vertical asymptote at $x = c$.

PROOF Consider the case for which $f(c) > 0$, and there exists $b > c$ such that $c < x < b$ implies $g(x) > 0$. Then for $M > 0$, choose δ_1 such that

$$0 < x - c < \delta_1 \quad \text{implies that} \quad \frac{f(c)}{2} < f(x) < \frac{3f(c)}{2}$$

and δ_2 such that

$$0 < x - c < \delta_2 \quad \text{implies that} \quad 0 < g(x) < \frac{f(c)}{2M}.$$

Now let δ be the smaller of δ_1 and δ_2. Then it follows that

$$0 < x - c < \delta \quad \text{implies that} \quad \frac{f(x)}{g(x)} > \frac{f(c)}{2}\left[\frac{2M}{f(c)}\right] = M.$$

So, it follows that

$$\lim_{x \to c^+} \frac{f(x)}{g(x)} = \infty$$

and the line $x = c$ is a vertical asymptote of the graph of h. ■

ALTERNATIVE FORM OF THE DERIVATIVE (PAGE 101)

The derivative of f at c is given by

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

provided this limit exists.

PROOF The derivative of f at c is given by

$$f'(c) = \lim_{\Delta x \to 0} \frac{f(c + \Delta x) - f(c)}{\Delta x}.$$

Let $x = c + \Delta x$. Then $x \to c$ as $\Delta x \to 0$. So, replacing $c + \Delta x$ by x, you have

$$f'(c) = \lim_{\Delta x \to 0} \frac{f(c + \Delta x) - f(c)}{\Delta x} = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}.$$ ■

THEOREM 2.10 THE CHAIN RULE (PAGE 131)

If $y = f(u)$ is a differentiable function of u, and $u = g(x)$ is a differentiable function of x, then $y = f(g(x))$ is a differentiable function of x and

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

or, equivalently,

$$\frac{d}{dx}[f(g(x))] = f'(g(x))g'(x).$$

PROOF In Section 2.4, you let $h(x) = f(g(x))$ and used the alternative form of the derivative to show that $h'(c) = f'(g(c))g'(c)$, provided $g(x) \neq g(c)$ for values of x other than c. Now consider a more general proof. Begin by considering the derivative of f.

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

For a fixed value of x, define a function η such that

$$\eta(\Delta x) = \begin{cases} 0, & \Delta x = 0 \\ \dfrac{\Delta y}{\Delta x} - f'(x), & \Delta x \neq 0. \end{cases}$$

Because the limit of $\eta(\Delta x)$ as $\Delta x \to 0$ doesn't depend on the value of $\eta(0)$, you have

$$\lim_{\Delta x \to 0} \eta(\Delta x) = \lim_{\Delta x \to 0}\left[\frac{\Delta y}{\Delta x} - f'(x)\right] = 0$$

and you can conclude that η is continuous at 0. Moreover, because $\Delta y = 0$ when $\Delta x = 0$, the equation

$$\Delta y = \Delta x \eta(\Delta x) + \Delta x f'(x)$$

is valid whether Δx is zero or not. Now, by letting $\Delta u = g(x + \Delta x) - g(x)$, you can use the continuity of g to conclude that

$$\lim_{\Delta x \to 0} \Delta u = \lim_{\Delta x \to 0} [g(x + \Delta x) - g(x)] = 0$$

which implies that

$$\lim_{\Delta x \to 0} \eta(\Delta u) = 0.$$

Finally,

$$\Delta y = \Delta u \eta(\Delta u) + \Delta u f'(u) \to \frac{\Delta y}{\Delta x} = \frac{\Delta u}{\Delta x}\eta(\Delta u) + \frac{\Delta u}{\Delta x} f'(u), \quad \Delta x \neq 0$$

and taking the limit as $\Delta x \to 0$, you have

$$\begin{aligned} \frac{dy}{dx} = \frac{du}{dx}\left[\lim_{\Delta x \to 0} \eta(\Delta u)\right] + \frac{du}{dx} f'(u) &= \frac{dy}{dx}(0) + \frac{du}{dx} f'(u) \\ &= \frac{du}{dx} f'(u) \\ &= \frac{du}{dx} \cdot \frac{dy}{du}. \end{aligned}$$

■

CONCAVITY INTERPRETATION (PAGE 190)

1. Let f be differentiable on an open interval I. If the graph of f is concave *upward* on I, then the graph of f lies *above* all of its tangent lines on I.
2. Let f be differentiable on an open interval I. If the graph of f is concave *downward* on I, then the graph of f lies *below* all of its tangent lines on I.

PROOF Assume that f is concave upward on $I = (a, b)$. Then, f' is increasing on (a, b). Let c be a point in the interval $I = (a, b)$. The equation of the tangent line to the graph of f at c is given by

$$g(x) = f(c) + f'(c)(x - c).$$

If x is in the open interval (c, b), then the directed distance from point $(x, f(x))$ (on the graph of f) to the point $(x, g(x))$ (on the tangent line) is given by

$$\begin{aligned} d &= f(x) - [f(c) + f'(c)(x - c)] \\ &= f(x) - f(c) - f'(c)(x - c). \end{aligned}$$

Moreover, by the Mean Value Theorem there exists a number z in (c, x) such that

$$f'(z) = \frac{f(x) - f(c)}{x - c}.$$

So, you have

$$\begin{aligned} d &= f(x) - f(c) - f'(c)(x - c) \\ &= f'(z)(x - c) - f'(c)(x - c) \\ &= [f'(z) - f'(c)](x - c). \end{aligned}$$

The second factor $(x - c)$ is positive because $c < x$. Moreover, because f' is increasing, it follows that the first factor $[f'(z) - f'(c)]$ is also positive. Therefore, $d > 0$ and you can conclude that the graph of f lies above the tangent line at x. If x is in the open interval (a, c), a similar argument can be given. This proves the first statement. The proof of the second statement is similar. ■

THEOREM 3.7 TEST FOR CONCAVITY (PAGE 191)

Let f be a function whose second derivative exists on an open interval I.

1. If $f''(x) > 0$ for all x in I, then the graph of f is concave upward in I.
2. If $f''(x) < 0$ for all x in I, then the graph of f is concave downward in I.

PROOF For Property 1, assume $f''(x) > 0$ for all x in (a, b). Then, by Theorem 3.5, f' is increasing on $[a, b]$. Thus, by the definition of concavity, the graph of f is concave upward on (a, b).

For Property 2, assume $f''(x) < 0$ for all x in (a, b). Then, by Theorem 3.5, f' is decreasing on $[a, b]$. Thus, by the definition of concavity, the graph of f is concave downward on (a, b). ■

THEOREM 3.10 LIMITS AT INFINITY (PAGE 199)

If r is a positive rational number and c is any real number, then

$$\lim_{x\to\infty} \frac{c}{x^r} = 0.$$

Furthermore, if x^r is defined when $x < 0$, then $\lim_{x\to-\infty} \frac{c}{x^r} = 0$.

PROOF Begin by proving that

$$\lim_{x\to\infty} \frac{1}{x} = 0.$$

For $\varepsilon > 0$, let $M = 1/\varepsilon$. Then, for $x > M$, you have

$$x > M = \frac{1}{\varepsilon} \quad\Rightarrow\quad \frac{1}{x} < \varepsilon \quad\Rightarrow\quad \left|\frac{1}{x} - 0\right| < \varepsilon.$$

So, by the definition of a limit at infinity, you can conclude that the limit of $1/x$ as $x \to \infty$ is 0. Now, using this result, and letting $r = m/n$, you can write the following.

$$\begin{aligned}
\lim_{x\to\infty} \frac{c}{x^r} &= \lim_{x\to\infty} \frac{c}{x^{m/n}} \\
&= c\left[\lim_{x\to\infty}\left(\frac{1}{\sqrt[n]{x}}\right)^m\right] \\
&= c\left(\lim_{x\to\infty}\sqrt[n]{\frac{1}{x}}\right)^m \\
&= c\left(\sqrt[n]{\lim_{x\to\infty}\frac{1}{x}}\right)^m \\
&= c\left(\sqrt[n]{0}\right)^m \\
&= 0
\end{aligned}$$

The proof of the second part of the theorem is similar. ■

THEOREM 4.2 SUMMATION FORMULAS (PAGE 260)

1. $\displaystyle\sum_{i=1}^{n} c = cn$
2. $\displaystyle\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$
3. $\displaystyle\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$
4. $\displaystyle\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$

PROOF The proof of Property 1 is straightforward. By adding c to itself n times, you obtain a sum of cn.

To prove Property 2, write the sum in increasing and decreasing order and add corresponding terms, as follows.

$$\begin{array}{rcccccccccc}
\displaystyle\sum_{i=1}^{n} i = & 1 & + & 2 & + & 3 & + \cdots + & (n-1) & + & n \\
& \downarrow & & & & \downarrow & & \downarrow & & \downarrow \\
\displaystyle\sum_{i=1}^{n} i = & n & + & (n-1) & + & (n-2) & + \cdots + & 2 & + & 1 \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow
\end{array}$$

$$2\sum_{i=1}^{n} i = \underbrace{(n+1) + (n+1) + (n+1) + \cdots + (n+1) + (n+1)}_{n \text{ terms}}$$

So,

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}.$$

To prove Property 3, use mathematical induction. First, if $n = 1$, the result is true because

$$\sum_{i=1}^{1} i^2 = 1^2 = 1 = \frac{1(1+1)(2+1)}{6}.$$

Now, assuming the result is true for $n = k$, you can show that it is true for $n = k + 1$, as follows.

$$\begin{aligned}
\sum_{i=1}^{k+1} i^2 &= \sum_{i=1}^{k} i^2 + (k+1)^2 \\
&= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \\
&= \frac{k+1}{6}(2k^2 + k + 6k + 6) \\
&= \frac{k+1}{6}[(2k+3)(k+2)] \\
&= \frac{(k+1)(k+2)[2(k+1)+1]}{6}
\end{aligned}$$

Property 4 can be proved using a similar argument with mathematical induction. ■

THEOREM 4.8 PRESERVATION OF INEQUALITY (PAGE 278)

1. If f is integrable and nonnegative on the closed interval $[a, b]$, then

$$0 \le \int_a^b f(x)\,dx.$$

2. If f and g are integrable on the closed interval $[a, b]$ and $f(x) \le g(x)$ for every x in $[a, b]$, then

$$\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx.$$

PROOF To prove Property 1, suppose, on the contrary, that

$$\int_a^b f(x)\,dx = I < 0.$$

Then, let $a = x_0 < x_1 < x_2 < \cdot \cdot \cdot < x_n = b$ be a partition of $[a, b]$, and let

$$R = \sum_{i=1}^{n} f(c_i)\,\Delta x_i$$

be a Riemann sum. Because $f(x) \geq 0$, it follows that $R \geq 0$. Now, for $\|\Delta\|$ sufficiently small, you have $|R - I| < -I/2$, which implies that

$$\sum_{i=1}^{n} f(c_i)\,\Delta x_i = R < I - \frac{I}{2} < 0$$

which is not possible. From this contradiction, you can conclude that

$$0 \leq \int_a^b f(x)\,dx.$$

To prove Property 2 of the theorem, note that $f(x) \leq g(x)$ implies that $g(x) - f(x) \geq 0$. So, you can apply the result of Property 1 to conclude that

$$\begin{aligned} 0 &\leq \int_a^b [g(x) - f(x)]\,dx \\ 0 &\leq \int_a^b g(x)\,dx - \int_a^b f(x)\,dx \\ \int_a^b f(x)\,dx &\leq \int_a^b g(x)\,dx. \end{aligned}$$

■

PROPERTIES OF THE NATURAL LOGARITHMIC FUNCTION (PAGE 325)

The natural logarithmic function is one-to-one.

$$\lim_{x\to 0^+} \ln x = -\infty \qquad \text{and} \qquad \lim_{x\to\infty} \ln x = \infty$$

PROOF Recall from Section P.3 that a function f is one-to-one if for x_1 and x_2 in its domain

$$x_1 \neq x_2 \quad \Rightarrow \quad f(x_1) \neq f(x_2).$$

Let $f(x) = \ln x$. Then $f'(x) = \dfrac{1}{x} > 0$ for $x > 0$. So f is increasing on its entire domain $(0, \infty)$ and therefore is strictly monotonic (see Section 3.3). Choose x_1 and x_2 in the domain of f such that $x_1 \neq x_2$. Because f is strictly monotonic, it follows that either

$$f(x_1) < f(x_2) \quad \text{or} \quad f(x_1) > f(x_2).$$

In either case, $f(x_1) \neq f(x_2)$. So, $f(x) = \ln x$ is one-to-one. To verify the limits, begin by showing that $\ln 2 \geq \frac{1}{2}$. From the Mean Value Theorem for Integrals, you can write

$$\ln 2 = \int_1^2 \frac{1}{x}\,dx = \frac{1}{c}(2 - 1) = \frac{1}{c}$$

where c is in $[1, 2]$.

This implies that

$$\begin{aligned} 1 \leq \quad c \quad &\leq 2 \\ 1 \geq \quad \frac{1}{c} \quad &\geq \frac{1}{2} \\ 1 \geq \ln 2 &\geq \frac{1}{2}. \end{aligned}$$

Now, let N be any positive (large) number. Because $\ln x$ is increasing, it follows that if $x > 2^{2N}$, then

$$\ln x > \ln 2^{2N} = 2N \ln 2.$$

However, because $\ln 2 \geq \frac{1}{2}$, it follows that

$$\ln x > 2N \ln 2 \geq 2N\left(\frac{1}{2}\right) = N.$$

This verifies the second limit. To verify the first limit, let $z = 1/x$. Then, $z \to \infty$ as $x \to 0^+$, and you can write

$$\begin{aligned} \lim_{x\to 0^+} \ln x &= \lim_{x\to 0^+} \left(-\ln \frac{1}{x}\right) \\ &= \lim_{z\to\infty} (-\ln z) \\ &= -\lim_{z\to\infty} \ln z \\ &= -\infty. \end{aligned}$$

THEOREM 5.8 CONTINUITY AND DIFFERENTIABILITY OF INVERSE FUNCTIONS (PAGE 347)

Let f be a function whose domain is an interval I. If f has an inverse function, then the following statements are true.

1. If f is continuous on its domain, then f^{-1} is continuous on its domain.
2. If f is increasing on its domain, then f^{-1} is increasing on its domain.
3. If f is decreasing on its domain, then f^{-1} is decreasing on its domain.
4. If f is differentiable on an interval containing c and $f'(c) \neq 0$, then f^{-1} is differentiable at $f(c)$.

PROOF To prove Property 1, first show that if f is continuous on I and has an inverse function, then f is strictly monotonic on I. Suppose that f were not strictly monotonic. Then there would exist numbers x_1, x_2, x_3 in I such that $x_1 < x_2 < x_3$, but $f(x_2)$ is not between $f(x_1)$ and $f(x_3)$. Without loss of generality, assume $f(x_1) < f(x_3) < f(x_2)$. By the Intermediate Value Theorem, there exists a number x_0 between x_1 and x_2 such that $f(x_0) = f(x_3)$. So, f is not one-to-one and cannot have an inverse function. So, f must be strictly monotonic.

Because f is continuous, the Intermediate Value Theorem implies that the set of values of f

$$\{f(x): x \in I\}$$

forms an interval J. Assume that a is an interior point of J. From the previous argument, $f^{-1}(a)$ is an interior point of I. Let $\varepsilon > 0$. There exists $0 < \varepsilon_1 < \varepsilon$ such that

$$I_1 = (f^{-1}(a) - \varepsilon_1, f^{-1}(a) + \varepsilon_1) \subseteq I.$$

Because f is strictly monotonic on I_1, the set of values $\{f(x): x \in I_1\}$ forms an interval $J_1 \subseteq J$. Let $\delta > 0$ such that $(a - \delta, a + \delta) \subseteq J_1$. Finally, if

$$|y - a| < \delta, \text{ then } |f^{-1}(y) - f^{-1}(a)| < \varepsilon_1 < \varepsilon.$$

So, f^{-1} is continuous at a. A similar proof can be given if a is an endpoint.

To prove Property 2, let y_1 and y_2 be in the domain of f^{-1}, with $y_1 < y_2$. Then, there exist x_1 and x_2 in the domain of f such that

$$f(x_1) = y_1 < y_2 = f(x_2).$$

Because f is increasing, $f(x_1) < f(x_2)$ holds precisely when $x_1 < x_2$. Therefore,

$$f^{-1}(y_1) = x_1 < x_2 = f^{-1}(y_2)$$

which implies that f^{-1} is increasing. (Property 3 can be proved in a similar way.)

Finally, to prove Property 4, consider the limit

$$(f^{-1})'(a) = \lim_{y \to a} \frac{f^{-1}(y) - f^{-1}(a)}{y - a}$$

where a is in the domain of f^{-1} and $f^{-1}(a) = c$. Because f is differentiable on an interval containing c, f is continuous on that interval, and so is f^{-1} at a. So, $y \to a$ implies that $x \to c$, and you have

$$\begin{aligned}
(f^{-1})'(a) &= \lim_{x \to c} \frac{x - c}{f(x) - f(c)} \\
&= \lim_{x \to c} \frac{1}{\left(\dfrac{f(x) - f(c)}{x - c}\right)} \\
&= \frac{1}{\lim\limits_{x \to c} \dfrac{f(x) - f(c)}{x - c}} \\
&= \frac{1}{f'(c)}.
\end{aligned}$$

So, $(f^{-1})'(a)$ exists, and f^{-1} is differentiable at $f(c)$. ■

THEOREM 5.9 THE DERIVATIVE OF AN INVERSE FUNCTION (PAGE 347)

Let f be a function that is differentiable on an interval I. If f has an inverse function g, then g is differentiable at any x for which $f'(g(x)) \neq 0$. Moreover,

$$g'(x) = \frac{1}{f'(g(x))}, \quad f'(g(x)) \neq 0.$$

PROOF From the proof of Theorem 5.8, letting $a = x$, you know that g is differentiable. Using the Chain Rule, differentiate both sides of the equation $x = f(g(x))$ to obtain

$$1 = f'(g(x)) \frac{d}{dx}[g(x)].$$

Because $f'(g(x)) \neq 0$, you can divide by this quantity to obtain

$$\frac{d}{dx}[g(x)] = \frac{1}{f'(g(x))}.$$ ■

THEOREM 5.10 OPERATIONS WITH EXPONENTIAL FUNCTIONS (PROPERTY 2) (PAGE 353)

2. $\dfrac{e^a}{e^b} = e^{a-b}$ (Let a and b be any real numbers.)

PROOF To prove Property 2, you can write

$$\ln\left(\frac{e^a}{e^b}\right) = \ln e^a - \ln e^b = a - b = \ln(e^{a-b})$$

Because the natural logarithmic function is one-to-one, you can conclude that

$$\frac{e^a}{e^b} = e^{a-b}.$$

■

THEOREM 5.15 A LIMIT INVOLVING *e* (PAGE 366)

$$\lim_{x\to\infty}\left(1 + \frac{1}{x}\right)^x = \lim_{x\to\infty}\left(\frac{x+1}{x}\right)^x = e$$

PROOF Let $y = \lim_{x\to\infty}\left(1 + \frac{1}{x}\right)^x$. Taking the natural logarithm of each side, you have

$$\ln y = \ln\left[\lim_{x\to\infty}\left(1 + \frac{1}{x}\right)^x\right].$$

Because the natural logarithmic function is continuous, you can write

$$\ln y = \lim_{x\to\infty}\left[x \ln\left(1 + \frac{1}{x}\right)\right] = \lim_{x\to\infty}\left\{\frac{\ln[1 + (1/x)]}{1/x}\right\}.$$

Letting $x = \dfrac{1}{t}$, you have

$$\begin{aligned}
\ln y &= \lim_{t\to 0^+}\frac{\ln(1+t)}{t} = \lim_{t\to 0^+}\frac{\ln(1+t) - \ln 1}{t}\\
&= \frac{d}{dx}\ln x \ \text{ at } \ x = 1\\
&= \frac{1}{x} \ \text{ at } \ x = 1\\
&= 1.
\end{aligned}$$

Finally, because $\ln y = 1$, you know that $y = e$, and you can conclude that

$$\lim_{x\to\infty}\left(1 + \frac{1}{x}\right)^x = e.$$

■

THEOREM 5.16 DERIVATIVES OF INVERSE TRIGONOMETRIC FUNCTIONS (arcsin *u* and arccos *u*) (PAGE 376)

Let u be a differentiable function of x.

$$\frac{d}{dx}[\arcsin u] = \frac{u'}{\sqrt{1-u^2}} \qquad \frac{d}{dx}[\arccos u] = \frac{-u'}{\sqrt{1-u^2}}$$

PROOF
Method 1: Apply Theorem 5.9.

Let $f(x) = \sin x$ and $g(x) = \arcsin x$. Because f is differentiable on $-\pi/2 \leq y \leq \pi/2$, you can apply Theorem 5.9.

$$g'(x) = \frac{1}{f'(g(x))} = \frac{1}{\cos(\arcsin x)} = \frac{1}{\sqrt{1 - \sin^2(\arcsin x)}} = \frac{1}{\sqrt{1 - x^2}}$$

If u is a differentiable function of x, then you can use the Chain Rule to write

$$\frac{d}{dx}[\arcsin u] = \frac{u'}{\sqrt{1 - u^2}}, \quad \text{where} \quad u' = \frac{du}{dx}.$$

Method 2: Use implicit differentiation.

Let $y = \arccos x$, $0 \leq y \leq \pi$. So, $\cos y = x$, and you can use implicit differentiation as follows.

$$\cos y = x$$

$$-\sin y \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{-1}{\sin y} = \frac{-1}{\sqrt{1 - \cos^2 y}} = \frac{-1}{\sqrt{1 - x^2}}$$

If u is a differentiable function of x, then you can use the Chain Rule to write

$$\frac{d}{dx}[\arccos u] = \frac{-u'}{\sqrt{1 - u^2}}, \quad \text{where} \quad u' = \frac{du}{dx}.$$ ■

THEOREM 8.3 THE EXTENDED MEAN VALUE THEOREM (PAGE 570)

If f and g are differentiable on an open interval (a, b) and continuous on $[a, b]$ such that $g'(x) \neq 0$ for any x in (a, b), then there exists a point c in (a, b) such that $\dfrac{f'(c)}{g'(c)} = \dfrac{f(b) - f(a)}{g(b) - g(a)}$.

PROOF You can assume that $g(a) \neq g(b)$, because otherwise, by Rolle's Theorem, it would follow that $g'(x) = 0$ for some x in (a, b). Now, define $h(x)$ as

$$h(x) = f(x) - \left[\frac{f(b) - f(a)}{g(b) - g(a)}\right] g(x).$$

Then

$$h(a) = f(a) - \left[\frac{f(b) - f(a)}{g(b) - g(a)}\right] g(a) = \frac{f(a)g(b) - f(b)g(a)}{g(b) - g(a)}$$

and

$$h(b) = f(b) - \left[\frac{f(b) - f(a)}{g(b) - g(a)}\right] g(b) = \frac{f(a)g(b) - f(b)g(a)}{g(b) - g(a)}$$

and by Rolle's Theorem there exists a point c in (a, b) such that

$$h'(c) = f'(c) - \frac{f(b) - f(a)}{g(b) - g(a)} g'(c) = 0$$

which implies that $\dfrac{f'(c)}{g'(c)} = \dfrac{f(b) - f(a)}{g(b) - g(a)}$. ■

THEOREM 8.4 L'HÔPITAL'S RULE (PAGE 570)

Let f and g be functions that are differentiable on an open interval (a, b) containing c, except possibly at c itself. Assume that $g'(x) \neq 0$ for all x in (a, b), except possibly at c itself. If the limit of $f(x)/g(x)$ as x approaches c produces the indeterminate form $0/0$, then

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}$$

provided the limit on the right exists (or is infinite). This result also applies if the limit of $f(x)/g(x)$ as x approaches c produces any one of the indeterminate forms ∞/∞, $(-\infty)/\infty$, $\infty/(-\infty)$, or $(-\infty)/(-\infty)$.

You can use the Extended Mean Value Theorem to prove L'Hôpital's Rule. Of the several different cases of this rule, the proof of only one case is illustrated. The remaining cases where $x \to c^-$ and $x \to c$ are left for you to prove.

PROOF Consider the case for which $\lim\limits_{x \to c^+} f(x) = 0$ and $\lim\limits_{x \to c^+} g(x) = 0$. Define the following new functions:

$$F(x) = \begin{cases} f(x), & x \neq c \\ 0, & x = c \end{cases} \quad \text{and} \quad G(x) = \begin{cases} g(x), & x \neq c \\ 0, & x = c \end{cases}.$$

For any x, $c < x < b$, F and G are differentiable on $(c, x]$ and continuous on $[c, x]$. You can apply the Extended Mean Value Theorem to conclude that there exists a number z in (c, x) such that

$$\frac{F'(z)}{G'(z)} = \frac{F(x) - F(c)}{G(x) - G(c)} = \frac{F(x)}{G(x)} = \frac{f'(z)}{g'(z)} = \frac{f(x)}{g(x)}.$$

Finally, by letting x approach c from the right, $x \to c^+$, you have $z \to c^+$ because $c < z < x$, and

$$\lim_{x \to c^+} \frac{f(x)}{g(x)} = \lim_{x \to c^+} \frac{f'(z)}{g'(z)} = \lim_{z \to c^+} \frac{f'(z)}{g'(z)} = \lim_{x \to c^+} \frac{f'(x)}{g'(x)}. \qquad \blacksquare$$

THEOREM 9.19 TAYLOR'S THEOREM (PAGE 656)

If a function f is differentiable through order $n + 1$ in an interval I containing c, then, for each x in I, there exists z between x and c such that

$$f(x) = f(c) + f'(c)(x - c) + \frac{f''(c)}{2!}(x - c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x - c)^n + R_n(x)$$

where $R_n(x) = \dfrac{f^{(n+1)}(z)}{(n + 1)!}(x - c)^{n+1}$.

PROOF To find $R_n(x)$, fix x in I $(x \neq c)$ and write $R_n(x) = f(x) - P_n(x)$ where $P_n(x)$ is the nth Taylor polynomial for $f(x)$. Then let g be a function of t defined by

$$g(t) = f(x) - f(t) - f'(t)(x - t) - \cdots - \frac{f^{(n)}(t)}{n!}(x - t)^n - R_n(x)\frac{(x - t)^{n+1}}{(x - c)^{n+1}}.$$

The reason for defining g in this way is that differentiation with respect to t has a telescoping effect. For example, you have

$$\frac{d}{dt}[-f(t) - f'(t)(x - t)] = -f'(t) + f'(t) - f''(t)(x - t) = -f''(t)(x - t).$$

The result is that the derivative $g'(t)$ simplifies to

$$g'(t) = -\frac{f^{(n+1)}(t)}{n!}(x - t)^n + (n + 1)R_n(x)\frac{(x - t)^n}{(x - c)^{n+1}}$$

for all t between c and x. Moreover, for a fixed x,

$$g(c) = f(x) - [P_n(x) + R_n(x)] = f(x) - f(x) = 0$$

and

$$g(x) = f(x) - f(x) - 0 - \cdot \cdot \cdot - 0 = f(x) - f(x) = 0.$$

Therefore, g satisfies the conditions of Rolle's Theorem, and it follows that there is a number z between c and x such that $g'(z) = 0$. Substituting z for t in the equation for $g'(t)$ and then solving for $R_n(x)$, you obtain

$$g'(z) = -\frac{f^{(n+1)}(z)}{n!}(x - z)^n + (n + 1)R_n(x)\frac{(x - z)^n}{(x - c)^{n+1}} = 0$$

$$R_n(x) = \frac{f^{(n+1)}(z)}{(n + 1)!}(x - c)^{n+1}.$$

Finally, because $g(c) = 0$, you have

$$0 = f(x) - f(c) - f'(c)(x - c) - \cdot \cdot \cdot - \frac{f^{(n)}(c)}{n!}(x - c)^n - R_n(x)$$

$$f(x) = f(c) + f'(c)(x - c) + \cdot \cdot \cdot + \frac{f^{(n)}(c)}{n!}(x - c)^n + R_n(x).$$ ■

THEOREM 9.20 CONVERGENCE OF A POWER SERIES (PAGE 662)

For a power series centered at c, precisely one of the following is true.

1. The series converges only at c.
2. There exists a real number $R > 0$ such that the series converges absolutely for $|x - c| < R$, and diverges for $|x - c| > R$.
3. The series converges absolutely for all x.

The number R is the **radius of convergence** of the power series. If the series converges only at c, the radius of convergence is $R = 0$, and if the series converges for all x, the radius of convergence is $R = \infty$. The set of all values of x for which the power series converges is the **interval of convergence** of the power series.

PROOF In order to simplify the notation, the theorem for the power series $\Sigma\, a_n x^n$ centered at $x = 0$ will be proved. The proof for a power series centered at $x = c$ follows easily. A key step in this proof uses the completeness property of the set of real numbers: If a nonempty set S of real numbers has an upper bound, then it must have a least upper bound (see page 603).

It must be shown that if a power series $\Sigma\, a_n x^n$ converges at $x = d$, $d \neq 0$, then it converges for all b satisfying $|b| < |d|$. Because $\Sigma\, a_n x^n$ converges, $\lim\limits_{x\to\infty} a_n d^n = 0$.

So, there exists $N > 0$ such that $a_n d^n < 1$ for all $n \geq N$. Then for $n \geq N$,

$$|a_n b^n| = \left|a_n b^n \frac{d^n}{d^n}\right| = |a_n d^n|\left|\frac{b^n}{d^n}\right| < \left|\frac{b^n}{d^n}\right|.$$

So, for $|b| < |d|$, $\left|\frac{b}{d}\right| < 1$, which implies that

$$\sum \left|\frac{b^n}{d^n}\right|$$

is a convergent geometric series. By the Comparison Test, the series $\sum a_n b^n$ converges.

Similarly, if the power series $\sum a_n x^n$ diverges at $x = b$, where $b \neq 0$, then it diverges for all d satisfying $|d| > |b|$. If $\sum a_n d^n$ converged, then the argument above would imply that $\sum a_n b^n$ converged as well.

Finally, to prove the theorem, suppose that neither Case 1 nor Case 3 is true. Then there exist points b and d such that $\sum a_n x^n$ converges at b and diverges at d. Let $S = \{x: \sum a_n x^n \text{ converges}\}$. S is nonempty because $b \in S$. If $x \in S$, then $|x| \leq |d|$, which shows that $|d|$ is an upper bound for the nonempty set S. By the completeness property, S has a least upper bound, R.

Now, if $|x| > R$, then $x \notin S$, so $\sum a_n x^n$ diverges. And if $|x| < R$, then $|x|$ is not an upper bound for S, so there exists b in S satisfying $|b| > |x|$. Since $b \in S$, $\sum a_n b^n$ converges, which implies that $\sum a_n x^n$ converges. ■

THEOREM 10.16 CLASSIFICATION OF CONICS BY ECCENTRICITY (PAGE 750)

Let F be a fixed point (*focus*) and let D be a fixed line (*directrix*) in the plane. Let P be another point in the plane and let e (*eccentricity*) be the ratio of the distance between P and F to the distance between P and D. The collection of all points P with a given eccentricity is a conic.

1. The conic is an ellipse if $0 < e < 1$.
2. The conic is a parabola if $e = 1$.
3. The conic is a hyperbola if $e > 1$.

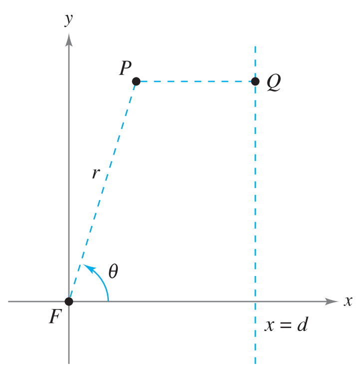


Figure A.1

PROOF If $e = 1$, then, by definition, the conic must be a parabola. If $e \neq 1$, then you can consider the focus F to lie at the origin and the directrix $x = d$ to lie to the right of the origin, as shown in Figure A.1. For the point $P = (r, \theta) = (x, y)$, you have $|PF| = r$ and $|PQ| = d - r\cos\theta$. Given that $e = |PF|/|PQ|$, it follows that

$$|PF| = |PQ|e \quad \Longrightarrow \quad r = e(d - r\cos\theta).$$

By converting to rectangular coordinates and squaring each side, you obtain

$$x^2 + y^2 = e^2(d - x)^2 = e^2(d^2 - 2dx + x^2).$$

Completing the square produces

$$\left(x + \frac{e^2 d}{1 - e^2}\right)^2 + \frac{y^2}{1 - e^2} = \frac{e^2 d^2}{(1 - e^2)^2}.$$

If $e < 1$, this equation represents an ellipse. If $e > 1$, then $1 - e^2 < 0$, and the equation represents a hyperbola. ■

THEOREM 13.4 SUFFICIENT CONDITION FOR DIFFERENTIABILITY (PAGE 919)

If f is a function of x and y, where f_x and f_y are continuous in an open region R, then f is differentiable on R.

PROOF Let S be the surface defined by $z = f(x, y)$, where f, f_x, and f_y are continuous at (x, y). Let A, B, and C be points on surface S, as shown in Figure A.2. From this figure, you can see that the change in f from point A to point C is given by

$$\begin{aligned}\Delta z &= f(x + \Delta x, y + \Delta y) - f(x, y)\\ &= [f(x + \Delta x, y) - f(x, y)] + [f(x + \Delta x, y + \Delta y) - f(x + \Delta x, y)]\\ &= \Delta z_1 + \Delta z_2.\end{aligned}$$

Between A and B, y is fixed and x changes. So, by the Mean Value Theorem, there is a value x_1 between x and $x + \Delta x$ such that

$$\Delta z_1 = f(x + \Delta x, y) - f(x, y) = f_x(x_1, y)\,\Delta x.$$

Similarly, between B and C, x is fixed and y changes, and there is a value y_1 between y and $y + \Delta y$ such that

$$\Delta z_2 = f(x + \Delta x, y + \Delta y) - f(x + \Delta x, y) = f_y(x + \Delta x, y_1)\,\Delta y.$$

By combining these two results, you can write

$$\Delta z = \Delta z_1 + \Delta z_2 = f_x(x_1, y)\Delta x + f_y(x + \Delta x, y_1)\,\Delta y.$$

If you define ε_1 and ε_2 as $\varepsilon_1 = f_x(x_1, y) - f_x(x, y)$ and $\varepsilon_2 = f_y(x + \Delta x, y_1) - f_y(x, y)$, it follows that

$$\begin{aligned}\Delta z = \Delta z_1 + \Delta z_2 &= [\varepsilon_1 + f_x(x, y)]\,\Delta x + [\varepsilon_2 + f_y(x, y)]\,\Delta y\\ &= [f_x(x, y)\,\Delta x + f_y(x, y)\,\Delta y] + \varepsilon_1\Delta x + \varepsilon_2\Delta y.\end{aligned}$$

By the continuity of f_x and f_y and the fact that $x \le x_1 \le x + \Delta x$ and $y \le y_1 \le y + \Delta y$, it follows that $\varepsilon_1 \to 0$ and $\varepsilon_2 \to 0$ as $\Delta x \to 0$ and $\Delta y \to 0$. Therefore, by definition, f is differentiable. ■

THEOREM 13.6 CHAIN RULE: ONE INDEPENDENT VARIABLE (PAGE 925)

Let $w = f(x, y)$, where f is a differentiable function of x and y. If $x = g(t)$ and $y = h(t)$, where g and h are differentiable functions of t, then w is a differentiable function of t, and

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}.$$

PROOF Because g and h are differentiable functions of t, you know that both Δx and Δy approach zero as Δt approaches zero. Moreover, because f is a differentiable function of x and y, you know that $\Delta w = (\partial w/\partial x)\,\Delta x + (\partial w/\partial y)\,\Delta y + \varepsilon_1\Delta x + \varepsilon_2\Delta y$, where both ε_1 and $\varepsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0, 0)$. So, for $\Delta t \neq 0$

$$\frac{\Delta w}{\Delta t} = \frac{\partial w}{\partial x}\frac{\Delta x}{\Delta t} + \frac{\partial w}{\partial y}\frac{\Delta y}{\Delta t} + \varepsilon_1\frac{\Delta x}{\Delta t} + \varepsilon_2\frac{\Delta y}{\Delta t}$$

from which it follows that

$$\frac{dw}{dt} = \lim_{\Delta t \to 0}\frac{\Delta w}{\Delta t} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + 0\left(\frac{dx}{dt}\right) + 0\left(\frac{dy}{dt}\right) = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}.$$ ■

B Integration Tables

Forms Involving u^n

1. $\displaystyle\int u^n\,du = \frac{u^{n+1}}{n+1} + C,\ n \neq -1$

2. $\displaystyle\int \frac{1}{u}\,du = \ln|u| + C$

Forms Involving $a + bu$

3. $\displaystyle\int \frac{u}{a+bu}\,du = \frac{1}{b^2}\bigl(bu - a\ln|a+bu|\bigr) + C$

4. $\displaystyle\int \frac{u}{(a+bu)^2}\,du = \frac{1}{b^2}\left(\frac{a}{a+bu} + \ln|a+bu|\right) + C$

5. $\displaystyle\int \frac{u}{(a+bu)^n}\,du = \frac{1}{b^2}\left[\frac{-1}{(n-2)(a+bu)^{n-2}} + \frac{a}{(n-1)(a+bu)^{n-1}}\right] + C,\quad n \neq 1, 2$

6. $\displaystyle\int \frac{u^2}{a+bu}\,du = \frac{1}{b^3}\left[-\frac{bu}{2}(2a - bu) + a^2\ln|a+bu|\right] + C$

7. $\displaystyle\int \frac{u^2}{(a+bu)^2}\,du = \frac{1}{b^3}\left(bu - \frac{a^2}{a+bu} - 2a\ln|a+bu|\right) + C$

8. $\displaystyle\int \frac{u^2}{(a+bu)^3}\,du = \frac{1}{b^3}\left[\frac{2a}{a+bu} - \frac{a^2}{2(a+bu)^2} + \ln|a+bu|\right] + C$

9. $\displaystyle\int \frac{u^2}{(a+bu)^n}\,du = \frac{1}{b^3}\left[\frac{-1}{(n-3)(a+bu)^{n-3}} + \frac{2a}{(n-2)(a+bu)^{n-2}} - \frac{a^2}{(n-1)(a+bu)^{n-1}}\right] + C,\quad n \neq 1, 2, 3$

10. $\displaystyle\int \frac{1}{u(a+bu)}\,du = \frac{1}{a}\ln\left|\frac{u}{a+bu}\right| + C$

11. $\displaystyle\int \frac{1}{u(a+bu)^2}\,du = \frac{1}{a}\left(\frac{1}{a+bu} + \frac{1}{a}\ln\left|\frac{u}{a+bu}\right|\right) + C$

12. $\displaystyle\int \frac{1}{u^2(a+bu)}\,du = -\frac{1}{a}\left(\frac{1}{u} + \frac{b}{a}\ln\left|\frac{u}{a+bu}\right|\right) + C$

13. $\displaystyle\int \frac{1}{u^2(a+bu)^2}\,du = -\frac{1}{a^2}\left[\frac{a+2bu}{u(a+bu)} + \frac{2b}{a}\ln\left|\frac{u}{a+bu}\right|\right] + C$

Forms Involving $a + bu + cu^2,\ b^2 \neq 4ac$

14. $$\int \frac{1}{a+bu+cu^2}\,du = \begin{cases} \dfrac{2}{\sqrt{4ac-b^2}}\arctan\dfrac{2cu+b}{\sqrt{4ac-b^2}} + C, & b^2 < 4ac \\[2ex] \dfrac{1}{\sqrt{b^2-4ac}}\ln\left|\dfrac{2cu+b-\sqrt{b^2-4ac}}{2cu+b+\sqrt{b^2-4ac}}\right| + C, & b^2 > 4ac \end{cases}$$

15. $\displaystyle\int \frac{u}{a+bu+cu^2}\,du = \frac{1}{2c}\left(\ln|a+bu+cu^2| - b\int \frac{1}{a+bu+cu^2}\,du\right)$

Forms Involving $\sqrt{a+bu}$

16. $\displaystyle\int u^n\sqrt{a+bu}\,du = \frac{2}{b(2n+3)}\left[u^n(a+bu)^{3/2} - na\int u^{n-1}\sqrt{a+bu}\,du\right]$

17. $\displaystyle\int \frac{1}{u\sqrt{a+bu}}\,du = \begin{cases} \dfrac{1}{\sqrt{a}}\ln\left|\dfrac{\sqrt{a+bu}-\sqrt{a}}{\sqrt{a+bu}+\sqrt{a}}\right| + C, & a > 0 \\ \dfrac{2}{\sqrt{-a}}\arctan\sqrt{\dfrac{a+bu}{-a}} + C, & a < 0 \end{cases}$

18. $\displaystyle\int \frac{1}{u^n\sqrt{a+bu}}\,du = \frac{-1}{a(n-1)}\left[\frac{\sqrt{a+bu}}{u^{n-1}} + \frac{(2n-3)b}{2}\int \frac{1}{u^{n-1}\sqrt{a+bu}}\,du\right],\; n \neq 1$

19. $\displaystyle\int \frac{\sqrt{a+bu}}{u}\,du = 2\sqrt{a+bu} + a\int \frac{1}{u\sqrt{a+bu}}\,du$

20. $\displaystyle\int \frac{\sqrt{a+bu}}{u^n}\,du = \frac{-1}{a(n-1)}\left[\frac{(a+bu)^{3/2}}{u^{n-1}} + \frac{(2n-5)b}{2}\int \frac{\sqrt{a+bu}}{u^{n-1}}\,du\right],\; n \neq 1$

21. $\displaystyle\int \frac{u}{\sqrt{a+bu}}\,du = \frac{-2(2a-bu)}{3b^2}\sqrt{a+bu} + C$

22. $\displaystyle\int \frac{u^n}{\sqrt{a+bu}}\,du = \frac{2}{(2n+1)b}\left(u^n\sqrt{a+bu} - na\int \frac{u^{n-1}}{\sqrt{a+bu}}\,du\right)$

Forms Involving $a^2 \pm u^2,\; a > 0$

23. $\displaystyle\int \frac{1}{a^2+u^2}\,du = \frac{1}{a}\arctan\frac{u}{a} + C$

24. $\displaystyle\int \frac{1}{u^2-a^2}\,du = -\int \frac{1}{a^2-u^2}\,du = \frac{1}{2a}\ln\left|\frac{u-a}{u+a}\right| + C$

25. $\displaystyle\int \frac{1}{(a^2 \pm u^2)^n}\,du = \frac{1}{2a^2(n-1)}\left[\frac{u}{(a^2 \pm u^2)^{n-1}} + (2n-3)\int \frac{1}{(a^2 \pm u^2)^{n-1}}\,du\right],\; n \neq 1$

Forms Involving $\sqrt{u^2 \pm a^2},\; a > 0$

26. $\displaystyle\int \sqrt{u^2 \pm a^2}\,du = \frac{1}{2}\left(u\sqrt{u^2 \pm a^2} \pm a^2\ln\left|u + \sqrt{u^2 \pm a^2}\right|\right) + C$

27. $\displaystyle\int u^2\sqrt{u^2 \pm a^2}\,du = \frac{1}{8}\left[u(2u^2 \pm a^2)\sqrt{u^2 \pm a^2} - a^4\ln\left|u + \sqrt{u^2 \pm a^2}\right|\right] + C$

28. $\displaystyle\int \frac{\sqrt{u^2+a^2}}{u}\,du = \sqrt{u^2+a^2} - a\ln\left|\frac{a+\sqrt{u^2+a^2}}{u}\right| + C$

29. $\displaystyle\int \frac{\sqrt{u^2-a^2}}{u}\,du = \sqrt{u^2-a^2} - a\,\text{arcsec}\,\frac{|u|}{a} + C$

30. $\displaystyle\int \frac{\sqrt{u^2 \pm a^2}}{u^2}\,du = \frac{-\sqrt{u^2 \pm a^2}}{u} + \ln\left|u + \sqrt{u^2 \pm a^2}\right| + C$

31. $\displaystyle\int \frac{1}{\sqrt{u^2 \pm a^2}}\,du = \ln\left|u + \sqrt{u^2 \pm a^2}\right| + C$

32. $\displaystyle\int \frac{1}{u\sqrt{u^2+a^2}}\,du = \frac{-1}{a}\ln\left|\frac{a+\sqrt{u^2+a^2}}{u}\right| + C$

33. $\displaystyle\int \frac{1}{u\sqrt{u^2-a^2}}\,du = \frac{1}{a}\,\text{arcsec}\,\frac{|u|}{a} + C$

34. $\displaystyle\int \frac{u^2}{\sqrt{u^2 \pm a^2}}\,du = \frac{1}{2}\left(u\sqrt{u^2 \pm a^2} \mp a^2\ln\left|u + \sqrt{u^2 \pm a^2}\right|\right) + C$

35. $\displaystyle\int \frac{1}{u^2\sqrt{u^2 \pm a^2}}\,du = \mp\frac{\sqrt{u^2 \pm a^2}}{a^2u} + C$

36. $\displaystyle\int \frac{1}{(u^2 \pm a^2)^{3/2}}\,du = \frac{\pm u}{a^2\sqrt{u^2 \pm a^2}} + C$

Forms Involving $\sqrt{a^2 - u^2},\; a > 0$

37. $\displaystyle\int \sqrt{a^2-u^2}\,du = \frac{1}{2}\left(u\sqrt{a^2-u^2} + a^2\arcsin\frac{u}{a}\right) + C$

38. $\displaystyle\int u^2\sqrt{a^2-u^2}\,du = \frac{1}{8}\left[u(2u^2-a^2)\sqrt{a^2-u^2} + a^4\arcsin\frac{u}{a}\right] + C$

39. $\displaystyle\int \frac{\sqrt{a^2 - u^2}}{u}\,du = \sqrt{a^2 - u^2} - a \ln\left|\frac{a + \sqrt{a^2 - u^2}}{u}\right| + C$

40. $\displaystyle\int \frac{\sqrt{a^2 - u^2}}{u^2}\,du = \frac{-\sqrt{a^2 - u^2}}{u} - \arcsin\frac{u}{a} + C$

41. $\displaystyle\int \frac{1}{\sqrt{a^2 - u^2}}\,du = \arcsin\frac{u}{a} + C$

42. $\displaystyle\int \frac{1}{u\sqrt{a^2 - u^2}}\,du = \frac{-1}{a}\ln\left|\frac{a + \sqrt{a^2 - u^2}}{u}\right| + C$

43. $\displaystyle\int \frac{u^2}{\sqrt{a^2 - u^2}}\,du = \frac{1}{2}\left(-u\sqrt{a^2 - u^2} + a^2 \arcsin\frac{u}{a}\right) + C$

44. $\displaystyle\int \frac{1}{u^2\sqrt{a^2 - u^2}}\,du = \frac{-\sqrt{a^2 - u^2}}{a^2 u} + C$

45. $\displaystyle\int \frac{1}{(a^2 - u^2)^{3/2}}\,du = \frac{u}{a^2\sqrt{a^2 - u^2}} + C$

Forms Involving $\sin u$ or $\cos u$

46. $\displaystyle\int \sin u\,du = -\cos u + C$

47. $\displaystyle\int \cos u\,du = \sin u + C$

48. $\displaystyle\int \sin^2 u\,du = \frac{1}{2}(u - \sin u \cos u) + C$

49. $\displaystyle\int \cos^2 u\,du = \frac{1}{2}(u + \sin u \cos u) + C$

50. $\displaystyle\int \sin^n u\,du = -\frac{\sin^{n-1} u \cos u}{n} + \frac{n-1}{n}\int \sin^{n-2} u\,du$

51. $\displaystyle\int \cos^n u\,du = \frac{\cos^{n-1} u \sin u}{n} + \frac{n-1}{n}\int \cos^{n-2} u\,du$

52. $\displaystyle\int u \sin u\,du = \sin u - u\cos u + C$

53. $\displaystyle\int u \cos u\,du = \cos u + u\sin u + C$

54. $\displaystyle\int u^n \sin u\,du = -u^n \cos u + n\int u^{n-1}\cos u\,du$

55. $\displaystyle\int u^n \cos u\,du = u^n \sin u - n\int u^{n-1}\sin u\,du$

56. $\displaystyle\int \frac{1}{1 \pm \sin u}\,du = \tan u \mp \sec u + C$

57. $\displaystyle\int \frac{1}{1 \pm \cos u}\,du = -\cot u \pm \csc u + C$

58. $\displaystyle\int \frac{1}{\sin u \cos u}\,du = \ln|\tan u| + C$

Forms Involving $\tan u$, $\cot u$, $\sec u$, $\csc u$

59. $\displaystyle\int \tan u\,du = -\ln|\cos u| + C$

60. $\displaystyle\int \cot u\,du = \ln|\sin u| + C$

61. $\displaystyle\int \sec u\,du = \ln|\sec u + \tan u| + C$

62. $\displaystyle\int \csc u\,du = \ln|\csc u - \cot u| + C$ or $\displaystyle\int \csc u\,du = -\ln|\csc u + \cot u| + C$

63. $\displaystyle\int \tan^2 u\,du = -u + \tan u + C$

64. $\displaystyle\int \cot^2 u\,du = -u - \cot u + C$

65. $\displaystyle\int \sec^2 u\,du = \tan u + C$

66. $\displaystyle\int \csc^2 u\,du = -\cot u + C$

67. $\displaystyle\int \tan^n u\,du = \frac{\tan^{n-1} u}{n-1} - \int \tan^{n-2} u\,du,\ n \neq 1$

68. $\displaystyle\int \cot^n u\,du = -\frac{\cot^{n-1} u}{n-1} - \int (\cot^{n-2} u)\,du,\ n \neq 1$

69. $\displaystyle\int \sec^n u\,du = \frac{\sec^{n-2} u \tan u}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2} u\,du,\ n \neq 1$

70. $\displaystyle\int \csc^n u\,du = -\frac{\csc^{n-2} u \cot u}{n-1} + \frac{n-2}{n-1}\int \csc^{n-2} u\,du,\ n \neq 1$

71. $\displaystyle\int \frac{1}{1 \pm \tan u}\,du = \frac{1}{2}\left(u \pm \ln|\cos u \pm \sin u|\right) + C$

72. $\displaystyle\int \frac{1}{1 \pm \cot u}\,du = \frac{1}{2}\left(u \mp \ln|\sin u \pm \cos u|\right) + C$

73. $\displaystyle\int \frac{1}{1 \pm \sec u}\,du = u + \cot u \mp \csc u + C$

74. $\displaystyle\int \frac{1}{1 \pm \csc u}\,du = u - \tan u \pm \sec u + C$

Forms Involving Inverse Trigonometric Functions

75. $\displaystyle\int \arcsin u\,du = u \arcsin u + \sqrt{1 - u^2} + C$

76. $\displaystyle\int \arccos u\,du = u \arccos u - \sqrt{1 - u^2} + C$

77. $\displaystyle\int \arctan u\,du = u \arctan u - \ln\sqrt{1 + u^2} + C$

78. $\displaystyle\int \operatorname{arccot} u\,du = u \operatorname{arccot} u + \ln\sqrt{1 + u^2} + C$

79. $\displaystyle\int \operatorname{arcsec} u\,du = u \operatorname{arcsec} u - \ln\left|u + \sqrt{u^2 - 1}\right| + C$

80. $\displaystyle\int \operatorname{arccsc} u\,du = u \operatorname{arccsc} u + \ln\left|u + \sqrt{u^2 - 1}\right| + C$

Forms Involving e^u

81. $\displaystyle\int e^u\,du = e^u + C$

82. $\displaystyle\int ue^u\,du = (u - 1)e^u + C$

83. $\displaystyle\int u^n e^u\,du = u^n e^u - n\int u^{n-1} e^u\,du$

84. $\displaystyle\int \frac{1}{1 + e^u}\,du = u - \ln(1 + e^u) + C$

85. $\displaystyle\int e^{au} \sin bu\,du = \frac{e^{au}}{a^2 + b^2}(a \sin bu - b \cos bu) + C$

86. $\displaystyle\int e^{au} \cos bu\,du = \frac{e^{au}}{a^2 + b^2}(a \cos bu + b \sin bu) + C$

Forms Involving $\ln u$

87. $\displaystyle\int \ln u\,du = u(-1 + \ln u) + C$

88. $\displaystyle\int u \ln u\,du = \frac{u^2}{4}(-1 + 2 \ln u) + C$

89. $\displaystyle\int u^n \ln u\,du = \frac{u^{n+1}}{(n + 1)^2}\left[-1 + (n + 1) \ln u\right] + C,\ n \neq -1$

90. $\displaystyle\int (\ln u)^2\,du = u\left[2 - 2 \ln u + (\ln u)^2\right] + C$

91. $\displaystyle\int (\ln u)^n\,du = u(\ln u)^n - n\int (\ln u)^{n-1}\,du$

Forms Involving Hyperbolic Functions

92. $\displaystyle\int \cosh u\,du = \sinh u + C$

93. $\displaystyle\int \sinh u\,du = \cosh u + C$

94. $\displaystyle\int \operatorname{sech}^2 u\,du = \tanh u + C$

95. $\displaystyle\int \operatorname{csch}^2 u\,du = -\coth u + C$

96. $\displaystyle\int \operatorname{sech} u \tanh u\,du = -\operatorname{sech} u + C$

97. $\displaystyle\int \operatorname{csch} u \coth u\,du = -\operatorname{csch} u + C$

Forms Involving Inverse Hyperbolic Functions (in logarithmic form)

98. $\displaystyle\int \frac{du}{\sqrt{u^2 \pm a^2}} = \ln\left(u + \sqrt{u^2 \pm a^2}\right) + C$

99. $\displaystyle\int \frac{du}{a^2 - u^2} = \frac{1}{2a} \ln\left|\frac{a + u}{a - u}\right| + C$

100. $\displaystyle\int \frac{du}{u\sqrt{a^2 \pm u^2}} = -\frac{1}{a} \ln \frac{a + \sqrt{a^2 \pm u^2}}{|u|} + C$

Answers to Odd-Numbered Exercises

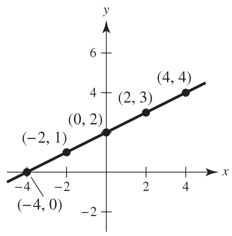
Chapter P

Section P.1 (page 8)

1. b **2.** d **3.** a **4.** c

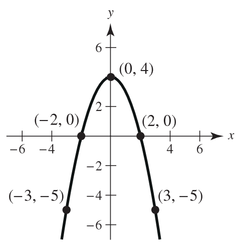
5. Answers will vary.

x	-4	-2	0	2	4
y	0	1	2	3	4



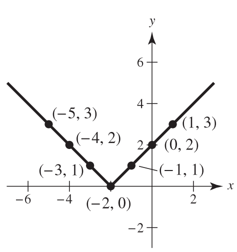
7. Answers will vary.

x	-3	-2	0	2	3
y	-5	0	4	0	-5



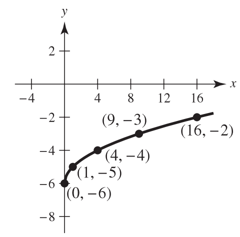
9. Answers will vary.

x	-5	-4	-3	-2	-1	0	1
y	3	2	1	0	1	2	3



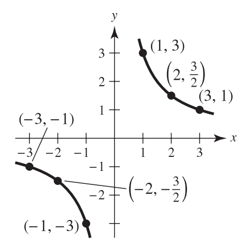
11. Answers will vary.

x	0	1	4	9	16
y	-6	-5	-4	-3	-2



13. Answers will vary.

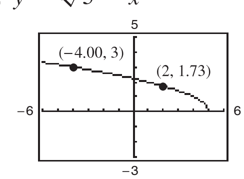
x	-3	-2	-1	0	1	2	3
y	-1	$-\frac{3}{2}$	-3	Undef.	3	$\frac{3}{2}$	1



15.

Xmin = -5
Xmax = 4
Xscl = 1
Ymin = -5
Ymax = 8
Yscl = 1

17. $y = \sqrt{5 - x}$



(a) $y \approx 1.73$ (b) $x = -4$

19. $(0, -5), \left(\frac{5}{2}, 0\right)$ **21.** $(0, -2), (-2, 0), (1, 0)$

23. $(0, 0), (4, 0), (-4, 0)$ **25.** $(4, 0)$ **27.** $(0, 0)$

29. Symmetric with respect to the y-axis

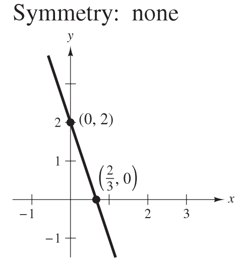
31. Symmetric with respect to the x-axis

33. Symmetric with respect to the origin **35.** No symmetry

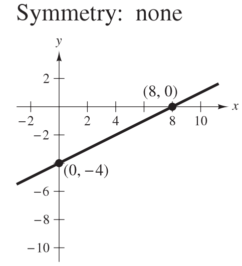
37. Symmetric with respect to the origin

39. Symmetric with respect to the y-axis

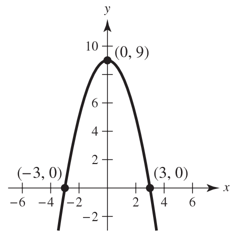
41. $y = 2 - 3x$
Symmetry: none



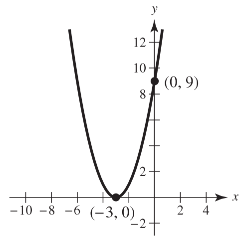
43. $y = \frac{1}{2}x - 4$
Symmetry: none



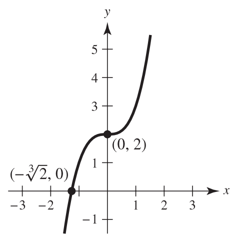
45. $y = 9 - x^2$
Symmetry: y-axis



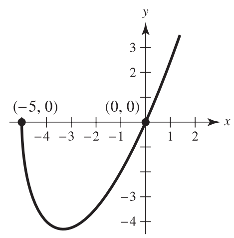
47. $y = (x + 3)^2$
Symmetry: none



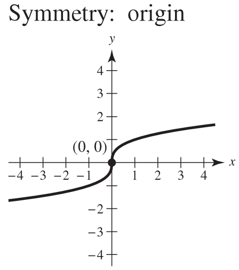
49. $y = x^3 + 2$
Symmetry: none



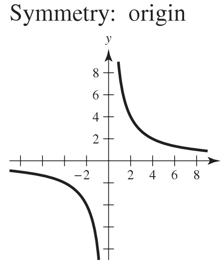
51. $y = x\sqrt{x + 5}$
Symmetry: none



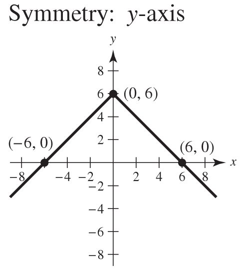
53. $x = y^3$
Symmetry: origin



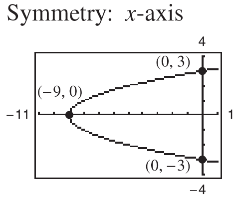
55. $y = 8/x$
Symmetry: origin



57. $y = 6 - |x|$
Symmetry: y-axis



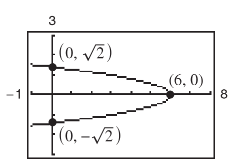
59. $y_1 = \sqrt{x + 9}$
$y_2 = -\sqrt{x + 9}$
Symmetry: x-axis



61. $y_1 = \sqrt{\dfrac{6 - x}{3}}$

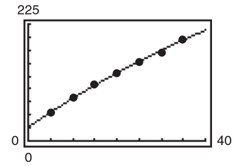
$y_2 = -\sqrt{\dfrac{6 - x}{3}}$

Symmetry: x-axis



63. $(3, 5)$ **65.** $(-1, 5), (2, 2)$ **67.** $(-1, -2), (2, 1)$
69. $(-1, -1), (0, 0), (1, 1)$
71. $(-1, -5), (0, -1), (2, 1)$ **73.** $(-2, 2), (-3, \sqrt{3})$

75. (a) $y = -0.027t^2 + 5.73t + 26.9$
(b)



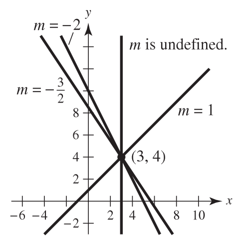
The model is a good fit for the data.
(c) 212.9

77. $x \approx 3133$ units **79.** $y = (x + 4)(x - 3)(x - 8)$
81. (a) Proof (b) Proof
83. False. $(4, -5)$ is not a point on the graph of $x = y^2 - 29$.
85. True **87.** $x^2 + (y - 4)^2 = 4$

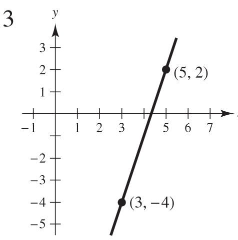
Section P.2 (page 16)

1. $m = 1$ **3.** $m = 0$ **5.** $m = -12$

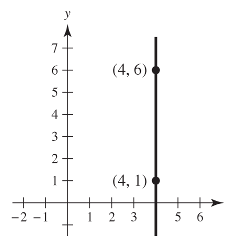
7.



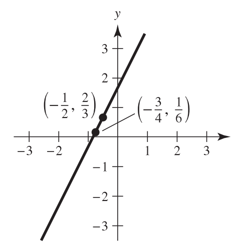
9. $m = 3$



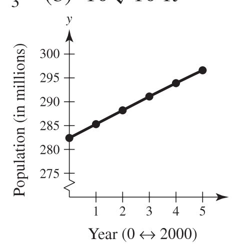
11. m is undefined.



13. $m = 2$

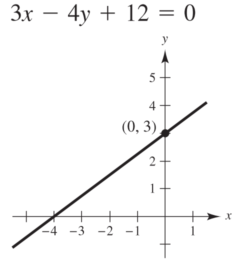


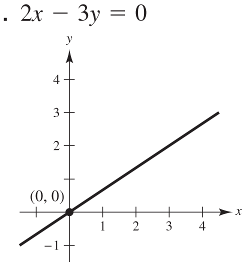
15. $(0, 2), (1, 2), (5, 2)$ **17.** $(0, 10), (2, 4), (3, 1)$
19. (a) $\frac{1}{3}$ (b) $10\sqrt{10}$ ft
21. (a)



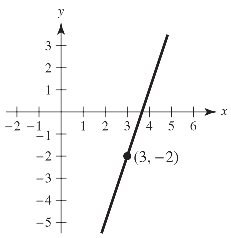
(b) Population increased least rapidly from 2004 to 2005.

23. $m = 4, (0, -3)$
25. $m = -\frac{1}{5}, (0, 4)$ **27.** m is undefined, no y-intercept
29. $3x - 4y + 12 = 0$ **31.** $2x - 3y = 0$

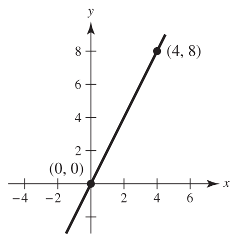




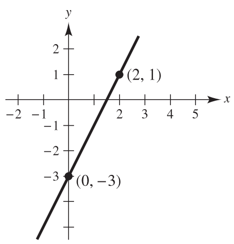
33. $3x - y - 11 = 0$



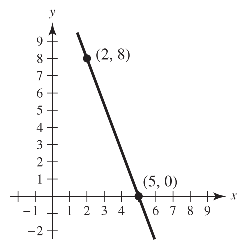
35. $2x - y = 0$



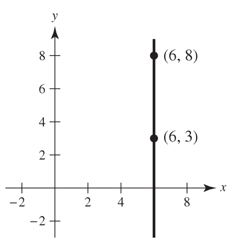
37. $2x - y - 3 = 0$



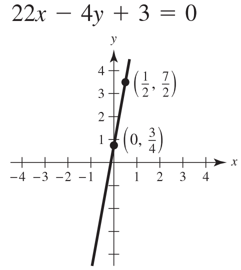
39. $8x + 3y - 40 = 0$



41. $x - 6 = 0$

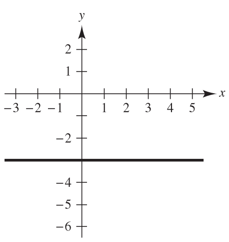


43. $22x - 4y + 3 = 0$

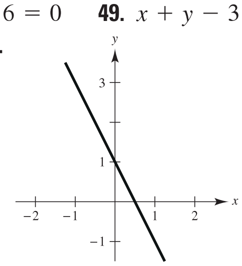


45. $x - 3 = 0$ **47.** $3x + 2y - 6 = 0$ **49.** $x + y - 3 = 0$

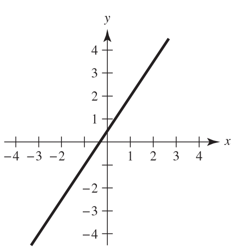
51.



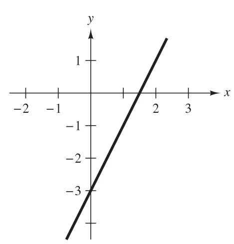
53.



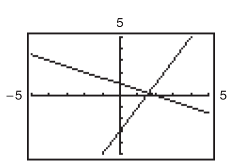
55.



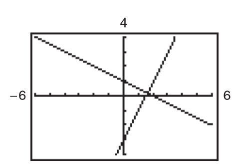
57.



59. (a)



(b)



The lines in (a) do not appear perpendicular, but they do in (b) because a square setting is used. The lines are perpendicular.

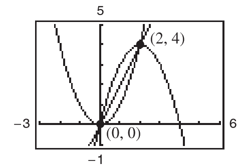
61. (a) $x + 7 = 0$ (b) $y + 2 = 0$

63. (a) $2x - y - 3 = 0$ (b) $x + 2y - 4 = 0$

65. (a) $40x - 24y - 9 = 0$ (b) $24x + 40y - 53 = 0$

67. $V = 250t - 150$ **69.** $V = -1600t + 30{,}000$

71. $y = 2x$ **73.** Not collinear, because $m_1 \neq m_2$

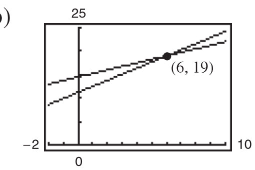


75. $\left(0, \dfrac{-a^2 + b^2 + c^2}{2c}\right)$ **77.** $\left(b, \dfrac{a^2 - b^2}{c}\right)$

79. $5F - 9C - 160 = 0$; $72°\text{F} \approx 22.2°\text{C}$

81. (a) $W_1 = 14.50 + 0.75x$, $W_2 = 11.20 + 1.30x$

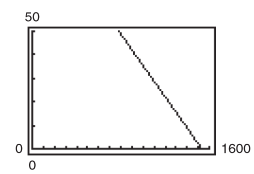
(b)



(c) When six units are produced, the wage for both options is \$19.00 per hour. Choose option 1 if you think you will produce less than six units and choose option 2 if you think you will produce more than six units.

83. (a) $x = (1530 - p)/15$

(b)



(c) 49 units

45 units

85. $12y + 5x - 169 = 0$ **87.** 2 **89.** $(5\sqrt{2})/2$ **91.** $2\sqrt{2}$

93. Proof **95.** Proof **97.** Proof **99.** True

Section P.3 (page 27)

1. (a) Domain of f: $[-4, 4]$; Range of f: $[-3, 5]$
Domain of g: $[-3, 3]$; Range of g: $[-4, 4]$
(b) $f(-2) = -1$; $g(3) = -4$
(c) $x = -1$ (d) $x = 1$ (e) $x = -1$, $x = 1$, and $x = 2$

3. (a) -4 (b) -25 (c) $7b - 4$ (d) $7x - 11$

5. (a) 5 (b) 0 (c) 1 (d) $4 + 2t - t^2$

7. (a) 1 (b) 0 (c) $-\frac{1}{2}$ **9.** $3x^2 + 3x\,\Delta x + (\Delta x)^2$, $\Delta x \neq 0$

11. $\left(\sqrt{x-1} - x + 1\right)/[(x-2)(x-1)]$
$= -1/\left[\sqrt{x-1}\left(1 + \sqrt{x-1}\right)\right]$, $x \neq 2$

13. Domain: $(-\infty, \infty)$; Range: $[0, \infty)$

15. Domain: $[0, \infty)$; Range: $[0, \infty)$

17. Domain: All real numbers t such that $t \neq 4n + 2$, where n is an integer; Range: $(-\infty, -1] \cup [1, \infty)$

19. Domain: $(-\infty, 0) \cup (0, \infty)$; Range: $(-\infty, 0) \cup (0, \infty)$

21. Domain: $[0, 1]$

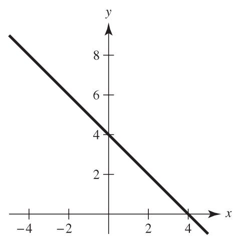
23. Domain: All real numbers x such that $x \neq 2n\pi$, where n is an integer

25. Domain: $(-\infty, -3) \cup (-3, \infty)$

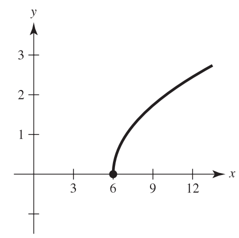
27. (a) -1 (b) 2 (c) 6 (d) $2t^2 + 4$
Domain: $(-\infty, \infty)$; Range: $(-\infty, 1) \cup [2, \infty)$

29. (a) 4 (b) 0 (c) -2 (d) $-b^2$
Domain: $(-\infty, \infty)$; Range: $(-\infty, 0] \cup [1, \infty)$

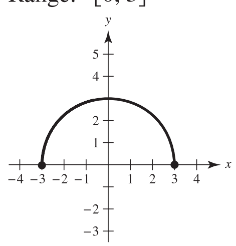
31. $f(x) = 4 - x$
Domain: $(-\infty, \infty)$
Range: $(-\infty, \infty)$



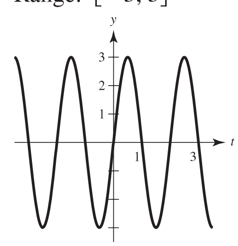
33. $h(x) = \sqrt{x - 6}$
Domain: $[6, \infty)$
Range: $[0, \infty)$



35. $f(x) = \sqrt{9 - x^2}$
Domain: $[-3, 3]$
Range: $[0, 3]$



37. $g(t) = 3 \sin \pi t$
Domain: $(-\infty, \infty)$
Range: $[-3, 3]$



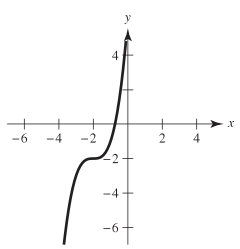
39. The student travels $\frac{1}{2}$ mile/minute during the first 4 minutes, is stationary for the next 2 minutes, and travels 1 mile/minute during the final 4 minutes.

41. y is not a function of x. **43.** y is a function of x.

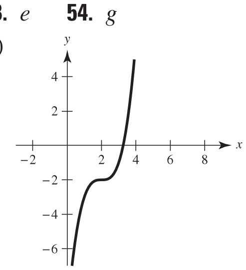
45. y is not a function of x. **47.** y is not a function of x.

49. d **50.** b **51.** c **52.** a **53.** e **54.** g

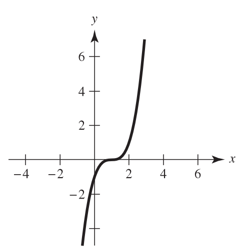
55. (a)



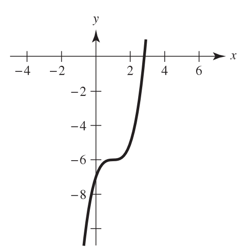
(b)



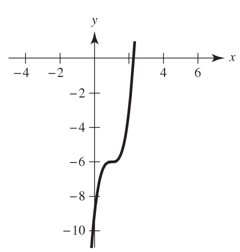
(c)



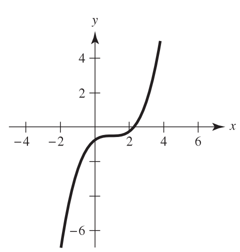
(d)



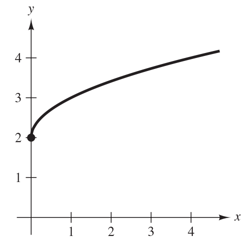
(e)



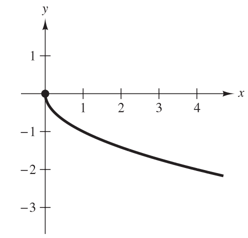
(f)



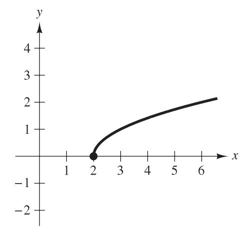
57. (a) Vertical translation



(b) Reflection about the x-axis



(c) Horizontal translation



59. (a) 0 (b) 0 (c) -1 (d) $\sqrt{15}$
(e) $\sqrt{x^2 - 1}$ (f) $x - 1$ $(x \geq 0)$

61. $(f \circ g)(x) = x$; Domain: $[0, \infty)$
$(g \circ f)(x) = |x|$; Domain: $(-\infty, \infty)$
No, their domains are different.

63. $(f \circ g)(x) = 3/(x^2 - 1)$; Domain: $(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$
$(g \circ f)(x) = (9/x^2) - 1$; Domain: $(-\infty, 0) \cup (0, \infty)$
No

65. (a) 4 (b) -2
(c) Undefined. The graph of g does not exist at $x = -5$.
(d) 3 (e) 2
(f) Undefined. The graph of f does not exist at $x = -4$.

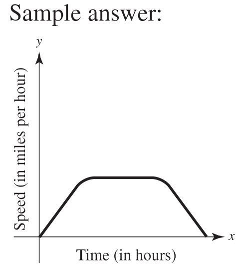
67. Answers will vary.
Example: $f(x) = \sqrt{x}$; $g(x) = x - 2$; $h(x) = 2x$

69. Even **71.** Odd **73.** (a) $\left(\frac{3}{2}, 4\right)$ (b) $\left(\frac{3}{2}, -4\right)$

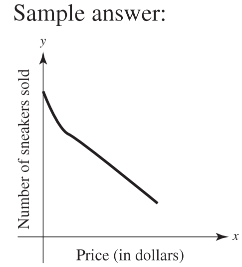
75. f is even. g is neither even nor odd. h is odd.

77. $f(x) = -5x - 6, -2 \leq x \leq 0$ **79.** $y = -\sqrt{-x}$

81. Answers will vary.
Sample answer:



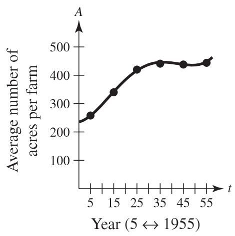
83. Answers will vary.
Sample answer:



85. $c = 25$

87. (a) $T(4) = 16°\text{C}, T(15) \approx 23°\text{C}$
(b) The changes in temperature occur 1 hour later.
(c) The temperatures are 1° lower.

89. (a)



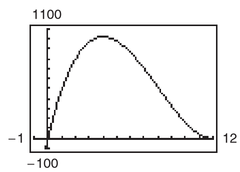
(b) $A(20) \approx 384$ acres/farm

91. $f(x) = |x| + |x - 2| = \begin{cases} 2x - 2, & \text{if } x \geq 2 \\ 2, & \text{if } 0 < x < 2 \\ -2x + 2, & \text{if } x \leq 0 \end{cases}$

93. Proof **95.** Proof

97. (a) $V(x) = x(24 - 2x)^2, 0 < x < 12$

(b)



$4 \times 16 \times 16$ cm

(c)

Height, x	*Length and Width*	*Volume, V*
1	$24 - 2(1)$	$1[24 - 2(1)]^2 = 484$
2	$24 - 2(2)$	$2[24 - 2(2)]^2 = 800$
3	$24 - 2(3)$	$3[24 - 2(3)]^2 = 972$
4	$24 - 2(4)$	$4[24 - 2(4)]^2 = 1024$
5	$24 - 2(5)$	$5[24 - 2(5)]^2 = 980$
6	$24 - 2(6)$	$6[24 - 2(6)]^2 = 864$

The dimensions of the box that yield a maximum volume are $4 \times 16 \times 16$ cm.

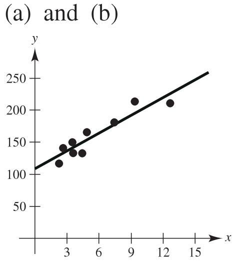
99. False. For example, if $f(x) = x^2$, then $f(-1) = f(1)$.

101. True **103.** Putnam Problem A1, 1988

Section P.4 (page 34)

1. Trigonometric **3.** No relationship

5. (a) and (b)

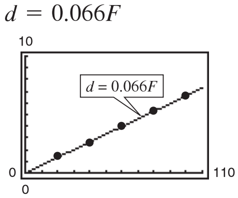


Approximately linear

(c) 136

7. (a) $d = 0.066F$

(b)

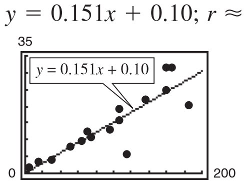


The model fits well.

(c) 3.63 cm

9. (a) $y = 0.151x + 0.10$; $r \approx 0.880$

(b)

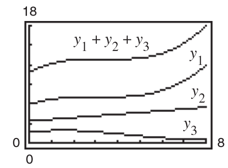


(c) Greater per capita energy consumption by a country tends to correspond to greater per capita gross national product of the country. The four countries that differ most from the linear model are Venezuela, South Korea, Hong Kong, and the United Kingdom.

(d) $y = 0.155x + 0.22$; $r \approx 0.984$

11. (a) $y_1 = 0.04040t^3 - 0.3695t^2 + 1.123t + 5.88$
$y_2 = 0.264t + 3.35$
$y_3 = 0.01439t^3 - 0.1886t^2 + 0.476t + 1.59$

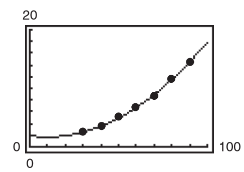
(b) $y_1 + y_2 + y_3 = 0.05479t^3 - 0.5581t^2 + 1.863t + 10.82$



About 47.5 cents/mi

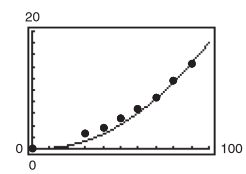
13. (a) $t = 0.002s^2 - 0.04s + 1.9$

(b)



(c) According to the model, the times required to attain speeds of less than 20 miles per hour are all about the same.

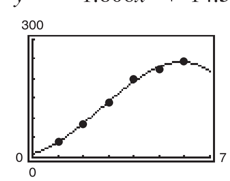
(d) $t = 0.002s^2 + 0.02s + 0.1$



(e) No. From the graph in part (b), you can see that the model from part (a) follows the data more closely than the model from part (d).

15. (a) $y = -1.806x^3 + 14.58x^2 + 16.4x + 10$

(b)

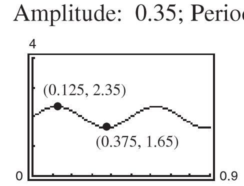


(c) 214 hp

17. (a) Yes. At time t there is one and only one displacement y.

(b) Amplitude: 0.35; Period: 0.5 (c) $y = 0.35 \sin(4\pi t) + 2$

(d)



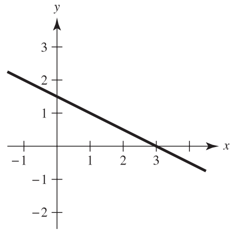
The model appears to fit the data well.

19. Answers will vary. **21.** Putnam Problem A2, 2004

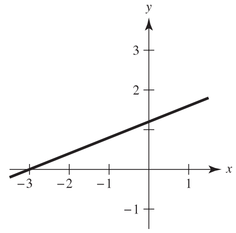
Review Exercises for Chapter P (page 37)

1. $\left(\frac{8}{5}, 0\right), (0, -8)$ **3.** $(3, 0), \left(0, \frac{3}{4}\right)$ **5.** y-axis symmetry

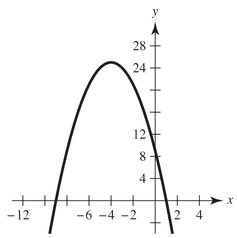
7.



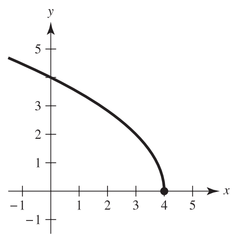
9.



11.



13.

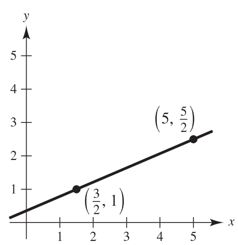


15.

Xmin = -5
Xmax = 5
Xscl = 1
Ymin = -30
Ymax = 10
Yscl = 5

17. $(-2, 3)$ **19.** $y = x^3 - 16x$

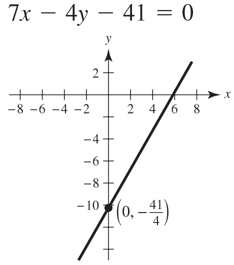
21.



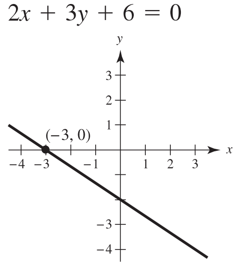
$m = \frac{3}{7}$

23. $t = \frac{1}{5}$

25. $y = \frac{7}{4}x - \frac{41}{4}$ or $7x - 4y - 41 = 0$



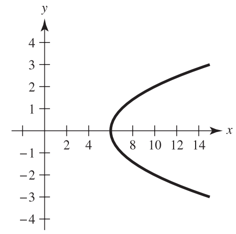
27. $y = -\frac{2}{3}x - 2$ or $2x + 3y + 6 = 0$



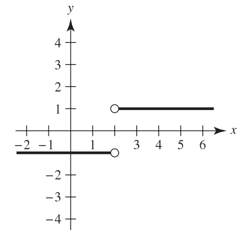
29. (a) $7x - 16y + 101 = 0$ (b) $5x - 3y + 30 = 0$ (c) $5x + 3y = 0$ (d) $x + 3 = 0$

31. $V = 12{,}500 - 850t$; \$9950

33. Not a function



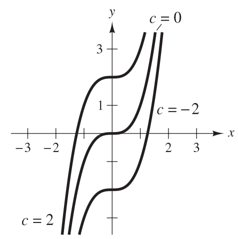
35. Function



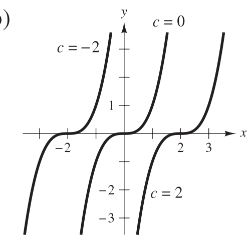
37. (a) Undefined (b) $-1/(1 + \Delta x), \Delta x \neq 0, -1$

39. (a) Domain: $[-6, 6]$; Range: $[0, 6]$
(b) Domain: $(-\infty, 5) \cup (5, \infty)$; Range: $(-\infty, 0) \cup (0, \infty)$
(c) Domain: $(-\infty, \infty)$; Range: $(-\infty, \infty)$

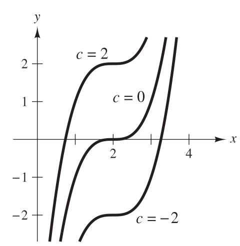
41. (a)



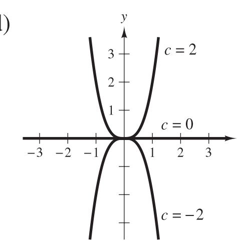
(b)



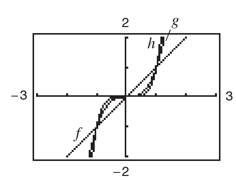
(c)

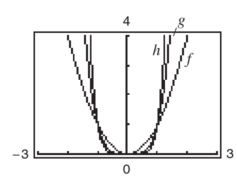


(d)



43. (a)

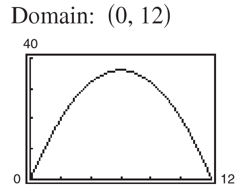




All the graphs pass through the origin. The graphs of the odd powers of x are symmetric with respect to the origin and the graphs of the even powers are symmetric with respect to the y-axis. As the powers increase, the graphs become flatter in the interval $-1 < x < 1$. Graphs of these equations with odd powers pass through Quadrants I and III. Graphs of these equations with even powers pass through Quadrants I and II.

(b) The graph of $y = x^7$ should pass through the origin and Quadrants I and III. It should be symmetric with respect to the origin and be fairly flat in the interval $(-1, 1)$. The graph of $y = x^8$ should pass through the origin and Quadrants I and II. It should be symmetric with respect to the y-axis and be fairly flat in the interval $(-1, 1)$.

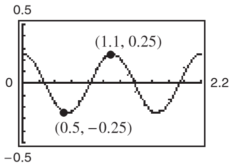
45. (a) $A = x(12 - x)$
(b) Domain: $(0, 12)$



(c) Maximum area: 36 in.^{2}; 6×6 in.

47. (a) Minimum degree: 3; Leading coefficient: negative
(b) Minimum degree: 4; Leading coefficient: positive
(c) Minimum degree: 2; Leading coefficient: negative
(d) Minimum degree: 5; Leading coefficient: positive

49. (a) Yes. For each time t there corresponds one and only one displacement y.
(b) Amplitude: 0.25; Period: 1.1 (c) $y \approx \frac{1}{4}\cos(5.7t)$
(d)

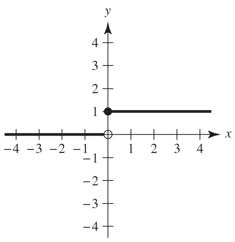


The model appears to fit the data.

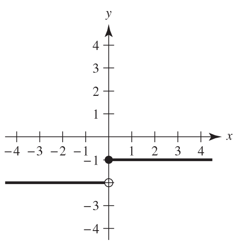
P.S. Problem Solving (page 39)

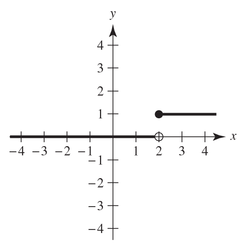
1. (a) Center: $(3, 4)$; Radius: 5
(b) $y = -\frac{3}{4}x$ (c) $y = \frac{3}{4}x - \frac{9}{2}$ (d) $\left(3, -\frac{9}{4}\right)$

3.

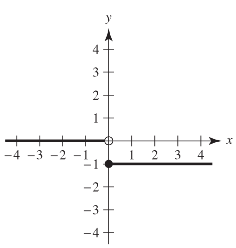


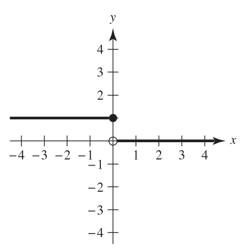
(a) $H(x) - 2 = \begin{cases} -1, & x \geq 0 \\ -2, & x < 0 \end{cases}$ (b) $H(x - 2) = \begin{cases} 1, & x \geq 2 \\ 0, & x < 2 \end{cases}$



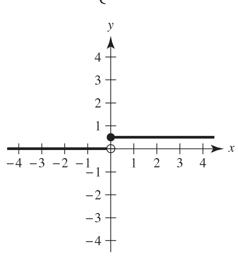


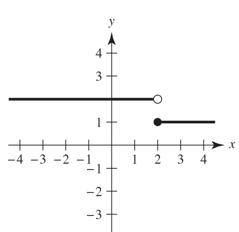
(c) $-H(x) = \begin{cases} -1, & x \geq 0 \\ 0, & x < 0 \end{cases}$ (d) $H(-x) = \begin{cases} 1, & x \leq 0 \\ 0, & x > 0 \end{cases}$



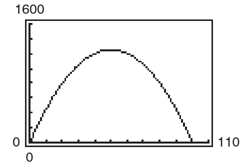


(e) $\frac{1}{2}H(x) = \begin{cases} \frac{1}{2}, & x \geq 0 \\ 0, & x < 0 \end{cases}$ (f) $-H(x - 2) + 2 = \begin{cases} 1, & x \geq 2 \\ 2, & x < 2 \end{cases}$





5. (a) $A(x) = x[(100 - x)/2]$; Domain: $(0, 100)$
(b)



Dimensions 50 m × 25 m yield maximum area of 1250 m^2.

(c) 50 m × 25 m; Area = 1250 m^2

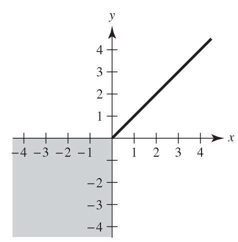
7. $T(x) = \left[2\sqrt{4 + x^2} + \sqrt{(3 - x)^2 + 1}\right]/4$

9. (a) 5, less (b) 3, greater (c) 4.1, less
(d) $4 + h$ (e) 4; Answers will vary.

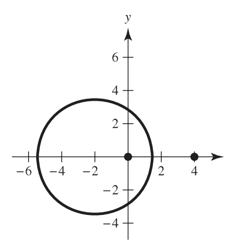
11. Using the definition of absolute value, you can rewrite the equation as

$$\begin{cases} 2y, & y > 0 \\ 0, & y \leq 0 \end{cases} = \begin{cases} 2x, & x > 0 \\ 0, & x \leq 0 \end{cases}.$$

For $x > 0$ and $y > 0$, you have $2y = 2x \rightarrow y = x$. For any $x \leq 0$, y is any $y \leq 0$. So, the graph of $y + |y| = x + |x|$ is as follows.

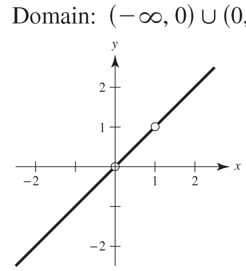


13. (a) $\left(x + \dfrac{4}{k - 1}\right)^2 + y^2 = \dfrac{16k}{(k - 1)^2}$
(b)



(c) As k becomes very large, $\dfrac{4}{k - 1} \rightarrow 0$ and $\dfrac{16k}{(k - 1)^2} \rightarrow 0$.
The center of the circle gets closer to $(0, 0)$, and its radius approaches 0.

15. (a) Domain: $(-\infty, 1) \cup (1, \infty)$; Range: $(-\infty, 0) \cup (0, \infty)$
(b) $f(f(x)) = \dfrac{x - 1}{x}$
Domain: $(-\infty, 0) \cup (0, 1) \cup (1, \infty)$
(c) $f(f(f(x))) = x$
Domain: $(-\infty, 0) \cup (0, 1) \cup (1, \infty)$
(d)



The graph is not a line because there are holes at $x = 0$ and $x = 1$.

Chapter 1

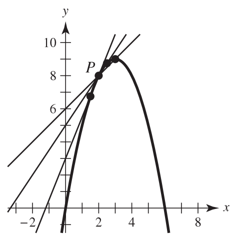
Section 1.1 (page 47)

1. Precalculus: 300 ft

3. Calculus: Slope of the tangent line at $x = 2$ is 0.16.

5. (a) Precalculus: 10 square units (b) Calculus: 5 square units

7. (a)



(b) $1; \frac{3}{2}; \frac{5}{2}$

(c) 2. Use points closer to P.

9. (a) Area ≈ 10.417; Area ≈ 9.145 (b) Use more rectangles.

11. (a) 5.66 (b) 6.11 (c) Increase the number of line segments.

Section 1.2 (page 54)

1.

x	3.9	3.99	3.999	4.001	4.01	4.1
$f(x)$	0.2041	0.2004	0.2000	0.2000	0.1996	0.1961

$$\lim_{x\to 4} \frac{x-4}{x^2-3x-4} \approx 0.2000 \left(\text{Actual limit is } \frac{1}{5}.\right)$$

3.

x	-0.1	-0.01	-0.001	0.001	0.01	0.1
$f(x)$	0.2050	0.2042	0.2041	0.2041	0.2040	0.2033

$$\lim_{x\to 0} \frac{\sqrt{x+6}-\sqrt{6}}{x} \approx 0.2041 \left(\text{Actual limit is } \frac{1}{2\sqrt{6}}.\right)$$

5.

x	2.9	2.99	2.999
$f(x)$	-0.0641	-0.0627	-0.0625

x	3.001	3.01	3.1
$f(x)$	-0.0625	-0.0623	-0.0610

$$\lim_{x\to 3} \frac{[1/(x+1)]-(1/4)}{x-3} \approx -0.0625 \left(\text{Actual limit is } -\frac{1}{16}.\right)$$

7.

x	-0.1	-0.01	-0.001
$f(x)$	0.9983	0.99998	1.0000

x	0.001	0.01	0.1
$f(x)$	1.0000	0.99998	0.9983

$$\lim_{x\to 0} \frac{\sin x}{x} \approx 1.0000 \text{ (Actual limit is 1.)}$$

9.

x	0.9	0.99	0.999	1.001	1.01	1.1
$f(x)$	0.2564	0.2506	0.2501	0.2499	0.2494	0.2439

$$\lim_{x\to 1} \frac{x-2}{x^2+x-6} \approx 0.2500 \left(\text{Actual limit is } \frac{1}{4}.\right)$$

11.

x	0.9	0.99	0.999	1.001	1.01	1.1
$f(x)$	0.7340	0.6733	0.6673	0.6660	0.6600	0.6015

$$\lim_{x\to 1} \frac{x^4-1}{x^6-1} \approx 0.6666 \left(\text{Actual limit is } \frac{2}{3}.\right)$$

13.

x	-0.1	-0.01	-0.001	0.001	0.01	0.1
$f(x)$	1.9867	1.9999	2.0000	2.0000	1.9999	1.9867

$$\lim_{x\to 0} \frac{\sin 2x}{x} \approx 2.0000 \text{ (Actual limit is 2.)}$$

15. 1 **17.** 2

19. Limit does not exist. The function approaches 1 from the right side of 2 but it approaches -1 from the left side of 2.

21. 0

23. Limit does not exist. The function oscillates between 1 and -1 as x approaches 0.

25. (a) 2

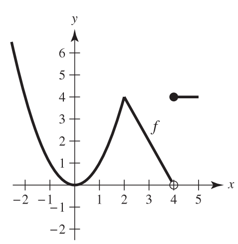
(b) Limit does not exist. The function approaches 1 from the right side of 1 but it approaches 3.5 from the left side of 1.

(c) Value does not exist. The function is undefined at $x = 4$.

(d) 2

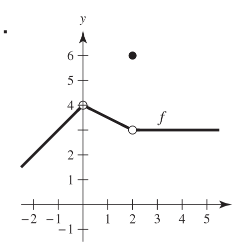
27. $\lim_{x\to c} f(x)$ exists for all points on the graph except where $c = -3$.

29.

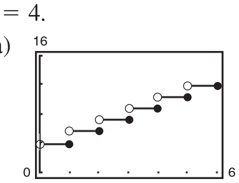


$\lim_{x\to c} f(x)$ exists for all points on the graph except where $c = 4$.

31.



33. (a)



(b)

t	3	3.3	3.4	3.5	3.6	3.7	4
C	11.57	12.36	12.36	12.36	12.36	12.36	12.36

$\lim_{t\to 3.5} C(t) = 12.36$

(c)

t	2	2.5	2.9	3	3.1	3.5	4
C	10.78	11.57	11.57	11.57	12.36	12.36	12.36

The limit does not exist because the limits from the right and left are not equal.

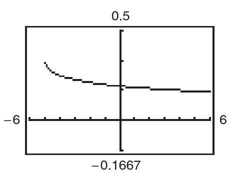
35. $\delta = 0.4$ **37.** $\delta = \frac{1}{11} \approx 0.091$

39. $L = 8$. Let $\delta = 0.01/3 \approx 0.0033$.

41. $L = 1$. Let $\delta = 0.01/5 = 0.002$.

43. 6 **45.** -3 **47.** 3 **49.** 0 **51.** 10 **53.** 2 **55.** 4

57.

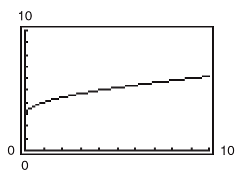


$\lim_{x\to 4} f(x) = \frac{1}{6}$

Domain: $[-5, 4) \cup (4, \infty)$

The graph has a hole at $x = 4$.

59.



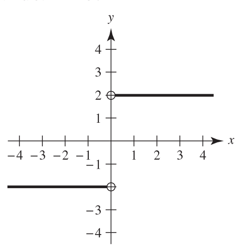
$\lim_{x\to 9} f(x) = 6$

Domain: $[0, 9) \cup (9, \infty)$

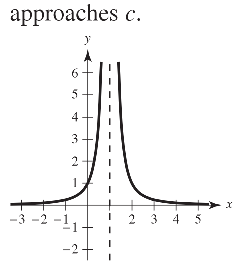
The graph has a hole at $x = 9$.

61. Answers will vary. Sample answer: As x approaches 8 from either side, $f(x)$ becomes arbitrarily close to 25.

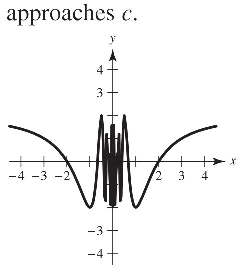
63. (i) The values of f approach different numbers as x approaches c from different sides of c.



(ii) The values of f increase or decrease without bound as x approaches c.



(iii) The values of f oscillate between two fixed numbers as x approaches c.



65. (a) $r = \dfrac{3}{\pi} \approx 0.9549$ cm

(b) $\dfrac{5.5}{2\pi} \le r \le \dfrac{6.5}{2\pi}$, or approximately $0.8754 < r < 1.0345$

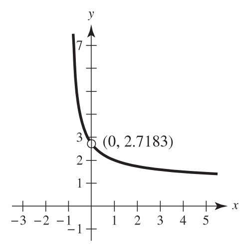
(c) $\lim_{r\to 3/\pi} 2\pi r = 6$; $\varepsilon = 0.5$; $\delta \approx 0.0796$

67.

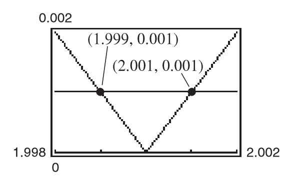
x	-0.001	-0.0001	-0.00001
$f(x)$	2.7196	2.7184	2.7183

x	0.00001	0.0001	0.001
$f(x)$	2.7183	2.7181	2.7169

$\lim_{x\to 0} f(x) \approx 2.7183$



69.



$\delta = 0.001$

$(1.999, 2.001)$

71. False. The existence or nonexistence of $f(x)$ at $x = c$ has no bearing on the existence of the limit of $f(x)$ as $x \to c$.

73. False. See Exercise 17.

75. Yes. As x approaches 0.25 from either side, $\sqrt{x}$ becomes arbitrarily close to 0.5.

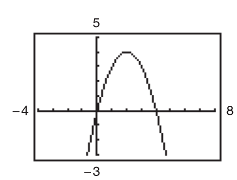
77. $\lim_{x\to 0} \dfrac{\sin nx}{x} = n$

79–81. Proofs **83.** Answers will vary.

85. Putnam Problem B1, 1986

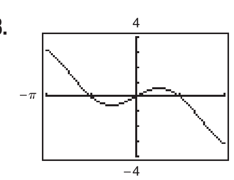
Section 1.3 (page 67)

1.



(a) 0 (b) -5

3.



(a) 0 (b) About 0.52 or $\pi/6$

5. 8 **7.** -1 **9.** 0 **11.** 7 **13.** 2 **15.** 1

17. $1/2$ **19.** $1/5$ **21.** 7 **23.** (a) 4 (b) 64 (c) 64

25. (a) 3 (b) 2 (c) 2 **27.** 1 **29.** $1/2$ **31.** 1

33. $1/2$ **35.** -1 **37.** (a) 10 (b) 5 (c) 6 (d) $3/2$

39. (a) 64 (b) 2 (c) 12 (d) 8

41. (a) -1 (b) -2

$g(x) = \dfrac{x^2 - x}{x}$ and $f(x) = x - 1$ agree except at $x = 0$.

43. (a) 2 (b) 0

$g(x) = \dfrac{x^3 - x}{x - 1}$ and $f(x) = x^2 + x$ agree except at $x = 1$.

45. -2

$f(x) = \dfrac{x^2 - 1}{x + 1}$ and $g(x) = x - 1$ agree except at $x = -1$.

47. 12

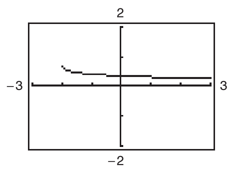
$f(x) = \dfrac{x^3 - 8}{x - 2}$ and $g(x) = x^2 + 2x + 4$ agree except at $x = 2$.

49. -1 **51.** $1/8$ **53.** $5/6$ **55.** $1/6$ **57.** $\sqrt{5}/10$

59. $-1/9$ **61.** 2 **63.** $2x - 2$

65. $1/5$ **67.** 0 **69.** 0 **71.** 0 **73.** 1 **75.** $3/2$

77.



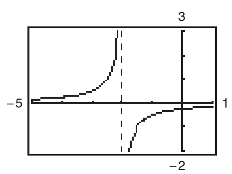
The graph has a hole at $x = 0$.

Answers will vary. Example:

x	-0.1	-0.01	-0.001	0.001	0.01	0.1
$f(x)$	0.358	0.354	0.354	0.354	0.353	0.349

$\lim_{x\to 0} \dfrac{\sqrt{x+2} - \sqrt{2}}{x} \approx 0.354 \left(\text{Actual limit is } \dfrac{1}{2\sqrt{2}} = \dfrac{\sqrt{2}}{4}.\right)$

79.



The graph has a hole at $x = 0$.

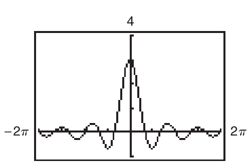
Answers will vary. Example:

x	-0.1	-0.01	-0.001
$f(x)$	-0.263	-0.251	-0.250

x	0.001	0.01	0.1
$f(x)$	-0.250	-0.249	-0.238

$\lim_{x\to 0} \dfrac{[1/(2 + x)] - (1/2)}{x} \approx -0.250 \left(\text{Actual limit is } -\dfrac{1}{4}.\right)$

81.



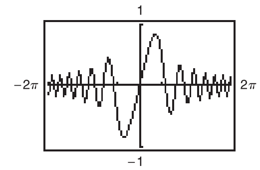
The graph has a hole at $t = 0$.

Answers will vary. Example:

t	-0.1	-0.01	0	0.01	0.1
$f(t)$	2.96	2.9996	?	2.9996	2.96

$\lim_{t\to 0} \dfrac{\sin 3t}{t} = 3$

83.



The graph has a hole at $x = 0$.

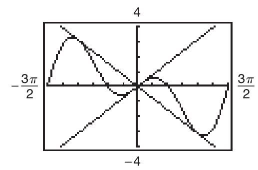
Answers will vary. Example:

x	-0.1	-0.01	-0.001	0	0.001	0.01	0.1
$f(x)$	-0.1	-0.01	-0.001	?	0.001	0.01	0.1

$\lim_{x\to 0} \dfrac{\sin x^2}{x} = 0$

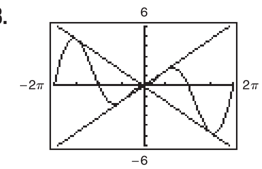
85. 3 **87.** $-1/(x + 3)^2$ **89.** 4

91.



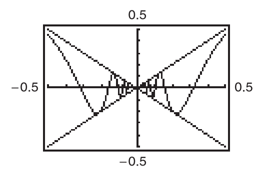
0

93.



0

95.



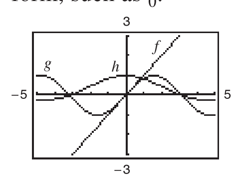
0

The graph has a hole at $x = 0$.

97. f and g agree at all but one point if c is a real number such that $f(x) = g(x)$ for all $x \neq c$.

99. An indeterminate form is obtained when evaluating a limit using direct substitution produces a meaningless fractional form, such as $\frac{0}{0}$.

101.



The magnitudes of $f(x)$ and $g(x)$ are approximately equal when x is close to 0. Therefore, their ratio is approximately 1.

103. -64 ft/sec (speed = 64 ft/sec) **105.** -29.4 m/sec

107. Let $f(x) = 1/x$ and $g(x) = -1/x$.

$\lim_{x\to 0} f(x)$ and $\lim_{x\to 0} g(x)$ do not exist. However,

$\lim_{x\to 0} [f(x) + g(x)] = \lim_{x\to 0} \left[\dfrac{1}{x} + \left(-\dfrac{1}{x}\right)\right] = \lim_{x\to 0} 0 = 0$

and therefore does exist.

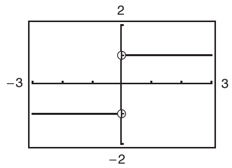
109–113. Proofs

115. Let $f(x) = \begin{cases} 4, & \text{if } x \geq 0 \\ -4, & \text{if } x < 0 \end{cases}$

$\lim_{x\to 0} |f(x)| = \lim_{x\to 0} 4 = 4$

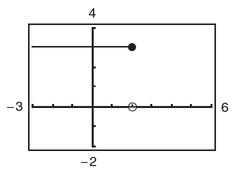
$\lim_{x\to 0} f(x)$ does not exist because for $x < 0$, $f(x) = -4$ and for $x \geq 0$, $f(x) = 4$.

117. False. The limit does not exist because the function approaches 1 from the right side of 0 and approaches -1 from the left side of 0. (See graph below.)



119. True.

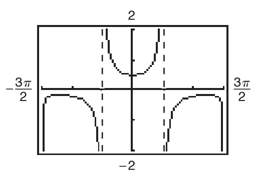
121. False. The limit does not exist because $f(x)$ approaches 3 from the left side of 2 and approaches 0 from the right side of 2. (See graph below.)



123. Proof

125. (a) All $x \neq 0, \dfrac{\pi}{2} + n\pi$

(b)



The domain is not obvious. The hole at $x = 0$ is not apparent from the graph.

(c) $\dfrac{1}{2}$ (d) $\dfrac{1}{2}$

127. The graphing utility was not set in *radian* mode.

Section 1.4 (page 78)

1. (a) 3 (b) 3 (c) 3; $f(x)$ is continuous on $(-\infty, \infty)$.

3. (a) 0 (b) 0 (c) 0; Discontinuity at $x = 3$

5. (a) -3 (b) 3 (c) Limit does not exist.
Discontinuity at $x = 2$

7. $\frac{1}{16}$ **9.** $\frac{1}{10}$

11. Limit does not exist. The function decreases without bound as x approaches -3 from the left.

13. -1 **15.** $-1/x^2$ **17.** $5/2$ **19.** 2

21. Limit does not exist. The function decreases without bound as x approaches π from the left and increases without bound as x approaches π from the right.

23. 8

25. Limit does not exist. The function approaches 5 from the left side of 3 but approaches 6 from the right side of 3.

27. Discontinuous at $x = -2$ and $x = 2$

29. Discontinuous at every integer

31. Continuous on $[-7, 7]$ **33.** Continuous on $[-1, 4]$

35. Nonremovable discontinuity at $x = 0$

37. Continuous for all real x

39. Nonremovable discontinuities at $x = -2$ and $x = 2$

41. Continuous for all real x

43. Nonremovable discontinuity at $x = 1$
Removable discontinuity at $x = 0$

45. Continuous for all real x

47. Removable discontinuity at $x = -2$
Nonremovable discontinuity at $x = 5$

49. Nonremovable discontinuity at $x = -7$

51. Continuous for all real x

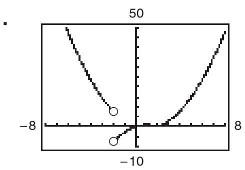
53. Nonremovable discontinuity at $x = 2$

55. Continuous for all real x

57. Nonremovable discontinuities at integer multiples of $\pi/2$

59. Nonremovable discontinuities at each integer

61.



$\lim_{x\to 0^+} f(x) = 0$

$\lim_{x\to 0^-} f(x) = 0$

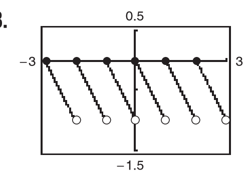
Discontinuity at $x = -2$

63. $a = 7$ **65.** $a = 2$ **67.** $a = -1,\ b = 1$

69. Continuous for all real x

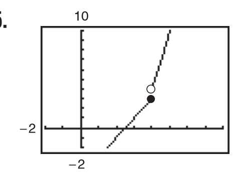
71. Nonremovable discontinuities at $x = 1$ and $x = -1$

73.



Nonremovable discontinuity at each integer

75.

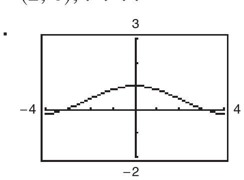


Nonremovable discontinuity at $x = 4$

77. Continuous on $(-\infty, \infty)$

79. Continuous on the open intervals . . . $(-6, -2)$, $(-2, 2)$, $(2, 6)$, ...

81.



The graph has a hole at $x = 0$. The graph appears to be continuous, but the function is not continuous on $[-4, 4]$. It is not obvious from the graph that the function has a discontinuity at $x = 0$.

83. Because $f(x)$ is continuous on the interval $[1, 2]$ and $f(1) = 37/12$ and $f(2) = -8/3$, by the Intermediate Value Theorem there exists a real number c in $[1, 2]$ such that $f(c) = 0$.

85. Because $f(x)$ is continuous on the interval $[0, \pi]$ and $f(0) = -3$ and $f(\pi) \approx 8.87$, by the Intermediate Value Theorem there exists a real number c in $[0, \pi]$ such that $f(c) = 0$.

87. 0.68, 0.6823 **89.** 0.56, 0.5636

91. $f(3) = 11$ **93.** $f(2) = 4$

95. (a) The limit does not exist at $x = c$.
(b) The function is not defined at $x = c$.
(c) The limit exists, but it is not equal to the value of the function at $x = c$.
(d) The limit does not exist at $x = c$.

97. If f and g are continuous for all real x, then so is $f + g$ (Theorem 1.11, part 2). However, f/g might not be continuous if $g(x) = 0$. For example, let $f(x) = x$ and $g(x) = x^2 - 1$. Then f and g are continuous for all real x, but f/g is not continuous at $x = \pm 1$.

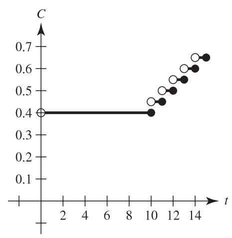
99. True

101. False. A rational function can be written as $P(x)/Q(x)$ where P and Q are polynomials of degree m and n, respectively. It can have, at most, n discontinuities.

103. $\lim_{t\to 4^-} f(t) \approx 28$; $\lim_{t\to 4^+} f(t) \approx 56$

At the end of day 3, the amount of chlorine in the pool is about 28 oz. At the beginning of day 4, the amount of chlorine in the pool is about 56 oz.

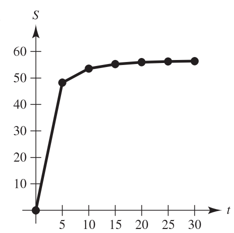
105. $C = \begin{cases} 0.40, & 0 < t \le 10 \\ 0.40 + 0.05[\![t - 9]\!], & t > 10, t \text{ is not an integer} \\ 0.40 + 0.05(t - 10), & t > 10, t \text{ is an integer} \end{cases}$



There is a nonremovable discontinuity at each integer greater than or equal to 10.

107–109. Proofs **111.** Answers will vary.

113. (a)

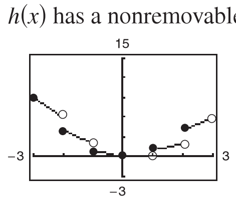


(b) There appears to be a limiting speed, and a possible cause is air resistance.

115. $c = \left(-1 \pm \sqrt{5}\right)/2$

117. Domain: $[-c^2, 0) \cup (0, \infty)$; Let $f(0) = 1/(2c)$

119. $h(x)$ has a nonremovable discontinuity at every integer except 0.



121. Putnam Problem B2, 1988

Section 1.5 (page 88)

1. $\lim_{x\to 4^+} \dfrac{1}{x - 4} = \infty$, $\lim_{x\to 4^-} \dfrac{1}{x - 4} = -\infty$

3. $\lim_{x\to 4^+} \dfrac{1}{(x - 4)^2} = \infty$, $\lim_{x\to 4^-} \dfrac{1}{(x - 4)^2} = \infty$

5. $\lim_{x\to -2^+} 2\left|\dfrac{x}{x^2 - 4}\right| = \infty$, $\lim_{x\to -2^-} 2\left|\dfrac{x}{x^2 - 4}\right| = \infty$

7. $\lim_{x\to -2^+} \tan(\pi x/4) = -\infty$, $\lim_{x\to -2^-} \tan(\pi x/4) = \infty$

9.

x	-3.5	-3.1	-3.01	-3.001
$f(x)$	0.31	1.64	16.6	167

x	-2.999	-2.99	-2.9	-2.5
$f(x)$	-167	-16.7	-1.69	-0.36

$\lim_{x\to -3^+} f(x) = -\infty$ $\lim_{x\to -3^-} f(x) = \infty$

11.

x	-3.5	-3.1	-3.01	-3.001
$f(x)$	3.8	16	151	1501

x	-2.999	-2.99	-2.9	-2.5
$f(x)$	-1499	-149	-14	-2.3

$\lim_{x\to -3^+} f(x) = -\infty$ $\lim_{x\to -3^-} f(x) = \infty$

13. $x = 0$ **15.** $x = \pm 2$ **17.** No vertical asymptote

19. $x = 2$, $x = -1$ **21.** $t = 0$ **23.** $x = -2, x = 1$

25. No vertical asymptote **27.** No vertical asymptote

29. $x = \frac{1}{2} + n$, n is an integer.

31. $t = n\pi$, n is a nonzero integer.

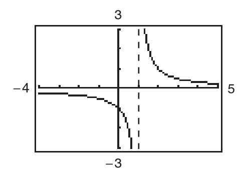
33. Removable discontinuity at $x = -1$

35. Vertical asymptote at $x = -1$ **37.** ∞ **39.** ∞

41. ∞ **43.** $-\frac{1}{5}$ **45.** $\frac{1}{2}$ **47.** $-\infty$ **49.** ∞ **51.** 0

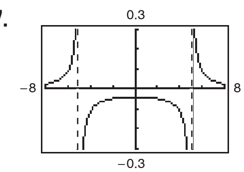
53. Limit does not exist.

55.



$\lim_{x\to 1^+} f(x) = \infty$

57.

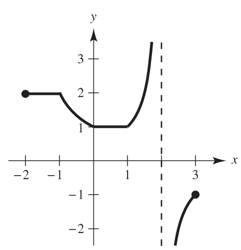


$\lim_{x\to 5^-} f(x) = -\infty$

59. Answers will vary.

61. Answers will vary. Example: $f(x) = \dfrac{x - 3}{x^2 - 4x - 12}$

63.



65. ∞

67. (a) $\frac{1}{3}(200\pi)$ ft/sec

(b) 200π ft/sec

(c) $\lim_{\theta\to(\pi/2)^-} [50\pi \sec^2 \theta] = \infty$

69. (a) Domain: $x > 25$

(b)

x	30	40	50	60
y	150	66.667	50	42.857

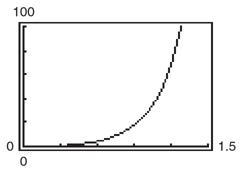
(c) $\lim_{x\to 25^+} \dfrac{25x}{x - 25} = \infty$

As x gets closer and closer to 25 mi/h, y becomes larger and larger.

71. (a) $A = 50 \tan \theta - 50\theta$; Domain: $(0, \pi/2)$

(b)

θ	0.3	0.6	0.9	1.2	1.5
$f(\theta)$	0.47	4.21	18.0	68.6	630.1



(c) $\lim_{\theta \to \pi/2^-} A = \infty$

73. False; let $f(x) = (x^2 - 1)/(x - 1)$

75. False; let $f(x) = \tan x$

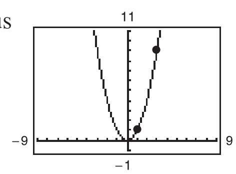
77. Let $f(x) = \dfrac{1}{x^2}$ and $g(x) = \dfrac{1}{x^4}$, and let $c = 0$. $\lim_{x \to 0} \dfrac{1}{x^2} = \infty$ and $\lim_{x \to 0} \dfrac{1}{x^4} = \infty$, but $\lim_{x \to 0} \left(\dfrac{1}{x^2} - \dfrac{1}{x^4}\right) = \lim_{x \to 0} \left(\dfrac{x^2 - 1}{x^4}\right) = -\infty \neq 0$.

79. Given $\lim_{x \to c} f(x) = \infty$, let $g(x) = 1$. Then $\lim_{x \to c} \dfrac{g(x)}{f(x)} = 0$ by Theorem 1.15.

81. Answers will vary.

Review Exercises for Chapter 1 (page 91)

1. Calculus



Estimate: 8.3

3.

x	-0.1	-0.01	-0.001
$f(x)$	-1.0526	-1.0050	-1.0005

x	0.001	0.01	0.1
$f(x)$	-0.9995	-0.9950	-0.9524

The estimate of the limit of $f(x)$, as x approaches zero, is -1.00.

5. 5; Proof **7.** -3; Proof **9.** (a) 4 (b) 5 **11.** 16

13. $\sqrt{6} \approx 2.45$ **15.** $-\frac{1}{4}$ **17.** $\frac{1}{2}$ **19.** -1 **21.** 75

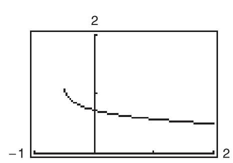
23. 0 **25.** $\sqrt{3}/2$ **27.** $-\frac{1}{2}$ **29.** $\frac{7}{12}$

31. (a)

x	1.1	1.01	1.001	1.0001
$f(x)$	0.5680	0.5764	0.5773	0.5773

$\lim_{x \to 1^+} f(x) \approx 0.5773$

(b)



The graph has a hole at $x = 1$.
$\lim_{x \to 1^+} f(x) \approx 0.5774$

(c) $\sqrt{3}/3$

33. -39.2 m/sec **35.** -1 **37.** 0

39. Limit does not exist. The limit as t approaches 1 from the left is 2 whereas the limit as t approaches 1 from the right is 1.

41. Continuous for all real x

43. Nonremovable discontinuity at each integer
Continuous on $(k, k + 1)$ for all integers k

45. Removable discontinuity at $x = 1$
Continuous on $(-\infty, 1) \cup (1, \infty)$

47. Nonremovable discontinuity at $x = 2$
Continuous on $(-\infty, 2) \cup (2, \infty)$

49. Nonremovable discontinuity at $x = -1$
Continuous on $(-\infty, -1) \cup (-1, \infty)$

51. Nonremovable discontinuity at each even integer
Continuous on $(2k, 2k + 2)$ for all integers k

53. $c = -\frac{1}{2}$ **55.** Proof

57. (a) -4 (b) 4 (c) Limit does not exist.

59. $x = 0$ **61.** $x = 10$ **63.** $-\infty$ **65.** $\frac{1}{3}$

67. $-\infty$ **69.** $-\infty$ **71.** $\frac{4}{5}$ **73.** ∞

75. (a) \$14,117.65 (b) \$80,000.00 (c) \$720,000.00 (d) ∞

P.S. Problem Solving (page 93)

1. (a) Perimeter $\triangle PAO = 1 + \sqrt{(x^2 - 1)^2 + x^2} + \sqrt{x^4 + x^2}$
Perimeter $\triangle PBO = 1 + \sqrt{x^4 + (x - 1)^2} + \sqrt{x^4 + x^2}$

(b)

x	4	2	1
Perimeter $\triangle PAO$	33.0166	9.0777	3.4142
Perimeter $\triangle PBO$	33.7712	9.5952	3.4142
$r(x)$	0.9777	0.9461	1.0000

x	0.1	0.01
Perimeter $\triangle PAO$	2.0955	2.0100
Perimeter $\triangle PBO$	2.0006	2.0000
$r(x)$	1.0475	1.0050

(c) 1

3. (a) Area (hexagon) $= (3\sqrt{3})/2 \approx 2.5981$
Area (circle) $= \pi \approx 3.1416$
Area (circle) $-$ Area (hexagon) ≈ 0.5435

(b) $A_n = (n/2)\sin(2\pi/n)$

(c)

n	6	12	24	48	96
A_n	2.5981	3.0000	3.1058	3.1326	3.1394

(d) 3.1416 or π

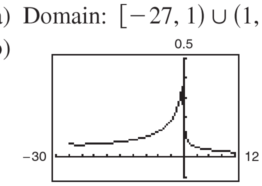
5. (a) $m = -\frac{12}{5}$ (b) $y = \frac{5}{12}x - \frac{169}{12}$

(c) $m_x = \dfrac{-\sqrt{169 - x^2} + 12}{x - 5}$

(d) $\frac{5}{12}$; It is the same as the slope of the tangent line found in (b).

7. (a) Domain: $[-27, 1) \cup (1, \infty)$

(b)

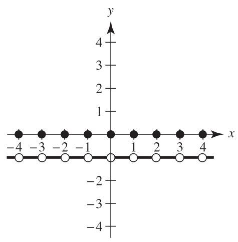


The graph has a hole at $x = 1$.

(c) $\frac{1}{14}$ (d) $\frac{1}{12}$

9. (a) g_1, g_4 (b) g_1 (c) g_1, g_3, g_4

11.



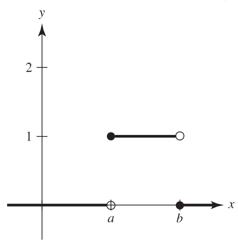
The graph jumps at every integer.

(a) $f(1) = 0,\ f(0) = 0,\ f\left(\frac{1}{2}\right) = -1,\ f(-2.7) = -1$

(b) $\lim\limits_{x\to 1^-} f(x) = -1,\ \lim\limits_{x\to 1^+} f(x) = -1,\ \lim\limits_{x\to 1/2} f(x) = -1$

(c) There is a discontinuity at each integer.

13. (a)



(b) (i) $\lim\limits_{x\to a^+} P_{a,b}(x) = 1$

(ii) $\lim\limits_{x\to a^-} P_{a,b}(x) = 0$

(iii) $\lim\limits_{x\to b^+} P_{a,b}(x) = 0$

(iv) $\lim\limits_{x\to b^-} P_{a,b}(x) = 1$

(c) Continuous for all positive real numbers except a and b

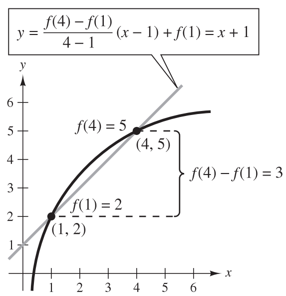
(d) The area under the graph of U and above the x-axis is 1.

Chapter 2

Section 2.1 (page 103)

1. (a) $m_1 = 0, m_2 = 5/2$ (b) $m_1 = -5/2, m_2 = 2$

3.



5. $m = -5$ **7.** $m = 4$

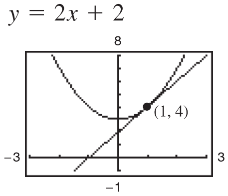
9. $m = 3$ **11.** $f'(x) = 0$ **13.** $f'(x) = -10$ **15.** $h'(s) = \frac{2}{3}$

17. $f'(x) = 2x + 1$ **19.** $f'(x) = 3x^2 - 12$

21. $f'(x) = \dfrac{-1}{(x - 1)^2}$ **23.** $f'(x) = \dfrac{1}{2\sqrt{x + 4}}$

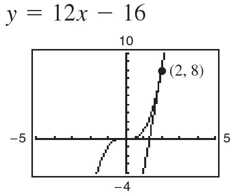
25. (a) Tangent line: $y = 2x + 2$

(b)



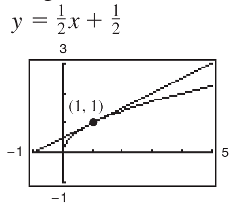
27. (a) Tangent line: $y = 12x - 16$

(b)



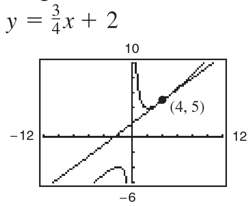
29. (a) Tangent line: $y = \frac{1}{2}x + \frac{1}{2}$

(b)



31. (a) Tangent line: $y = \frac{3}{4}x + 2$

(b)

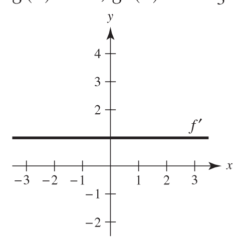


33. $y = 2x - 1$ **35.** $y = 3x - 2;\ y = 3x + 2$

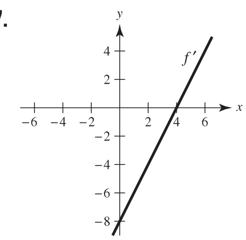
37. $y = -\frac{1}{2}x + \frac{3}{2}$ **39.** b **40.** d **41.** a **42.** c

43. $g(4) = 5;\ g'(4) = -\frac{5}{3}$

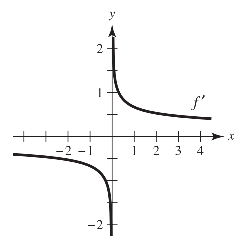
45.



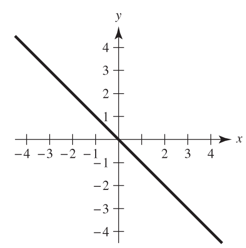
47.



49.



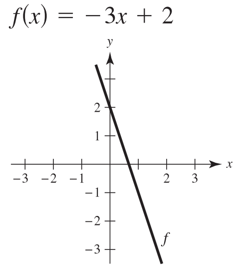
51. Answers will vary.
Sample answer: $y = -x$



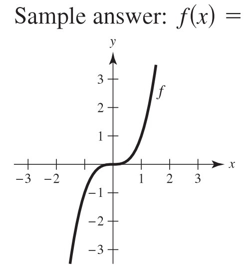
53. $f(x) = 5 - 3x$
$c = 1$

55. $f(x) = -x^2$
$c = 6$

57. $f(x) = -3x + 2$

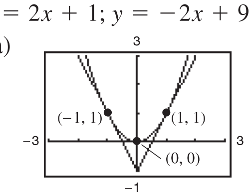


59. Answers will vary.
Sample answer: $f(x) = x^3$



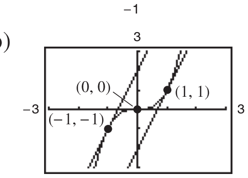
61. $y = 2x + 1;\ y = -2x + 9$

63. (a)



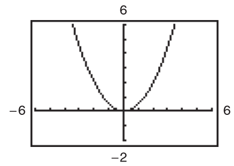
For this function, the slopes of the tangent lines are always distinct for different values of x.

(b)



For this function, the slopes of the tangent lines are sometimes the same.

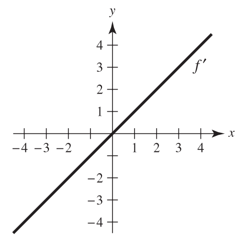
65. (a)



$f'(0) = 0, f'(\frac{1}{2}) = \frac{1}{2}, f'(1) = 1, f'(2) = 2$

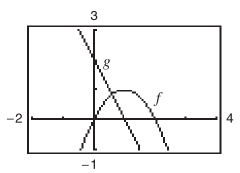
(b) $f'(-\frac{1}{2}) = -\frac{1}{2}, f'(-1) = -1, f'(-2) = -2$

(c)



(d) $f'(x) = x$

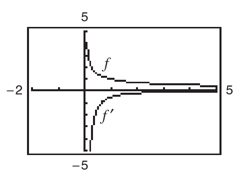
67.



$g(x) \approx f'(x)$

69. $f(2) = 4; f(2.1) = 3.99; f'(2) \approx -0.1$

71.



As x approaches infinity, the graph of f approaches a line of slope 0. Thus $f'(x)$ approaches 0.

73. 6 **75.** 4 **77.** $g(x)$ is not differentiable at $x = 0$.

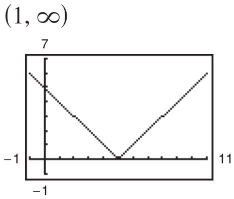
79. $f(x)$ is not differentiable at $x = 6$.

81. $h(x)$ is not differentiable at $x = -7$.

83. $(-\infty, 3) \cup (3, \infty)$ **85.** $(-\infty, -4) \cup (-4, \infty)$

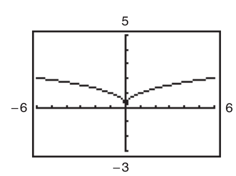
87. $(1, \infty)$

89.



$(-\infty, 5) \cup (5, \infty)$

91.



$(-\infty, 0) \cup (0, \infty)$

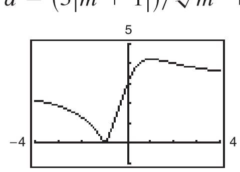
93. The derivative from the left is -1 and the derivative from the right is 1, so f is not differentiable at $x = 1$.

95. The derivatives from both the right and the left are 0, so $f'(1) = 0$.

97. f is differentiable at $x = 2$.

99. (a) $d = (3|m + 1|)/\sqrt{m^2 + 1}$

(b)



Not differentiable at $m = -1$

101. False. The slope is $\lim_{\Delta x \to 0} \dfrac{f(2 + \Delta x) - f(2)}{\Delta x}$.

103. False. For example: $f(x) = |x|$. The derivative from the left and the derivative from the right both exist but are not equal.

105. Proof

Section 2.2 (page 115)

1. (a) $\frac{1}{2}$ (b) 3 **3.** 0 **5.** $7x^6$ **7.** $-5/x^6$ **9.** $1/(5x^{4/5})$

11. 1 **13.** $-4t + 3$ **15.** $2x + 12x^2$ **17.** $3t^2 + 10t - 3$

19. $\dfrac{\pi}{2}\cos\theta + \sin\theta$ **21.** $2x + \dfrac{1}{2}\sin x$ **23.** $-\dfrac{1}{x^2} - 3\cos x$

	Function	*Rewrite*	*Derivative*	*Simplify*
25.	$y = \dfrac{5}{2x^2}$	$y = \dfrac{5}{2}x^{-2}$	$y' = -5x^{-3}$	$y' = -\dfrac{5}{x^3}$
27.	$y = \dfrac{6}{(5x)^3}$	$y = \dfrac{6}{125}x^{-3}$	$y' = -\dfrac{18}{125}x^{-4}$	$y' = -\dfrac{18}{125x^4}$
29.	$y = \dfrac{\sqrt{x}}{x}$	$y = x^{-1/2}$	$y' = -\dfrac{1}{2}x^{-3/2}$	$y' = -\dfrac{1}{2x^{3/2}}$

31. -2 **33.** 0 **35.** 8 **37.** 3 **39.** $2x + 6/x^3$

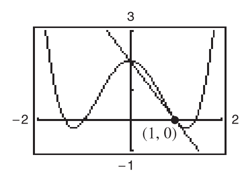
41. $2t + 12/t^4$ **43.** $8x + 3$ **45.** $(x^3 - 8)/x^3$

47. $3x^2 + 1$ **49.** $\dfrac{1}{2\sqrt{x}} - \dfrac{2}{x^{2/3}}$ **51.** $\dfrac{4}{5s^{1/5}} - \dfrac{2}{3s^{1/3}}$

53. $\dfrac{3}{\sqrt{x}} - 5\sin x$

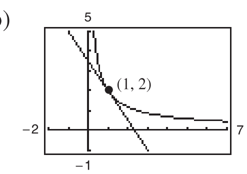
55. (a) $2x + y - 2 = 0$

(b)



57. (a) $3x + 2y - 7 = 0$

(b)

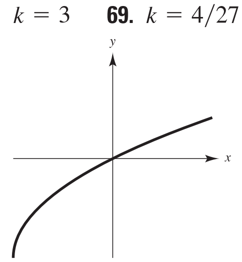


59. $(-1, 2), (0, 3), (1, 2)$ **61.** No horizontal tangents

63. (π, π) **65.** $k = -1, k = -9$

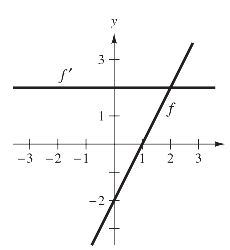
67. $k = 3$ **69.** $k = 4/27$

71.



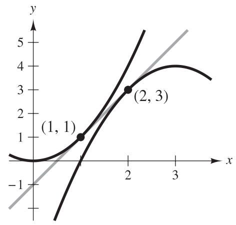
73. $g'(x) = f'(x)$

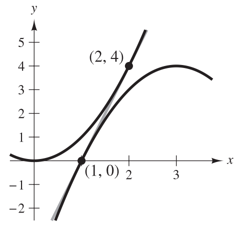
75.



The rate of change of f is constant and therefore f' is a constant function.

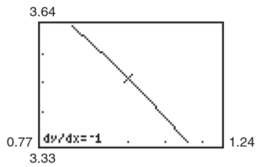
77. $y = 2x - 1$ $\quad$ $y = 4x - 4$





79. $f'(x) = 3 + \cos x \neq 0$ for all x. **81.** $x - 4y + 4 = 0$

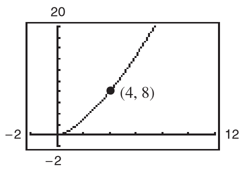
83.



$f'(1)$ appears to be close to -1.

$f'(1) = -1$

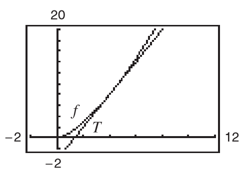
85. (a)



(3.9, 7.7019),
$S(x) = 2.981x - 3.924$

(b) $T(x) = 3(x - 4) + 8 = 3x - 4$
The slope (and equation) of the secant line approaches that of the tangent line at (4, 8) as you choose points closer and closer to (4, 8).

(c)



The approximation becomes less accurate.

(d)

Δx	-3	-2	-1	-0.5	-0.1	0
$f(4 + \Delta x)$	1	2.828	5.196	6.548	7.702	8
$T(4 + \Delta x)$	-1	2	5	6.5	7.7	8

Δx	0.1	0.5	1	2	3
$f(4 + \Delta x)$	8.302	9.546	11.180	14.697	18.520
$T(4 + \Delta x)$	8.3	9.5	11	14	17

87. False. Let $f(x) = x$ and $g(x) = x + 1$.

89. False. $dy/dx = 0$ **91.** True

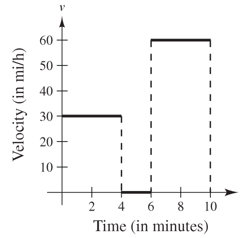
93. Average rate: 4
Instantaneous rates:
$f'(1) = 4; f'(2) = 4$

95. Average rate: $\frac{1}{2}$
Instantaneous rates:
$f'(1) = 1; f'(2) = \frac{1}{4}$

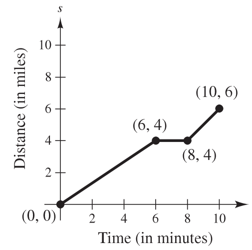
97. (a) $s(t) = -16t^2 + 1362$; $v(t) = -32t$ (b) -48 ft/sec
(c) $s'(1) = -32$ ft/sec; $s'(2) = -64$ ft/sec
(d) $t = \dfrac{\sqrt{1362}}{4} \approx 9.226$ sec (e) -295.242 ft/sec

99. $v(5) = 71$ m/sec; $v(10) = 22$ m/sec

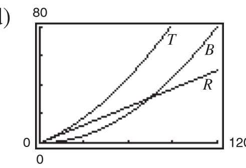
101.



103.



105. (a) $R(v) = 0.417v - 0.02$
(b) $B(v) = 0.0056v^2 + 0.001v + 0.04$
(c) $T(v) = 0.0056v^2 + 0.418v + 0.02$
(d)



(e) $T'(v) = 0.0112v + 0.418$
$T'(40) = 0.866$
$T'(80) = 1.314$
$T'(100) = 1.538$

(f) Stopping distance increases at an increasing rate.

107. $V'(6) = 108\ \text{cm}^3/\text{cm}$ **109.** Proof

111. (a) The rate of change of the number of gallons of gasoline sold when the price is \$2.979
(b) In general, the rate of change when $p = 2.979$ should be negative. As prices go up, sales go down.

113. $y = 2x^2 - 3x + 1$ **115.** $9x + y = 0, 9x + 4y + 27 = 0$

117. $a = \frac{1}{3}, b = -\frac{4}{3}$

119. $f_1(x) = |\sin x|$ is differentiable for all $x \neq n\pi$, n an integer.
$f_2(x) = \sin|x|$ is differentiable for all $x \neq 0$.

Section 2.3 (page 126)

1. $2(2x^3 - 6x^2 + 3x - 6)$ **3.** $(1 - 5t^2)/(2\sqrt{t})$

5. $x^2(3\cos x - x\sin x)$ **7.** $(1 - x^2)/(x^2 + 1)^2$

9. $(1 - 5x^3)/[2\sqrt{x}(x^3 + 1)^2]$ **11.** $(x\cos x - 2\sin x)/x^3$

13. $f'(x) = (x^3 + 4x)(6x + 2) + (3x^2 + 2x - 5)(3x^2 + 4)$
$= 15x^4 + 8x^3 + 21x^2 + 16x - 20$
$f'(0) = -20$

15. $f'(x) = \dfrac{x^2 - 6x + 4}{(x - 3)^2}$
$f'(1) = -\dfrac{1}{4}$

17. $f'(x) = \cos x - x\sin x$
$f'\left(\dfrac{\pi}{4}\right) = \dfrac{\sqrt{2}}{8}(4 - \pi)$

	Function	*Rewrite*	*Differentiate*	*Simplify*
19.	$y = \dfrac{x^2 + 3x}{7}$	$y = \dfrac{1}{7}x^2 + \dfrac{3}{7}x$	$y' = \dfrac{2}{7}x + \dfrac{3}{7}$	$y' = \dfrac{2x + 3}{7}$
21.	$y = \dfrac{6}{7x^2}$	$y = \dfrac{6}{7}x^{-2}$	$y' = -\dfrac{12}{7}x^{-3}$	$y' = -\dfrac{12}{7x^3}$
23.	$y = \dfrac{4x^{3/2}}{x}$	$y = 4x^{1/2}$, $x > 0$	$y' = 2x^{-1/2}$	$y' = \dfrac{2}{\sqrt{x}}$, $x > 0$

25. $\dfrac{(x^2 - 1)(-3 - 2x) - (4 - 3x - x^2)(2x)}{(x^2 - 1)^2} = \dfrac{3}{(x + 1)^2}, x \neq 1$

27. $1 - 12/(x + 3)^2 = (x^2 + 6x - 3)/(x + 3)^2$

29. $\dfrac{3}{2}x^{-1/2} + \dfrac{1}{2}x^{-3/2} = (3x + 1)/2x^{3/2}$

31. $6s^2(s^3 - 2)$ **33.** $-(2x^2 - 2x + 3)/[x^2(x - 3)^2]$

35. $(6x^2 + 5)(x - 3)(x + 2) + (2x^3 + 5x)(1)(x + 2)$
$\quad + (2x^3 + 5x)(x - 3)(1)$
$\quad = 10x^4 - 8x^3 - 21x^2 - 10x - 30$

37. $\dfrac{(x^2 - c^2)(2x) - (x^2 + c^2)(2x)}{(x^2 - c^2)^2} = -\dfrac{4xc^2}{(x^2 - c^2)^2}$

39. $t(t \cos t + 2 \sin t)$ **41.** $-(t \sin t + \cos t)/t^2$

43. $-1 + \sec^2 x = \tan^2 x$ **45.** $\dfrac{1}{4t^{3/4}} - 6 \csc t \cot t$

47. $\dfrac{-6 \cos^2 x + 6 \sin x - 6 \sin^2 x}{4 \cos^2 x} = \dfrac{3}{2}(-1 + \tan x \sec x - \tan^2 x)$
$\quad = \dfrac{3}{2} \sec x(\tan x - \sec x)$

49. $\csc x \cot x - \cos x = \cos x \cot^2 x$ **51.** $x(x \sec^2 x + 2 \tan x)$

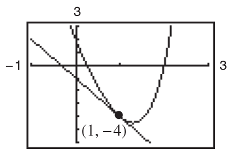
53. $2x \cos x + 2 \sin x - x^2 \sin x + 2x \cos x$
$\quad = 4x \cos x + (2 - x^2) \sin x$

55. $\left(\dfrac{x + 1}{x + 2}\right)(2) + (2x - 5)\left[\dfrac{(x + 2)(1) - (x + 1)(1)}{(x + 2)^2}\right]$
$\quad = \dfrac{2x^2 + 8x - 1}{(x + 2)^2}$

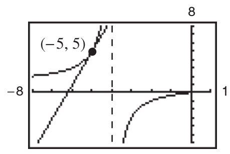
57. $\dfrac{1 - \sin \theta + \theta \cos \theta}{(1 - \sin \theta)^2}$ **59.** $y' = \dfrac{-2 \csc x \cot x}{(1 - \csc x)^2}, \quad -4\sqrt{3}$

61. $h'(t) = \sec t(t \tan t - 1)/t^2, \quad 1/\pi^2$

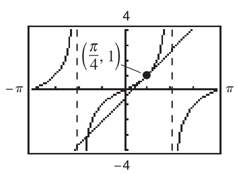
63. (a) $y = -3x - 1$
(b)



65. (a) $y = 4x + 25$
(b)



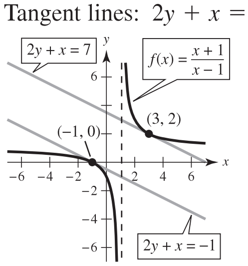
67. (a) $4x - 2y - \pi + 2 = 0$
(b)



69. $2y + x - 4 = 0$

71. $25y - 12x + 16 = 0$ **73.** $(1, 1)$ **75.** $(0, 0), (2, 4)$

77. Tangent lines: $2y + x = 7$; $2y + x = -1$



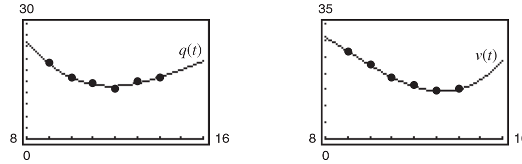
79. $f(x) + 2 = g(x)$ **81.** (a) $p'(1) = 1$ (b) $q'(4) = -1/3$

83. $(18t + 5)/(2\sqrt{t})$ cm²/sec

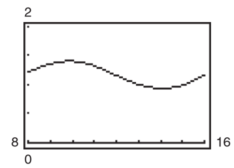
85. (a) −\$38.13 thousand/100 components
(b) −\$10.37 thousand/100 components
(c) −\$3.80 thousand/100 components
The cost decreases with increasing order size.

87. 31.55 bacteria/h **89.** Proof

91. (a) $q(t) = -0.0546t^3 + 2.529t^2 - 36.89t + 186.6$
$v(t) = 0.0796t^3 - 2.162t^2 + 15.32t + 5.9$
(b)



(c) $A = \dfrac{0.0796t^3 - 2.162t^2 + 15.32t + 5.9}{-0.0546t^3 + 2.529t^2 - 36.89t + 186.6}$



A represents the average value (in billions of dollars) per one million personal computers.

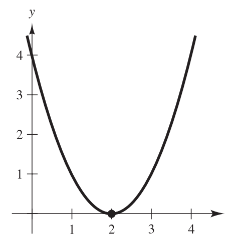
(d) $A'(t)$ represents the rate of change of the average value per one million personal computers for the given year.

93. $12x^2 + 12x - 6$ **95.** $3/\sqrt{x}$ **97.** $2/(x - 1)^3$

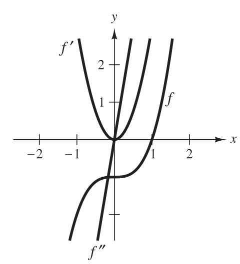
99. $2 \cos x - x \sin x$ **101.** $2x$ **103.** $1/\sqrt{x}$

105. 0 **107.** −10

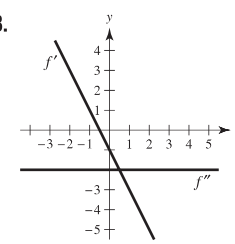
109. Answers will vary. For example: $f(x) = (x - 2)^2$



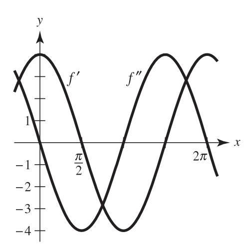
111.



113.



115.



117. $v(3) = 27$ m/sec
$a(3) = -6$ m/sec²
The speed of the object is decreasing.

119.

t	0	1	2	3	4
$s(t)$	0	57.75	99	123.75	132
$v(t)$	66	49.5	33	16.5	0
$a(t)$	−16.5	−16.5	−16.5	−16.5	−16.5

The average velocity on [0, 1] is 57.75, on [1, 2] is 41.25, on [2, 3] is 24.75, and on [3, 4] is 8.25.

121. $f^{(n)}(x) = n(n-1)(n-2)\cdots(2)(1) = n!$

123. (a) $f''(x) = g(x)h''(x) + 2g'(x)h'(x) + g''(x)h(x)$

$f'''(x) = g(x)h'''(x) + 3g'(x)h''(x) + 3g''(x)h'(x) + g'''(x)h(x)$

$f^{(4)}(x) = g(x)h^{(4)}(x) + 4g'(x)h'''(x) + 6g''(x)h''(x) + 4g'''(x)h'(x) + g^{(4)}(x)h(x)$

(b) $f^{(n)}(x) = g(x)h^{(n)}(x) + \dfrac{n!}{1!(n-1)!}g'(x)h^{(n-1)}(x) + \dfrac{n!}{2!(n-2)!}g''(x)h^{(n-2)}(x) + \cdots + \dfrac{n!}{(n-1)!1!}g^{(n-1)}(x)h'(x) + g^{(n)}(x)h(x)$

125. $n = 1$: $f'(x) = x\cos x + \sin x$

$n = 2$: $f'(x) = x^2\cos x + 2x\sin x$

$n = 3$: $f'(x) = x^3\cos x + 3x^2\sin x$

$n = 4$: $f'(x) = x^4\cos x + 4x^3\sin x$

General rule: $f'(x) = x^n\cos x + nx^{(n-1)}\sin x$

127. $y' = -1/x^2$, $y'' = 2/x^3$,

$x^3y'' + 2x^2y' = x^3(2/x^3) + 2x^2(-1/x^2)$

$= 2 - 2 = 0$

129. $y' = 2\cos x$, $y'' = -2\sin x$,

$y'' + y = -2\sin x + 2\sin x + 3 = 3$

131. False. $dy/dx = f(x)g'(x) + g(x)f'(x)$ **133.** True

135. True **137.** $f(x) = 3x^2 - 2x - 1$

139. $f'(x) = 2|x|$; $f''(0)$ does not exist. **141.** Proof

Section 2.4 (page 137)

	$y = f(g(x))$	$u = g(x)$	$y = f(u)$
1.	$y = (5x-8)^4$	$u = 5x - 8$	$y = u^4$
3.	$y = \sqrt{x^3 - 7}$	$u = x^3 - 7$	$y = \sqrt{u}$
5.	$y = \csc^3 x$	$u = \csc x$	$y = u^3$

7. $12(4x-1)^2$ **9.** $-108(4-9x)^3$

11. $\frac{1}{2}(5-t)^{-1/2}(-1) = -1/\left(2\sqrt{5-t}\right)$

13. $\frac{1}{3}(6x^2+1)^{-2/3}(12x) = 4x/\sqrt[3]{(6x^2+1)^2}$

15. $\frac{1}{2}(9-x^2)^{-3/4}(-2x) = -x/\sqrt[4]{(9-x^2)^3}$ **17.** $-1/(x-2)^2$

19. $-2(t-3)^{-3}(1) = -2/(t-3)^3$ **21.** $-1/\left[2\sqrt{(x+2)^3}\right]$

23. $x^2[4(x-2)^3(1)] + (x-2)^4(2x) = 2x(x-2)^3(3x-2)$

25. $x\left(\dfrac{1}{2}\right)(1-x^2)^{-1/2}(-2x) + (1-x^2)^{1/2}(1) = \dfrac{1-2x^2}{\sqrt{1-x^2}}$

27. $\dfrac{(x^2+1)^{1/2}(1) - x(1/2)(x^2+1)^{-1/2}(2x)}{x^2+1} = \dfrac{1}{\sqrt{(x^2+1)^3}}$

29. $\dfrac{-2(x+5)(x^2+10x-2)}{(x^2+2)^3}$ **31.** $\dfrac{-9(1-2v)^2}{(v+1)^4}$

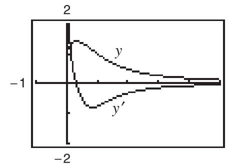
33. $2((x^2+3)^5 + x)(5(x^2+3)^4(2x) + 1)$

$= 20x(x^2+3)^9 + 2(x^2+3)^5 + 20x^2(x^2+3)^4 + 2x$

35. $\dfrac{1}{2}(2 + (2 + x^{1/2})^{1/2})^{-1/2}\left(\dfrac{1}{2}(2 + x^{1/2})^{-1/2}\right)\left(\dfrac{1}{2}x^{-1/2}\right)$

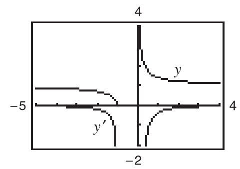
$= \dfrac{1}{8\sqrt{x}\left(\sqrt{2+\sqrt{x}}\right)\left(\sqrt{2+\sqrt{2+\sqrt{x}}}\right)}$

37. $(1 - 3x^2 - 4x^{3/2})/[2\sqrt{x}(x^2+1)^2]$



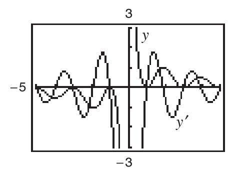
The zero of y' corresponds to the point on the graph of the function where the tangent line is horizontal.

39. $-\dfrac{\sqrt{\dfrac{x+1}{x}}}{2x(x+1)}$



y' has no zeros.

41. $-[\pi x\sin(\pi x) + \cos(\pi x) + 1]/x^2$



The zeros of y' correspond to the points on the graph of the function where the tangent lines are horizontal.

43. (a) 1 (b) 2; The slope of $\sin ax$ at the origin is a.

45. $-4\sin 4x$ **47.** $15\sec^2 3x$ **49.** $2\pi^2 x\cos(\pi x)^2$

51. $2\cos 4x$ **53.** $(-1 - \cos^2 x)/\sin^3 x$

55. $8\sec^2 x\tan x$ **57.** $10\tan 5\theta\sec^2 5\theta$

59. $\sin 2\theta\cos 2\theta = \dfrac{1}{2}\sin 4\theta$

61. $\dfrac{6\pi\sin(\pi t - 1)}{\cos^3(\pi t - 1)}$ **63.** $\dfrac{1}{2\sqrt{x}} + 2x\cos(2x)^2$

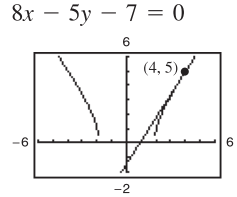
65. $2\sec^2 2x\cos(\tan 2x)$

67. $s'(t) = \dfrac{t+3}{\sqrt{t^2+6t-2}}, \dfrac{6}{5}$ **69.** $f'(x) = \dfrac{-15x^2}{(x^3-2)^2}, -\dfrac{3}{5}$

71. $f'(t) = \dfrac{-5}{(t-1)^2}, -5$ **73.** $y' = -12\sec^3 4x\tan 4x, 0$

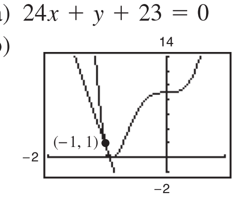
75. (a) $8x - 5y - 7 = 0$

(b)



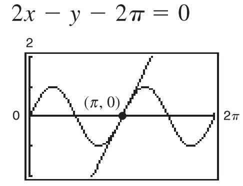
77. (a) $24x + y + 23 = 0$

(b)



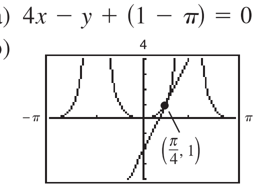
79. (a) $2x - y - 2\pi = 0$

(b)



81. (a) $4x - y + (1 - \pi) = 0$

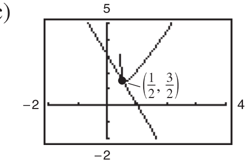
(b)



83. (a) $g'(1/2) = -3$

(b) $3x + y - 3 = 0$

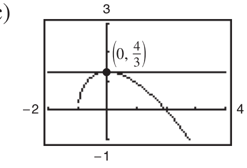
(c)



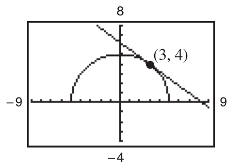
85. (a) $s'(0) = 0$

(b) $y = \frac{4}{3}$

(c)



87. $3x + 4y - 25 = 0$



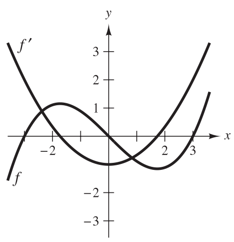
89. $\left(\dfrac{\pi}{6}, \dfrac{3\sqrt{3}}{2}\right), \left(\dfrac{5\pi}{6}, -\dfrac{3\sqrt{3}}{2}\right), \left(\dfrac{3\pi}{2}, 0\right)$ **91.** $2940(2 - 7x)^2$

93. $\dfrac{2}{(x - 6)^3}$

95. $2(\cos x^2 - 2x^2 \sin x^2)$ **97.** $h''(x) = 18x + 6, 24$

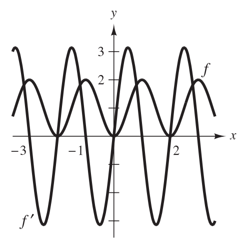
99. $f''(x) = -4x^2 \cos(x^2) - 2\sin(x^2), 0$

101.



The zeros of f' correspond to the points where the graph of f has horizontal tangents.

103.



The zeros of f' correspond to the points where the graph of f has horizontal tangents.

105. The rate of change of g is three times as fast as the rate of change of f.

107. (a) $g'(x) = f'(x)$ (b) $h'(x) = 2f'(x)$
(c) $r'(x) = -3f'(-3x)$ (d) $s'(x) = f'(x + 2)$

x	-2	-1	0	1	2	3
$f'(x)$	4	$\frac{2}{3}$	$-\frac{1}{3}$	-1	-2	-4
$g'(x)$	4	$\frac{2}{3}$	$-\frac{1}{3}$	-1	-2	-4
$h'(x)$	8	$\frac{4}{3}$	$-\frac{2}{3}$	-2	-4	-8
$r'(x)$		12	1			
$s'(x)$	$-\frac{1}{3}$	-1	-2	-4		

109. (a) $\frac{1}{2}$
(b) $s'(5)$ does not exist because g is not differentiable at 6.

111. (a) 1.461 (b) -1.016

113. 0.2 rad, 1.45 rad/sec **115.** 0.04224 cm/sec

117. (a) $x = -1.637t^3 + 19.31t^2 - 0.5t - 1$
(b) $\dfrac{dC}{dt} = -294.66t^2 + 2317.2t - 30$
(c) Because x, the number of units produced in t hours, is not a linear function, and therefore the cost with respect to time t is not linear.

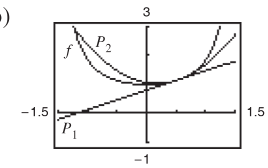
119. (a) Yes, if $f(x + p) = f(x)$ for all x, then $f'(x + p) = f'(x)$, which shows that f' is periodic as well.
(b) Yes, if $g(x) = f(2x)$, then $g'(x) = 2f'(2x)$. Because f' is periodic, so is g'.

121. (a) 0
(b) $f'(x) = 2\sec x \cdot \sec x \tan x = 2\sec^2 x \tan x$
$g'(x) = 2\tan x \sec^2 x = 2\sec^2 x \tan x$
$f'(x) = g'(x)$

123. Proof **125.** $f'(x) = 2x\left(\dfrac{x^2 - 9}{|x^2 - 9|}\right), x \neq \pm 3$

127. $f'(x) = \cos x \sin x/|\sin x|, x \neq k\pi$

129. (a) $P_1(x) = 2/3(x - \pi/6) + 2/\sqrt{3}$
$P_2(x) = 5/(3\sqrt{3})(x - \pi/6)^2 + 2/3(x - \pi/6) + 2/\sqrt{3}$
(b)



(c) P_2
(d) The accuracy worsens as you move away from $x = \pi/6$.

131. False. If $f(x) = \sin^2 2x$, then $f'(x) = 2(\sin 2x)(2\cos 2x)$.

133. Putnam Problem A1, 1967

Section 2.5 (page 146)

1. $-x/y$ **3.** $-\sqrt{y/x}$ **5.** $(y - 3x^2)/(2y - x)$

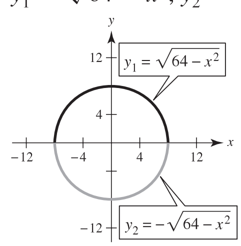
7. $(1 - 3x^2y^3)/(3x^3y^2 - 1)$

9. $(6xy - 3x^2 - 2y^2)/(4xy - 3x^2)$

11. $\cos x/[4\sin(2y)]$ **13.** $(\cos x - \tan y - 1)/(x \sec^2 y)$

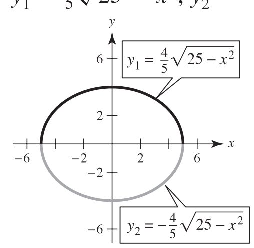
15. $[y\cos(xy)]/[1 - x\cos(xy)]$

17. (a) $y_1 = \sqrt{64 - x^2}$; $y_2 = -\sqrt{64 - x^2}$
(b)



(c) $y' = \mp \dfrac{x}{\sqrt{64 - x^2}} = -\dfrac{x}{y}$
(d) $y' = -\dfrac{x}{y}$

19. (a) $y_1 = \frac{4}{5}\sqrt{25 - x^2}$; $y_2 = -\frac{4}{5}\sqrt{25 - x^2}$
(b)



(c) $y' = \mp \dfrac{4x}{5\sqrt{25 - x^2}} = -\dfrac{16x}{25y}$
(d) $y' = -\dfrac{16x}{25y}$

21. $-\dfrac{y}{x}, -\dfrac{1}{6}$ **23.** $\dfrac{98x}{y(x^2 + 49)^2}$, Undefined **25.** $-\sqrt[3]{\dfrac{y}{x}}, -\dfrac{1}{2}$

27. $-\sin^2(x + y)$ or $-\dfrac{x^2}{x^2 + 1}, 0$ **29.** $-\dfrac{1}{2}$ **31.** 0

33. $y = -x + 7$ **35.** $y = -x + 2$

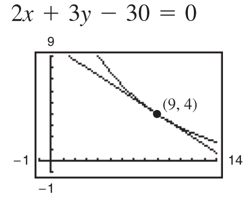
37. $y = \sqrt{3}x/6 + 8\sqrt{3}/3$ **39.** $y = -\frac{2}{11}x + \frac{30}{11}$

41. (a) $y = -2x + 4$ (b) Answers will vary.

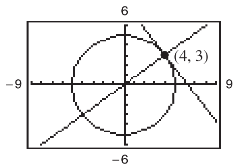
43. $\cos^2 y, -\dfrac{\pi}{2} < y < \dfrac{\pi}{2}, \dfrac{1}{1 + x^2}$ **45.** $-4/y^3$

47. $-36/y^3$ **49.** $(3x)/(4y)$

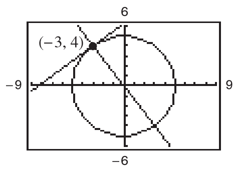
51. $2x + 3y - 30 = 0$



53. At $(4, 3)$:
Tangent line: $4x + 3y - 25 = 0$
Normal line: $3x - 4y = 0$



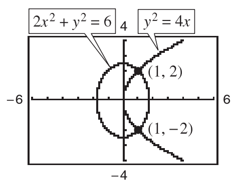
At $(-3, 4)$:
Tangent line: $3x - 4y + 25 = 0$
Normal line: $4x + 3y = 0$



55. $x^2 + y^2 = r^2 \Rightarrow y' = -x/y \Rightarrow y/x =$ slope of normal line. Then for (x_0, y_0) on the circle, $x_0 \neq 0$, an equation of the normal line is $y = (y_0/x_0)x$, which passes through the origin. If $x_0 = 0$, the normal line is vertical and passes through the origin.

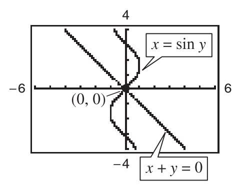
57. Horizontal tangents: $(-4, 0), (-4, 10)$
Vertical tangents: $(0, 5), (-8, 5)$

59.



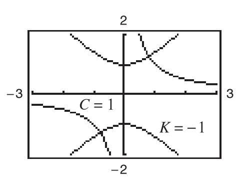
At $(1, 2)$:
Slope of ellipse: -1
Slope of parabola: 1
At $(1, -2)$:
Slope of ellipse: 1
Slope of parabola: -1

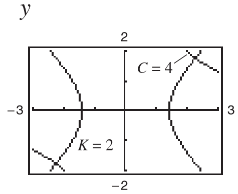
61.



At $(0, 0)$:
Slope of line: -1
Slope of sine curve: 1

63. Derivatives: $\dfrac{dy}{dx} = -\dfrac{y}{x}, \dfrac{dy}{dx} = \dfrac{x}{y}$



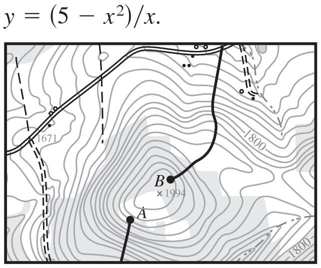


65. (a) $\dfrac{dy}{dx} = \dfrac{3x^3}{y}$ (b) $y\dfrac{dy}{dt} = 3x^3\dfrac{dx}{dt}$

67. (a) $\dfrac{dy}{dx} = \dfrac{-3\cos \pi x}{\sin \pi y}$ (b) $-\sin \pi y\left(\dfrac{dy}{dt}\right) = 3\cos \pi x\left(\dfrac{dx}{dt}\right)$

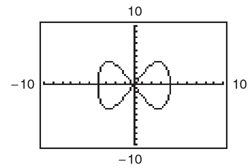
69. Answers will vary. In the explicit form of a function, the variable is explicitly written as a function of x. In an implicit equation, the function is only implied by an equation. An example of an implicit function is $x^2 + xy = 5$. In explicit form it would be $y = (5 - x^2)/x$.

71.

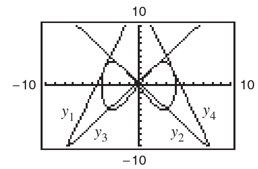


Use starting point B.

73. (a)



(b)



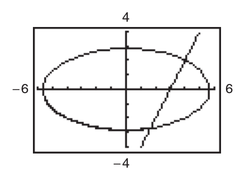
(c) $\left(\dfrac{8\sqrt{7}}{7}, 5\right)$

$y_1 = \frac{1}{3}\left[\left(\sqrt{7} + 7\right)x + \left(8\sqrt{7} + 23\right)\right]$
$y_2 = -\frac{1}{3}\left[\left(-\sqrt{7} + 7\right)x - \left(23 - 8\sqrt{7}\right)\right]$
$y_3 = -\frac{1}{3}\left[\left(\sqrt{7} - 7\right)x - \left(23 - 8\sqrt{7}\right)\right]$
$y_4 = -\frac{1}{3}\left[\left(\sqrt{7} + 7\right)x - \left(8\sqrt{7} + 23\right)\right]$

75. Proof **77.** $(6, -8), (-6, 8)$

79. $y = -\dfrac{\sqrt{3}}{2}x + 2\sqrt{3}, y = \dfrac{\sqrt{3}}{2}x - 2\sqrt{3}$

81. (a) $y = 2x - 6$
(b)



(c) $\left(\frac{28}{17}, -\frac{46}{17}\right)$

Section 2.6 (page 154)

1. (a) $\frac{3}{4}$ (b) 20 **3.** (a) $-\frac{5}{8}$ (b) $\frac{3}{2}$
5. (a) -8 cm/sec (b) 0 cm/sec (c) 8 cm/sec
7. (a) 8 cm/sec (b) 4 cm/sec (c) 2 cm/sec
9. In a linear function, if x changes at a constant rate, so does y. However, unless $a = 1$, y does not change at the same rate as x.
11. $(4x^3 + 6x)/\sqrt{x^4 + 3x^2 + 1}$
13. (a) 64π cm^2/min (b) 256π cm^2/min
15. (a) Proof
(b) When $\theta = \dfrac{\pi}{6}, \dfrac{dA}{dt} = \dfrac{\sqrt{3}}{8}s^2$. When $\theta = \dfrac{\pi}{3}, \dfrac{dA}{dt} = \dfrac{1}{8}s^2$.
(c) If s and $d\theta/dt$ are constant, dA/dt is proportional to $\cos\theta$.
17. (a) $2/(9\pi)$ cm/min (b) $1/(18\pi)$ cm/min
19. (a) 144 cm^2/sec (b) 720 cm^2/sec **21.** $8/(405\pi)$ ft/min
23. (a) 12.5% (b) $\frac{1}{144}$ m/min
25. (a) $-\frac{7}{12}$ ft/sec; $-\frac{3}{2}$ ft/sec; $-\frac{48}{7}$ ft/sec
(b) $\frac{527}{24}$ ft^2/sec (c) $\frac{1}{12}$ rad/sec
27. Rate of vertical change: $\frac{1}{5}$ m/sec
Rate of horizontal change: $-\sqrt{3}/15$ m/sec
29. (a) -750 mi/h (b) 30 min
31. $-50/\sqrt{85} \approx -5.42$ ft/sec **33.** (a) $\frac{25}{3}$ ft/sec (b) $\frac{10}{3}$ ft/sec
35. (a) 12 sec (b) $\frac{1}{2}\sqrt{3}$ m (c) $\left(\sqrt{5}\pi\right)/120$ m/sec

37. Evaporation rate proportional to $S \Longrightarrow \dfrac{dV}{dt} = k(4\pi r^2)$

$V = \left(\dfrac{4}{3}\right)\pi r^3 \Longrightarrow \dfrac{dV}{dt} = 4\pi r^2 \dfrac{dr}{dt}$. So $k = \dfrac{dr}{dt}$.

39. 0.6 ohm/sec **41.** $\dfrac{dv}{dt} = \dfrac{16r}{v}\sec^2\theta\dfrac{d\theta}{dt}, \dfrac{d\theta}{dt} = \dfrac{v}{16r}\cos^2\theta\dfrac{dv}{dt}$

43. $\dfrac{2\sqrt{21}}{525} \approx 0.017$ rad/sec

45. (a) $\dfrac{200\pi}{3}$ ft/sec (b) 200π ft/sec (c) About 427.43π ft/sec

47. About 84.9797 mi/h

49. (a) $\dfrac{dy}{dt} = 3\dfrac{dx}{dt}$ means that y changes three times as fast as x changes.

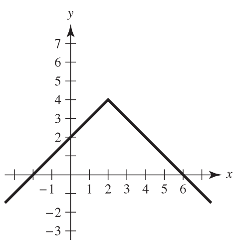
(b) y changes slowly when $x \approx 0$ or $x \approx L$. y changes more rapidly when x is near the middle of the interval.

51. -18.432 ft/sec^2 **53.** About -97.96 m/sec

Review Exercises for Chapter 2 (page 158)

1. $f'(x) = 2x - 4$ **3.** $f'(x) = -2/(x-1)^2$

5. f is differentiable at all $x \neq 3$.

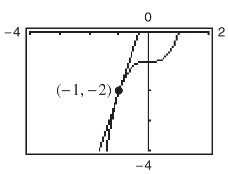
7. 

(a) Yes

(b) No, because the derivatives from the left and right are not equal.

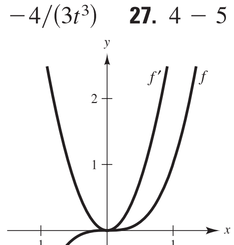
9. $-\frac{3}{2}$

11. (a) $y = 3x + 1$ **13.** 8

(b) 

15. 0 **17.** $8x^7$ **19.** $52t^3$ **21.** $3x^2 - 22x$ **23.** $\dfrac{3}{\sqrt{x}} + \dfrac{1}{\sqrt[3]{x^2}}$

25. $-4/(3t^3)$ **27.** $4 - 5\cos\theta$ **29.** $-3\sin\theta - (\cos\theta)/4$

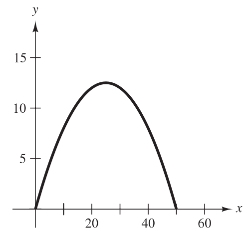
31. 

$f' > 0$ where the slopes of tangent lines to the graph of f are positive.

33. (a) 50 vibrations/sec/lb

(b) 33.33 vibrations/sec/lb

35. 1354.24 ft or 412.77 m

37. (a) 

(b) 50

(c) $x = 25$

(d) $y' = 1 - 0.04x$

x	0	10	25	30	50
y'	1	0.6	0	-0.2	-1

(e) $y'(25) = 0$

39. (a) $x'(t) = 2t - 3$ (b) $(-\infty, 1.5)$ (c) $x = -\frac{1}{4}$ (d) 1

41. $4(5x^3 - 15x^2 - 11x - 8)$ **43.** $\sqrt{x}\cos x + \sin x/(2\sqrt{x})$

45. $-(x^2+1)/(x^2-1)^2$ **47.** $(8x)/(9 - 4x^2)^2$

49. $\dfrac{4x^3\cos x + x^4 \sin x}{\cos^2 x}$ **51.** $3x^2\sec x\tan x + 6x\sec x$

53. $-x\sin x$ **55.** $y = 4x - 3$ **57.** $y = 0$

59. $v(4) = 20$ m/sec; $a(4) = -8$ m/sec^2

61. $-48t$ **63.** $\frac{225}{4}\sqrt{x}$ **65.** $6\sec^2\theta\tan\theta$

67. $y'' + y = -(2\sin x + 3\cos x) + (2\sin x + 3\cos x) = 0$

69. $\dfrac{2(x+5)(-x^2-10x+3)}{(x^2+3)^3}$

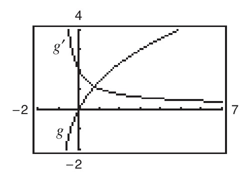
71. $s(s^2-1)^{3/2}(8s^3 - 3s + 25)$

73. $-45\sin(9x+1)$ **75.** $\frac{1}{2}(1 - \cos 2x) = \sin^2 x$

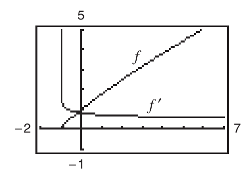
77. $\sin^{1/2}x\cos x - \sin^{5/2}x\cos x = \cos^3 x\sqrt{\sin x}$

79. $\dfrac{(x+2)(\pi\cos\pi x) - \sin\pi x}{(x+2)^2}$ **81.** -2 **83.** 0

85. $(x+2)/(x+1)^{3/2}$ **87.** $5/[6(t+1)^{1/6}]$

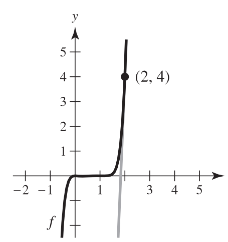


g' is not equal to zero for any x.



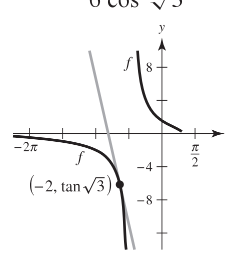
f' has no zeros.

89. (a) $f'(2) = 24$ (b) $y = 24t - 44$

(c) 

91. (a) $f'(-2) = -\dfrac{1}{2\sqrt{3}\cos^2\sqrt{3}} \approx -11.1983$

(b) $y = -\dfrac{\sqrt{3}(x+2)}{6\cos^2\sqrt{3}} + \tan\sqrt{3}$

(c) 

93. $14 - 4\cos 2x$ **95.** $2\csc^2 x \cot x$

97. $[8(2t+1)]/(1-t)^4$

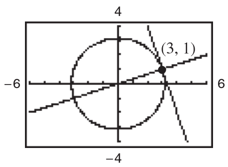
99. $18\sec^2 3\theta \tan 3\theta + \sin(\theta - 1)$

101. (a) $-18.667°/\text{h}$ (b) $-7.284°/\text{h}$
(c) $-3.240°/\text{h}$ (d) $-0.747°/\text{h}$

103. $-\dfrac{2x+3y}{3(x+y^2)}$ **105.** $\dfrac{\sqrt{y}(2\sqrt{x}-\sqrt{y})}{\sqrt{x}(\sqrt{x}+8\sqrt{y})} = \dfrac{2x-9y}{9x-32y}$

107. $\dfrac{y\sin x + \sin y}{\cos x - x\cos y}$

109. Tangent line: $3x + y - 10 = 0$
Normal line: $x - 3y = 0$



111. (a) $2\sqrt{2}$ units/sec (b) 4 units/sec (c) 8 units/sec

113. $\frac{2}{25}$ m/min **115.** -38.34 m/sec

P.S. Problem Solving (page 161)

1. (a) $r = \frac{1}{2}$; $x^2 + \left(y - \frac{1}{2}\right)^2 = \frac{1}{4}$
(b) Center: $\left(0, \frac{5}{4}\right)$; $x^2 + \left(y - \frac{5}{4}\right)^2 = 1$

3. (a) $P_1(x) = 1$ (b) $P_2(x) = 1 - \frac{1}{2}x^2$
(c)

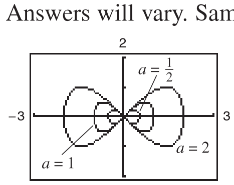
x	-1.0	-0.1	-0.001	0	0.001
cos x	0.5403	0.9950	1.000	1	1.000
$P_2(x)$	0.5	0.995	1.000	1	1.000

x	0.1	1.0
cos x	0.9950	0.5403
$P_2(x)$	0.995	0.5

$P_2(x)$ is a good approximation of $f(x) = \cos x$ when x is very close to 0.
(d) $P_3(x) = x - \frac{1}{6}x^3$

5. $p(x) = 2x^3 + 4x^2 - 5$

7. (a) Graph $\begin{cases} y_1 = \dfrac{1}{a}\sqrt{x^2(a^2 - x^2)} \\ y_2 = -\dfrac{1}{a}\sqrt{x^2(a^2 - x^2)} \end{cases}$ as separate equations.
(b) Answers will vary. Sample answer:



The intercepts will always be $(0, 0)$, $(a, 0)$, and $(-a, 0)$, and the maximum and minimum y-values appear to be $\pm\frac{1}{2}a$.
(c) $\left(\dfrac{a\sqrt{2}}{2}, \dfrac{a}{2}\right)$, $\left(\dfrac{a\sqrt{2}}{2}, -\dfrac{a}{2}\right)$, $\left(-\dfrac{a\sqrt{2}}{2}, \dfrac{a}{2}\right)$, $\left(-\dfrac{a\sqrt{2}}{2}, -\dfrac{a}{2}\right)$

9. (a) When the man is 90 ft from the light, the tip of his shadow is $112\frac{1}{2}$ ft from the light. The tip of the child's shadow is $111\frac{1}{9}$ ft from the light, so the man's shadow extends $1\frac{7}{18}$ ft beyond the child's shadow.
(b) When the man is 60 ft from the light, the tip of his shadow is 75 ft from the light. The tip of the child's shadow is $77\frac{7}{9}$ ft from the light, so the child's shadow extends $2\frac{7}{9}$ ft beyond the man's shadow.
(c) $d = 80$ ft
(d) Let x be the distance of the man from the light and let s be the distance from the light to the tip of the shadow.
If $0 < x < 80$, $ds/dt = -50/9$.
If $x > 80$, $ds/dt = -25/4$.
There is a discontinuity at $x = 80$.

11. Proof. The graph of L is a line passing through the origin $(0, 0)$.

13. (a)

$z°$	0.1	0.01	0.0001
$\dfrac{\sin z}{z}$	0.0174532837	0.0174532924	0.0174532925

(b) $\pi/180$ (c) $(\pi/180)\cos z$
(d) $S(90) = 1$, $C(180) = -1$; $(\pi/180)C(z)$
(e) Answers will vary.

15. (a) j would be the rate of change of acceleration.
(b) $j = 0$. Acceleration is constant, so there is no change in acceleration.
(c) a: position function, d: velocity function,
b: acceleration function, c: jerk function

Chapter 3

Section 3.1 (page 169)

1. $f'(0) = 0$ **3.** $f'(2) = 0$ **5.** $f'(-2)$ is undefined.

7. 2, absolute maximum (and relative maximum)

9. 1, absolute maximum (and relative maximum);
2, absolute minimum (and relative minimum);
3, absolute maximum (and relative maximum)

11. $x = 0, x = 2$ **13.** $t = 8/3$ **15.** $x = \pi/3, \pi, 5\pi/3$

17. Minimum: $(2, 1)$
Maximum: $(-1, 4)$

19. Minimum: $(1, -1)$
Maximum: $(4, 8)$

21. Minimum: $\left(-1, -\frac{5}{2}\right)$
Maximum: $(2, 2)$

23. Minimum: $(0, 0)$
Maximum: $(-1, 5)$

25. Minimum: $(0, 0)$
Maxima: $\left(-1, \frac{1}{4}\right)$ and $\left(1, \frac{1}{4}\right)$

27. Minimum: $(1, -1)$
Maximum: $\left(0, -\frac{1}{2}\right)$

29. Minimum: $(-1, -1)$
Maximum: $(3, 3)$

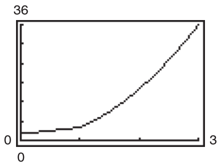
31. Minimum value is -2 for $-2 \le x < -1$.
Maximum: $(2, 2)$

33. Minimum: $\left(1/6, \sqrt{3}/2\right)$
Maximum: $(0, 1)$

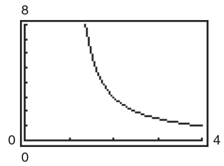
35. Minimum: $(\pi, -3)$
Maxima: $(0, 3)$ and $(2\pi, 3)$

37. (a) Minimum: $(0, -3)$;
Maximum: $(2, 1)$
(b) Minimum: $(0, -3)$
(c) Maximum: $(2, 1)$
(d) No extrema

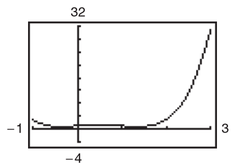
39. (a) Minimum: $(1, -1)$;
Maximum: $(-1, 3)$
(b) Maximum: $(3, 3)$
(c) Minimum: $(1, -1)$
(d) Minimum: $(1, -1)$

41. 

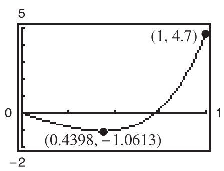
Minimum: $(0, 2)$
Maximum: $(3, 36)$

43. 

Minimum: $(4, 1)$

45. 

Minima: $\left(\dfrac{-\sqrt{3}+1}{2}, \dfrac{3}{4}\right)$ and $\left(\dfrac{\sqrt{3}+1}{2}, \dfrac{3}{4}\right)$
Maximum: $(3, 31)$

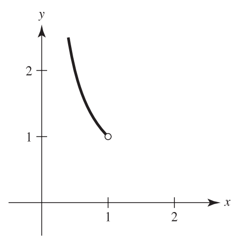
47. (a) 

(b) Minimum: $(0.4398, -1.0613)$

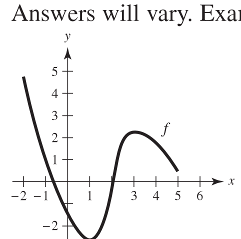
49. Maximum: $\left|f''\left(\sqrt[3]{-10+\sqrt{108}}\right)\right| = f''\left(\sqrt{3}-1\right) \approx 1.47$

51. Maximum: $|f^{(4)}(0)| = \frac{56}{81}$

53. Answers will vary. Let $f(x) = 1/x$. f is continuous on $(0, 1)$ but does not have a maximum or minimum.



55. Answers will vary. Example:



57. (a) Yes (b) No **59.** (a) No (b) Yes

61. Maximum: $P(12) = 72$; No. P is decreasing for $I > 12$.

63. $\theta = \operatorname{arcsec}\sqrt{3} \approx 0.9553$ rad

65. True **67.** True **69.** Proof **71.** Putnam Problem B3, 2004

Section 3.2 (page 176)

1. $f(-1) = f(1) = 1$; f is not continuous on $[-1, 1]$.

3. $f(0) = f(2) = 0$; f is not differentiable on $(0, 2)$.

5. $(2, 0), (-1, 0)$; $f'\left(\frac{1}{2}\right) = 0$ **7.** $(0, 0), (-4, 0)$; $f'\left(-\frac{8}{3}\right) = 0$

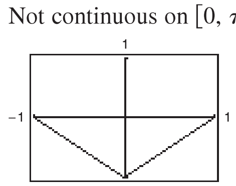
9. $f'(-1) = 0$ **11.** $f'\left(\frac{3}{2}\right) = 0$

13. $f'\left(\dfrac{6-\sqrt{3}}{3}\right) = 0$; $f'\left(\dfrac{6+\sqrt{3}}{3}\right) = 0$

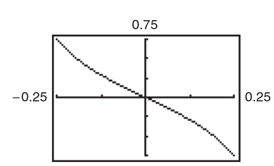
15. Not differentiable at $x = 0$ **17.** $f'\left(-2+\sqrt{5}\right) = 0$

19. $f'(\pi/2) = 0$; $f'(3\pi/2) = 0$ **21.** $f'(0.249) \approx 0$

23. Not continuous on $[0, \pi]$

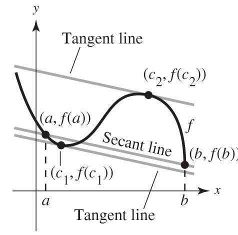
25. 

Rolle's Theorem does not apply.

27. 

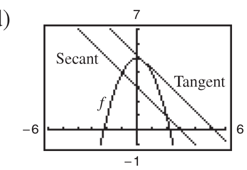
Rolle's Theorem does not apply.

29. (a) $f(1) = f(2) = 38$
(b) Velocity $= 0$ for some t in $(1, 2)$; $t = \frac{3}{2}$ sec

31. 

33. The function is not continuous on $[0, 6]$.

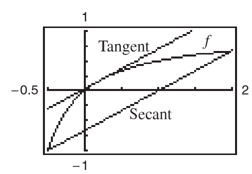
35. The function is not continuous on $[0, 6]$.

37. (a) Secant line: $x + y - 3 = 0$ (b) $c = \frac{1}{2}$
(c) Tangent line: $4x + 4y - 21 = 0$
(d) 

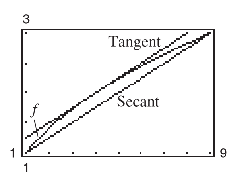
39. $f'(-1/2) = -1$ **41.** $f'\left(1/\sqrt{3}\right) = 3, f'\left(-1/\sqrt{3}\right) = 3$

43. $f'(8/27) = 1$ **45.** f is not differentiable at $x = -\frac{1}{2}$.

47. $f'(\pi/2) = 0$

49. (a)–(c) 

(b) $y = \frac{2}{3}(x - 1)$
(c) $y = \frac{1}{3}\left(2x + 5 - 2\sqrt{6}\right)$

51. (a)–(c) 

(b) $y = \frac{1}{4}x + \frac{3}{4}$
(c) $y = \frac{1}{4}x + 1$

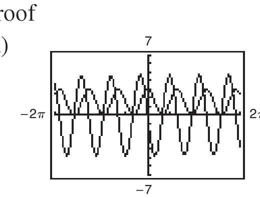
53. (a) -14.7 m/sec (b) 1.5 sec

55. No. Let $f(x) = x^2$ on $[-1, 2]$.

57. No. $f(x)$ is not continuous on $[0, 1]$. So it does not satisfy the hypothesis of Rolle's Theorem.

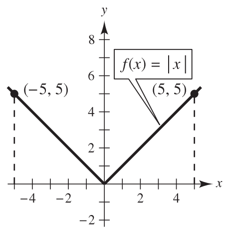
59. By the Mean Value Theorem, there is a time when the speed of the plane must equal the average speed of 454.5 miles/hour. The speed was 400 miles/hour when the plane was accelerating to 454.5 miles/hour and decelerating from 454.5 miles/hour.

61. Proof

63. (a) 

(b) Yes; yes
(c) Because $f(-1) = f(1) = 0$, Rolle's Theorem applies on $[-1, 1]$. Because $f(1) = 0$ and $f(2) = 3$, Rolle's Theorem does not apply on $[1, 2]$.
(d) $\lim_{x \to 3^-} f'(x) = 0$; $\lim_{x \to 3^+} f'(x) = 0$

65.



67. Proof **69.** Proof

71. $a = 6, b = 1, c = 2$ **73.** $f(x) = 5$ **75.** $f(x) = x^2 - 1$

77. False. f is not continuous on $[-1, 1]$. **79.** True

81–89. Proofs

Section 3.3 (page 186)

1. (a) $(0, 6)$ (b) $(6, 8)$

3. Increasing on $(3, \infty)$; Decreasing on $(-\infty, 3)$

5. Increasing on $(-\infty, -2)$ and $(2, \infty)$; Decreasing on $(-2, 2)$

7. Increasing on $(-\infty, -1)$; Decreasing on $(-1, \infty)$

9. Increasing on $(1, \infty)$; Decreasing on $(-\infty, 1)$

11. Increasing on $(-2\sqrt{2}, 2\sqrt{2})$
Decreasing on $(-4, -2\sqrt{2})$ and $(2\sqrt{2}, 4)$

13. Increasing on $(0, \pi/2)$ and $(3\pi/2, 2\pi)$;
Decreasing on $(\pi/2, 3\pi/2)$

15. Increasing on $(0, 7\pi/6)$ and $(11\pi/6, 2\pi)$;
Decreasing on $(7\pi/6, 11\pi/6)$

17. (a) Critical number: $x = 2$
(b) Increasing on $(2, \infty)$; Decreasing on $(-\infty, 2)$
(c) Relative minimum: $(2, -4)$

19. (a) Critical number: $x = 1$
(b) Increasing on $(-\infty, 1)$; Decreasing on $(1, \infty)$
(c) Relative maximum: $(1, 5)$

21. (a) Critical numbers: $x = -2, 1$
(b) Increasing on $(-\infty, -2)$ and $(1, \infty)$; Decreasing on $(-2, 1)$
(c) Relative maximum: $(-2, 20)$; Relative minimum: $(1, -7)$

23. (a) Critical numbers: $x = -\frac{5}{3}, 1$
(b) Increasing on $(-\infty, -\frac{5}{3})$, $(1, \infty)$
Decreasing on $(-\frac{5}{3}, 1)$
(c) Relative maximum: $(-\frac{5}{3}, \frac{256}{27})$
Relative minimum: $(1, 0)$

25. (a) Critical numbers: $x = \pm 1$
(b) Increasing on $(-\infty, -1)$ and $(1, \infty)$; Decreasing on $(-1, 1)$
(c) Relative maximum: $(-1, \frac{4}{5})$; Relative minimum: $(1, -\frac{4}{5})$

27. (a) Critical number: $x = 0$
(b) Increasing on $(-\infty, \infty)$
(c) No relative extrema

29. (a) Critical number: $x = -2$
(b) Increasing on $(-2, \infty)$; Decreasing on $(-\infty, -2)$
(c) Relative minimum: $(-2, 0)$

31. (a) Critical number: $x = 5$
(b) Increasing on $(-\infty, 5)$; Decreasing on $(5, \infty)$
(c) Relative maximum: $(5, 5)$

33. (a) Critical numbers: $x = \pm\sqrt{2}/2$; Discontinuity: $x = 0$
(b) Increasing on $(-\infty, -\sqrt{2}/2)$ and $(\sqrt{2}/2, \infty)$
Decreasing on $(-\sqrt{2}/2, 0)$ and $(0, \sqrt{2}/2)$
(c) Relative maximum: $(-\sqrt{2}/2, -2\sqrt{2})$
Relative minimum: $(\sqrt{2}/2, 2\sqrt{2})$

35. (a) Critical number: $x = 0$; Discontinuities: $x = \pm 3$
(b) Increasing on $(-\infty, -3)$ and $(-3, 0)$
Decreasing on $(0, 3)$ and $(3, \infty)$
(c) Relative maximum: $(0, 0)$

37. (a) Critical numbers: $x = -3, 1$; Discontinuity: $x = -1$
(b) Increasing on $(-\infty, -3)$ and $(1, \infty)$
Decreasing on $(-3, -1)$ and $(-1, 1)$
(c) Relative maximum: $(-3, -8)$; Relative minimum: $(1, 0)$

39. (a) Critical number: $x = 0$
(b) Increasing on $(-\infty, 0)$; Decreasing on $(0, \infty)$
(c) No relative extrema

41. (a) Critical number: $x = 1$
(b) Increasing on $(-\infty, 1)$; Decreasing on $(1, \infty)$
(c) Relative maximum: $(1, 4)$

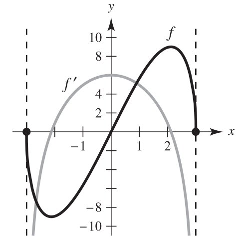
43. (a) Critical numbers: $x = \pi/6, 5\pi/6$
Increasing on $(0, \pi/6)$, $(5\pi/6, 2\pi)$
Decreasing on $(\pi/6, 5\pi/6)$
(b) Relative maximum: $(\pi/6, (\pi + 6\sqrt{3})/12)$
Relative minimum: $(5\pi/6, (5\pi - 6\sqrt{3})/12)$

45. (a) Critical numbers: $x = \pi/4, 5\pi/4$
Increasing on $(0, \pi/4)$, $(5\pi/4, 2\pi)$
Decreasing on $(\pi/4, 5\pi/4)$
(b) Relative maximum: $(\pi/4, \sqrt{2})$
Relative minimum: $(5\pi/4, -\sqrt{2})$

47. (a) Critical numbers:
$x = \pi/4, \pi/2, 3\pi/4, \pi, 5\pi/4, 3\pi/2, 7\pi/4$
Increasing on $(\pi/4, \pi/2)$, $(3\pi/4, \pi)$, $(5\pi/4, 3\pi/2)$, $(7\pi/4, 2\pi)$
Decreasing on $(0, \pi/4)$, $(\pi/2, 3\pi/4)$, $(\pi, 5\pi/4)$, $(3\pi/2, 7\pi/4)$
(b) Relative maxima: $(\pi/2, 1)$, $(\pi, 1)$, $(3\pi/2, 1)$
Relative minima: $(\pi/4, 0)$, $(3\pi/4, 0)$, $(5\pi/4, 0)$, $(7\pi/4, 0)$

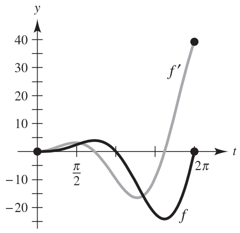
49. (a) Critical numbers: $\pi/2, 7\pi/6, 3\pi/2, 11\pi/6$
Increasing on $\left(0, \dfrac{\pi}{2}\right), \left(\dfrac{7\pi}{6}, \dfrac{3\pi}{2}\right), \left(\dfrac{11\pi}{6}, 2\pi\right)$
Decreasing on $\left(\dfrac{\pi}{2}, \dfrac{7\pi}{6}\right), \left(\dfrac{3\pi}{2}, \dfrac{11\pi}{6}\right)$
(b) Relative maxima: $\left(\dfrac{\pi}{2}, 2\right), \left(\dfrac{3\pi}{2}, 0\right)$
Relative minima: $\left(\dfrac{7\pi}{6}, -\dfrac{1}{4}\right), \left(\dfrac{11\pi}{6}, -\dfrac{1}{4}\right)$

51. (a) $f'(x) = 2(9 - 2x^2)/\sqrt{9 - x^2}$
(b)



(c) Critical numbers: $x = \pm 3\sqrt{2}/2$
(d) $f' > 0$ on $(-3\sqrt{2}/2, 3\sqrt{2}/2)$
$f' < 0$ on $(-3, -3\sqrt{2}/2)$, $(3\sqrt{2}/2, 3)$

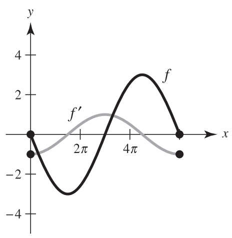
53. (a) $f'(t) = t(t\cos t + 2\sin t)$

(b) 

(c) Critical numbers: $t = 2.2889, 5.0870$

(d) $f' > 0$ on $(0, 2.2889)$, $(5.0870, 2\pi)$
$f' < 0$ on $(2.2889, 5.0870)$

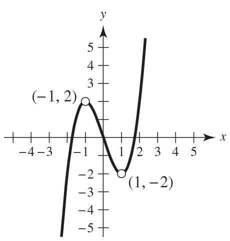
55. (a) $f'(x) = -\cos(x/3)$

(b) 

(c) Critical numbers: $x = 3\pi/2, 9\pi/2$

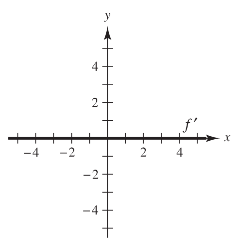
(d) $f' > 0$ on $\left(\frac{3\pi}{2}, \frac{9\pi}{2}\right)$
$f' < 0$ on $\left(0, \frac{3\pi}{2}\right), \left(\frac{9\pi}{2}, 6\pi\right)$

57. $f(x)$ is symmetric with respect to the origin.
Zeros: $(0, 0), (\pm\sqrt{3}, 0)$

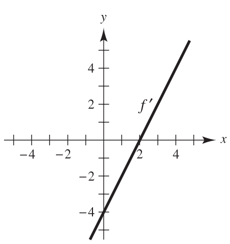


$g(x)$ is continuous on $(-\infty, \infty)$ and $f(x)$ has holes at $x = 1$ and $x = -1$.

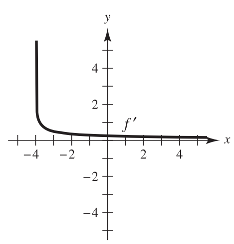
59.



61.



63.



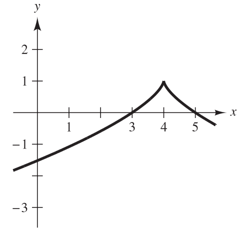
65. (a) Increasing on $(2, \infty)$; Decreasing on $(-\infty, 2)$
(b) Relative minimum: $x = 2$

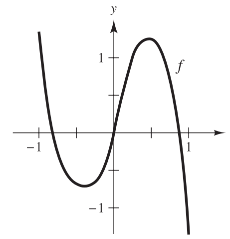
67. (a) Increasing on $(-\infty, -1)$ and $(0, 1)$; Decreasing on $(-1, 0)$ and $(1, \infty)$
(b) Relative maxima: $x = -1$ and $x = 1$
Relative minimum: $x = 0$

69. (a) Critical numbers: $x = -1, x = 1, x = 2$
(b) Relative maximum at $x = 1$, relative minimum at $x = 2$, and neither at $x = -1$

71. $g'(0) < 0$ **73.** $g'(-6) < 0$ **75.** $g'(0) > 0$

77. Answers will vary. Sample answer:



79. (a) 

(b) Critical numbers: $x \approx -0.40$ and $x \approx 0.48$
(c) Relative maximum: $(0.48, 1.25)$
Relative minimum: $(-0.40, 0.75)$

81. (a) $s'(t) = 9.8(\sin\theta)t$; speed $= |9.8(\sin\theta)t|$
(b)

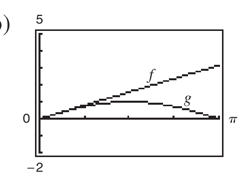
θ	0	$\pi/4$	$\pi/3$	$\pi/2$	$2\pi/3$	$3\pi/4$	π
$s'(t)$	0	$4.9\sqrt{2}t$	$4.9\sqrt{3}t$	$9.8t$	$4.9\sqrt{3}t$	$4.9\sqrt{2}t$	0

The speed is maximum at $\theta = \pi/2$.

83. (a)

x	0.5	1	1.5	2	2.5	3
$f(x)$	0.5	1	1.5	2	2.5	3
$g(x)$	0.48	0.84	1.00	0.91	0.60	0.14

$f(x) > g(x)$

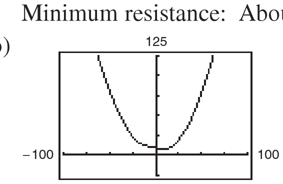
(b) 

$f(x) > g(x)$

(c) Proof

85. $r = 2R/3$

87. (a) $\dfrac{dR}{dT} = \dfrac{0.004T^3 - 4}{2\sqrt{0.001T^4 - 4T + 100}}$
Critical number: $T = 10$
Minimum resistance: About 8.3666 ohms

(b) 

Minimum resistance: About 8.3666 ohms

89. (a) $v(t) = 6 - 2t$ (b) $(0, 3)$ (c) $(3, \infty)$ (d) $t = 3$

91. (a) $v(t) = 3t^2 - 10t + 4$
(b) $\left(0, \left(5 - \sqrt{13}\right)/3\right)$ and $\left(\left(5 + \sqrt{13}\right)/3, \infty\right)$
(c) $\left(\dfrac{5 - \sqrt{13}}{3}, \dfrac{5 + \sqrt{13}}{3}\right)$ (d) $t = \dfrac{5 \pm \sqrt{13}}{3}$

93. Answers will vary.

95. (a) Minimum degree: 3
(b) $a_3(0)^3 + a_2(0)^2 + a_1(0) + a_0 = 0$
$a_3(2)^3 + a_2(2)^2 + a_1(2) + a_0 = 2$
$3a_3(0)^2 + 2a_2(0) + a_1 = 0$
$3a_3(2)^2 + 2a_2(2) + a_1 = 0$
(c) $f(x) = -\frac{1}{2}x^3 + \frac{3}{2}x^2$

97. (a) Minimum degree: 4
(b) $a_4(0)^4 + a_3(0)^3 + a_2(0)^2 + a_1(0) + a_0 = 0$
$a_4(2)^4 + a_3(2)^3 + a_2(2)^2 + a_1(2) + a_0 = 4$
$a_4(4)^4 + a_3(4)^3 + a_2(4)^2 + a_1(4) + a_0 = 0$
$4a_4(0)^3 + 3a_3(0)^2 + 2a_2(0) + a_1 = 0$
$4a_4(2)^3 + 3a_3(2)^2 + 2a_2(2) + a_1 = 0$
$4a_4(4)^3 + 3a_3(4)^2 + 2a_2(4) + a_1 = 0$
(c) $f(x) = \frac{1}{4}x^4 - 2x^3 + 4x^2$

99. True **101.** False. Let $f(x) = x^3$.

103. False. Let $f(x) = x^3$. There is a critical number at $x = 0$, but not a relative extremum.

105–107. Proofs

Section 3.4 (page 195)

1. $f' > 0, f'' > 0$ **3.** $f' < 0, f'' < 0$

5. Concave upward: $(-\infty, \infty)$

7. Concave upward: $(-\infty, 1)$; Concave downward: $(1, \infty)$

9. Concave upward: $(-\infty, 2)$; Concave downward: $(2, \infty)$

11. Concave upward: $(-\infty, -2), (2, \infty)$
Concave downward: $(-2, 2)$

13. Concave upward: $(-\infty, -1), (1, \infty)$
Concave downward: $(-1, 1)$

15. Concave upward: $(-2, 2)$
Concave downward: $(-\infty, -2), (2, \infty)$

17. Concave upward: $(-\pi/2, 0)$; Concave downward: $(0, \pi/2)$

19. Points of inflection: $(-2, -8), (0, 0)$
Concave upward: $(-\infty, -2), (0, \infty)$
Concave downward: $(-2, 0)$

21. Point of inflection: $(2, 8)$; Concave downward: $(-\infty, 2)$
Concave upward: $(2, \infty)$

23. Points of inflection: $\left(\pm 2\sqrt{3}/3, -20/9\right)$
Concave upward: $\left(-\infty, -2\sqrt{3}/3\right), \left(2\sqrt{3}/3, \infty\right)$
Concave downward: $\left(-2\sqrt{3}/3, 2\sqrt{3}/3\right)$

25. Points of inflection: $(2, -16), (4, 0)$
Concave upward: $(-\infty, 2), (4, \infty)$; Concave downward: $(2, 4)$

27. Concave upward: $(-3, \infty)$

29. Points of inflection: $\left(-\sqrt{3}/3, 3\right), \left(\sqrt{3}/3, 3\right)$
Concave upward: $\left(-\infty, -\sqrt{3}/3\right), \left(\sqrt{3}/3, \infty\right)$
Concave downward: $\left(-\sqrt{3}/3, \sqrt{3}/3\right)$

31. Point of inflection: $(2\pi, 0)$
Concave upward: $(2\pi, 4\pi)$; Concave downward: $(0, 2\pi)$

33. Concave upward: $(0, \pi), (2\pi, 3\pi)$
Concave downward: $(\pi, 2\pi), (3\pi, 4\pi)$

35. Points of inflection: $(\pi, 0), (1.823, 1.452), (4.46, -1.452)$
Concave upward: $(1.823, \pi), (4.46, 2\pi)$
Concave downward: $(0, 1.823), (\pi, 4.46)$

37. Relative minimum: $(5, 0)$ **39.** Relative maximum: $(3, 9)$

41. Relative maximum: $(0, 3)$; Relative minimum: $(2, -1)$

43. Relative minimum: $(3, -25)$

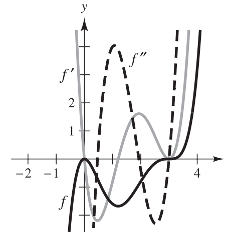
45. Relative maximum: $(2.4, 268.74)$; Relative minimum: $(0, 0)$

47. Relative minimum: $(0, -3)$

49. Relative maximum: $(-2, -4)$; Relative minimum: $(2, 4)$

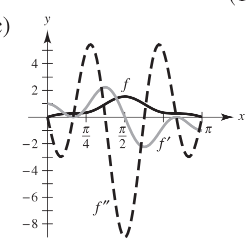
51. No relative extrema, because f is nonincreasing.

53. (a) $f'(x) = 0.2x(x - 3)^2(5x - 6)$
$f''(x) = 0.4(x - 3)(10x^2 - 24x + 9)$
(b) Relative maximum: $(0, 0)$
Relative minimum: $(1.2, -1.6796)$
Points of inflection: $(0.4652, -0.7048)$, $(1.9348, -0.9048)$, $(3, 0)$
(c)



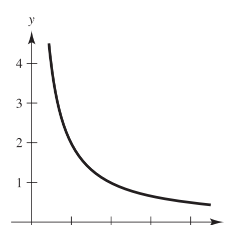
f is increasing when f' is positive, and decreasing when f' is negative. f is concave upward when f'' is positive, and concave downward when f'' is negative.

55. (a) $f'(x) = \cos x - \cos 3x + \cos 5x$
$f''(x) = -\sin x + 3\sin 3x - 5\sin 5x$
(b) Relative maximum: $(\pi/2, 1.53333)$
Points of inflection: $(\pi/6, 0.2667), (1.1731, 0.9637)$, $(1.9685, 0.9637), (5\pi/6, 0.2667)$
(c)

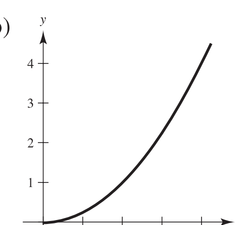


f is increasing when f' is positive, and decreasing when f' is negative. f is concave upward when f'' is positive, and concave downward when f'' is negative.

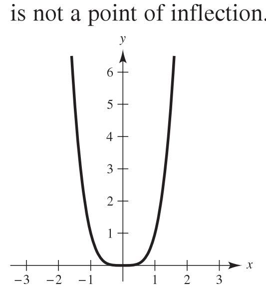
57. (a)



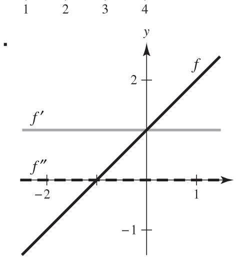
(b)



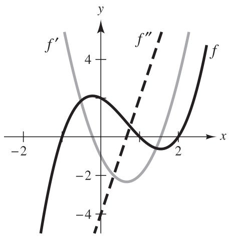
59. Answers will vary. Example:
$f(x) = x^4$; $f''(0) = 0$, but $(0, 0)$ is not a point of inflection.



61.



63.



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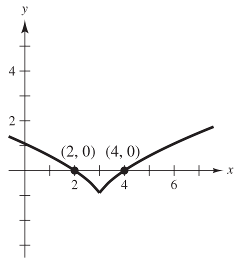
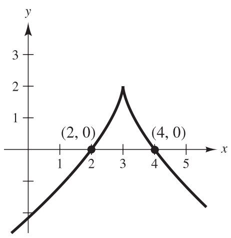
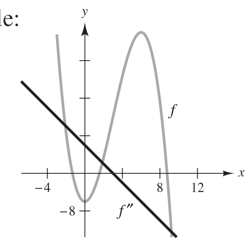


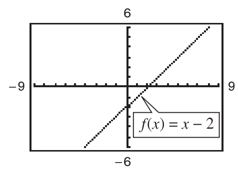
67.

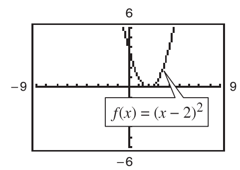


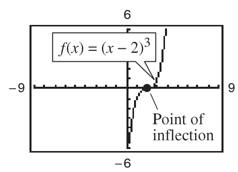
69. Example:

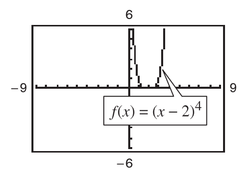


71. (a) $f(x) = (x-2)^n$ has a point of inflection at $(2, 0)$ if n is odd and $n \geq 3$.









(b) Proof

73. $f(x) = \frac{1}{2}x^3 - 6x^2 + \frac{45}{2}x - 24$

75. (a) $f(x) = \frac{1}{32}x^3 + \frac{3}{16}x^2$ (b) Two miles from touchdown

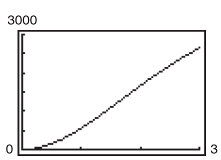
77. $x = \left(\dfrac{15 - \sqrt{33}}{16}\right)L \approx 0.578L$ **79.** $x = 100$ units

81. (a)

t	0.5	1	1.5	2	2.5	3
S	151.5	555.6	1097.6	1666.7	2193.0	2647.1

$1.5 < t < 2$

(b)



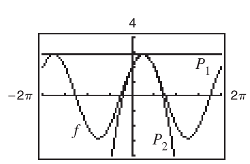
$t \approx 1.5$

(c) About 1.633 yr

83. $P_1(x) = 2\sqrt{2}$

$P_2(x) = 2\sqrt{2} - \sqrt{2}(x - \pi/4)^2$

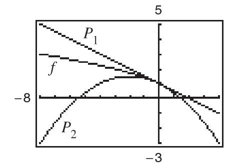
The values of f, P_1, and P_2 and their first derivatives are equal when $x = \pi/4$. The approximations worsen as you move away from $x = \pi/4$.



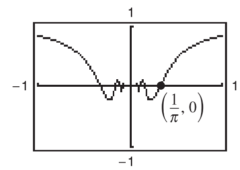
85. $P_1(x) = 1 - x/2$

$P_2(x) = 1 - x/2 - x^2/8$

The values of f, P_1, and P_2 and their first derivatives are equal when $x = 0$. The approximations worsen as you move away from $x = 0$.



87.



89. Proof **91.** True

93. False. f is concave upward at $x = c$ if $f''(c) > 0$. **95.** Proof

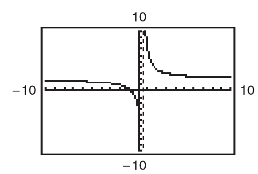
Section 3.5 (page 205)

1. f **2.** c **3.** d **4.** a **5.** b **6.** e

7.

x	10^0	10^1	10^2	10^3
$f(x)$	7	2.2632	2.0251	2.0025

x	10^4	10^5	10^6
$f(x)$	2.0003	2.0000	2.0000

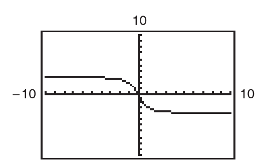


$$\lim_{x\to\infty} \frac{4x+3}{2x-1} = 2$$

9.

x	10^0	10^1	10^2	10^3
$f(x)$	-2	-2.9814	-2.9998	-3.0000

x	10^4	10^5	10^6
$f(x)$	-3.0000	-3.0000	-3.0000

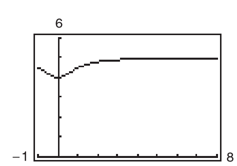


$$\lim_{x\to\infty} \frac{-6x}{\sqrt{4x^2+5}} = -3$$

11.

x	10^0	10^1	10^2	10^3
$f(x)$	4.5000	4.9901	4.9999	5.0000

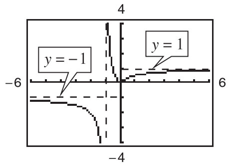
x	10^4	10^5	10^6
$f(x)$	5.0000	5.0000	5.0000



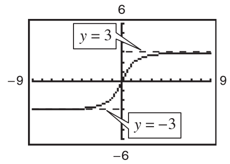
$$\lim_{x\to\infty} \left(5 - \frac{1}{x^2+1}\right) = 5$$

13. (a) ∞ (b) 5 (c) 0 **15.** (a) 0 (b) 1 (c) ∞
17. (a) 0 (b) $-\frac{2}{3}$ (c) $-\infty$ **19.** 4 **21.** $\frac{2}{3}$ **23.** 0
25. $-\infty$ **27.** -1 **29.** -2 **31.** $\frac{1}{2}$ **33.** ∞
35. 0 **37.** 0

39.



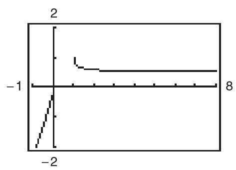
41.



43. 1 **45.** 0 **47.** $\frac{1}{6}$

49.

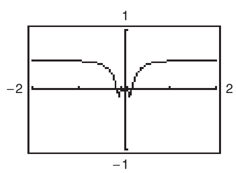
x	10^0	10^1	10^2	10^3	10^4	10^5	10^6
$f(x)$	1.000	0.513	0.501	0.500	0.500	0.500	0.500



$$\lim_{x\to\infty}\left[x - \sqrt{x(x-1)}\right] = \frac{1}{2}$$

51.

x	10^0	10^1	10^2	10^3	10^4	10^5	10^6
$f(x)$	0.479	0.500	0.500	0.500	0.500	0.500	0.500

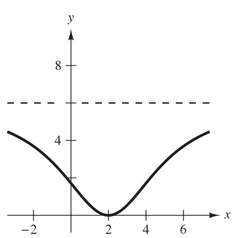


The graph has a hole at $x = 0$.

$$\lim_{x\to\infty} x \sin\frac{1}{2x} = \frac{1}{2}$$

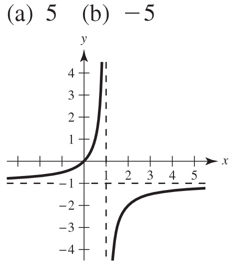
53. As x becomes large, $f(x)$ approaches 4.

55. Answers will vary. Example: let $f(x) = \dfrac{-6}{0.1(x-2)^2 + 1} + 6$.



57. (a) 5 (b) -5

59.



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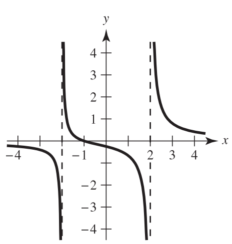

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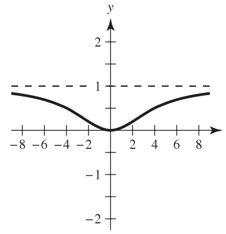

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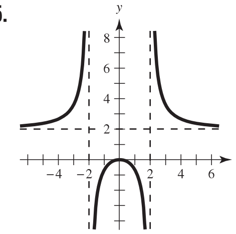

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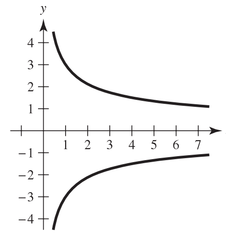

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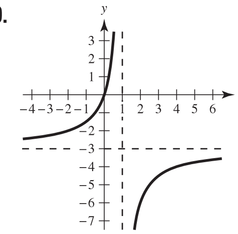

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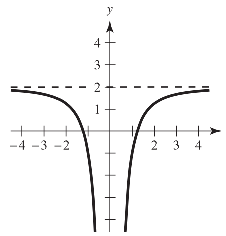

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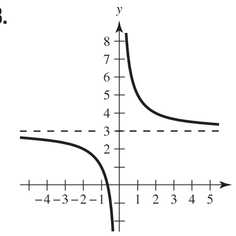

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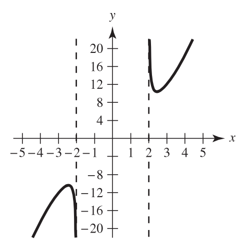

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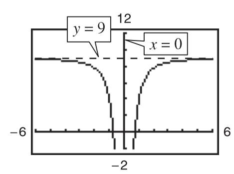

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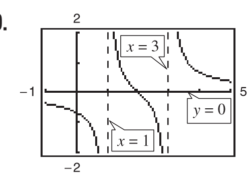

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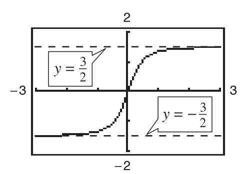
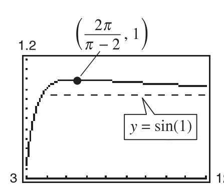
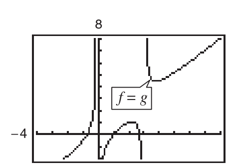


83.

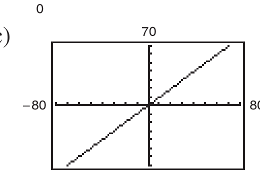


85. (a)



(b) Proof

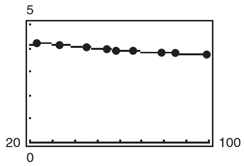
(c)



The slant asymptote $y = x$

87. 100% **89.** $\lim_{t\to\infty} N(t) = +\infty$; $\lim_{t\to\infty} E(t) = c$

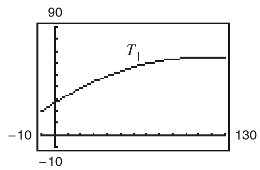
91. (a)



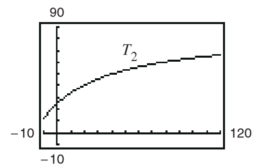
(b) Yes. $\lim_{t\to\infty} y = 3.351$

93. (a) $T_1 = -0.003t^2 + 0.68t + 26.6$

(b)



(c)

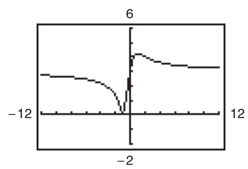


(d) $T_1(0) \approx 26.6°$, $T_2(0) \approx 25.0°$ (e) 86

(f) The limiting temperature is 86°.
No. T_1 has no horizontal asymptote.

95. (a) $d(m) = \dfrac{|3m + 3|}{\sqrt{m^2 + 1}}$

(b)



(c) $\lim_{m\to\infty} d(m) = 3$

$\lim_{m\to-\infty} d(m) = 3$

As m approaches $\pm\infty$, the distance approaches 3.

97. (a) $\lim_{x\to\infty} f(x) = 2$ (b) $x_1 = \sqrt{\dfrac{4 - 2\varepsilon}{\varepsilon}}, x_2 = -\sqrt{\dfrac{4 - 2\varepsilon}{\varepsilon}}$

(c) $\sqrt{\dfrac{4 - 2\varepsilon}{\varepsilon}}$ (d) $-\sqrt{\dfrac{4 - 2\varepsilon}{\varepsilon}}$

99. (a) Answers will vary. $M = \dfrac{5\sqrt{33}}{11}$

(b) Answers will vary. $M = \dfrac{29\sqrt{177}}{59}$

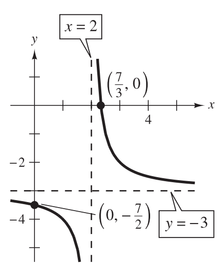
101–105. Proofs

107. False. Let $f(x) = \dfrac{2x}{\sqrt{x^2 + 2}}$. $f'(x) > 0$ for all real numbers.

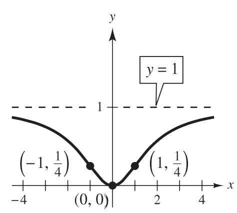
Section 3.6 (page 215)

1. d **2.** c **3.** a **4.** b

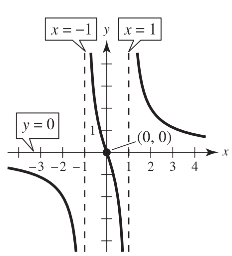
5.



7.



9.



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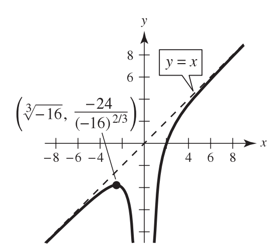

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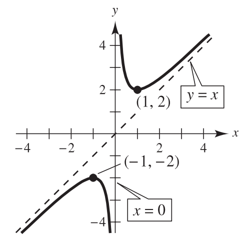

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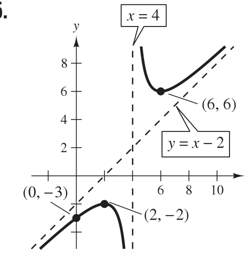
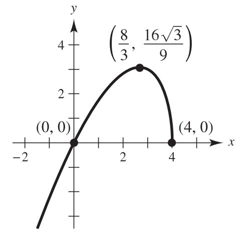


17.



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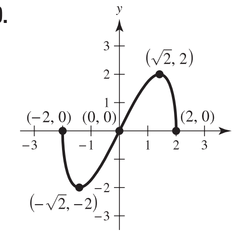
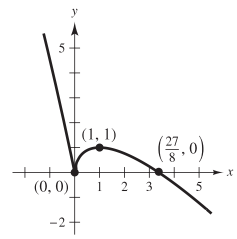
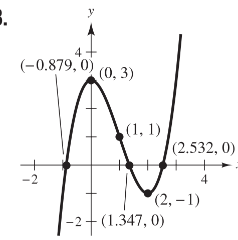


21.



23.



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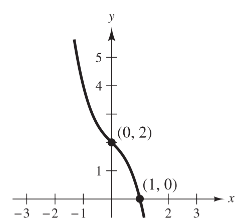
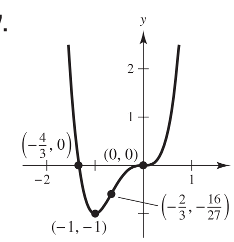
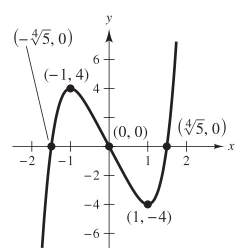


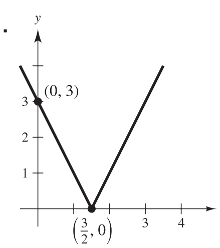
27.



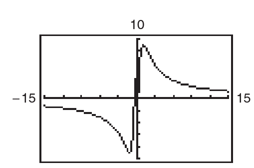
29.



31.

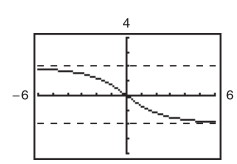


33.

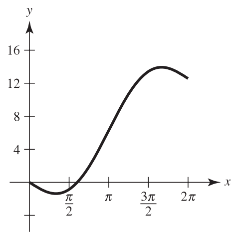


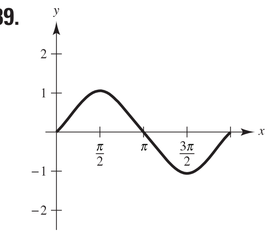
Minimum: $(-1.10, -9.05)$
Maximum: $(1.10, 9.05)$
Points of inflection:
$(-1.84, -7.86), (1.84, 7.86)$
Vertical asymptote: $x = 0$
Horizontal asymptote: $y = 0$

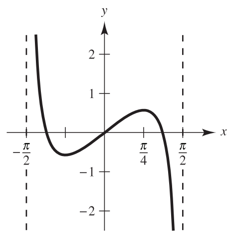
35.

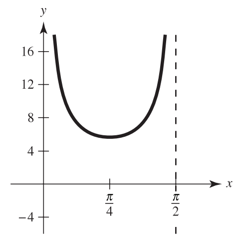


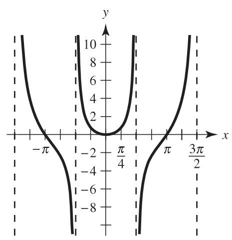
Point of inflection: $(0, 0)$
Horizontal asymptotes: $y = \pm 2$

37. 

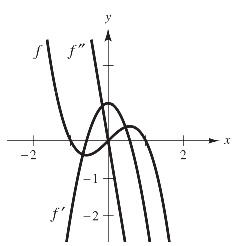
39. 

41. 

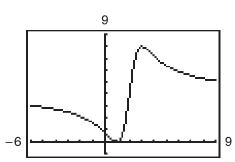
43. 

45. 

47. f is decreasing on $(2, 8)$ and therefore $f(3) > f(5)$.

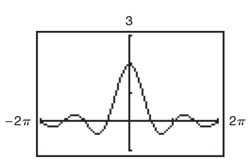
49. 

The zeros of f' correspond to the points where the graph of f has horizontal tangents. The zero of f'' corresponds to the point where the graph of f' has a horizontal tangent.

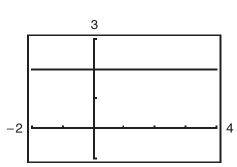
51. 

The graph crosses the horizontal asymptote $y = 4$.

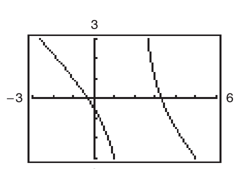
The graph of a function f does not cross its vertical asymptote $x = c$ because $f(c)$ does not exist.

53. 

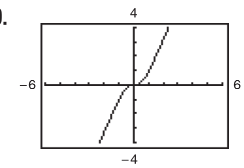
The graph has a hole at $x = 0$.
The graph crosses the horizontal asymptote $y = 0$.
The graph of a function f does not cross its vertical asymptote $x = c$ because $f(c)$ does not exist.

55. 

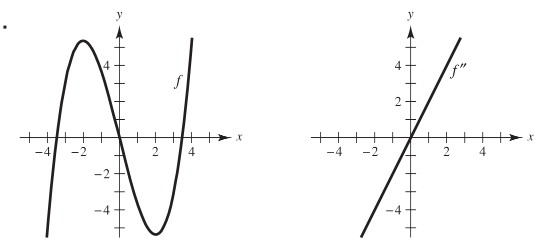
The graph has a hole at $x = 3$. The rational function is not reduced to lowest terms.

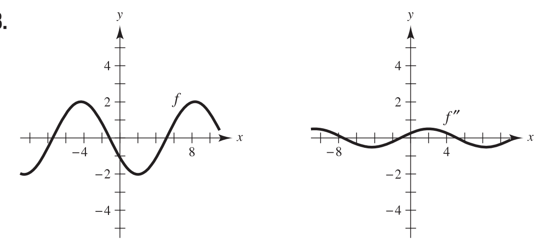
57. 

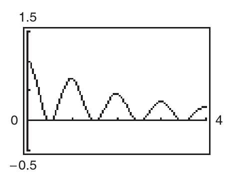
The graph appears to approach the line $y = -x + 1$, which is the slant asymptote.

59. 

The graph appears to approach the line $y = 2x$, which is the slant asymptote.

61. 

63. 

65. (a) 

The graph has holes at $x = 0$ and at $x = 4$.
Visually approximated critical numbers: $\frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, 3, \frac{7}{2}$

(b) $f'(x) = \dfrac{-x\cos^2(\pi x)}{(x^2+1)^{3/2}} - \dfrac{2\pi\sin(\pi x)\cos(\pi x)}{\sqrt{x^2+1}}$

Approximate critical numbers: $\frac{1}{2}, 0.97, \frac{3}{2}, 1.98, \frac{5}{2}, 2.98, \frac{7}{2}$
The critical numbers where maxima occur appear to be integers in part (a), but by approximating them using f' you can see that they are not integers.

67. Answers will vary. Example: $y = 1/(x - 3)$

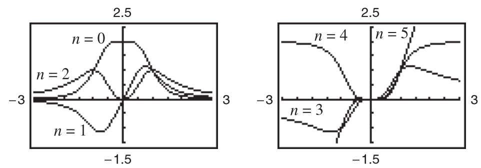
69. Answers will vary. Example: $y = (3x^2 - 7x - 5)/(x - 3)$

71. (a) $f'(x) = 0$ for $x = \pm 2$; $f'(x) > 0$ for $(-\infty, -2), (2, \infty)$
$f'(x) < 0$ for $(-2, 2)$
(b) $f''(x) = 0$ for $x = 0$; $f''(x) > 0$ for $(0, \infty)$
$f''(x) < 0$ for $(-\infty, 0)$
(c) $(0, \infty)$
(d) f' is minimum for $x = 0$.
f is decreasing at the greatest rate at $x = 0$.

73. Answers will vary. Sample answer: The graph has a vertical asymptote at $x = b$. If a and b are both positive, or both negative, then the graph of f approaches ∞ as x approaches b, and the graph has a minimum at $x = -b$. If a and b have opposite signs, then the graph of f approaches $-\infty$ as x approaches b, and the graph has a maximum at $x = -b$.

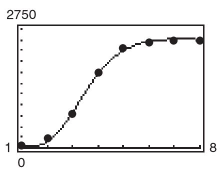
75. (a) If n is even, f is symmetric with respect to the y-axis.
If n is odd, f is symmetric with respect to the origin.
(b) $n = 0, 1, 2, 3$ (c) $n = 4$
(d) When $n = 5$, the slant asymptote is $y = 2x$.

(e)



n	0	1	2	3	4	5
M	1	2	3	2	1	0
N	2	3	4	5	2	3

77. (a)



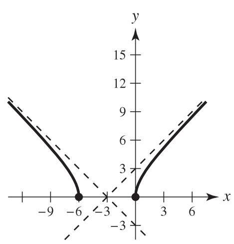
(b) 2434

(c) The number of bacteria reaches its maximum early on the seventh day.

(d) The rate of increase in the number of bacteria is greatest in the early part of the third day.

(e) 13,250/7

79. $y = x + 3$, $y = -x - 3$



Section 3.7 (page 223)

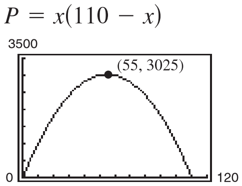
1. (a) and (b)

First Number x	Second Number	Product P
10	$110 - 10$	$10(110 - 10) = 1000$
20	$110 - 20$	$20(110 - 20) = 1800$
30	$110 - 30$	$30(110 - 30) = 2400$
40	$110 - 40$	$40(110 - 40) = 2800$
50	$110 - 50$	$50(110 - 50) = 3000$
60	$110 - 60$	$60(110 - 60) = 3000$
70	$110 - 70$	$70(110 - 70) = 2800$
80	$110 - 80$	$80(110 - 80) = 2400$
90	$110 - 90$	$90(110 - 90) = 1800$
100	$110 - 100$	$100(110 - 100) = 1000$

The maximum is attained near $x = 50$ and 60.

(c) $P = x(110 - x)$

(d)



(e) 55 and 55

3. $S/2$ and $S/2$ **5.** 21 and 7 **7.** 54 and 27

9. $l = w = 20$ m **11.** $l = w = 4\sqrt{2}$ ft **13.** $(1, 1)$

15. $\left(\frac{7}{2}, \sqrt{\frac{7}{2}}\right)$

17. Dimensions of page: $\left(2 + \sqrt{30}\right)$ in. $\times \left(2 + \sqrt{30}\right)$ in.

19. $x = Q_0/2$ **21.** 700×350 m

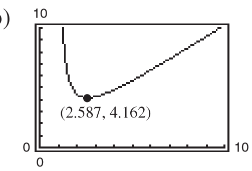
23. (a) Proof (b) $V_1 = 99$ in.^{3}, $V_2 = 125$ in.^{3}, $V_3 = 117$ in.^{3}

(c) $5 \times 5 \times 5$ in.

25. Rectangular portion: $16/(\pi + 4) \times 32/(\pi + 4)$ ft

27. (a) $L = \sqrt{x^2 + 4 + \dfrac{8}{x - 1} + \dfrac{4}{(x - 1)^2}}, \quad x > 1$

(b)

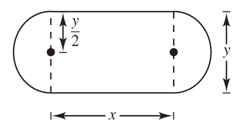


Minimum when $x \approx 2.587$

(c) $(0, 0)$, $(2, 0)$, $(0, 4)$

29. Width: $5\sqrt{2}/2$; Length: $5\sqrt{2}$

31. (a)



(b)

Length x	Width y	Area xy
10	$2/\pi(100 - 10)$	$(10)(2/\pi)(100 - 10) \approx 573$
20	$2/\pi(100 - 20)$	$(20)(2/\pi)(100 - 20) \approx 1019$
30	$2/\pi(100 - 30)$	$(30)(2/\pi)(100 - 30) \approx 1337$
40	$2/\pi(100 - 40)$	$(40)(2/\pi)(100 - 40) \approx 1528$
50	$2/\pi(100 - 50)$	$(50)(2/\pi)(100 - 50) \approx 1592$
60	$2/\pi(100 - 60)$	$(60)(2/\pi)(100 - 60) \approx 1528$

The maximum area of the rectangle is approximately 1592 m^2.

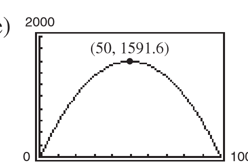
(c) $A = 2/\pi(100x - x^2), \quad 0 < x < 100$

(d) $\dfrac{dA}{dx} = \dfrac{2}{\pi}(100 - 2x)$

$= 0$ when $x = 50$

The maximum value is approximately 1592 when $x = 50$.

(e)



33. $18 \times 18 \times 36$ in. **35.** $32\pi r^3/81$

37. No. The volume changes because the shape of the container changes when squeezed.

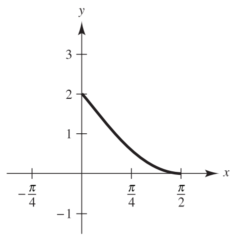
39. $r = \sqrt[3]{21/(2\pi)} \approx 1.50$ ($h = 0$, so the solid is a sphere.)

41. Side of square: $\dfrac{10\sqrt{3}}{9 + 4\sqrt{3}}$; Side of triangle: $\dfrac{30}{9 + 4\sqrt{3}}$

43. $w = \left(20\sqrt{3}\right)/3$ in., $h = \left(20\sqrt{6}\right)/3$ in. **45.** $\theta = \pi/4$

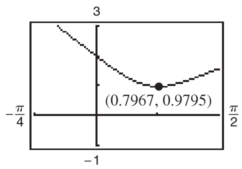
47. $h = \sqrt{2}$ ft **49.** One mile from the nearest point on the coast

51. Proof

53. 

(a) Origin to y-intercept: 2
Origin to x-intercept: $\pi/2$

(b) $d = \sqrt{x^2 + (2 - 2\sin x)^2}$



(c) Minimum distance is 0.9795 when $x \approx 0.7967$.

55. $F = kW/\sqrt{k^2 + 1}$; $\theta = \arctan k$

57. (a)

Base 1	Base 2	Altitude	Area
8	$8 + 16\cos 10°$	$8\sin 10°$	≈ 22.1
8	$8 + 16\cos 20°$	$8\sin 20°$	≈ 42.5
8	$8 + 16\cos 30°$	$8\sin 30°$	≈ 59.7
8	$8 + 16\cos 40°$	$8\sin 40°$	≈ 72.7
8	$8 + 16\cos 50°$	$8\sin 50°$	≈ 80.5
8	$8 + 16\cos 60°$	$8\sin 60°$	≈ 83.1

(b)

Base 1	Base 2	Altitude	Area
8	$8 + 16\cos 70°$	$8\sin 70°$	≈ 80.7
8	$8 + 16\cos 80°$	$8\sin 80°$	≈ 74.0
8	$8 + 16\cos 90°$	$8\sin 90°$	≈ 64.0

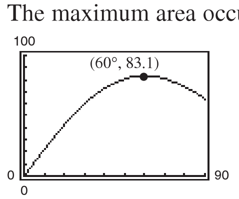
The maximum cross-sectional area is approximately 83.1 ft^2.

(c) $A = 64(1 + \cos\theta)\sin\theta$, $0° < \theta < 90°$

(d) $\dfrac{dA}{d\theta} = 64(2\cos\theta - 1)(\cos\theta + 1)$

$= 0$ when $\theta = 60°, 180°, 300°$

The maximum area occurs when $\theta = 60°$.

(e) 

59. 4045 units **61.** $y = \frac{64}{141}x$; $S_1 \approx 6.1$ mi

63. $y = \frac{3}{10}x$; $S_3 \approx 4.50$ mi **65.** Putnam Problem A1, 1986

Section 3.8 (page 233)

In the answers for Exercises 1 and 3, the values in the tables have been rounded for convenience. Because a calculator or a computer program calculates internally using more digits than they display, you may produce slightly different values than those shown in the tables.

1.

n	x_n	$f(x_n)$	$f'(x_n)$	$\dfrac{f(x_n)}{f'(x_n)}$	$x_n - \dfrac{f(x_n)}{f'(x_n)}$
1	2.2000	−0.1600	4.4000	−0.0364	2.2364
2	2.2364	0.0015	4.4728	0.0003	2.2361

3.

n	x_n	$f(x_n)$	$f'(x_n)$	$\dfrac{f(x_n)}{f'(x_n)}$	$x_n - \dfrac{f(x_n)}{f'(x_n)}$
1	1.6	−0.0292	−0.9996	0.0292	1.5708
2	1.5708	0	−1	0	1.5708

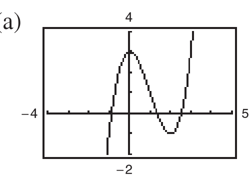
5. −1.587 **7.** 0.682 **9.** 1.250, 5.000

11. 0.900, 1.100, 1.900 **13.** 1.935 **15.** 0.569

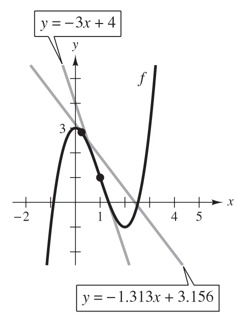
17. 4.493 **19.** (a) Proof (b) $\sqrt{5} \approx 2.236$; $\sqrt{7} \approx 2.646$

21. $f'(x_1) = 0$ **23.** $2 = x_1 = x_3 = \ldots$; $1 = x_2 = x_4 = \ldots$

25. 0.74 **27.** Proof

29. (a) 

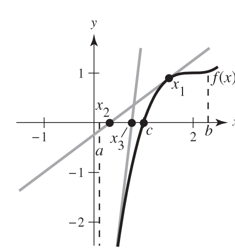
(b) 1.347 (c) 2.532

(d) 

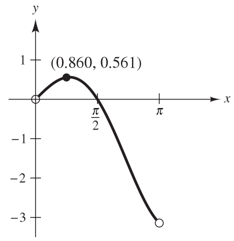
x-intercept of $y = -3x + 4$ is $\frac{4}{3}$.
x-intercept of $y = -1.313x + 3.156$ is approximately 2.404.

(e) If the initial estimate $x = x_1$ is not sufficiently close to the desired zero of a function, the x-intercept of the corresponding tangent line to the function may approximate a second zero of the function.

31. Answers will vary. Sample answer: If f is a function continuous on $[a, b]$ and differentiable on (a, b), where $c \in [a, b]$ and $f(c) = 0$, Newton's Method uses tangent lines to approximate c. First, estimate an initial x_1 close to c. (See graph.) Then determine x_2 using $x_2 = x_1 - f(x_1)/f'(x_1)$. Calculate a third estimate x_3 using $x_3 = x_2 - f(x_2)/f'(x_2)$. Continue this process until $|x_n - x_{n+1}|$ is within the desired accuracy and let x_{n+1} be the final approximation of c.



33. 0.860



35. (1.939, 0.240)

37. $x \approx 1.563$ mi **39.** 15.1, 26.8 **41.** False: let $f(x) = \dfrac{x^2 - 1}{x - 1}$.
43. True **45.** 0.217

Section 3.9 (page 240)

1. $T(x) = 4x - 4$

x	1.9	1.99	2	2.01	2.1
$f(x)$	3.610	3.960	4	4.040	4.410
$T(x)$	3.600	3.960	4	4.040	4.400

3. $T(x) = 80x - 128$

x	1.9	1.99	2	2.01	2.1
$f(x)$	24.761	31.208	32	32.808	40.841
$T(x)$	24.000	31.200	32	32.800	40.000

5. $T(x) = (\cos 2)(x - 2) + \sin 2$

x	1.9	1.99	2	2.01	2.1
$f(x)$	0.946	0.913	0.909	0.905	0.863
$T(x)$	0.951	0.913	0.909	0.905	0.868

7. $\Delta y = 0.331$; $dy = 0.3$ **9.** $\Delta y = -0.039$; $dy = -0.040$

11. $6x\,dx$ **13.** $-\dfrac{3}{(2x - 1)^2}\,dx$ **15.** $\dfrac{1 - 2x^2}{\sqrt{1 - x^2}}\,dx$

17. $(3 - \sin 2x)\,dx$ **19.** $-\pi \sin\left(\dfrac{6\pi x - 1}{2}\right) dx$

21. (a) 0.9 (b) 1.04 **23.** (a) 1.05 (b) 0.98

25. (a) 8.035 (b) 7.95 **27.** $\pm\frac{5}{8}$ in.^{2} **29.** $\pm 8\pi$ in.^{2}

31. (a) $\frac{5}{6}$% (b) 1.25%

33. (a) $\pm 5.12\pi$ in.^{3} (b) $\pm 1.28\pi$ in.^{2} (c) 0.75%, 0.5%

35. 80π cm^3 **37.** (a) $\frac{1}{4}$% (b) 216 sec = 3.6 min

39. (a) 0.87% (b) 2.16% **41.** 6407 ft

43. $f(x) = \sqrt{x}$, $dy = \dfrac{1}{2\sqrt{x}}\,dx$

$f(99.4) \approx \sqrt{100} + \dfrac{1}{2\sqrt{100}}(-0.6) = 9.97$

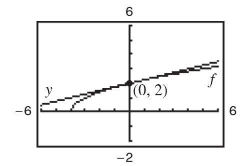
Calculator: 9.97

45. $f(x) = \sqrt[4]{x}$, $dy = \dfrac{1}{4x^{3/4}}\,dx$

$f(624) \approx \sqrt[4]{625} + \dfrac{1}{4(625)^{3/4}}(-1) = 4.998$

Calculator: 4.998

47. $y - f(0) = f'(0)(x - 0)$
$y - 2 = \frac{1}{4}x$
$y = 2 + x/4$



49. The value of dy becomes closer to the value of Δy as Δx decreases.

51. (a) $f(x) = \sqrt{x}$; $dy = \dfrac{1}{2\sqrt{x}}\,dx$

$f(4.02) \approx \sqrt{4} + \dfrac{1}{2\sqrt{4}}(0.02) = 2 + \dfrac{1}{4}(0.02)$

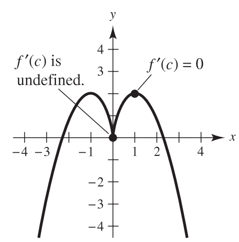
(b) $f(x) = \tan x$; $dy = \sec^2 x\,dx$

$f(0.05) \approx \tan 0 + \sec^2(0)\,(0.05) = 0 + 1(0.05)$

53. True **55.** True

Review Exercises for Chapter 3 (page 242)

1. Let f be defined at c. If $f'(c) = 0$ or if f' is undefined at c, then c is a critical number of f.

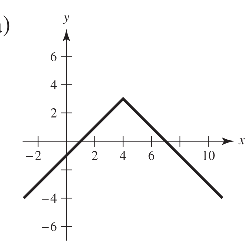


3. Maximum: (0, 0)
Minimum: $\left(-\frac{5}{2}, -\frac{25}{4}\right)$

5. Maximum: $(2\pi, 17.57)$
Minimum: (2.73, 0.88)

7. $f(0) \neq f(4)$ **9.** Not continuous on $[-2, 2]$

11. (a)



(b) f is not differentiable at $x = 4$.

13. $f'\left(\dfrac{2744}{729}\right) = \dfrac{3}{7}$ **15.** f is not differentiable at $x = 5$.

17. $f'(0) = 1$ **19.** $c = \dfrac{x_1 + x_2}{2}$

21. Critical number: $x = -\frac{3}{2}$
Increasing on $\left(-\frac{3}{2}, \infty\right)$; Decreasing on $\left(-\infty, -\frac{3}{2}\right)$

23. Critical numbers: $x = 1, \frac{7}{3}$
Increasing on $(-\infty, 1)$, $\left(\frac{7}{3}, \infty\right)$; Decreasing on $\left(1, \frac{7}{3}\right)$

25. Critical number: $x = 1$
Increasing on $(1, \infty)$; Decreasing on $(0, 1)$

27. Relative maximum: $\left(-\dfrac{\sqrt{15}}{6}, \dfrac{5\sqrt{15}}{9}\right)$

Relative minimum: $\left(\dfrac{\sqrt{15}}{6}, -\dfrac{5\sqrt{15}}{9}\right)$

29. Relative minimum: $(2, -12)$

31. (a) $y = \frac{1}{4}$ in.; $v = 4$ in./sec (b) Proof
(c) Period: $\pi/6$; Frequency: $6/\pi$

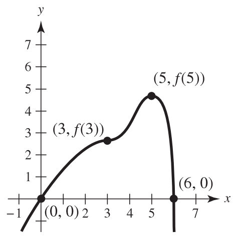
33. $(3, -54)$; Concave upward: $(3, \infty)$;
Concave downward: $(-\infty, 3)$

35. $(\pi/2, \pi/2), (3\pi/2, 3\pi/2)$; Concave upward: $(\pi/2, 3\pi/2)$
Concave downward: $(0, \pi/2), (3\pi/2, 2\pi)$

37. Relative minimum: $(-9, 0)$

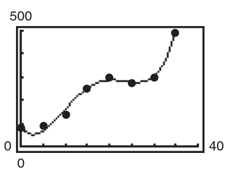
39. Relative maxima: $\left(\sqrt{2}/2, 1/2\right), \left(-\sqrt{2}/2, 1/2\right)$
Relative minimum: $(0, 0)$

41.



43. Increasing and concave down

45. (a) $D = 0.00430t^4 - 0.2856t^3 + 5.833t^2 - 26.85t + 87.1$
(b)



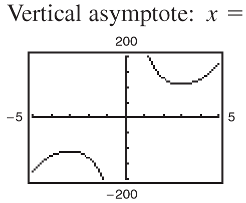
(c) Maximum in 2005; Minimum in 1972 (d) 2005

47. 8 **49.** $\frac{2}{3}$ **51.** $-\infty$ **53.** 0 **55.** 6

57. Vertical asymptote: $x = 0$; Horizontal asymptote: $y = -2$

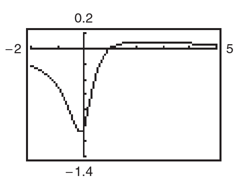
59. Vertical asymptote: $x = 4$; Horizontal asymptote: $y = 2$

61.



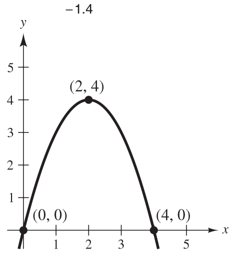
Vertical asymptote: $x = 0$
Relative minimum: $(3, 108)$
Relative maximum: $(-3, -108)$

63.

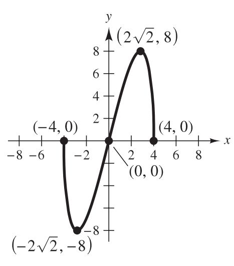


Horizontal asymptote: $y = 0$
Relative minimum:
$(-0.155, -1.077)$
Relative maximum:
$(2.155, 0.077)$

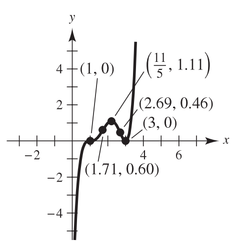
65.



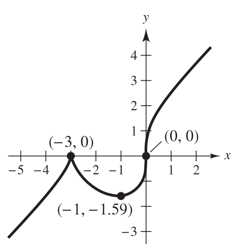
67.



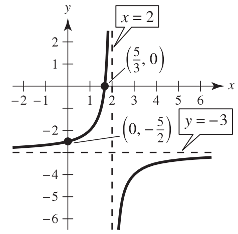
69.



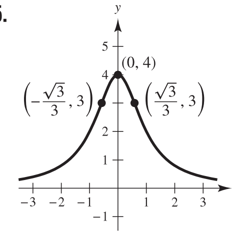
71.



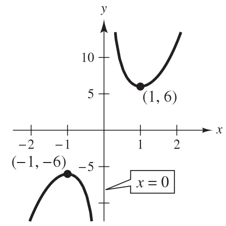
73.



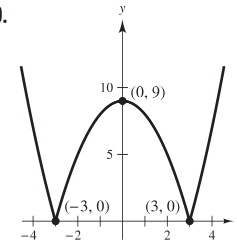
75.



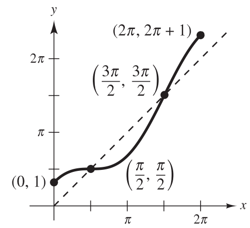
77.



79.



81.



83. (a) and (b) Maximum: $(1, 3)$
Minimum: $(1, 1)$

85. $t \approx 4.92 \approx$ 4:55 P.M.; $d \approx 64$ km

87. $(0, 0), (5, 0), (0, 10)$ **89.** Proof **91.** 14.05 ft

93. $3(3^{2/3} + 2^{2/3})^{3/2} \approx 21.07$ ft **95.** $v \approx 54.77$ mi/h

97. $-1.532, -0.347, 1.879$ **99.** $-1.164, 1.453$

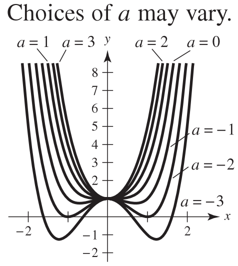
101. $dy = (1 - \cos x + x \sin x)\, dx$

103. $dS = \pm 1.8\pi \text{ cm}^2, \dfrac{dS}{S} \times 100 \approx \pm 0.56\%$

$dV = \pm 8.1\pi \text{ cm}^3, \dfrac{dV}{V} \times 100 \approx \pm 0.83\%$

P.S. Problem Solving (page 245)

1. Choices of a may vary.



(a) One relative minimum at $(0, 1)$ for $a \geq 0$
(b) One relative maximum at $(0, 1)$ for $a < 0$
(c) Two relative minima for $a < 0$ when $x = \pm\sqrt{-a/2}$
(d) If $a < 0$, there are three critical points; if $a \geq 0$, there is only one critical point.

3. All c where c is a real number **5–7.** Proofs

9. About 9.19 ft

11. Minimum: $\left(\sqrt{2} - 1\right)d$; There is no maximum.

13. (a)–(c) Proofs

15. (a)

x	0	0.5	1	2
$\sqrt{1 + x}$	1	1.2247	1.4142	1.7321
$\frac{1}{2}x + 1$	1	1.25	1.5	2

(b) Proof

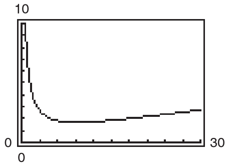
17. (a)

v	20	40	60	80	100
s	5.56	11.11	16.67	22.22	27.78
d	5.1	13.7	27.2	44.2	66.4

$d(s) = 0.071s^2 + 0.389s + 0.727$

(b) The distance between the back of the first vehicle and the front of the second vehicle is $d(s)$, the safe stopping distance. The first vehicle passes the given point in $5.5/s$ seconds, and the second vehicle takes $d(s)/s$ additional seconds. So, $T = d(s)/s + 5.5/s$.

(c)

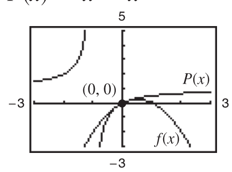


$s \approx 9.365$ m/sec

(d) $s \approx 9.365$ m/sec; 1.719 sec; 33.714 km/h (e) 10.597 m

19. (a) $P(x) = x - x^2$

(b)



Chapter 4

Section 4.1 (page 255)

1–3. Proofs **5.** $y = 3t^3 + C$ **7.** $y = \frac{2}{5}x^{5/2} + C$

	Original Integral	*Rewrite*	*Integrate*	*Simplify*
9.	$\int \sqrt[3]{x}\,dx$	$\int x^{1/3}\,dx$	$\frac{x^{4/3}}{4/3} + C$	$\frac{3}{4}x^{4/3} + C$
11.	$\int \frac{1}{x\sqrt{x}}\,dx$	$\int x^{-3/2}\,dx$	$\frac{x^{-1/2}}{-1/2} + C$	$-\frac{2}{\sqrt{x}} + C$
13.	$\int \frac{1}{2x^3}\,dx$	$\frac{1}{2}\int x^{-3}\,dx$	$\frac{1}{2}\left(\frac{x^{-2}}{-2}\right) + C$	$-\frac{1}{4x^2} + C$

15. $\frac{1}{2}x^2 + 7x + C$ **17.** $x^2 - x^3 + C$ **19.** $\frac{1}{6}x^6 + x + C$

21. $\frac{2}{5}x^{5/2} + x^2 + x + C$ **23.** $\frac{3}{5}x^{5/3} + C$ **25.** $-1/(4x^4) + C$

27. $\frac{2}{3}x^{3/2} + 12x^{1/2} + C = \frac{2}{3}x^{1/2}(x + 18) + C$

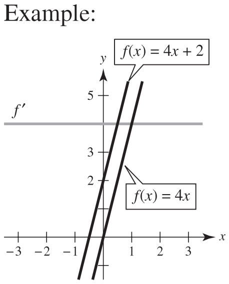
29. $x^3 + \frac{1}{2}x^2 - 2x + C$

31. $\frac{2}{7}y^{7/2} + C$ **33.** $x + C$ **35.** $5 \sin x - 4 \cos x + C$

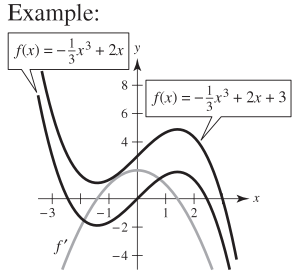
37. $t + \csc t + C$ **39.** $\tan \theta + \cos \theta + C$ **41.** $\tan y + C$

43. $-\csc x + C$

45. Answers will vary.
Example:

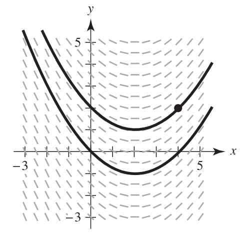


47. Answers will vary.
Example:

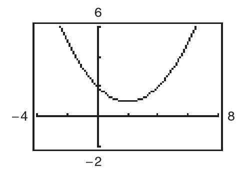


49. $y = x^2 - x + 1$

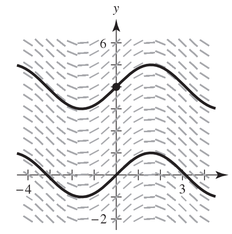
51. (a) Answers will vary.
Example:



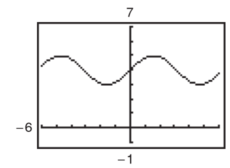
(b) $y = \frac{1}{4}x^2 - x + 2$



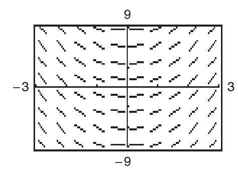
53. (a) Answers will vary.
Example:



(b) $y = \sin x + 4$

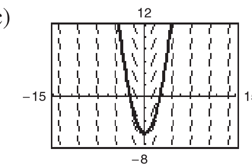


55. (a)



(b) $y = x^2 - 6$

(c)



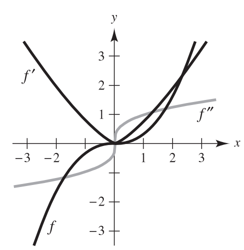
57. $f(x) = 3x^2 + 8$ **59.** $h(t) = 2t^4 + 5t - 11$

61. $f(x) = x^2 + x + 4$ **63.** $f(x) = -4\sqrt{x} + 3x$

65. (a) $h(t) = \frac{3}{4}t^2 + 5t + 12$ (b) 69 cm

67. When you evaluate the integral $\int f(x)\,dx$, you are finding a function $F(x)$ that is an antiderivative of $f(x)$. So, there is no difference.

69.



71. 62.25 ft **73.** $v_0 \approx 187.617$ ft/sec

75. $v(t) = -9.8t + C_1 = -9.8t + v_0$
$f(t) = -4.9t^2 + v_0t + C_2 = -4.9t^2 + v_0t + s_0$

77. 7.1 m **79.** 320 m; -32 m/sec

81. (a) $v(t) = 3t^2 - 12t + 9$; $a(t) = 6t - 12$
(b) $(0, 1), (3, 5)$ (c) -3

83. $a(t) = -1/(2t^{3/2})$; $x(t) = 2\sqrt{t} + 2$

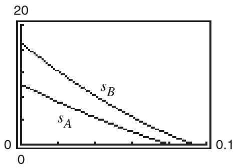
85. (a) 1.18 m/sec² (b) 190 m

87. (a) 300 ft (b) 60 ft/sec $\approx$ 41 mi/h

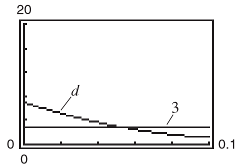
89. (a) Airplane A: $s_A = \frac{625}{2}t^2 - 150t + 10$

Airplane B: $s_B = \frac{49{,}275}{68}t^2 - 250t + 17$

(b)



(c) $d = \dfrac{28{,}025}{68}t^2 - 100t + 7$

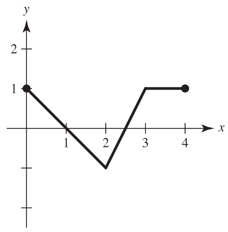


Yes, $d < 3$ for $t > 0.0505$ h

91. True **93.** True

95. False. f has an infinite number of antiderivatives, each differing by a constant.

97.



99. Proof

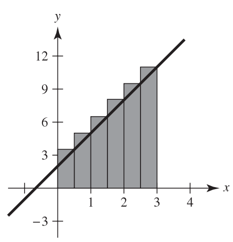
Section 4.2 (page 267)

1. 75 **3.** $\dfrac{158}{85}$ **5.** $4c$ **7.** $\displaystyle\sum_{i=1}^{11} \frac{1}{5i}$ **9.** $\displaystyle\sum_{j=1}^{6} \left[7\left(\frac{j}{6}\right) + 5\right]$

11. $\displaystyle\frac{2}{n}\sum_{i=1}^{n} \left[\left(\frac{2i}{n}\right)^3 - \left(\frac{2i}{n}\right)\right]$ **13.** $\displaystyle\frac{3}{n}\sum_{i=1}^{n} \left[2\left(1 + \frac{3i}{n}\right)^2\right]$ **15.** 84

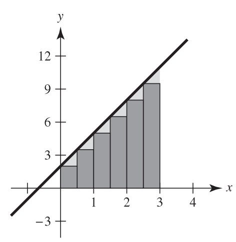
17. 1200 **19.** 2470 **21.** 12,040 **23.** 2930

25. (a)



Area $\approx$ 21.75

(b)



Area $\approx$ 17.25

27. $13 <$ (Area of region) < 15

29. $55 <$ (Area of region) < 74.5

31. $0.7908 <$ (Area of region) < 1.1835

33. The area of the shaded region falls between 12.5 square units and 16.5 square units.

35. The area of the shaded region falls between 7 square units and 11 square units.

37. $\frac{81}{4}$ **39.** 9 **41.** $A \approx S \approx 0.768$; $A \approx s \approx 0.518$ **43.** $A \approx S \approx 0.746$; $A \approx s \approx 0.646$

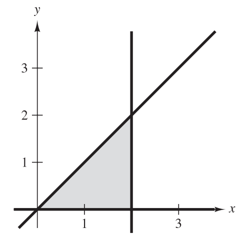
45. $(n + 2)/n$
$n = 10$: $S = 1.2$
$n = 100$: $S = 1.02$
$n = 1000$: $S = 1.002$
$n = 10{,}000$: $S = 1.0002$

47. $[2(n + 1)(n - 1)]/n^2$
$n = 10$: $S = 1.98$
$n = 100$: $S = 1.9998$
$n = 1000$: $S = 1.999998$
$n = 10{,}000$: $S = 1.99999998$

49. $\displaystyle\lim_{n\to\infty}\left[\frac{12(n + 1)}{n}\right] = 12$ **51.** $\displaystyle\lim_{n\to\infty}\frac{1}{6}\left(\frac{2n^3 - 3n^2 + n}{n^3}\right) = \frac{1}{3}$

53. $\displaystyle\lim_{n\to\infty}[(3n + 1)/n] = 3$

55. (a)



(b) $\Delta x = (2 - 0)/n = 2/n$

(c) $\displaystyle s(n) = \sum_{i=1}^{n} f(x_{i-1})\,\Delta x = \sum_{i=1}^{n} [(i - 1)(2/n)](2/n)$

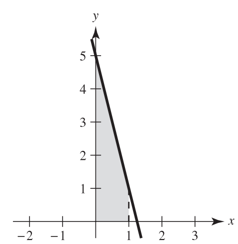
(d) $\displaystyle S(n) = \sum_{i=1}^{n} f(x_i)\,\Delta x = \sum_{i=1}^{n} [i(2/n)](2/n)$

(e)

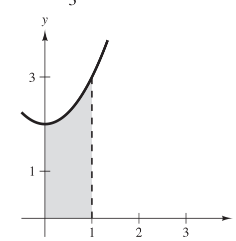
n	5	10	50	100
$s(n)$	1.6	1.8	1.96	1.98
$S(n)$	2.4	2.2	2.04	2.02

(f) $\displaystyle\lim_{n\to\infty}\sum_{i=1}^{n}[(i - 1)(2/n)](2/n) = 2$; $\displaystyle\lim_{n\to\infty}\sum_{i=1}^{n}[i(2/n)](2/n) = 2$

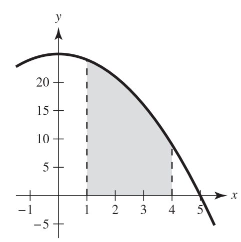
57. $A = 3$



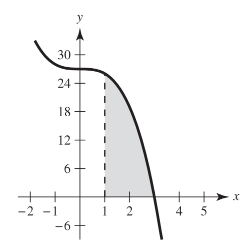
59. $A = \frac{7}{3}$



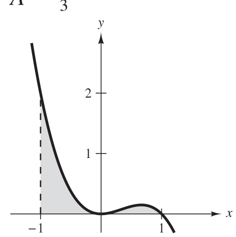
61. $A = 54$



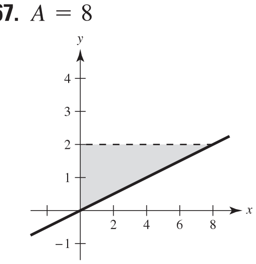
63. $A = 34$



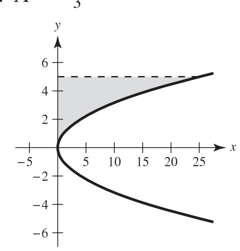
65. $A = \frac{2}{3}$



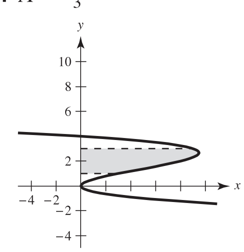
67. $A = 8$



69. $A = \frac{125}{3}$



71. $A = \frac{44}{3}$



73. $\frac{69}{8}$ **75.** 0.345

77.

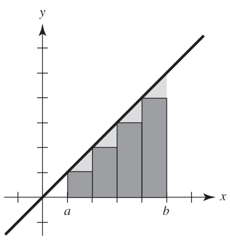
n	4	8	12	16	20
Approximate Area	5.3838	5.3523	5.3439	5.3403	5.3384

79.

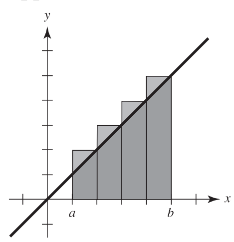
n	4	8	12	16	20
Approximate Area	2.2223	2.2387	2.2418	2.2430	2.2435

81. b

83. You can use the line $y = x$ bounded by $x = a$ and $x = b$. The sum of the areas of the inscribed rectangles in the figure below is the lower sum.

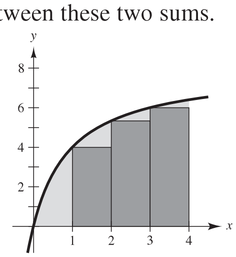


The sum of the areas of the circumscribed rectangles in the figure below is the upper sum.



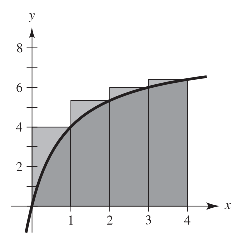
The rectangles in the first graph do not contain all of the area of the region, and the rectangles in the second graph cover more than the area of the region. The exact value of the area lies between these two sums.

85. (a)



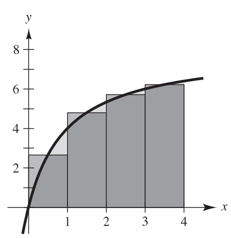
$s(4) = \frac{46}{3}$

(b)



$S(4) = \frac{326}{15}$

(c)



$M(4) = \frac{6112}{315}$

(d) Proof

(e)

n	4	8	20	100	200
$s(n)$	15.333	17.368	18.459	18.995	19.060
$S(n)$	21.733	20.568	19.739	19.251	19.188
$M(n)$	19.403	19.201	19.137	19.125	19.125

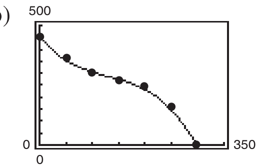
(f) Because f is an increasing function, $s(n)$ is always increasing and $S(n)$ is always decreasing.

87. True

89. Suppose there are n rows and $n + 1$ columns. The stars on the left total $1 + 2 + \cdots + n$, as do the stars on the right. There are $n(n + 1)$ stars in total. So, $2[1 + 2 + \cdots + n] = n(n + 1)$ and $1 + 2 + \cdots + n = [n(n + 1)]/2$.

91. (a) $y = (-4.09 \times 10^{-5})x^3 + 0.016x^2 - 2.67x + 452.9$

(b)



(c) $76{,}897.5 \text{ ft}^2$

93. Proof

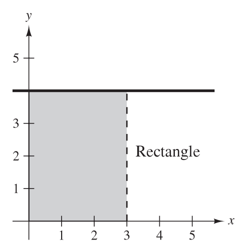
Section 4.3 (page 278)

1. $2\sqrt{3} \approx 3.464$ **3.** 32 **5.** 0 **7.** $\frac{10}{3}$ **9.** $\int_{-1}^{5} (3x + 10)\, dx$

11. $\int_0^3 \sqrt{x^2 + 4}\, dx$ **13.** $\int_0^4 5\, dx$ **15.** $\int_{-4}^{4} (4 - |x|)\, dx$

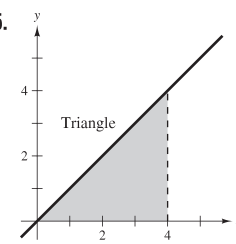
17. $\int_{-5}^{5} (25 - x^2)\, dx$ **19.** $\int_0^{\pi/2} \cos x\, dx$ **21.** $\int_0^2 y^3\, dy$

23.



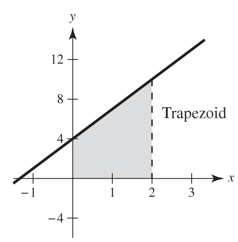
$A = 12$

25.



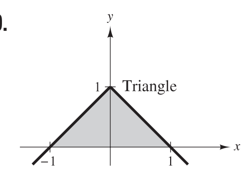
$A = 8$

27.



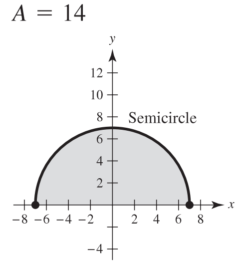
$A = 14$

29.



$A = 1$

31.



$A = 49\pi/2$

33. -6 **35.** 48 **37.** -12

39. 16 **41.** (a) 13 (b) -10 (c) 0 (d) 30

43. (a) 8 (b) -12 (c) -4 (d) 30 **45.** $-48, 88$

47. (a) $-\pi$ (b) 4 (c) $-(1 + 2\pi)$ (d) $3 - 2\pi$ (e) $5 + 2\pi$ (f) $23 - 2\pi$

49. (a) 14 (b) 4 (c) 8 (d) 0 **51.** 81

53. $\sum_{i=1}^{n} f(x_i)\, \Delta x > \int_1^5 f(x)\, dx$

55. No. There is a discontinuity at $x = 4$. **57.** a **59.** d

61.

n	4	8	12	16	20
$L(n)$	3.6830	3.9956	4.0707	4.1016	4.1177
$M(n)$	4.3082	4.2076	4.1838	4.1740	4.1690
$R(n)$	3.6830	3.9956	4.0707	4.1016	4.1177

63.

n	4	8	12	16	20
$L(n)$	0.5890	0.6872	0.7199	0.7363	0.7461
$M(n)$	0.7854	0.7854	0.7854	0.7854	0.7854
$R(n)$	0.9817	0.8836	0.8508	0.8345	0.8247

65. True **67.** True

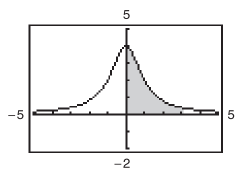
69. False: $\int_0^2 (-x)\,dx = -2$ **71.** 272 **73.** Proof

75. No. No matter how small the subintervals, the number of both rational and irrational numbers within each subinterval is infinite and $f(c_i) = 0$ or $f(c_i) = 1$.

77. $a = -1$ and $b = 1$ maximize the integral. **79.** $\frac{1}{3}$

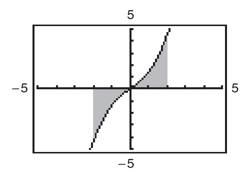
Section 4.4 (page 293)

1.



Positive

3.



Zero

5. 12 **7.** -2 **9.** $-\frac{10}{3}$ **11.** $\frac{1}{3}$ **13.** $\frac{1}{2}$ **15.** $\frac{2}{3}$ **17.** -4

19. $-\frac{1}{18}$ **21.** $-\frac{27}{20}$ **23.** $\frac{25}{2}$ **25.** $\frac{64}{3}$ **27.** $\pi + 2$

29. $\pi/4$ **31.** $2\sqrt{3}/3$ **33.** 0 **35.** $\frac{1}{6}$ **37.** 1 **39.** $\frac{52}{3}$

41. 20 **43.** $\frac{32}{3}$ **45.** $3\sqrt[3]{2}/2 \approx 1.8899$

47. $\frac{1444}{225} \approx 6.4178$ **49.** $\pm\arccos\sqrt{\pi}/2 \approx \pm 0.4817$

51. Average value $= 6$
$x = \pm\sqrt{3} \approx \pm 1.7321$

53. Average value $= \frac{1}{4}$
$x = \sqrt[3]{2}/2 \approx 0.6300$

55. Average value $= 2/\pi$
$x \approx 0.690, x \approx 2.451$

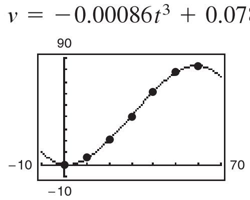
57. About 540 ft

59. (a) 8 (b) $\frac{4}{3}$ (c) $\int_1^7 f(x)\,dx = 20$; Average value $= \frac{10}{3}$

61. (a) $F(x) = 500\sec^2 x$ (b) $1500\sqrt{3}/\pi \approx 827$ N

63. About 0.5318 L

65. (a) $v = -0.00086t^3 + 0.0782t^2 - 0.208t + 0.10$
(b)



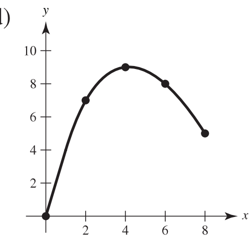
(c) 2475.6 m

67. $F(x) = 2x^2 - 7x$
$F(2) = -6$
$F(5) = 15$
$F(8) = 72$

69. $F(x) = -20/x + 20$
$F(2) = 10$
$F(5) = 16$
$F(8) = \frac{35}{2}$

71. $F(x) = \sin x - \sin 1$
$F(2) = \sin 2 - \sin 1 \approx 0.0678$
$F(5) = \sin 5 - \sin 1 \approx -1.8004$
$F(8) = \sin 8 - \sin 1 \approx 0.1479$

73. (a) $g(0) = 0, g(2) \approx 7, g(4) \approx 9, g(6) \approx 8, g(8) \approx 5$
(b) Increasing: $(0, 4)$; Decreasing: $(4, 8)$
(c) A maximum occurs at $x = 4$.
(d)

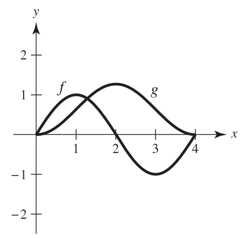


75. $\frac{1}{2}x^2 + 2x$ **77.** $\frac{3}{4}x^{4/3} - 12$ **79.** $\tan x - 1$

81. $x^2 - 2x$ **83.** $\sqrt{x^4 + 1}$ **85.** $x\cos x$ **87.** 8

89. $\cos x\sqrt{\sin x}$ **91.** $3x^2 \sin x^6$

93.



An extremum of g occurs at $x = 2$.

95. (a) $C(x) = 1000(12x^{5/4} + 125)$
(b) $C(1) = \$137{,}000$
$C(5) \approx \$214{,}721$
$C(10) \approx \$338{,}394$

97. (a) $\frac{3}{2}$ ft to the right (b) $\frac{113}{10}$ ft **99.** (a) 0 ft (b) $\frac{63}{2}$ ft

101. (a) 2 ft to the right (b) 2 ft **103.** 28 units **105.** 8190 L

107. $f(x) = x^{-2}$ has a nonremovable discontinuity at $x = 0$.

109. $f(x) = \sec^2 x$ has a nonremovable discontinuity at $x = \pi/2$.

111. $2/\pi \approx 63.7\%$ **113.** True

115. $f'(x) = \dfrac{1}{(1/x)^2 + 1}\left(-\dfrac{1}{x^2}\right) + \dfrac{1}{x^2 + 1} = 0$

Because $f'(x) = 0$, $f(x)$ is constant.

117. (a) 0 (b) 0 (c) $xf(x) + \int_0^x f(t)\,dt$ (d) 0

Section 4.5 (page 306)

	$\int f(g(x))g'(x)\,dx$	$u = g(x)$	$du = g'(x)\,dx$
1.	$\int (8x^2 + 1)^2(16x)\,dx$	$8x^2 + 1$	$16x\,dx$
3.	$\int \dfrac{x}{\sqrt{x^2 + 1}}\,dx$	$x^2 + 1$	$2x\,dx$
5.	$\int \tan^2 x \sec^2 x\,dx$	$\tan x$	$\sec^2 x\,dx$

7. No **9.** Yes **11.** $\frac{1}{5}(1 + 6x)^5 + C$

13. $\frac{2}{3}(25 - x^2)^{3/2} + C$ **15.** $\frac{1}{12}(x^4 + 3)^3 + C$

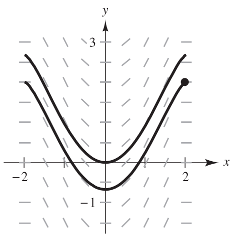
17. $\frac{1}{15}(x^3 - 1)^5 + C$ **19.** $\frac{1}{3}(t^2 + 2)^{3/2} + C$

21. $-\frac{15}{8}(1 - x^2)^{4/3} + C$ **23.** $1/[4(1 - x^2)^2] + C$

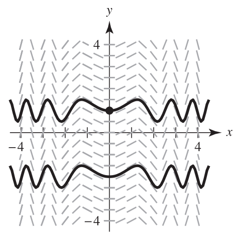
25. $-1/[3(1 + x^3)] + C$ **27.** $-\sqrt{1 - x^2} + C$

29. $-\frac{1}{4}(1 + 1/t)^4 + C$ **31.** $\sqrt{2x} + C$

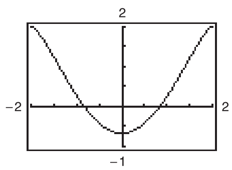
33. $\frac{2}{5}x^{5/2} + \frac{10}{3}x^{3/2} - 16x^{1/2} + C = \frac{1}{15}\sqrt{x}(6x^2 + 50x - 240) + C$
35. $\frac{1}{4}t^4 - 4t^2 + C$
37. $6y^{3/2} - \frac{2}{5}y^{5/2} + C = \frac{2}{5}y^{3/2}(15 - y) + C$
39. $2x^2 - 4\sqrt{16 - x^2} + C$ **41.** $-1/[2(x^2 + 2x - 3)] + C$
43. (a) Answers will vary. Example:



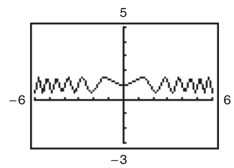
45. (a) Answers will vary. Example:



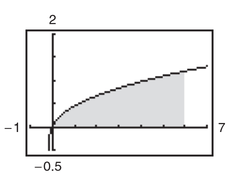
(b) $y = -\frac{1}{3}(4 - x^2)^{3/2} + 2$



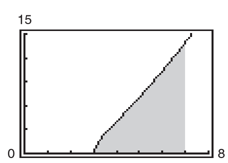
(b) $y = \frac{1}{2}\sin x^2 + 1$



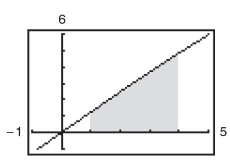
47. $-\cos(\pi x) + C$ **49.** $-\frac{1}{4}\cos 4x + C$ **51.** $-\sin(1/\theta) + C$
53. $\frac{1}{4}\sin^2 2x + C$ or $-\frac{1}{4}\cos^2 2x + C_1$ or $-\frac{1}{8}\cos 4x + C_2$
55. $\frac{1}{5}\tan^5 x + C$ **57.** $\frac{1}{2}\tan^2 x + C$ or $\frac{1}{2}\sec^2 x + C_1$
59. $-\cot x - x + C$ **61.** $f(x) = 2\cos(x/2) + 4$
63. $f(x) = -\frac{1}{2}\cos 4x - 1$ **65.** $f(x) = \frac{1}{12}(4x^2 - 10)^3 - 8$
67. $\frac{2}{5}(x + 6)^{5/2} - 4(x + 6)^{3/2} + C = \frac{2}{5}(x + 6)^{3/2}(x - 4) + C$
69. $-\left[\frac{2}{3}(1 - x)^{3/2} - \frac{4}{5}(1 - x)^{5/2} + \frac{2}{7}(1 - x)^{7/2}\right] + C =$
$-\frac{2}{105}(1 - x)^{3/2}(15x^2 + 12x + 8) + C$
71. $\frac{1}{8}\left[\frac{2}{5}(2x - 1)^{5/2} + \frac{4}{3}(2x - 1)^{3/2} - 6(2x - 1)^{1/2}\right] + C =$
$\left(\sqrt{2x - 1}/15\right)(3x^2 + 2x - 13) + C$
73. $-x - 1 - 2\sqrt{x + 1} + C$ or $-\left(x + 2\sqrt{x + 1}\right) + C_1$
75. 0 **77.** $12 - \frac{8}{9}\sqrt{2}$ **79.** 2 **81.** $\frac{1}{2}$ **83.** $\frac{4}{15}$ **85.** $3\sqrt{3}/4$
87. $f(x) = (2x^3 + 1)^3 + 3$ **89.** $f(x) = \sqrt{2x^2 - 1} - 3$
91. $1209/28$ **93.** 4 **95.** $2\left(\sqrt{3} - 1\right)$
97. $\frac{14}{3}$



99. $\frac{144}{5}$

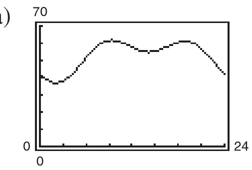


101. 9.21



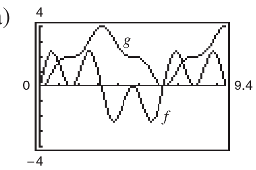
103. $\frac{272}{15}$ **105.** $\frac{2}{3}$ **107.** (a) $\frac{64}{3}$ (b) $\frac{128}{3}$ (c) $-\frac{64}{3}$ (d) 64
109. $2\int_0^3 (4x^2 - 6)\,dx = 36$
111. If $u = 5 - x^2$, then $du = -2x\,dx$ and
$\int x(5 - x^2)^3\,dx = -\frac{1}{2}\int(5 - x^2)^3(-2x)\,dx = -\frac{1}{2}\int u^3\,du.$

113. 16 **115.** \$250,000
117. (a) Relative minimum: $(6.7, 0.7)$ or July
Relative maximum: $(1.3, 5.1)$ or February
(b) 36.68 in. (c) 3.99 in.
119. (a)

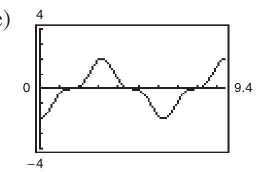


Maximum flow: $R \approx 61.713$ at $t = 9.36$.

(b) 1272 thousand gallons
121. (a) $P_{0.50,\,0.75} \approx 35.3\%$ (b) $b \approx 58.6\%$
123. (a) \$9.17 (b) \$3.14
125. (a)



(b) g is nonnegative because the graph of f is positive at the beginning, and generally has more positive sections than negative ones.
(c) The points on g that correspond to the extrema of f are points of inflection of g.
(d) No, some zeros of f, such as $x = \pi/2$, do not correspond to extrema of g. The graph of g continues to increase after $x = \pi/2$ because f remains above the x-axis.
(e)



The graph of h is that of g shifted 2 units downward.

127. (a) Proof (b) Proof
129. False. $\int (2x + 1)^2\,dx = \frac{1}{6}(2x + 1)^3 + C$
131. True **133.** True **135–137.** Proofs
139. Putnam Problem A1, 1958

Section 4.6 (page 316)

	Trapezoidal	*Simpson's*	*Exact*
1.	2.7500	2.6667	2.6667
3.	4.2500	4.0000	4.0000
5.	20.2222	20.0000	20.0000
7.	12.6640	12.6667	12.6667
9.	0.3352	0.3334	0.3333

	Trapezoidal	*Simpson's*	*Graphing Utility*
11.	3.2833	3.2396	3.2413
13.	0.3415	0.3720	0.3927
15.	0.5495	0.5483	0.5493
17.	−0.0975	−0.0977	−0.0977
19.	0.1940	0.1860	0.1858

21. Trapezoidal: Linear (1st-degree) polynomials
Simpson's: Quadratic (2nd-degree) polynomials
23. (a) 1.500 (b) 0.000 **25.** (a) 0.01 (b) 0.0005
27. (a) 0.1615 (b) 0.0066 **29.** (a) $n = 366$ (b) $n = 26$
31. (a) $n = 77$ (b) $n = 8$ **33.** (a) $n = 287$ (b) $n = 16$
35. (a) $n = 130$ (b) $n = 12$ **37.** (a) $n = 643$ (b) $n = 48$
39. (a) 24.5 (b) 25.67 **41.** Answers will vary.

43.

n	$L(n)$	$M(n)$	$R(n)$	$T(n)$	$S(n)$
4	0.8739	0.7960	0.6239	0.7489	0.7709
8	0.8350	0.7892	0.7100	0.7725	0.7803
10	0.8261	0.7881	0.7261	0.7761	0.7818
12	0.8200	0.7875	0.7367	0.7783	0.7826
16	0.8121	0.7867	0.7496	0.7808	0.7836
20	0.8071	0.7864	0.7571	0.7821	0.7841

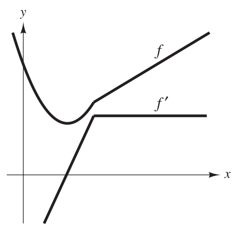
45. 0.701 **47.** 17.476

49. (a) Trapezoidal Rule: 12.518; Simpson's Rule: 12.592
(b) $y = -1.37266x^3 + 4.0092x^2 - 0.620x + 4.28$
$\int_0^2 y\,dx \approx 12.521$

51. 3.14159 **53.** 7435 m^2 **55.** 2.477

Review Exercises for Chapter 4 (page 318)

1.

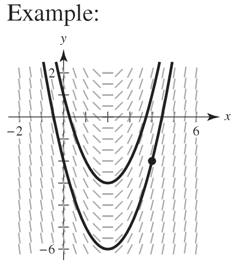


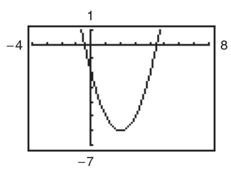
3. $\frac{4}{3}x^3 + \frac{1}{2}x^2 + 3x + C$

5. $x^2/2 - 4/x^2 + C$ **7.** $x^2 + 9\cos x + C$

9. $y = 1 - 3x^2$

11. (a) Answers will vary. (b) $y = x^2 - 4x - 2$
Example:





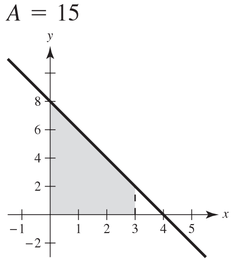
13. 240 ft/sec **15.** (a) 3 sec; 144 ft (b) $\frac{3}{2}$ sec (c) 108 ft

17. $\sum_{n=1}^{10} \frac{1}{3n}$ **19.** 420 **21.** 3310

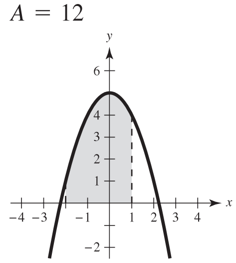
23. (a) $\sum_{i=1}^{10} (2i - 1)$ (b) $\sum_{i=1}^{n} i^3$ (c) $\sum_{i=1}^{10} (4i + 2)$

25. $9.038 <$ (Area of region) < 13.038

27. $A = 15$

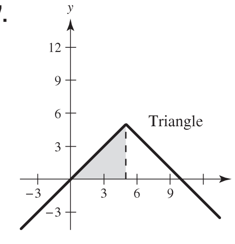


29. $A = 12$



31. $\frac{27}{2}$ **33.** $\int_4^6 (2x - 3)\,dx$ **35.** $\int_{-4}^0 (2x + 8)\,dx$

37.

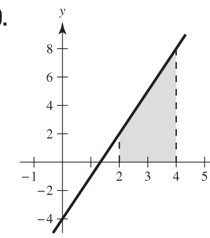


$A = \frac{25}{2}$

39. (a) 17 (b) 7 (c) 9 (d) 84

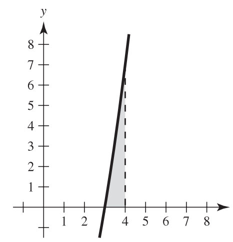
41. 56 **43.** 0 **45.** $\frac{422}{5}$ **47.** $(\sqrt{2} + 2)/2$

49.



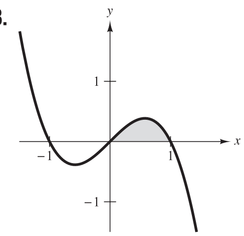
$A = 10$

51.



$A = \frac{10}{3}$

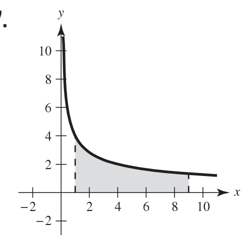
53.



$A = \frac{1}{4}$

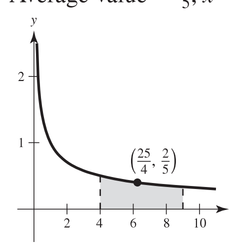
55. $-\cos 2 + 1 \approx 1.416$

57.



$A = 16$

59. Average value $= \frac{2}{5}$, $x = \frac{25}{4}$



61. $x^2\sqrt{1 + x^3}$ **63.** $x^2 + 3x + 2$

65. $-\frac{1}{7}x^7 + \frac{9}{5}x^5 - 9x^3 + 27x + C$ **67.** $\frac{2}{3}\sqrt{x^3 + 3} + C$

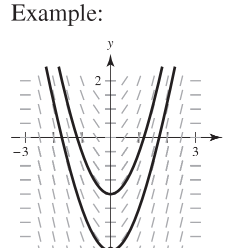
69. $-\frac{1}{30}(1 - 3x^2)^5 + C = \frac{1}{30}(3x^2 - 1)^5 + C$

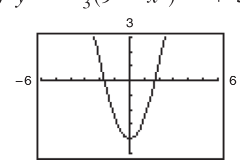
71. $\frac{1}{4}\sin^4 x + C$ **73.** $-2\sqrt{1 - \sin\theta} + C$

75. $\frac{1}{3\pi}(1 + \sec \pi x)^3 + C$

77. 21/4 **79.** 2 **81.** $28\pi/15$ **83.** 2

85. (a) Answers will vary. (b) $y = -\frac{1}{3}(9 - x^2)^{3/2} + 5$
Example:





87. $\frac{468}{7}$

89. (a) $\int_0^{12}[2.880 + 2.125\sin(0.578t + 0.745)]\,dt \approx 36.63$ in.
(b) 2.22 in.

91. Trapezoidal Rule: 0.285
Simpson's Rule: 0.284
Graphing Utility: 0.284

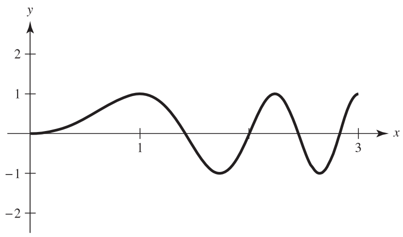
93. Trapezoidal Rule: 0.637
Simpson's Rule: 0.685
Graphing Utility: 0.704

P.S. Problem Solving (page 321)

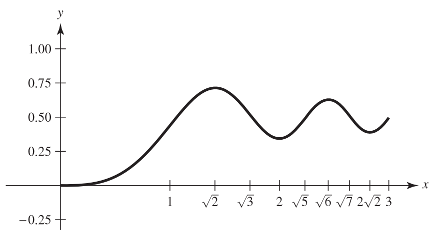
1. (a) $L(1) = 0$ (b) $L'(x) = 1/x,\ L'(1) = 1$
(c) $x \approx 2.718$ (d) Proof

3. (a) $\lim_{n\to\infty}\left[\frac{32}{n^5}\sum_{i=1}^{n} i^4 - \frac{64}{n^4}\sum_{i=1}^{n} i^3 + \frac{32}{n^3}\sum_{i=1}^{n} i^2\right]$
(b) $(16n^4 - 16)/(15n^4)$ (c) $16/15$

5. (a)

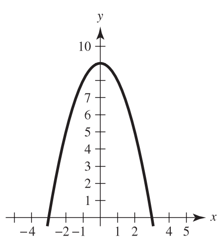


(b)



(c) Relative maxima at $x = \sqrt{2},\ \sqrt{6}$
Relative minima at $x = 2,\ 2\sqrt{2}$
(d) Points of inflection at $x = 1,\ \sqrt{3},\ \sqrt{5},\ \sqrt{7}$

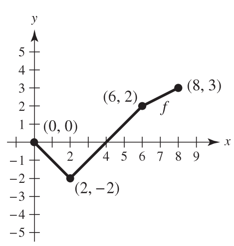
7. (a)



Area = 36

(b) Base = 6, height = 9
Area $= \frac{2}{3}bh = \frac{2}{3}(6)(9) = 36$
(c) Proof

9. (a)



(b)

x	0	1	2	3	4	5	6	7	8
$F(x)$	0	$-\frac{1}{2}$	-2	$-\frac{7}{2}$	-4	$-\frac{7}{2}$	-2	$\frac{1}{4}$	3

(c) $x = 4, 8$ (d) $x = 2$

11. Proof **13.** $\frac{2}{3}$ **15.** $1 \le \int_0^1 \sqrt{1 + x^4}\,dx \le \sqrt{2}$

17. (a) Proof (b) Proof (c) Proof

19. (a) $R(n), I, T(n), L(n)$
(b) $S(4) = \frac{1}{3}[f(0) + 4f(1) + 2f(2) + 4f(3) + f(4)] \approx 5.42$

21. $a = -4,\ b = 4$

Chapter 5

Section 5.1 (page 331)

1.

x	0.5	1.5	2	2.5
$\int_1^x (1/t)\,dt$	-0.6932	0.4055	0.6932	0.9163

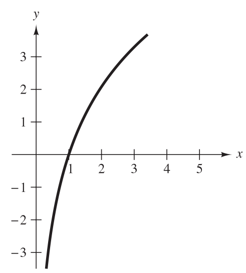
x	3	3.5	4
$\int_1^x (1/t)\,dt$	1.0987	1.2529	1.3865

3. (a) 3.8067 (b) $\ln 45 = \int_1^{45}\frac{1}{t}\,dt \approx 3.8067$

5. (a) -0.2231 (b) $\ln 0.8 = \int_1^{0.8}\frac{1}{t}\,dt \approx -0.2231$

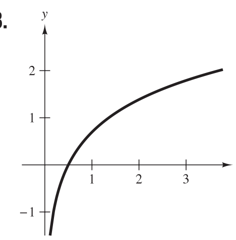
7. b **8.** d **9.** a **10.** c

11.



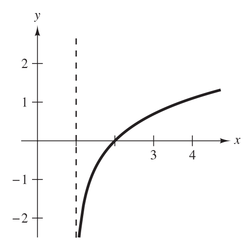
Domain: $x > 0$

13.



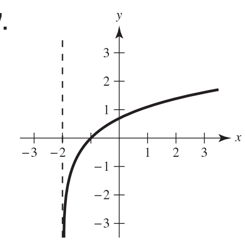
Domain: $x > 0$

15.



Domain: $x > 1$

17.



Domain: $x > -2$

19. (a) 1.7917 (b) -0.4055 (c) 4.3944 (d) 0.5493

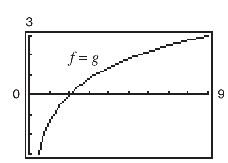
21. $\ln x - \ln 4$ **23.** $\ln x + \ln y - \ln z$

25. $\ln x + \frac{1}{2}\ln(x^2 + 5)$ **27.** $\frac{1}{2}[\ln(x - 1) - \ln x]$

29. $\ln z + 2\ln(z - 1)$

31. $\ln\frac{x - 2}{x + 2}$ **33.** $\ln\sqrt[3]{\frac{x(x + 3)^2}{x^2 - 1}}$ **35.** $\ln\left(9/\sqrt{x^2 + 1}\right)$

37. (a)



(b) $f(x) = \ln\frac{x^2}{4} = \ln x^2 - \ln 4$
$= 2\ln x - \ln 4$
$= g(x)$

39. $-\infty$ **41.** $\ln 4 \approx 1.3863$ **43.** $y = 3x - 3$

45. $y = 4x - 4$ **47.** $1/x$ **49.** $2/x$ **51.** $4(\ln x)^3/x$

53. $2/(t + 1)$ **55.** $\dfrac{2x^2 - 1}{x(x^2 - 1)}$ **57.** $\dfrac{1 - x^2}{x(x^2 + 1)}$

59. $\dfrac{1 - 2\ln t}{t^3}$ **61.** $\dfrac{2}{x \ln x^2} = \dfrac{1}{x \ln x}$ **63.** $\dfrac{1}{1 - x^2}$

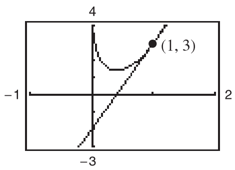
65. $\dfrac{-4}{x(x^2 + 4)}$ **67.** $\dfrac{\sqrt{x^2 + 1}}{x^2}$ **69.** $\cot x$

71. $-\tan x + \dfrac{\sin x}{\cos x - 1}$ **73.** $\dfrac{3 \cos x}{(\sin x - 1)(\sin x + 2)}$

75. $[\ln(2x) + 1]/x$

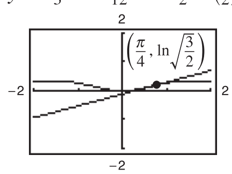
77. (a) $5x - y - 2 = 0$

(b)



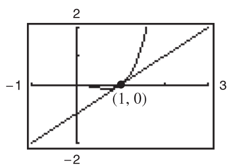
79. (a) $y = \frac{1}{3}x - \frac{1}{12}\pi + \frac{1}{2}\ln\left(\frac{3}{2}\right)$

(b)



81. (a) $y = x - 1$

(b)



83. $2xy/(3 - 2y^2)$

85. $\dfrac{y(1 - 6x^2)}{1 + y}$ **87.** $y = x - 1$

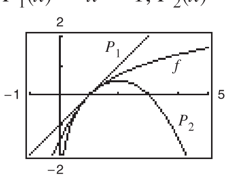
89. $xy'' + y' = x(-2/x^2) + (2/x) = 0$

91. Relative minimum: $\left(1, \frac{1}{2}\right)$

93. Relative minimum: $(e^{-1}, -e^{-1})$

95. Relative minimum: (e, e); Point of inflection: $(e^2, e^2/2)$

97. $P_1(x) = x - 1$; $P_2(x) = x - 1 - \frac{1}{2}(x - 1)^2$



The values of f, P_1, and P_2 and their first derivatives agree at $x = 1$.

99. $x \approx 0.567$ **101.** $(2x^2 + 1)/\sqrt{x^2 + 1}$

103. $\dfrac{3x^3 + 15x^2 - 8x}{2(x + 1)^3\sqrt{3x - 2}}$ **105.** $\dfrac{(2x^2 + 2x - 1)\sqrt{x - 1}}{(x + 1)^{3/2}}$

107. The domain of the natural logarithmic function is $(0, \infty)$ and the range is $(-\infty, \infty)$. The function is continuous, increasing, and one-to-one, and its graph is concave downward. In addition, if a and b are positive numbers and n is rational, then $\ln(1) = 0$, $\ln(a \cdot b) = \ln a + \ln b$, $\ln(a^n) = n \ln a$, and $\ln(a/b) = \ln a - \ln b$.

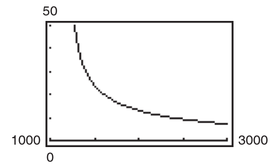
109. (a) Yes. If the graph of g is increasing , then $g'(x) > 0$. Since $f(x) > 0$, you know that $f'(x) = g'(x)f(x)$ and thus $f'(x) > 0$. Therefore, the graph of f is increasing.

(b) No. Let $f(x) = x^2 + 1$ (positive and concave up) and let $g(x) = \ln(x^2 + 1)$ (not concave up).

111. False; $\ln x + \ln 25 = \ln 25x$.

113. False; π is a constant, so $\dfrac{d}{dx}[\ln \pi] = 0$.

115. (a)



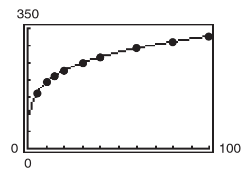
(b) 30 yr; \$503,434.80

(c) 20 yr; \$386,685.60

(d) When $x = 1398.43$, $dt/dx \approx -0.0805$.
When $x = 1611.19$, $dt/dx \approx -0.0287$.

(e) Two benefits of a higher monthly payment are a shorter term and the total amount paid is lower.

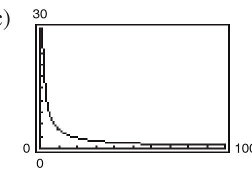
117. (a)



(b) $T'(10) \approx 4.75°/\text{lb/in.}^2$

$T'(70) \approx 0.97°/\text{lb/in.}^2$

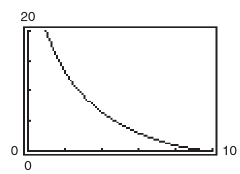
(c)



$\lim\limits_{p \to \infty} T'(p) = 0$

Answers will vary.

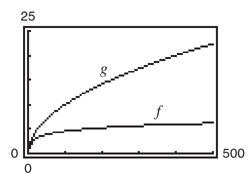
119. (a)



(b) When $x = 5$, $dy/dx = -\sqrt{3}$.
When $x = 9$, $dy/dx = -\sqrt{19}/9$.

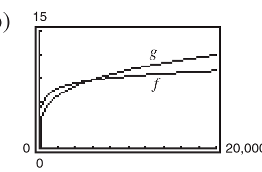
(c) $\lim\limits_{x \to 10^-} \dfrac{dy}{dx} = 0$

121. (a)



For $x > 4$, $g'(x) > f'(x)$. g is increasing at a faster rate than f for large values of x.

(b)



For $x > 256$, $g'(x) > f'(x)$. g is increasing at a faster rate than f for large values of x.

$f(x) = \ln x$ increases very slowly for large values of x.

Section 5.2 (page 340)

1. $5 \ln|x| + C$ **3.** $\ln|x + 1| + C$ **5.** $\frac{1}{2}\ln|2x + 5| + C$

7. $\frac{1}{2}\ln|x^2 - 3| + C$ **9.** $\ln|x^4 + 3x| + C$

11. $x^2/2 - \ln(x^4) + C$ **13.** $\frac{1}{3}\ln|x^3 + 3x^2 + 9x| + C$

15. $\frac{1}{2}x^2 - 4x + 6\ln|x + 1| + C$ **17.** $\frac{1}{3}x^3 + 5\ln|x - 3| + C$

19. $\frac{1}{3}x^3 - 2x + \ln\sqrt{x^2 + 2} + C$ **21.** $\frac{1}{3}(\ln x)^3 + C$

23. $2\sqrt{x + 1} + C$ **25.** $2\ln|x - 1| - 2/(x - 1) + C$

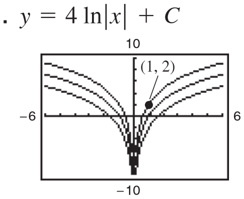
27. $\sqrt{2x} - \ln\left|1 + \sqrt{2x}\right| + C$

29. $x + 6\sqrt{x} + 18\ln\left|\sqrt{x} - 3\right| + C$ **31.** $3\ln\left|\sin\dfrac{\theta}{3}\right| + C$

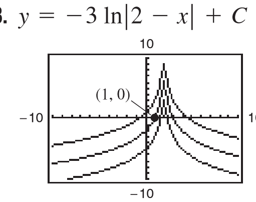
33. $-\frac{1}{2}\ln|\csc 2x + \cot 2x| + C$ **35.** $\frac{1}{3}\sin 3\theta - \theta + C$

37. $\ln|1 + \sin t| + C$ **39.** $\ln|\sec x - 1| + C$

41. $y = 4\ln|x| + C$

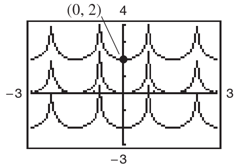


43. $y = -3\ln|2 - x| + C$

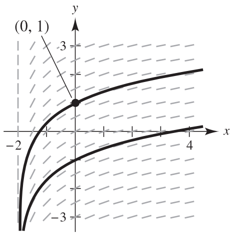


The graph has a hole at $x = 2$.

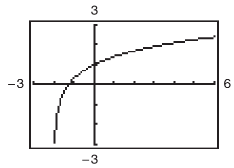
45. $s = -\frac{1}{2}\ln|\cos 2\theta| + C$ **47.** $f(x) = -2\ln x + 3x - 2$



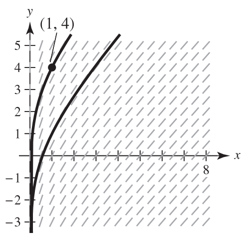
49. (a)



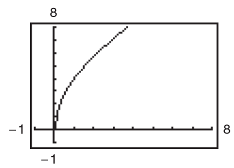
(b) $y = \ln\left(\dfrac{x+2}{2}\right) + 1$



51. (a)



(b) $y = \ln x + x + 3$



53. $\frac{5}{3}\ln 13 \approx 4.275$ **55.** $\frac{7}{3}$ **57.** $-\ln 3 \approx -1.099$

59. $\ln\left|\dfrac{2-\sin 2}{1-\sin 1}\right| \approx 1.929$ **61.** $2\left[\sqrt{x} - \ln\left(1+\sqrt{x}\right)\right] + C$

63. $\ln\left(\dfrac{\sqrt{x}-1}{\sqrt{x}+1}\right) + 2\sqrt{x} + C$ **65.** $\ln\left(\sqrt{2}+1\right) - \dfrac{\sqrt{2}}{2} \approx 0.174$

67. $1/x$ **69.** $1/x$ **71.** d **73.** $6\ln 3$ **75.** $\frac{1}{2}\ln 2$

77. $\frac{15}{2} + 8\ln 2 \approx 13.045$

79. $(12/\pi)\ln\left(2+\sqrt{3}\right) \approx 5.03$

81. Trapezoidal Rule: 20.2
Simpson's Rule: 19.4667

83. Trapezoidal Rule: 5.3368
Simpson's Rule: 5.3632

85. Power Rule **87.** Log Rule **89.** $x = 2$ **91.** Proof

93. $-\ln|\cos x| + C = \ln|1/\cos x| + C = \ln|\sec x| + C$

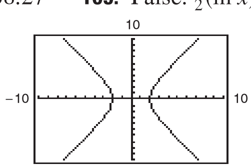
95. $\ln|\sec x + \tan x| + C = \ln\left|\dfrac{\sec^2 x - \tan^2 x}{\sec x - \tan x}\right| + C$
$= -\ln|\sec x - \tan x| + C$

97. 1 **99.** $1/(e-1) \approx 0.582$

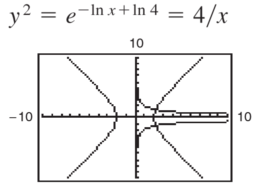
101. $P(t) = 1000(12\ln|1 + 0.25t| + 1)$; $P(3) \approx 7715$

103. $168.27 **105.** False. $\frac{1}{2}(\ln x) = \ln x^{1/2}$ **107.** True

109. (a)



(b) Answers will vary.
Example:
$y^2 = e^{-\ln x + \ln 4} = 4/x$



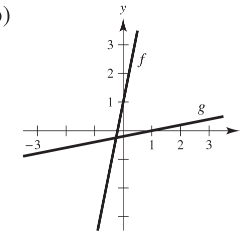
(c) Answers will vary.

111. Proof

Section 5.3 (page 349)

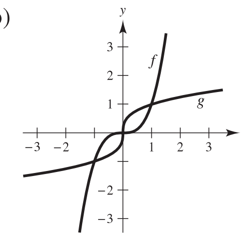
1. (a) $f(g(x)) = 5[(x-1)/5] + 1 = x$
$g(f(x)) = [(5x+1) - 1]/5 = x$

(b)



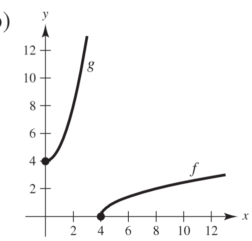
3. (a) $f(g(x)) = \left(\sqrt[3]{x}\right)^3 = x$; $g(f(x)) = \sqrt[3]{x^3} = x$

(b)



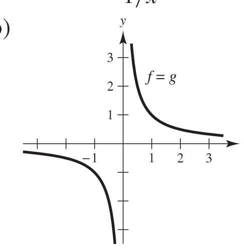
5. (a) $f(g(x)) = \sqrt{x^2 + 4 - 4} = x$;
$g(f(x)) = \left(\sqrt{x-4}\right)^2 + 4 = x$

(b)



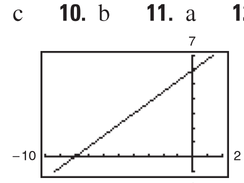
7. (a) $f(g(x)) = \dfrac{1}{1/x} = x$; $g(f(x)) = \dfrac{1}{1/x} = x$

(b)



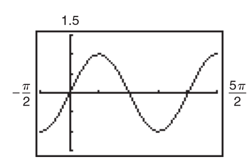
9. c **10.** b **11.** a **12.** d

13.



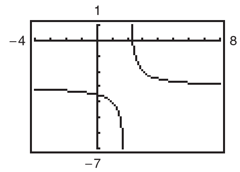
One-to-one, inverse exists.

15.



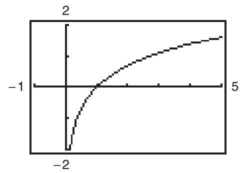
Not one-to-one, inverse does not exist.

17.



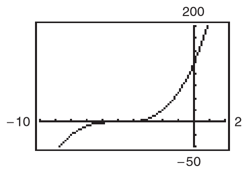
One-to-one, inverse exists.

19.



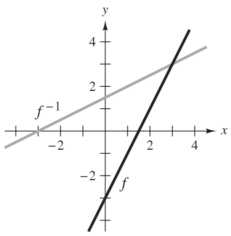
One-to-one, inverse exists.

21.



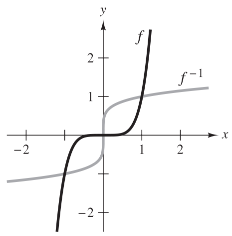
One-to-one, inverse exists.

23. (a) $f^{-1}(x) = (x + 3)/2$
(b)



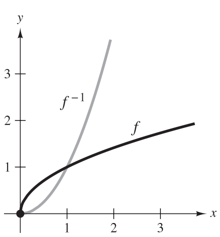
(c) f and f^{-1} are symmetric about $y = x$.
(d) Domain of f and f^{-1}: all real numbers
Range of f and f^{-1}: all real numbers

25. (a) $f^{-1}(x) = x^{1/5}$
(b)



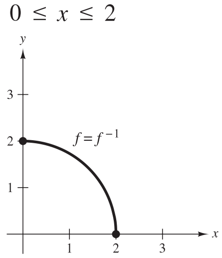
(c) f and f^{-1} are symmetric about $y = x$.
(d) Domain of f and f^{-1}: all real numbers
Range of f and f^{-1}: all real numbers

27. (a) $f^{-1}(x) = x^2,\ x \geq 0$
(b)



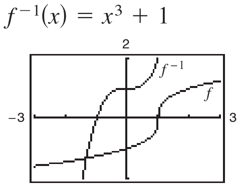
(c) f and f^{-1} are symmetric about $y = x$.
(d) Domain of f and f^{-1}: $x \geq 0$
Range of f and f^{-1}: $y \geq 0$

29. (a) $f^{-1}(x) = \sqrt{4 - x^2}$, $0 \leq x \leq 2$
(b)



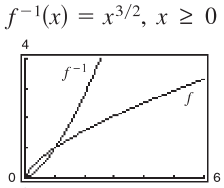
(c) f and f^{-1} are symmetric about $y = x$.
(d) Domain of f and f^{-1}: $0 \leq x \leq 2$
Range of f and f^{-1}: $0 \leq y \leq 2$

31. (a) $f^{-1}(x) = x^3 + 1$
(b)



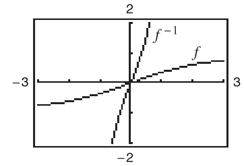
(c) f and f^{-1} are symmetric about $y = x$.
(d) Domain of f and f^{-1}: all real numbers
Range of f and f^{-1}: all real numbers

33. (a) $f^{-1}(x) = x^{3/2},\ x \geq 0$
(b)



(c) f and f^{-1} are symmetric about $y = x$.
(d) Domain of f and f^{-1}: $x \geq 0$
Range of f and f^{-1}: $y \geq 0$

35. (a) $f^{-1}(x) = \sqrt{7}x/\sqrt{1 - x^2},\ -1 < x < 1$
(b)

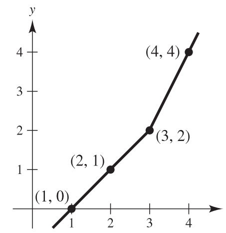


(c) f and f^{-1} are symmetric about $y = x$.
(d) Domain of f: all real numbers
Domain of f^{-1}: $-1 < x < 1$
Range of f: $-1 < y < 1$
Range of f^{-1}: all real numbers

37.

x	0	1	2	4
$f(x)$	1	2	3	4

x	1	2	3	4
$f^{-1}(x)$	0	1	2	4



39. (a) Proof
(b) $y = \frac{20}{7}(80 - x)$
x: total cost
y: number of pounds of the less expensive commodity
(c) $[62.5, 80]$ (d) 20 lb

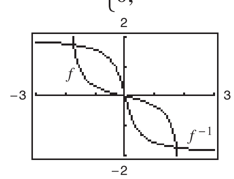
41. Inverse exists. **43.** Inverse does not exist.

45. Inverse exists. **47.** $f'(x) = 2(x - 4) > 0$ on $(4, \infty)$

49. $f'(x) = -8/x^3 < 0$ on $(0, \infty)$

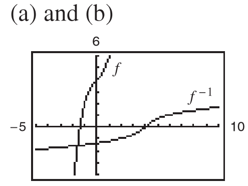
51. $f'(x) = -\sin x < 0$ on $(0, \pi)$

53. $f^{-1}(x) = \begin{cases} \left[1 - \sqrt{1 + 16x^2}\right]/(2x), & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$



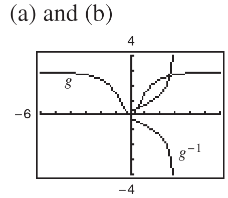
The graph of f^{-1} is a reflection of the graph of f in the line $y = -x$.

55. (a) and (b)



(c) f is one-to-one and has an inverse function.

57. (a) and (b)



(c) g is not one-to-one and does not have an inverse function.

59. One-to-one
$f^{-1}(x) = x^2 + 2,\ x \geq 0$

61. One-to-one
$f^{-1}(x) = 2 - x,\ x \geq 0$

63. $f^{-1}(x) = \sqrt{x} + 3,\ x \geq 0$ (Answer is not unique.)

65. $f^{-1}(x) = x - 3,\ x \geq 0$ (Answer is not unique.)

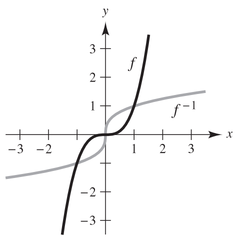
67. Inverse exists. Volume is an increasing function, and therefore is one-to-one. The inverse function gives the time t corresponding to the volume V.

69. Inverse does not exist. **71.** $1/27$ **73.** $1/5$

75. $2\sqrt{3}/3$ **77.** -2 **79.** $1/13$

81. (a) Domain of f: $(-\infty, \infty)$ (b) Range of f: $(-\infty, \infty)$
Domain of f^{-1}: $(-\infty, \infty)$ Range of f^{-1}: $(-\infty, \infty)$

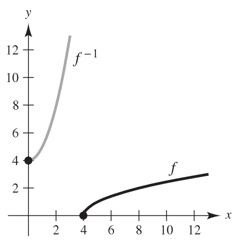
(c)



(d) $f'\left(\frac{1}{2}\right) = \frac{3}{4}$, $(f^{-1})'\left(\frac{1}{8}\right) = \frac{4}{3}$

83. (a) Domain of f: $[4, \infty)$ (b) Range of f: $[0, \infty)$
Domain of f^{-1}: $[0, \infty)$ Range of f^{-1}: $[4, \infty)$

(c)



(d) $f'(5) = \frac{1}{2}$, $(f^{-1})'(1) = 2$

85. $-\frac{1}{11}$ **87.** 32 **89.** 600

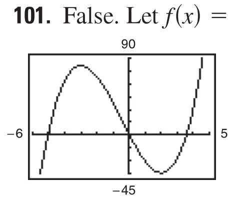
91. $(g^{-1} \circ f^{-1})(x) = (x + 1)/2$ **93.** $(f \circ g)^{-1}(x) = (x + 1)/2$

95. Let $y = f(x)$ be one-to-one. Solve for x as a function of y. Interchange x and y to get $y = f^{-1}(x)$. Let the domain of f^{-1} be the range of f. Verify that $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$.
Example: $f(x) = x^3$; $y = x^3$; $x = \sqrt[3]{y}$; $y = \sqrt[3]{x}$; $f^{-1}(x) = \sqrt[3]{x}$

97. Many x-values yield the same y-value. For example, $f(\pi) = 0 = f(0)$. The graph is not continuous at $[(2n - 1)\pi]/2$ where n is an integer.

99. $\frac{1}{4}$ **101.** False. Let $f(x) = x^2$. **103.** True

105. (a)



f does not pass the horizontal line test.

(b) $c = 2$

107–109. Proofs **111.** Proof; concave upward

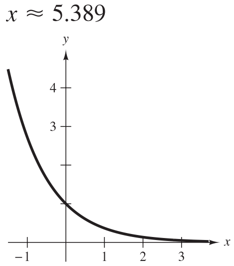
113. Proof; $\sqrt{5}/5$

115. (a) Proof (b) $f^{-1}(x) = \dfrac{b - dx}{cx - a}$
(c) $a = -d$, or $b = c = 0$, $a = d$

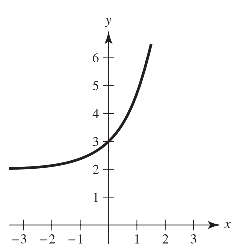
Section 5.4 (page 358)

1. $x = 4$ **3.** $x \approx 2.485$ **5.** $x = 0$ **7.** $x \approx 0.511$
9. $x \approx 8.862$ **11.** $x \approx 7.389$ **13.** $x \approx 10.389$
15. $x \approx 5.389$

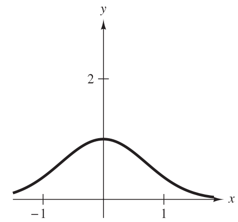
17.



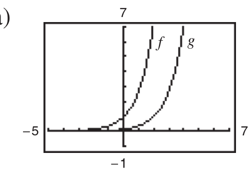
19.



21.

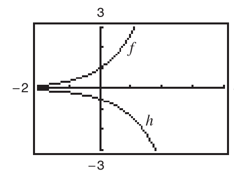


23. (a)



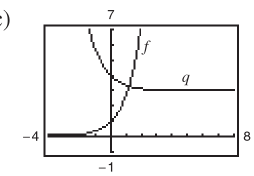
Translation two units to the right

(b)



Reflection in the x-axis and a vertical shrink

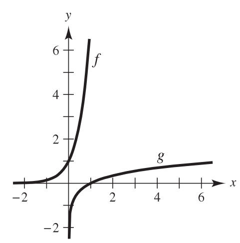
(c)



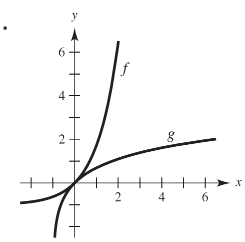
Reflection in the y-axis and a translation three units upward

25. c **26.** d **27.** a **28.** b

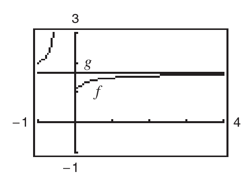
29.



31.



33.



$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = e^{0.5}$

35. $2.7182805 < e$ **37.** (a) $y = 3x + 1$ (b) $y = -3x + 1$

39. $2e^{2x}$ **41.** $e^{\sqrt{x}}/(2\sqrt{x})$ **43.** e^{x-4} **45.** $e^x\left(\dfrac{1}{x} + \ln x\right)$

47. $e^x(x^3 + 3x^2)$ **49.** $3(e^{-t} + e^t)^2(e^t - e^{-t})$

51. $2e^{2x}/(1 + e^{2x})$ **53.** $-2(e^x - e^{-x})/(e^x + e^{-x})^2$

55. $-2e^x/(e^x - 1)^2$ **57.** $2e^x \cos x$ **59.** $\cos(x)/x$

61. $y = -x + 2$ **63.** $y = -4(x + 1)$ **65.** $y = ex$

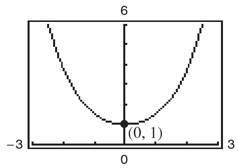
67. $y = (1/e)x - 1/e$ **69.** $\dfrac{10 - e^y}{xe^y + 3}$

71. $y = (-e - 1)x + 1$ **73.** $3(6x + 5)e^{-3x}$

75. $y'' - y = 0$
$4e^{-x} - 4e^{-x} = 0$

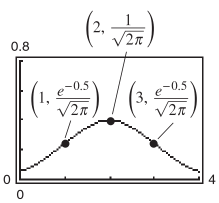
77. $y'' - 2y' + 3y = 0$
$e^x\left[-\cos\sqrt{2}x - \sin\sqrt{2}x - 2\sqrt{2}\sin\sqrt{2}x + 2\sqrt{2}\cos\sqrt{2}x\right] -$
$2e^x\left[-\sqrt{2}\sin\sqrt{2}x + \sqrt{2}\cos\sqrt{2}x + \cos\sqrt{2}x + \sin\sqrt{2}x\right] +$
$3e^x\left[\cos\sqrt{2}x + \sin\sqrt{2}x\right] = 0$
$0 = 0$

79. Relative minimum: $(0, 1)$

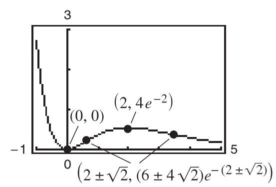


81. Relative maximum: $\left(2, 1/\sqrt{2\pi}\right)$

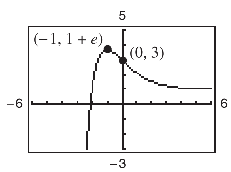
Points of inflection: $\left(1, \dfrac{e^{-0.5}}{\sqrt{2\pi}}\right), \left(3, \dfrac{e^{-0.5}}{\sqrt{2\pi}}\right)$



83. Relative minimum: $(0, 0)$
Relative maximum: $(2, 4e^{-2})$
Points of inflection: $\left(2 \pm \sqrt{2}, \left(6 \pm 4\sqrt{2}\right)e^{-(2 \pm \sqrt{2})}\right)$

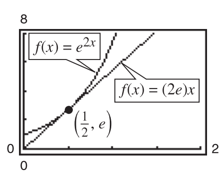


85. Relative maximum: $(-1, 1 + e)$
Point of inflection: $(0, 3)$

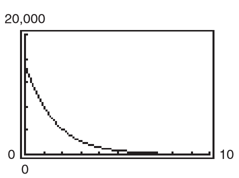


87. $A = \sqrt{2}e^{-1/2}$

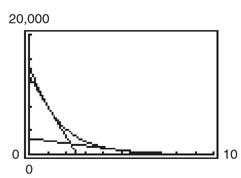
89. $\left(\frac{1}{2}, e\right)$



91. (a)



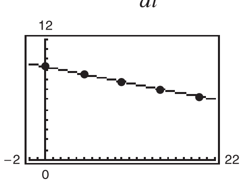
(c)



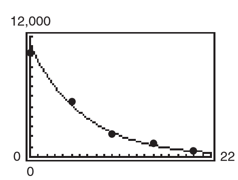
(b) When $t = 1$, $\dfrac{dV}{dt} \approx -5028.84$.

When $t = 5$, $\dfrac{dV}{dt} \approx -406.89$.

93. (a)



(c)

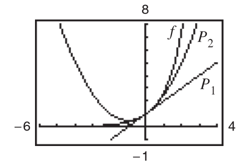


$\ln P = -0.1499h + 9.3018$

(b) $P = 10{,}957.7e^{-0.1499h}$

(d) $h = 5$: -776
$h = 18$: -111

95. $P_1 = 1 + x$; $P_2 = 1 + x + \frac{1}{2}x^2$



The values of f, P_1, and P_2 and their first derivatives agree at $x = 0$.

97. $12! = 479{,}001{,}600$
Stirling's Formula: $12! \approx 475{,}687{,}487$

99. $e^{5x} + C$ **101.** $\frac{1}{2}e^{2x-1} + C$ **103.** $\frac{1}{3}e^{x^3} + C$

105. $2e^{\sqrt{x}} + C$

107. $x - \ln(e^x + 1) + C_1$ or $-\ln(1 + e^{-x}) + C_2$

109. $-\frac{2}{3}(1 - e^x)^{3/2} + C$ **111.** $\ln|e^x - e^{-x}| + C$

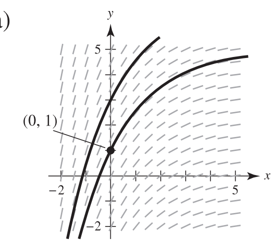
113. $-\frac{5}{2}e^{-2x} + e^{-x} + C$ **115.** $\ln|\cos e^{-x}| + C$

117. $(e^2 - 1)/(2e^2)$ **119.** $(e - 1)/(2e)$ **121.** $(e/3)(e^2 - 1)$

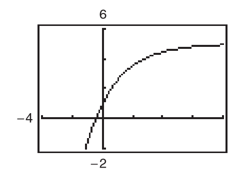
123. $\ln\left(\dfrac{1 + e^6}{2}\right)$ **125.** $(1/\pi)[e^{\sin(\pi^2/2)} - 1]$

127. $[1/(2a)]e^{ax^2} + C$ **129.** $f(x) = \frac{1}{2}(e^x + e^{-x})$

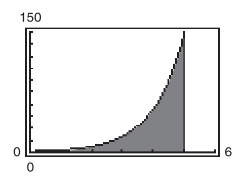
131. (a)

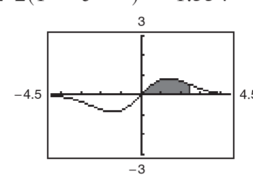


(b) $y = -4e^{-x/2} + 5$



133. $e^5 - 1 \approx 147.413$ **135.** $2(1 - e^{-3/2}) \approx 1.554$





137. Midpoint Rule: 92.190; Trapezoidal Rule: 93.837; Simpson's Rule: 92.7385

139. The probability that a given battery will last between 48 months and 60 months is approximately 47.72%.

141. (a) $t = \dfrac{1}{2k}\ln\dfrac{B}{A}$

(b) $x''(t) = k^2(Ae^{kt} + Be^{-kt})$, k^2 is the constant of proportionality.

143. $f(x) = e^x$
The domain of $f(x)$ is $(-\infty, \infty)$ and the range of $f(x)$ is $(0, \infty)$. $f(x)$ is continuous, increasing, one-to-one, and concave upward on its entire domain.
$\lim\limits_{x \to -\infty} e^x = 0$ and $\lim\limits_{x \to \infty} e^x = \infty$

145. (a) Log Rule (b) Substitution

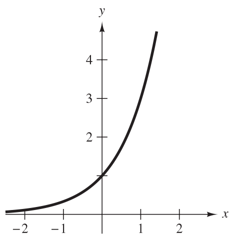
147. $\displaystyle\int_0^x e^t\,dt \ge \int_0^x 1\,dt$; $e^x - 1 \ge x$; $e^x \ge x + 1$ for $x \ge 0$

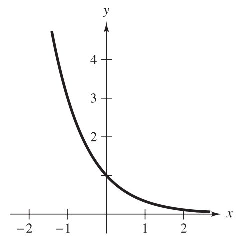
149. $x \approx 0.567$ **151.** Proof

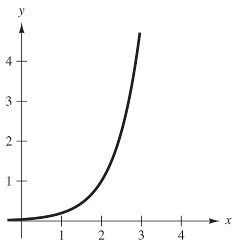
Section 5.5 (page 368)

1. -3 **3.** 0 **5.** (a) $\log_2 8 = 3$ (b) $\log_3(1/3) = -1$

7. (a) $10^{-2} = 0.01$ (b) $\left(\frac{1}{2}\right)^{-3} = 8$

9. 

11. 

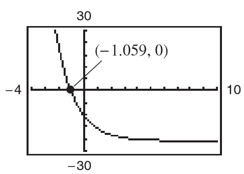
13. 

15. d **16.** c **17.** b **18.** a

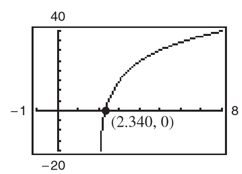
19. (a) $x = 3$ (b) $x = -1$ **21.** (a) $x = \frac{1}{3}$ (b) $x = \frac{1}{16}$

23. (a) $x = -1, 2$ (b) $x = \frac{1}{3}$ **25.** 1.965 **27.** -6.288

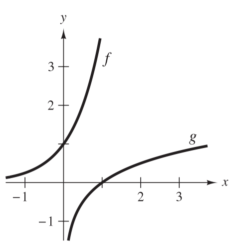
29. 12.253 **31.** 33.000 **33.** ± 11.845

35. 

$(-1.059, 0)$

37. 

$(2.340, 0)$

39. 

41. $(\ln 4)4^x$ **43.** $(-4 \ln 5)5^{-4x}$

45. $9^x(x \ln 9 + 1)$ **47.** $t2^t(t \ln 2 + 2)$

49. $-2^{-\theta}[(\ln 2) \cos \pi\theta + \pi \sin \pi\theta]$

51. $5/[(\ln 4)(5x + 1)]$ **53.** $2/[(\ln 5)(t - 4)]$

55. $x/[(\ln 5)(x^2 - 1)]$ **57.** $(x - 2)/[(\ln 2)x(x - 1)]$

59. $(3x - 2)/[(2x \ln 3)(x - 1)]$

61. $5(1 - \ln t)/(t^2 \ln 2)$ **63.** $y = -2x \ln 2 - 2 \ln 2 + 2$

65. $y = [1/(27 \ln 3)]x + 3 - 1/\ln 3$ **67.** $2(1 - \ln x)x^{(2/x)-2}$

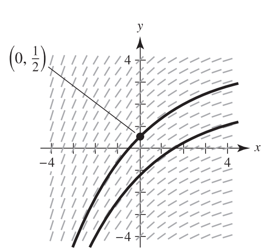
69. $(x - 2)^{x+1}[(x + 1)/(x - 2) + \ln(x - 2)]$

71. $y = x$ **73.** $y = \dfrac{\cos e}{e}x - \cos e + 1$ **75.** $3^x/\ln 3 + C$

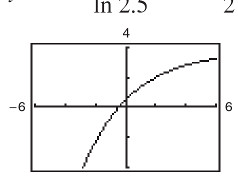
77. $\dfrac{1}{3}x^3 - \dfrac{2^{-x}}{\ln 2} + C$ **79.** $[-1/(2 \ln 5)](5^{-x^2}) + C$

81. $\ln(3^{2x} + 1)/(2 \ln 3) + C$ **83.** $7/(2 \ln 2)$

85. $4/\ln 5 - 2/\ln 3$ **87.** $26/\ln 3$

89. (a) 

(b) $y = \dfrac{3(1 - 0.4^{x/3})}{\ln 2.5} + \dfrac{1}{2}$



91. (a) $x > 0$ (b) 10^x (c) $3 \le f(x) \le 4$
(d) $0 < x < 1$ (e) 10 (f) 100^n

93. (a) ax^{a-1} (b) $(\ln a)a^x$ (c) $x^x(1 + \ln x)$ (d) 0

95. (a) \$40.64 (b) $C'(1) \approx 0.051P$, $C'(8) \approx 0.072P$
(c) ln 1.05

97.

n	1	2	4	12
A	\$1410.60	\$1414.78	\$1416.91	\$1418.34

n	365	Continuous
A	\$1419.04	\$1419.07

99.

n	1	2	4	12
A	\$4321.94	\$4399.79	\$4440.21	\$4467.74

n	365	Continuous
A	\$4481.23	\$4481.69

101.

t	1	10	20	30
P	\$95,122.94	\$60,653.07	\$36,787.94	\$22,313.02

t	40	50
P	\$13,533.53	\$8208.50

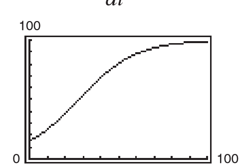
103.

t	1	10	20	30
P	\$95,132.82	\$60,716.10	\$36,864.45	\$22,382.66

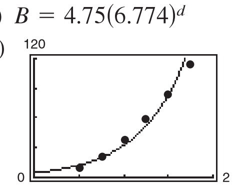
t	40	50
P	\$13,589.88	\$8251.24

105. c

107. (a) 6.7 million ft^3/acre
(b) $t = 20$: $\dfrac{dV}{dt} = 0.073$; $t = 60$: $\dfrac{dV}{dt} = 0.040$

109. (a) 

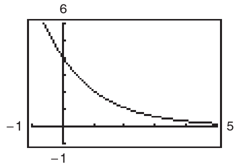
(b) 16.7%
(c) $x \approx 38.8$ or 38,800 egg masses
(d) $x \approx 27.75$ or 27,750 egg masses

111. (a) $B = 4.75(6.774)^d$
(b) 

(c) When $d = 0.8$, the rate of growth is 41.99.
When $d = 1.5$, the rate of growth is 160.21.

113. (a) 5.67; 5.67; 5.67

(b)



(c) $f(t) = g(t) = h(t)$. No, because the definite integrals of two functions over a given interval may be equal even though the functions are not equal.

115. $y = 1200(0.6^t)$ **117.** e **119.** e^2

121. False: e is an irrational number. **123.** True **125.** True

127. (a) $(2^3)^2 = 2^6 = 64$

$2^{(3^2)} = 2^9 = 512$

(b) No. $f(x) = (x^x)^x = x^{(x^2)}$ and $g(x) = x^{(x^x)}$

(c) $f'(x) = x^{x^2}(x + 2x \ln x)$

$g'(x) = x^{x^x + x - 1}[x(\ln x)^2 + x \ln x + 1]$

129. Proof

131. (a) $\dfrac{dy}{dx} = \dfrac{y^2 - yx \ln y}{x^2 - xy \ln x}$

(b) (i) 1 when $c \neq 0, c \neq e$ (ii) -3.1774 (iii) -0.3147

(c) (e, e)

133. Putnam Problem A15, 1940

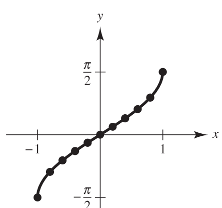
Section 5.6 (page 379)

1. (a)

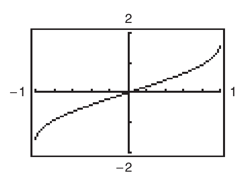
x	-1	-0.8	-0.6	-0.4	-0.2
y	-1.57	-0.93	-0.64	-0.41	-0.20

x	0	0.2	0.4	0.6	0.8	1
y	0	0.20	0.41	0.64	0.93	1.57

(b)



(c)



(d) Intercept: (0, 0);
Symmetry: origin

3. $(-\sqrt{2}/2, 3\pi/4), (1/2, \pi/3), (\sqrt{3}/2, \pi/6)$

5. $\pi/6$ **7.** $\pi/3$ **9.** $\pi/6$ **11.** $-\pi/4$ **13.** 2.50

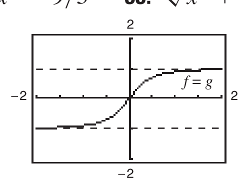
15. $\arccos(1/1.269) \approx 0.66$ **17.** (a) $3/5$ (b) $5/3$

19. (a) $-\sqrt{3}$ (b) $-\frac{13}{5}$ **21.** x **23.** $\sqrt{1 - x^2}/x$ **25.** $1/x$

27. $\sqrt{1 - 4x^2}$ **29.** $\sqrt{x^2 - 1}/|x|$

31. $\sqrt{x^2 - 9}/3$ **33.** $\sqrt{x^2 + 2}/x$

35. (a)



(b) Proof

(c) Horizontal asymptotes:
$y = -1, y = 1$

37. $x = \frac{1}{3}\left[\sin\left(\frac{1}{2}\right) + \pi\right] \approx 1.207$ **39.** $x = \frac{1}{3}$

41. (a) and (b) Proofs **43.** $2/\sqrt{2x - x^2}$

45. $-3/\sqrt{4 - x^2}$ **47.** $e^x/(1 + e^{2x})$

49. $\left(3x - \sqrt{1 - 9x^2} \arcsin 3x\right)/\left(x^2\sqrt{1 - 9x^2}\right)$

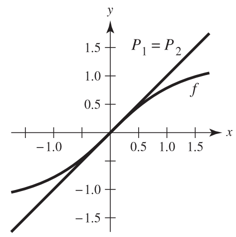
51. $-t/\sqrt{1 - t^2}$ **53.** $2 \arccos x$ **55.** $1/(1 - x^4)$

57. $\arcsin x$ **59.** $x^2/\sqrt{16 - x^2}$ **61.** $2/(1 + x^2)^2$

63. $y = \frac{1}{3}\left(4\sqrt{3}x - 2\sqrt{3} + \pi\right)$

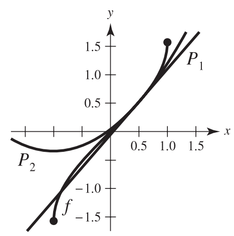
65. $y = \frac{1}{4}x + (\pi - 2)/4$ **67.** $y = (2\pi - 4)x + 4$

69. $P_1(x) = x$; $P_2(x) = x$



71. $P_1(x) = \dfrac{\pi}{6} + \dfrac{2\sqrt{3}}{3}\left(x - \dfrac{1}{2}\right)$

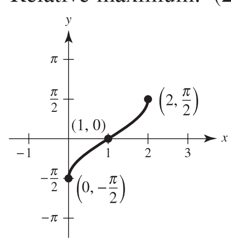
$P_2(x) = \dfrac{\pi}{6} + \dfrac{2\sqrt{3}}{3}\left(x - \dfrac{1}{2}\right) + \dfrac{2\sqrt{3}}{9}\left(x - \dfrac{1}{2}\right)^2$



73. Relative maximum: $(1.272, -0.606)$
Relative minimum: $(-1.272, 3.747)$

75. Relative maximum: $(2, 2.214)$

77.

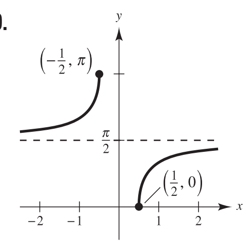


Maximum: $\left(2, \dfrac{\pi}{2}\right)$

Minimum: $\left(0, -\dfrac{\pi}{2}\right)$

Point of inflection: $(1, 0)$

79.



Maximum: $\left(-\dfrac{1}{2}, \pi\right)$

Minimum: $\left(\dfrac{1}{2}, 0\right)$

Asymptote: $y = \dfrac{\pi}{2}$

81. $y = -2\pi x/(\pi + 8) + 1 - \pi^2/(2\pi + 16)$

83. $y = -x + \sqrt{2}$

85. If the domains were not restricted, the trigonometric functions would have no inverses because they would not be one-to-one.

87. If $x > 0$, $y = \operatorname{arccot} x = \arctan \dfrac{1}{x}$; If $x < 0$, $y = \arctan \dfrac{1}{x} + \pi$.

89. (a) $\arcsin(\arcsin 0.5) \approx 0.551$

$\arcsin(\arcsin 1)$ does not exist.

(b) $\sin(-1) \leq x \leq \sin(1)$

91. False. The range of arccos is $[0, \pi]$. **93.** True **95.** True

97. (a) $\theta = \operatorname{arccot}(x/5)$

(b) $x = 10$: 16 rad/h; $x = 3$: 58.824 rad/h

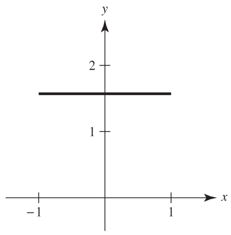
99. (a) $h(t) = -16t^2 + 256$; $t = 4$ sec

(b) $t = 1$: -0.0520 rad/sec; $t = 2$: -0.1116 rad/sec

101. $50\sqrt{2} \approx 70.71$ ft **103.** (a) and (b) Proofs

105. $k \leq -1$ or $k \geq 1$

107. (a)

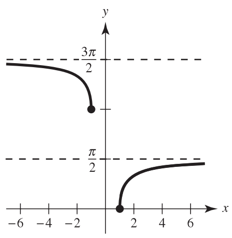


(b) The graph is a horizontal line at $\dfrac{\pi}{2}$.

(c) Proof

109. $c = 2$

111. (a)



(b) Proof

Section 5.7 (page 387)

1. $\arcsin \dfrac{x}{3} + C$ **3.** $\dfrac{7}{4} \arctan \dfrac{x}{4} + C$ **5.** $\operatorname{arcsec}|2x| + C$

7. $\arcsin(x + 1) + C$ **9.** $\frac{1}{2} \arcsin t^2 + C$

11. $\dfrac{1}{10} \arctan \dfrac{t^2}{5} + C$ **13.** $\dfrac{1}{4} \arctan (e^{2x}/2) + C$

15. $\arcsin\left(\dfrac{\tan x}{5}\right) + C$ **17.** $\frac{1}{2}x^2 - \frac{1}{2}\ln(x^2 + 1) + C$

19. $2 \arcsin \sqrt{x} + C$ **21.** $\frac{1}{2}\ln(x^2 + 1) - 3 \arctan x + C$

23. $8 \arcsin[(x - 3)/3] - \sqrt{6x - x^2} + C$ **25.** $\pi/6$

27. $\pi/6$ **29.** $\frac{1}{2}(\sqrt{3} - 2) \approx -0.134$

31. $\dfrac{1}{5} \arctan \dfrac{3}{5} \approx 0.108$ **33.** $\arctan 5 - \dfrac{\pi}{4} \approx 0.588$

35. $\pi/4$ **37.** $\frac{1}{32}\pi^2 \approx 0.308$ **39.** $\pi/2$

41. $\ln|x^2 + 6x + 13| - 3 \arctan[(x + 3)/2] + C$

43. $\arcsin[(x + 2)/2] + C$ **45.** $-\sqrt{-x^2 - 4x} + C$

47. $4 - 2\sqrt{3} + \frac{1}{6}\pi \approx 1.059$ **49.** $\frac{1}{2}\arctan(x^2 + 1) + C$

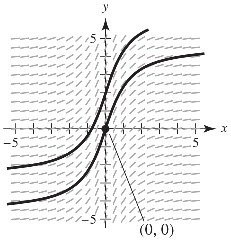
51. $2\sqrt{e^t - 3} - 2\sqrt{3} \arctan\left(\sqrt{e^t - 3}/\sqrt{3}\right) + C$ **53.** $\pi/6$

55. a and b **57.** a, b, and c

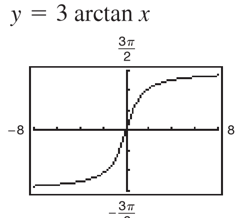
59. No. This integral does not correspond to any of the basic integration rules.

61. $y = \arcsin(x/2) + \pi$

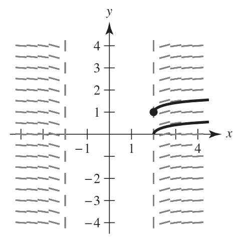
63. (a)



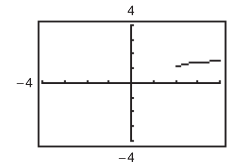
(b) $y = 3 \arctan x$



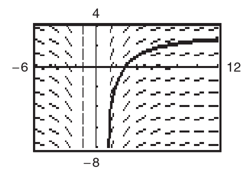
65. (a)



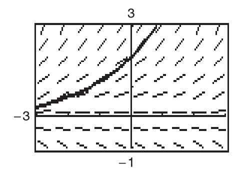
(b) $y = \frac{1}{2} \operatorname{arcsec}(x/2) + 1$, $x \geq 2$



67.



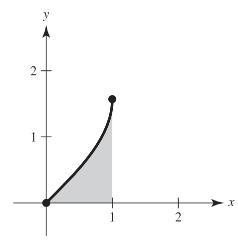
69.



71. $\pi/3$ **73.** $\pi/8$ **75.** $3\pi/2$

77. (a) Proof (b) $\ln\left(\sqrt{6}/2\right) + \left(9\pi - 4\pi\sqrt{3}\right)/36$

79. (a)



(b) 0.5708

(c) $(\pi - 2)/2$

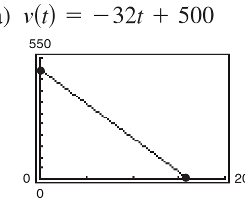
81. (a) $F(x)$ represents the average value of $f(x)$ over the interval $[x, x + 2]$. Maximum at $x = -1$.

(b) Maximum at $x = -1$.

83. False. $\displaystyle\int \frac{dx}{3x\sqrt{9x^2 - 16}} = \frac{1}{12} \operatorname{arcsec} \frac{|3x|}{4} + C$

85. True **87–89.** Proofs

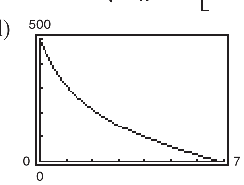
91. (a) $v(t) = -32t + 500$



(b) $s(t) = -16t^2 + 500t$; 3906.25 ft

(c) $v(t) = \sqrt{\dfrac{32}{k}} \tan\left[\arctan\left(500\sqrt{\dfrac{k}{32}}\right) - \sqrt{32k}\,t\right]$

(d)



$t_0 = 6.86$ sec

(e) 1088 ft

(f) When air resistance is taken into account, the maximum height of the object is not as great.

Section 5.8 (page 398)

1. (a) 10.018 (b) -0.964 **3.** (a) $\frac{4}{3}$ (b) $\frac{13}{12}$

5. (a) 1.317 (b) 0.962 **7–15.** Proofs

17. $\cosh x = \sqrt{13}/2$; $\tanh x = 3\sqrt{13}/13$; $\operatorname{csch} x = 2/3$; $\operatorname{sech} x = 2\sqrt{13}/13$; $\coth x = \sqrt{13}/3$

19. $3 \cosh 3x$ **21.** $-10x[\operatorname{sech}(5x^2)\tanh(5x^2)]$ **23.** $\coth x$

25. $\operatorname{csch} x$ **27.** $\sinh^2 x$ **29.** $\operatorname{sech} t$

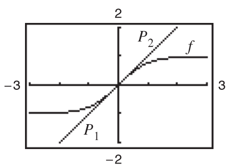
31. $y = -2x + 2$ **33.** $y = 1 - 2x$

35. Relative maxima: $(\pm\pi, \cosh \pi)$; Relative minimum: $(0, -1)$

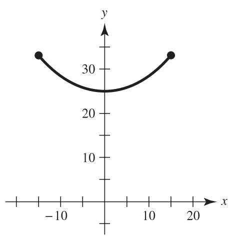
37. Relative maximum: $(1.20, 0.66)$
Relative minimum: $(-1.20, -0.66)$

39. $y = a \sinh x$; $y' = a \cosh x$; $y'' = a \sinh x$; $y''' = a \cosh x$; Therefore, $y''' - y' = 0$.

41. $P_1(x) = x$; $P_2(x) = x$



43. (a)



(b) 33.146 units; 25 units
(c) $m = \sinh(1) \approx 1.175$

45. $\frac{1}{2} \sinh 2x + C$ **47.** $-\frac{1}{2} \cosh(1 - 2x) + C$

49. $\frac{1}{3} \cosh^3(x - 1) + C$ **51.** $\ln|\sinh x| + C$

53. $-\coth(x^2/2) + C$ **55.** $\operatorname{csch}(1/x) + C$

57. $\frac{1}{2} \arctan x^2 + C$ **59.** $\ln(5/4)$ **61.** $\frac{1}{5} \ln 3$

63. $\pi/4$ **65.** $3/\sqrt{9x^2 - 1}$ **67.** $\dfrac{1}{2\sqrt{x}(1 - x)}$ **69.** $|\sec x|$

71. $\dfrac{-2 \operatorname{csch}^{-1} x}{|x|\sqrt{1 + x^2}}$ **73.** $2 \sinh^{-1}(2x)$ **75.** Answers will vary.

77. $\cosh x$, $\operatorname{sech} x$ **79.** ∞ **81.** 1 **83.** 0 **85.** 1

87. $\dfrac{\sqrt{3}}{18} \ln\left|\dfrac{1 + \sqrt{3}x}{1 - \sqrt{3}x}\right| + C$ **89.** $\ln\left(\sqrt{e^{2x} + 1} - 1\right) - x + C$

91. $2 \sinh^{-1}\sqrt{x} + C = 2 \ln\left(\sqrt{x} + \sqrt{1 + x}\right) + C$

93. $\dfrac{1}{4} \ln\left|\dfrac{x - 4}{x}\right| + C$ **95.** $\dfrac{1}{2\sqrt{6}} \ln\left|\dfrac{\sqrt{2}(x + 1) + \sqrt{3}}{\sqrt{2}(x + 1) - \sqrt{3}}\right| + C$

97. $\ln\left(\dfrac{3 + \sqrt{5}}{2}\right)$ **99.** $\dfrac{\ln 7}{12}$ **101.** $\dfrac{1}{4} \arcsin\left(\dfrac{4x - 1}{9}\right) + C$

103. $-\dfrac{x^2}{2} - 4x - \dfrac{10}{3} \ln\left|\dfrac{x - 5}{x + 1}\right| + C$

105. $8 \arctan(e^2) - 2\pi \approx 5.207$ **107.** $\frac{5}{2} \ln\left(\sqrt{17} + 4\right) \approx 5.237$

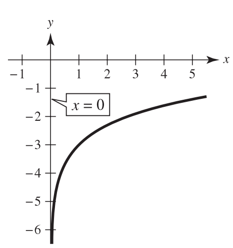
109. (a) $\ln\left(\sqrt{3} + 2\right)$ (b) $\sinh^{-1}\sqrt{3}$

111. $\frac{52}{31}$ kg **113.** $-\sqrt{a^2 - x^2}/x$ **115–123.** Proofs

125. Putnam Problem 8, 1939

Review Exercises for Chapter 5 (page 401)

1.



Vertical asymptote: $x = 0$

3. $\frac{1}{5}[\ln(2x + 1) + \ln(2x - 1) - \ln(4x^2 + 1)]$

5. $\ln\left(3\sqrt[3]{4 - x^2}/x\right)$ **7.** $e^4 - 1 \approx 53.598$

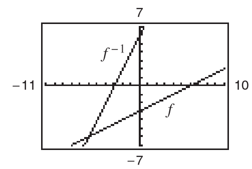
9. $1/(2x)$ **11.** $(1 + 2 \ln x)/\left(2\sqrt{\ln x}\right)$

13. $\dfrac{dy}{dx} = \dfrac{1}{b^2}\left(b - \dfrac{ab}{a + bx}\right) = \dfrac{x}{a + bx}$ **15.** $y = -x + 1$

17. $\frac{1}{7} \ln|7x - 2| + C$ **19.** $-\ln|1 + \cos x| + C$

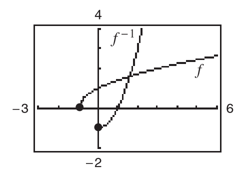
21. $3 + \ln 2$ **23.** $\ln\left(2 + \sqrt{3}\right)$

25. (a) $f^{-1}(x) = 2x + 6$
(b)



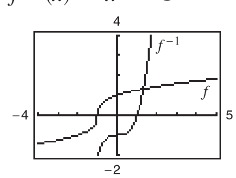
(c) Proof
(d) Domain of f and f^{-1}: all real numbers
Range of f and f^{-1}: all real numbers

27. (a) $f^{-1}(x) = x^2 - 1, \quad x \geq 0$
(b)



(c) Proof
(d) Domain of f: $x \geq -1$
Domain of f^{-1}: $x \geq 0$
Range of f: $y \geq 0$
Range of f^{-1}: $y \geq -1$

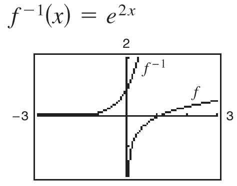
29. (a) $f^{-1}(x) = x^3 - 1$
(b)



(c) Proof
(d) Domain of f and f^{-1}: all real numbers
Range of f and f^{-1}: all real numbers

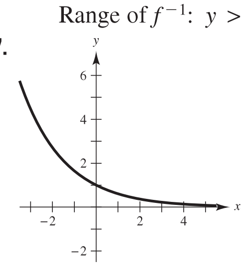
31. $1/\left[3\left(\sqrt[3]{-3}\right)^2\right] \approx 0.160$ **33.** $3/4$

35. (a) $f^{-1}(x) = e^{2x}$
(b)



(c) Proof
(d) Domain of f: $x > 0$
Domain of f^{-1}: all real numbers
Range of f: all real numbers
Range of f^{-1}: $y > 0$

37.



39. $te^t(t + 2)$

41. $(e^{2x} - e^{-2x})/\sqrt{e^{2x} + e^{-2x}}$ **43.** $x(2 - x)/e^x$

45. $y = -4x + 4$ **47.** $-y/[x(2y + \ln x)]$

49. $(1 - e^{-3})/6 \approx 0.158$ **51.** $(e^{4x} - 3e^{2x} - 3)/(3e^x) + C$

53. $-\frac{1}{2}e^{1-x^2} + C$ **55.** $\ln(e^2 + e + 1) \approx 2.408$

57. $y = e^x(a \cos 3x + b \sin 3x)$

$y' = e^x[(-3a + b) \sin 3x + (a + 3b) \cos 3x]$

$y'' = e^x[(-6a - 8b) \sin 3x + (-8a + 6b) \cos 3x]$

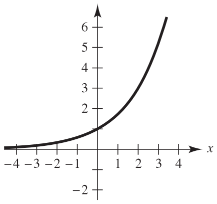
$y'' - 2y' + 10y$

$= e^x\{[(-6a - 8b) - 2(-3a + b) + 10b] \sin 3x +$

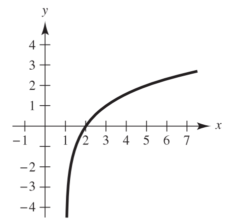
$[(-8a + 6b) - 2(a + 3b) + 10a] \cos 3x\} = 0$

59. $-\frac{1}{2}(e^{-16} - 1) \approx 0.500$

61.



63.

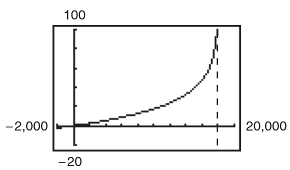


65. $3^{x-1} \ln 3$ **67.** $x^{2x+1}(2 \ln x + 2 + 1/x)$

69. $-1/[\ln 3(2 - 2x)]$ **71.** $5^{(x+1)^2}/(2 \ln 5) + C$

73. (a) Domain: $0 \leq h < 18{,}000$

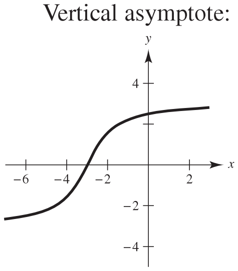
(b)



(c) $t = 0$

Vertical asymptote: $h = 18{,}000$

75.



77. (a) $1/2$ (b) $\sqrt{3}/2$

79. $(1 - x^2)^{-3/2}$ **81.** $\dfrac{x}{|x|\sqrt{x^2 - 1}} + \operatorname{arcsec} x$ **83.** $(\arcsin x)^2$

85. $\frac{1}{2}\arctan(e^{2x}) + C$ **87.** $\frac{1}{2}\arcsin x^2 + C$

89. $\frac{1}{4}[\arctan(x/2)]^2 + C$ **91.** $\frac{2}{3}\pi + \sqrt{3} - 2 \approx 1.826$

93. $y = A \sin\left(t\sqrt{k/m}\right)$

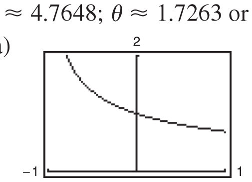
95. $y' = x\left(\dfrac{2}{1 - 4x^2}\right) + \tanh^{-1} 2x = \dfrac{2x}{1 - 4x^2} + \tanh^{-1} 2x$

97. $\frac{1}{3}\tanh x^3 + C$

P.S. Problem Solving (page 403)

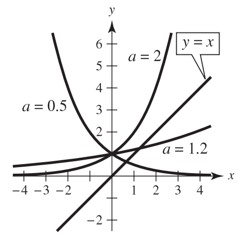
1. $a \approx 4.7648$; $\theta \approx 1.7263$ or $98.9°$

3. (a)



(b) 1 (c) Proof

5.



$y = 0.5^x$ and $y = 1.2^x$ intersect the line $y = x$; $0 < a < e^{1/e}$

7. $e - 1$

9. (a) Area of region $A = (\sqrt{3} - \sqrt{2})/2 \approx 0.1589$

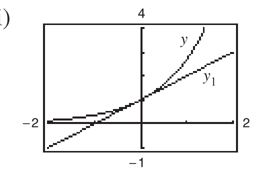
Area of region $B = \pi/12 \approx 0.2618$

(b) $\frac{1}{24}[3\pi\sqrt{2} - 12(\sqrt{3} - \sqrt{2}) - 2\pi] \approx 0.1346$

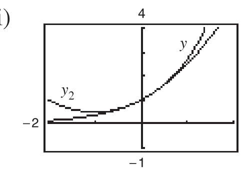
(c) 1.2958 (d) 0.6818

11. Proof **13.** $2 \ln \frac{3}{2} \approx 0.8109$

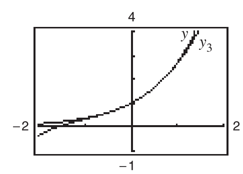
15. (a) (i)



(ii)

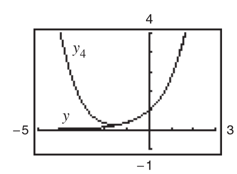


(iii)



(b) Pattern: $y_n = 1 + \dfrac{x}{1!} + \dfrac{x^2}{2!} + \cdots + \dfrac{x^n}{n!} + \cdots$

$y_4 = 1 + \dfrac{x}{1!} + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \dfrac{x^4}{4!}$



(c) The pattern implies that $e^x = 1 + \dfrac{x}{1!} + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \cdots$.

Chapter 6

Section 6.1 (page 411)

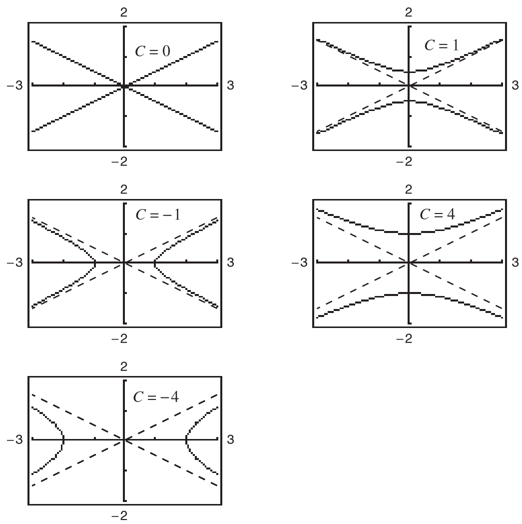
1–11. Proofs **13.** Not a solution **15.** Solution

17. Solution **19.** Solution **21.** Not a solution

23. Solution **25.** Not a solution **27.** Not a solution

29. $y = 3e^{-x/2}$ **31.** $4y^2 = x^3$

33.



35. $y = 3e^{-2x}$ **37.** $y = 2 \sin 3x - \frac{1}{3} \cos 3x$

39. $y = -2x + \frac{1}{2}x^3$ **41.** $2x^3 + C$

43. $y = \frac{1}{2} \ln(1 + x^2) + C$ **45.** $y = x - \ln x^2 + C$

47. $y = -\frac{1}{2} \cos 2x + C$

49. $y = \frac{2}{5}(x - 6)^{5/2} + 4(x - 6)^{3/2} + C$

51. $y = \frac{1}{2}e^{x^2} + C$

53.

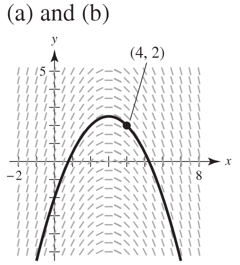
x	-4	-2	0	2	4	8
y	2	0	4	4	6	8
dy/dx	-4	Undef.	0	1	$\frac{4}{3}$	2

55.

x	-4	-2	0	2	4	8
y	2	0	4	4	6	8
dy/dx	$-2\sqrt{2}$	-2	0	0	$-2\sqrt{2}$	-8

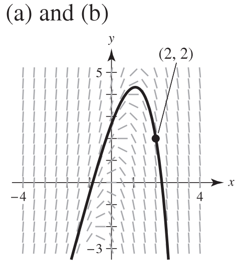
57. b **58.** c **59.** d **60.** a

61. (a) and (b)



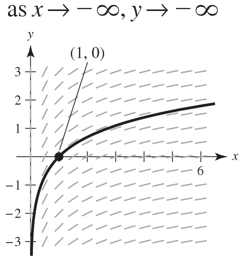
(c) As $x \to \infty$, $y \to -\infty$; as $x \to -\infty$, $y \to -\infty$

63. (a) and (b)



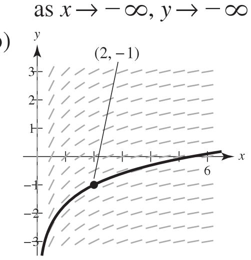
(c) As $x \to \infty$, $y \to -\infty$; as $x \to -\infty$, $y \to -\infty$

65. (a)



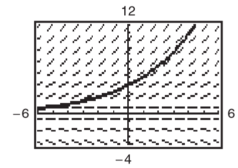
As $x \to \infty$, $y \to \infty$

(b)

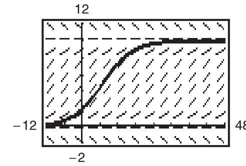


As $x \to \infty$, $y \to \infty$

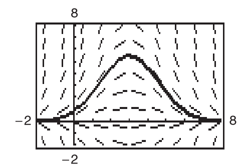
67. (a) and (b)



69. (a) and (b)



71. (a) and (b)



73.

n	0	1	2	3	4	5	6
x_n	0	0.1	0.2	0.3	0.4	0.5	0.6
y_n	2	2.2	2.43	2.693	2.992	3.332	3.715

n	7	8	9	10
x_n	0.7	0.8	0.9	1.0
y_n	4.146	4.631	5.174	5.781

75.

n	0	1	2	3	4	5	6
x_n	0	0.05	0.1	0.15	0.2	0.25	0.3
y_n	3	2.7	2.438	2.209	2.010	1.839	1.693

n	7	8	9	10
x_n	0.35	0.4	0.45	0.5
y_n	1.569	1.464	1.378	1.308

77.

n	0	1	2	3	4	5	6
x_n	0	0.1	0.2	0.3	0.4	0.5	0.6
y_n	1	1.1	1.212	1.339	1.488	1.670	1.900

n	7	8	9	10
x_n	0.7	0.8	0.9	1.0
y_n	2.213	2.684	3.540	5.958

79.

x	0	0.2	0.4	0.6	0.8	1
$y(x)$ (exact)	3.0000	3.6642	4.4755	5.4664	6.6766	8.1548
$y(x)$ ($h = 0.2$)	3.0000	3.6000	4.3200	5.1840	6.2208	7.4650
$y(x)$ ($h = 0.1$)	3.0000	3.6300	4.3923	5.3147	6.4308	7.7812

81.

x	0	0.2	0.4	0.6	0.8	1
$y(x)$ (exact)	0.0000	0.2200	0.4801	0.7807	1.1231	1.5097
$y(x)$ ($h = 0.2$)	0.0000	0.2000	0.4360	0.7074	1.0140	1.3561
$y(x)$ ($h = 0.1$)	0.0000	0.2095	0.4568	0.7418	1.0649	1.4273

83. (a) $y(1) = 112.7141°$; $y(2) = 96.3770°$; $y(3) = 86.5954°$
(b) $y(1) = 113.2441°$; $y(2) = 97.0158°$; $y(3) = 87.1729°$
(c) Euler's Method: $y(1) = 112.9828°$; $y(2) = 96.6998°$; $y(3) = 86.8863°$
Exact solution: $y(1) = 113.2441°$; $y(2) = 97.0158°$; $y(3) = 87.1729°$
The approximations are better using $h = 0.05$.

85. The general solution is a family of curves that satisfies the differential equation. A particular solution is one member of the family that satisfies given conditions.

87. Begin with a point (x_0, y_0) that satisfies the initial condition $y(x_0) = y_0$. Then, using a small step size h, calculate the point $(x_1, y_1) = (x_0 + h, y_0 + hF(x_0, y_0))$. Continue generating the sequence of points $(x_n + h, y_n + hF(x_n, y_n))$ or (x_{n+1}, y_{n+1}).

89. False: $y = x^3$ is a solution of $xy' - 3y = 0$, but $y = x^3 + 1$ is not a solution.

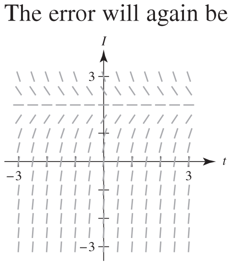
91. True

93. (a)

x	0	0.2	0.4	0.6	0.8	1
y	4	2.6813	1.7973	1.2048	0.8076	0.5413
y_1	4	2.56	1.6384	1.0486	0.6711	0.4295
y_2	4	2.4	1.44	0.864	0.5184	0.3110
e_1	0	0.1213	0.1589	0.1562	0.1365	0.1118
e_2	0	0.2813	0.3573	0.3408	0.2892	0.2303
r		0.4312	0.4447	0.4583	0.4720	0.4855

(b) If h is halved, then the error is approximately halved because r is approximately 0.5.
(c) The error will again be halved.

95. (a)



(b) $\lim_{t \to \infty} I(t) = 2$

97. $\omega = \pm 4$ **99.** Putnam Problem 3, Morning Session, 1954

Section 6.2 (page 420)

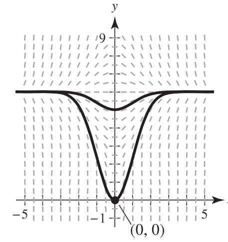
1. $y = \frac{1}{2}x^2 + 3x + C$ **3.** $y = Ce^x - 3$ **5.** $y^2 - 5x^2 = C$

7. $y = Ce^{(2x^{3/2})/3}$ **9.** $y = C(1 + x^2)$

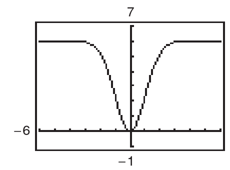
11. $dQ/dt = k/t^2$
$Q = -k/t + C$

13. $dN/ds = k(500 - s)$
$N = -(k/2)(500 - s)^2 + C$

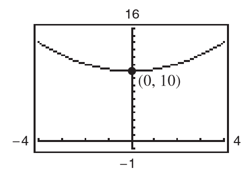
15. (a)



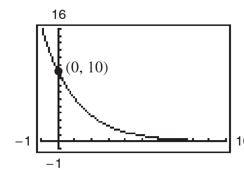
(b) $y = 6 - 6e^{-x^2/2}$



17. $y = \frac{1}{4}t^2 + 10$



19. $y = 10e^{-t/2}$



21. $dy/dx = ky$
$y = 6e^{(1/4)\ln(5/2)x} \approx 6e^{0.2291x}$
$y(8) \approx 37.5$

23. $dV/dt = kV$
$V = 20{,}000e^{(1/4)\ln(5/8)t} \approx 20{,}000e^{-0.1175t}$
$V(6) \approx 9882$

25. $y = (1/2)e^{[(\ln 10)/5]t} \approx (1/2)e^{0.4605t}$

27. $y = 5(5/2)^{1/4}e^{[\ln(2/5)/4]t} \approx 6.2872e^{-0.2291t}$

29. C is the initial value of y, and k is the proportionality constant.

31. Quadrants I and III; dy/dx is positive when both x and y are positive (Quadrant I) or when both x and y are negative (Quadrant III).

33. Amount after 1000 yr: 12.96 g;
Amount after 10,000 yr: 0.26 g

35. Initial quantity: 7.63 g;
Amount after 1000 yr: 4.95 g

37. Amount after 1000 yr: 4.43 g;
Amount after 10,000 yr: 1.49 g

39. Initial quantity: 2.16 g;
Amount after 10,000 yr: 1.62 g

41. 95.76%

43. Time to double: 11.55 yr; Amount after 10 yr: \$7288.48

45. Annual rate: 8.94%; Amount after 10 yr: \$1833.67

47. Annual rate: 9.50%; Time to double: 7.30 yr

49. \$224,174.18 **51.** \$61,377.75

53. (a) 10.24 yr (b) 9.93 yr (c) 9.90 yr (d) 9.90 yr

55. (a) 8.50 yr (b) 8.18 yr (c) 8.16 yr (d) 8.15 yr

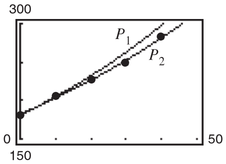
57. (a) $P = 2.40e^{-0.006t}$ (b) 2.19 million
(c) Because $k < 0$, the population is decreasing.

59. (a) $P = 5.66e^{0.024t}$ (b) 8.11 million
(c) Because $k > 0$, the population is increasing.

61. (a) $P = 23.55e^{0.036t}$ (b) 40.41 million
(c) Because $k > 0$, the population is increasing.

63. (a) $N = 100.1596(1.2455)^t$ (b) 6.3 h

65. (a) $N \approx 30(1 - e^{-0.0502t})$ (b) 36 days

67. (a) $P_1 \approx 181e^{0.01245t} \approx 181(1.01253)^t$
(b) $P_2 = 182.3248(1.01091)^t$
(c)


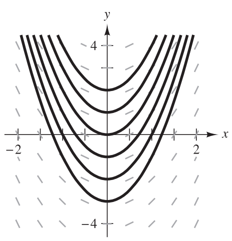
(d) 2011

P_2 is a better approximation.

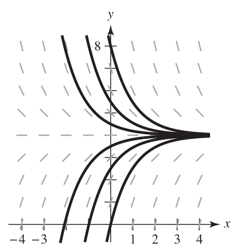
69. (a) 20 dB (b) 70 dB (c) 95 dB (d) 120 dB
71. 2024 ($t = 16$) **73.** 379.2°F
75. False. The rate of growth dy/dx is proportional to y.
77. True

Section 6.3 (page 431)

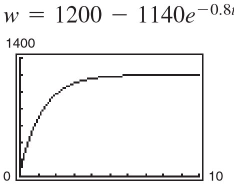
1. $y^2 - x^2 = C$ **3.** $15y^2 + 2x^3 = C$ **5.** $r = Ce^{0.75s}$
7. $y = C(x + 2)^3$ **9.** $y^2 = C - 8 \cos x$
11. $y = -\frac{1}{4}\sqrt{1 - 4x^2} + C$ **13.** $y = Ce^{(\ln x)^2/2}$
15. $y^2 = 4e^x + 5$ **17.** $y = e^{-(x^2+2x)/2}$
19. $y^2 = 4x^2 + 3$ **21.** $u = e^{(1 - \cos v^2)/2}$ **23.** $P = P_0 e^{kt}$
25. $4y^2 - x^2 = 16$ **27.** $y = \frac{1}{3}\sqrt{x}$ **29.** $f(x) = Ce^{-x/2}$
31. Homogeneous of degree 3 **33.** Homogeneous of degree 3
35. Not homogeneous **37.** Homogeneous of degree 0
39. $|x| = C(x - y)^2$ **41.** $|y^2 + 2xy - x^2| = C$
43. $y = Ce^{-x^2/(2y^2)}$ **45.** $e^{y/x} = 1 + \ln x^2$ **47.** $x = e^{\sin(y/x)}$

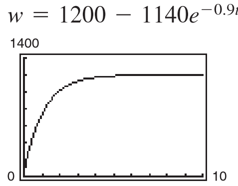
49.


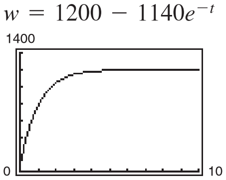
$y = \frac{1}{2}x^2 + C$

51.


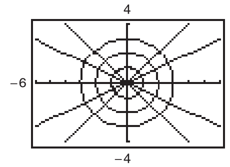
$y = 4 + Ce^{-x}$

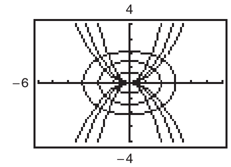
53. (a) $y \approx 0.1602$ (b) $y = 5e^{-3x^2}$ (c) $y \approx 0.2489$
55. (a) $y \approx 3.0318$ (b) $y^3 - 4y = x^2 + 12x - 13$ (c) $y = 3$
57. 97.9% of the original amount
59. (a) $dy/dx = k(y - 4)$ (b) a (c) Proof
60. (a) $dy/dx = k(x - 4)$ (b) b (c) Proof
61. (a) $dy/dx = ky(y - 4)$ (b) c (c) Proof
62. (a) $dy/dx = ky^2$ (b) d (c) Proof
63. (a) $w = 1200 - 1140e^{-0.8t}$


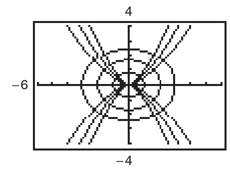
$w = 1200 - 1140e^{-0.9t}$


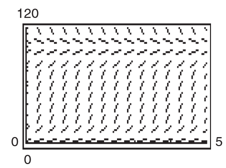
$w = 1200 - 1140e^{-t}$


(b) 1.31 yr; 1.16 yr; 1.05 yr (c) 1200 lb

65. Circles: $x^2 + y^2 = C$
Lines: $y = Kx$
Graphs will vary.


67. Parabolas: $x^2 = Cy$
Ellipses: $x^2 + 2y^2 = K$
Graphs will vary.


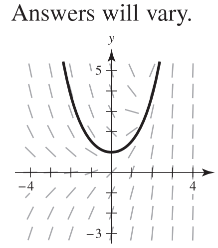
69. Curves: $y^2 = Cx^3$
Ellipses: $2x^2 + 3y^2 = K$
Graphs will vary.


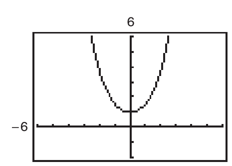
71. d **72.** a **73.** b **74.** c
75. (a) 0.75 (b) 2100 (c) 70 (d) 4.49 yr
(e) $dP/dt = 0.75P(1 - P/2100)$
77. (a) 3 (b) 100
(c)


(d) 50

79. $y = 36/(1 + 8e^{-t})$ **81.** $y = 120/(1 + 14e^{-0.8t})$
83. (a) $P = \dfrac{200}{1 + 7e^{-0.2640t}}$ (b) 70 panthers (c) 7.37 yr
(d) $dP/dt = 0.2640P(1 - P/200)$; 65.6 (e) 100 yr
85. Answers will vary.
87. Two families of curves are mutually orthogonal if each curve in the first family intersects each curve in the second family at right angles.
89. Proof
91. False. $y' = x/y$ is separable, but $y = 0$ is not a solution.
93. False: $f(tx, ty) \neq t^n f(x, y)$. **95.** Putnam Problem A2, 1988

Section 6.4 (page 440)

1. Linear; can be written in the form $dy/dx + P(x)y = Q(x)$
3. Not linear; cannot be written in the form $dy/dx + P(x)y = Q(x)$
5. $y = 2x^2 + x + C/x$ **7.** $y = -16 + Ce^x$
9. $y = -1 + Ce^{\sin x}$ **11.** $y = (x^3 - 3x + C)/[3(x - 1)]$
13. $y = e^{x^3}(x + C)$
15. (a) Answers will vary.


(c)


(b) $y = \frac{1}{2}(e^x + e^{-x})$
17. $y = 1 + 4/e^{\tan x}$ **19.** $y = \sin x + (x + 1)\cos x$
21. $xy = 4$ **23.** $y = -2 + x \ln|x| + 12x$

25. $P = -N/k + (N/k + P_0)e^{kt}$

27. (a) \$4,212,796.94 (b) \$31,424,909.75

29. (a) $\dfrac{dQ}{dt} = q - kQ$ (b) $Q = \dfrac{q}{k} + \left(Q_0 - \dfrac{q}{k}\right)e^{-kt}$ (c) $\dfrac{q}{k}$

31. Proof

33. (a) $Q = 25e^{-t/20}$ (b) $-20 \ln\left(\frac{3}{5}\right) \approx 10.2$ min (c) 0

35. (a) $t = 50$ min (b) $100 - \dfrac{25}{\sqrt{2}} \approx 82.32$ lb

(c) $t = 50$ min; $200 - \dfrac{50}{\sqrt{2}} \approx 164.64$ lb

37. $v(t) = -159.47(1 - e^{-0.2007t})$; -159.47 ft/sec

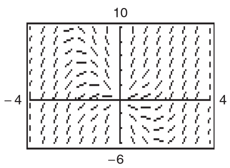
39. $I = \dfrac{E_0}{R} + Ce^{-Rt/L}$ **41.** $\dfrac{dy}{dx} + P(x)y = Q(x)$; $u(x) = e^{\int P(x)\,dx}$

43. c **44.** d **45.** a **46.** b

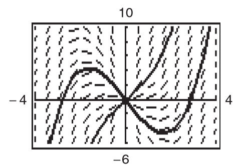
47. $1/y^2 = Ce^{2x^3} + \frac{1}{3}$ **49.** $y = 1/(Cx - x^2)$

51. $1/y^2 = 2x + Cx^2$ **53.** $y^{2/3} = 2e^x + Ce^{2x/3}$

55. (a)



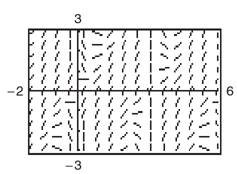
(c)



(b) $(-2, 4)$: $y = \frac{1}{2}x(x^2 - 8)$

$(2, 8)$: $y = \frac{1}{2}x(x^2 + 4)$

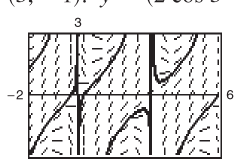
57. (a)



(b) $(1, 1)$: $y = (2 \cos 1 + \sin 1) \csc x - 2 \cot x$

$(3, -1)$: $y = (2 \cos 3 - \sin 3) \csc x - 2 \cot x$

(c)



59. $2e^x + e^{-2y} = C$ **61.** $y = Ce^{-\sin x} + 1$

63. $x^3y^2 + x^4y = C$ **65.** $y = [e^x(x - 1) + C]/x^2$

67. $x^4y^4 - 2x^2 = C$ **69.** $y = \frac{12}{5}x^2 + C/x^3$

71. False. $y' + xy = x^2$ is linear.

Review Exercises for Chapter 6 (page 443)

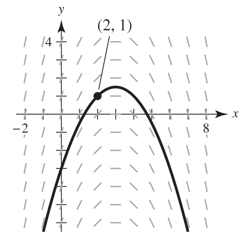
1. Yes **3.** $y = \frac{4}{3}x^3 + 7x + C$ **5.** $y = \frac{1}{2} \sin 2x + C$

7. $y = \frac{2}{5}(x - 5)^{5/2} + \frac{10}{3}(x - 5)^{3/2} + C$ **9.** $y = -e^{2-x} + C$

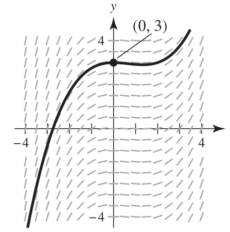
11.

x	-4	-2	0	2	4	8
y	2	0	4	4	6	8
dy/dx	-10	-4	-4	0	2	8

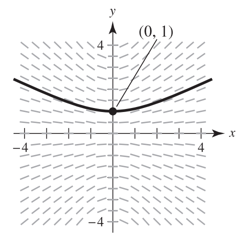
13. (a) and (b)



15. (a) and (b)



17. (a) and (b)



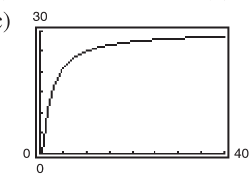
19. $y = 8x - \frac{1}{2}x^2 + C$

21. $y = -3 - 1/(x + C)$ **23.** $y = Ce^x/(2 + x)^2$

25. $y \approx \frac{3}{4}e^{0.379t}$ **27.** $y \approx 5e^{-0.680t}$ **29.** About 7.79 in.

31. (a) $S \approx 30e^{-1.7918/t}$ (b) 20,965 units

(c)

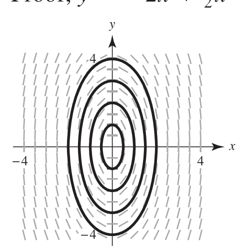


33. About 37.5 yr **35.** $y = \frac{1}{5}x^5 + 7 \ln|x| + C$

37. $y = Ce^{8x^2}$ **39.** $x/(x^2 - y^2) = C$

41. Proof; $y = -2x + \frac{1}{2}x^3$

43.



Graphs will vary.
$4x^2 + y^2 = C$

45. (a) 0.55 (b) 5250 (c) 150 (d) 6.41 yr

(e) $\dfrac{dP}{dt} = 0.55P\left(1 - \dfrac{P}{5250}\right)$

47. $y = \dfrac{80}{1 + 9e^{-t}}$

49. (a) $P(t) = \dfrac{20{,}400}{1 + 16e^{-0.553t}}$ (b) 17,118 trout (c) 4.94 yr

51. $y = -10 + Ce^x$ **53.** $y = e^{x/4}\left(\frac{1}{4}x + C\right)$

55. $y = (x + C)/(x - 2)$

57. $y = Ce^{3x} - \frac{1}{13}(2 \cos 2x + 3 \sin 2x)$

59. $y = e^{5x}/10 + Ce^{-5x}$ **61.** $y = 1/(1 + x + Ce^x)$

63. $y^{-2} = Cx^2 + 2/(3x)$

65. Answers will vary. Sample answer:
$(x^2 + 3y^2)\,dx - 2xy\,dy = 0$; $x^3 = C(x^2 + y^2)$

67. Answers will vary. Sample answer:
$x^3y' + 2x^2y = 1$; $x^2y = \ln|x| + C$

P.S. Problem Solving (page 445)

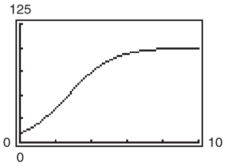
1. (a) $y = 1/(1 - 0.01t)^{100}$; $T = 100$

(b) $y = 1/\left[\left(\dfrac{1}{y_0}\right)^{\varepsilon} - k\varepsilon t\right]^{1/\varepsilon}$; Explanations will vary.

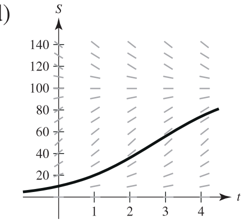
3. (a) $dS/dt = kS(L - S)$; $S = 100/(1 + 9e^{-0.8109t})$

(b) 2.7 mo

(c)



(d)



(e) Sales will decrease toward the line $S = L$.

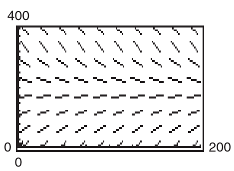
5. Proof; The graph of the logistics function is just a shift of the graph of the hyperbolic tangent.

7. 1481.45 sec $\approx$ 24 min, 41 sec

9. 2575.95 sec $\approx$ 42 min, 56 sec

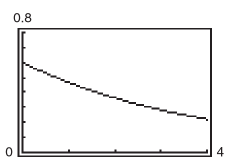
11. (a) $s = 184.21 - Ce^{-0.019t}$

(b)

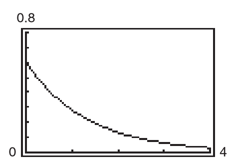


(c) As $t \to \infty$, $Ce^{-0.019t} \to 0$, and $s \to 184.21$.

13. (a) $C = 0.6e^{-0.25t}$



(b) $C = 0.6e^{-0.75t}$



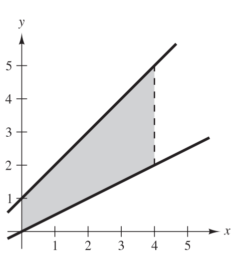
Chapter 7

Section 7.1 (page 454)

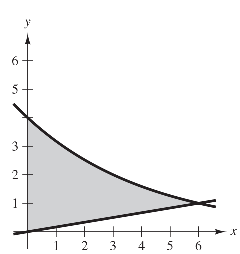
1. $-\displaystyle\int_0^6 (x^2 - 6x)\,dx$ **3.** $\displaystyle\int_0^3 (-2x^2 + 6x)\,dx$

5. $-6\displaystyle\int_0^1 (x^3 - x)\,dx$

7.



9.



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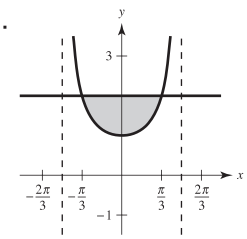

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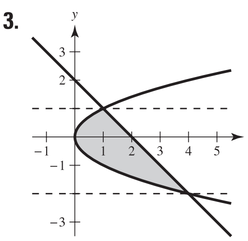
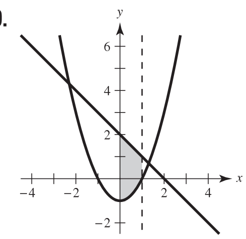


15. d

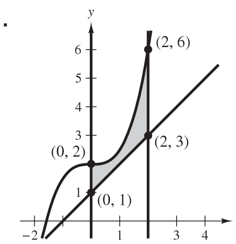
17. (a) $\frac{125}{6}$ (b) $\frac{125}{6}$ (c) Integrating with respect to y; Answers will vary.

19.



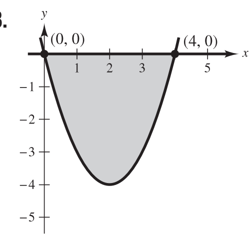
$\frac{13}{6}$

21.



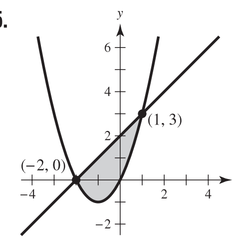
2

23.



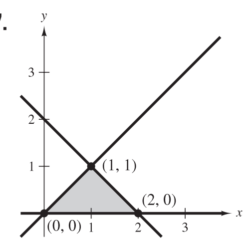
$\frac{32}{3}$

25.



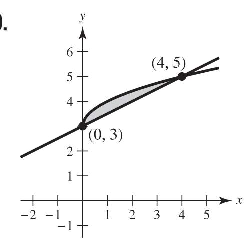
$\frac{9}{2}$

27.



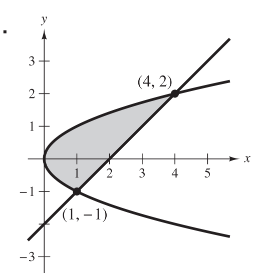
1

29.



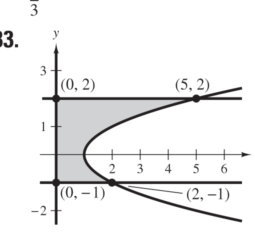
$\frac{4}{3}$

31.

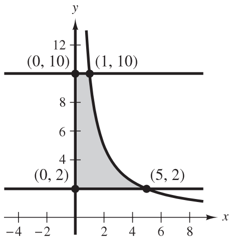


$\frac{9}{2}$

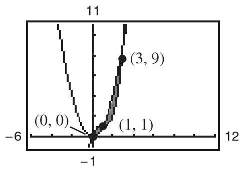
33.



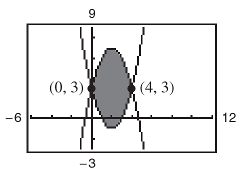
6

35.


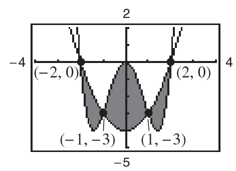
$10 \ln 5 \approx 16.094$

37. (a)


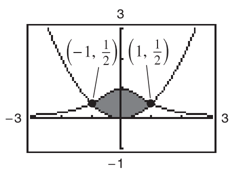
(b) $\frac{37}{12}$

39. (a)


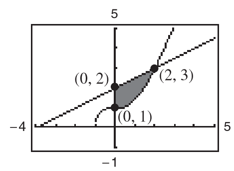
(b) $\frac{64}{3}$

41. (a)


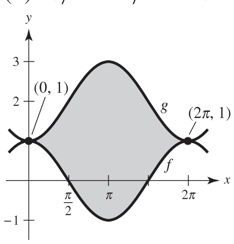
(b) 8

43. (a)


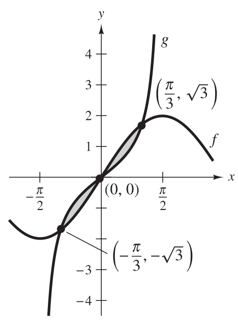
(b) $\pi/2 - 1/3 \approx 1.237$

45. (a)


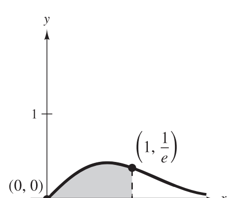
(b) ≈ 1.759

47.


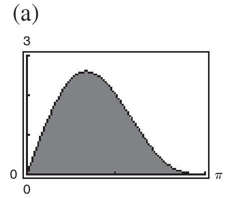
$4\pi \approx 12.566$

49.


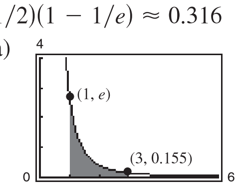
$2(1 - \ln 2) \approx 0.614$

51.


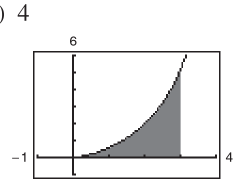
$(1/2)(1 - 1/e) \approx 0.316$

53. (a)


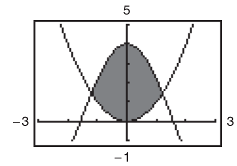
(b) 4

55. (a)


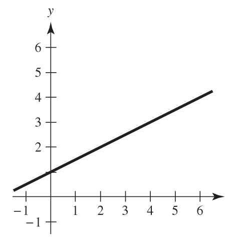
(b) About 1.323

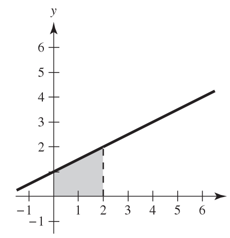
57. (a)


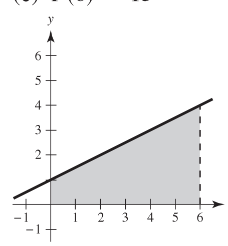
(b) The function is difficult to integrate.
(c) About 4.7721

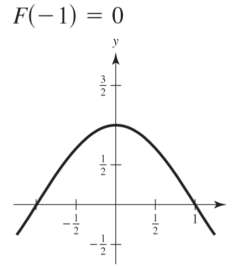
59. (a)


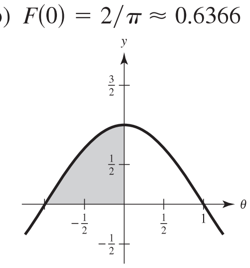
(b) The intersections are difficult to find.
(c) About 6.3043

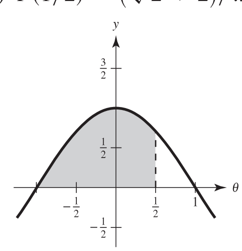
61. $F(x) = \frac{1}{4}x^2 + x$
(a) $F(0) = 0$


(b) $F(2) = 3$


(c) $F(6) = 15$


63. $F(\alpha) = (2/\pi)[\sin(\pi\alpha/2) + 1]$
(a) $F(-1) = 0$


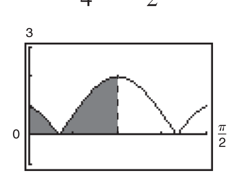
(b) $F(0) = 2/\pi \approx 0.6366$


(c) $F(1/2) = \left(\sqrt{2} + 2\right)/\pi \approx 1.0868$


65. 14 **67.** 16

69. Answers will vary. Sample answers:
(a) About 966 ft^2 (b) About 1004 ft^2

71. $A = \dfrac{3\sqrt{3}}{4} - \dfrac{1}{2} \approx 0.7990$ **73.** $\int_{-2}^{1} [x^3 - (3x - 2)]\, dx = \frac{27}{4}$



75. $\displaystyle\int_0^1 \left[\frac{1}{x^2+1} - \left(-\frac{1}{2}x + 1\right)\right] dx \approx 0.0354$

77. Answers will vary. Example: $x^4 - 2x^2 + 1 \le 1 - x^2$ on $[-1, 1]$

$$\int_{-1}^{1} [(1 - x^2) - (x^4 - 2x^2 + 1)]\, dx = \tfrac{4}{15}$$

79. Offer 2 is better because the cumulative salary (area under the curve) is greater.

81. (a) The integral $\int_0^5 [v_1(t) - v_2(t)]\, dt = 10$ means that the first car traveled 10 more meters than the second car between 0 and 5 seconds.

The integral $\int_0^{10} [v_1(t) - v_2(t)]\, dt = 30$ means that the first car traveled 30 more meters than the second car between 0 and 10 seconds.

The integral $\int_{20}^{30} [v_1(t) - v_2(t)]\, dt = -5$ means that the second car traveled 5 more meters than the first car between 20 and 30 seconds.

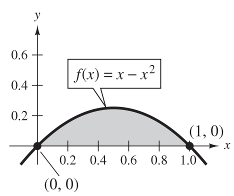
(b) No. You do not know when both cars started or the initial distance between the cars.

(c) The car with velocity v_1 is ahead by 30 meters.

(d) Car 1 is ahead by 8 meters.

83. $b = 9\left(1 - 1/\sqrt[3]{4}\right) \approx 3.330$ **85.** $a = 4 - 2\sqrt{2} \approx 1.172$

87. Answers will vary. Sample answer: $\frac{1}{6}$



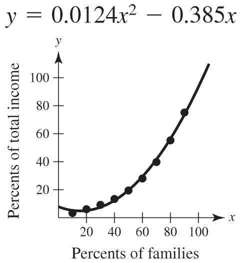
89. (a) $(-2, -11), (0, 7)$ (b) $y = 9x + 7$

(c) 3.2, 6.4, 3.2; The area between the two inflection points is the sum of the areas between the other two regions.

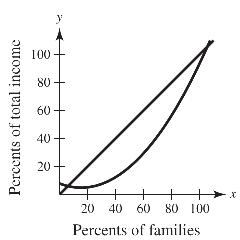
91. \$6.825 billion

93. (a) $y = 0.0124x^2 - 0.385x + 7.85$

(b)



(c)



(d) About 2006.7

95. $\frac{16}{3}\left(4\sqrt{2} - 5\right) \approx 3.503$

97. (a) About 6.031 m^2 (b) About 12.062 m^3 (c) 60,310 lb

99. True

101. False. Let $f(x) = x$ and $g(x) = 2x - x^2$. f and g intersect at $(1, 1)$, the midpoint of $[0, 2]$, but

$$\int_a^b [f(x) - g(x)]\, dx = \int_0^2 [x - (2x - x^2)]\, dx = \tfrac{2}{3} \ne 0.$$

103. $\sqrt{3}/2 + 7\pi/24 + 1 \approx 2.7823$

105. Putnam Problem A1, 1993

Section 7.2 (page 465)

1. $\pi\displaystyle\int_0^1 (-x + 1)^2\, dx = \frac{\pi}{3}$ **3.** $\pi\displaystyle\int_1^4 \left(\sqrt{x}\right)^2 dx = \frac{15\pi}{2}$

5. $\pi\displaystyle\int_0^1 [(x^2)^2 - (x^5)^2]\, dx = \frac{6\pi}{55}$ **7.** $\pi\displaystyle\int_0^4 \left(\sqrt{y}\right)^2 dy = 8\pi$

9. $\pi\displaystyle\int_0^1 (y^{3/2})^2\, dy = \frac{\pi}{4}$

11. (a) $9\pi/2$ (b) $\left(36\pi\sqrt{3}\right)/5$ (c) $\left(24\pi\sqrt{3}\right)/5$ (d) $\left(84\pi\sqrt{3}\right)/5$

13. (a) $32\pi/3$ (b) $64\pi/3$ **15.** 18π

17. $\pi\left(48 \ln 2 - \frac{27}{4}\right) \approx 83.318$ **19.** $124\pi/3$ **21.** $832\pi/15$

23. $\pi \ln 5$ **25.** $2\pi/3$ **27.** $(\pi/2)(1 - 1/e^2) \approx 1.358$

29. $277\pi/3$ **31.** 8π **33.** $\pi^2/2 \approx 4.935$

35. $(\pi/2)(e^2 - 1) \approx 10.036$ **37.** 1.969 **39.** 15.4115

41. $\pi/3$ **43.** $2\pi/15$ **45.** $\pi/2$ **47.** $\pi/6$

49. (a) The area appears to be close to 1 and therefore the volume (area squared $\times$ π) is near 3.

51. A sine curve on $[0, \pi/2]$ revolved about the x-axis

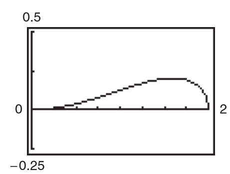
53. The parabola $y = 4x - x^2$ is a horizontal translation of the parabola $y = 4 - x^2$. Therefore, their volumes are equal.

55. (a) This statement is true. Explanations will vary.

(b) This statement is false. Explanations will vary.

57. 18π **59.** Proof **61.** $\pi r^2 h[1 - (h/H) + h^2/(3H^2)]$

63.

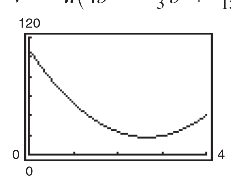


$\pi/30$

65. (a) 60π (b) 50π

67. (a) $V = \pi\left(4b^2 - \frac{64}{3}b + \frac{512}{15}\right)$

(b)



$b \approx 2.67$

(c) $b = \frac{8}{3} \approx 2.67$

69. (a) ii; right circular cylinder of radius r and height h

(b) iv; ellipsoid whose underlying ellipse has the equation $(x/b)^2 + (y/a)^2 = 1$

(c) iii; sphere of radius r

(d) i; right circular cone of radius r and height h

(e) v; torus of cross-sectional radius r and other radius R

71. (a) $\frac{81}{10}$ (b) $\frac{9}{2}$ **73.** $\frac{16}{3}r^3$ **75.** $V = \frac{4}{3}\pi(R^2 - r^2)^{3/2}$

77. 19.7443 **79.** (a) $\frac{2}{3}r^3$ (b) $\frac{2}{3}r^3 \tan\theta$; As $\theta \to 90°$, $V \to \infty$.

Section 7.3 (page 474)

1. $2\pi\displaystyle\int_0^2 x^2\, dx = \frac{16\pi}{3}$ **3.** $2\pi\displaystyle\int_0^4 x\sqrt{x}\, dx = \frac{128\pi}{5}$

5. $2\pi\displaystyle\int_0^3 x^3\, dx = \frac{81}{2}\pi$ **7.** $2\pi\displaystyle\int_0^2 x(4x - 2x^2)\, dx = \frac{16\pi}{3}$

9. $2\pi\int_0^2 x(x^2 - 4x + 4)\,dx = \frac{8\pi}{3}$

11. $2\pi\int_2^4 x\sqrt{x-2}\,dx = \frac{128\pi}{15}\sqrt{2}$

13. $2\pi\int_0^1 x\left(\frac{1}{\sqrt{2\pi}}e^{-x^2/2}\right)dx = \sqrt{2\pi}\left(1 - \frac{1}{\sqrt{e}}\right) \approx 0.986$

15. $2\pi\int_0^2 y(2 - y)\,dy = \frac{8\pi}{3}$

17. $2\pi\left[\int_0^{1/2} y\,dy + \int_{1/2}^1 y\left(\frac{1}{y} - 1\right)dy\right] = \frac{\pi}{2}$

19. $2\pi\left[\int_0^8 y^{4/3}\,dy\right] = \frac{768\pi}{7}$

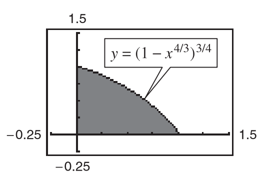
21. $2\pi\int_0^2 y(4 - 2y)\,dy = 16\pi/3$ **23.** 64π **25.** 16π

27. Shell method; it is much easier to put x in terms of y rather than vice versa.

29. (a) $128\pi/7$ (b) $64\pi/5$ (c) $96\pi/5$

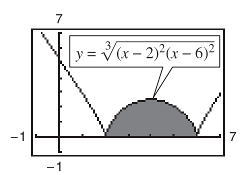
31. (a) $\pi a^3/15$ (b) $\pi a^3/15$ (c) $4\pi a^3/15$

33. (a)



(b) 1.506

35. (a)



(b) 187.25

37. d **39.** a, c, b

41. Both integrals yield the volume of the solid generated by revolving the region bounded by the graphs of $y = \sqrt{x-1}$, $y = 0$, and $x = 5$ about the x-axis.

43. (a) The rectangles would be vertical.
(b) The rectangles would be horizontal.

45. Diameter $= 2\sqrt{4 - 2\sqrt{3}} \approx 1.464$ **47.** $4\pi^2$

49. (a) Region bounded by $y = x^2$, $y = 0$, $x = 0$, $x = 2$
(b) Revolved about the y-axis

51. (a) Region bounded by $x = \sqrt{6 - y}$, $y = 0$, $x = 0$
(b) Revolved about $y = -2$

53. (a) Proof (b) (i) $V = 2\pi$ (ii) $V = 6\pi^2$

55. Proof

57. (a) $R_1(n) = n/(n + 1)$ (b) $\lim_{n\to\infty} R_1(n) = 1$
(c) $V = \pi ab^{n+2}[n/(n + 2)]$; $R_2(n) = n/(n + 2)$
(d) $\lim_{n\to\infty} R_2(n) = 1$
(e) As $n \to \infty$, the graph approaches the line $x = b$.

59. (a) and (b) About 121,475 ft^3 **61.** $c = 2$

63. (a) $64\pi/3$ (b) $2048\pi/35$ (c) $8192\pi/105$

Section 7.4 (page 485)

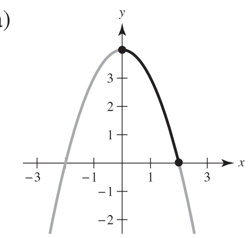
1. (a) and (b) 17 **3.** $\frac{5}{3}$ **5.** $\frac{2}{3}\left(2\sqrt{2} - 1\right) \approx 1.219$

7. $5\sqrt{5} - 2\sqrt{2} \approx 8.352$ **9.** 309.3195

11. $\ln\left[\left(\sqrt{2} + 1\right)/\left(\sqrt{2} - 1\right)\right] \approx 1.763$

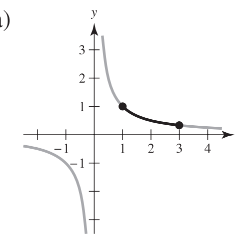
13. $\frac{1}{2}(e^2 - 1/e^2) \approx 3.627$ **15.** $\frac{76}{3}$

17. (a)



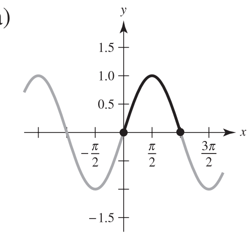
(b) $\int_0^2 \sqrt{1 + 4x^2}\,dx$
(c) About 4.647

19. (a)



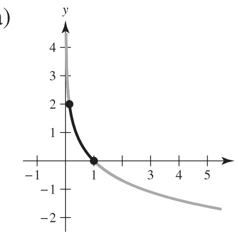
(b) $\int_1^3 \sqrt{1 + \frac{1}{x^4}}\,dx$
(c) About 2.147

21. (a)



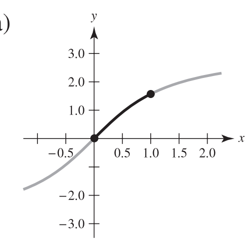
(b) $\int_0^{\pi} \sqrt{1 + \cos^2 x}\,dx$
(c) About 3.820

23. (a)



(b) $\int_0^2 \sqrt{1 + e^{-2y}}\,dy$
$= \int_{e^{-2}}^1 \sqrt{1 + \frac{1}{x^2}}\,dx$
(c) About 2.221

25. (a)



(b) $\int_0^1 \sqrt{1 + \left(\frac{2}{1 + x^2}\right)^2}\,dx$
(c) About 1.871

27. b **29.** (a) 64.125 (b) 64.525 (c) 64.666 (d) 64.672

31. $20[\sinh 1 - \sinh(-1)] \approx 47.0$ m **33.** About 1480

35. $3 \arcsin \frac{2}{3} \approx 2.1892$

37. $2\pi\int_0^3 \frac{1}{3}x^3\sqrt{1 + x^4}\,dx = \frac{\pi}{9}\left(82\sqrt{82} - 1\right) \approx 258.85$

39. $2\pi\int_1^2 \left(\frac{x^3}{6} + \frac{1}{2x}\right)\left(\frac{x^2}{2} + \frac{1}{2x^2}\right)dx = \frac{47\pi}{16} \approx 9.23$

41. $2\pi\int_{-1}^1 2\,dx = 8\pi \approx 25.13$

43. $2\pi\int_1^8 x\sqrt{1 + \frac{1}{9x^{4/3}}}\,dx = \frac{\pi}{27}\left(145\sqrt{145} - 10\sqrt{10}\right) \approx 199.48$

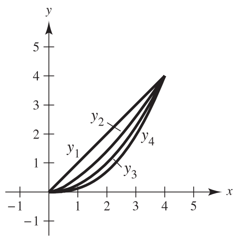
45. $2\pi\int_0^2 x\sqrt{1 + \frac{x^2}{4}}\,dx = \frac{\pi}{3}\left(16\sqrt{2} - 8\right) \approx 15.318$

47. 14.424

49. A rectifiable curve is a curve with a finite arc length.

51. The integral formula for the area of a surface of revolution is derived from the formula for the lateral surface area of the frustum of a right circular cone. The formula is $S = 2\pi rL$, where $r = \frac{1}{2}(r_1 + r_2)$, which is the average radius of the frustum, and L is the length of a line segment on the frustum. The representative element is $2\pi f(d_i)\sqrt{1 + (\Delta y_i/\Delta x_i)^2}\,\Delta x_i$.

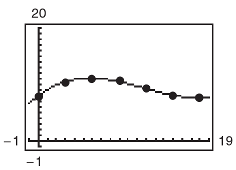
53. (a)



(b) y_1, y_2, y_3, y_4
(c) $s_1 \approx 5.657$; $s_2 \approx 5.759$; $s_3 \approx 5.916$; $s_4 \approx 6.063$

55. 20π **57.** $6\pi(3 - \sqrt{5}) \approx 14.40$

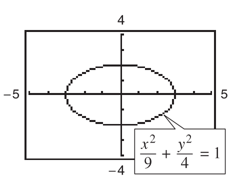
59. (a) Answers will vary. Sample answer: 5207.62 in.^{3}
(b) Answers will vary. Sample answer: 1168.64 in.^{2}
(c) $r = 0.0040y^3 - 0.142y^2 + 1.23y + 7.9$



(d) 5279.64 in.^{3}; 1179.5 in.^{2}

61. (a) $\pi(1 - 1/b)$ (b) $2\pi \int_1^b \sqrt{x^4 + 1}/x^3\, dx$
(c) $\lim\limits_{b\to\infty} V = \lim\limits_{b\to\infty} \pi(1 - 1/b) = \pi$
(d) Because $\dfrac{\sqrt{x^4 + 1}}{x^3} > \dfrac{\sqrt{x^4}}{x^3} = \dfrac{1}{x} > 0$ on $[1, b]$,
you have $\displaystyle\int_1^b \frac{\sqrt{x^4 + 1}}{x^3}\, dx > \int_1^b \frac{1}{x}\, dx = \Big[\ln x\Big]_1^b = \ln b$
and $\lim\limits_{b\to\infty} \ln b \to \infty$. So, $\lim\limits_{b\to\infty} 2\pi \displaystyle\int_1^b \frac{\sqrt{x^4 + 1}}{x^3}\, dx = \infty$.

63. (a)



(b) $\displaystyle\int_0^3 \sqrt{1 + \frac{4x^2}{81 - 9x^2}}\, dx$

(c) You cannot evaluate this definite integral because the integrand is not defined at $x = 3$. Simpson's Rule will not work for the same reason.

65. Fleeing object: $\frac{2}{3}$ unit
Pursuer: $\displaystyle\frac{1}{2}\int_0^1 \frac{x + 1}{\sqrt{x}}\, dx = \frac{4}{3} = 2\left(\frac{2}{3}\right)$

67. $384\pi/5$ **69.** Proof **71.** Proof

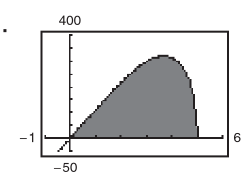
Section 7.5 (page 495)

1. 2000 ft-lb **3.** 896 N-m
5. 40.833 in.-lb ≈ 3.403 ft-lb **7.** 8750 N-cm = 87.5 N-m
9. 160 in.-lb ≈ 13.3 ft-lb **11.** 37.125 ft-lb
13. (a) 487.805 mile-tons ≈ 5.151×10^9 ft-lb
(b) 1395.349 mile-tons ≈ 1.473×10^{10} ft-lb
15. (a) 2.93×10^4 mile-tons ≈ 3.10×10^{11} ft-lb
(b) 3.38×10^4 mile-tons ≈ 3.57×10^{11} ft-lb
17. (a) 2496 ft-lb (b) 9984 ft-lb **19.** $470{,}400\pi$ N-m
21. 2995.2π ft-lb **23.** $20{,}217.6\pi$ ft-lb **25.** 2457π ft-lb
27. 600 ft-lb **29.** 450 ft-lb **31.** 168.75 ft-lb
33. If an object is moved a distance D in the direction of an applied constant force F, then the work W done by the force is defined as $W = FD$.
35. The situation in part (a) requires more work. There is no work required for part (b) because the distance is 0.
37. (a) 54 ft-lb (b) 160 ft-lb (c) 9 ft-lb (d) 18 ft-lb
39. $2000 \ln(3/2) \approx 810.93$ ft-lb **41.** $3k/4$
43. 3249.4 ft-lb **45.** 10,330.3 ft-lb

Section 7.6 (page 506)

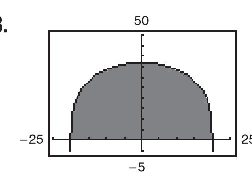
1. $\bar{x} = -\frac{4}{3}$ **3.** $\bar{x} = 12$ **5.** (a) $\bar{x} = 16$ (b) $\bar{x} = -2$
7. $x = 6$ ft **9.** $(\bar{x}, \bar{y}) = \left(\frac{10}{9}, -\frac{1}{9}\right)$ **11.** $(\bar{x}, \bar{y}) = \left(2, \frac{48}{25}\right)$
13. $M_x = \rho/3$, $M_y = 4\rho/3$, $(\bar{x}, \bar{y}) = (4/3, 1/3)$
15. $M_x = 4\rho$, $M_y = 64\rho/5$, $(\bar{x}, \bar{y}) = (12/5, 3/4)$
17. $M_x = \rho/35$, $M_y = \rho/20$, $(\bar{x}, \bar{y}) = (3/5, 12/35)$
19. $M_x = 99\rho/5$, $M_y = 27\rho/4$, $(\bar{x}, \bar{y}) = (3/2, 22/5)$
21. $M_x = 192\rho/7$, $M_y = 96\rho$, $(\bar{x}, \bar{y}) = (5, 10/7)$
23. $M_x = 0$, $M_y = 256\rho/15$, $(\bar{x}, \bar{y}) = (8/5, 0)$
25. $M_x = 27\rho/4$, $M_y = -27\rho/10$, $(\bar{x}, \bar{y}) = (-3/5, 3/2)$
27. $A = \displaystyle\int_0^2 (2x - x^2)\, dx = \frac{4}{3}$
$M_x = \displaystyle\int_0^2 \left(\frac{2x + x^2}{2}\right)(2x - x^2)\, dx = \frac{32}{15}$
$M_y = \displaystyle\int_0^2 x(2x - x^2)\, dx = \frac{4}{3}$
29. $A = \displaystyle\int_0^3 (2x + 4)\, dx = 21$
$M_x = \displaystyle\int_0^3 \left(\frac{2x + 4}{2}\right)(2x + 4)\, dx = 78$
$M_y = \displaystyle\int_0^3 x(2x + 4)\, dx = 36$

31.



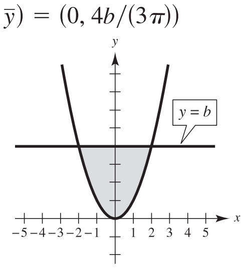
$(\bar{x}, \bar{y}) = (3.0, 126.0)$

33.



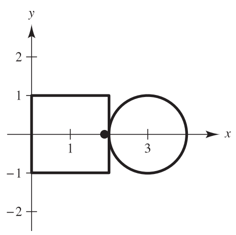
$(\bar{x}, \bar{y}) = (0, 16.2)$

35. $(\bar{x}, \bar{y}) = \left(\dfrac{b}{3}, \dfrac{c}{3}\right)$ **37.** $(\bar{x}, \bar{y}) = \left(\dfrac{(a + 2b)c}{3(a + b)}, \dfrac{a^2 + ab + b^2}{3(a + b)}\right)$
39. $(\bar{x}, \bar{y}) = (0, 4b/(3\pi))$
41. (a)



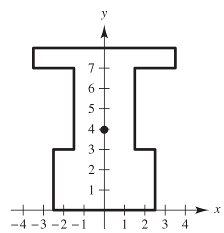
(b) $\bar{x} = 0$ by symmetry
(c) $M_y = \displaystyle\int_{-\sqrt{b}}^{\sqrt{b}} x(b - x^2)\, dx = 0$ because $x(b - x^2)$ is an odd function.
(d) $\bar{y} > b/2$ because the area is greater for $y > b/2$.
(e) $\bar{y} = (3/5)b$
43. (a) $(\bar{x}, \bar{y}) = (0, 12.98)$
(b) $y = (-1.02 \times 10^{-5})x^4 - 0.0019x^2 + 29.28$
(c) $(\bar{x}, \bar{y}) = (0, 12.85)$

45.



$(\bar{x}, \bar{y}) = \left(\dfrac{4 + 3\pi}{4 + \pi}, 0\right)$

47.



$(\bar{x}, \bar{y}) = \left(0, \dfrac{135}{34}\right)$

49. $(\bar{x}, \bar{y}) = \left(\dfrac{2 + 3\pi}{2 + \pi}, 0\right)$ **51.** $160\pi^2 \approx 1579.14$

53. $128\pi/3 \approx 134.04$

55. The center of mass $(\bar{x}, \bar{y})$ is $\bar{x} = M_y/m$ and $\bar{y} = M_x/m$, where:
1. $m = m_1 + m_2 + \cdots + m_n$ is the total mass of the system.
2. $M_y = m_1 x_1 + m_2 x_2 + \cdots + m_n x_n$ is the moment about the y-axis.
3. $M_x = m_1 y_1 + m_2 y_2 + \cdots + m_n y_n$ is the moment about the x-axis.

57. See Theorem 7.1 on page 505. **59.** $(\bar{x}, \bar{y}) = (0, 2r/\pi)$

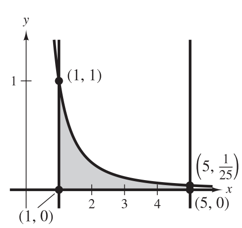
61. $(\bar{x}, \bar{y}) = \left(\dfrac{n+1}{n+2}, \dfrac{n+1}{4n+2}\right)$; As $n \to \infty$, the region shrinks toward the line segments $y = 0$ for $0 \le x \le 1$ and $x = 1$ for $0 \le y \le 1$; $(\bar{x}, \bar{y}) \to \left(1, \dfrac{1}{4}\right)$.

Section 7.7 (page 513)

1. 1497.6 lb **3.** 4992 lb **5.** 748.8 lb **7.** 1123.2 lb
9. 748.8 lb **11.** 1064.96 lb **13.** 117,600 N
15. 2,381,400 N **17.** 2814 lb **19.** 6753.6 lb **21.** 94.5 lb
23. Proof **25.** Proof **27.** 960 lb
29. Answers will vary. Sample answer (using Simpson's Rule): 3010.8 lb
31. 8213.0 lb
33. $3\sqrt{2}/2 \approx 2.12$ ft; The pressure increases with increasing depth.
35. Because you are measuring total force against a region between two depths.

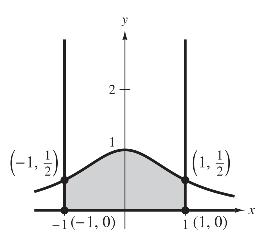
Review Exercises for Chapter 7 (page 515)

1.



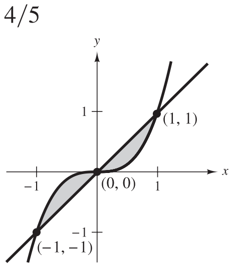
$4/5$

3.



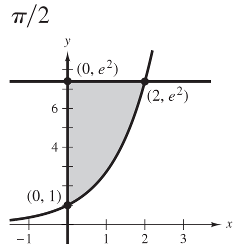
$\pi/2$

5.



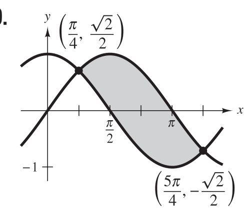
$\frac{1}{2}$

7.



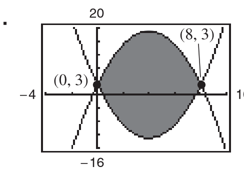
$e^2 + 1$

9.



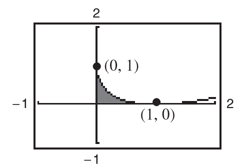
$2\sqrt{2}$

11.



$\frac{512}{3}$

13.



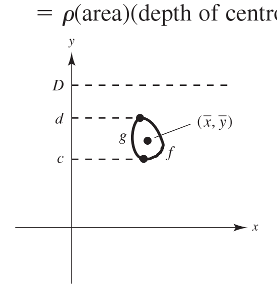
$\frac{1}{6}$

15. $\displaystyle\int_0^2 [0 - (y^2 - 2y)]\, dy = \int_{-1}^0 2\sqrt{x+1}\, dx = \frac{4}{3}$

17. $\displaystyle\int_0^2 \left[1 - \left(1 - \frac{x}{2}\right)\right] dx + \int_2^3 [1 - (x - 2)]\, dx$
$\displaystyle = \int_0^1 [(y + 2) - (2 - 2y)]\, dy = \frac{3}{2}$

19. (a) 9920 ft^2 (b) $10{,}413\frac{1}{3}$ ft^2
21. (a) 9π (b) 18π (c) 9π (d) 36π
23. (a) 64π (b) 48π **25.** $\pi^2/4$
27. $(4\pi/3)(20 - 9\ln 3) \approx 42.359$
29. (a) $\dfrac{4}{15}$ (b) $\dfrac{\pi}{12}$ (c) $\dfrac{32\pi}{105}$ **31.** 1.958 ft
33. $\frac{8}{15}\left(1 + 6\sqrt{3}\right) \approx 6.076$ **35.** 4018.2 ft
37. 15π **39.** 62.5 in.-lb $\approx$ 5.208 ft-lb
41. $122{,}980\pi$ ft-lb $\approx$ 193.2 foot-tons **43.** 200 ft-lb
45. $a = 15/4$ **47.** $(\bar{x}, \bar{y}) = (a/5, a/5)$ **49.** $(\bar{x}, \bar{y}) = (0, 2a^2/5)$
51. $(\bar{x}, \bar{y}) = \left(\dfrac{2(9\pi + 49)}{3(\pi + 9)}, 0\right)$
53. Let D = surface of liquid; ρ = weight per cubic volume.

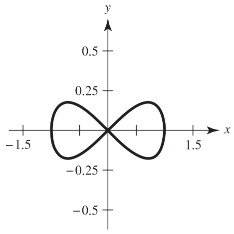
$$F = \rho\int_c^d (D - y)[f(y) - g(y)]\, dy$$
$$= \rho\left[\int_c^d D[f(y) - g(y)]\, dy - \int_c^d y[f(y) - g(y)]\, dy\right]$$
$$= \rho\left[\int_c^d [f(y) - g(y)]\, dy\right]\left[D - \frac{\int_c^d y[f(y) - g(y)]\, dy}{\int_c^d [f(y) - g(y)]\, dy}\right]$$
$$= \rho(\text{area})(D - \bar{y})$$
$$= \rho(\text{area})(\text{depth of centroid})$$



P.S. Problem Solving (page 517)

1. 3 **3.** $y = 0.2063x$

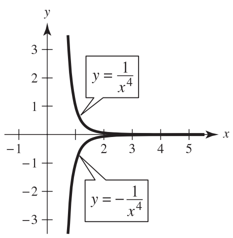
5. $\dfrac{5\sqrt{2}\pi}{3}$



7. $V = 2\pi\left[d + \frac{1}{2}\sqrt{w^2 + l^2}\right]lw$ **9.** $f(x) = 2e^{x/2} - 2$

11. 89.3%

13.



(a) $(\bar{x}, \bar{y}) = \left(\dfrac{63}{43}, 0\right)$

(b) $(\bar{x}, \bar{y}) = \left(\dfrac{3b(b+1)}{2(b^2+b+1)}, 0\right)$

(c) $\left(\dfrac{3}{2}, 0\right)$

15. Consumer surplus: 1600; Producer surplus: 400

17. Wall at shallow end: 9984 lb
Wall at deep end: 39,936 lb
Side wall: 19,968 + 26,624 = 46,592 lb

Chapter 8

Section 8.1 (page 524)

1. b **3.** c

5. $\displaystyle\int u^n\,du$; $u = 5x - 3, n = 4$ **7.** $\displaystyle\int \frac{du}{u}$; $u = 1 - 2\sqrt{x}$ **9.** $\displaystyle\int \frac{du}{\sqrt{a^2 - u^2}}$; $u = t, a = 1$

11. $\displaystyle\int \sin u\,du$; $u = t^2$ **13.** $\displaystyle\int e^u\,du$; $u = \sin x$ **15.** $2(x-5)^7 + C$

17. $-7/[6(z-10)^6] + C$ **19.** $\frac{1}{2}v^2 - 1/[6(3v-1)^2] + C$

21. $-\frac{1}{3}\ln|-t^3 + 9t + 1| + C$ **23.** $\frac{1}{2}x^2 + x + \ln|x-1| + C$

25. $\ln(1 + e^x) + C$ **27.** $\dfrac{x}{15}(48x^4 + 200x^2 + 375) + C$

29. $\sin(2\pi x^2)/(4\pi) + C$ **31.** $-(1/\pi)\csc \pi x + C$

33. $\frac{1}{11}e^{11x} + C$ **35.** $2\ln(1 + e^x) + C$ **37.** $(\ln x)^2 + C$

39. $-\ln(1 - \sin x) + C = \ln|\sec x(\sec x + \tan x)| + C$

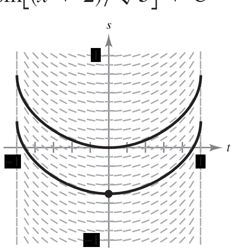
41. $\csc\theta + \cot\theta + C = (1 + \cos\theta)/\sin\theta + C$

43. $-\frac{1}{4}\arcsin(4t + 1) + C$ **45.** $\frac{1}{2}\ln|\cos(2/t)| + C$

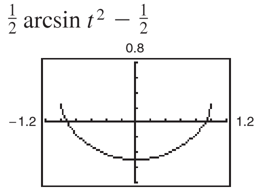
47. $6\arcsin[(x-5)/5] + C$ **49.** $\frac{1}{4}\arctan[(2x+1)/8] + C$

51. $\arcsin\left[(x+2)/\sqrt{5}\right] + C$

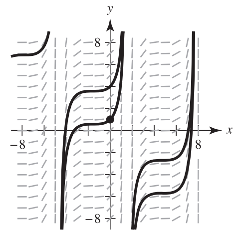
53. (a)



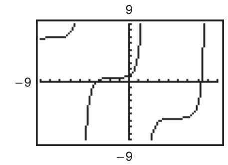
(b) $\frac{1}{2}\arcsin t^2 - \frac{1}{2}$



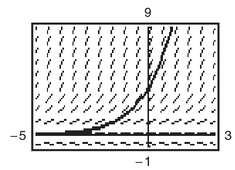
55. (a)



(b) $2\tan x + 2\sec x - x - 1 + C$



57. $y = 4e^{0.8x}$



59. $y = \frac{1}{2}e^{2x} + 10e^x + 25x + C$

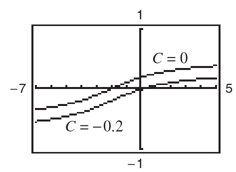
61. $r = 10\arcsin e^t + C$ **63.** $y = \frac{1}{2}\arctan(\tan x/2) + C$

65. $\frac{1}{2}$ **67.** $\frac{1}{2}(1 - e^{-1}) \approx 0.316$ **69.** 8 **71.** $\pi/18$

73. $18\sqrt{6}/5 \approx 8.82$ **75.** $\frac{3}{2}\ln\left(\frac{34}{9}\right) + \frac{2}{3}\arctan\left(\frac{5}{3}\right) \approx 2.68$

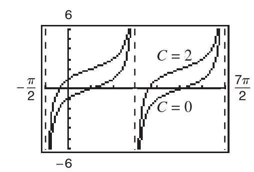
77. $\frac{4}{3} \approx 1.333$

79. $\frac{1}{3}\arctan\left[\frac{1}{3}(x+2)\right] + C$
Graphs will vary.
Example:



One graph is a vertical translation of the other.

81. $\tan\theta - \sec\theta + C$
Graphs will vary.
Example:



One graph is a vertical translation of the other.

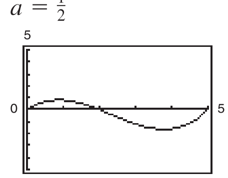
83. Power Rule: $\displaystyle\int u^n\,du = \frac{u^{n+1}}{n+1} + C$; $u = x^2 + 1, n = 3$

85. Log Rule: $\displaystyle\int \frac{du}{u} = \ln|u| + C$; $u = x^2 + 1$

87. $a = \sqrt{2}, b = \dfrac{\pi}{4}$; $-\dfrac{1}{\sqrt{2}}\ln\left|\csc\left(x + \dfrac{\pi}{4}\right) + \cot\left(x + \dfrac{\pi}{4}\right)\right| + C$

89. $a = \frac{1}{2}$

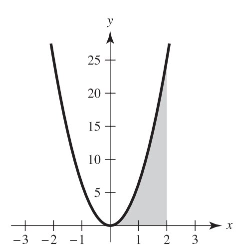
91.



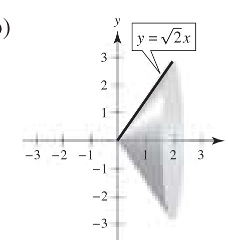
Negative; more area below the x-axis than above

93. a

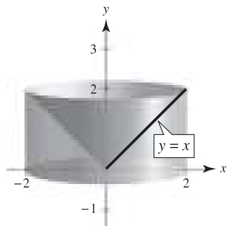
95. (a)



(b)



(c)



97. (a) $\pi(1 - e^{-1}) \approx 1.986$ (b) $b = \sqrt{\ln\left(\dfrac{3\pi}{3\pi - 4}\right)} \approx 0.743$

99. $\ln\left(\sqrt{2} + 1\right) \approx 0.8814$

101. $(8\pi/3)\left(10\sqrt{10} - 1\right) \approx 256.545$

103. $\frac{1}{3} \arctan 3 \approx 0.416$ **105.** About 1.0320

107. (a) $\frac{1}{3} \sin x(\cos^2 x + 2)$

(b) $\frac{1}{15} \sin x(3 \cos^4 x + 4 \cos^2 x + 8)$

(c) $\frac{1}{35} \sin x(5 \cos^6 x + 6 \cos^4 x + 8 \cos^2 x + 16)$

(d) $\displaystyle\int \cos^{15} x\, dx = \int (1 - \sin^2 x)^7 \cos x\, dx$

You would expand $(1 - \sin^2 x)^7$.

109. Proof

Section 8.2 (page 533)

1. $u = x, dv = e^{2x}\,dx$ **3.** $u = (\ln x)^2, dv = dx$

5. $u = x, dv = \sec^2 x\, dx$ **7.** $\frac{1}{16}x^4(4 \ln x - 1) + C$

9. $\dfrac{1}{9} \sin 3x - \dfrac{1}{3}x \cos 3x + C$ **11.** $-\dfrac{1}{16e^{4x}}(4x + 1) + C$

13. $e^x(x^3 - 3x^2 + 6x - 6) + C$

15. $\frac{1}{3}e^{x^3} + C$ **17.** $\frac{1}{4}[2(t^2 - 1) \ln|t + 1| - t^2 + 2t] + C$

19. $\frac{1}{3}(\ln x)^3 + C$ **21.** $e^{2x}/[4(2x + 1)] + C$

23. $(x - 1)^2 e^x + C$ **25.** $\frac{2}{15}(x - 5)^{3/2}(3x + 10) + C$

27. $x \sin x + \cos x + C$

29. $(6x - x^3) \cos x + (3x^2 - 6) \sin x + C$

31. $-t \csc t - \ln|\csc t + \cot t| + C$

33. $x \arctan x - \frac{1}{2}\ln(1 + x^2) + C$

35. $\frac{1}{5}e^{2x}(2 \sin x - \cos x) + C$

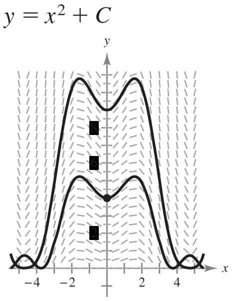
37. $\frac{1}{5}e^{-x}(2 \sin 2x - \cos 2x) + C$ **39.** $y = \frac{1}{2}e^{x^2} + C$

41. $y = \dfrac{2}{5}t^2\sqrt{3 + 5t} - \dfrac{8t}{75}(3 + 5t)^{3/2} + \dfrac{16}{1875}(3 + 5t)^{5/2} + C$

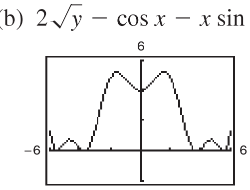
$= \dfrac{2}{625}\sqrt{3 + 5t}(25t^2 - 20t + 24) + C$

43. $\sin y = x^2 + C$

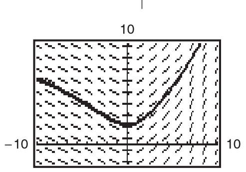
45. (a)



(b) $2\sqrt{y} - \cos x - x \sin x = 3$



47.



49. $2e^{3/2} + 4 \approx 12.963$

51. $\dfrac{\pi}{8} - \dfrac{1}{4} \approx 0.143$

53. $\left(\pi - 3\sqrt{3} + 6\right)/6 \approx 0.658$

55. $\frac{1}{2}[e(\sin 1 - \cos 1) + 1] \approx 0.909$

57. $\frac{4}{3}\sqrt{2} \ln 2 - \frac{8}{9}\sqrt{2} + \frac{4}{9} \approx 0.494$

59. $8 \operatorname{arcsec} 4 + \sqrt{3}/2 - \sqrt{15}/2 - 2\pi/3 \approx 7.380$

61. $(e^{2x}/4)(2x^2 - 2x + 1) + C$

63. $(3x^2 - 6) \sin x - (x^3 - 6x) \cos x + C$

65. $x \tan x + \ln|\cos x| + C$ **67.** $2\left(\sin\sqrt{x} - \sqrt{x} \cos \sqrt{x}\right) + C$

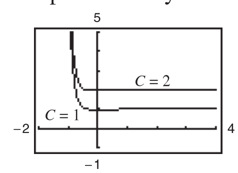
69. $\frac{128}{15}$ **71.** $\frac{1}{2}(x^4 e^{x^2} - 2x^2 e^{x^2} + 2e^{x^2}) + C$

73. $\frac{1}{2}x[\cos(\ln x) + \sin(\ln x)] + C$ **75.** Product Rule

77. In order for the integration by parts technique to be efficient, you want dv to be the most complicated portion of the integrand and you want u to be the portion of the integrand whose derivative is a function simpler than u. If you let $u = \sin x$, then du is not a simpler function.

79. (a) $-(e^{-4t}/128)(32t^3 + 24t^2 + 12t + 3) + C$

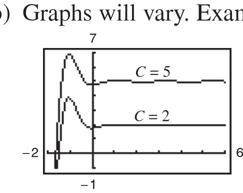
(b) Graphs will vary. Example:



(c) One graph is a vertical translation of the other.

81. (a) $\frac{1}{13}(2e^{-\pi} + 3) \approx 0.2374$

(b) Graphs will vary. Example:



(c) One graph is a vertical translation of the other.

83. $\frac{2}{5}(2x - 3)^{3/2}(x + 1) + C$ **85.** $\frac{1}{3}\sqrt{4 + x^2}(x^2 - 8) + C$

87. $n = 0$: $x(\ln x - 1) + C$

$n = 1$: $\frac{1}{4}x^2(2 \ln x - 1) + C$

$n = 2$: $\frac{1}{9}x^3(3 \ln x - 1) + C$

$n = 3$: $\frac{1}{16}x^4(4 \ln x - 1) + C$

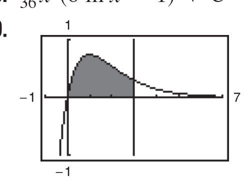
$n = 4$: $\frac{1}{25}x^5(5 \ln x - 1) + C$

$\displaystyle\int x^n \ln x\, dx = \frac{x^{n+1}}{(n + 1)^2}[(n + 1) \ln x - 1] + C$

89–93. Proofs

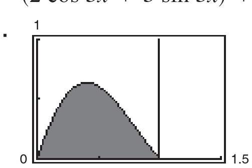
95. $\frac{1}{36}x^6(6 \ln x - 1) + C$ **97.** $\frac{1}{13}e^{2x}(2 \cos 3x + 3 \sin 3x) + C$

99.



$2 - \dfrac{8}{e^3} \approx 1.602$

101.



$\dfrac{\pi}{1 + \pi^2}\left(\dfrac{1}{e} + 1\right) \approx 0.395$

103. (a) 1 (b) $\pi(e - 2) \approx 2.257$ (c) $\frac{1}{2}\pi(e^2 + 1) \approx 13.177$

(d) $\left(\dfrac{e^2 + 1}{4}, \dfrac{e - 2}{2}\right) \approx (2.097, 0.359)$

105. In Example 6, we showed that the centroid of an equivalent region was $(1, \pi/8)$. By symmetry, the centroid of this region is $(\pi/8, 1)$.

107. $[7/(10\pi)](1 - e^{-4\pi}) \approx 0.223$ **109.** \$931,265

111. Proof **113.** $b_n = [8h/(n\pi)^2] \sin(n\pi/2)$

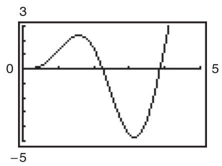
115. Shell: $V = \pi\left[b^2 f(b) - a^2 f(a) - \int_a^b x^2 f'(x)\,dx\right]$

Disk: $V = \pi\left[b^2 f(b) - a^2 f(a) - \int_{f(a)}^{f(b)} [f^{-1}(y)]^2\,dy\right]$

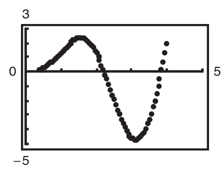
Both methods yield the same volume because $x = f^{-1}(y)$, $f'(x)\,dx = dy$, if $y = f(a)$ then $x = a$, and if $y = f(b)$ then $x = b$.

117. (a) $y = \frac{1}{4}(3 \sin 2x - 6x \cos 2x)$

(b)

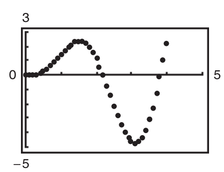


(c) You obtain the following points.



n	x_n	y_n
0	0	0
1	0.05	0
2	0.10	7.4875×10^{-4}
3	0.15	0.0037
4	0.20	0.0104
⋮	⋮	⋮
80	4.00	1.3181

(d) You obtain the following points.



n	x_n	y_n
0	0	0
1	0.1	0
2	0.2	0.0060
3	0.3	0.0293
4	0.4	0.0801
⋮	⋮	⋮
40	4.0	1.0210

119. The graph of $y = x \sin x$ is below the graph of $y = x$ on $[0, \pi/2]$.

Section 8.3 (page 542)

1. c **2.** a **3.** d **4.** b **5.** $-\frac{1}{6}\cos^6 x + C$

7. $\frac{1}{16}\sin^8 2x + C$ **9.** $-\frac{1}{3}\cos^3 x + \frac{1}{5}\cos^5 x + C$

11. $-\frac{1}{3}(\cos 2\theta)^{3/2} + \frac{1}{7}(\cos 2\theta)^{7/2} + C$

13. $\frac{1}{12}(6x + \sin 6x) + C$

15. $\frac{3}{8}\alpha + \frac{1}{12}\sin 6\alpha + \frac{1}{96}\sin 12\alpha + C$

17. $\frac{1}{8}(2x^2 - 2x \sin 2x - \cos 2x) + C$ **19.** $\frac{16}{35}$

21. $63\pi/512$ **23.** $5\pi/32$ **25.** $\frac{1}{7}\ln|\sec 7x + \tan 7x| + C$

27. $\frac{1}{15}\tan 5x(3 + \tan^2 5x) + C$

29. $\left(\sec \pi x \tan \pi x + \ln|\sec \pi x + \tan \pi x|\right)/(2\pi) + C$

31. $\frac{1}{2}\tan^4(x/2) - \tan^2(x/2) - 2\ln|\cos(x/2)| + C$

33. $\frac{1}{2}\tan^2 x + C$ **35.** $\frac{1}{3}\tan^3 x + \frac{1}{5}\tan^5 x + C$

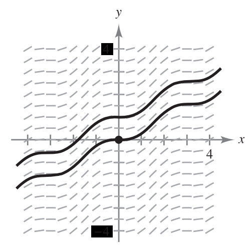
37. $\frac{1}{24}\sec^6 4x + C$ **39.** $\frac{1}{7}\sec^7 x - \frac{1}{5}\sec^5 x + C$

41. $\ln|\sec x + \tan x| - \sin x + C$

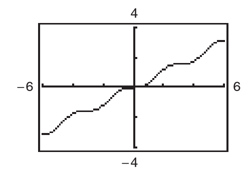
43. $(12\pi\theta - 8\sin 2\pi\theta + \sin 4\pi\theta)/(32\pi) + C$

45. $y = \frac{1}{9}\sec^3 3x - \frac{1}{3}\sec 3x + C$

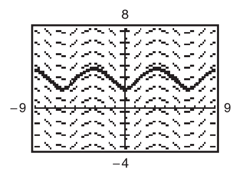
47. (a)



(b) $y = \frac{1}{2}x - \frac{1}{4}\sin 2x$



49.



51. $\frac{1}{16}(2 \sin 4x + \sin 8x) + C$

53. $\frac{1}{12}(3\cos 2x - \cos 6x) + C$ **55.** $\frac{1}{8}(2\sin 2\theta - \sin 4\theta) + C$

57. $\frac{1}{4}(\ln|\csc^2 2x| - \cot^2 2x) + C$ **59.** $-\frac{1}{2}\cot 2x - \frac{1}{6}\cot^3 2x + C$

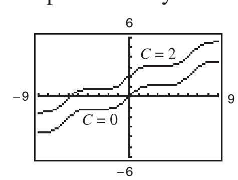
61. $\ln|\csc t - \cot t| + \cos t + C$

63. $\ln|\csc x - \cot x| + \cos x + C$ **65.** $t - 2\tan t + C$

67. π **69.** $3(1 - \ln 2)$ **71.** $\ln 2$ **73.** 4

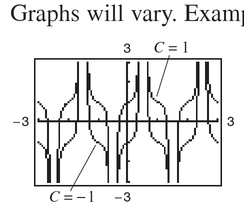
75. $\frac{1}{16}(6x + 8\sin x + \sin 2x) + C$

Graphs will vary. Example:



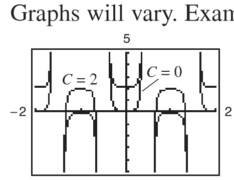
77. $\left[\sec^3 \pi x \tan \pi x + \frac{3}{2}\left(\sec \pi x \tan \pi x + \ln|\sec \pi x + \tan \pi x|\right)\right]/(4\pi) + C$

Graphs will vary. Example:



79. $(\sec^5 \pi x)/(5\pi) + C$ **81.** $2\sqrt{2}/7$ **83.** $3\pi/16$

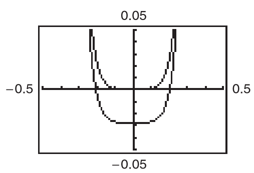
Graphs will vary. Example:



85. (a) Save one sine factor and convert the remaining factors to cosines. Then expand and integrate.
(b) Save one cosine factor and convert the remaining factors to sines. Then expand and integrate.
(c) Make repeated use of the power reducing formulas to convert the integrand to odd powers of the cosine. Then proceed as in part (b).

87. (a) $\frac{1}{2}\sin^2 x + C$ (b) $-\frac{1}{2}\cos^2 x + C$
(c) $\frac{1}{2}\sin^2 x + C$ (d) $-\frac{1}{4}\cos 2x + C$
The answers are all the same, they are just written in different forms. Using trigonometric identities, you can rewrite each answer in the same form.

89. (a) $\frac{1}{18}\tan^6 3x + \frac{1}{12}\tan^4 3x + C_1$, $\frac{1}{18}\sec^6 3x - \frac{1}{12}\sec^4 3x + C_2$
(b)



(c) Proof

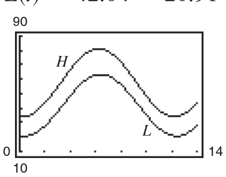
91. $\frac{1}{3}$ **93.** 1 **95.** $2\pi(1 - \pi/4) \approx 1.348$

97. (a) $\pi^2/2$ (b) $(\bar{x}, \bar{y}) = (\pi/2, \pi/8)$ **99–101.** Proofs

103. $-\frac{1}{15}\cos x(3\sin^4 x + 4\sin^2 x + 8) + C$

105. $\dfrac{5}{6\pi}\tan\dfrac{2\pi x}{5}\left(\sec^2\dfrac{2\pi x}{5} + 2\right) + C$

107. (a) $H(t) \approx 57.72 - 23.36\cos(\pi t/6) - 2.75\sin(\pi t/6)$
(b) $L(t) \approx 42.04 - 20.91\cos(\pi t/6) - 4.33\sin(\pi t/6)$
(c)



The maximum difference is at $t \approx 4.9$, or late spring.

109. Proof

Section 8.4 (page 551)

1. $x = 3\tan\theta$ **3.** $x = 4\sin\theta$ **5.** $x/\left(16\sqrt{16 - x^2}\right) + C$

7. $4\ln\left|\left(4 - \sqrt{16 - x^2}\right)/x\right| + \sqrt{16 - x^2} + C$

9. $\ln\left|x + \sqrt{x^2 - 25}\right| + C$

11. $\frac{1}{15}(x^2 - 25)^{3/2}(3x^2 + 50) + C$

13. $\frac{1}{3}(1 + x^2)^{3/2} + C$ **15.** $\frac{1}{2}[\arctan x + x/(1 + x^2)] + C$

17. $\frac{1}{2}x\sqrt{9 + 16x^2} + \frac{9}{8}\ln\left|4x + \sqrt{9 + 16x^2}\right| + C$

19. $\frac{25}{4}\arcsin(2x/5) + \frac{1}{2}x\sqrt{25 - 4x^2} + C$

21. $\sqrt{x^2 + 36} + C$ **23.** $\arcsin(x/4) + C$

25. $4\arcsin(x/2) + x\sqrt{4 - x^2} + C$ **27.** $\ln\left|x + \sqrt{x^2 - 4}\right| + C$

29. $-\dfrac{(1 - x^2)^{3/2}}{3x^3} + C$ **31.** $-\dfrac{1}{3}\ln\left|\dfrac{\sqrt{4x^2 + 9} + 3}{2x}\right| + C$

33. $3/\sqrt{x^2 + 3} + C$ **35.** $\frac{1}{3}(1 + e^{2x})^{3/2} + C$

37. $\frac{1}{2}\left(\arcsin e^x + e^x\sqrt{1 - e^{2x}}\right) + C$

39. $\frac{1}{4}\left[x/(x^2 + 2) + \left(1/\sqrt{2}\right)\arctan\left(x/\sqrt{2}\right)\right] + C$

41. $x \operatorname{arcsec} 2x - \frac{1}{2}\ln\left|2x + \sqrt{4x^2 - 1}\right| + C$

43. $\arcsin[(x - 2)/2] + C$

45. $\sqrt{x^2 + 6x + 12} - 3\ln\left|\sqrt{x^2 + 6x + 12} + (x + 3)\right| + C$

47. (a) and (b) $\sqrt{3} - \pi/3 \approx 0.685$

49. (a) and (b) $9\left(2 - \sqrt{2}\right) \approx 5.272$

51. (a) and (b) $-(9/2)\ln\left(2\sqrt{7}/3 - 4\sqrt{3}/3 - \sqrt{21}/3 + 8/3\right) + 9\sqrt{3} - 2\sqrt{7} \approx 12.644$

53. $\sqrt{x^2 - 9} - 3\arctan\left(\sqrt{x^2 - 9}/3\right) + 1$

55. $\frac{1}{2}(x - 15)\sqrt{x^2 + 10x + 9} + 33\ln\left|\sqrt{x^2 + 10x + 9} + (x + 5)\right| + C$

57. $\frac{1}{2}\left(x\sqrt{x^2 - 1} + \ln\left|x + \sqrt{x^2 - 1}\right|\right) + C$

59. (a) Let $u = a\sin\theta$, $\sqrt{a^2 - u^2} = a\cos\theta$, where $-\pi/2 \le \theta \le \pi/2$.
(b) Let $u = a\tan\theta$, $\sqrt{a^2 + u^2} = a\sec\theta$, where $-\pi/2 < \theta < \pi/2$.
(c) Let $u = a\sec\theta$, $\sqrt{u^2 - a^2} = \tan\theta$ if $u > a$ and $\sqrt{u^2 - a^2} = -\tan\theta$ if $u < -a$, where $0 \le \theta < \pi/2$ or $\pi/2 < \theta \le \pi$.

61. Trigonometric substitution: $x = \sec\theta$ **63.** True

65. False: $\displaystyle\int_0^{\sqrt{3}} \frac{dx}{(1 + x^2)^{3/2}} = \int_0^{\pi/3} \cos\theta\, d\theta$

67. πab **69.** (a) $5\sqrt{2}$ (b) $25(1 - \pi/4)$ (c) $r^2(1 - \pi/4)$

71. $6\pi^2$ **73.** $\ln\left[\dfrac{5\left(\sqrt{2} + 1\right)}{\sqrt{26} + 1}\right] + \sqrt{26} - \sqrt{2} \approx 4.367$

75. Length of one arch of sine curve: $y = \sin x$, $y' = \cos x$

$$L_1 = \int_0^{\pi} \sqrt{1 + \cos^2 x}\, dx$$

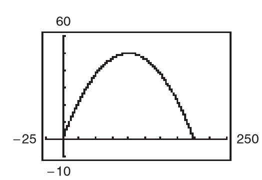
Length of one arch of cosine curve: $y = \cos x$, $y' = -\sin x$

$$L_2 = \int_{-\pi/2}^{\pi/2} \sqrt{1 + \sin^2 x}\, dx$$

$$= \int_{-\pi/2}^{\pi/2} \sqrt{1 + \cos^2(x - \pi/2)}\, dx, \quad u = x - \pi/2,\ du = dx$$

$$= \int_{-\pi}^{0} \sqrt{1 + \cos^2 u}\, du = \int_0^{\pi} \sqrt{1 + \cos^2 u}\, du = L_1$$

77. (a)



(b) 200

(c) $100\sqrt{2} + 50\ln\left[\left(\sqrt{2} + 1\right)/\left(\sqrt{2} - 1\right)\right] \approx 229.559$

79. $(0, 0.422)$ **81.** $(\pi/32)\left[102\sqrt{2} - \ln\left(3 + 2\sqrt{2}\right)\right] \approx 13.989$

83. (a) 187.2π lb (b) $62.4\pi d$ lb

85. Proof **87.** $12 + 9\pi/2 - 25\arcsin(3/5) \approx 10.050$

89. Putnum Problem A5, 2005

Section 8.5 (page 561)

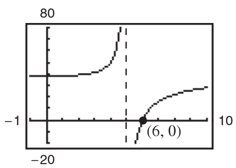
1. $\dfrac{A}{x} + \dfrac{B}{x - 8}$ **3.** $\dfrac{A}{x} + \dfrac{Bx + C}{x^2 + 10}$ **5.** $\dfrac{A}{x} + \dfrac{B}{x - 6}$

7. $\frac{1}{6}\ln|(x - 3)/(x + 3)| + C$ **9.** $\ln|(x - 1)/(x + 4)| + C$

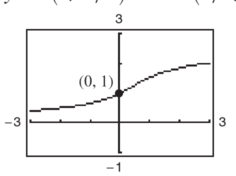
11. $\frac{3}{2}\ln|2x - 1| - 2\ln|x + 1| + C$

13. $5\ln|x - 2| - \ln|x + 2| - 3\ln|x| + C$

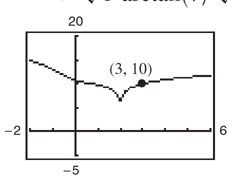
15. $x^2 + \frac{3}{2}\ln|x - 4| - \frac{1}{2}\ln|x + 2| + C$
17. $1/x + \ln|x^4 + x^3| + C$
19. $2\ln|x - 2| - \ln|x| - 3/(x - 2) + C$
21. $\ln|(x^2 + 1)/x| + C$
23. $\frac{1}{6}\left[\ln|(x - 2)/(x + 2)| + \sqrt{2}\arctan\left(x/\sqrt{2}\right)\right] + C$
25. $\frac{1}{16}\ln|(4x^2 - 1)/(4x^2 + 1)| + C$
27. $\ln|x + 1| + \sqrt{2}\arctan\left[(x - 1)/\sqrt{2}\right] + C$
29. $\ln 3$ **31.** $\frac{1}{2}\ln(8/5) - \pi/4 + \arctan 2 \approx 0.557$
33. $y = 5\ln|x - 5| - 5x/(x - 5) + 30$



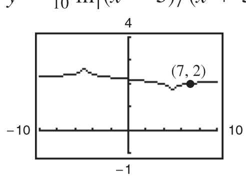
35. $y = \left(\sqrt{2}/2\right)\arctan\left(x/\sqrt{2}\right) - 1/[2(x^2 + 2)] + 5/4$



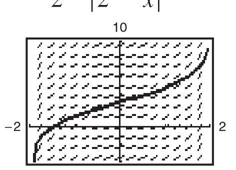
37. $y = \ln|x - 2| + \frac{1}{2}\ln|x^2 + x + 1|$
$- \sqrt{3}\arctan\left[(2x + 1)/\sqrt{3}\right] - \frac{1}{2}\ln 13$
$+ \sqrt{3}\arctan\left(7/\sqrt{3}\right) + 10$



39. $y = \frac{1}{10}\ln|(x - 5)/(x + 5)| + \frac{1}{10}\ln 6 + 2$



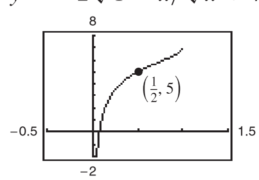
41. $\ln\left|\dfrac{\cos x}{\cos x - 1}\right| + C$ **43.** $\ln\left|\dfrac{\sin x}{1 + \sin x}\right| + C$
45. $\ln\left|\dfrac{\tan x + 2}{\tan x + 3}\right| + C$ **47.** $\dfrac{1}{5}\ln\left|\dfrac{e^x - 1}{e^x + 4}\right| + C$
49. $2\sqrt{x} + 2\ln\left|\dfrac{\sqrt{x} - 2}{\sqrt{x} + 2}\right| + C$ **51–53.** Proofs
55. $y = \dfrac{3}{2}\ln\left|\dfrac{2 + x}{2 - x}\right| + 3$ **57.** First divide x^3 by $(x - 5)$.



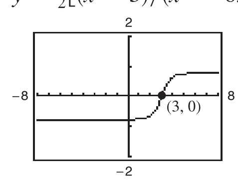
59. $12\ln\left(\frac{9}{8}\right) \approx 1.4134$ **61.** $6 - \frac{7}{4}\ln 7 \approx 2.5947$
63. 4.90 or \$490,000
65. $V = 2\pi\left(\arctan 3 - \frac{3}{10}\right) \approx 5.963$; $(\bar{x}, \bar{y}) \approx (1.521, 0.412)$
67. $x = n[e^{(n+1)kt} - 1]/[n + e^{(n+1)kt}]$ **69.** $\pi/8$

Section 8.6 (page 567)

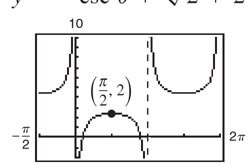
1. $-\frac{1}{2}x(10 - x) + 25\ln|5 + x| + C$
3. $\frac{1}{2}\left[e^x\sqrt{e^{2x} + 1} + \ln\left(e^x + \sqrt{e^{2x} + 1}\right)\right] + C$
5. $-\sqrt{1 - x^2}/x + C$
7. $\frac{1}{24}(3x + \sin 3x\cos 3x + 2\cos^3 3x\sin 3x) + C$
9. $-2\left(\cot\sqrt{x} + \csc\sqrt{x}\right) + C$ **11.** $x - \frac{1}{2}\ln(1 + e^{2x}) + C$
13. $\frac{1}{64}x^8(8\ln x - 1) + C$
15. (a) and (b) $\frac{1}{27}e^{3x}(9x^2 - 6x + 2) + C$
17. (a) and (b) $\ln|(x + 1)/x| - 1/x + C$
19. $\frac{1}{2}\left[(x^2 + 1)\operatorname{arccsc}(x^2 + 1) + \ln\left(x^2 + 1 + \sqrt{x^4 + 2x^2}\right)\right] + C$
21. $\sqrt{x^2 - 4}/(4x) + C$ **23.** $\frac{4}{25}[\ln|2 - 5x| + 2/(2 - 5x)] + C$
25. $e^x\arccos(e^x) - \sqrt{1 - e^{2x}} + C$
27. $\frac{1}{2}(x^2 + \cot x^2 + \csc x^2) + C$
29. $\left(\sqrt{2}/2\right)\arctan\left[(1 + \sin\theta)/\sqrt{2}\right] + C$
31. $-\sqrt{2 + 9x^2}/(2x) + C$
33. $\frac{1}{4}\left(2\ln|x| - 3\ln|3 + 2\ln|x||\right) + C$
35. $(3x - 10)/[2(x^2 - 6x + 10)] + \frac{3}{2}\arctan(x - 3) + C$
37. $\frac{1}{2}\ln\left|x^2 - 3 + \sqrt{x^4 - 6x^2 + 5}\right| + C$
39. $-\frac{1}{3}\sqrt{4 - x^2}(x^2 + 8) + C$
41. $2/(1 + e^x) - 1/[2(1 + e^x)^2] + \ln(1 + e^x) + C$
43. $\frac{1}{2}(e - 1) \approx 0.8591$ **45.** $\frac{32}{5}\ln 2 - \frac{31}{25} \approx 3.1961$
47. $\pi/2$ **49.** $\pi^3/8 - 3\pi + 6 \approx 0.4510$ **51–55.** Proofs
57. $y = -2\sqrt{1 - x}/\sqrt{x} + 7$



59. $y = \frac{1}{2}[(x - 3)/(x^2 - 6x + 10) + \arctan(x - 3)]$



61. $y = -\csc\theta + \sqrt{2} + 2$



63. $\dfrac{1}{\sqrt{5}}\ln\left|\dfrac{2\tan(\theta/2) - 3 - \sqrt{5}}{2\tan(\theta/2) - 3 + \sqrt{5}}\right| + C$ **65.** $\ln 2$
67. $\frac{1}{2}\ln(3 - 2\cos\theta) + C$ **69.** $-2\cos\sqrt{\theta} + C$ **71.** $4\sqrt{3}$
73. (a) $\displaystyle\int x\ln x\,dx = \frac{1}{2}x^2\ln x - \frac{1}{4}x^2 + C$

$$\int x^2\ln x\,dx = \frac{1}{3}x^3\ln x - \frac{1}{9}x^3 + C$$

$$\int x^3\ln x\,dx = \frac{1}{4}x^4\ln x - \frac{1}{16}x^4 + C$$

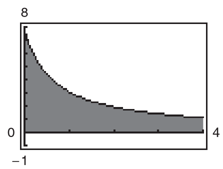
(b) $\int x^n \ln x\,dx = x^{n+1}\ln x/(n+1) - x^{n+1}/(n+1)^2 + C$

75. False. Substitutions may first have to be made to rewrite the integral in a form that appears in the table.

77. $32\pi^2$ **79.** 1919.145 ft-lb

81. (a) $V = 80\ln\left(\sqrt{10}+3\right) \approx 145.5\text{ ft}^3$
$W = 11{,}840\ln\left(\sqrt{10}+3\right) \approx 21{,}530.4\text{ lb}$
(b) $(0, 1.19)$

83. (a) $k = 30/\ln 7 \approx 15.42$ **85.** Putnam Problem A3, 1980
(b)



Section 8.7 (page 576)

1.

x	-0.1	-0.01	-0.001	0.001	0.01	0.1
$f(x)$	1.3177	1.3332	1.3333	1.3333	1.3332	1.3177

$\frac{4}{3}$

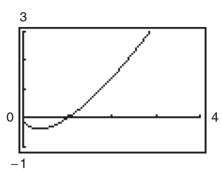
3.

x	1	10	10^2	10^3	10^4	10^5
$f(x)$	0.9900	90,483.7	3.7×10^9	4.5×10^{10}	0	0

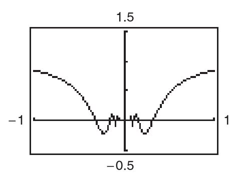
0

5. $\frac{3}{8}$ **7.** $\frac{1}{8}$ **9.** $\frac{5}{3}$ **11.** 4 **13.** 0 **15.** 2
17. ∞ **19.** $\frac{11}{4}$ **21.** $\frac{3}{5}$ **23.** 1 **25.** $\frac{5}{4}$ **27.** ∞
29. 0 **31.** 1 **33.** 0 **35.** 0 **37.** ∞
39. $\frac{5}{9}$ **41.** 1 **43.** ∞

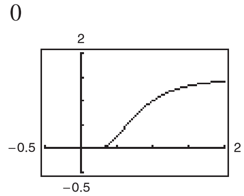
45. (a) Not indeterminate
(b) ∞
(c)



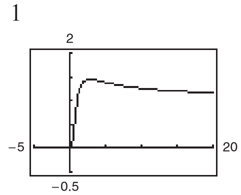
47. (a) $0 \cdot \infty$
(b) 1
(c)



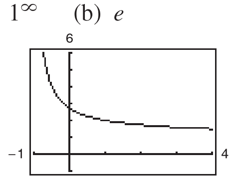
49. (a) Not indeterminate
(b) 0
(c)



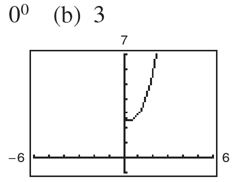
51. (a) ∞^0
(b) 1
(c)



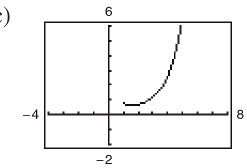
53. (a) 1^∞ (b) e
(c)



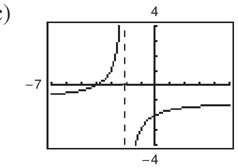
55. (a) 0^0 (b) 3
(c)



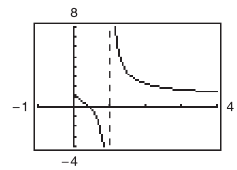
57. (a) 0^0 (b) 1
(c)



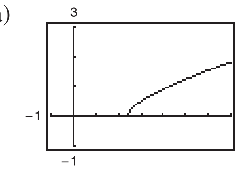
59. (a) $\infty - \infty$ (b) $-\frac{3}{2}$
(c)



61. (a) $\infty - \infty$ (b) ∞
(c)

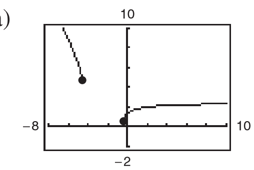


63. (a)



(b) $\frac{1}{2}$

65. (a)



(b) $\frac{5}{2}$

67. $\frac{0}{0}, \frac{\infty}{\infty}, 0 \cdot \infty, 1^\infty, 0^0, \infty - \infty$

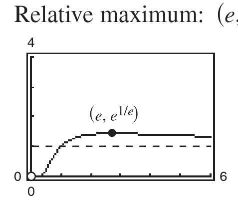
69. Answers will vary. Examples:
(a) $f(x) = x^2 - 25,\ g(x) = x - 5$
(b) $f(x) = (x-5)^2,\ g(x) = x^2 - 25$
(c) $f(x) = x^2 - 25,\ g(x) = (x-5)^3$

71.

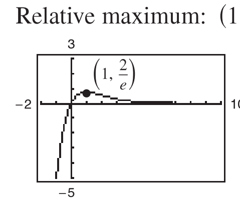
x	10	10^2	10^4	10^6	10^8	10^{10}
$\dfrac{(\ln x)^4}{x}$	2.811	4.498	0.720	0.036	0.001	0.000

73. 0 **75.** 0 **77.** 0

79. Horizontal asymptote: $y = 1$
Relative maximum: $(e, e^{1/e})$

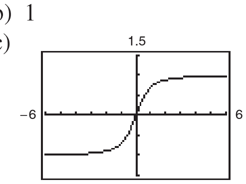


81. Horizontal asymptote: $y = 0$
Relative maximum: $(1, 2/e)$

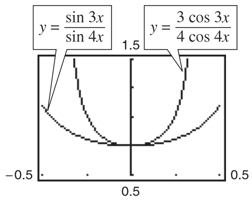


83. Limit is not of the form $0/0$ or ∞/∞.
85. Limit is not of the form $0/0$ or ∞/∞.
87. Limit is not of the form $0/0$ or ∞/∞.

89. (a) $\displaystyle\lim_{x\to\infty}\frac{x}{\sqrt{x^2+1}} = \lim_{x\to\infty}\frac{\sqrt{x^2+1}}{x} = \lim_{x\to\infty}\frac{x}{\sqrt{x^2+1}}$
Applying L'Hôpital's Rule twice results in the original limit, so L'Hôpital's Rule fails.
(b) 1
(c)



91.



As $x \to 0$, the graphs get closer together (they both approach 0.75).

By L'Hôpital's Rule,

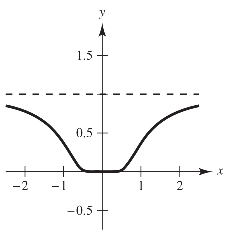
$$\lim_{x\to 0} \frac{\sin 3x}{\sin 4x} = \lim_{x\to 0} \frac{3\cos 3x}{4\cos 4x} = \frac{3}{4}.$$

93. $v = 32t + v_0$ **95.** Proof **97.** $c = \frac{2}{3}$ **99.** $c = \pi/4$

101. False: L'Hôpital's Rule does not apply, because $\lim_{x\to 0}(x^2 + x + 1) \neq 0$. **103.** True

105. $\frac{3}{4}$ **107.** $\frac{4}{3}$ **109.** $a = 1, b = \pm 2$ **111.** Proof

113.



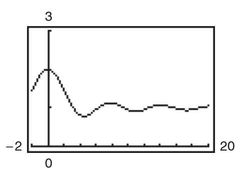
$g'(0) = 0$

115. (a) $0 \cdot \infty$ (b) 0

117. Proof

119. (a)–(c) 2

121. (a)



(b) $\lim_{x\to\infty} h(x) = 1$

(c) No

123. Putnam Problem A1, 1956

Section 8.8 (page 587)

1. Improper; $0 \le \frac{3}{5} \le 1$ **3.** Not improper; continuous on $[0, 1]$

5. Not improper; continuous on $[0, 2]$

7. Improper; infinite limits of integration

9. Infinite discontinuity at $x = 0$; 4

11. Infinite discontinuity at $x = 1$; diverges

13. Infinite limit of integration; $\frac{1}{4}$

15. Infinite discontinuity at $x = 0$; diverges

17. Infinite limit of integration; converges to 1 **19.** $\frac{1}{2}$

21. Diverges **23.** Diverges **25.** 2 **27.** $\frac{1}{2}$

29. $1/[2(\ln 4)^2]$ **31.** π **33.** $\pi/4$ **35.** Diverges

37. Diverges **39.** 6 **41.** $-\frac{1}{4}$ **43.** Diverges **45.** $\pi/3$

47. $\ln(2 + \sqrt{3})$ **49.** 0 **51.** $\pi/6$ **53.** $2\pi\sqrt{6}/3$

55. $p > 1$ **57.** Proof **59.** Diverges **61.** Converges

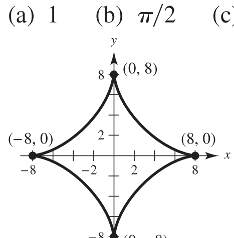
63. Converges **65.** Diverges **67.** Diverges **69.** Converges

71. An integral with infinite integration limits, an integral with an infinite discontinuity at or between the integration limits

73. The improper integral diverges. **75.** e **77.** π

79. (a) 1 (b) $\pi/2$ (c) 2π

81.



Perimeter = 48

83. $8\pi^2$ **85.** (a) $W = 20{,}000$ mile-tons (b) 4000 mi

87. (a) Proof (b) $P = 43.53\%$ (c) $E(x) = 7$

89. (a) \$757,992.41 (b) \$837,995.15 (c) \$1,066,666.67

91. $P = \left[2\pi NI\left(\sqrt{r^2 + c^2} - c\right)\right]/\left(kr\sqrt{r^2 + c^2}\right)$

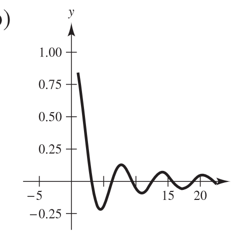
93. False. Let $f(x) = 1/(x + 1)$. **95.** True

97. (a) and (b) Proofs

(c) The definition of the improper integral $\int_{-\infty}^{\infty}$ is not $\lim_{a\to\infty} \int_{-a}^{a}$ but rather if you rewrite the integral that diverges, you can find that the integral converges.

99. (a) $\int_1^{\infty} \frac{1}{x^n}\,dx$ will converge if $n > 1$ and diverge if $n \le 1$.

(b)



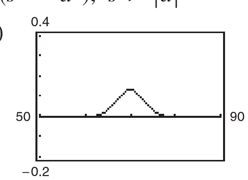
(c) Converges

101. (a) $\Gamma(1) = 1, \Gamma(2) = 1, \Gamma(3) = 2$ (b) Proof
(c) $\Gamma(n) = (n - 1)!$

103. $1/s,\ s > 0$ **105.** $2/s^3,\ s > 0$ **107.** $s/(s^2 + a^2),\ s > 0$

109. $s/(s^2 - a^2),\ s > |a|$

111. (a)



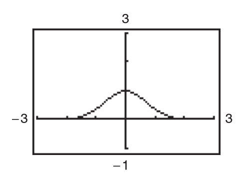
(b) About 0.2525
(c) 0.2525; same by symmetry

113. $c = 1$; $\ln(2)$

115. $8\pi[(\ln 2)^2/3 - (\ln 4)/9 + 2/27] \approx 2.01545$

117. $\int_0^1 2\sin(u^2)\,du$; 0.6278

119. (a)



(b) Proof

Review Exercises for Chapter 8 (page 591)

1. $\frac{1}{3}(x^2 - 36)^{3/2} + C$ **3.** $\frac{1}{2}\ln|x^2 - 49| + C$

5. $\ln(2) + \frac{1}{2} \approx 1.1931$ **7.** $100 \arcsin(x/10) + C$

9. $\frac{1}{9}e^{3x}(3x - 1) + C$

11. $\frac{1}{13}e^{2x}(2\sin 3x - 3\cos 3x) + C$

13. $\frac{2}{15}(x - 1)^{3/2}(3x + 2) + C$

15. $-\frac{1}{2}x^2\cos 2x + \frac{1}{2}x\sin 2x + \frac{1}{4}\cos 2x + C$

17. $\frac{1}{16}\left[(8x^2 - 1)\arcsin 2x + 2x\sqrt{1 - 4x^2}\right] + C$

19. $\sin(\pi x - 1)[\cos^2(\pi x - 1) + 2]/(3\pi) + C$

21. $\frac{2}{3}[\tan^3(x/2) + 3\tan(x/2)] + C$ **23.** $\tan\theta + \sec\theta + C$

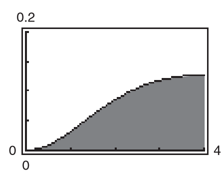
25. $3\pi/16 + \frac{1}{2} \approx 1.0890$ **27.** $3\sqrt{4 - x^2}/x + C$

29. $\frac{1}{3}(x^2 + 4)^{1/2}(x^2 - 8) + C$ **31.** π

33. (a), (b), and (c) $\frac{1}{3}\sqrt{4+x^2}(x^2-8)+C$
35. $6\ln|x+3|-5\ln|x-4|+C$
37. $\frac{1}{4}[6\ln|x-1|-\ln(x^2+1)+6\arctan x]+C$
39. $x-\frac{64}{11}\ln|x+8|+\frac{9}{11}\ln|x-3|+C$
41. $\frac{1}{25}[4/(4+5x)+\ln|4+5x|]+C$ **43.** $1-\sqrt{2}/2$
45. $\frac{1}{2}\ln|x^2+4x+8|-\arctan[(x+2)/2]+C$
47. $\ln|\tan \pi x|/\pi + C$ **49.** Proof
51. $\frac{1}{8}(\sin 2\theta - 2\theta\cos 2\theta)+C$
53. $\frac{4}{3}[x^{3/4}-3x^{1/4}+3\arctan(x^{1/4})]+C$
55. $2\sqrt{1-\cos x}+C$ **57.** $\sin x\ln(\sin x)-\sin x+C$
59. $\frac{5}{2}\ln|(x-5)/(x+5)|+C$
61. $y=x\ln|x^2+x|-2x+\ln|x+1|+C$ **63.** $\frac{1}{5}$
65. $\frac{1}{2}(\ln 4)^2\approx 0.961$ **67.** π **69.** $\frac{128}{15}$
71. $(\bar{x},\bar{y})=(0,4/(3\pi))$ **73.** 3.82 **75.** 0 **77.** ∞ **79.** 1
81. $1000e^{0.09}\approx 1094.17$ **83.** Converges; $\frac{32}{3}$ **85.** Diverges
87. Converges; 1 **89.** Converges; $\pi/4$
91. (a) \$6,321,205.59 (b) \$10,000,000
93. (a) 0.4581 (b) 0.0135

P.S. Problem Solving (page 593)

1. (a) $\frac{4}{3}, \frac{16}{15}$ (b) Proof **3.** ln 3 **5.** Proof
7. (a)



Area ≈ 0.2986

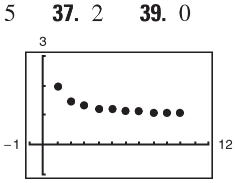
(b) $\ln 3-\frac{4}{5}$
(c) $\ln 3-\frac{4}{5}$

9. $\ln 3-\frac{1}{2}\approx 0.5986$ **11.** Proof **13.** About 0.8670
15. (a) ∞ (b) 0 (c) $-\frac{2}{3}$
The form $0\cdot\infty$ is indeterminant.
17. $\dfrac{1/12}{x}+\dfrac{1/42}{x-3}+\dfrac{1/10}{x-1}+\dfrac{111/140}{x+4}$
19. Proof **21.** About 0.0158

Chapter 9

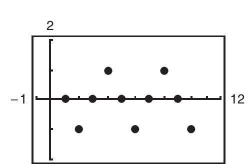
Section 9.1 (page 604)

1. 3, 9, 27, 81, 243 **3.** $-\frac{1}{4}, \frac{1}{16}, -\frac{1}{64}, \frac{1}{256}, -\frac{1}{1024}$
5. 1, 0, −1, 0, 1 **7.** $-1, -\frac{1}{4}, \frac{1}{9}, \frac{1}{16}, -\frac{1}{25}$ **9.** $5, \frac{19}{4}, \frac{43}{9}, \frac{77}{16}, \frac{121}{25}$
11. 3, 4, 6, 10, 18 **13.** 32, 16, 8, 4, 2
15. c **16.** a **17.** d **18.** b **19.** b **20.** c
21. a **22.** d **23.** 14, 17; add 3 to preceding term
25. 80, 160; multiply preceding term by 2.
27. $\frac{3}{16}, -\frac{3}{32}$; multiply preceding term by $-\frac{1}{2}$
29. $11\cdot 10\cdot 9=990$ **31.** $n+1$ **33.** $1/[(2n+1)(2n)]$
35. 5 **37.** 2 **39.** 0
41.



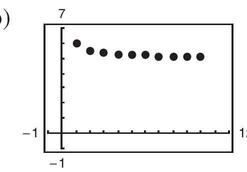
Converges to 1

43.



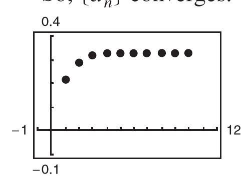
Diverges

45. Converges to −1 **47.** Converges to 0
49. Diverges **51.** Converges to $\frac{3}{2}$ **53.** Converges to 0
55. Converges to 0 **57.** Converges to 0 **59.** Converges to 0
61. Diverges **63.** Converges to 0 **65.** Converges to 0
67. Converges to 1 **69.** Converges to e^k **71.** Converges to 0
73. Answers will vary. Sample answer: $3n-2$
75. Answers will vary. Sample answer: n^2-2
77. Answers will vary. Sample answer: $(n+1)/(n+2)$
79. Answers will vary. Sample answer: $(n+1)/n$
81. Answers will vary. Sample answer: $n/[(n+1)(n+2)]$
83. Answers will vary. Sample answer:
$$\frac{(-1)^{n-1}}{1\cdot 3\cdot 5\cdots(2n-1)}=\frac{(-1)^{n-1}2^n n!}{(2n)!}$$
85. Answers will vary. Sample answer: $(2n)!$
87. Monotonic, bounded **89.** Monotonic, bounded
91. Not monotonic, bounded **93.** Monotonic, bounded
95. Not monotonic, bounded **97.** Not monotonic, bounded
99. (a) $\left|5+\frac{1}{n}\right|\le 6 \Rightarrow$ bounded
$a_n > a_{n+1} \Rightarrow$ monotonic
So, $\{a_n\}$ converges.
(b)



Limit = 5

101. (a) $\left|\frac{1}{3}\left(1-\frac{1}{3^n}\right)\right|<\frac{1}{3}\Rightarrow$ bounded
$a_n < a_{n+1} \Rightarrow$ monotonic
So, $\{a_n\}$ converges.
(b)



Limit = $\frac{1}{3}$

103. $\{a_n\}$ has a limit because it is bounded and monotonic; since $2\le a_n\le 4$, $2\le L\le 4$.
105. (a) No; $\lim_{n\to\infty} A_n$ does not exist.
(b)

n	1	2	3	4
A_n	\$10,045.83	\$10,091.88	\$10,138.13	\$10,184.60

n	5	6	7
A_n	\$10,231.28	\$10,278.17	\$10,325.28

n	8	9	10
A_n	\$10,372.60	\$10,420.14	\$10,467.90

107. No. A sequence is said to converge when its terms approach a real number.
109. The graph on the left represents a sequence with alternating signs because the terms alternate from being above the x-axis to being below the x-axis.

111. (a) $\$4{,}500{,}000{,}000(0.8)^n$

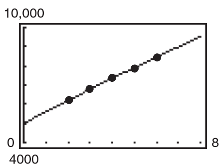
(b)

Year	1	2
Budget	\$3,600,000,000	\$2,880,000,000

Year	3	4
Budget	\$2,304,000,000	\$1,843,200,000

(c) Converges to 0

113. (a) $a_n = -5.364n^2 + 608.04n + 4998.3$



(b) \$11,522.4 billion

115. (a) $a_9 = a_{10} = 1{,}562{,}500/567$ (b) Decreasing
(c) Factorials increase more rapidly than exponentials.

117. 1, 1.4142, 1.4422, 1.4142, 1.3797, 1.3480; Converges to 1

119. True **121.** True **123.** True

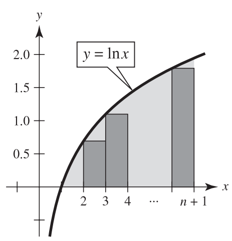
125. (a) 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144
(b) 1, 2, 1.5, 1.6667, 1.6, 1.6250, 1.6154, 1.6190, 1.6176, 1.6182
(c) Proof (d) $\rho = \left(1 + \sqrt{5}\right)/2 \approx 1.6180$

127. (a) 1.4142, 1.8478, 1.9616, 1.9904, 1.9976
(b) $a_n = \sqrt{2 + a_{n-1}}$ (c) $\lim_{n\to\infty} a_n = 2$

129. (a) Proof (b) Proof (c) $\lim_{n\to\infty} a_n = \left(1 + \sqrt{1 + 4k}\right)/2$

131. (a) Proof (b) Proof

133. (a) Proof
(b)



(c) Proof (d) Proof

(e) $\dfrac{\sqrt[20]{20!}}{20} \approx 0.4152$; $\dfrac{\sqrt[50]{50!}}{50} \approx 0.3897$; $\dfrac{\sqrt[100]{100!}}{100} \approx 0.3799$

135. Proof

137. Answers will vary. Sample answer: $a_n = (-1)^n$

139. Proof **141.** Putnam Problem A1, 1990

Section 9.2 (page 614)

1. 1, 1.25, 1.361, 1.424, 1.464

3. 3, −1.5, 5.25, −4.875, 10.3125

5. 3, 4.5, 5.25, 5.625, 5.8125

7. $\{a_n\}$ converges, $\sum a_n$ diverges

9. Geometric series: $r = \frac{7}{6} > 1$

11. Geometric series: $r = 1.055 > 1$ **13.** $\lim_{n\to\infty} a_n = 1 \neq 0$

15. $\lim_{n\to\infty} a_n = 1 \neq 0$ **17.** $\lim_{n\to\infty} a_n = \frac{1}{2} \neq 0$ **19.** c; 3

20. b; 3 **21.** a; 3 **22.** d; 3 **23.** f; $\frac{34}{9}$ **24.** e; $\frac{5}{3}$

25. Geometric series: $r = \frac{5}{6} < 1$

27. Geometric series: $r = 0.9 < 1$

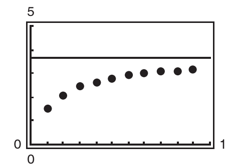
29. Telescoping series: $a_n = 1/n - 1/(n + 1)$; Converges to 1.

31. (a) $\frac{11}{3}$

(b)

n	5	10	20	50	100
S_n	2.7976	3.1643	3.3936	3.5513	3.6078

(c)



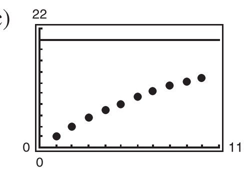
(d) The terms of the series decrease in magnitude relatively slowly, and the sequence of partial sums approaches the sum of the series relatively slowly.

33. (a) 20

(b)

n	5	10	20	50	100
S_n	8.1902	13.0264	17.5685	19.8969	19.9995

(c)



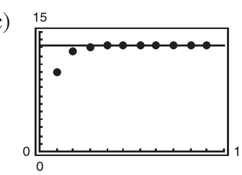
(d) The terms of the series decrease in magnitude relatively slowly, and the sequence of partial sums approaches the sum of the series relatively slowly.

35. (a) $\frac{40}{3}$

(b)

n	5	10	20	50	100
S_n	13.3203	13.3333	13.3333	13.3333	13.3333

(c)



(d) The terms of the series decrease in magnitude relatively rapidly, and the sequence of partial sums approaches the sum of the series relatively rapidly.

37. 2 **39.** $\frac{3}{4}$ **41.** $\frac{3}{4}$ **43.** 4 **45.** $\frac{10}{9}$ **47.** $\frac{9}{4}$ **49.** $\frac{1}{2}$

51. $\dfrac{\sin(1)}{1 - \sin(1)}$ **53.** (a) $\displaystyle\sum_{n=0}^{\infty} \frac{4}{10}(0.1)^n$ (b) $\dfrac{4}{9}$

55. (a) $\displaystyle\sum_{n=0}^{\infty} \frac{81}{100}(0.01)^n$ (b) $\dfrac{9}{11}$

57. (a) $\displaystyle\sum_{n=0}^{\infty} \frac{3}{40}(0.01)^n$ (b) $\dfrac{5}{66}$ **59.** Diverges **61.** Diverges

63. Converges **65.** Converges **67.** Diverges

69. Converges **71.** Diverges **73.** Diverges **75.** Diverges

77. See definitions on page 608.

79. The series given by

$$\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + \cdots + ar^n + \cdots, a \neq 0$$

is a geometric series with ratio r. When $0 < |r| < 1$, the series converges to the sum $\displaystyle\sum_{n=0}^{\infty} ar^n = \frac{a}{1 - r}$.

81. The series in (a) and (b) are the same. The series in (c) is different unless $a_1 = a_2 = \cdots = a$ is constant.

83. $-2 < x < 2$; $x/(2 - x)$ **85.** $0 < x < 2$; $(x - 1)/(2 - x)$

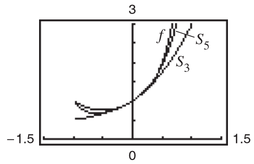
87. $-1 < x < 1$; $1/(1 + x)$

89. x: $(-\infty, -1) \cup (1, \infty)$; $x/(x - 1)$ **91.** $c = \left(\sqrt{3} - 1\right)/2$

93. Neither statement is true. The formula is valid for $-1 < x < 1$.

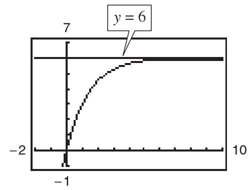
95. (a) x (b) $f(x) = 1/(1 - x)$, $|x| < 1$

(c)



Answers will vary.

97.



Horizontal asymptote: $y = 6$
The horizontal asymptote is the sum of the series.

99. The required terms for the two series are $n = 100$ and $n = 5$, respectively. The second series converges at a higher rate.

101. $160{,}000(1 - 0.95^n)$ units

103. $\sum_{i=0}^{\infty} 200(0.75)^i$; Sum = \$800 million

105. 152.42 feet **107.** $\frac{1}{8}$; $\sum_{n=0}^{\infty} \frac{1}{2}\left(\frac{1}{2}\right)^n = \frac{1/2}{1 - 1/2} = 1$

109. (a) $-1 + \sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n = -1 + \frac{a}{1 - r} = -1 + \frac{1}{1 - 1/2} = 1$
(b) No (c) 2

111. (a) 126 in.^{2} (b) 128 in.^{2}

113. The \$2,000,000 sweepstakes has a present value of \$1,146,992.12. After accruing interest over the 20-year period, it attains its full value.

115. (a) \$5,368,709.11 (b) \$10,737,418.23 (c) \$21,474,836.47

117. (a) \$14,773.59 (b) \$14,779.65

119. (a) \$91,373.09 (b) \$91,503.32 **121.** \$4,751,275.79

123. False. $\lim_{n\to\infty} \frac{1}{n} = 0$, but $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

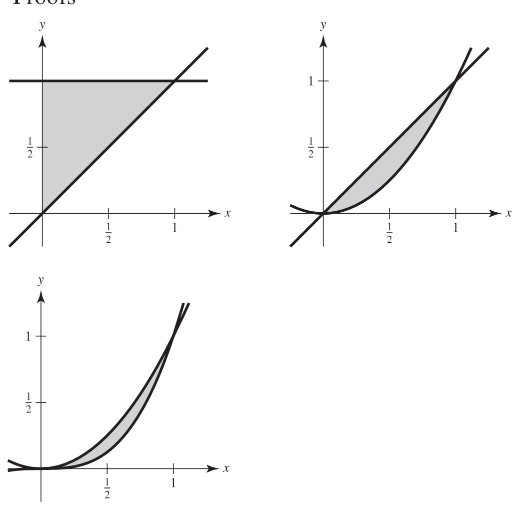
125. False. $\sum_{n=1}^{\infty} ar^n = \left(\frac{a}{1 - r}\right) - a$ The formula requires that the geometric series begins with $n = 0$.

127. True **129.** Proof

131. Answers will vary. Example: $\sum_{n=0}^{\infty} 1$, $\sum_{n=0}^{\infty} (-1)$

133–137. Proofs

139. (a)



(b) $\int_0^1 (1 - x)\,dx = \frac{1}{2}$

$\int_0^1 (x - x^2)\,dx = \frac{1}{6}$

$\int_0^1 (x^2 - x^3)\,dx = \frac{1}{12}$

(c) $a_n = \frac{1}{n} - \frac{1}{n + 1}$ and $\sum_{n=1}^{\infty} a_n = 1$; The sum of all the shaded regions is the area of the square, 1.

141. H = half-life of the drug
n = number of equal doses
P = number of units of the drug
t = equal time intervals
The total amount of the drug in the patient's system at the time the last dose is given is
$T_n = P + Pe^{kt} + Pe^{2kt} + \cdots + Pe^{(n-1)kt}$
where $k = -(\ln 2)/H$. One time interval after the last dose is administered is given by
$T_{n+1} = Pe^{kt} + Pe^{2kt} + Pe^{3kt} + \cdots + Pe^{nkt}$
and so on. Because $k < 0$, $T_{n+s} \to 0$ as $s \to \infty$.

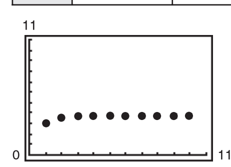
143. Putnam Problem A1, 1966

Section 9.3 (page 622)

1. Diverges **3.** Converges **5.** Converges **7.** Converges
9. Diverges **11.** Diverges **13.** Diverges **15.** Converges
17. Converges **19.** Converges **21.** Diverges
23. Diverges **25.** Diverges **27.** $f(x)$ is not positive for $x \geq 1$.
29. $f(x)$ is not always decreasing. **31.** Converges **33.** Diverges
35. Diverges **37.** Diverges **39.** Converges **41.** Converges
43. c; diverges **44.** f; diverges **45.** b; converges
46. a; diverges **47.** d; converges **48.** e; converges

49. (a)

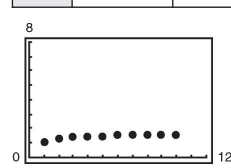
n	5	10	20	50	100
S_n	3.7488	3.75	3.75	3.75	3.75



The partial sums approach the sum 3.75 very quickly.

(b)

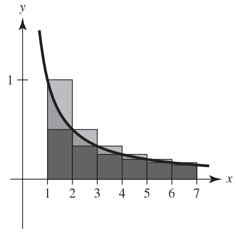
n	5	10	20	50	100
S_n	1.4636	1.5498	1.5962	1.6251	1.635



The partial sums approach the sum $\pi^2/6 \approx 1.6449$ more slowly than the series in part (a).

51. See Theorem 9.10 on page 619. Answers will vary. For example, convergence or divergence can be determined for the series $\sum_{n=1}^{\infty} \frac{1}{n^2 + 1}$.

53. No. Because $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, $\sum_{n=10{,}000}^{\infty} \frac{1}{n}$ also diverges. The convergence or divergence of a series is not determined by the first finite number of terms of the series.

55. 

$$\sum_{n=1}^{6} a_n \ge \int_1^7 f(x)\,dx \ge \sum_{n=2}^{7} a_n$$

57. $p > 1$ **59.** $p > 1$ **61.** $p > 1$ **63.** Diverges
65. Converges **67.** Proof
69. $S_6 \approx 1.0811$, $R_6 \approx 0.0015$ **71.** $S_{10} \approx 0.9818$, $R_{10} \approx 0.0997$ **73.** $S_4 \approx 0.4049$, $R_4 \approx 5.6 \times 10^{-8}$
75. $N \ge 7$ **77.** $N \ge 2$ **79.** $N \ge 1000$

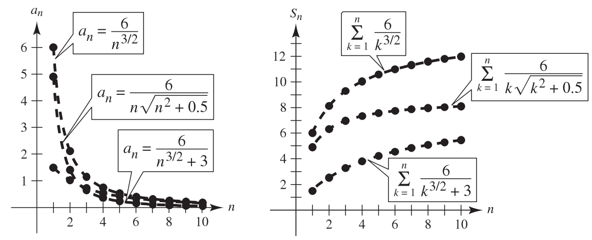
81. (a) $\sum_{n=2}^{\infty} \frac{1}{n^{1.1}}$ converges by the p-Series Test because $1.1 > 1$.
$\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ diverges by the Integral Test because $\int_2^{\infty} \frac{1}{x \ln x}\,dx$ diverges.
(b) $\sum_{n=2}^{\infty} \frac{1}{n^{1.1}} = 0.4665 + 0.2987 + 0.2176 + 0.1703 + 0.1393 + \cdots$
$\sum_{n=2}^{\infty} \frac{1}{n \ln n} = 0.7213 + 0.3034 + 0.1803 + 0.1243 + 0.0930 + \cdots$
(c) $n \ge 3.431 \times 10^{15}$

83. (a) Let $f(x) = 1/x$. f is positive, continuous, and decreasing on $[1, \infty)$.
$$S_n - 1 \le \int_1^n \frac{1}{x}\,dx = \ln n$$
$$S_n \ge \int_1^{n+1} \frac{1}{x}\,dx = \ln(n+1)$$
So, $\ln(n+1) \le S_n \le 1 + \ln n$.
(b) $\ln(n+1) - \ln n \le S_n - \ln n \le 1$.
Also, $\ln(n+1) - \ln n > 0$ for $n \ge 1$. So, $0 \le S_n - \ln n \le 1$, and the sequence $\{a_n\}$ is bounded.
(c) $a_n - a_{n+1} = [S_n - \ln n] - [S_{n+1} - \ln(n+1)]$
$$= \int_n^{n+1} \frac{1}{x}\,dx - \frac{1}{n+1} \ge 0$$
So, $a_n \ge a_{n+1}$.
(d) Because the sequence is bounded and monotonic, it converges to a limit, γ.
(e) 0.5822

85. (a) Diverges (b) Diverges
(c) $\sum_{n=2}^{\infty} x^{\ln n}$ converges for $x < 1/e$.

87. Diverges **89.** Converges **91.** Converges **93.** Diverges
95. Diverges **97.** Converges

Section 9.4 (page 630)

1. (a) 

(b) $\sum_{n=1}^{\infty} \frac{6}{n^{3/2}}$; Converges
(c) The magnitudes of the terms are less than the magnitudes of the terms of the p-series. Therefore, the series converges.
(d) The smaller the magnitudes of the terms, the smaller the magnitudes of the terms of the sequence of partial sums.

3. Converges **5.** Diverges **7.** Converges **9.** Diverges
11. Converges **13.** Converges **15.** Diverges **17.** Diverges
19. Converges **21.** Converges **23.** Converges
25. Diverges **27.** Diverges **29.** Diverges; p-Series Test
31. Converges; Direct Comparison Test with $\sum_{n=1}^{\infty} \left(\frac{1}{5}\right)^n$
33. Diverges; nth-Term Test **35.** Converges; Integral Test

37. $\lim_{n\to\infty} \frac{a_n}{1/n} = \lim_{n\to\infty} na_n$
$\lim_{n\to\infty} na_n \ne 0$, but is finite.
The series diverges by the Limit Comparison Test.

39. Diverges **41.** Converges

43. $\lim_{n\to\infty} n\left(\frac{n^3}{5n^4+3}\right) = \frac{1}{5} \ne 0$
So, $\sum_{n=1}^{\infty} \frac{n^3}{5n^4+3}$ diverges.

45. Diverges **47.** Converges

49. Convergence or divergence is dependent on the form of the general term for the series and not necessarily on the magnitudes of the terms.

51. See Theorem 9.13 on page 628. Answers will vary. For example,
$\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}-1}$ diverges because $\lim_{n\to\infty} \frac{1/\sqrt{n}-1}{1/\sqrt{n}} = 1$ and
$\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}}$ diverges (p-series).

53. (a) Proof
(b)

n	5	10	20	50	100
S_n	1.1839	1.2087	1.2212	1.2287	1.2312

(c) 0.1226 (d) 0.0277

55. False. Let $a_n = 1/n^3$ and $b_n = 1/n^2$.

57. True **59.** True **61.** Proof **63.** $\sum_{n=1}^{\infty} \frac{1}{n^2}$, $\sum_{n=1}^{\infty} \frac{1}{n^3}$

65–71. Proofs **73.** Putnam Problem B4, 1988

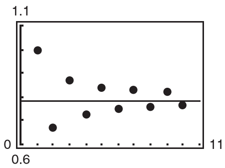
Section 9.5 (page 638)

1. d **2.** f **3.** a **4.** b **5.** e **6.** c

7. (a)

n	1	2	3	4	5
S_n	1.0000	0.6667	0.8667	0.7238	0.8349

n	6	7	8	9	10
S_n	0.7440	0.8209	0.7543	0.8131	0.7605

(b)


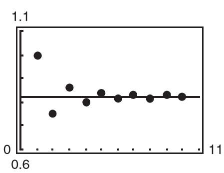
(c) The points alternate sides of the horizontal line $y = \pi/4$ that represents the sum of the series. The distances between the successive points and the line decrease.

(d) The distance in part (c) is always less than the magnitude of the next term of the series.

9. (a)

n	1	2	3	4	5
S_n	1.0000	0.7500	0.8611	0.7986	0.8386

n	6	7	8	9	10
S_n	0.8108	0.8312	0.8156	0.8280	0.8180

(b)


(c) The points alternate sides of the horizontal line $y = \pi^2/12$ that represents the sum of the series. The distances between the successive points and the line decrease.

(d) The distance in part (c) is always less than the magnitude of the next term of the series.

11. Converges **13.** Converges **15.** Diverges
17. Converges **19.** Diverges **21.** Converges **23.** Diverges
25. Diverges **27.** Diverges **29.** Converges
31. Converges **33.** Converges **35.** Converges
37. $0.7305 \le S \le 0.7361$ **39.** $2.3713 \le S \le 2.4937$
41. (a) 7 terms (Note that the sum begins with $n = 0$.) (b) 0.368
43. (a) 3 terms (Note that the sum begins with $n = 0$.) (b) 0.842
45. (a) 1000 terms (b) 0.693 **47.** 10 **49.** 7
51. Converges absolutely **53.** Converges absolutely
55. Converges absolutely **57.** Converges conditionally
59. Diverges **61.** Converges conditionally
63. Converges absolutely **65.** Converges absolutely
67. Converges conditionally **69.** Converges absolutely
71. An alternating series is a series whose terms alternate in sign.
73. $|S - S_N| = |R_N| \le a_{N+1}$
75. Graph (b). The partial sums alternate above and below the horizontal line representing the sum.
77. True **79.** $p > 0$
81. Proof; The converse is false. For example: Let $a_n = 1/n$.
83. $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, hence so does $\sum_{n=1}^{\infty} \frac{1}{n^4}$.
85. (a) No; $a_{n+1} \le a_n$ is not satisfied for all n. For example, $\frac{1}{9} < \frac{1}{8}$.
(b) Yes; 0.5
87. Converges; p-Series Test **89.** Diverges; nth-Term Test
91. Converges; Geometric Series Test
93. Converges; Integral Test
95. Converges; Alternating Series Test
97. The first term of the series is 0, not 1. You cannot regroup series terms arbitrarily.
99. Putnam Problem 2, afternoon session, 1954

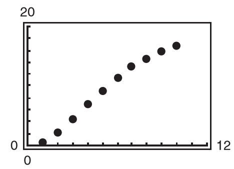
Section 9.6 (page 647)

1–3. Proofs **5.** d **6.** c **7.** f **8.** b **9.** a **10.** e

11. (a) Proof

(b)

n	5	10	15	20	25
S_n	9.2104	16.7598	18.8016	19.1878	19.2491

(c)


(d) 19.26

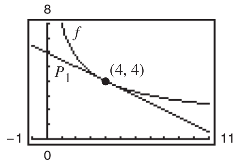
(e) The more rapidly the terms of the series approach 0, the more rapidly the sequence of partial sums approaches the sum of the series.

13. Converges **15.** Diverges **17.** Diverges
19. Converges **21.** Diverges **23.** Converges
25. Diverges **27.** Converges **29.** Converges
31. Diverges **33.** Converges **35.** Converges
37. Converges **39.** Diverges **41.** Converges
43. Diverges **45.** Converges **47.** Converges
49. Converges **51.** Converges; Alternating Series Test
53. Converges; p-Series Test **55.** Diverges; nth-Term Test
57. Diverges; Geometric Series Test
59. Converges; Limit Comparison Test with $b_n = 1/2^n$
61. Converges; Direct Comparison Test with $b_n = 1/3^n$
63. Converges; Ratio Test **65.** Converges; Ratio Test
67. Converges; Ratio Test **69.** a and c **71.** a and b
73. $\sum_{n=0}^{\infty} \frac{n+1}{7^{n+1}}$ **75.** (a) 9 (b) -0.7769
77. Diverges; $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| > 1$
79. Converges; $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$
81. Diverges; $\lim a_n \ne 0$ **83.** Converges **85.** Converges
87. $(-3, 3)$ **89.** $(-2, 0]$ **91.** $x = 0$
93. See Theorem 9.17 on page 641.
95. No; the series $\sum_{n=1}^{\infty} \frac{1}{n + 10{,}000}$ diverges.
97. Absolutely; by Theorem 9.17 **99–105.** Proofs
107. (a) Diverges (b) Converges (c) Converges
(d) Converges for all integers $x \ge 2$
109. Answers will vary.
111. Putnam Problem 7, morning session, 1951

Section 9.7 (page 658)

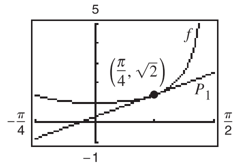
1. d **2.** c **3.** a **4.** b

5. $P_1 = -\frac{1}{2}x + 6$



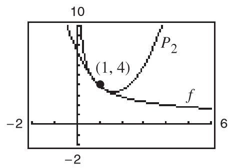
P_1 is the first-degree Taylor polynomial for f at 4.

7. $P_1 = \sqrt{2}x + \sqrt{2}(4 - \pi)/4$



P_1 is the first-degree Taylor polynomial for f at $\pi/4$.

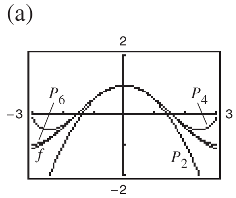
9.



x	0	0.8	0.9	1	1.1
$f(x)$	Error	4.4721	4.2164	4.0000	3.8139
$P_2(x)$	7.5000	4.4600	4.2150	4.0000	3.8150

x	1.2	2
$f(x)$	3.6515	2.8284
$P_2(x)$	3.6600	3.5000

11. (a)



(b) $f^{(2)}(0) = -1$ $\quad P_2^{(2)}(0) = -1$
$f^{(4)}(0) = 1$ $\quad P_4^{(4)}(0) = 1$
$f^{(6)}(0) = -1$ $\quad P_6^{(6)}(0) = -1$

(c) $f^{(n)}(0) = P_n^{(n)}(0)$

13. $1 + 3x + \frac{9}{2}x^2 + \frac{9}{2}x^3 + \frac{27}{8}x^4$

15. $1 - \frac{1}{2}x + \frac{1}{8}x^2 - \frac{1}{48}x^3 + \frac{1}{384}x^4$

17. $x - \frac{1}{6}x^3 + \frac{1}{120}x^5$ **19.** $x + x^2 + \frac{1}{2}x^3 + \frac{1}{6}x^4$

21. $1 - x + x^2 - x^3 + x^4 - x^5$ **23.** $1 + \frac{1}{2}x^2$

25. $2 - 2(x - 1) + 2(x - 1)^2 - 2(x - 1)^3$

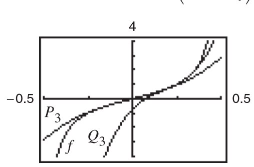
27. $2 + \frac{1}{4}(x - 4) - \frac{1}{64}(x - 4)^2 + \frac{1}{512}(x - 4)^3$

29. $\ln 2 + \frac{1}{2}(x - 2) - \frac{1}{8}(x - 2)^2 + \frac{1}{24}(x - 2)^3 - \frac{1}{64}(x - 2)^4$

31. (a) $P_3(x) = \pi x + \dfrac{\pi^3}{3}x^3$

(b)

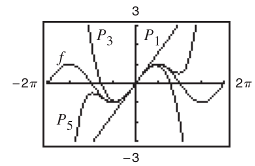
$$Q_3(x) = 1 + 2\pi\left(x - \frac{1}{4}\right) + 2\pi^2\left(x - \frac{1}{4}\right)^2 + \frac{8\pi^3}{3}\left(x - \frac{1}{4}\right)^3$$



33. (a)

x	0	0.25	0.50	0.75	1.00
$\sin x$	0	0.2474	0.4794	0.6816	0.8415
$P_1(x)$	0	0.25	0.50	0.75	1.00
$P_3(x)$	0	0.2474	0.4792	0.6797	0.8333
$P_5(x)$	0	0.2474	0.4794	0.6817	0.8417

(b)



(c) As the distance increases, the polynomial approximation becomes less accurate.

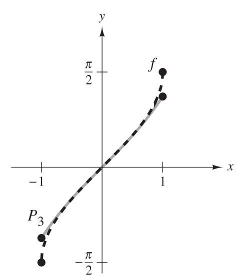
35. (a) $P_3(x) = x + \frac{1}{6}x^3$

(b)

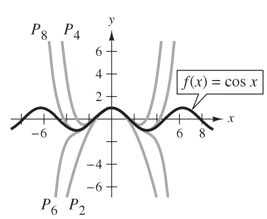
x	−0.75	−0.50	−0.25	0	0.25
$f(x)$	−0.848	−0.524	−0.253	0	0.253
$P_3(x)$	−0.820	−0.521	−0.253	0	0.253

x	0.50	0.75
$f(x)$	0.524	0.848
$P_3(x)$	0.521	0.820

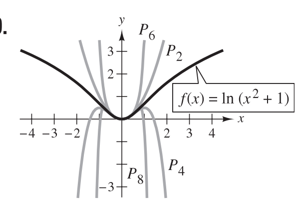
(c)



37.



39.



41. 4.3984 **43.** 0.7419 **45.** $R_4 \le 2.03 \times 10^{-5}$; 0.000001

47. $R_3 \le 7.82 \times 10^{-3}$; 0.00085 **49.** 3 **51.** 5

53. $n = 9$; $\ln(1.5) \approx 0.4055$ **55.** $n = 16$; $e^{-\pi(1.3)} \approx 0.01684$

57. $-0.3936 < x < 0$ **59.** $-0.9467 < x < 0.9467$

61. The graph of the approximating polynomial P and the elementary function f both pass through the point $(c, f(c))$, and the slope of the graph of P is the same as the slope of the graph of f at the point $(c, f(c))$. If P is of degree n, then the first n derivatives of f and P agree at c. This allows for the graph of P to resemble the graph of f near the point $(c, f(c))$.

63. See "Definitions of nth Taylor Polynomial and nth Maclaurin Polynomial" on page 652.

65. As the degree of the polynomial increases, the graph of the Taylor polynomial becomes a better and better approximation of the function within the interval of convergence. Therefore, the accuracy is increased.

67. (a) $f(x) \approx P_4(x) = 1 + x + (1/2)x^2 + (1/6)x^3 + (1/24)x^4$
$g(x) \approx Q_5(x) = x + x^2 + (1/2)x^3 + (1/6)x^4 + (1/24)x^5$
$Q_5(x) = xP_4(x)$
(b) $g(x) \approx P_6(x) = x^2 - x^4/3! + x^6/5!$
(c) $g(x) \approx P_4(x) = 1 - x^2/3! + x^4/5!$

69. (a) $Q_2(x) = -1 + (\pi^2/32)(x + 2)^2$
(b) $R_2(x) = -1 + (\pi^2/32)(x - 6)^2$
(c) No. Horizontal translations of the result in part (a) are possible only at $x = -2 + 8n$ (where n is an integer) because the period of f is 8.

71. Proof

73. As you move away from $x = c$, the Taylor polynomial becomes less and less accurate.

Section 9.8 (page 668)

1. 0 **3.** 2 **5.** $R = 1$ **7.** $R = \frac{1}{4}$ **9.** $R = \infty$
11. $(-4, 4)$ **13.** $(-1, 1]$ **15.** $(-\infty, \infty)$ **17.** $x = 0$
19. $(-4, 4)$ **21.** $(-5, 13]$ **23.** $(0, 2]$ **25.** $(0, 6)$
27. $\left(-\frac{1}{2}, \frac{1}{2}\right)$ **29.** $(-\infty, \infty)$ **31.** $(-1, 1)$ **33.** $x = 3$
35. $R = c$ **37.** $(-k, k)$ **39.** $(-1, 1)$

41. $\displaystyle\sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!}$ **43.** $\displaystyle\sum_{n=1}^{\infty} \frac{x^{2n-1}}{(2n-1)!}$

45. (a) $(-3, 3)$ (b) $(-3, 3)$ (c) $(-3, 3)$ (d) $[-3, 3)$
47. (a) $(0, 2]$ (b) $(0, 2)$ (c) $(0, 2)$ (d) $[0, 2]$
49. c; $S_1 = 1, S_2 = 1.33$ **50.** a; $S_1 = 1, S_2 = 1.67$
51. b; diverges **52.** d; alternating
53. b **54.** c **55.** d **56.** a

57. A series of the form
$$\sum_{n=0}^{\infty} a_n(x - c)^n = a_0 + a_1(x - c) + a_2(x - c)^2 + \cdots + a_n(x - c)^n + \cdots$$
is called a power series centered at c, where c is a constant.

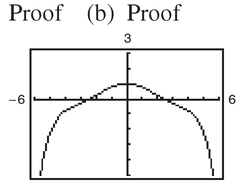
59. 1. A single point 2. An interval centered at c
3. The entire real line

61. Answers will vary.

63. (a) For $f(x)$: $(-\infty, \infty)$; For $g(x)$: $(-\infty, \infty)$
(b) Proof (c) Proof (d) $f(x) = \sin x$; $g(x) = \cos x$

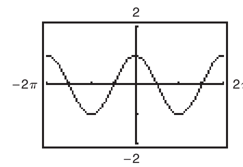
65–69. Proofs

71. (a) Proof (b) Proof
(c)



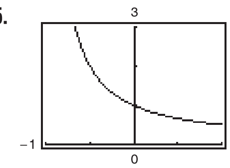
(d) 0.92

73.



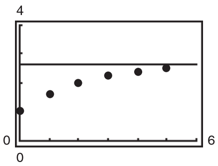
$f(x) = \cos x$

75.

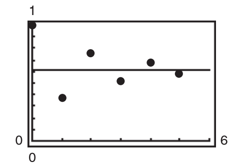


$f(x) = 1/(1 + x)$

77. (a) $\frac{8}{3}$



(b) $\frac{8}{13}$



(c) The alternating series converges more rapidly. The partial sums of the series of positive terms approach the sum from below. The partial sums of the alternating series alternate sides of the horizontal line representing the sum.

(d)

M	10	100	1000	10,000
N	5	14	24	35

79. False. Let $a_n = (-1)^n/(n2^n)$ **81.** True **83.** Proof
85. (a) $(-1, 1)$ (b) $f(x) = (c_0 + c_1x + c_2x^2)/(1 - x^3)$
87. Proof

Section 9.9 (page 676)

1. $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{4^{n+1}}$ **3.** $\displaystyle\sum_{n=0}^{\infty} \frac{3(-1)^n x^n}{4^{n+1}}$ **5.** $\displaystyle\sum_{n=0}^{\infty} \frac{(x-1)^n}{2^{n+1}}$ **7.** $\displaystyle\sum_{n=0}^{\infty} (3x)^n$
$(-1, 3)$ $\left(-\frac{1}{3}, \frac{1}{3}\right)$

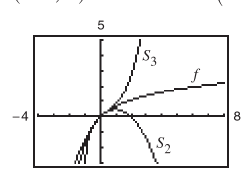
9. $\displaystyle -\frac{5}{9}\sum_{n=0}^{\infty}\left[\frac{2}{9}(x + 3)\right]^n$ **11.** $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n 2^{n+1} x^n}{3^{n+1}}$
$\left(-\dfrac{15}{2}, \dfrac{3}{2}\right)$ $\left(-\dfrac{3}{2}, \dfrac{3}{2}\right)$

13. $\displaystyle\sum_{n=0}^{\infty}\left[\frac{1}{(-3)^n} - 1\right]x^n$ **15.** $\displaystyle\sum_{n=0}^{\infty} x^n[1 + (-1)^n] = 2\sum_{n=0}^{\infty} x^{2n}$
$(-1, 1)$ $(-1, 1)$

17. $\displaystyle 2\sum_{n=0}^{\infty} x^{2n}$ **19.** $\displaystyle\sum_{n=1}^{\infty} n(-1)^n x^{n-1}$ **21.** $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1}$
$(-1, 1)$ $(-1, 1)$ $(-1, 1]$

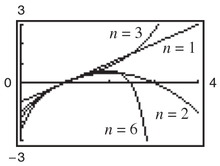
23. $\displaystyle\sum_{n=0}^{\infty} (-1)^n x^{2n}$ **25.** $\displaystyle\sum_{n=0}^{\infty} (-1)^n (2x)^{2n}$
$(-1, 1)$ $\left(-\frac{1}{2}, \frac{1}{2}\right)$

27.



x	0.0	0.2	0.4	0.6	0.8	1.0
S_2	0.000	0.180	0.320	0.420	0.480	0.500
$\ln(x + 1)$	0.000	0.182	0.336	0.470	0.588	0.693
S_3	0.000	0.183	0.341	0.492	0.651	0.833

29. (a)



(b) $\ln x, 0 < x \leq 2, R = 1$
(c) -0.6931
(d) $\ln(0.5)$; The error is approximately 0.

31. c **32.** d **33.** a **34.** b **35.** 0.245 **37.** 0.125

39. $\sum_{n=1}^{\infty} nx^{n-1},\ -1 < x < 1$ **41.** $\sum_{n=0}^{\infty} (2n + 1)x^n,\ -1 < x < 1$

43. $E(n) = 2$. Because the probability of obtaining a head on a single toss is $\frac{1}{2}$, it is expected that, on average, a head will be obtained in two tosses.

45. Because $\dfrac{1}{1 + x} = \dfrac{1}{1 - (-x)}$, substitute $(-x)$ into the geometric series.

47. Because $\dfrac{5}{1 + x} = 5\left(\dfrac{1}{1 - (-x)}\right)$, substitute $(-x)$ into the geometric series and then multiply the series by 5.

49. Proof **51.** (a) Proof (b) 3.14

53. $\ln \frac{3}{2} \approx 0.4055$; See Exercise 21.

55. $\ln \frac{7}{5} \approx 0.3365$; See Exercise 53.

57. $\arctan \frac{1}{2} \approx 0.4636$; See Exercise 56.

59. $f(x) = \arctan x$ is an odd function (symmetric to the origin).

61. The series in Exercise 56 converges to its sum at a lower rate because its terms approach 0 at a much lower rate.

63. The series converges on the interval $(-5, 3)$ and perhaps also at one or both endpoints.

65. $\sqrt{3}\pi/6$ **67.** $S_1 = 0.3183098862,\ 1/\pi \approx 0.3183098862$

Section 9.10 (page 687)

1. $\sum_{n=0}^{\infty} \dfrac{(2x)^n}{n!}$ **3.** $\dfrac{\sqrt{2}}{2} \sum_{n=0}^{\infty} \dfrac{(-1)^{n(n+1)/2}}{n!}\left(x - \dfrac{\pi}{4}\right)^n$

5. $\sum_{n=0}^{\infty} (-1)^n (x - 1)^n$ **7.** $\sum_{n=0}^{\infty} \dfrac{(-1)^n(x - 1)^{n+1}}{n + 1}$

9. $\sum_{n=0}^{\infty} \dfrac{(-1)^n(3x)^{2n+1}}{(2n + 1)!}$ **11.** $1 + x^2/2! + 5x^4/4! + \cdots$

13–15. Proofs **17.** $\sum_{n=0}^{\infty} (-1)^n(n + 1)x^n$

19. $1 + \sum_{n=1}^{\infty} \dfrac{1 \cdot 3 \cdot 5 \cdots (2n - 1)x^n}{2^n n!}$

21. $\dfrac{1}{2}\left[1 + \sum_{n=1}^{\infty} \dfrac{(-1)^n 1 \cdot 3 \cdot 5 \cdots (2n - 1)x^{2n}}{2^{3n} n!}\right]$

23. $1 + \dfrac{x}{2} + \sum_{n=2}^{\infty} \dfrac{(-1)^{n+1}\, 1 \cdot 3 \cdot 5 \cdots (2n - 3)x^n}{2^n n!}$

25. $1 + \dfrac{x^2}{2} + \sum_{n=2}^{\infty} \dfrac{(-1)^{n+1} 1 \cdot 3 \cdot 5 \cdots (2n - 3)x^{2n}}{2^n n!}$

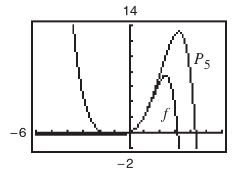
27. $\sum_{n=0}^{\infty} \dfrac{x^{2n}}{2^n n!}$ **29.** $\sum_{n=1}^{\infty} \dfrac{(-1)^{n-1}\, x^n}{n}$ **31.** $\sum_{n=0}^{\infty} \dfrac{(-1)^n(3x)^{2n+1}}{(2n + 1)!}$

33. $\sum_{n=0}^{\infty} \dfrac{(-1)^n\, 4^{2n}\, x^{2n}}{(2n)!}$ **35.** $\sum_{n=0}^{\infty} \dfrac{(-1)^n x^{3n}}{(2n)!}$ **37.** $\sum_{n=0}^{\infty} \dfrac{x^{2n+1}}{(2n + 1)!}$

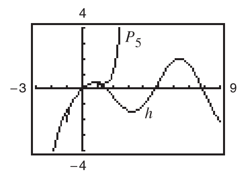
39. $\dfrac{1}{2}\left[1 + \sum_{n=0}^{\infty} \dfrac{(-1)^n(2x)^{2n}}{(2n)!}\right]$ **41.** $\sum_{n=0}^{\infty} \dfrac{(-1)^n x^{2n+2}}{(2n + 1)!}$

43. $\begin{cases} \sum_{n=0}^{\infty} \dfrac{(-1)^n x^{2n}}{(2n + 1)!}, & x \neq 0 \\ 1, & x = 0 \end{cases}$ **45.** Proof

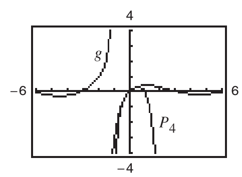
47. $P_5(x) = x + x^2 + \frac{1}{3}x^3 - \frac{1}{30}x^5$



49. $P_5(x) = x - \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{3}{40}x^5$



51. $P_4(x) = x - x^2 + \frac{5}{6}x^3 - \frac{5}{6}x^4$



53. c; $f(x) = x \sin x$ **54.** d; $f(x) = x \cos x$ **55.** a; $f(x) = xe^x$

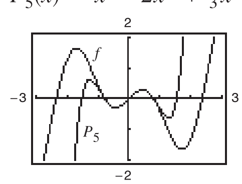
56. b; $f(x) = x^2\left(\dfrac{1}{1 + x}\right)$ **57.** $\sum_{n=0}^{\infty} \dfrac{(-1)^{(n+1)} x^{2n+3}}{(2n + 3)(n + 1)!}$

59. 0.6931 **61.** 7.3891 **63.** 0 **65.** 1 **67.** 0.8075

69. 0.9461 **71.** 0.4872 **73.** 0.2010 **75.** 0.7040

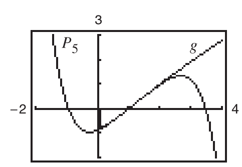
77. 0.3412

79. $P_5(x) = x - 2x^3 + \frac{2}{3}x^5$



$\left[-\frac{3}{4}, \frac{3}{4}\right]$

81. $P_5(x) = (x - 1) - \frac{1}{24}(x - 1)^3 + \frac{1}{24}(x - 1)^4 - \frac{71}{1920}(x - 1)^5$



$\left[\frac{1}{4}, 2\right]$

83. See "Guidelines for Finding a Taylor Series" on page 682.

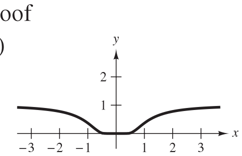
85. The binomial series is given by

$$(1 + x)^k = 1 + kx + \frac{k(k - 1)}{2!}x^2 + \frac{k(k - 1)(k - 2)}{3!}x^3 + \cdots.$$

The radius of convergence is $R = 1$.

87. Proof

89. (a)



(b) Proof

(c) $\sum_{n=0}^{\infty} 0x^n = 0 \neq f(x)$

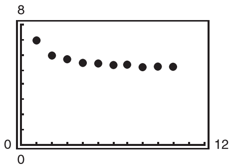
91. Proof **93.** 10 **95.** -0.0390625

97. $\sum_{n=0}^{\infty} \binom{k}{n} x^n$ **99.** Proof

Review Exercises for Chapter 9 (page 690)

1. $a_n = 1/(n! + 1)$ **3.** a **4.** c **5.** d **6.** b

7.



Converges to 5

9. Converges to 3 **11.** Diverges **13.** Converges to 0
15. Converges to 0 **17.** Converges to 0

19. (a)

n	1	2	3	4
A_n	\$8100.00	\$8201.25	\$8303.77	\$8407.56

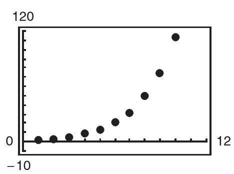
n	5	6	7	8
A_n	\$8512.66	\$8619.07	\$8726.80	\$8835.89

(b) \$13,148.96

21. (a)

n	5	10	15	20	25
S_n	13.2	113.3	873.8	6648.5	50,500.3

(b)



23. (a)

n	5	10	15	20	25
S_n	0.4597	0.4597	0.4597	0.4597	0.4597

(b)

25. 3 **27.** 5.5 **29.** (a) $\sum_{n=0}^{\infty} (0.09)(0.01)^n$ (b) $\frac{1}{11}$

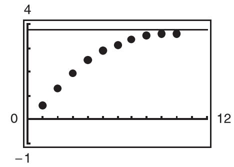
31. Diverges **33.** Diverges **35.** $45\frac{1}{3}$ m **37.** \$7630.70
39. Converges **41.** Diverges **43.** Diverges
45. Converges **47.** Diverges **49.** Converges
51. Converges **53.** Diverges **55.** Diverges
57. Converges **59.** Diverges

61. (a) Proof

(b)

n	5	10	15	20	25
S_n	2.8752	3.6366	3.7377	3.7488	3.7499

(c)



(d) 3.75

63. (a)

N	5	10	20	30	40
$\sum_{n=1}^{N} \frac{1}{n^2}$	1.4636	1.5498	1.5962	1.6122	1.6202
$\int_N^{\infty} \frac{1}{x^2}\,dx$	0.2000	0.1000	0.0500	0.0333	0.0250

(b)

N	5	10	20	30	40
$\sum_{n=1}^{N} \frac{1}{n^5}$	1.0367	1.0369	1.0369	1.0369	1.0369
$\int_N^{\infty} \frac{1}{x^5}\,dx$	0.0004	0.0000	0.0000	0.0000	0.0000

The series in part (b) converges more rapidly. This is evident from the integrals that give the remainders of the partial sums.

65. $P_3(x) = 1 - 3x + \frac{9}{2}x^2 - \frac{9}{2}x^3$ **67.** 0.996 **69.** 0.559

71. (a) 4 (b) 6 (c) 5 (d) 10 **73.** $(-10, 10)$

75. $[1, 3]$ **77.** Converges only at $x = 2$ **79.** Proof

81. $\sum_{n=0}^{\infty} \frac{2}{3}\left(\frac{x}{3}\right)^n$ **83.** $\sum_{n=0}^{\infty} \frac{2}{9}(n + 1)\left(\frac{x}{3}\right)^n$

85. $f(x) = \frac{3}{3 - 2x}, \left(-\frac{3}{2}, \frac{3}{2}\right)$ **87.** $\frac{\sqrt{2}}{2}\sum_{n=0}^{\infty} \frac{(-1)^{n(n+1)/2}}{n!}\left(x - \frac{3\pi}{4}\right)^n$

89. $\sum_{n=0}^{\infty} \frac{(x \ln 3)^n}{n!}$ **91.** $-\sum_{n=0}^{\infty} (x + 1)^n$

93. $1 + x/5 - 2x^2/25 + 6x^3/125 - 21x^4/625 + \cdots$

95. $\ln \frac{5}{4} \approx 0.2231$ **97.** $e^{1/2} \approx 1.6487$ **99.** $\cos \frac{2}{3} \approx 0.7859$

101. The series in Exercise 45 converges to its sum at a lower rate because its terms approach 0 at a lower rate.

103. (a)–(c) $1 + 2x + 2x^2 + \frac{4}{3}x^3$

105. $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n + 1)(2n + 1)!}$ **107.** $\sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{(n + 1)^2}$ **109.** 0

P.S. Problem Solving (page 693)

1. (a) 1 (b) Answers will vary. Example: $0, \frac{1}{3}, \frac{2}{3}$ (c) 0

3. Proof **5.** (a) Proof (b) Yes (c) Any distance

7. For $a = b$, the series converges conditionally. For no values of a and b does the series converge absolutely.

9. 665,280 **11.** (a) Proof (b) Diverges

13. Proof **15.** (a) Proof (b) Proof

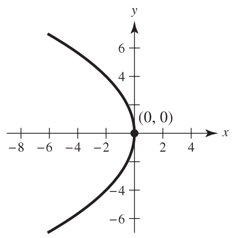
17. (a) The height is infinite. (b) The surface area is infinite. (c) Proof

Chapter 10

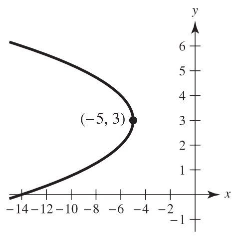
Section 10.1 (page 706)

1. h **2.** a **3.** e **4.** b **5.** f **6.** g **7.** c **8.** d

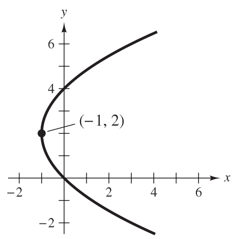
9. Vertex: $(0, 0)$
Focus: $(-2, 0)$
Directrix: $x = 2$



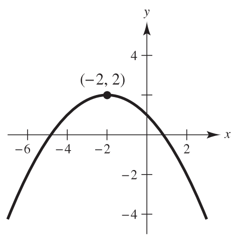
11. Vertex: $(-5, 3)$
Focus: $\left(-\frac{21}{4}, 3\right)$
Directrix: $x = -\frac{19}{4}$



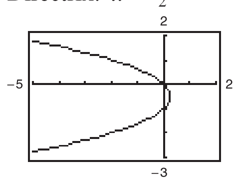
13. Vertex: $(-1, 2)$
Focus: $(0, 2)$
Directrix: $x = -2$



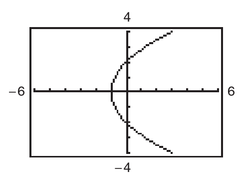
15. Vertex: $(-2, 2)$
Focus: $(-2, 1)$
Directrix: $y = 3$



17. Vertex: $\left(\frac{1}{4}, -\frac{1}{2}\right)$
Focus: $\left(0, -\frac{1}{2}\right)$
Directrix: $x = \frac{1}{2}$



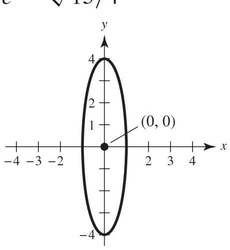
19. Vertex: $(-1, 0)$
Focus: $(0, 0)$
Directrix: $x = -2$



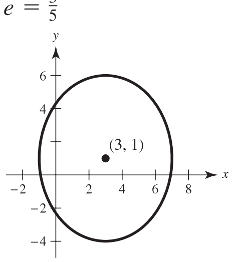
21. $y^2 - 8y + 8x - 24 = 0$ **23.** $x^2 - 32y + 160 = 0$

25. $x^2 + y - 4 = 0$ **27.** $5x^2 - 14x - 3y + 9 = 0$

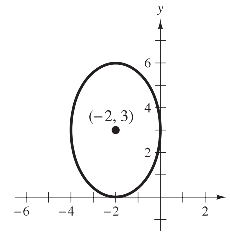
29. Center: $(0, 0)$
Foci: $\left(0, \pm\sqrt{15}\right)$
Vertices: $(0, \pm 4)$
$e = \sqrt{15}/4$



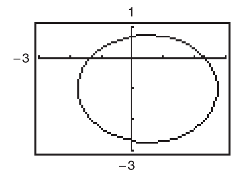
31. Center: $(3, 1)$
Foci: $(3, 4), (3, -2)$
Vertices: $(3, 6), (3, -4)$
$e = \frac{3}{5}$



33. Center: $(-2, 3)$
Foci: $\left(-2, 3 \pm \sqrt{5}\right)$
Vertices: $(-2, 6), (-2, 0)$
$e = \sqrt{5}/3$

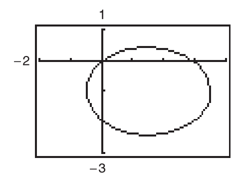


35. Center: $\left(\frac{1}{2}, -1\right)$
Foci: $\left(\frac{1}{2} \pm \sqrt{2}, -1\right)$
Vertices: $\left(\frac{1}{2} \pm \sqrt{5}, -1\right)$
To obtain the graph, solve for y and get



$y_1 = -1 + \sqrt{(57 + 12x - 12x^2)/20}$ and
$y_2 = -1 - \sqrt{(57 + 12x - 12x^2)/20}$.
Graph these equations in the same viewing window.

37. Center: $\left(\frac{3}{2}, -1\right)$
Foci: $\left(\frac{3}{2} \pm \sqrt{2}, -1\right)$
Vertices: $\left(-\frac{1}{2}, -1\right), \left(\frac{7}{2}, -1\right)$
To obtain the graph, solve for y and get

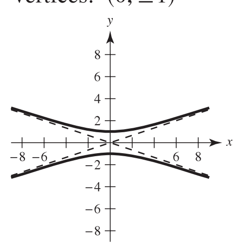


$y_1 = -1 + \sqrt{(7 + 12x - 4x^2)/8}$ and
$y_2 = -1 - \sqrt{(7 + 12x - 4x^2)/8}$.
Graph these equations in the same viewing window.

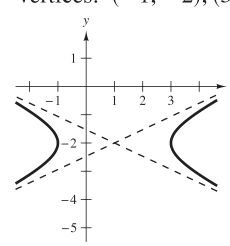
39. $x^2/36 + y^2/11 = 1$ **41.** $(x - 3)^2/9 + (y - 5)^2/16 = 1$

43. $x^2/16 + 7y^2/16 = 1$

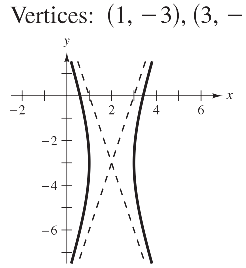
45. Center: $(0, 0)$
Foci: $\left(0, \pm\sqrt{10}\right)$
Vertices: $(0, \pm 1)$



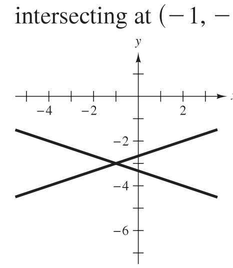
47. Center: $(1, -2)$
Foci: $\left(1 \pm \sqrt{5}, -2\right)$
Vertices: $(-1, -2), (3, -2)$



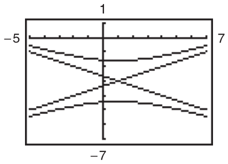
49. Center: $(2, -3)$
Foci: $\left(2 \pm \sqrt{10}, -3\right)$
Vertices: $(1, -3), (3, -3)$



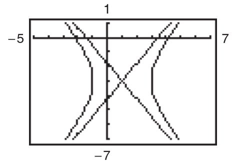
51. Degenerate hyperbola
Graph is two lines
$y = -3 \pm \frac{1}{3}(x + 1)$
intersecting at $(-1, -3)$.



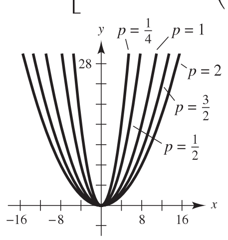
53. Center: $(1, -3)$
Foci: $\left(1, -3 \pm 2\sqrt{5}\right)$
Vertices: $\left(1, -3 \pm \sqrt{2}\right)$
Asymptotes:
$y = \frac{1}{3}x - \frac{1}{3} - 3;$
$y = -\frac{1}{3}x + \frac{1}{3} - 3$



55. Center: $(1, -3)$
Foci: $\left(1 \pm \sqrt{10}, -3\right)$
Vertices: $(-1, -3), (3, -3)$
Asymptotes:
$y = \sqrt{6}x/2 - \sqrt{6}/2 - 3;$
$y = -\sqrt{6}x/2 + \sqrt{6}/2 - 3$

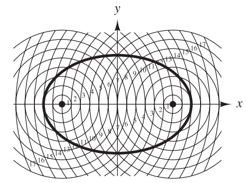


57. $x^2/1 - y^2/25 = 1$ **59.** $y^2/9 - (x - 2)^2/(9/4) = 1$
61. $y^2/4 - x^2/12 = 1$ **63.** $(x - 3)^2/9 - (y - 2)^2/4 = 1$
65. (a) $\left(6, \sqrt{3}\right)$: $2x - 3\sqrt{3}y - 3 = 0$
$\left(6, -\sqrt{3}\right)$: $2x + 3\sqrt{3}y - 3 = 0$
(b) $\left(6, \sqrt{3}\right)$: $9x + 2\sqrt{3}y - 60 = 0$
$\left(6, -\sqrt{3}\right)$: $9x - 2\sqrt{3}y - 60 = 0$
67. Ellipse **69.** Parabola **71.** Circle
73. Circle **75.** Hyperbola
77. (a) A parabola is the set of all points (x, y) that are equidistant from a fixed line and a fixed point not on the line.
(b) For directrix $y = k - p$: $(x - h)^2 = 4p(y - k)$
For directrix $x = h - p$: $(y - k)^2 = 4p(x - h)$
(c) If P is a point on a parabola, then the tangent line to the parabola at P makes equal angles with the line passing through P and the focus, and with the line passing through P parallel to the axis of the parabola.
79. (a) A hyperbola is the set of all points (x, y) for which the absolute value of the difference between the distances from two distinct fixed points is constant.
(b) Transverse axis is horizontal: $\dfrac{(x - h)^2}{a^2} - \dfrac{(y - k)^2}{b^2} = 1$
Transverse axis is vertical: $\dfrac{(y - k)^2}{a^2} - \dfrac{(x - h)^2}{b^2} = 1$
(c) Transverse axis is horizontal:
$y = k + (b/a)(x - h)$ and $y = k - (b/a)(x - h)$
Transverse axis is vertical:
$y = k + (a/b)(x - h)$ and $y = k - (a/b)(x - h)$
81. $\frac{9}{4}$ m **83.** $y = 2ax_0x - ax_0^2$ **85.** (a) Proof (b) Proof
87. $x_0 = 2\sqrt{3}/3$; Distance from hill: $2\sqrt{3}/3 - 1$
89. $[16(4 + 3\sqrt{3} - 2\pi)]/3 \approx 15.536$ ft^2
91. (a) $y = (1/180)x^2$
(b) $10\left[2\sqrt{13} + 9\ln\left(\dfrac{2 + \sqrt{13}}{3}\right)\right] \approx 128.4$ m
93.

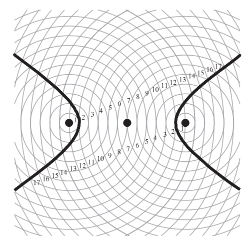


As p increases, the graph of $x^2 = 4py$ gets wider.

95. (a) $L = 2a$
(b) The thumbtacks are located at the foci, and the length of string is the constant sum of distances from the foci.
97.



99. Proof **101.** $e \approx 0.1776$
103. $e \approx 0.9671$ **105.** $\left(0, \frac{25}{3}\right)$
107. Minor-axis endpoints: $(-6, -2), (0, -2)$
Major-axis endpoints: $(-3, -6), (-3, 2)$
109. (a) Area $= 2\pi$
(b) Volume $= 8\pi/3$
Surface area $= [2\pi(9 + 4\sqrt{3}\pi)]/9 \approx 21.48$
(c) Volume $= 16\pi/3$
Surface area $= \dfrac{4\pi\left[6 + \sqrt{3}\ln\left(2 + \sqrt{3}\right)\right]}{3} \approx 34.69$
111. 37.96 **113.** 40 **115.** $(x - 6)^2/9 - (y - 2)^2/7 = 1$
117.



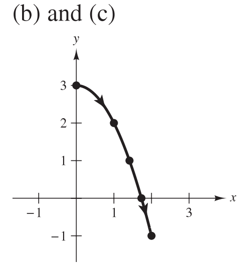
119. Proof
121. $x = \left(-90 + 96\sqrt{2}\right)/7 \approx 6.538$
$y = \left(160 - 96\sqrt{2}\right)/7 \approx 3.462$
123. There are four points of intersection.
At $\left(\dfrac{\sqrt{2}ac}{\sqrt{2a^2 - b^2}}, \dfrac{b^2}{\sqrt{2}\sqrt{2a^2 - b^2}}\right)$, the slopes of the tangent lines are $y'_e = -c/a$ and $y'_h = a/c$.
Because the slopes are negative reciprocals, the tangent lines are perpendicular. Similarly, the curves are perpendicular at the other three points of intersection.
125. False. See the definition of a parabola. **127.** True
129. True **131.** Putnam Problem B4, 1976

Section 10.2 (page 718)

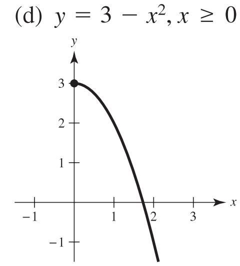
1. (a)

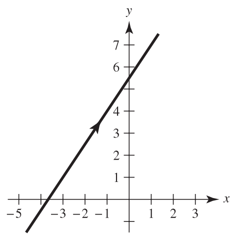
t	0	1	2	3	4
x	0	1	$\sqrt{2}$	$\sqrt{3}$	2
y	3	2	1	0	-1

(b) and (c)

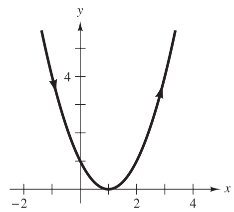


(d) $y = 3 - x^2, x \geq 0$

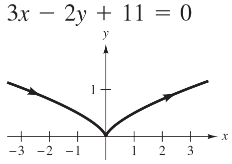


3.


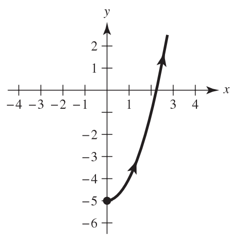
$3x - 2y + 11 = 0$

5.


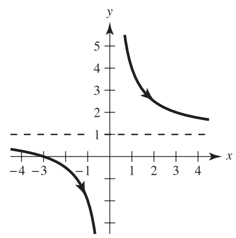
$y = (x - 1)^2$

7.


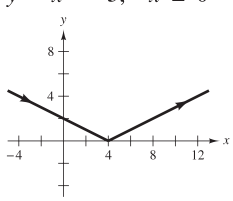
$y = \frac{1}{2}x^{2/3}$

9.


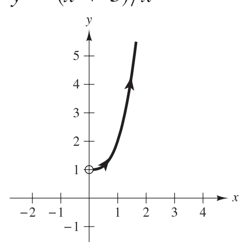
$y = x^2 - 5, \quad x \geq 0$

11.


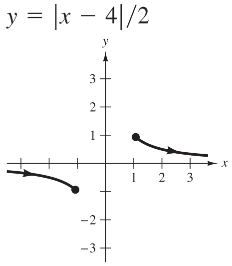
$y = (x + 3)/x$

13.


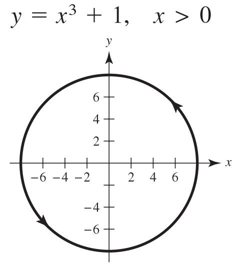
$y = |x - 4|/2$

15.


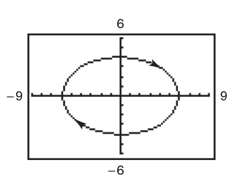
$y = x^3 + 1, \quad x > 0$

17.


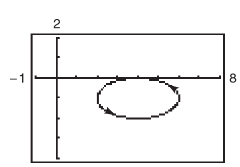
$y = 1/x, \quad |x| \geq 1$

19.


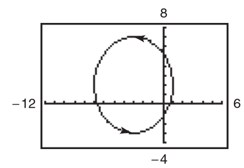
$x^2 + y^2 = 64$

21.


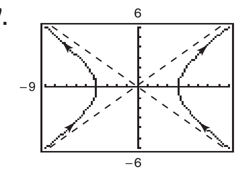
$\dfrac{x^2}{36} + \dfrac{y^2}{16} = 1$

23.


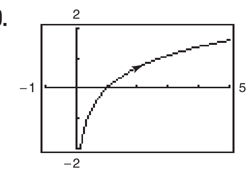
$\dfrac{(x - 4)^2}{4} + \dfrac{(y + 1)^2}{1} = 1$

25.


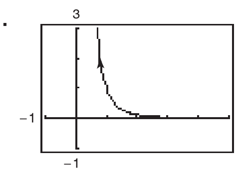
$\dfrac{(x + 3)^2}{16} + \dfrac{(y - 2)^2}{25} = 1$

27.


$\dfrac{x^2}{16} - \dfrac{y^2}{9} = 1$

29.


$y = \ln x$

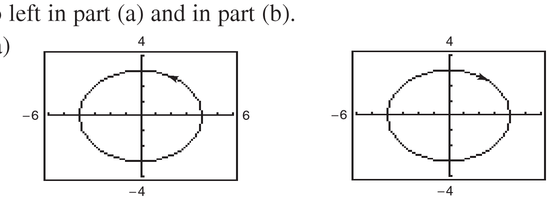
31.


$y = 1/x^3, \quad x > 0$

33. Each curve represents a portion of the line $y = 2x + 1$.

	Domain	*Orientation*	*Smooth*
(a)	$-\infty < x < \infty$	Up	Yes
(b)	$-1 \leq x \leq 1$	Oscillates	No, $\dfrac{dx}{d\theta} = \dfrac{dy}{d\theta} = 0$ when $\theta = 0, \pm\pi, \pm 2\pi, \ldots$
(c)	$0 < x < \infty$	Down	Yes
(d)	$0 < x < \infty$	Up	Yes

35. (a) and (b) represent the parabola $y = 2(1 - x^2)$ for $-1 \leq x \leq 1$. The curve is smooth. The orientation is from right to left in part (a) and in part (b).

37. (a)


(b) The orientation is reversed. (c) The orientation is reversed.

(d) Answers will vary. For example,

$x = 2 \sec t \qquad x = 2 \sec(-t)$

$y = 5 \sin t \qquad y = 5 \sin(-t)$

have the same graphs, but their orientations are reversed.

39. $y - y_1 = \dfrac{y_2 - y_1}{x_2 - x_1}(x - x_1)$

41. $\dfrac{(x - h)^2}{a^2} + \dfrac{(y - k)^2}{b^2} = 1$

43. $x = 4t$

$y = -7t$

(Solution is not unique.)

45. $x = 3 + 2 \cos \theta$

$y = 1 + 2 \sin \theta$

(Solution is not unique.)

47. $x = 10 \cos \theta$

$y = 6 \sin \theta$

(Solution is not unique.)

49. $x = 4 \sec \theta$

$y = 3 \tan \theta$

(Solution is not unique.)

51. $x = t$

$y = 6t - 5;$

$x = t + 1$

$y = 6t + 1$

(Solution is not unique.)

53. $x = t$

$y = t^3;$

$x = \tan t$

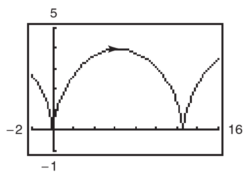
$y = \tan^3 t$

(Solution is not unique.)

55. $x = t + 3, y = 2t + 1$

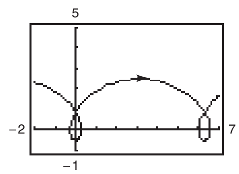
57. $x = t, y = t^2$

59.



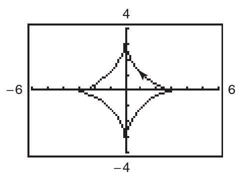
Not smooth at $\theta = 2n\pi$

61.



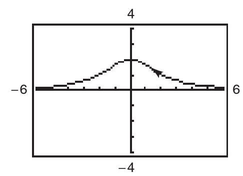
Smooth everywhere

63.



Not smooth at $\theta = \frac{1}{2}n\pi$

65.



Smooth everywhere

67. Each point (x, y) in the plane is determined by the plane curve $x = f(t)$, $y = g(t)$. For each t, plot (x, y). As t increases, the curve is traced out in a specific direction called the orientation of the curve.

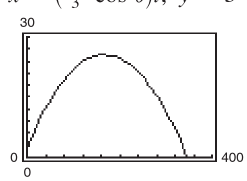
69. $x = a\theta - b \sin \theta$; $y = a - b \cos \theta$

71. False. The graph of the parametric equations is the portion of the line $y = x$ when $x \geq 0$.

73. True

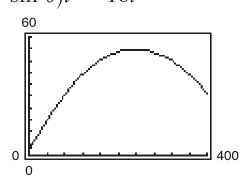
75. (a) $x = \left(\frac{440}{3} \cos \theta\right)t$; $y = 3 + \left(\frac{440}{3} \sin \theta\right)t - 16t^2$

(b)



Not a home run

(c)



Home run

(d) 19.4°

Section 10.3 (page 727)

1. $-3/t$ **3.** -1

5. $\dfrac{dy}{dx} = \dfrac{3}{4}, \dfrac{d^2y}{dx^2} = 0$; Neither concave upward nor concave downward

7. $dy/dx = 2t + 3$, $d^2y/dx^2 = 2$
At $t = -1$, $dy/dx = 1$, $d^2y/dx^2 = 2$; Concave upward

9. $dy/dx = -\cot \theta$, $d^2y/dx^2 = -(\csc \theta)^3/4$
At $\theta = \pi/4$, $dy/dx = -1$, $d^2y/dx^2 = -\sqrt{2}/2$;
Concave downward

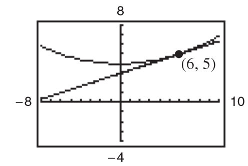
11. $dy/dx = 2 \csc \theta$, $d^2y/dx^2 = -2 \cot^3 \theta$
At $\theta = \pi/6$, $dy/dx = 4$, $d^2y/dx^2 = -6\sqrt{3}$;
Concave downward

13. $dy/dx = -\tan \theta$, $d^2y/dx^2 = \sec^4 \theta \csc \theta/3$
At $\theta = \pi/4$, $dy/dx = -1$, $d^2y/dx^2 = 4\sqrt{2}/3$;
Concave upward

15. $\left(-2/\sqrt{3}, 3/2\right)$: $3\sqrt{3}x - 8y + 18 = 0$
$(0, 2)$: $y - 2 = 0$
$\left(2\sqrt{3}, 1/2\right)$: $\sqrt{3}x + 8y - 10 = 0$

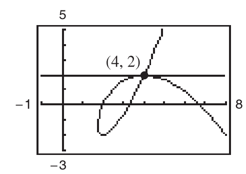
17. $(0, 0)$: $2y - x = 0$
$(-3, -1)$: $y + 1 = 0$
$(-3, 3)$: $2x - y + 9 = 0$

19. (a) and (d)



(b) At $t = 1$, $dx/dt = 6$, $dy/dt = 2$, and $dy/dx = 1/3$.
(c) $y = \frac{1}{3}x + 3$

21. (a) and (d)



(b) At $t = -1$, $dx/dt = -3$, $dy/dt = 0$, and $dy/dx = 0$.
(c) $y = 2$

23. $y = \pm\frac{3}{4}x$ **25.** $y = 3x - 5$ and $y = 1$

27. Horizontal: $(1, 0), (-1, \pi), (1, -2\pi)$
Vertical: $(\pi/2, 1), (-3\pi/2, -1), (5\pi/2, 1)$

29. Horizontal: $(4, 0)$
Vertical: None

31. Horizontal: $(5, -2), (3, 2)$
Vertical: None

33. Horizontal: $(0, 3), (0, -3)$
Vertical: $(3, 0), (-3, 0)$

35. Horizontal: $(5, -1), (5, -3)$
Vertical: $(8, -2), (2, -2)$

37. Horizontal: None
Vertical: $(1, 0), (-1, 0)$

39. Concave downward: $-\infty < t < 0$
Concave upward: $0 < t < \infty$

41. Concave upward: $t > 0$

43. Concave downward: $0 < t < \pi/2$
Concave upward: $\pi/2 < t < \pi$

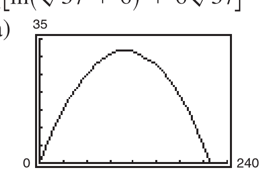
45. $\displaystyle\int_1^3 \sqrt{4t^2 - 3t + 9}\, dt$ **47.** $\displaystyle\int_{-2}^2 \sqrt{e^{2t} + 4}\, dt$

49. $4\sqrt{13} \approx 14.422$ **51.** $70\sqrt{5} \approx 156.525$

53. $\sqrt{2}(1 - e^{-\pi/2}) \approx 1.12$

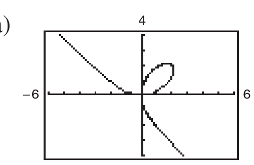
55. $\frac{1}{12}\left[\ln\left(\sqrt{37} + 6\right) + 6\sqrt{37}\right] \approx 3.249$ **57.** $6a$ **59.** $8a$

61. (a)



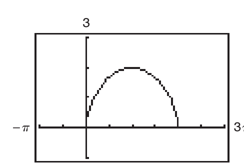
(b) 219.2 ft
(c) 230.8 ft

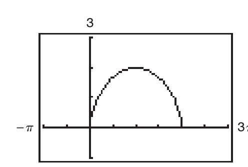
63. (a)



(b) $(0, 0)$, $\left(4\sqrt[3]{2}/3, 4\sqrt[3]{4}/3\right)$
(c) About 6.557

65. (a)





(b) The average speed of the particle on the second path is twice the average speed of the particle on the first path.
(c) 4π

67. $S = 2\pi \int_0^4 \sqrt{10}(t + 2)\, dt = 32\pi\sqrt{10} \approx 317.907$

69. $S = 2\pi \int_0^{\pi/2} \left(\sin\theta \cos\theta \sqrt{4\cos^2\theta + 1}\right) d\theta = \dfrac{(5\sqrt{5} - 1)\pi}{6}$
≈ 5.330

71. (a) $27\pi\sqrt{13}$ (b) $18\pi\sqrt{13}$ **73.** 50π **75.** $12\pi a^2/5$

77. See Theorem 10.7, Parametric Form of the Derivative, on page 721.

79. 6

81. (a) $S = 2\pi \int_a^b g(t) \sqrt{\left(\dfrac{dx}{dt}\right)^2 + \left(\dfrac{dy}{dt}\right)^2}\, dt$
(b) $S = 2\pi \int_a^b f(t) \sqrt{\left(\dfrac{dx}{dt}\right)^2 + \left(\dfrac{dy}{dt}\right)^2}\, dt$

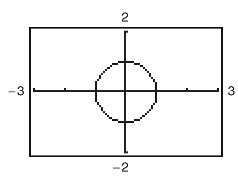
83. Proof **85.** $3\pi/2$ **87.** d **88.** b **89.** f **90.** c

91. a **92.** e **93.** $\left(\frac{3}{4}, \frac{8}{5}\right)$ **95.** 288π

97. (a) $dy/dx = \sin\theta/(1 - \cos\theta)$; $d^2y/dx^2 = -1/[a(\cos\theta - 1)^2]$
(b) $y = (2 + \sqrt{3})[x - a(\pi/6 - \frac{1}{2})] + a(1 - \sqrt{3}/2)$
(c) $(a(2n + 1)\pi, 2a)$
(d) Concave downward on $(0, 2\pi)$, $(2\pi, 4\pi)$, etc.
(e) $s = 8a$

99. Proof

101. (a)



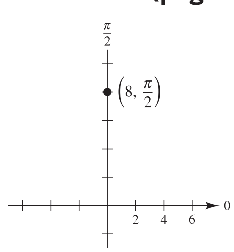
(b) Circle of radius 1 and center at $(0, 0)$ except the point $(-1, 0)$
(c) As t increases from -20 to 0, the speed increases, and as t increases from 0 to 20, the speed decreases.

103. False: $\dfrac{d^2y}{dx^2} = \dfrac{\dfrac{d}{dt}\left[\dfrac{g'(t)}{f'(t)}\right]}{f'(t)} = \dfrac{f'(t)g''(t) - g'(t)f''(t)}{[f'(t)]^3}$.

105. About 982 ft

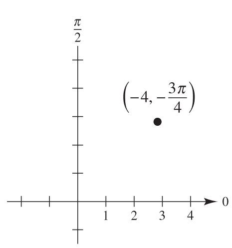
Section 10.4 (page 738)

1.



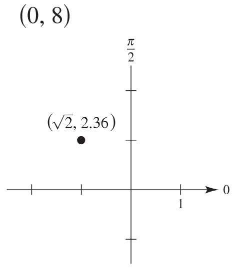
(0, 8)

3.



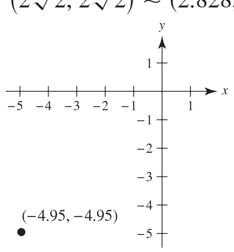
$(2\sqrt{2}, 2\sqrt{2}) \approx (2.828, 2.828)$

5.

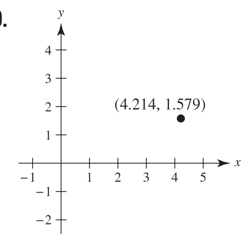


$(-1.004, 0.996)$

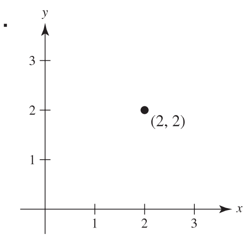
7.



9.

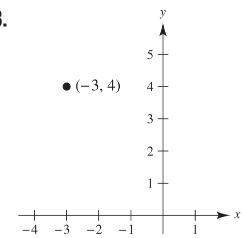


11.



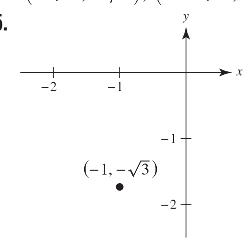
$(2\sqrt{2}, \pi/4), (-2\sqrt{2}, 5\pi/4)$

13.



$(5, 2.214), (-5, 5.356)$

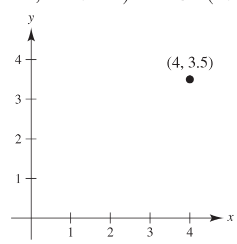
15.



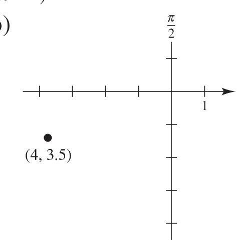
$(2, 4\pi/3), (-2, \pi/3)$

17. $(3.606, -0.588)$ **19.** $(3.052, 0.960)$

21. (a)

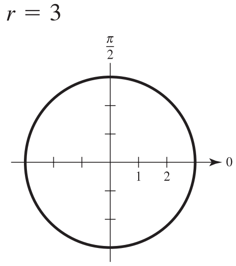


(b)

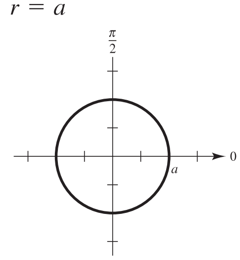


23. c **24.** b **25.** a **26.** d

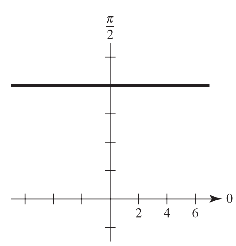
27. $r = 3$



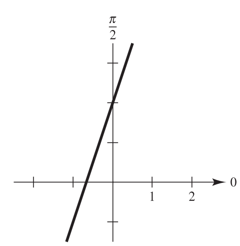
29. $r = a$



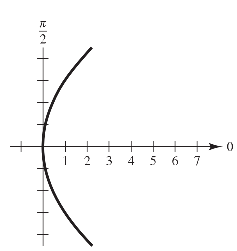
31. $r = 8\csc\theta$



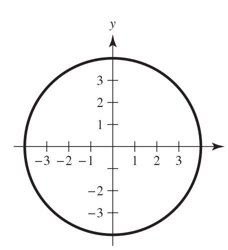
33. $r = \dfrac{-2}{3\cos\theta - \sin\theta}$



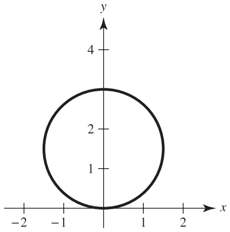
35. $r = 9\csc^2\theta\cos\theta$



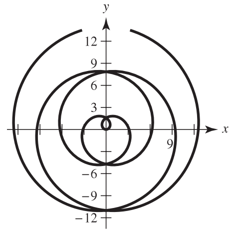
37. $x^2 + y^2 = 16$



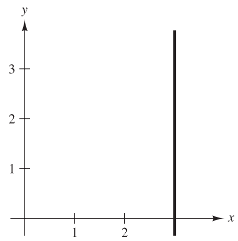
39. $x^2 + y^2 - 3y = 0$



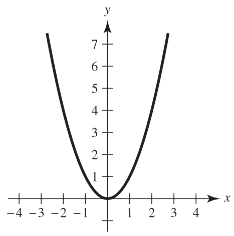
41. $\sqrt{x^2 + y^2} = \arctan(y/x)$



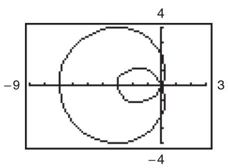
43. $x - 3 = 0$



45. $x^2 - y = 0$

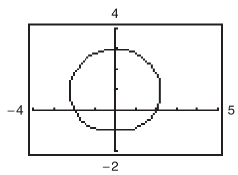


47.



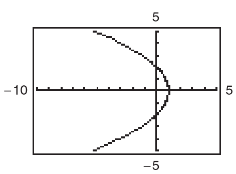
$0 \le \theta < 2\pi$

49.



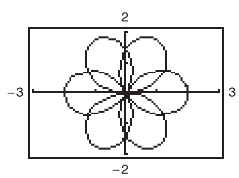
$0 \le \theta < 2\pi$

51.



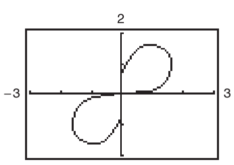
$-\pi < \theta < \pi$

53.



$0 \le \theta < 4\pi$

55.



$0 \le \theta < \pi/2$

57. $(x - h)^2 + (y - k)^2 = h^2 + k^2$
Radius: $\sqrt{h^2 + k^2}$
Center: (h, k)

59. $\sqrt{17}$ **61.** About 5.6

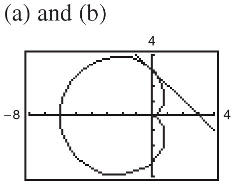
63. $\dfrac{dy}{dx} = \dfrac{2\cos\theta(3\sin\theta + 1)}{6\cos^2\theta - 2\sin\theta - 3}$

$(5, \pi/2)$: $dy/dx = 0$

$(2, \pi)$: $dy/dx = -2/3$

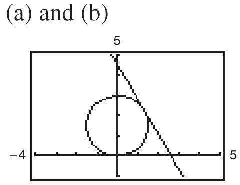
$(-1, 3\pi/2)$: $dy/dx = 0$

65. (a) and (b)



(c) $dy/dx = -1$

67. (a) and (b)

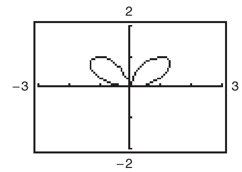


(c) $dy/dx = -\sqrt{3}$

69. Horizontal: $(2, 3\pi/2)$, $\left(\frac{1}{2}, \pi/6\right)$, $\left(\frac{1}{2}, 5\pi/6\right)$
Vertical: $\left(\frac{3}{2}, 7\pi/6\right)$, $\left(\frac{3}{2}, 11\pi/6\right)$

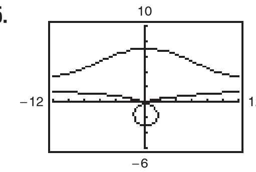
71. $(5, \pi/2)$, $(1, 3\pi/2)$

73.



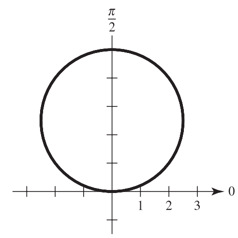
(0, 0), (1.4142, 0.7854), (1.4142, 2.3562)

75.



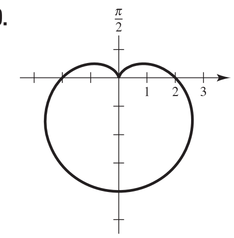
(7, 1.5708), (3, 4.7124)

77.



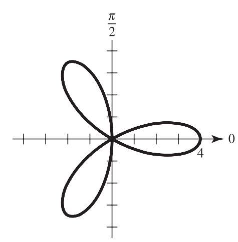
$\theta = 0$

79.



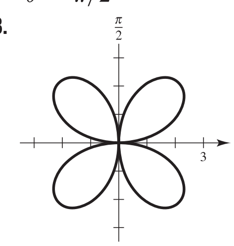
$\theta = \pi/2$

81.



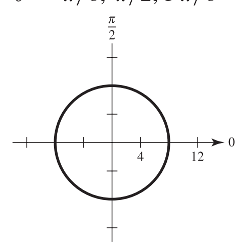
$\theta = \pi/6, \pi/2, 5\pi/6$

83.

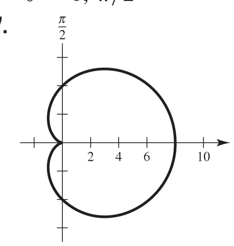


$\theta = 0, \pi/2$

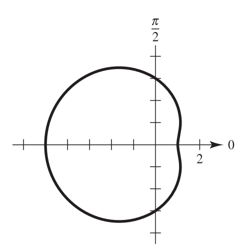
85.



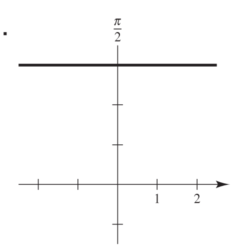
87.



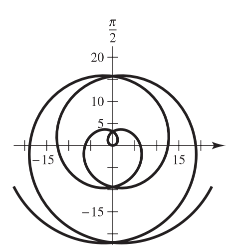
89.



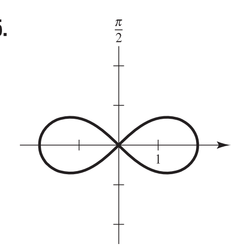
91.



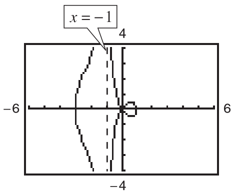
93.



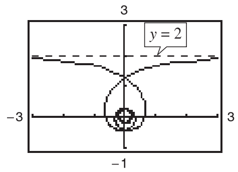
95.



97.



99.



101. The rectangular coordinate system is a collection of points of the form (x, y), where x is the directed distance from the y-axis to the point and y is the directed distance from the x-axis to the point. Every point has a unique representation.

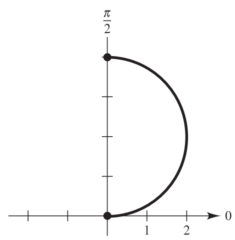
The polar coordinate system is a collection of points of the form (r, θ), where r is the directed distance from the origin O to a point P and θ is the directed angle, measured counterclockwise, from the polar axis to the segment $\overline{OP}$. Polar coordinates do not have unique representations.

103. Slope of tangent line to graph of $r = f(\theta)$ at (r, θ) is

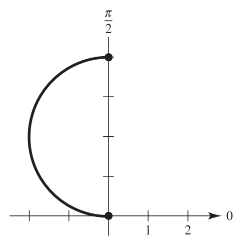
$$\frac{dy}{dx} = \frac{f(\theta)\cos\theta + f'(\theta)\sin\theta}{-f(\theta)\sin\theta + f'(\theta)\cos\theta}.$$

If $f(\alpha) = 0$ and $f'(\alpha) \neq 0$, then $\theta = \alpha$ is tangent at the pole.

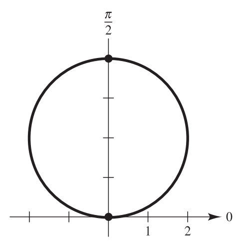
105. (a)



(b)



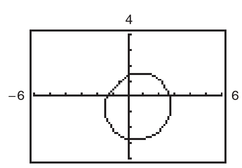
(c)



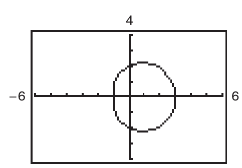
107. Proof

109. (a) $r = 2 - \sin(\theta - \pi/4)$

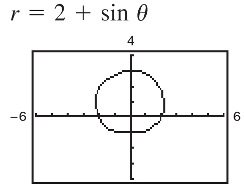
$= 2 - \dfrac{\sqrt{2}(\sin\theta - \cos\theta)}{2}$



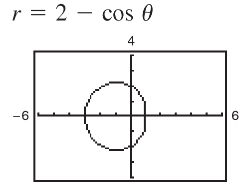
(b) $r = 2 + \cos\theta$



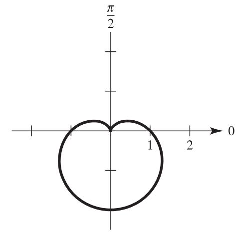
(c) $r = 2 + \sin\theta$



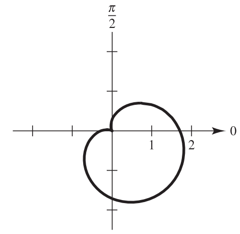
(d) $r = 2 - \cos\theta$



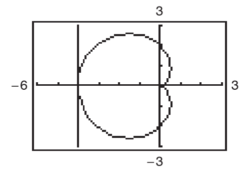
111. (a)



(b)

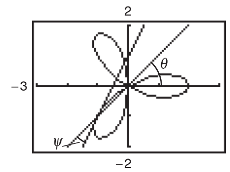


113.



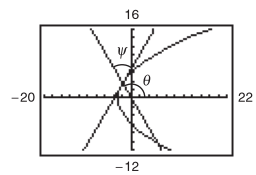
$\psi = \pi/2$

115.



$\psi = \arctan\frac{1}{3} \approx 18.4°$

117.



$\psi = \pi/3, 60°$

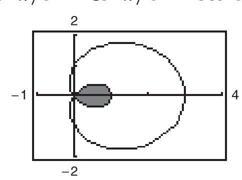
119. True **121.** True

Section 10.5 (page 747)

1. $8\int_0^{\pi/2} \sin^2\theta\, d\theta$ **3.** $\frac{1}{2}\int_{\pi/2}^{3\pi/2} (3 - 2\sin\theta)^2\, d\theta$ **5.** 9π

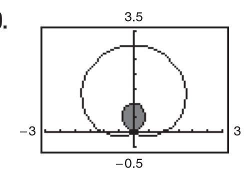
7. $\pi/3$ **9.** $\pi/8$ **11.** $3\pi/2$ **13.** 27π **15.** 4

17.



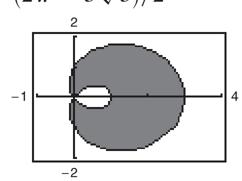
$(2\pi - 3\sqrt{3})/2$

19.



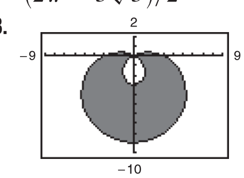
$(2\pi - 3\sqrt{3})/2$

21.



$\pi + 3\sqrt{3}$

23.



$9\pi + 27\sqrt{3}$

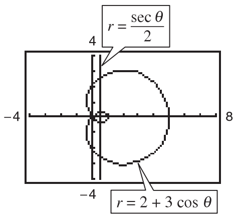
25. $(1, \pi/2), (1, 3\pi/2), (0, 0)$

27. $\left(\dfrac{2 - \sqrt{2}}{2}, \dfrac{3\pi}{4}\right), \left(\dfrac{2 + \sqrt{2}}{2}, \dfrac{7\pi}{4}\right), (0, 0)$

29. $\left(\dfrac{3}{2}, \dfrac{\pi}{6}\right), \left(\dfrac{3}{2}, \dfrac{5\pi}{6}\right), (0, 0)$ **31.** $(2, 4), (-2, -4)$

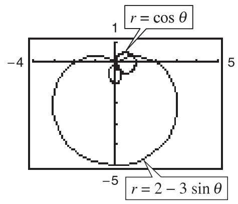
33. $(1, \pi/12), (1, 5\pi/12), (1, 7\pi/12), (1, 11\pi/12),$
$(1, 13\pi/12), (1, 17\pi/12), (1, 19\pi/12), (1, 23\pi/12)$

35.



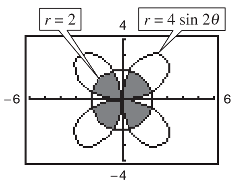
$(-0.581, \pm 2.607)$, $(2.581, \pm 1.376)$

37.



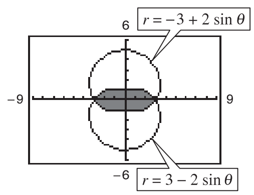
$(0, 0)$, $(0.935, 0.363)$, $(0.535, -1.006)$
The graphs reach the pole at different times (θ-values).

39.



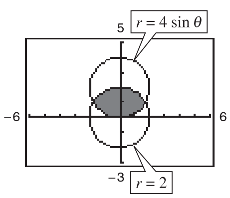
$\frac{4}{3}(4\pi - 3\sqrt{3})$

41.



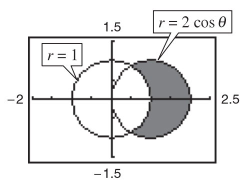
$11\pi - 24$

43.



$\frac{2}{3}(4\pi - 3\sqrt{3})$

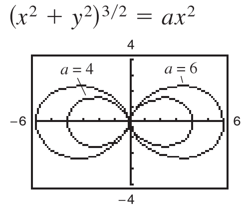
45.



$\pi/3 + \sqrt{3}/2$

47. $5\pi a^2/4$ **49.** $(a^2/2)(\pi - 2)$

51. (a) $(x^2 + y^2)^{3/2} = ax^2$
(b)

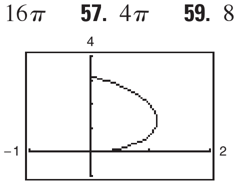


(c) $15\pi/2$

53. The area enclosed by the function is $\pi a^2/4$ if n is odd and is $\pi a^2/2$ if n is even.

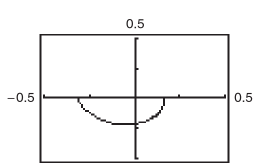
55. 16π **57.** 4π **59.** 8

61.



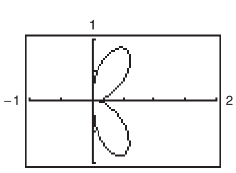
About 4.16

63.



About 0.71

65.



About 4.39

67. 36π

69. $\dfrac{2\pi\sqrt{1 + a^2}}{1 + 4a^2}(e^{\pi a} - 2a)$ **71.** 21.87

73. You will only find simultaneous points of intersection. There may be intersection points that do not occur with the same coordinates in the two graphs.

75. (a) $S = 2\pi\int_{\alpha}^{\beta} f(\theta)\sin\theta\sqrt{f(\theta)^2 + f'(\theta)^2}\,d\theta$
(b) $S = 2\pi\int_{\alpha}^{\beta} f(\theta)\cos\theta\sqrt{f(\theta)^2 + f'(\theta)^2}\,d\theta$

77. $40\pi^2$

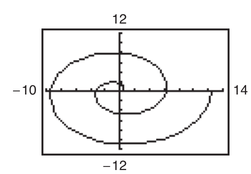
79. (a) 16π
(b)

θ	0.2	0.4	0.6	0.8	1.0	1.2	1.4
A	6.32	12.14	17.06	20.80	23.27	24.60	25.08

(c) and (d) For $\frac{1}{4}$ of area $(4\pi \approx 12.57)$: 0.42
For $\frac{1}{2}$ of area $(8\pi \approx 25.13)$: $1.57(\pi/2)$
For $\frac{3}{4}$ of area $(12\pi \approx 37.70)$: 2.73
(e) No. The results do not depend on the radius. Answers will vary.

81. Circle

83. (a)



The graph becomes larger and more spread out. The graph is reflected over the y-axis.

(b) $(an\pi, n\pi)$ where $n = 1, 2, 3, \ldots$
(c) About 21.26 (d) $4/3\pi^3$

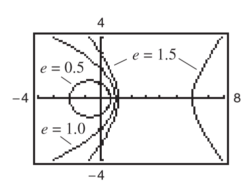
85. $r = \sqrt{2}\cos\theta$

87. False. The graphs of $f(\theta) = 1$ and $g(\theta) = -1$ coincide.

89. Proof

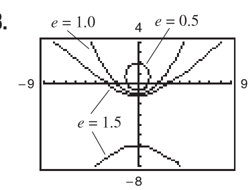
Section 10.6 (page 755)

1.



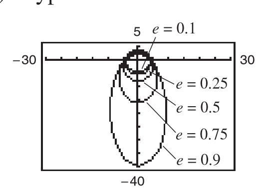
(a) Parabola (b) Ellipse
(c) Hyperbola

3.



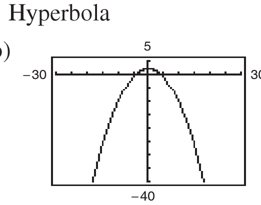
(a) Parabola (b) Ellipse
(c) Hyperbola

5. (a)

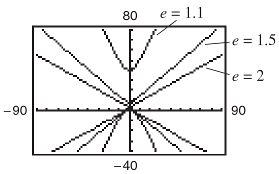


Ellipse
As $e \to 1^-$, the ellipse becomes more elliptical, and as $e \to 0^+$, it becomes more circular.

(b)



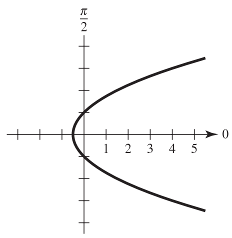
Parabola

(c) 

Hyperbola
As $e \to 1^+$, the hyperbola opens more slowly, and as $e \to \infty$, it opens more rapidly.

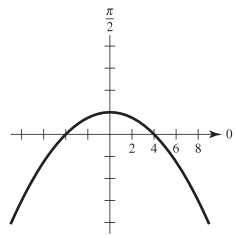
7. c **8.** f **9.** a **10.** e **11.** b **12.** d

13. $e = 1$
Distance $= 1$



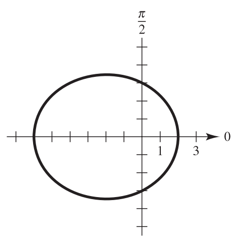
Parabola

15. $e = 1$
Distance $= 4$



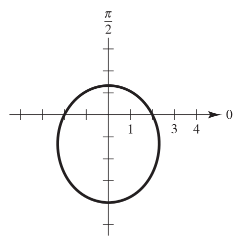
Parabola

17. $e = \frac{1}{2}$
Distance $= 6$



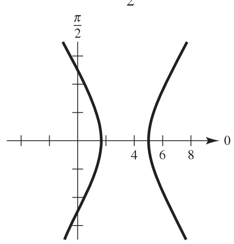
Ellipse

19. $e = \frac{1}{2}$
Distance $= 4$



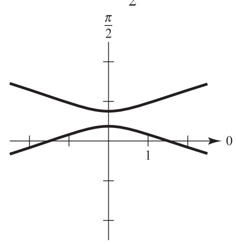
Ellipse

21. $e = 2$
Distance $= \frac{5}{2}$



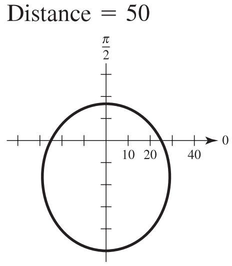
Hyperbola

23. $e = 3$
Distance $= \frac{1}{2}$

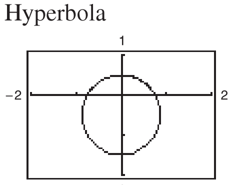


Hyperbola

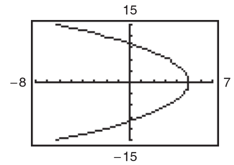
25. $e = \frac{1}{2}$
Distance $= 50$



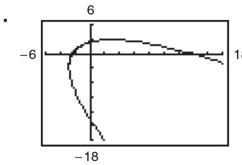
Ellipse

27. 

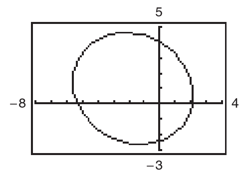
Ellipse
$e = \frac{1}{2}$

29. 

Parabola
$e = 1$

31. 

Rotated $\pi/4$ radian counterclockwise.

33. 

Rotated $\pi/6$ radian clockwise.

35. $r = \dfrac{8}{8 + 5\cos\left(\theta + \dfrac{\pi}{6}\right)}$

37. $r = 3/(1 - \cos\theta)$ **39.** $r = 1/(2 + \sin\theta)$

41. $r = 2/(1 + 2\cos\theta)$ **43.** $r = 2/(1 - \sin\theta)$

45. $r = 16/(5 + 3\cos\theta)$ **47.** $r = 9/(4 - 5\sin\theta)$

49. $r = 4/(2 + \cos\theta)$

51. If $0 < e < 1$, the conic is an ellipse.
If $e = 1$, the conic is a parabola.
If $e > 1$, the conic is a hyperbola.

53. If the foci are fixed and $e \to 0$, then $d \to \infty$. To see this, compare the ellipses

$$r = \frac{1/2}{1 + (1/2)\cos\theta},\ e = \frac{1}{2},\ d = 1 \text{ and}$$
$$r = \frac{5/16}{1 + (1/4)\cos\theta},\ e = \frac{1}{4},\ d = \frac{5}{4}.$$

55. Proof

57. $r^2 = \dfrac{9}{1 - (16/25)\cos^2\theta}$ **59.** $r^2 = \dfrac{-16}{1 - (25/9)\cos^2\theta}$

61. About 10.88 **63.** 3.37 **65.** $\dfrac{7979.21}{1 - 0.9372\cos\theta}$; 11,015 mi

67. $r = \dfrac{149{,}558{,}278.0560}{1 - 0.0167\cos\theta}$
Perihelion: 147,101,680 km
Aphelion: 152,098,320 km

69. $r = \dfrac{4{,}497{,}667{,}328}{1 - 0.0086\cos\theta}$
Perihelion: 4,459,317,200 km
Aphelion: 4,536,682,800 km

71. Answers will vary. Sample answers:
(a) 3.591×10^{18} km^2; 9.322 yr
(b) $\alpha \approx 0.361 + \pi$; Larger angle with the smaller ray to generate an equal area
(c) Part (a): 1.583×10^9 km; 1.698×10^8 km/yr
Part (b): 1.610×10^9 km; 1.727×10^8 km/yr

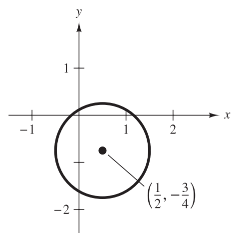
73. Proof

75. Let $r_1 = ed/(1 + \sin\theta)$ and $r_2 = ed/(1 - \sin\theta)$.
The points of intersection of r_1 and r_2 are $(ed, 0)$ and (ed, π). The slopes of the tangent lines to r_1 are -1 at $(ed, 0)$ and 1 at (ed, π). The slopes of the tangent lines to r_2 are 1 at $(ed, 0)$ and -1 at (ed, π). Therefore, at $(ed, 0)$, $m_1 m_2 = -1$, and at (ed, π), $m_1 m_2 = -1$, and the curves intersect at right angles.

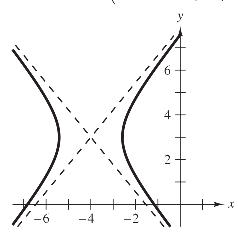
Review Exercises for Chapter 10 (page 758)

1. e **2.** c **3.** b **4.** d **5.** a **6.** f

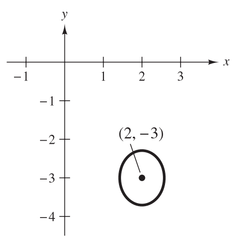
7. Circle
Center: $\left(\frac{1}{2}, -\frac{3}{4}\right)$
Radius: 1



9. Hyperbola
Center: $(-4, 3)$
Vertices: $\left(-4 \pm \sqrt{2}, 3\right)$



11. Ellipse
Center: $(2, -3)$
Vertices: $\left(2, -3 \pm \sqrt{2}/2\right)$

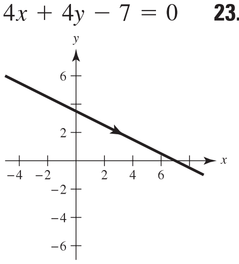


13. $y^2 - 4y - 12x + 4 = 0$ **15.** $(x - 1)^2/36 + y^2/20 = 1$

17. $x^2/49 - y^2/32 = 1$ **19.** About 15.87

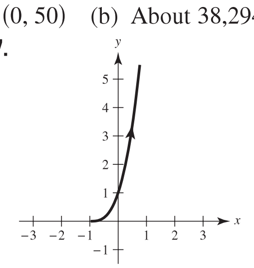
21. $4x + 4y - 7 = 0$ **23.** (a) $(0, 50)$ (b) About 38,294.49

25.



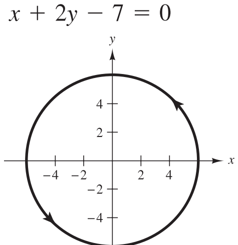
$x + 2y - 7 = 0$

27.



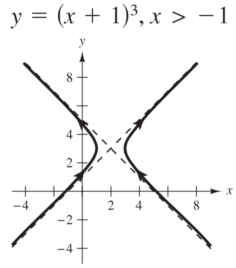
$y = (x + 1)^3, x > -1$

29.



$x^2 + y^2 = 36$

31.

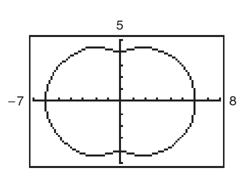


$(x - 2)^2 - (y - 3)^2 = 1$

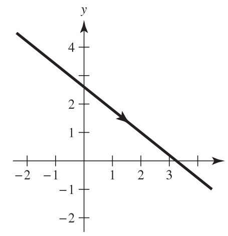
33. Answers will vary. Sample answer:
$x = 5t - 2$
$y = 6 - 4t$

35. $x = 4 \cos \theta - 3$
$y = 4 + 3 \sin \theta$

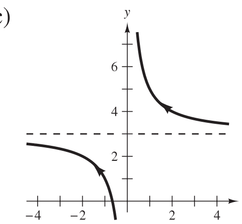
37.



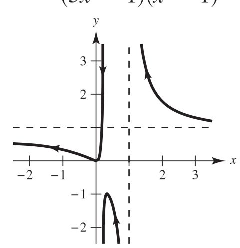
39. (a) $dy/dx = -\frac{4}{5}$;
Horizontal tangents: None
(b) $y = (-4x + 13)/5$
(c)



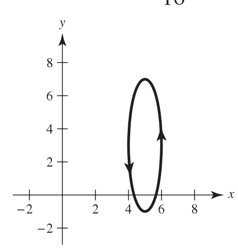
41. (a) $dy/dx = -2t^2$;
Horizontal tangents: None
(b) $y = 3 + 2/x$
(c)



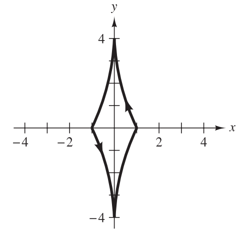
43. (a) $\dfrac{dy}{dx} = \dfrac{(t - 1)(2t + 1)^2}{t^2(t - 2)^2}$;
Horizontal tangent: $\left(\dfrac{1}{3}, -1\right)$
(b) $y = \dfrac{4x^2}{(5x - 1)(x - 1)}$
(c)



45. (a) $\dfrac{dy}{dx} = -4 \cot \theta$;
Horizontal tangents: $(5, 7), (5, -1)$
(b) $(x - 5)^2 + \dfrac{(y - 3)^2}{16} = 1$
(c)



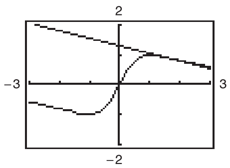
47. (a) $\dfrac{dy}{dx} = -4 \tan \theta$;
Horizontal tangents: None
(b) $x^{2/3} + (y/4)^{2/3} = 1$
(c)



49. Horizontal: $(5, 0)$
Vertical: None

51. Horizontal: $(2, 2), (2, 0)$
Vertical: $(4, 1), (0, 1)$

53. (a) and (c)

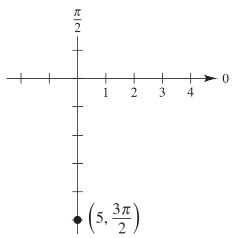


(b) $dx/d\theta = -4,\ dy/d\theta = 1,\ dy/dx = -\frac{1}{4}$

55. $\frac{1}{2}\pi^2 r$

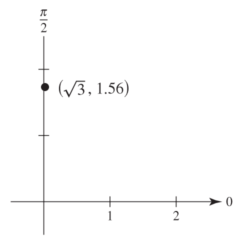
57. (a) $s = 12\pi\sqrt{10} \approx 119.215$ **59.** $A = 3\pi$
(b) $s = 4\pi\sqrt{10} \approx 39.738$

61.



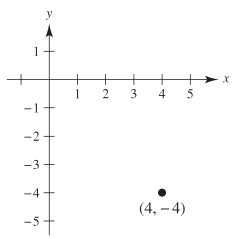
Rectangular: $(0, -5)$

63.



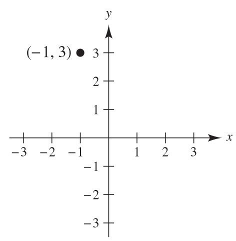
Rectangular: $(0.0187, 1.7320)$

65.



$\left(4\sqrt{2}, \frac{7\pi}{4}\right), \left(-4\sqrt{2}, \frac{3\pi}{4}\right)$

67.



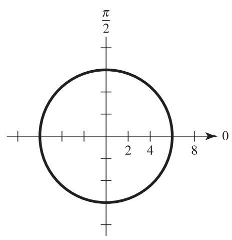
$\left(\sqrt{10}, 1.89\right), \left(-\sqrt{10}, 5.03\right)$

69. $x^2 + y^2 - 3x = 0$ **71.** $(x^2 + y^2 + 2x)^2 = 4(x^2 + y^2)$

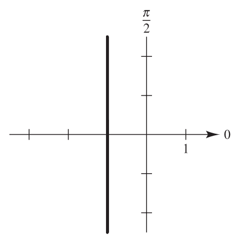
73. $(x^2 + y^2)^2 = x^2 - y^2$ **75.** $y^2 = x^2[(4 - x)/(4 + x)]$

77. $r = a\cos^2\theta\sin\theta$ **79.** $r^2 = a^2\theta^2$

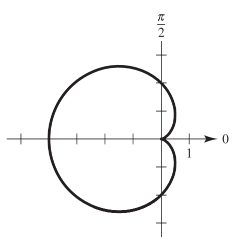
81. Circle



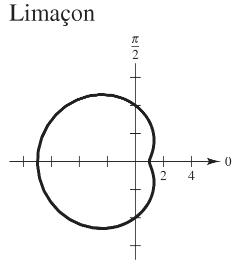
83. Line



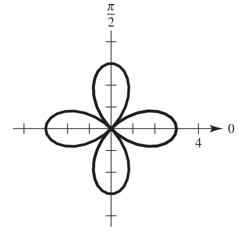
85. Cardioid



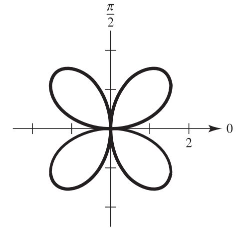
87. Limaçon



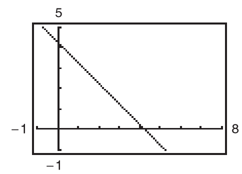
89. Rose curve



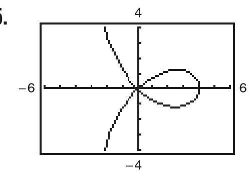
91. Rose curve



93.



95.

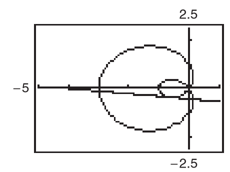


97. (a) $\theta = \pm\pi/3$

(b) Vertical: $(-1, 0), (3, \pi), \left(\frac{1}{2}, \pm 1.318\right)$

Horizontal: $(-0.686, \pm 0.568), (2.186, \pm 2.206)$

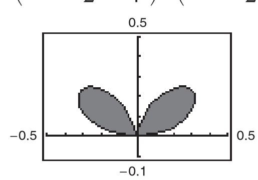
(c)



99. Proof **101.** $\dfrac{9\pi}{20}$ **103.** $\dfrac{9\pi}{2}$ **105.** 4

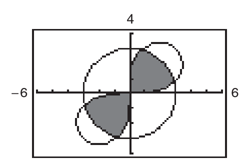
107. $\left(1 + \dfrac{\sqrt{2}}{2}, \dfrac{3\pi}{4}\right), \left(1 - \dfrac{\sqrt{2}}{2}, \dfrac{7\pi}{4}\right), (0, 0)$

109.



$$A = 2\left(\frac{1}{2}\right)\int_0^{\pi/2} \sin^2\theta\cos^4\theta\, d\theta \approx 0.10$$

111.

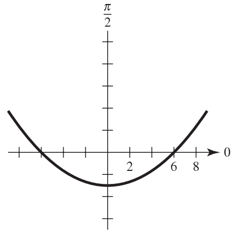


$$A = 2\left[\frac{1}{2}\int_0^{\pi/12} 18\sin 2\theta + \frac{1}{2}\int_{\pi/12}^{5\pi/12} 9\, d\theta + \frac{1}{2}\int_{5\pi/12}^{\pi/2} 18\sin 2\theta\, d\theta\right]$$
$$\approx 1.2058 + 9.4248 + 1.2058 = 11.8364$$

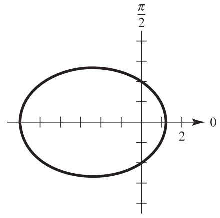
113. $4a$

115. $S = 2\pi\displaystyle\int_0^{\pi/2} (1 + 4\cos\theta)\sin\theta\sqrt{17 + 8\cos\theta}\, d\theta$
$= 34\pi\sqrt{17}/5 \approx 88.08$

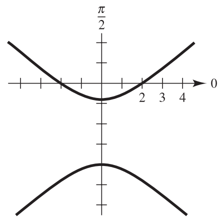
117. Parabola



119. Ellipse



121. Hyperbola

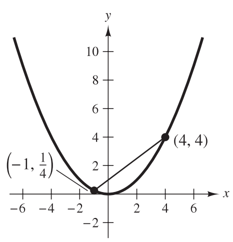


123. $r = 10 \sin \theta$

125. $r = 4/(1 - \cos \theta)$ **127.** $r = 5/(3 - 2 \cos \theta)$

P.S. Problem Solving (page 761)

1. (a)



(b) and (c) Proofs

3. Proof

5. (a) $r = 2a \tan \theta \sin \theta$

(b) $x = 2at^2/(1 + t^2)$

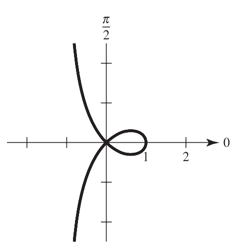
$y = 2at^3/(1 + t^2)$

(c) $y^2 = x^3/(2a - x)$

7. (a) $y^2 = x^2[(1 - x)/(1 + x)]$

(b) $r = \cos 2\theta \cdot \sec \theta$

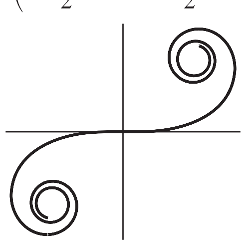
(c)



(d) $y = x, y = -x$

(e) $\left(\dfrac{\sqrt{5} - 1}{2}, \pm \dfrac{\sqrt{5} - 1}{2}\sqrt{-2 + \sqrt{5}}\right)$

9. (a)



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(b) Proof (c) $a, 2\pi$

11. $A = \frac{1}{2}ab$ **13.** $r^2 = 2 \cos 2\theta$

15. (a) First plane: $x_1 = \cos 70°(150 - 375t)$

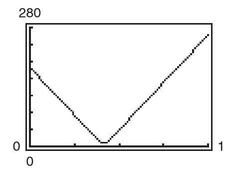
$y_1 = \sin 70°(150 - 375t)$

Second plane: $x_2 = \cos 45°(450t - 190)$

$y_2 = \sin 45°(190 - 450t)$

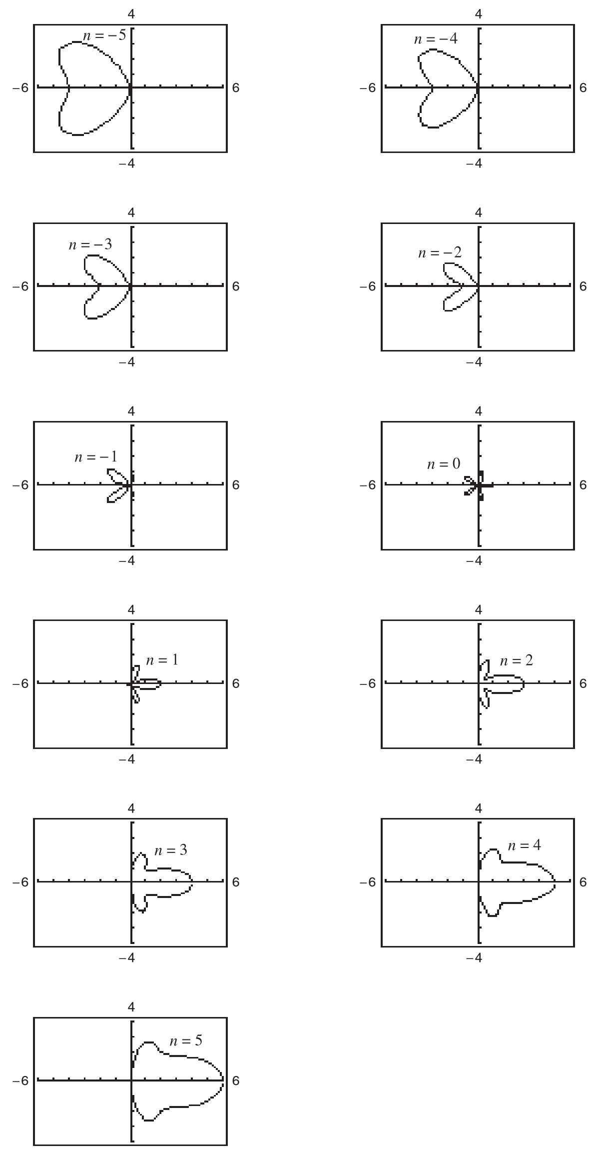
(b) $\{[\cos 45°(450t - 190) - \cos 70°(150 - 375t)]^2 + [\sin 45°(190 - 450t) - \sin 70°(150 - 375t)]^2\}^{1/2}$

(c)



0.4145 h; Yes

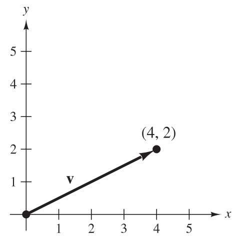
17.

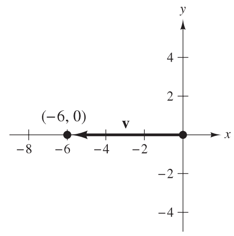


$n = 1, 2, 3, 4, 5$ produce "bells"; $n = -1, -2, -3, -4, -5$ produce "hearts."

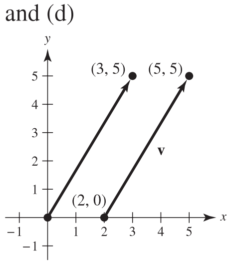
Chapter 11

Section 11.1 (page 771)

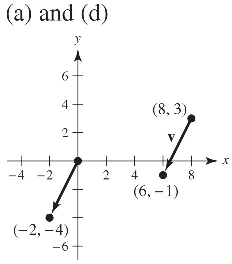
1. (a) $\langle 4, 2\rangle$
(b)


3. (a) $\langle -6, 0\rangle$
(b)


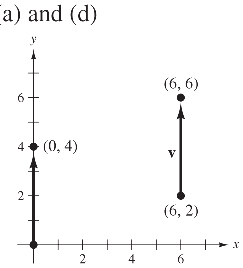
5. $\mathbf{u} = \mathbf{v} = \langle 2, 4\rangle$ **7.** $\mathbf{u} = \mathbf{v} = \langle 6, -5\rangle$

9. (a) and (d)


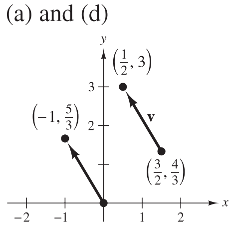
(b) $\langle 3, 5\rangle$
(c) $\mathbf{v} = 3\mathbf{i} + 5\mathbf{j}$

11. (a) and (d)


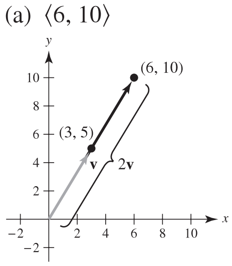
(b) $\langle -2, -4\rangle$
(c) $\mathbf{v} = -2\mathbf{i} - 4\mathbf{j}$

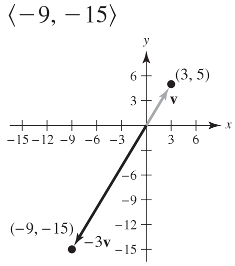
13. (a) and (d)


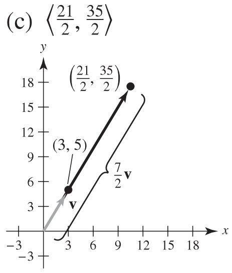
(b) $\langle 0, 4\rangle$ (c) $\mathbf{v} = 4\mathbf{j}$

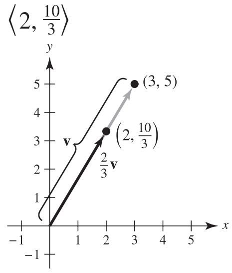
15. (a) and (d)


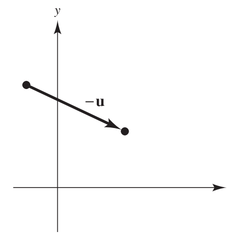
(b) $\left\langle -1, \frac{5}{3}\right\rangle$
(c) $\mathbf{v} = -\mathbf{i} + \frac{5}{3}\mathbf{j}$

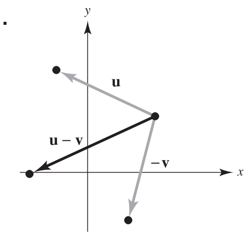
17. (a) $\langle 6, 10\rangle$


(b) $\langle -9, -15\rangle$


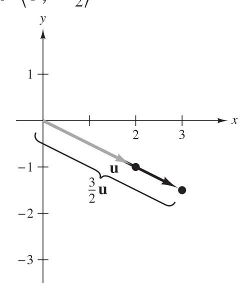
(c) $\left\langle \frac{21}{2}, \frac{35}{2}\right\rangle$


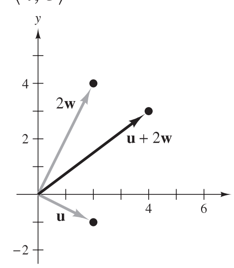
(d) $\left\langle 2, \frac{10}{3}\right\rangle$


19.


21.


23. (a) $\left\langle \frac{8}{3}, 6\right\rangle$ (b) $\langle -2, -14\rangle$ (c) $\langle 18, -7\rangle$

25. $\left\langle 3, -\frac{3}{2}\right\rangle$


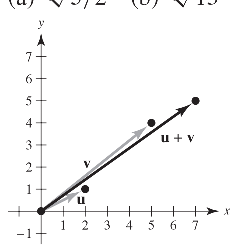
27. $\langle 4, 3\rangle$


29. $(3, 5)$ **31.** 7 **33.** 5 **35.** $\sqrt{61}$

37. $\left\langle \sqrt{17}/17, 4\sqrt{17}/17\right\rangle$ **39.** $\left\langle 3\sqrt{34}/34, 5\sqrt{34}/34\right\rangle$

41. (a) $\sqrt{2}$ (b) $\sqrt{5}$ (c) 1 (d) 1 (e) 1 (f) 1

43. (a) $\sqrt{5}/2$ (b) $\sqrt{13}$ (c) $\sqrt{85}/2$ (d) 1 (e) 1 (f) 1

45.


$\|\mathbf{u}\| + \|\mathbf{v}\| = \sqrt{5} + \sqrt{41}$ and $\|\mathbf{u} + \mathbf{v}\| = \sqrt{74}$
$\sqrt{74} \le \sqrt{5} + \sqrt{41}$

47. $\langle 0, 6\rangle$ **49.** $\left\langle -\sqrt{5}, 2\sqrt{5}\right\rangle$ **51.** $\langle 3, 0\rangle$ **53.** $\left\langle -\sqrt{3}, 1\right\rangle$

55. $\left\langle \dfrac{2 + 3\sqrt{2}}{2}, \dfrac{3\sqrt{2}}{2}\right\rangle$ **57.** $\langle 2\cos 4 + \cos 2, 2\sin 4 + \sin 2\rangle$

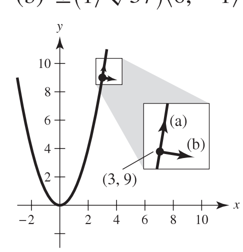
59. Answers will vary. Example: A scalar is a single real number such as 2. A vector is a line segment having both direction and magnitude. The vector $\left\langle \sqrt{3}, 1\right\rangle$, given in component form, has a direction of $\pi/6$ and a magnitude of 2.

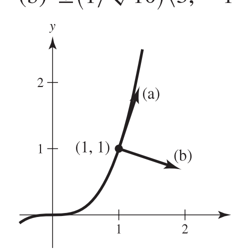
61. (a) Vector; has magnitude and direction
(b) Scalar; has only magnitude

63. $a = 1, b = 1$ **65.** $a = 1, b = 2$ **67.** $a = \frac{2}{3}, b = \frac{1}{3}$

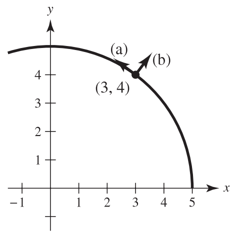
69. (a) $\pm\left(1/\sqrt{37}\right)\langle 1, 6\rangle$
(b) $\pm\left(1/\sqrt{37}\right)\langle 6, -1\rangle$

71. (a) $\pm\left(1/\sqrt{10}\right)\langle 1, 3\rangle$
(b) $\pm\left(1/\sqrt{10}\right)\langle 3, -1\rangle$





73. (a) $\pm\frac{1}{5}\langle -4, 3\rangle$
(b) $\pm\frac{1}{5}\langle 3, 4\rangle$



75. $\left\langle -\sqrt{2}/2, \sqrt{2}/2\right\rangle$

77. (a)–(c) Answers will vary.
(d) Magnitude ≈ 63.5, direction $\approx -8.26°$

79. 1.33, 132.5° **81.** 10.7°, 584.6 lb **83.** 71.3°, 228.5 lb

85. (a) $\theta = 0°$ (b) $\theta = 180°$
(c) No, the resultant can only be less than or equal to the sum.

87. $(-4, -1), (6, 5), (10, 3)$

89. Tension in cable $AC \approx 2638.2$ lb
Tension in cable $BC \approx 1958.1$ lb

91. Horizontal: 1193.43 ft/sec
Vertical: 125.43 ft/sec

93. 38.3° north of west
882.9 km/h

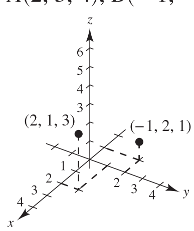
95. True **97.** True **99.** False. $\|a\mathbf{i} + b\mathbf{j}\| = \sqrt{2}|a|$

101–103. Proofs **105.** $x^2 + y^2 = 25$

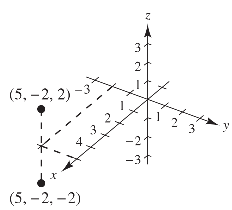
Section 11.2 (page 780)

1. $A(2, 3, 4), B(-1, -2, 2)$

3.



5.



7. $(-3, 4, 5)$ **9.** $(12, 0, 0)$ **11.** 0

13. Six units above the xy-plane

15. Three units behind the yz-plane

17. To the left of the xz-plane

19. Within three units of the xz-plane

21. Three units below the xy-plane, and below either quadrant I or III

23. Above the xy-plane and above quadrants II or IV, *or* below the xy-plane and below quadrants I or III

25. $\sqrt{69}$ **27.** $\sqrt{61}$ **29.** $7, 7\sqrt{5}, 14$; Right triangle

31. $\sqrt{41}, \sqrt{41}, \sqrt{14}$; Isosceles triangle

33. $(0, 0, 9), (2, 6, 12), (6, 4, -3)$

35. $\left(\frac{3}{2}, -3, 5\right)$ **37.** $(x - 0)^2 + (y - 2)^2 + (z - 5)^2 = 4$

39. $(x - 1)^2 + (y - 3)^2 + (z - 0)^2 = 10$

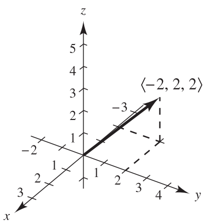
41. $(x - 1)^2 + (y + 3)^2 + (z + 4)^2 = 25$
Center: $(1, -3, -4)$
Radius: 5

43. $\left(x - \frac{1}{3}\right)^2 + (y + 1)^2 + z^2 = 1$
Center: $\left(\frac{1}{3}, -1, 0\right)$
Radius: 1

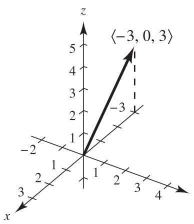
45. A solid sphere with center $(0, 0, 0)$ and radius 6

47. Interior of sphere of radius 4 centered at $(2, -3, 4)$

49. (a) $\langle -2, 2, 2\rangle$
(b) $\mathbf{v} = -2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$
(c)



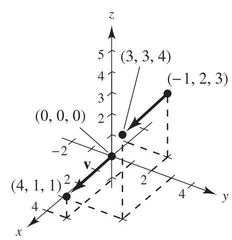
51. (a) $\langle -3, 0, 3\rangle$
(b) $\mathbf{v} = -3\mathbf{i} + 3\mathbf{k}$
(c)



53. $\mathbf{v} = \langle 1, -1, 6\rangle$
$\|\mathbf{v}\| = \sqrt{38}$
$\mathbf{u} = \dfrac{1}{\sqrt{38}}\langle 1, -1, 6\rangle$

55. $\mathbf{v} = \langle -1, 0, -1\rangle$
$\|\mathbf{v}\| = \sqrt{2}$
$\mathbf{u} = \dfrac{1}{\sqrt{2}}\langle -1, 0, -1\rangle$

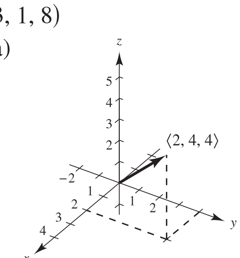
57. (a) and (d)



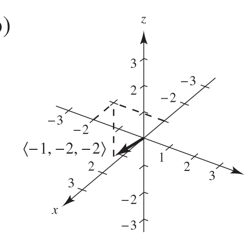
(b) $\langle 4, 1, 1\rangle$ (c) $\mathbf{v} = 4\mathbf{i} + \mathbf{j} + \mathbf{k}$

59. $(3, 1, 8)$

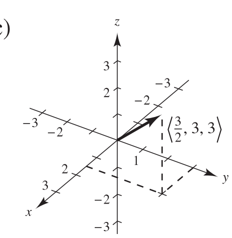
61. (a)



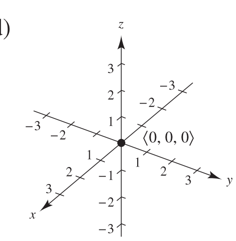
(b)



(c)



(d)



63. $\langle -1, 0, 4\rangle$ **65.** $\langle 6, 12, 6\rangle$ **67.** $\left\langle \frac{7}{2}, 3, \frac{5}{2}\right\rangle$

69. a and b **71.** a **73.** Collinear **75.** Not collinear

77. $\overrightarrow{AB} = \langle 1, 2, 3\rangle$
$\overrightarrow{CD} = \langle 1, 2, 3\rangle$
$\overrightarrow{BD} = \langle -2, 1, 1\rangle$
$\overrightarrow{AC} = \langle -2, 1, 1\rangle$
Because $\overrightarrow{AB} = \overrightarrow{CD}$ and $\overrightarrow{BD} = \overrightarrow{AC}$, the given points form the vertices of a parallelogram.

79. 0 **81.** $\sqrt{34}$ **83.** $\sqrt{14}$

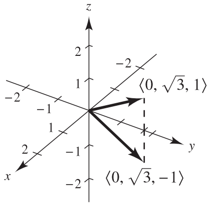
85. (a) $\frac{1}{3}\langle 2, -1, 2\rangle$ (b) $-\frac{1}{3}\langle 2, -1, 2\rangle$

87. (a) $\left(1/\sqrt{38}\right)\langle 3, 2, -5\rangle$ (b) $-\left(1/\sqrt{38}\right)\langle 3, 2, -5\rangle$

89. (a)–(d) Answers will vary.
(e) $\mathbf{u} + \mathbf{v} = \langle 4, 7.5, -2\rangle$
$\|\mathbf{u} + \mathbf{v}\| \approx 8.732$
$\|\mathbf{u}\| \approx 5.099$
$\|\mathbf{v}\| \approx 9.014$

91. $\pm\frac{7}{3}$ **93.** $\left\langle 0, 10/\sqrt{2}, 10/\sqrt{2}\right\rangle$ **95.** $\left\langle 1, -1, \frac{1}{2}\right\rangle$

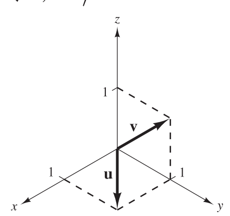
97.



$\left\langle 0, \sqrt{3}, \pm 1\right\rangle$

99. $(2, -1, 2)$

101. (a)



(b) $a = 0, a + b = 0, b = 0$
(c) $a = 1, a + b = 2, b = 1$
(d) Not possible

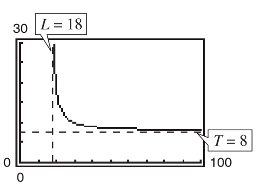
103. x_0 is directed distance to yz-plane.
y_0 is directed distance to xz-plane.
z_0 is directed distance to xy-plane.

105. $(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = r^2$ **107.** **0**

109. (a) $T = 8L/\sqrt{L^2 - 18^2}, L > 18$
(b)

L	20	25	30	35	40	45	50
T	18.4	11.5	10	9.3	9.0	8.7	8.6

(c)



(d) Proof (e) 30 in.

111. $\left(\sqrt{3}/3\right)\langle 1, 1, 1\rangle$

113. Tension in cable AB: 202.919 N
Tension in cable AC: 157.909 N
Tension in cable AD: 226.521 N

115. $\left(x - \frac{4}{3}\right)^2 + (y - 3)^2 + \left(z + \frac{1}{3}\right)^2 = \frac{44}{9}$

Section 11.3 (page 789)

1. (a) 17 (b) 25 (c) 25 (d) $\langle -17, 85\rangle$ (e) 34

3. (a) -26 (b) 52 (c) 52 (d) $\langle 78, -52\rangle$ (e) -52

5. (a) 2 (b) 29 (c) 29 (d) $\langle 0, 12, 10\rangle$ (e) 4

7. (a) 1 (b) 6 (c) 6 (d) $\mathbf{i} - \mathbf{k}$ (e) 2

9. 20 **11.** $\pi/2$ **13.** $\arccos\left[-1/\left(5\sqrt{2}\right)\right] \approx 98.1°$

15. $\arccos\left(\sqrt{2}/3\right) \approx 61.9°$ **17.** $\arccos\left(-8\sqrt{13}/65\right) \approx 116.3°$

19. Neither **21.** Orthogonal **23.** Neither

25. Orthogonal **27.** Right triangle; answers will vary.

29. Acute triangle; answers will vary.

31. $\cos\alpha = \frac{1}{3}$
$\cos\beta = \frac{2}{3}$
$\cos\gamma = \frac{2}{3}$

33. $\cos\alpha = 0$
$\cos\beta = 3/\sqrt{13}$
$\cos\gamma = -2/\sqrt{13}$

35. $\alpha \approx 43.3°, \beta \approx 61.0°, \gamma \approx 119.0°$

37. $\alpha \approx 100.5°, \beta \approx 24.1°, \gamma \approx 68.6°$

39. Magnitude: 124.310 lb
$\alpha \approx 29.48°, \beta \approx 61.39°, \gamma \approx 96.53°$

41. $\alpha = 90°, \beta = 45°, \gamma = 45°$ **43.** (a) $\langle 2, 8\rangle$ (b) $\langle 4, -1\rangle$

45. (a) $\left\langle \frac{5}{2}, \frac{1}{2}\right\rangle$ (b) $\left\langle -\frac{1}{2}, \frac{5}{2}\right\rangle$ **47.** (a) $\langle -2, 2, 2\rangle$ (b) $\langle 2, 1, 1\rangle$

49. (a) $\left\langle 0, \frac{33}{25}, \frac{44}{25}\right\rangle$ (b) $\left\langle 2, -\frac{8}{25}, \frac{6}{25}\right\rangle$

51. See "Definition of Dot Product," page 783.

53. (a) and (b) are defined. (c) and (d) are not defined because it is not possible to find the dot product of a scalar and a vector or to add a scalar to a vector.

55. See Figure 11.29 on page 787.

57. Yes.

$$\left\|\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{v}\|^2}\mathbf{v}\right\| = \left\|\frac{\mathbf{v}\cdot\mathbf{u}}{\|\mathbf{u}\|^2}\mathbf{u}\right\|$$
$$|\mathbf{u}\cdot\mathbf{v}|\frac{\|\mathbf{v}\|}{\|\mathbf{v}\|^2} = |\mathbf{v}\cdot\mathbf{u}|\frac{\|\mathbf{u}\|}{\|\mathbf{u}\|^2}$$
$$\frac{1}{\|\mathbf{v}\|} = \frac{1}{\|\mathbf{u}\|}$$
$$\|\mathbf{u}\| = \|\mathbf{v}\|$$

59. \$12,351.25; Total revenue **61.** (a)–(c) Answers will vary.

63. Answers will vary. **65.** $\mathbf{u}$

67. Answers will vary. Example: $\langle 12, 2\rangle$ and $\langle -12, -2\rangle$

69. Answers will vary. Example: $\langle 2, 0, 3\rangle$ and $\langle -2, 0, -3\rangle$

71. (a) 8335.1 lb (b) 47,270.8 lb

73. 425 ft-lb **75.** 2900.2 km-N

77. False. For example, $\langle 1, 1\rangle \cdot \langle 2, 3\rangle = 5$ and $\langle 1, 1\rangle \cdot \langle 1, 4\rangle = 5$, but $\langle 2, 3\rangle \neq \langle 1, 4\rangle$.

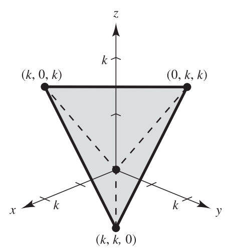
79. $\arccos\left(1/\sqrt{3}\right) \approx 54.7°$

81. (a) $(0, 0), (1, 1)$
(b) To $y = x^2$ at $(1, 1)$: $\left\langle \pm\sqrt{5}/5, \pm 2\sqrt{5}/5\right\rangle$
To $y = x^{1/3}$ at $(1, 1)$: $\left\langle \pm 3\sqrt{10}/10, \pm\sqrt{10}/10\right\rangle$
To $y = x^2$ at $(0, 0)$: $\langle \pm 1, 0\rangle$
To $y = x^{1/3}$ at $(0, 0)$: $\langle 0, \pm 1\rangle$
(c) At $(1, 1)$: $\theta = 45°$
At $(0, 0)$: $\theta = 90°$

83. (a) $(-1, 0), (1, 0)$
(b) To $y = 1 - x^2$ at $(1, 0)$: $\langle \pm\sqrt{5}/5, \mp 2\sqrt{5}/5 \rangle$
To $y = x^2 - 1$ at $(1, 0)$: $\langle \pm\sqrt{5}/5, \pm 2\sqrt{5}/5 \rangle$
To $y = 1 - x^2$ at $(-1, 0)$: $\langle \pm\sqrt{5}/5, \pm 2\sqrt{5}/5 \rangle$
To $y = x^2 - 1$ at $(-1, 0)$: $\langle \pm\sqrt{5}/5, \mp 2\sqrt{5}/5 \rangle$
(c) At $(1, 0)$: $\theta = 53.13°$
At $(-1, 0)$: $\theta = 53.13°$

85. Proof

87. (a)

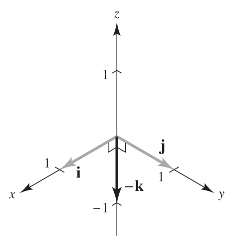


(b) $k\sqrt{2}$ (c) $60°$ (d) $109.5°$

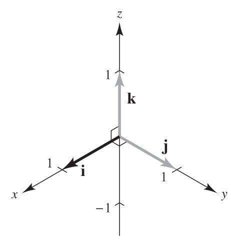
89–91. Proofs

Section 11.4 (page 798)

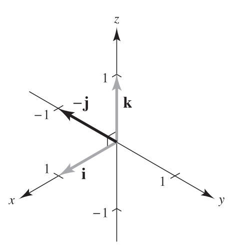
1. $-\mathbf{k}$



3. $\mathbf{i}$



5. $-\mathbf{j}$

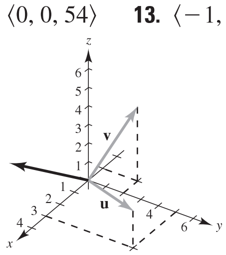


7. (a) $20\mathbf{i} + 10\mathbf{j} - 16\mathbf{k}$
(b) $-20\mathbf{i} - 10\mathbf{j} + 16\mathbf{k}$
(c) $\mathbf{0}$

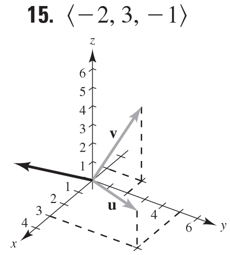
9. (a) $17\mathbf{i} - 33\mathbf{j} - 10\mathbf{k}$
(b) $-17\mathbf{i} + 33\mathbf{j} + 10\mathbf{k}$
(c) $\mathbf{0}$

11. $\langle 0, 0, 54 \rangle$ **13.** $\langle -1, -1, -1 \rangle$ **15.** $\langle -2, 3, -1 \rangle$

17.



19.



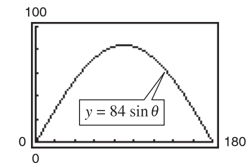
21. $\langle -73.5, 5.5, 44.75 \rangle, \left\langle -\dfrac{2.94}{\sqrt{11.8961}}, \dfrac{0.22}{\sqrt{11.8961}}, \dfrac{1.79}{\sqrt{11.8961}} \right\rangle$

23. $\langle -3.6, -1.4, 1.6 \rangle, \left\langle -\dfrac{1.8}{\sqrt{4.37}}, -\dfrac{0.7}{\sqrt{4.37}}, \dfrac{0.8}{\sqrt{4.37}} \right\rangle$

25. Answers will vary. **27.** 1 **29.** $6\sqrt{5}$ **31.** $9\sqrt{5}$

33. $\dfrac{11}{2}$ **35.** $\dfrac{\sqrt{16{,}742}}{2}$ **37.** $10 \cos 40° \approx 7.66$ ft-lb

39. (a) $84 \sin \theta$



(b) $42\sqrt{2} \approx 59.40$
(c) $\theta = 90°$; This is what should be expected. When $\theta = 90°$, the pipe wrench is horizontal.

41. 1 **43.** 6 **45.** 2 **47.** 75

49. At least one of the vectors is the zero vector.

51. See "Definition of Cross Product of Two Vectors in Space," page 792.

53. The magnitude of the cross product will increase by a factor of 4.

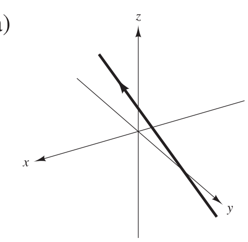
55. False. The cross product of two vectors is not defined in a two-dimensional coordinate system.

57. False. Let $\mathbf{u} = \langle 1, 0, 0 \rangle$, $\mathbf{v} = \langle 1, 0, 0 \rangle$, and $\mathbf{w} = \langle -1, 0, 0 \rangle$. Then $\mathbf{u} \times \mathbf{v} = \mathbf{u} \times \mathbf{w} = \mathbf{0}$, but $\mathbf{v} \neq \mathbf{w}$.

59–67. Proofs

Section 11.5 (page 807)

1. (a)



(b) $P = (1, 2, 2)$, $Q = (10, -1, 17)$, $\overrightarrow{PQ} = \langle 9, -3, 15 \rangle$
(There are many correct answers.) The components of the vector and the coefficients of t are proportional because the line is parallel to $\overrightarrow{PQ}$.
(c) $\left(-\frac{1}{5}, \frac{12}{5}, 0\right), (7, 0, 12), \left(0, \frac{7}{3}, \frac{1}{3}\right)$

3. (a) Yes (b) No

	Parametric Equations (a)	*Symmetric Equations (b)*	*Direction Numbers*
5.	$x = 3t$ $y = t$ $z = 5t$	$\dfrac{x}{3} = y = \dfrac{z}{5}$	3, 1, 5
7.	$x = -2 + 2t$ $y = 4t$ $z = 3 - 2t$	$\dfrac{x+2}{2} = \dfrac{y}{4} = \dfrac{z-3}{-2}$	2, 4, −2
9.	$x = 1 + 3t$ $y = -2t$ $z = 1 + t$	$\dfrac{x-1}{3} = \dfrac{y}{-2} = \dfrac{z-1}{1}$	3, −2, 1

	Parametric Equations (a)	*Symmetric Equations (b)*	*Direction Numbers*
11.	$x = 5 + 17t$ $y = -3 - 11t$ $z = -2 - 9t$	$\dfrac{x-5}{17} = \dfrac{y+3}{-11} = \dfrac{z+2}{-9}$	$17, -11, -9$
13.	$x = 7 - 10t$ $y = -2 + 2t$ $z = 6$	Not possible	$-10, 2, 0$

15. $x = 2$, $y = 3$, $z = 4 + t$ **17.** $x = 2 + 3t$, $y = 3 + 2t$, $z = 4 - t$ **19.** $x = 5 + 2t$, $y = -3 - t$, $z = -4 + 3t$

21. $x = 2 - t$
$y = 1 + t$
$z = 2 + t$

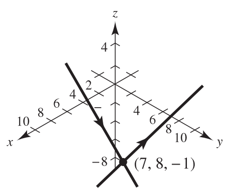
23. $P(3, -1, -2)$; $\mathbf{v} = \langle -1, 2, 0 \rangle$

25. $P(7, -6, -2)$; $\mathbf{v} = \langle 4, 2, 1 \rangle$

27. $L_1 = L_2$ and is parallel to L_3. **29.** L_1 and L_3 are identical.

31. $(2, 3, 1)$; $\cos\theta = 7\sqrt{17}/51$ **33.** Not intersecting

35.



$(7, 8, -1)$

37. (a) $P = (0, 0, -1)$, $Q = (0, -2, 0)$, $R = (3, 4, -1)$
$\overrightarrow{PQ} = \langle 0, -2, 1 \rangle$, $\overrightarrow{PR} = \langle 3, 4, 0 \rangle$
(There are many correct answers.)
(b) $\overrightarrow{PQ} \times \overrightarrow{PR} = \langle -4, 3, 6 \rangle$
The components of the cross product are proportional to the coefficients of the variables in the equation. The cross product is parallel to the normal vector.

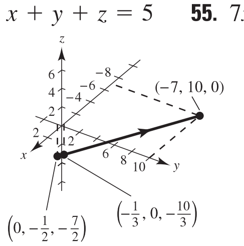
39. (a) Yes (b) Yes **41.** $y - 3 = 0$

43. $2x + 3y - z = 10$ **45.** $2x - y - 2z + 6 = 0$

47. $3x - 19y - 2z = 0$ **49.** $4x - 3y + 4z = 10$ **51.** $z = 3$

53. $x + y + z = 5$ **55.** $7x + y - 11z = 5$ **57.** $y - z = -1$

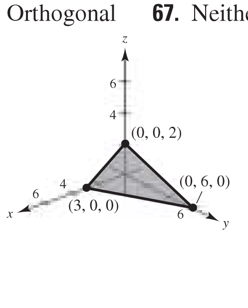
59.



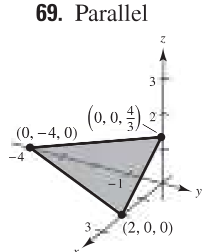
61. $x - z = 0$ **63.** $9x - 3y + 2z - 21 = 0$

65. Orthogonal **67.** Neither; 83.5° **69.** Parallel

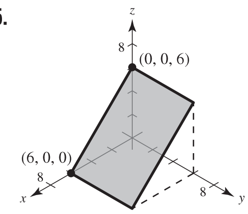
71.



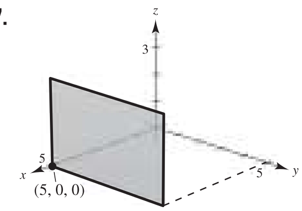
73.



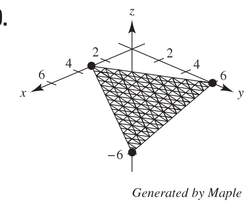
75.



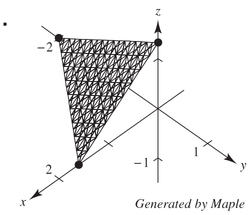
77.



79.



81.



83. P_1 and P_2 are parallel. **85.** $P_1 = P_4$ and is parallel to P_2.

87. The planes have intercepts at $(c, 0, 0)$, $(0, c, 0)$, and $(0, 0, c)$ for each value of c.

89. If $c = 0$, $z = 0$ is the xy-plane; If $c \neq 0$, the plane is parallel to the x-axis and passes through $(0, 0, 0)$ and $(0, 1, -c)$.

91. (a) $\theta \approx 65.91°$
(b) $x = 2$
$y = 1 + t$
$z = 1 + 2t$

93. $(2, -3, 2)$; The line does not lie in the plane.

95. Not intersecting **97.** $6\sqrt{14}/7$ **99.** $11\sqrt{6}/6$

101. $2\sqrt{26}/13$ **103.** $27\sqrt{94}/188$ **105.** $\sqrt{2533}/17$

107. $7\sqrt{3}/3$ **109.** $\sqrt{66}/3$

111. Parametric equations: $x = x_1 + at$, $y = y_1 + bt$, and $z = z_1 + ct$
Symmetric equations: $\dfrac{x - x_1}{a} = \dfrac{y - y_1}{b} = \dfrac{z - z_1}{c}$
You need a vector $\mathbf{v} = \langle a, b, c \rangle$ parallel to the line and a point $P(x_1, y_1, z_1)$ on the line.

113. Simultaneously solve the two linear equations representing the planes and substitute the values back into one of the original equations. Then choose a value for t and form the corresponding parametric equations for the line of intersection.

115. (a) Parallel if vector $\langle a_1, b_1, c_1 \rangle$ is a scalar multiple of $\langle a_2, b_2, c_2 \rangle$; $\theta = 0$.
(b) Perpendicular if $a_1a_2 + b_1b_2 + c_1c_2 = 0$; $\theta = \pi/2$.

117. $cbx + acy + abz = abc$

119. Sphere: $(x - 3)^2 + (y + 2)^2 + (z - 5)^2 = 16$

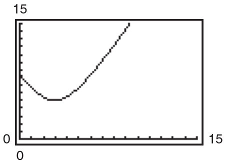
121. (a)

Year	1999	2000	2001	2002
***z* (approx.)**	6.25	6.05	5.94	5.76

Year	2003	2004	2005
***z* (approx.)**	5.66	5.56	5.56

The approximations are close to the actual values.
(b) Answers will vary.

123. (a) $\sqrt{70}$ in.

(b)


(c) The distance is never zero.
(d) 5 in.

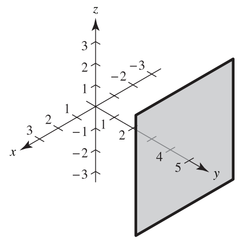
125. $\left(\frac{77}{13}, \frac{48}{13}, -\frac{23}{13}\right)$ **127.** $\left(-\frac{1}{2}, -\frac{9}{4}, \frac{1}{4}\right)$ **129.** True **131.** True

133. False. Plane $7x + y - 11z = 5$ and plane $5x + 2y - 4z = 1$ are perpendicular to plane $2x - 3y + z = 3$ but are not parallel.

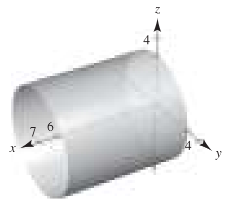
Section 11.6 (page 820)

1. c **2.** e **3.** f **4.** b **5.** d **6.** a

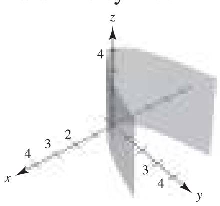
7. Plane



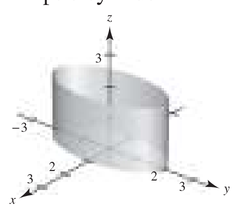
9. Right circular cylinder



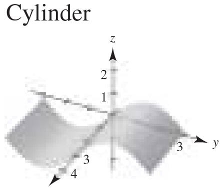
11. Parabolic cylinder



13. Elliptic cylinder

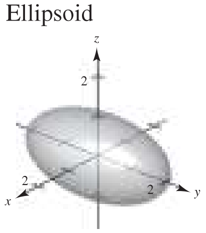


15. Cylinder

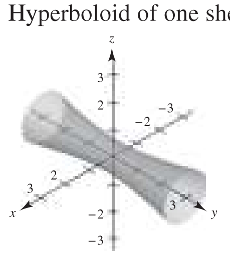


17. (a) (20, 0, 0)
(b) (10, 10, 20)
(c) (0, 0, 20)
(d) (0, 20, 0)

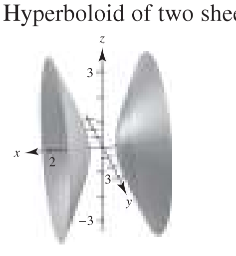
19. Ellipsoid



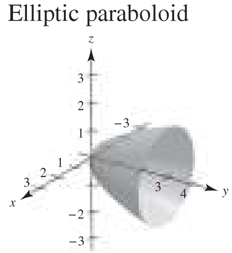
21. Hyperboloid of one sheet



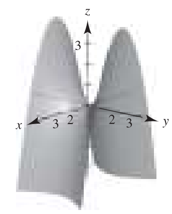
23. Hyperboloid of two sheets



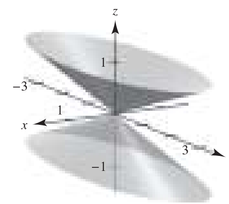
25. Elliptic paraboloid



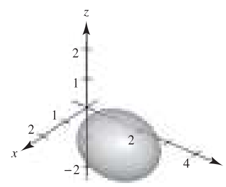
27. Hyperbolic paraboloid



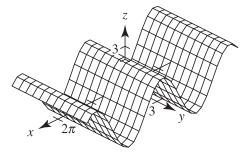
29. Elliptic cone



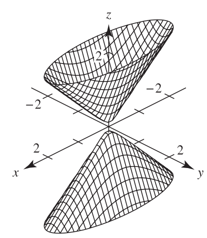
31. Ellipsoid



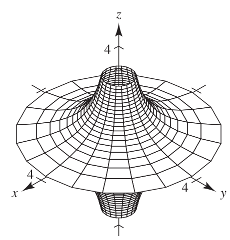
33.



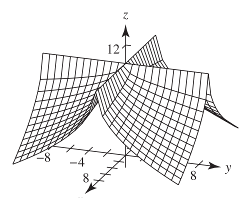
35.



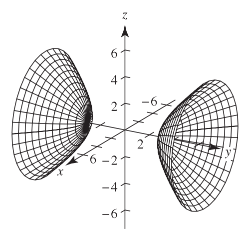
37.



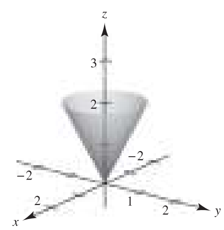
39.



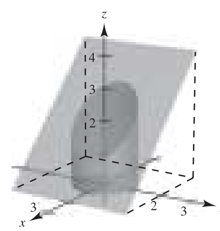
41.



43.



45.



47. $x^2 + z^2 = 4y$ **49.** $4x^2 + 4y^2 = z^2$ **51.** $y^2 + z^2 = 4/x^2$

53. $y = \sqrt{2z}$ (or $x = \sqrt{2z}$)

55. Let C be a curve in a plane and let L be a line not in a parallel plane. The set of all lines parallel to L and intersecting C is called a cylinder. C is called the generating curve of the cylinder, and the parallel lines are called rulings.

57. See pages 814 and 815. **59.** $128\pi/3$

61. (a) Major axis: $4\sqrt{2}$
Minor axis: 4
Foci: $(0, \pm 2, 2)$

(b) Major axis: $8\sqrt{2}$
Minor axis: 8
Foci: $(0, \pm 4, 8)$

63. $x^2 + z^2 = 8y$; Elliptic paraboloid

65. $x^2/3963^2 + y^2/3963^2 + z^2/3950^2 = 1$

67. $x = at, y = -bt, z = 0;$
$x = at, y = bt + ab^2, z = 2abt + a^2b^2$

69. True

71. False. A trace of an ellipsoid can be a single point.

73. The Klein bottle does not have both an "inside" and an "outside." It is formed by inserting the small open end through the side of the bottle and making it contiguous with the top of the bottle.

Section 11.7 (page 827)

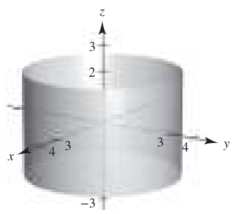
1. $(-7, 0, 5)$ **3.** $(3\sqrt{2}/2, 3\sqrt{2}/2, 1)$ **5.** $(-2\sqrt{3}, -2, 3)$

7. $(5, \pi/2, 1)$ **9.** $(2\sqrt{2}, -\pi/4, -4)$ **11.** $(2, \pi/3, 4)$

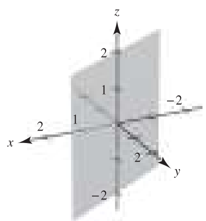
13. $z = 4$ **15.** $r^2 + z^2 = 17$ **17.** $r = \sec\theta\tan\theta$

19. $r^2\sin^2\theta = 10 - z^2$

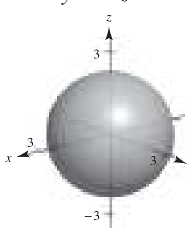
21. $x^2 + y^2 = 9$



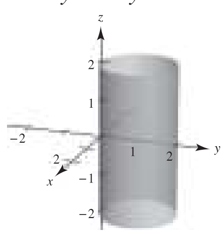
23. $x - \sqrt{3}y = 0$



25. $x^2 + y^2 + z^2 = 5$



27. $x^2 + y^2 - 2y = 0$



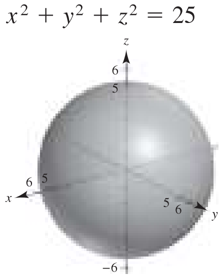
29. $(4, 0, \pi/2)$ **31.** $(4\sqrt{2}, 2\pi/3, \pi/4)$ **33.** $(4, \pi/6, \pi/6)$

35. $(\sqrt{6}, \sqrt{2}, 2\sqrt{2})$ **37.** $(0, 0, 12)$ **39.** $\left(\frac{5}{2}, \frac{5}{2}, -5\sqrt{2}/2\right)$

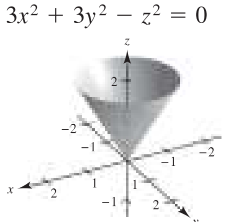
41. $\rho = 2\csc\phi\csc\theta$ **43.** $\rho = 7$

45. $\rho = 4\csc\phi$ **47.** $\tan^2\phi = 2$

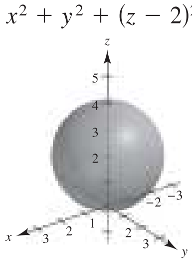
49. $x^2 + y^2 + z^2 = 25$



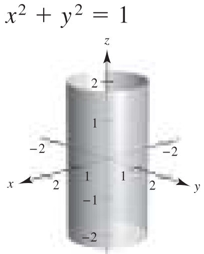
51. $3x^2 + 3y^2 - z^2 = 0$



53. $x^2 + y^2 + (z - 2)^2 = 4$



55. $x^2 + y^2 = 1$



57. $(4, \pi/4, \pi/2)$ **59.** $(4\sqrt{2}, \pi/2, \pi/4)$

61. $(2\sqrt{13}, -\pi/6, \arccos[3/\sqrt{13}])$ **63.** $(13, \pi, \arccos[5/13])$

65. $(10, \pi/6, 0)$ **67.** $(36, \pi, 0)$

69. $(3\sqrt{3}, -\pi/6, 3)$ **71.** $(4, 7\pi/6, 4\sqrt{3})$

	Rectangular	*Cylindrical*	*Spherical*
73.	$(4, 6, 3)$	$(7.211, 0.983, 3)$	$(7.810, 0.983, 1.177)$
75.	$(4.698, 1.710, 8)$	$(5, \pi/9, 8)$	$(9.434, 0.349, 0.559)$
77.	$(-7.071, 12.247, 14.142)$	$(14.142, 2.094, 14.142)$	$(20, 2\pi/3, \pi/4)$
79.	$(3, -2, 2)$	$(3.606, -0.588, 2)$	$(4.123, -0.588, 1.064)$
81.	$\left(\frac{5}{2}, \frac{4}{3}, -\frac{3}{2}\right)$	$(2.833, 0.490, -1.5)$	$(3.206, 0.490, 2.058)$
83.	$(-3.536, 3.536, -5)$	$(5, 3\pi/4, -5)$	$(7.071, 2.356, 2.356)$
85.	$(2.804, -2.095, 6)$	$(-3.5, 2.5, 6)$	$(6.946, 5.642, 0.528)$
87.	$(-1.837, 1.837, 1.5)$	$(2.598, 2.356, 1.5)$	$(3, 3\pi/4, \pi/3)$

89. d **90.** e **91.** c **92.** a **93.** f **94.** b

95. Rectangular to cylindrical:
$r^2 = x^2 + y^2$, $\tan\theta = y/x$, $z = z$
Cylindrical to rectangular:
$x = r\cos\theta$, $y = r\sin\theta$, $z = z$

97. Rectangular to spherical:
$\rho^2 = x^2 + y^2 + z^2$, $\tan\theta = y/x$, $\phi = \arccos\left(z/\sqrt{x^2 + y^2 + z^2}\right)$
Spherical to rectangular:
$x = \rho\sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta$, $z = \rho\cos\phi$

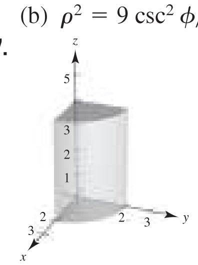
99. (a) $r^2 + z^2 = 25$ (b) $\rho = 5$

101. (a) $r^2 + (z - 1)^2 = 1$ (b) $\rho = 2\cos\phi$

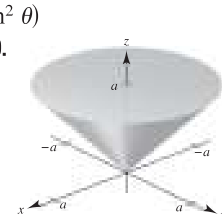
103. (a) $r = 4\sin\theta$ (b) $\rho = 4\sin\theta/\sin\phi = 4\sin\theta\csc\phi$

105. (a) $r^2 = 9/(\cos^2\theta - \sin^2\theta)$
(b) $\rho^2 = 9\csc^2\phi/(\cos^2\theta - \sin^2\theta)$

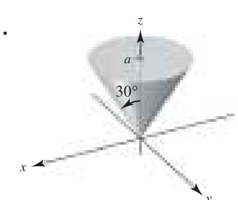
107.



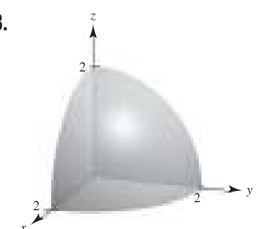
109.



111.



113.



115. Rectangular: $0 \le x \le 10$
$0 \le y \le 10$
$0 \le z \le 10$

117. Spherical: $4 \le \rho \le 6$

119. Cylindrical: $r^2 + z^2 \le 9$, $r \le 3\cos\theta$, $0 \le \theta \le \pi$

121. False. $r = z$ represents a cone.

123. False. See page 823. **125.** Ellipse

Review Exercises for Chapter 11 (page 829)

1. (a) $\mathbf{u} = \langle 3, -1\rangle$ (b) $\mathbf{u} = 3\mathbf{i} - \mathbf{j}$ (c) $2\sqrt{5}$ (d) $10\mathbf{i}$
$\mathbf{v} = \langle 4, 2\rangle$

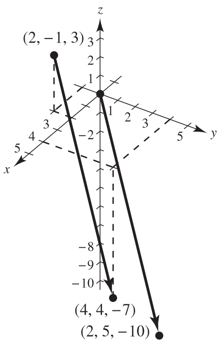
3. $\mathbf{v} = \langle 4, 4\sqrt{3}\rangle$ **5.** $(-5, 4, 0)$

7. Above the xy-plane and to the right of the xz-plane *or* below the xy-plane and to the left of the xz-plane

9. $(x-3)^2+(y+2)^2+(z-6)^2=\frac{225}{4}$

11. $(x-2)^2+(y-3)^2+z^2=9$
Center: $(2, 3, 0)$
Radius: 3

13. (a) and (d) (b) $\mathbf{u}=\langle 2, 5, -10\rangle$
(c) $\mathbf{u}=2\mathbf{i}+5\mathbf{j}-10\mathbf{k}$



15. Collinear **17.** $(1/\sqrt{38})\langle 2, 3, 5\rangle$

19. (a) $\mathbf{u}=\langle -1, 4, 0\rangle$, $\mathbf{v}=\langle -3, 0, 6\rangle$ (b) 3 (c) 45

21. Orthogonal **23.** $\theta=\arccos\left(\dfrac{\sqrt{2}+\sqrt{6}}{4}\right)=15°$ **25.** π

27. Answers will vary. Example: $\langle -6, 5, 0\rangle$, $\langle 6, -5, 0\rangle$

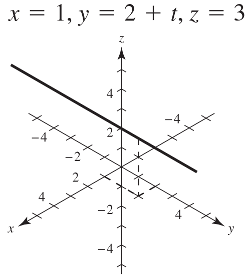
29. $\mathbf{u}\cdot\mathbf{u}=14=\|\mathbf{u}\|^2$ **31.** $\left\langle -\frac{15}{14}, \frac{5}{7}, -\frac{5}{14}\right\rangle$

33. $(1/\sqrt{5})(-2\mathbf{i}-\mathbf{j})$ or $(1/\sqrt{5})(2\mathbf{i}+\mathbf{j})$

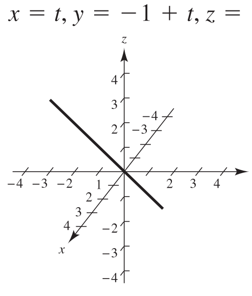
35. 4 **37.** $\sqrt{285}$ **39.** 100 sec 20° ≈ 106.4 lb

41. (a) $x=3+6t$, $y=11t$, $z=2+4t$
(b) $(x-3)/6=y/11=(z-2)/4$

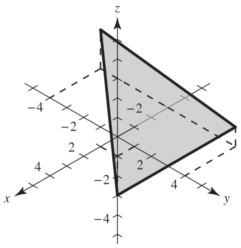
43. (a) $x=1$, $y=2+t$, $z=3$ (b) None
(c)

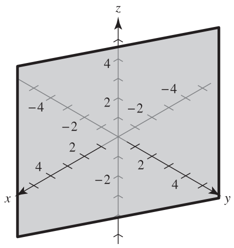


45. (a) $x=t$, $y=-1+t$, $z=1$ (b) $x=y+1$, $z=1$
(c)



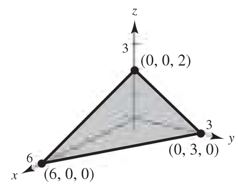
47. $27x+4y+32z+33=0$ **49.** $x+2y=1$



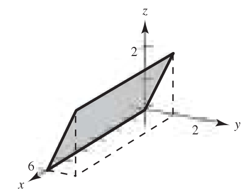


51. $\frac{8}{7}$ **53.** $\sqrt{35}/7$

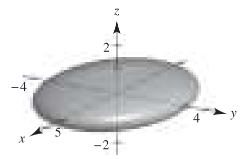
55. Plane



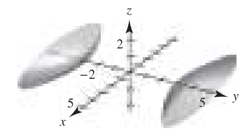
57. Plane



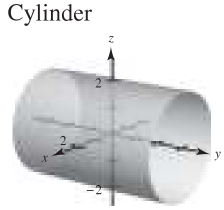
59. Ellipsoid



61. Hyperboloid of two sheets



63. Cylinder



65. Let $y=2\sqrt{x}$ and revolve around the x-axis.

67. $x^2+z^2=2y$

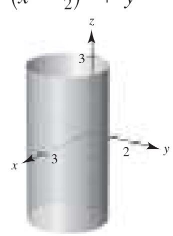
69. (a) $(4, 3\pi/4, 2)$ (b) $\left(2\sqrt{5}, 3\pi/4, \arccos\left[\sqrt{5}/5\right]\right)$

71. $\left(50\sqrt{5}, -\pi/6, \arccos\left[1/\sqrt{5}\right]\right)$

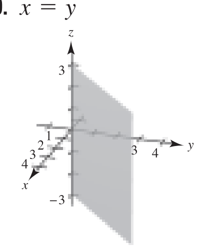
73. $\left(25\sqrt{2}/2, -\pi/4, -25\sqrt{2}/2\right)$

75. (a) $r^2\cos 2\theta=2z$ (b) $\rho=2\sec 2\theta\cos\phi\csc^2\phi$

77. $\left(x-\frac{5}{2}\right)^2+y^2=\frac{25}{4}$



79. $x=y$

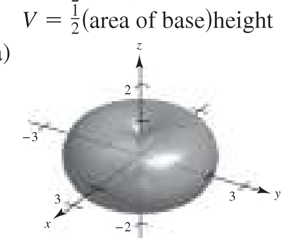


P.S. Problem Solving (page 831)

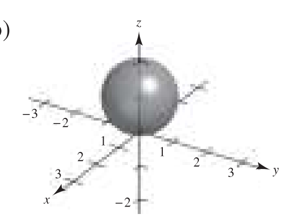
1–3. Proofs **5.** (a) $3\sqrt{2}/2\approx 2.12$ (b) $\sqrt{5}\approx 2.24$

7. (a) $\pi/2$ (b) $\frac{1}{2}(\pi abk)k$
(c) $V=\frac{1}{2}(\pi ab)k^2$
$V=\frac{1}{2}$(area of base)height

9. (a)



(b)

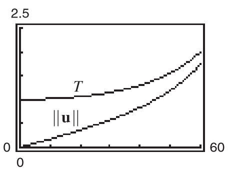


11. Proof

13. (a) Tension: $2\sqrt{3}/3 \approx 1.1547$ lb
Magnitude of $\mathbf{u}$: $\sqrt{3}/3 \approx 0.5774$ lb
(b) $T = \sec\theta$; $\|\mathbf{u}\| = \tan\theta$; Domain: $0° \leq \theta \leq 90°$
(c)

θ	0°	10°	20°	30°
T	1	1.0154	1.0642	1.1547
$\|\mathbf{u}\|$	0	0.1763	0.3640	0.5774

θ	40°	50°	60°
T	1.3054	1.5557	2
$\|\mathbf{u}\|$	0.8391	1.1918	1.7321

(d) 

(e) Both are increasing functions.

(f) $\lim_{\theta\to\pi/2^-} T = \infty$ and $\lim_{\theta\to\pi/2^-} \|\mathbf{u}\| = \infty$
Yes. As θ increases, both T and $\|\mathbf{u}\|$ increase.

15. $\langle 0, 0, \cos\alpha\sin\beta - \cos\beta\sin\alpha\rangle$; Proof

17. $D = \dfrac{|\overrightarrow{PQ}\cdot\mathbf{n}|}{\|\mathbf{n}\|}$
$= \dfrac{|\mathbf{w}\cdot(\mathbf{u}\times\mathbf{v})|}{\|\mathbf{u}\times\mathbf{v}\|} = \dfrac{|(\mathbf{u}\times\mathbf{v})\cdot\mathbf{w}|}{\|\mathbf{u}\times\mathbf{v}\|} = \dfrac{|\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})|}{\|\mathbf{u}\times\mathbf{v}\|}$

19. Proof

Chapter 12

Section 12.1 (page 839)

1. $(-\infty, -1)\cup(-1, \infty)$ **3.** $(0, \infty)$

5. $[0, \infty)$ **7.** $(-\infty, \infty)$

9. (a) $\frac{1}{2}\mathbf{i}$ (b) $\mathbf{j}$ (c) $\frac{1}{2}(s+1)^2\mathbf{i} - s\mathbf{j}$ (d) $\frac{1}{2}\Delta t(\Delta t + 4)\mathbf{i} - \Delta t\mathbf{j}$

11. (a) $\ln 2\mathbf{i} + \dfrac{1}{2}\mathbf{j} + 6\mathbf{k}$ (b) Not possible
(c) $\ln(t-4)\mathbf{i} + \dfrac{1}{t-4}\mathbf{j} + 3(t-4)\mathbf{k}$
(d) $\ln(1+\Delta t)\mathbf{i} - \dfrac{\Delta t}{1+\Delta t}\mathbf{j} + 3\Delta t\mathbf{k}$

13. $\sqrt{t(1+25t)}$

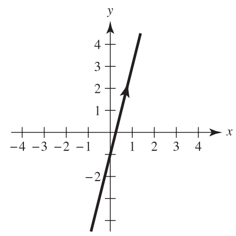
15. $\mathbf{r}(t) = 3t\mathbf{i} + t\mathbf{j} + 2t\mathbf{k}$
$x = 3t,\ y = t,\ z = 2t$

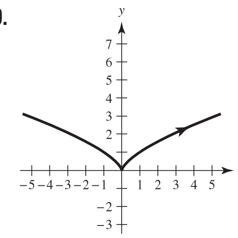
17. $\mathbf{r}(t) = (-2+t)\mathbf{i} + (5-t)\mathbf{j} + (-3+12t)\mathbf{k}$
$x = -2 + t,\ y = 5 - t,\ z = -3 + 12t$

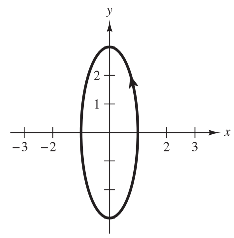
19. $t^2(5t-1)$; No, the dot product is a scalar.

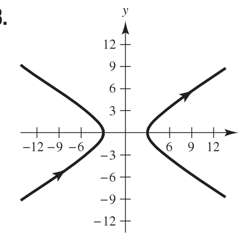
21. b **22.** c **23.** d **24.** a

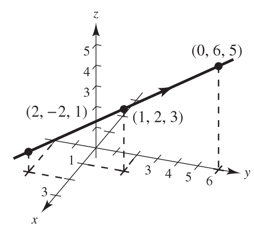
25. (a) $(-20, 0, 0)$ (b) $(10, 20, 10)$
(c) $(0, 0, 20)$ (d) $(20, 0, 0)$

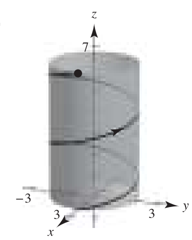
27. 

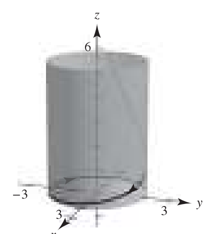
29. 

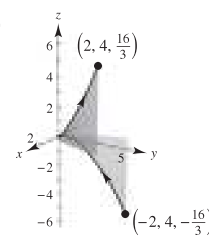
31. 

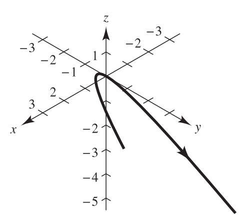
33. 

35. 

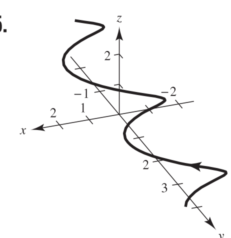
37. 

39. 

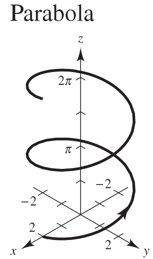
41. 

43. 

Parabola

45. 

Helix

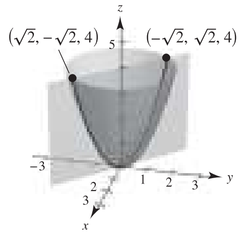
47. 

(a) The helix is translated two units back on the x-axis.
(b) The height of the helix increases at a greater rate.
(c) The orientation of the graph is reversed.
(d) The axis of the helix is the x-axis.
(e) The radius of the helix is increased from 2 to 6.

49–55. Answers will vary.

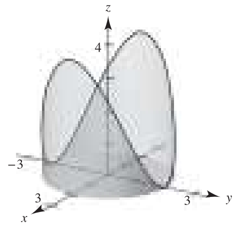
57. Answers will vary. Sample answer:
$\mathbf{r}_1(t) = t\mathbf{i} + t^2\mathbf{j},\quad 0 \leq t \leq 2$
$\mathbf{r}_2(t) = (2-t)\mathbf{i} + 4\mathbf{j},\quad 0 \leq t \leq 2$
$\mathbf{r}_3(t) = (4-t)\mathbf{j},\quad 0 \leq t \leq 4$

59.



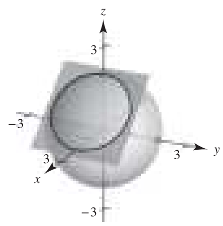
$\mathbf{r}(t) = t\mathbf{i} - t\mathbf{j} + 2t^2\mathbf{k}$

61.



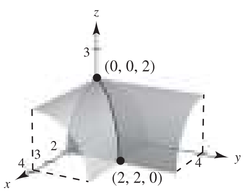
$\mathbf{r}(t) = 2 \sin t\mathbf{i} + 2 \cos t\mathbf{j} + 4 \sin^2 t\mathbf{k}$

63.



$\mathbf{r}(t) = (1 + \sin t)\mathbf{i} + \sqrt{2} \cos t\mathbf{j} + (1 - \sin t)\mathbf{k}$ and
$\mathbf{r}(t) = (1 + \sin t)\mathbf{i} - \sqrt{2} \cos t\mathbf{j} + (1 - \sin t)\mathbf{k}$

65.

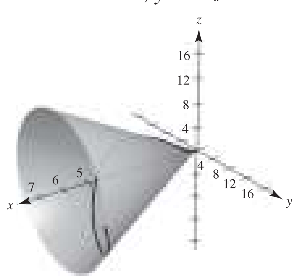


$\mathbf{r}(t) = t\mathbf{i} + t\mathbf{j} + \sqrt{4 - t^2}\mathbf{k}$

67. Let $x = t$, $y = 2t \cos t$, and $z = 2t \sin t$. Then

$$y^2 + z^2 = (2t \cos t)^2 + (2t \sin t)^2 = 4t^2 \cos^2 t + 4t^2 \sin^2 t = 4t^2(\cos^2 t + \sin^2 t) = 4t^2.$$

Because $x = t$, $y^2 + z^2 = 4x^2$.



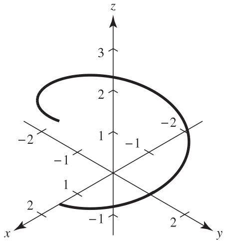
69. $\pi\mathbf{i} - \mathbf{j}$ **71.** $\mathbf{0}$ **73.** $\mathbf{i} + \mathbf{j} + \mathbf{k}$

75. $(-\infty, 0), (0, \infty)$ **77.** $[-1, 1]$

79. $(-\pi/2 + n\pi, \pi/2 + n\pi)$, n is an integer.

81. (a) $\mathbf{s}(t) = t^2\mathbf{i} + (t - 3)\mathbf{j} + (t + 3)\mathbf{k}$
(b) $\mathbf{s}(t) = (t^2 - 2)\mathbf{i} + (t - 3)\mathbf{j} + t\mathbf{k}$
(c) $\mathbf{s}(t) = t^2\mathbf{i} + (t + 2)\mathbf{j} + t\mathbf{k}$

83. Answers will vary. Sample answer:



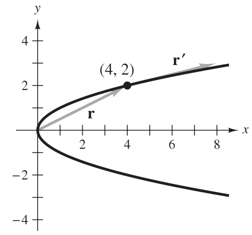
$\mathbf{r}(t) = 1.5 \cos t\mathbf{i} + 1.5 \sin t\mathbf{j} + \dfrac{1}{\pi}t\mathbf{k}$, $0 \le t \le 2\pi$

85–87. Proofs **89.** Yes; Yes **91.** Not necessarily

93. True **95.** True

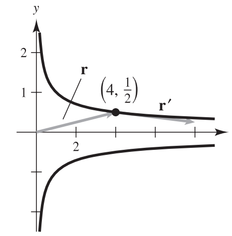
Section 12.2 (page 848)

1. $\mathbf{r}(2) = 4\mathbf{i} + 2\mathbf{j}$
$\mathbf{r}'(2) = 4\mathbf{i} + \mathbf{j}$



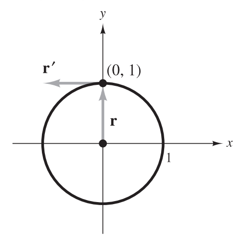
$\mathbf{r}'(t_0)$ is tangent to the curve at t_0.

3. $\mathbf{r}(2) = 4\mathbf{i} + \frac{1}{2}\mathbf{j}$
$\mathbf{r}'(2) = 4\mathbf{i} - \frac{1}{4}\mathbf{j}$



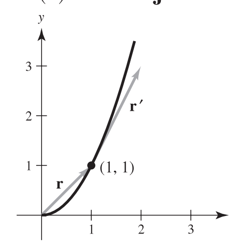
$\mathbf{r}'(t_0)$ is tangent to the curve at t_0.

5. $\mathbf{r}(\pi/2) = \mathbf{j}$
$\mathbf{r}'(\pi/2) = -\mathbf{i}$



$\mathbf{r}'(t_0)$ is tangent to the curve at t_0.

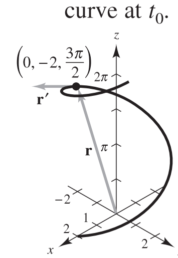
7. $\mathbf{r}(0) = \mathbf{i} + \mathbf{j}$
$\mathbf{r}'(0) = \mathbf{i} + 2\mathbf{j}$



$\mathbf{r}'(t_0)$ is tangent to the curve at t_0.

9. $\mathbf{r}\left(\dfrac{3\pi}{2}\right) = -2\mathbf{j} + \left(\dfrac{3\pi}{2}\right)\mathbf{k}$

$\mathbf{r}'\left(\dfrac{3\pi}{2}\right) = 2\mathbf{i} + \mathbf{k}$



11. $3t^2\mathbf{i} - 3\mathbf{j}$ **13.** $-2 \sin t\mathbf{i} + 5 \cos t\mathbf{j}$

15. $6\mathbf{i} - 14t\mathbf{j} + 3t^2\mathbf{k}$ **17.** $-3a \sin t \cos^2 t\mathbf{i} + 3a \sin^2 t \cos t\mathbf{j}$

19. $-e^{-t}\mathbf{i} + (5te^t + 5e^t)\mathbf{k}$

21. $\langle \sin t + t \cos t, \cos t - t \sin t, 1\rangle$

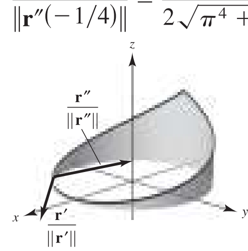
23. (a) $3t^2\mathbf{i} + t\mathbf{j}$ (b) $6t\mathbf{i} + \mathbf{j}$ (c) $18t^3 + t$

25. (a) $-4 \sin t\mathbf{i} + 4 \cos t\mathbf{j}$ (b) $-4 \cos t\mathbf{i} - 4 \sin t\mathbf{j}$ (c) 0

27. (a) $t\mathbf{i} - \mathbf{j} + \frac{1}{2}t^2\mathbf{k}$ (b) $\mathbf{i} + t\mathbf{k}$ (c) $t^3/2 + t$

29. (a) $\langle t \cos t, t \sin t, 1\rangle$
(b) $\langle \cos t - t \sin t, \sin t + t \cos t, 0\rangle$ (c) t

31. $$\frac{\mathbf{r}'(-1/4)}{\|\mathbf{r}'(-1/4)\|} = \frac{1}{\sqrt{4\pi^2 + 1}}\left(\sqrt{2}\pi\mathbf{i} + \sqrt{2}\pi\mathbf{j} - \mathbf{k}\right)$$
$$\frac{\mathbf{r}''(-1/4)}{\|\mathbf{r}''(-1/4)\|} = \frac{1}{2\sqrt{\pi^4 + 4}}\left(-\sqrt{2}\pi^2\mathbf{i} + \sqrt{2}\pi^2\mathbf{j} + 4\mathbf{k}\right)$$



33. $(-\infty, 0), (0, \infty)$ **35.** $(n\pi/2, (n+1)\pi/2)$

37. $(-\infty, \infty)$ **39.** $(-\infty, 0), (0, \infty)$

41. $(-\pi/2 + n\pi, \pi/2 + n\pi)$, n is an integer.

43. (a) $\mathbf{i} + 3\mathbf{j} + 2t\mathbf{k}$ (b) $2\mathbf{k}$ (c) $8t + 9t^2 + 5t^4$
(d) $-\mathbf{i} + (9 - 2t)\mathbf{j} + (6t - 3t^2)\mathbf{k}$
(e) $8t^3\mathbf{i} + (12t^2 - 4t^3)\mathbf{j} + (3t^2 - 24t)\mathbf{k}$
(f) $(10 + 2t^2)/\sqrt{10 + t^2}$

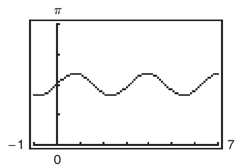
45. (a) $7t^6$ (b) $12t^5\mathbf{i} - 5t^4\mathbf{j}$

47. $\theta(t) = \arccos\left(\dfrac{-7 \sin t \cos t}{\sqrt{9\sin^2 t + 16\cos^2 t}\,\sqrt{9\cos^2 t + 16\sin^2 t}}\right)$

Maximum: $\theta\left(\dfrac{\pi}{4}\right) = \theta\left(\dfrac{5\pi}{4}\right) \approx 1.855$

Minimum: $\theta\left(\dfrac{3\pi}{4}\right) = \theta\left(\dfrac{7\pi}{4}\right) \approx 1.287$

Orthogonal: $\dfrac{n\pi}{2}$, n is an integer.



49. $\mathbf{r}'(t) = 3\mathbf{i} - 2t\mathbf{j}$ **51.** $\mathbf{r}'(t) = 2t\mathbf{i} + 2\mathbf{k}$

53. $t^2\mathbf{i} + t\mathbf{j} + t\mathbf{k} + \mathbf{C}$ **55.** $\ln t\,\mathbf{i} + t\mathbf{j} - \frac{2}{5}t^{5/2}\mathbf{k} + \mathbf{C}$

57. $(t^2 - t)\mathbf{i} + t^4\mathbf{j} + 2t^{3/2}\mathbf{k} + \mathbf{C}$

59. $\tan t\,\mathbf{i} + \arctan t\,\mathbf{j} + \mathbf{C}$ **61.** $4\mathbf{i} + \frac{1}{2}\mathbf{j} - \mathbf{k}$

63. $a\mathbf{i} + a\mathbf{j} + (\pi/2)\mathbf{k}$

65. $2\mathbf{i} + (e^2 - 1)\mathbf{j} - (e^2 + 1)\mathbf{k}$

67. $2e^{2t}\mathbf{i} + 3(e^t - 1)\mathbf{j}$ **69.** $600\sqrt{3}t\mathbf{i} + (-16t^2 + 600t)\mathbf{j}$

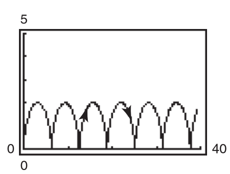
71. $((2 - e^{-t^2})/2)\mathbf{i} + (e^{-t} - 2)\mathbf{j} + (t + 1)\mathbf{k}$

73. See "Definition of the Derivative of a Vector-Valued Function" and Figure 12.8 on page 842.

75. The three components of $\mathbf{u}$ are increasing functions of t at $t = t_0$.

77–83. Proofs

85. (a)



The curve is a cycloid.

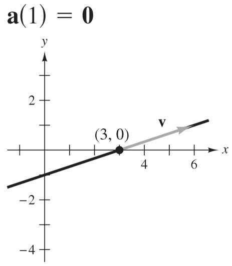
(b) The maximum of $\|\mathbf{r}'\|$ is 2; the minimum of $\|\mathbf{r}'\|$ is 0. The maximum and the minimum of $\|\mathbf{r}''\|$ is 1.

87. Proof **89.** True

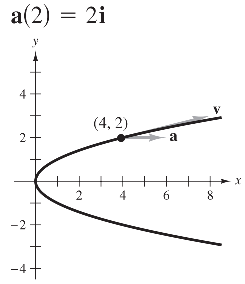
91. False: Let $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + \mathbf{k}$, then $d/dt[\|\mathbf{r}(t)\|] = 0$, but $\|\mathbf{r}'(t)\| = 1$.

Section 12.3 (page 856)

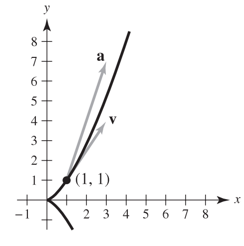
1. $\mathbf{v}(1) = 3\mathbf{i} + \mathbf{j}$
$\mathbf{a}(1) = \mathbf{0}$



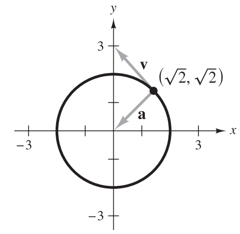
3. $\mathbf{v}(2) = 4\mathbf{i} + \mathbf{j}$
$\mathbf{a}(2) = 2\mathbf{i}$



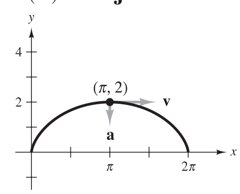
5. $\mathbf{v}(1) = 2\mathbf{i} + 3\mathbf{j}$
$\mathbf{a}(1) = 2\mathbf{i} + 6\mathbf{j}$



7. $\mathbf{v}(\pi/4) = -\sqrt{2}\mathbf{i} + \sqrt{2}\mathbf{j}$
$\mathbf{a}(\pi/4) = -\sqrt{2}\mathbf{i} - \sqrt{2}\mathbf{j}$



9. $\mathbf{v}(\pi) = 2\mathbf{i}$
$\mathbf{a}(\pi) = -\mathbf{j}$



11. $\mathbf{v}(t) = \mathbf{i} + 5\mathbf{j} + 3\mathbf{k}$
$\|\mathbf{v}(t)\| = \sqrt{35}$
$\mathbf{a}(t) = \mathbf{0}$

13. $\mathbf{v}(t) = \mathbf{i} + 2t\mathbf{j} + t\mathbf{k}$
$\|\mathbf{v}(t)\| = \sqrt{1 + 5t^2}$
$\mathbf{a}(t) = 2\mathbf{j} + \mathbf{k}$

15. $\mathbf{v}(t) = \mathbf{i} + \mathbf{j} - \left(t/\sqrt{9 - t^2}\right)\mathbf{k}$
$\|\mathbf{v}(t)\| = \sqrt{(18 - t^2)/(9 - t^2)}$
$\mathbf{a}(t) = (-9/(9 - t^2)^{3/2})\mathbf{k}$

17. $\mathbf{v}(t) = 4\mathbf{i} - 3\sin t\,\mathbf{j} + 3\cos t\,\mathbf{k}$
$\|\mathbf{v}(t)\| = 5$
$\mathbf{a}(t) = -3\cos t\,\mathbf{j} - 3\sin t\,\mathbf{k}$

19. $\mathbf{v}(t) = (e^t\cos t - e^t\sin t)\mathbf{i} + (e^t\sin t + e^t\cos t)\mathbf{j} + e^t\mathbf{k}$
$\|\mathbf{v}(t)\| = e^t\sqrt{3}$
$\mathbf{a}(t) = -2e^t\sin t\,\mathbf{i} + 2e^t\cos t\,\mathbf{j} + e^t\mathbf{k}$

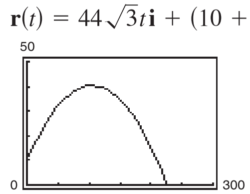
21. (a) $x = 1 + t$ (b) $(1.100, -1.200, 0.325)$
$y = -1 - 2t$
$z = \frac{1}{4} + \frac{3}{4}t$

23. $\mathbf{v}(t) = t(\mathbf{i} + \mathbf{j} + \mathbf{k})$
$\mathbf{r}(t) = (t^2/2)(\mathbf{i} + \mathbf{j} + \mathbf{k})$
$\mathbf{r}(2) = 2(\mathbf{i} + \mathbf{j} + \mathbf{k})$

25. $\mathbf{v}(t) = \left(t^2/2 + \frac{9}{2}\right)\mathbf{j} + \left(t^2/2 - \frac{1}{2}\right)\mathbf{k}$
$\mathbf{r}(t) = \left(t^3/6 + \frac{9}{2}t - \frac{14}{3}\right)\mathbf{j} + \left(t^3/6 - \frac{1}{2}t + \frac{1}{3}\right)\mathbf{k}$
$\mathbf{r}(2) = \frac{17}{3}\mathbf{j} + \frac{2}{3}\mathbf{k}$

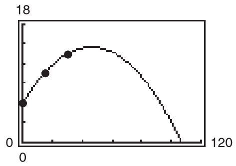
27. $\mathbf{v}(t) = -\sin t\,\mathbf{i} + \cos t\,\mathbf{j} + \mathbf{k}$
$\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + t\mathbf{k}$
$\mathbf{r}(2) = (\cos 2)\mathbf{i} + (\sin 2)\mathbf{j} + 2\mathbf{k}$

29. $\mathbf{r}(t) = 44\sqrt{3}t\mathbf{i} + (10 + 44t - 16t^2)\mathbf{j}$



31. $v_0 = 40\sqrt{6}$ ft/sec; 78 ft **33.** Proof

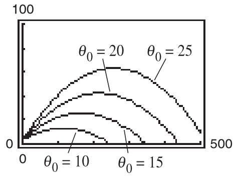
35. (a) $y = -0.004x^2 + 0.37x + 6$
$\mathbf{r}(t) = t\mathbf{i} + (-0.004t^2 + 0.37t + 6)\mathbf{j}$

(b) 

(c) 14.56 ft

(d) Initial velocity: 67.4 ft/sec; $\theta \approx 20.14°$

37. (a) $\mathbf{r}(t) = \left(\frac{440}{3}\cos\theta_0\right)t\mathbf{i} + \left[3 + \left(\frac{440}{3}\sin\theta_0\right)t - 16t^2\right]\mathbf{j}$

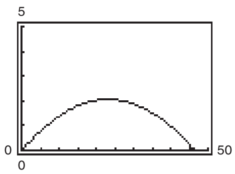
(b) 

The minimum angle appears to be $\theta_0 = 20°$.

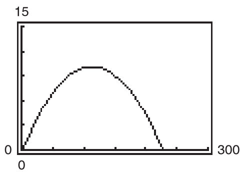
(c) $\theta_0 \approx 19.38°$

39. (a) $v_0 = 28.78$ ft/sec; $\theta = 58.28°$ (b) $v_0 \approx 32$ ft/sec

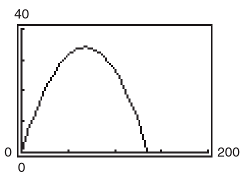
41. 1.91°

43. (a) 

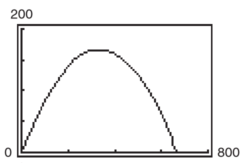
Maximum height: 2.1 ft
Range: 46.6 ft

(b) 

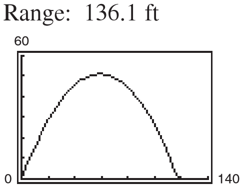
Maximum height: 10.0 ft
Range: 227.8 ft

(c) 

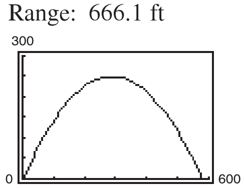
Maximum height: 34.0 ft
Range: 136.1 ft

(d) 

Maximum height: 166.5 ft
Range: 666.1 ft

(e) 

Maximum height: 51.0 ft
Range: 117.9 ft

(f) 

Maximum height: 249.8 ft
Range: 576.9 ft

45. Maximum height: 129.1 m
Range: 886.3 m

47. $\mathbf{v}(t) = b\omega[(1 - \cos\omega t)\mathbf{i} + \sin\omega t\mathbf{j}]$
$\mathbf{a}(t) = b\omega^2(\sin\omega t\mathbf{i} + \cos\omega t\mathbf{j})$

(a) $\|\mathbf{v}(t)\| = 0$ when $\omega t = 0, 2\pi, 4\pi, \ldots$

(b) $\|\mathbf{v}(t)\|$ is maximum when $\omega t = \pi, 3\pi, \ldots$

49. $\mathbf{v}(t) = -b\omega\sin\omega t\mathbf{i} + b\omega\cos\omega t\mathbf{j}$
$\mathbf{v}(t)\cdot\mathbf{r}(t) = 0$

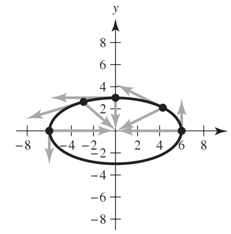
51. $\mathbf{a}(t) = -b\omega^2(\cos\omega t\mathbf{i} + \sin\omega t\mathbf{j}) = -\omega^2\mathbf{r}(t)$; $\mathbf{a}(t)$ is a negative multiple of a unit vector from $(0, 0)$ to $(\cos\omega t, \sin\omega t)$, so $\mathbf{a}(t)$ is directed toward the origin.

53. $8\sqrt{10}$ ft/sec **55–57.** Proofs

59. (a) $\mathbf{v}(t) = -6\sin t\mathbf{i} + 3\cos t\mathbf{j}$
$\|\mathbf{v}(t)\| = 3\sqrt{3\sin^2 t + 1}$
$\mathbf{a}(t) = -6\cos t\mathbf{i} - 3\sin t\mathbf{j}$

(b)

t	0	$\pi/4$	$\pi/2$	$2\pi/3$	π
Speed	3	$3\sqrt{10}/2$	6	$3\sqrt{13}/2$	3

(c) 

(d) The speed is increasing when the angle between $\mathbf{v}$ and $\mathbf{a}$ is in the interval $[0, \pi/2)$, and decreasing when the angle is in the interval $(\pi/2, \pi]$.

61. The velocity of an object involves both magnitude and direction of motion, whereas speed involves only magnitude.

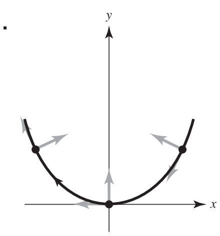
63. (a) Velocity: $\mathbf{r}_2'(t) = 2\mathbf{r}_1'(2t)$
Acceleration: $\mathbf{r}_2''(t) = 4\mathbf{r}_1''(2t)$

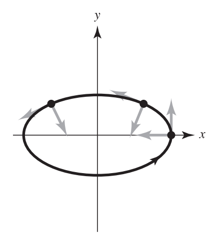
(b) In general, if $\mathbf{r}_3(t) = \mathbf{r}_1(\omega t)$, then:
Velocity: $\mathbf{r}_3'(t) = \omega\mathbf{r}_1'(\omega t)$
Acceleration: $\mathbf{r}_3''(t) = \omega^2\mathbf{r}_1''(\omega t)$

65. False; acceleration is the derivative of the velocity.

67. True

Section 12.4 (page 865)

1. 

3. 

5. $\mathbf{T}(1) = \left(\sqrt{2}/2\right)(\mathbf{i} + \mathbf{j})$ **7.** $\mathbf{T}(\pi/4) = \left(\sqrt{2}/2\right)(-\mathbf{i} + \mathbf{j})$

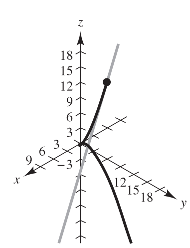
9. $\mathbf{T}(e) = (3e\mathbf{i} - \mathbf{j})/\sqrt{9e^2 + 1} \approx 0.9926\mathbf{i} - 0.1217\mathbf{j}$

11. $\mathbf{T}(0) = \left(\sqrt{2}/2\right)(\mathbf{i} + \mathbf{k})$
$x = t$
$y = 0$
$z = t$

13. $\mathbf{T}(0) = \left(\sqrt{10}/10\right)(3\mathbf{j} + \mathbf{k})$
$x = 3$
$y = 3t$
$z = t$

15. $\mathbf{T}(\pi/4) = \frac{1}{2}\langle -\sqrt{2}, \sqrt{2}, 0\rangle$
$x = \sqrt{2} - \sqrt{2}\,t$
$y = \sqrt{2} + \sqrt{2}\,t$
$z = 4$

17. $\mathbf{T}(3) = \frac{1}{19}\langle 1, 6, 18\rangle$
$x = 3 + t$
$y = 9 + 6t$
$z = 18 + 18t$



19. Tangent line: $x = 1 + t, y = t, z = 1 + \frac{1}{2}t$
$\mathbf{r}(1.1) \approx \langle 1.1, 0.1, 1.05\rangle$

21. 1.2° **23.** $\mathbf{N}(2) = \left(\sqrt{5}/5\right)(-2\mathbf{i} + \mathbf{j})$

25. $\mathbf{N}(2) = \left(-\sqrt{5}/5\right)(2\mathbf{i} - \mathbf{j})$

27. $\mathbf{N}(1) = \left(-\sqrt{14}/14\right)(\mathbf{i} - 2\mathbf{j} + 3\mathbf{k})$

29. $\mathbf{N}(3\pi/4) = \left(\sqrt{2}/2\right)(\mathbf{i} - \mathbf{j})$

31. $\mathbf{v}(t) = 4\mathbf{i}$
$\mathbf{a}(t) = \mathbf{0}$
$\mathbf{T}(t) = \mathbf{i}$
$\mathbf{N}(t)$ is undefined. The path is a line and the speed is constant.

33. $\mathbf{v}(t) = 8t\,\mathbf{i}$
$\mathbf{a}(t) = 8\mathbf{i}$
$\mathbf{T}(t) = \mathbf{i}$
$\mathbf{N}(t)$ is undefined. The path is a line and the speed is variable.

35. $\mathbf{T}(1) = (\sqrt{2}/2)(\mathbf{i} - \mathbf{j})$
$\mathbf{N}(1) = (\sqrt{2}/2)(\mathbf{i} + \mathbf{j})$
$a_{\mathbf{T}} = -\sqrt{2}$
$a_{\mathbf{N}} = \sqrt{2}$

37. $\mathbf{T}(1) = (-\sqrt{5}/5)(\mathbf{i} - 2\mathbf{j})$
$\mathbf{N}(1) = (-\sqrt{5}/5)(2\mathbf{i} + \mathbf{j})$
$a_{\mathbf{T}} = 14\sqrt{5}/5$
$a_{\mathbf{N}} = 8\sqrt{5}/5$

39. $\mathbf{T}(0) = (\sqrt{5}/5)(\mathbf{i} - 2\mathbf{j})$
$\mathbf{N}(0) = (\sqrt{5}/5)(2\mathbf{i} + \mathbf{j})$
$a_{\mathbf{T}} = -7\sqrt{5}/5$
$a_{\mathbf{N}} = 6\sqrt{5}/5$

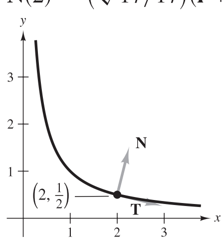
41. $\mathbf{T}(\pi/2) = (\sqrt{2}/2)(-\mathbf{i} + \mathbf{j})$
$\mathbf{N}(\pi/2) = (-\sqrt{2}/2)(\mathbf{i} + \mathbf{j})$
$a_{\mathbf{T}} = \sqrt{2}e^{\pi/2}$
$a_{\mathbf{N}} = \sqrt{2}e^{\pi/2}$

43. $\mathbf{T}(t_0) = (\cos \omega t_0)\mathbf{i} + (\sin \omega t_0)\mathbf{j}$
$\mathbf{N}(t_0) = (-\sin \omega t_0)\mathbf{i} + (\cos \omega t_0)\mathbf{j}$
$a_{\mathbf{T}} = \omega^2$
$a_{\mathbf{N}} = \omega^3 t_0$

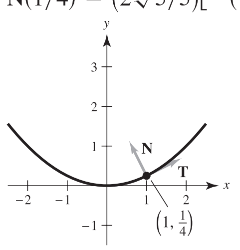
45. $\mathbf{T}(t) = -\sin(\omega t)\mathbf{i} + \cos(\omega t)\mathbf{j}$
$\mathbf{N}(t) = -\cos(\omega t)\mathbf{i} - \sin(\omega t)\mathbf{j}$
$a_{\mathbf{T}} = 0$
$a_{\mathbf{N}} = a\omega^2$

47. $\|\mathbf{v}(t)\| = a\omega$; The speed is constant because $a_{\mathbf{T}} = 0$.

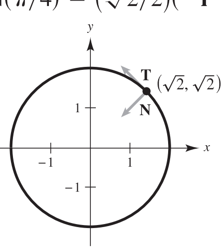
49. $\mathbf{r}(2) = 2\mathbf{i} + \frac{1}{2}\mathbf{j}$
$\mathbf{T}(2) = (\sqrt{17}/17)(4\mathbf{i} - \mathbf{j})$
$\mathbf{N}(2) = (\sqrt{17}/17)(\mathbf{i} + 4\mathbf{j})$



51. $\mathbf{r}(1/4) = \mathbf{i} + (1/4)\mathbf{j}$
$\mathbf{T}(1/4) = (\sqrt{5}/5)(2\mathbf{i} + \mathbf{j})$
$\mathbf{N}(1/4) = (2\sqrt{5}/5)[-(1/2)\mathbf{i} + \mathbf{j}]$



53. $\mathbf{r}(\pi/4) = \sqrt{2}\,\mathbf{i} + \sqrt{2}\,\mathbf{j}$
$\mathbf{T}(\pi/4) = (\sqrt{2}/2)(-\mathbf{i} + \mathbf{j})$
$\mathbf{N}(\pi/4) = (\sqrt{2}/2)(-\mathbf{i} - \mathbf{j})$



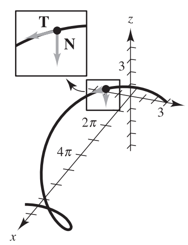
55. $\mathbf{T}(1) = (\sqrt{14}/14)(\mathbf{i} + 2\mathbf{j} - 3\mathbf{k})$
$\mathbf{N}(1)$ is undefined.
$a_{\mathbf{T}}$ is undefined.
$a_{\mathbf{N}}$ is undefined.

57. $\mathbf{T}(\pi/3) = (\sqrt{5}/5)[-(\sqrt{3}/2)\mathbf{i} + (1/2)\mathbf{j} + 2\mathbf{k}]$
$\mathbf{N}(\pi/3) = -(1/2)\mathbf{i} - (\sqrt{3}/2)\mathbf{j}$
$a_{\mathbf{T}} = 0$
$a_{\mathbf{N}} = 1$

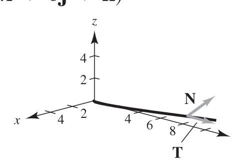
59. $\mathbf{T}(1) = (\sqrt{6}/6)(\mathbf{i} + 2\mathbf{j} + \mathbf{k})$
$\mathbf{N}(1) = (\sqrt{30}/30)(-5\mathbf{i} + 2\mathbf{j} + \mathbf{k})$
$a_{\mathbf{T}} = 5\sqrt{6}/6$
$a_{\mathbf{N}} = \sqrt{30}/6$

61. $\mathbf{T}(0) = (\sqrt{3}/3)(\mathbf{i} + \mathbf{j} + \mathbf{k})$
$\mathbf{N}(0) = (\sqrt{2}/2)(\mathbf{i} - \mathbf{j})$
$a_{\mathbf{T}} = \sqrt{3}$
$a_{\mathbf{N}} = \sqrt{2}$

63. $\mathbf{T}(\pi/2) = \frac{1}{5}(4\mathbf{i} - 3\mathbf{j})$
$\mathbf{N}(\pi/2) = -\mathbf{k}$
$a_{\mathbf{T}} = 0$
$a_{\mathbf{N}} = 3$



65. $\mathbf{T}(2) = (\sqrt{149}/149)(\mathbf{i} + 12\mathbf{j} + 2\mathbf{k})$
$\mathbf{N}(2) = (\sqrt{5513}/5513)(-74\mathbf{i} + 6\mathbf{j} + \mathbf{k})$
$a_{\mathbf{T}} = 74\sqrt{149}/149$
$a_{\mathbf{N}} = \sqrt{5513}/149$



67. Let C be a smooth curve represented by $\mathbf{r}$ on an open interval I. The unit tangent vector $\mathbf{T}(t)$ at t is defined as

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|},\ \mathbf{r}'(t) \neq 0.$$

The principal unit normal vector $\mathbf{N}(t)$ at t is defined as

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|},\ \mathbf{T}'(t) \neq 0.$$

The tangential and normal components of acceleration are defined as $\mathbf{a}(t) = a_{\mathbf{T}}\mathbf{T}(t) + a_{\mathbf{N}}\mathbf{N}(t)$.

69. (a) The particle's motion is in a straight line.
(b) The particle's speed is constant.

71. (a) $t = \frac{1}{2}$: $a_{\mathbf{T}} = \sqrt{2}\pi^2/2,\ a_{\mathbf{N}} = \sqrt{2}\pi^2/2$
$t = 1$: $a_{\mathbf{T}} = 0,\ a_{\mathbf{N}} = \pi^2$
$t = \frac{3}{2}$: $a_{\mathbf{T}} = -\sqrt{2}\pi^2/2,\ a_{\mathbf{N}} = \sqrt{2}\pi^2/2$
(b) $t = \frac{1}{2}$: Increasing because $a_{\mathbf{T}} > 0$.
$t = 1$: Maximum because $a_{\mathbf{T}} = 0$.
$t = \frac{3}{2}$: Decreasing because $a_{\mathbf{T}} < 0$.

73. $\mathbf{T}(\pi/2) = (\sqrt{17}/17)(-4\mathbf{i} + \mathbf{k})$
$\mathbf{N}(\pi/2) = -\mathbf{j}$
$\mathbf{B}(\pi/2) = (\sqrt{17}/17)(\mathbf{i} + 4\mathbf{k})$

75. $\mathbf{T}(\pi/4) = (\sqrt{2}/2)(\mathbf{j} - \mathbf{k})$
$\mathbf{N}(\pi/4) = -(\sqrt{2}/2)(\mathbf{j} + \mathbf{k})$
$\mathbf{B}(\pi/4) = -\mathbf{i}$

77. $\mathbf{T}(\pi/3) = \left(\sqrt{5}/5\right)\left(\mathbf{i} - \sqrt{3}\mathbf{j} + \mathbf{k}\right)$
$\mathbf{N}(\pi/3) = -\frac{1}{2}\left(\sqrt{3}\mathbf{i} + \mathbf{j}\right)$
$\mathbf{B}(\pi/3) = \left(\sqrt{5}/10\right)\left(\mathbf{i} - \sqrt{3}\mathbf{j} - 4\mathbf{k}\right)$

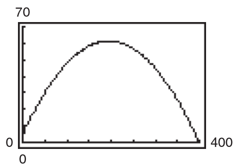
79. $a_{\mathbf{T}} = \dfrac{-32(v_0 \sin\theta - 32t)}{\sqrt{v_0^2\cos^2\theta + (v_0\sin\theta - 32t)^2}}$

$a_{\mathbf{N}} = \dfrac{32v_0\cos\theta}{\sqrt{v_0^2\cos^2\theta + (v_0\sin\theta - 32t)^2}}$

At maximum height, $a_{\mathbf{T}} = 0$ and $a_{\mathbf{N}} = 32$.

81. (a) $\mathbf{r}(t) = 60\sqrt{3}t\mathbf{i} + (5 + 60t - 16t^2)\mathbf{j}$

(b)



Maximum height $\approx$ 61.245 ft
Range $\approx$ 398.186 ft

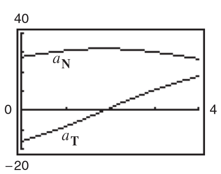
(c) $\mathbf{v}(t) = 60\sqrt{3}\mathbf{i} + (60 - 32t)\mathbf{j}$
$\|\mathbf{v}(t)\| = 8\sqrt{16t^2 - 60t + 225}$
$\mathbf{a}(t) = -32\mathbf{j}$

(d)

t	0.5	1.0	1.5
Speed	112.85	107.63	104.61

t	2.0	2.5	3.0
Speed	104	105.83	109.98

(e)



The speed is decreasing when $a_{\mathbf{T}}$ and $a_{\mathbf{N}}$ have opposite signs.

83. (a) $4\sqrt{625\pi^2 + 1} \approx 314$ mi/h
(b) $a_{\mathbf{T}} = 0$, $a_{\mathbf{N}} = 1000\pi^2$
$a_{\mathbf{T}} = 0$ because the speed is constant.

85. (a) The centripetal component is quadrupled.
(b) The centripetal component is halved.

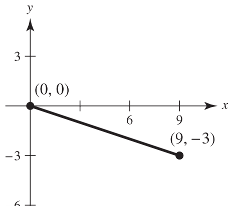
87. 4.82 mi/sec **89.** 4.67 mi/sec

91. False; centripetal acceleration may occur with constant speed.

93. (a) Proof (b) Proof **95–97.** Proofs

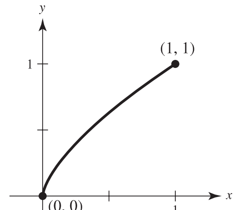
Section 12.5 (page 877)

1.



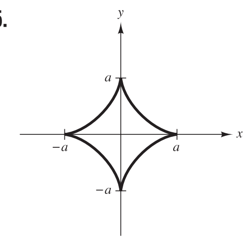
$3\sqrt{10}$

3.



$\left(13\sqrt{13} - 8\right)/27$

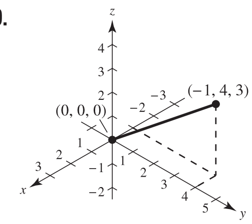
5.



$6a$

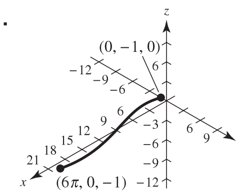
7. (a) $\mathbf{r}(t) = \left(50t\sqrt{2}\right)\mathbf{i} + \left(3 + 50t\sqrt{2} - 16t^2\right)\mathbf{j}$
(b) $\frac{649}{8} \approx 81$ ft (c) 315.5 ft (d) 362.9 ft

9.



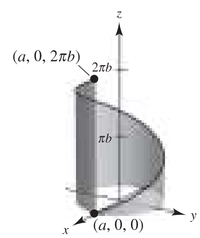
$\sqrt{26}$

11.



$3\sqrt{17}\pi/2$

13.



$2\pi\sqrt{a^2 + b^2}$

15. 8.37

17. (a) $2\sqrt{21} \approx 9.165$ (b) 9.529
(c) Increase the number of line segments. (d) 9.571

19. (a) $s = \sqrt{5}t$ (b) $\mathbf{r}(s) = 2\cos\dfrac{s}{\sqrt{5}}\mathbf{i} + 2\sin\dfrac{s}{\sqrt{5}}\mathbf{j} + \dfrac{s}{\sqrt{5}}\mathbf{k}$
(c) $s = \sqrt{5}$: (1.081, 1.683, 1.000)
$s = 4$: (−0.433, 1.953, 1.789)
(d) Proof

21. 0 **23.** $\frac{2}{5}$ **25.** 0 **27.** $\sqrt{2}/2$ **29.** 1

31. $\frac{1}{4}$ **33.** $1/a$ **35.** $\sqrt{2}/\left(4a\sqrt{1 - \cos\omega t}\right)$

37. $\sqrt{5}/(1 + 5t^2)^{3/2}$ **39.** $\frac{3}{25}$ **41.** $\frac{12}{125}$ **43.** $7\sqrt{26}/676$

45. $K = 0$, $1/K$ is undefined.

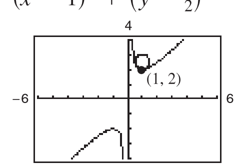
47. $K = 4/17^{3/2}$, $1/K = 17^{3/2}/4$ **49.** $K = 4$, $1/K = 1/4$

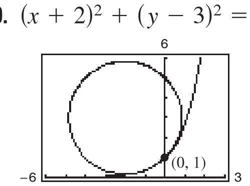
51. $K = 1/a$, $1/K = a$

53. $K = 12/145^{3/2}$, $1/K = 145^{3/2}/12$

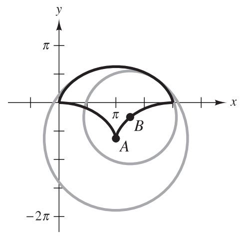
55. (a) $(x - \pi/2)^2 + y^2 = 1$
(b) Because the curvature is not as great, the radius of the curvature is greater.

57. $(x - 1)^2 + \left(y - \frac{5}{2}\right)^2 = \left(\frac{1}{2}\right)^2$ **59.** $(x + 2)^2 + (y - 3)^2 = 8$





61.



63. (a) $(1, 3)$ (b) 0

65. (a) $K \to \infty$ as $x \to 0$ (No maximum) (b) 0

67. (a) $\left(1/\sqrt{2}, -\ln 2/2\right)$ (b) 0

69. (a) $(\pm\ \text{arcsinh}(1), 1)$ (b) 0

71. $(0, 1)$ **73.** $(\pi/2 + K\pi, 0)$

75. (a) $s = \displaystyle\int_a^b \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}\, dt = \int_a^b \|\mathbf{r}'(t)\|\, dt$

(b) Plane: $K = \left\| \dfrac{d\mathbf{T}}{ds} \right\| = \|\mathbf{T}'(s)\|$

Space: $K = \dfrac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} = \dfrac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3}$

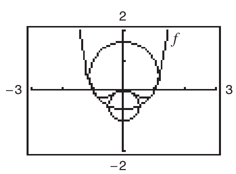
77. $K = |y''|$; Yes, for example, $y = x^4$ has a curvature of 0 at its relative minimum $(0, 0)$. The curvature is positive at any other point on the curve.

79. Proof

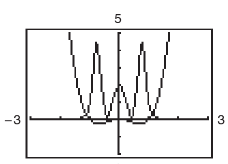
81. (a) $K = \dfrac{2|6x^2 - 1|}{(16x^6 - 16x^4 + 4x^2 + 1)^{3/2}}$

(b) $x = 0$: $x^2 + \left(y + \dfrac{1}{2}\right)^2 = \dfrac{1}{4}$

$x = 1$: $x^2 + \left(y - \dfrac{1}{2}\right)^2 = \dfrac{5}{4}$



(c)



The curvature tends to be greatest near the extrema of the function and decreases as $x \to \pm\infty$. However, f and K do not have the same critical numbers.

Critical numbers of f: $x = 0, \pm\sqrt{2}/2 \approx \pm 0.7071$

Critical numbers of K: $x = 0, \pm 0.7647, \pm 0.4082$

83. (a) 12.25 units (b) $\frac{1}{2}$ **85–87.** Proofs

89. (a) 0 (b) 0 **91.** $\frac{1}{4}$ **93.** Proof

95. $K = [1/(4a)]|\csc(\theta/2)|$ **97.** 3327.5 lb

Minimum: $K = 1/(4a)$

There is no maximum.

99. Proof

101. False. See Exploration on page 869. **103.** True

105–111. Proofs

Review Exercises for Chapter 12 (page 881)

1. (a) All reals except $(\pi/2) + n\pi$, n is an integer

(b) Continuous except at $t = (\pi/2) + n\pi$, n is an integer

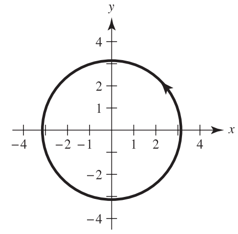
3. (a) $(0, \infty)$ (b) Continuous for all $t > 0$

5. (a) $\mathbf{i} - \sqrt{2}\mathbf{k}$ (b) $-3\mathbf{i} + 4\mathbf{j}$

(c) $(2c - 1)\mathbf{i} + (c - 1)^2\mathbf{j} - \sqrt{c + 1}\,\mathbf{k}$

(d) $2\Delta t\,\mathbf{i} + \Delta t(\Delta t + 2)\mathbf{j} - \left(\sqrt{\Delta t + 3} - \sqrt{3}\right)\mathbf{k}$

7.



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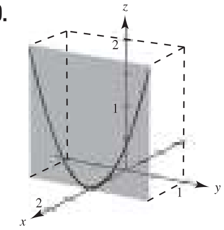

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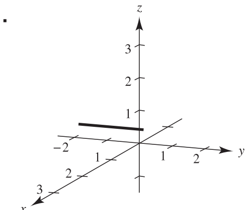
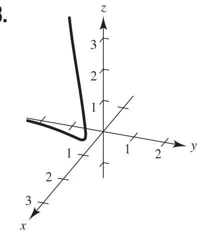


13.



15. $\mathbf{r}_1(t) = 3t\,\mathbf{i} + 4t\,\mathbf{j}$, $0 \le t \le 1$

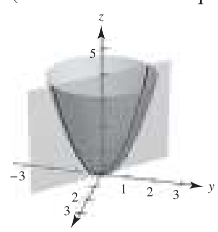
$\mathbf{r}_2(t) = 3\mathbf{i} + (4 - t)\mathbf{j}$, $0 \le t \le 4$

$\mathbf{r}_3(t) = (3 - t)\mathbf{i}$, $0 \le t \le 3$

17. $\mathbf{r}(t) = \langle -2 + 7t, -3 + 4t, 8 - 10t \rangle$

(Answer is not unique.)

19.



$x = t, y = -t, z = 2t^2$

21. $4\mathbf{i} + \mathbf{k}$

23. (a) $3\mathbf{i} + \mathbf{j}$ (b) $\mathbf{0}$ (c) $4t + 3t^2$

(d) $-5\mathbf{i} + (2t - 2)\mathbf{j} + 2t^2\mathbf{k}$

(e) $(10t - 1)/\sqrt{10t^2 - 2t + 1}$

(f) $\left(\frac{8}{3}t^3 - 2t^2\right)\mathbf{i} - 8t^3\mathbf{j} + (9t^2 - 2t + 1)\mathbf{k}$

25. $x(t)$ and $y(t)$ are increasing functions at $t = t_0$, and $z(t)$ is a decreasing function at $t = t_0$.

27. $\sin t\,\mathbf{i} + (t \sin t + \cos t)\mathbf{j} + \mathbf{C}$

29. $\frac{1}{2}\left(t\sqrt{1 + t^2} + \ln\left|t + \sqrt{1 + t^2}\right|\right) + \mathbf{C}$

31. $\frac{32}{3}\mathbf{j}$ **33.** $2(e - 1)\mathbf{i} - 8\mathbf{j} - 2\mathbf{k}$

35. $\mathbf{r}(t) = (t^2 + 1)\mathbf{i} + (e^t + 2)\mathbf{j} - (e^{-t} + 4)\mathbf{k}$

37. $\mathbf{v}(t) = 4\mathbf{i} + 3t^2\mathbf{j} - \mathbf{k}$

$\|\mathbf{v}(t)\| = \sqrt{17 + 9t^4}$

$\mathbf{a}(t) = 6t\mathbf{j}$

39. $\mathbf{v}(t) = \langle -3\cos^2 t \sin t, 3\sin^2 t \cos t, 3 \rangle$

$\|\mathbf{v}(t)\| = 3\sqrt{\sin^2 t \cos^2 t + 1}$

$\mathbf{a}(t) = \langle 3\cos t(2\sin^2 t - \cos^2 t), 3\sin t(2\cos^2 t - \sin^2 t), 0 \rangle$

41. $x(t) = t, y(t) = 16 + 8t, z(t) = 2 + \frac{1}{2}t$

$\mathbf{r}(4.1) \approx \langle 0.1, 16.8, 2.05 \rangle$

43. About 191.0 ft **45.** About 38.1 m/sec

47. $\mathbf{v} = -\mathbf{i} + 3\mathbf{j}$
$\|\mathbf{v}\| = \sqrt{10}$
$\mathbf{a} = \mathbf{0}$
$\mathbf{a} \cdot \mathbf{T} = 0$
$\mathbf{a} \cdot \mathbf{N}$ does not exist.

49. $\mathbf{v} = \mathbf{i} + \left[1/\left(2\sqrt{t}\right)\right]\mathbf{j}$
$\|\mathbf{v}\| = \sqrt{4t + 1}/\left(2\sqrt{t}\right)$
$\mathbf{a} = \left[-1/\left(4t\sqrt{t}\right)\right]\mathbf{j}$
$\mathbf{a} \cdot \mathbf{T} = -1/\left(4t\sqrt{t}\sqrt{4t + 1}\right)$
$\mathbf{a} \cdot \mathbf{N} = 1/\left(2t\sqrt{4t + 1}\right)$

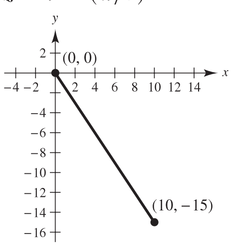
51. $\mathbf{v} = e^t\mathbf{i} - e^{-t}\mathbf{j}$
$\|\mathbf{v}\| = \sqrt{e^{2t} + e^{-2t}}$
$\mathbf{a} = e^t\mathbf{i} + e^{-t}\mathbf{j}$
$\mathbf{a} \cdot \mathbf{T} = \dfrac{e^{2t} - e^{-2t}}{\sqrt{e^{2t} + e^{-2t}}}$
$\mathbf{a} \cdot \mathbf{N} = \dfrac{2}{\sqrt{e^{2t} + e^{-2t}}}$

53. $\mathbf{v} = \mathbf{i} + 2t\mathbf{j} + t\mathbf{k}$
$\|\mathbf{v}\| = \sqrt{1 + 5t^2}$
$\mathbf{a} = 2\mathbf{j} + \mathbf{k}$
$\mathbf{a} \cdot \mathbf{T} = \dfrac{5t}{\sqrt{1 + 5t^2}}$
$\mathbf{a} \cdot \mathbf{N} = \dfrac{\sqrt{5}}{\sqrt{1 + 5t^2}}$

55. $x = -\sqrt{3}t + 1$
$y = t + \sqrt{3}$
$z = t + (\pi/3)$

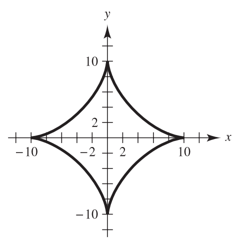
57. 4.58 mi/sec

59.



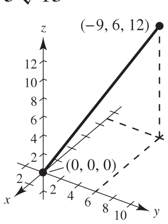
$5\sqrt{13}$

61.



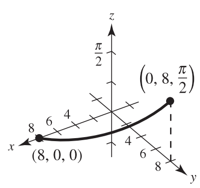
60

63.



$3\sqrt{29}$

65.



$\sqrt{65}\pi/2$

67. 0 **69.** $\left(2\sqrt{5}\right)/(4 + 5t^2)^{3/2}$ **71.** $\sqrt{2}/3$

73. $K = \sqrt{17}/289$; $r = 17\sqrt{17}$ **75.** $K = \sqrt{2}/4$; $r = 2\sqrt{2}$

77. The curvature changes abruptly from zero to a nonzero constant at the points B and C.

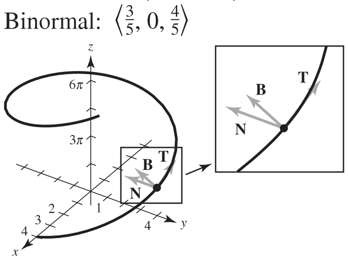
P.S. Problem Solving (page 883)

1. (a) a (b) πa (c) $K = \pi a$

3. Initial speed: 447.21 ft/sec; $\theta \approx 63.43°$

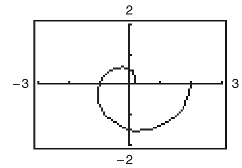
5–7. Proofs

9. Unit tangent: $\left\langle -\frac{4}{5}, 0, \frac{3}{5}\right\rangle$
Unit normal: $\langle 0, -1, 0\rangle$
Binormal: $\left\langle \frac{3}{5}, 0, \frac{4}{5}\right\rangle$



11. (a) Proof (b) Proof

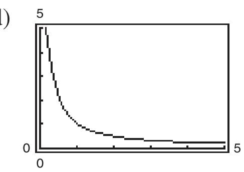
13. (a)



(b) 6.766

(c) $K = [\pi(\pi^2t^2 + 2)]/(\pi^2t^2 + 1)^{3/2}$
$K(0) = 2\pi$
$K(1) = [\pi(\pi^2 + 2)]/(\pi^2 + 1)^{3/2} \approx 1.04$
$K(2) \approx 0.51$

(d)



(e) $\lim\limits_{t\to\infty} K = 0$

(f) As $t \to \infty$, the graph spirals outward and the curvature decreases.

Chapter 13

Section 13.1 (page 894)

1. Not a function because for some values of x and y (for example $x = y = 0$), there are two z-values.

3. z is a function of x and y. **5.** z is not a function of x and y.

7. (a) 6 (b) -4 (c) 150 (d) $5y$ (e) $2x$ (f) $5t$

9. (a) 5 (b) $3e^2$ (c) $2/e$ (d) $5e^y$ (e) xe^2 (f) te^t

11. (a) $\frac{2}{3}$ (b) 0 (c) $-\frac{3}{2}$ (d) $-\frac{10}{3}$

13. (a) $\sqrt{2}$ (b) $3 \sin 1$ (c) $-3\sqrt{3}/2$ (d) 4

15. (a) -4 (b) -6 (c) $-\frac{25}{4}$ (d) $\frac{9}{4}$

17. (a) $2, \Delta x \neq 0$ (b) $2y + \Delta y, \Delta y \neq 0$

19. Domain: $\{(x, y)\colon\ x$ is any real number, y is any real number$\}$
Range: $z \geq 0$

21. Domain: $\{(x, y)\colon\ y \geq 0\}$
Range: all real numbers

23. Domain: $\{(x, y)\colon\ x \neq 0, y \neq 0\}$
Range: all real numbers

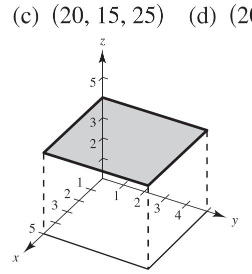
25. Domain: $\{(x, y)\colon\ x^2 + y^2 \leq 4\}$
Range: $0 \leq z \leq 2$

27. Domain: $\{(x, y)\colon\ -1 \leq x + y \leq 1\}$
Range: $0 \leq z \leq \pi$

29. Domain: $\{(x, y)\colon\ y < -x + 4\}$
Range: all real numbers

31. (a) $(20, 0, 0)$ (b) $(-15, 10, 20)$
(c) $(20, 15, 25)$ (d) $(20, 20, 0)$

33.



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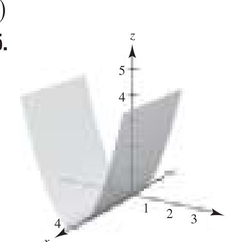

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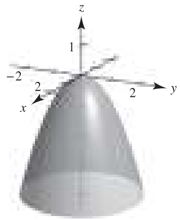

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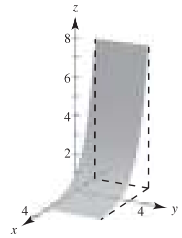

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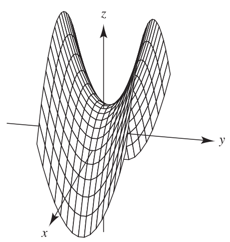

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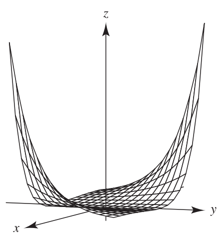
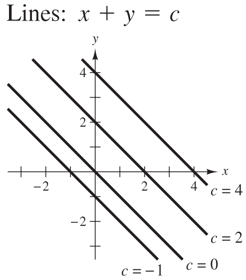
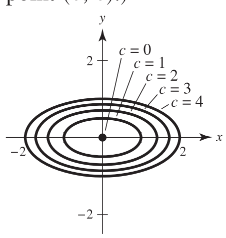


45. c **46.** d **47.** b **48.** a

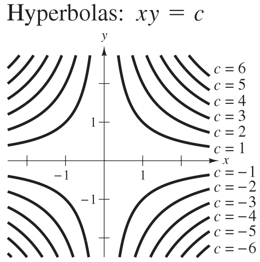
49. Lines: $x + y = c$



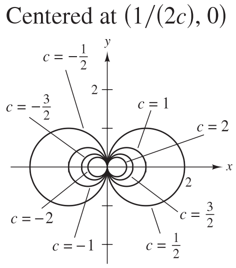
51. Ellipses: $x^2 + 4y^2 = c$ (except $x^2 + 4y^2 = 0$ is the point $(0, 0)$.)



53. Hyperbolas: $xy = c$



55. Circles passing through $(0, 0)$ Centered at $(1/(2c), 0)$



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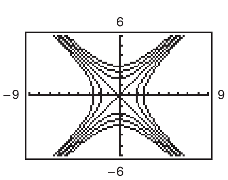

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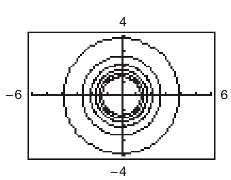


61. The graph of a function of two variables is the set of all points (x, y, z) for which $z = f(x, y)$ and (x, y) is in the domain of f. The graph can be interpreted as a surface in space. Level curves are the scalar fields $f(x, y) = c$, where c is a constant.

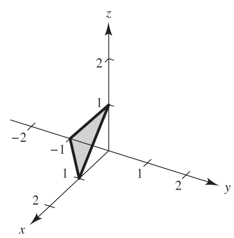
63. $f(x, y) = x/y$; the level curves are the lines $y = (1/c)x$.

65. The surface may be shaped like a saddle. For example, let $f(x, y) = xy$. The graph is not unique; any vertical translation will produce the same level curves.

67.

	Inflation Rate		
Tax Rate	**0**	**0.03**	**0.05**
0	\$1790.85	\$1332.56	\$1099.43
0.28	\$1526.43	\$1135.80	\$937.09
0.35	\$1466.07	\$1090.90	\$900.04

69.



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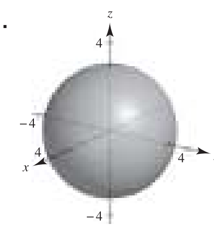
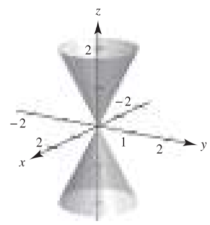
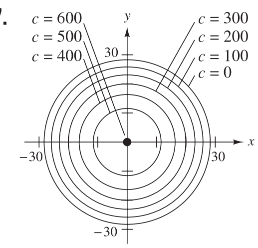


73.



75. (a) 243 board-ft
(b) 507 board-ft

77.



79. Proof

81. $C = 1.20xy + 1.50(xz + yz)$

83. (a) $k = \frac{520}{3}$
(b) $P = 520T/(3V)$
The level curves are lines.

85. (a) C (b) A (c) B

87. (a) No; the level curves are uneven and sporadically spaced.
(b) Use more colors.

89. False: let $f(x, y) = 4$. **91.** True

Section 13.2 (page 904)

1–3. Proofs **5.** 1 **7.** 12 **9.** 9, continuous

11. e^2, continuous **13.** 0, continuous for $y \neq 0$

15. $\frac{1}{2}$, continuous except at $(0, 0)$ **17.** 0, continuous

19. 0, continuous for $xy \neq 1$, $|xy| \leq 1$

21. $2\sqrt{2}$, continuous for $x + y + z \geq 0$ **23.** 0

25. Limit does not exist. **27.** 4 **29.** Limit does not exist.

31. Limit does not exist. **33.** 0

35. Limit does not exist. **37.** Continuous, 1

39.

(x, y)	$(1, 0)$	$(0.5, 0)$	$(0.1, 0)$	$(0.01, 0)$	$(0.001, 0)$
$f(x, y)$	0	0	0	0	0

$y = 0$: 0

(x, y)	$(1, 1)$	$(0.5, 0.5)$	$(0.1, 0.1)$
$f(x, y)$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$

(x, y)	$(0.01, 0.01)$	$(0.001, 0.001)$
$f(x, y)$	$\frac{1}{2}$	$\frac{1}{2}$

$y = x$: $\frac{1}{2}$
Limit does not exist.
Continuous except at $(0, 0)$

41.

(x, y)	$(1, 1)$	$(0.25, 0.5)$	$(0.01, 0.1)$
$f(x, y)$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$

(x, y)	$(0.0001, 0.01)$	$(0.000001, 0.001)$
$f(x, y)$	$-\frac{1}{2}$	$-\frac{1}{2}$

$x = y^2$: $-\frac{1}{2}$

(x, y)	$(-1, 1)$	$(-0.25, 0.5)$	$(-0.01, 0.1)$
$f(x, y)$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$

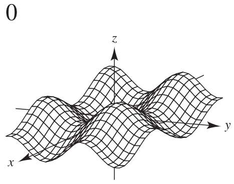
(x, y)	$(-0.0001, 0.01)$	$(-0.000001, 0.001)$
$f(x, y)$	$\frac{1}{2}$	$\frac{1}{2}$

$x = -y^2$: $\frac{1}{2}$
Limit does not exist.
Continuous except at $(0, 0)$

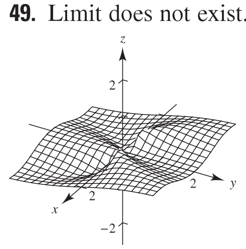
43. f is continuous. g is continuous except at $(0, 0)$. g has a removable discontinuity at $(0, 0)$.

45. f is continuous. g is continuous except at $(0, 0)$. g has a removable discontinuity at $(0, 0)$.

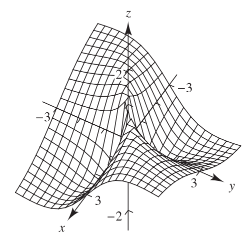
47. 0



49. Limit does not exist.



51. Limit does not exist.



53. 0 **55.** 0 **57.** 1 **59.** 1 **61.** 0

63. Continuous except at $(0, 0, 0)$ **65.** Continuous

67. Continuous **69.** Continuous

71. Continuous for $y \neq 2x/3$ **73.** (a) $2x$ (b) -4

75. (a) $1/y$ (b) $-x/y^2$ **77.** (a) $3 + y$ (b) $x - 2$

79. True **81.** False: let $f(x, y) = \begin{cases} \ln(x^2 + y^2), & x \neq 0, y \neq 0 \\ 0, & x = 0, y = 0 \end{cases}$.

83. (a) $(1 + a^2)/a$, $a \neq 0$ (b) Limit does not exist.
(c) No, the limit does not exist. Different paths result in different limits.

85. 0 **87.** $\pi/2$ **89.** Proof

91. See "Definition of the Limit of a Function of Two Variables," on page 899; show that the value of $\lim_{(x, y)\to(x_0, y_0)} f(x, y)$ is not the same for two different paths to (x_0, y_0).

93. (a) True. To find the first limit, you substitute $(2, 3)$ for (x, y). To find the second limit, you first substitute 3 for y to find a function of x. Then you substitute 2 for x.
(b) False. The convergence of one path does not imply the convergence of all paths.
(c) False. Let $f(x, y) = 4\left(\dfrac{(x - 2)^2 - (y - 3)^2}{(x - 2)^2 + (y - 3)^2}\right)^2$.
(d) True. When you multiply 0 by any real number, you always get 0.

Section 13.3 (page 914)

1. $f_x(4, 1) < 0$ **3.** $f_y(4, 1) > 0$

5. No. Because you are finding the partial derivative with respect to y, you consider x to be constant. So, the denominator is considered a constant and does not contain any variables.

7. Yes. Because you are finding the partial derivative with respect to x, you consider y to be constant. So, both the numerator and denominator contain variables.

9. $f_x(x, y) = 2$
$f_y(x, y) = -5$

11. $f_x(x, y) = 2xy^3$
$f_y(x, y) = 3x^2y^2$

13. $\partial z/\partial x = \sqrt{y}$
$\partial z/\partial y = x/\left(2\sqrt{y}\right)$

15. $\partial z/\partial x = 2x - 4y$
$\partial z/\partial y = -4x + 6y$

17. $\partial z/\partial x = ye^{xy}$
$\partial z/\partial y = xe^{xy}$

19. $\partial z/\partial x = 2xe^{2y}$
$\partial z/\partial y = 2x^2e^{2y}$

21. $\partial z/\partial x = 1/x$
$\partial z/\partial y = -1/y$

23. $\partial z/\partial x = 2x/(x^2 + y^2)$
$\partial z/\partial y = 2y/(x^2 + y^2)$

25. $\partial z/\partial x = (x^3 - 3y^3)/(x^2y)$
$\partial z/\partial y = (-x^3 + 12y^3)/(2xy^2)$

27. $h_x(x, y) = -2xe^{-(x^2+y^2)}$
$h_y(x, y) = -2ye^{-(x^2+y^2)}$

29. $f_x(x, y) = x/\sqrt{x^2 + y^2}$
$f_y(x, y) = y/\sqrt{x^2 + y^2}$

31. $\partial z/\partial x = -y \sin xy$
$\partial z/\partial y = -x \sin xy$

33. $\partial z/\partial x = 2 \sec^2(2x - y)$
$\partial z/\partial y = -\sec^2(2x - y)$

35. $\partial z/\partial x = ye^y \cos xy$
$\partial z/\partial y = e^y(x \cos xy + \sin xy)$

37. $\partial z/\partial x = 2\cosh(2x + 3y)$
$\partial z/\partial y = 3\cosh(2x + 3y)$

39. $f_x(x, y) = 1 - x^2$
$f_y(x, y) = y^2 - 1$

41. $f_x(x, y) = 3$
$f_y(x, y) = 2$

43. $f_x(x, y) = 1/\left(2\sqrt{x + y}\right)$
$f_y(x, y) = 1/\left(2\sqrt{x + y}\right)$

45. $\partial z/\partial x = -1$
$\partial z/\partial y = 0$

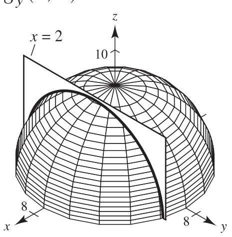
47. $\partial z/\partial x = -1$
$\partial z/\partial y = \frac{1}{2}$

49. $\partial z/\partial x = \frac{1}{4}$
$\partial z/\partial y = \frac{1}{4}$

51. $\partial z/\partial x = -\frac{1}{4}$
$\partial z/\partial y = \frac{1}{4}$

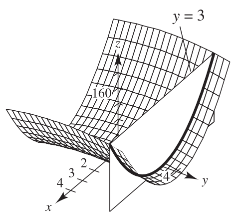
53. $g_x(1, 1) = -2$
$g_y(1, 1) = -2$

55.



$-\frac{1}{2}$

57.



18

59. $H_x(x, y, z) = \cos(x + 2y + 3z)$
$H_y(x, y, z) = 2\cos(x + 2y + 3z)$
$H_z(x, y, z) = 3\cos(x + 2y + 3z)$

61. $\dfrac{\partial w}{\partial x} = \dfrac{x}{\sqrt{x^2 + y^2 + z^2}}$
$\dfrac{\partial w}{\partial y} = \dfrac{y}{\sqrt{x^2 + y^2 + z^2}}$
$\dfrac{\partial w}{\partial z} = \dfrac{z}{\sqrt{x^2 + y^2 + z^2}}$

63. $F_x(x, y, z) = \dfrac{x}{x^2 + y^2 + z^2}$
$F_y(x, y, z) = \dfrac{y}{x^2 + y^2 + z^2}$
$F_z(x, y, z) = \dfrac{z}{x^2 + y^2 + z^2}$

65. $f_x = 3; f_y = 1; f_z = 2$

67. $f_x = 1; f_y = 1; f_z = 1$

69. $f_x = 0; f_y = 0; f_z = 1$

71. $\dfrac{\partial^2 z}{\partial x^2} = 0$
$\dfrac{\partial^2 z}{\partial y^2} = 6x$
$\dfrac{\partial^2 z}{\partial y \partial x} = \dfrac{\partial^2 z}{\partial x \partial y} = 6y$

73. $\dfrac{\partial^2 z}{\partial x^2} = 2$
$\dfrac{\partial^2 z}{\partial y^2} = 6$
$\dfrac{\partial^2 z}{\partial y \partial x} = \dfrac{\partial^2 z}{\partial x \partial y} = -2$

75. $\dfrac{\partial^2 z}{\partial x^2} = \dfrac{y^2}{(x^2 + y^2)^{3/2}}$
$\dfrac{\partial^2 z}{\partial y^2} = \dfrac{x^2}{(x^2 + y^2)^{3/2}}$
$\dfrac{\partial^2 z}{\partial y \partial x} = \dfrac{\partial^2 z}{\partial x \partial y} = \dfrac{-xy}{(x^2 + y^2)^{3/2}}$

77. $\dfrac{\partial^2 z}{\partial x^2} = e^x \tan y$
$\dfrac{\partial^2 z}{\partial y^2} = 2e^x \sec^2 y \tan y$
$\dfrac{\partial^2 z}{\partial y \partial x} = \dfrac{\partial^2 z}{\partial x \partial y} = e^x \sec^2 y$

79. $\dfrac{\partial^2 z}{\partial x^2} = -y^2 \cos xy$
$\dfrac{\partial^2 z}{\partial y^2} = -x^2 \cos xy$
$\dfrac{\partial^2 z}{\partial y \partial x} = \dfrac{\partial^2 z}{\partial x \partial y} = -xy \cos xy - \sin xy$

81. $x = 2, y = -2$ **83.** $x = -6, y = 4$

85. $x = 1, y = 1$ **87.** $x = 0, y = 0$

89. $\partial z/\partial x = \sec y$
$\partial z/\partial y = x \sec y \tan y$
$\partial^2 z/\partial x^2 = 0$
$\partial^2 z/\partial y^2 = x \sec y(\sec^2 y + \tan^2 y)$
$\partial^2 z/\partial y \partial x = \partial^2 z/\partial x \partial y = \sec y \tan y$
No values of x and y exist such that $f_x(x, y) = f_y(x, y) = 0$.

91. $\partial z/\partial x = (y^2 - x^2)/[x(x^2 + y^2)]$
$\partial z/\partial y = -2y/(x^2 + y^2)$
$\partial^2 z/\partial x^2 = (x^4 - 4x^2y^2 - y^4)/[x^2(x^2 + y^2)^2]$
$\partial^2 z/\partial y^2 = 2(y^2 - x^2)/(x^2 + y^2)^2$
$\partial^2 z/\partial y \partial x = \partial^2 z/\partial x \partial y = 4xy/(x^2 + y^2)^2$
No values of x and y exist such that $f_x(x, y) = f_y(x, y) = 0$.

93. $f_{xyy}(x, y, z) = f_{yxy}(x, y, z) = f_{yyx}(x, y, z) = 0$

95. $f_{xyy}(x, y, z) = f_{yxy}(x, y, z) = f_{yyx}(x, y, z) = z^2 e^{-x} \sin yz$

97. $\partial^2 z/\partial x^2 + \partial^2 z/\partial y^2 = 0 + 0 = 0$

99. $\partial^2 z/\partial x^2 + \partial^2 z/\partial y^2 = e^x \sin y - e^x \sin y = 0$

101. $\partial^2 z/\partial t^2 = -c^2 \sin(x - ct) = c^2(\partial^2 z/\partial x^2)$

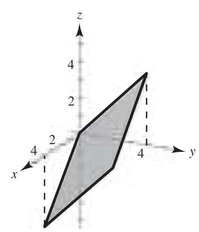
103. $\partial^2 z/\partial t^2 = -c^2/(x + ct)^2 = c^2(\partial^2 z/\partial x^2)$

105. $\partial z/\partial t = -e^{-t} \cos x/c = c^2(\partial^2 z/\partial x^2)$

107. Yes, $f(x, y) = \cos(3x - 2y)$. **109.** 0

111. If $z = f(x, y)$, then to find f_x you consider y constant and differentiate with respect to x. Similarly, to find f_y, you consider x constant and differentiate with respect to y.

113.



115. The mixed partial derivatives are equal. See Theorem 13.3.

117. (a) 72 (b) 72

119. $IQ_M = \dfrac{100}{C}, IQ_M(12, 10) = 10$

IQ increases at a rate of 10 points per year of mental age when the mental age is 12 and the chronological age is 10.

$IQ_C = -\dfrac{100M}{C^2}, IQ_C(12, 10) = -12$

IQ decreases at a rate of 12 points per year of chronological age when the mental age is 12 and the chronological age is 10.

121. An increase in either the charge for food and housing or the tuition will cause a decrease in the number of applicants.

123. $\partial T/\partial x = -2.4°/\text{m}, \partial T/\partial y = -9°/\text{m}$

125. $T = PV/(nR) \implies \partial T/\partial P = V/(nR)$
$P = nRT/V \implies \partial P/\partial V = -nRT/V^2$
$V = nRT/P \implies \partial V/\partial T = nR/P$
$\partial T/\partial P \cdot \partial P/\partial V \cdot \partial V/\partial T = -nRT/(VP) = -nRT/(nRT) = -1$

127. (a) $\partial z/\partial x = -0.92; \partial z/\partial y = 1.03$
(b) As the consumption of flavored milk (x) increases, the consumption of plain light and skim milks (z) decreases. As the consumption of plain reduced-fat milk (y) decreases, the consumption of plain light and skim milks also decreases.

129. False; Let $z = x + y + 1$. **131.** True

133. (a) $f_x(x, y) = \dfrac{y(x^4 + 4x^2y^2 - y^4)}{(x^2 + y^2)^2}$

$f_y(x, y) = \dfrac{x(x^4 - 4x^2y^2 - y^4)}{(x^2 + y^2)^2}$

(b) $f_x(0, 0) = 0,\ f_y(0, 0) = 0$

(c) $f_{xy}(0, 0) = -1,\ f_{yx}(0, 0) = 1$

(d) f_{xy} or f_{yx} or both are not continuous at $(0, 0)$.

135. (a) $f_x(0, 0) = 1, f_y(0, 0) = 1$

(b) $f_x(x, y)$ and $f_y(x, y)$ do not exist when $y = -x$.

Section 13.4 (page 923)

1. $dz = 4xy^3\,dx + 6x^2y^2\,dy$

3. $dz = 2(x\,dx + y\,dy)/(x^2 + y^2)^2$

5. $dz = (\cos y + y \sin x)\,dx - (x \sin y + \cos x)\,dy$

7. $dz = (e^x \sin y)\,dx + (e^x \cos y)\,dy$

9. $dw = 2z^3y \cos x\,dx + 2z^3 \sin x\,dy + 6z^2 y \sin x\,dz$

11. (a) $f(2, 1) = 1, f(2.1, 1.05) = 1.05, \Delta z = 0.05$

(b) $dz = 0.05$

13. (a) $f(2, 1) = 11, f(2.1, 1.05) = 10.4875, \Delta z = -0.5125$

(b) $dz = -0.5$

15. (a) $f(2, 1) = e^2 \approx 7.3891, f(2.1, 1.05) = 1.05e^{2.1} \approx 8.5745,$ $\Delta z \approx 1.1854$

(b) $dz \approx 1.1084$

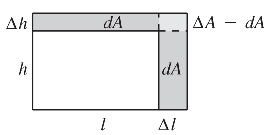
17. 0.44 **19.** -0.012

21. If $z = f(x, y)$ and Δx and Δy are increments of x and y, and x and y are independent variables, then the total differential of the dependent variable z is

$dz = (\partial z/\partial x)\,dx + (\partial z/\partial y)\,dy = f_x(x, y)\,\Delta x + f_y(x, y)\,\Delta y.$

23. The approximation of Δz by dz is called a linear approximation, where dz represents the change in height of a plane that is tangent to the surface at the point $P(x_0, y_0)$.

25. $dA = h\,dl + l\,dh$



$\Delta A - dA = dl\,dh$

27.

Δr	Δh	dV	ΔV	$\Delta V - dV$
0.1	0.1	8.3776	8.5462	0.1686
0.1	−0.1	5.0265	5.0255	−0.0010
0.001	0.002	0.1005	0.1006	0.0001
−0.0001	0.0002	−0.0034	−0.0034	0.0000

29. (a) $dz = -0.92\,dx + 1.03\,dy$

(b) $dz = \pm 0.4875;\ dz/z \approx 8.1\%$

31. 10% **33.** $dC = \pm 2.4418;\ dC/C = 19\%$

35. (a) $V = 18 \sin \theta$ ft^3; $\theta = \pi/2$

(b) 1.047 ft^3

37. 10% **39.** $L \approx 8.096 \times 10^{-4} \pm 6.6 \times 10^{-6}$ microhenrys

41. Answers will vary.
Example:
$\varepsilon_1 = \Delta x$
$\varepsilon_2 = 0$

43. Answers will vary.
Example:
$\varepsilon_1 = y\,\Delta x$
$\varepsilon_2 = 2x\,\Delta x + (\Delta x)^2$

45–47. Proofs

Section 13.5 (page 931)

1. $26t$ **3.** $e^t(\sin t + \cos t)$ **5.** (a) and (b) $-e^{-t}$

7. (a) and (b) $2e^{2t}$ **9.** (a) and (b) $3(2t^2 - 1)$

11. $-11\sqrt{29}/29 \approx -2.04$ **13.** $\dfrac{4}{(e^t + e^{-t})^2}$; 1

15. $\partial w/\partial s = 4s, 4$
$\partial w/\partial t = 4t, 0$

17. $\partial w/\partial s = 5 \cos(5s - t), 0$
$\partial w/\partial t = -\cos(5s - t), 0$

19. $\partial w/\partial r = 2r/\theta^2$
$\partial w/\partial \theta = -2r^2/\theta^3$

21. $\partial w/\partial r = 0$
$\partial w/\partial \theta = 1$

23. $\dfrac{\partial w}{\partial s} = t^2(3s^2 - t^2)$

$\dfrac{\partial w}{\partial t} = 2st(s^2 - 2t^2)$

25. $\dfrac{\partial w}{\partial s} = te^{s^2 - t^2}(2s^2 + 1)$

$\dfrac{\partial w}{\partial t} = se^{s^2 - t^2}(1 - 2t^2)$

27. $\dfrac{y - 2x + 1}{2y - x + 1}$ **29.** $-\dfrac{x^2 + y^2 + x}{x^2 + y^2 + y}$

31. $\dfrac{\partial z}{\partial x} = \dfrac{-x}{z}$

$\dfrac{\partial z}{\partial y} = \dfrac{-y}{z}$

33. $\dfrac{\partial z}{\partial x} = -\dfrac{x}{y + z}$

$\dfrac{\partial z}{\partial y} = -\dfrac{z}{y + z}$

35. $\dfrac{\partial z}{\partial x} = \dfrac{-\sec^2(x + y)}{\sec^2(y + z)}$

$\dfrac{\partial z}{\partial y} = -1 - \dfrac{\sec^2(x + y)}{\sec^2(y + z)}$

37. $\dfrac{\partial z}{\partial x} = -\dfrac{(ze^{xz} + y)}{xe^{xz}}$

$\dfrac{\partial z}{\partial y} = -e^{-xz}$

39. $\dfrac{\partial w}{\partial x} = -\dfrac{y + w}{x - z}$

$\dfrac{\partial w}{\partial y} = -\dfrac{x + z}{x - z}$

$\dfrac{\partial w}{\partial z} = \dfrac{w - y}{x - z}$

41. $\dfrac{\partial w}{\partial x} = \dfrac{y \sin xy}{z}$

$\dfrac{\partial w}{\partial y} = \dfrac{x \sin xy - z \cos yz}{z}$

$\dfrac{\partial w}{\partial z} = -\dfrac{y \cos yz + w}{z}$

43. (a) $f(tx, ty) = \dfrac{(tx)(ty)}{\sqrt{(tx)^2 + (ty)^2}} = t\left(\dfrac{xy}{\sqrt{x^2 + y^2}}\right) = tf(x, y);\ n = 1$

(b) $xf_x(x, y) + yf_y(x, y) = \dfrac{xy}{\sqrt{x^2 + y^2}} = 1f(x, y)$

45. (a) $f(tx, ty) = e^{tx/ty} = e^{x/y} = f(x, y);\ n = 0$

(b) $xf_x(x, y) + yf_y(x, y) = \dfrac{xe^{x/y}}{y} - \dfrac{xe^{x/y}}{y} = 0$

47. 47 **49.** $dw/dt = (\partial w/\partial x \cdot dx/dt) + (\partial w/\partial y \cdot dy/dt)$

51. $\dfrac{dy}{dx} = -\dfrac{f_x(x, y)}{f_y(x, y)}$

$\dfrac{\partial z}{\partial x} = -\dfrac{f_x(x, y, z)}{f_z(x, y, z)}$

$\dfrac{\partial z}{\partial y} = -\dfrac{f_y(x, y, z)}{f_z(x, y, z)}$

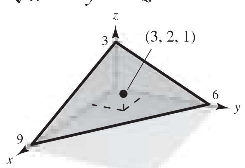
53. 4608π in.^{3}/min; 624π in.^{2}/min

55. $\dfrac{dT}{dt} = \dfrac{1}{mR}\left[V\dfrac{dp}{dt} + p\dfrac{dV}{dt}\right]$ **57.** $28m$ cm^2/sec

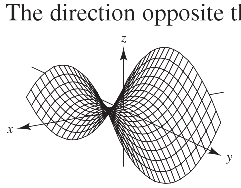
59. Proof **61.** (a) Proof (b) Proof **63–65.** Proofs

Section 13.6 (page 942)

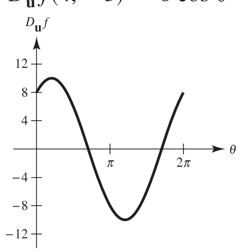
1. 1 **3.** -1 **5.** $-e$ **7.** $-\frac{7}{25}$ **9.** $2\sqrt{3}/3$ **11.** $\frac{8}{3}$
13. $\sqrt{2}(x + y)$ **15.** $\left[(2 + \sqrt{3})/2\right]\cos(2x + y)$ **17.** 6
19. $-8/\sqrt{5}$ **21.** $3\mathbf{i} + 10\mathbf{j}$ **23.** $4\mathbf{i} - \mathbf{j}$ **25.** $6\mathbf{i} - 10\mathbf{j} - 8\mathbf{k}$
27. $3\sqrt{2}$ **29.** $2\sqrt{5}/5$ **31.** $2[(x + y)\mathbf{i} + x\mathbf{j}]; 2\sqrt{2}$
33. $\tan y\mathbf{i} + x\sec^2 y\mathbf{j}, \sqrt{17}$ **35.** $e^{-x}(-y\mathbf{i} + \mathbf{j}); \sqrt{26}$
37. $\dfrac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{\sqrt{x^2 + y^2 + z^2}}, 1$ **39.** $yz(yz\mathbf{i} + 2xz\mathbf{j} + 2xy\mathbf{k}); \sqrt{33}$
41.



43. (a) $-5\sqrt{2}/12$ (b) $\frac{3}{5}$ (c) $-\frac{1}{5}$ (d) $-11\sqrt{10}/60$
45. $\sqrt{13}/6$
47. (a) Answers will vary. Example: $-4\mathbf{i} + \mathbf{j}$
(b) $-\frac{2}{5}\mathbf{i} + \frac{1}{10}\mathbf{j}$ (c) $\frac{2}{5}\mathbf{i} - \frac{1}{10}\mathbf{j}$
The direction opposite that of the gradient
49. (a)



(b) $D_{\mathbf{u}}f(4, -3) = 8\cos\theta + 6\sin\theta$



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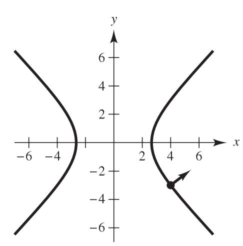
(c) $\theta \approx 2.21, \theta \approx 5.36$
Directions in which there is no change in f
(d) $\theta \approx 0.64, \theta \approx 3.79$
Directions of greatest rate of change in f
(e) 10; Magnitude of the greatest rate of change
(f)



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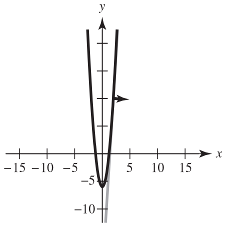
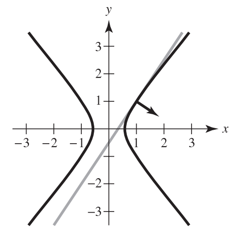
Orthogonal to the level curve

51. $-2\mathbf{i} - 3\mathbf{j}$ **53.** $3\mathbf{i} - \mathbf{j}$
55. (a) $16\mathbf{i} - \mathbf{j}$ (b) $(\sqrt{257}/257)(16\mathbf{i} - \mathbf{j})$ (c) $y = 16x - 22$
(d)



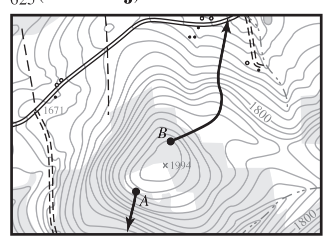
57. (a) $6\mathbf{i} - 4\mathbf{j}$ (b) $(\sqrt{13}/13)(3\mathbf{i} - 2\mathbf{j})$ (c) $y = \frac{3}{2}x - \frac{1}{2}$
(d)



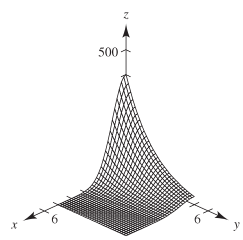
59. The directional derivative of $z = f(x, y)$ in the direction of $\mathbf{u} = \cos\theta\mathbf{i} + \sin\theta\mathbf{j}$ is

$$D_{\mathbf{u}}f(x, y) = \lim_{t\to 0}\frac{f(x + t\cos\theta, y + t\sin\theta) - f(x, y)}{t}$$

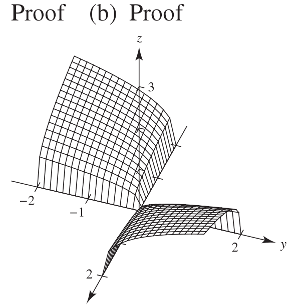
if the limit exists.
61. See the definition on page 936. See the properties on page 937.
63. The gradient vector is normal to the level curves.
65. $\frac{1}{625}(7\mathbf{i} - 24\mathbf{j})$
67.



69. $y^2 = 10x$
71. (a)

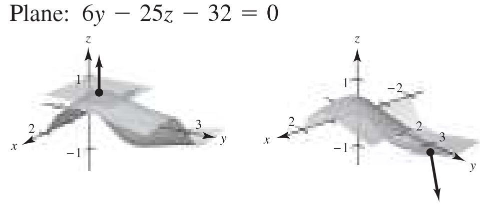


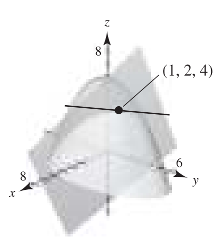
(b) There is no change in heat in directions perpendicular to the gradient: $\pm(\mathbf{i} - 6\mathbf{j})$.
(c) The greatest increase is in the direction of the gradient: $-3\mathbf{i} - \frac{1}{2}\mathbf{j}$.
73. True **75.** True **77.** $f(x, y, z) = e^x\cos y + \frac{1}{2}z^2 + C$
79. (a) Proof (b) Proof
(c)



Section 13.7 (page 951)

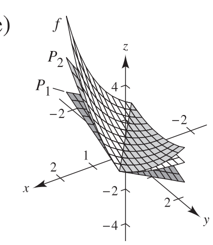
1. The level surface can be written as $3x - 5y + 3z = 15$, which is an equation of a plane in space.
3. The level surface can be written as $4x^2 + 9y^2 - 4z^2 = 0$, which is an elliptic cone that lies on the z-axis.

5. $\frac{1}{13}(3\mathbf{i} + 4\mathbf{j} + 12\mathbf{k})$ **7.** $(\sqrt{6}/6)(\mathbf{i} + \mathbf{j} + 2\mathbf{k})$
9. $(\sqrt{145}/145)(12\mathbf{i} - \mathbf{k})$ **11.** $\frac{1}{13}(4\mathbf{i} + 3\mathbf{j} + 12\mathbf{k})$
13. $(\sqrt{3}/3)(\mathbf{i} - \mathbf{j} + \mathbf{k})$ **15.** $(\sqrt{113}/113)(-\mathbf{i} - 6\sqrt{3}\mathbf{j} + 2\mathbf{k})$
17. $4x + 2y - z = 2$ **19.** $3x + 4y - 5z = 0$
21. $2x - 2y - z = 2$ **23.** $2x + 3y + 3z = 6$
25. $3x + 4y - 25z = 25(1 - \ln 5)$ **27.** $x - 4y + 2z = 18$
29. $6x - 3y - 2z = 11$
31. $x + y + z = 9$
$x - 3 = y - 3 = z - 3$
33. $2x + 4y + z = 14$
$\dfrac{x-1}{2} = \dfrac{y-2}{4} = \dfrac{z-4}{1}$
35. $6x - 4y - z = 5$
$\dfrac{x-3}{6} = \dfrac{y-2}{-4} = \dfrac{z-5}{-1}$
37. $10x + 5y + 2z = 30$
$\dfrac{x-1}{10} = \dfrac{y-2}{5} = \dfrac{z-5}{2}$
39. $x - y + 2z = \pi/2$
$\dfrac{(x-1)}{1} = \dfrac{(y-1)}{-1} = \dfrac{z-(\pi/4)}{2}$
41. (a) $\dfrac{x-1}{1} = \dfrac{y-1}{-1} = \dfrac{z-1}{1}$ (b) $\dfrac{1}{2}$, not orthogonal
43. (a) $\dfrac{x-3}{4} = \dfrac{y-3}{4} = \dfrac{z-4}{-3}$ (b) $\dfrac{16}{25}$, not orthogonal
45. (a) $\dfrac{x-3}{1} = \dfrac{y-1}{5} = \dfrac{z-2}{-4}$ (b) 0, orthogonal
47. 86.0° **49.** 77.4° **51.** $(0, 3, 12)$ **53.** $(2, 2, -4)$
55. $(0, 0, 0)$ **57.** Proof **59.** (a) Proof (b) Proof
61. $(-2, 1, -1)$ or $(2, -1, 1)$
63. $F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0$
65. Answers will vary.
67. (a) Line: $x = 1, y = 1, z = 1 - t$
Plane: $z = 1$
(b) Line: $x = -1, y = 2 + \frac{6}{25}t, z = -\frac{4}{5} - t$
Plane: $6y - 25z - 32 = 0$
(c)


69. (a) $x = 1 + t$
$y = 2 - 2t$
$z = 4$
$\theta \approx 48.2°$
(b)


71. $F(x, y, z) = \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} - 1$
$F_x(x, y, z) = 2x/a^2$
$F_y(x, y, z) = 2y/b^2$
$F_z(x, y, z) = 2z/c^2$
Plane: $\dfrac{2x_0}{a^2}(x - x_0) + \dfrac{2y_0}{b^2}(y - y_0) + \dfrac{2z_0}{c^2}(z - z_0) = 0$
$\dfrac{x_0x}{a^2} + \dfrac{y_0y}{b^2} + \dfrac{z_0z}{c^2} = 1$
73. $F(x, y, z) = a^2x^2 + b^2y^2 - z^2$
$F_x(x, y, z) = 2a^2x$
$F_y(x, y, z) = 2b^2y$
$F_z(x, y, z) = -2z$
Plane: $2a^2x_0(x - x_0) + 2b^2y_0(y - y_0) - 2z_0(z - z_0) = 0$
$a^2x_0x + b^2y_0y - z_0z = 0$
Therefore, the plane passes through the origin.
75. (a) $P_1(x, y) = 1 + x - y$
(b) $P_2(x, y) = 1 + x - y + \frac{1}{2}x^2 - xy + \frac{1}{2}y^2$
(c) If $x = 0$, $P_2(0, y) = 1 - y + \frac{1}{2}y^2$.
This is the second-degree Taylor polynomial for e^{-y}.
If $y = 0$, $P_2(x, 0) = 1 + x + \frac{1}{2}x^2$.
This is the second-degree Taylor polynomial for e^x.
(d)

x	y	$f(x, y)$	$P_1(x, y)$	$P_2(x, y)$
0	0	1	1	1
0	0.1	0.9048	0.9000	0.9050
0.2	0.1	1.1052	1.1000	1.1050
0.2	0.5	0.7408	0.7000	0.7450
1	0.5	1.6487	1.5000	1.6250

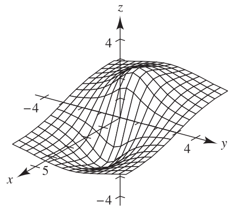
(e)


77. Proof

Section 13.8 (page 960)

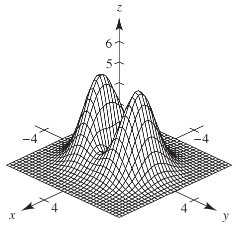
1. Relative minimum: $(1, 3, 0)$ **3.** Relative minimum: $(0, 0, 1)$
5. Relative minimum: $(-1, 3, -4)$
7. Relative minimum: $(1, 1, 11)$
9. Relative maximum: $(5, -1, 2)$
11. Relative minimum: $(3, -4, -5)$
13. Relative minimum: $(0, 0, 0)$
15. Relative maximum: $(0, 0, 4)$

17.



Relative maximum: $(-1, 0, 2)$
Relative minimum: $(1, 0, -2)$

19.



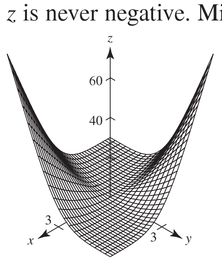
Relative minimum: $(0, 0, 0)$
Relative maxima: $(0, \pm 1, 4)$
Saddle points: $(\pm 1, 0, 1)$

21. Relative maximum: $(40, 40, 3200)$

23. Saddle point: $(0, 0, 0)$ **25.** Saddle point: $(1, -1, -1)$

27. There are no critical numbers.

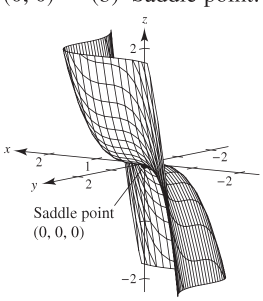
29. z is never negative. Minimum: $z = 0$ when $x = y \neq 0$.



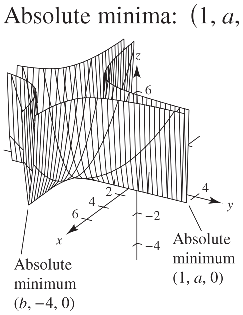
31. Insufficient information **33.** Saddle point

35. $-4 < f_{xy}(3, 7) < 4$

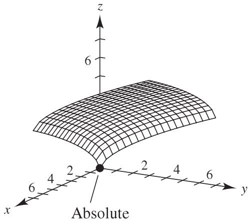
37. (a) $(0, 0)$ (b) Saddle point: $(0, 0, 0)$ (c) $(0, 0)$
(d)



39. (a) $(1, a), (b, -4)$ (b) Absolute minima: $(1, a, 0), (b, -4, 0)$
(c) $(1, a), (b, -4)$ (d)



41. (a) $(0, 0)$ (b) Absolute minimum: $(0, 0, 0)$ (c) $(0, 0)$
(d)



43. Relative minimum: $(0, 3, -1)$

45. Absolute maximum: $(4, 0, 21)$
Absolute minimum: $(4, 2, -11)$

47. Absolute maximum: $(0, 1, 10)$
Absolute minimum: $(1, 2, 5)$

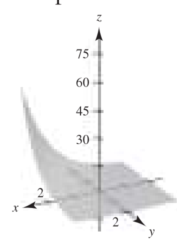
49. Absolute maxima: $(\pm 2, 4, 28)$
Absolute minimum: $(0, 1, -2)$

51. Absolute maxima: $(-2, -1, 9), (2, 1, 9)$
Absolute minima: $(x, -x, 0),\ |x| \leq 1$

53. Absolute maximum: $(1, 1, 1)$
Absolute minimum: $(0, 0, 0)$

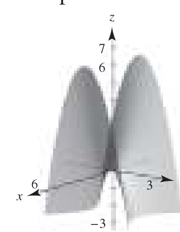
55. Point A is a saddle point.

57. Answers will vary.
Sample answer:



No extrema

59. Answers will vary.
Sample answer:



Saddle point

61. False. Let $f(x, y) = 1 - |x| - |y|$ at the point $(0, 0, 1)$.

63. False. Let $f(x, y) = x^2y^2$ (see Example 4 on page 958).

Section 13.9 (page 966)

1. $\sqrt{3}$ **3.** $\sqrt{7}$ **5.** $x = y = z = 3$ **7.** 10, 10, 10

9. 9 ft × 9 ft × 8.25 ft; \$26.73

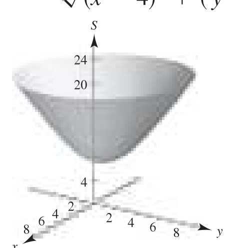
11. Let $a + b + c = k$.
$V = 4\pi abc/3 = \frac{4}{3}\pi ab(k - a - b) = \frac{4}{3}\pi(kab - a^2b - ab^2)$
$$\left.\begin{aligned} V_a &= \tfrac{4}{3}\pi(kb - 2ab - b^2) = 0 \\ V_b &= \tfrac{4}{3}\pi(ka - a^2 - 2ab) = 0 \end{aligned}\right\} \begin{aligned} kb - 2ab - b^2 &= 0 \\ kb - a^2 - 2ab &= 0 \end{aligned}$$
So, $a = b$ and $b = k/3$. Thus, $a = b = c = k/3$.

13. Let x, y, and z be the length, width, and height, respectively, and let V_0 be the given volume. Then $V_0 = xyz$ and $z = V_0/xy$. The surface area is
$S = 2xy + 2yz + 2xz = 2(xy + V_0/x + V_0/y)$.
$$\left.\begin{aligned} S_x &= 2(y - V_0/x^2) = 0 \\ S_y &= 2(x - V_0/y^2) = 0 \end{aligned}\right\} \begin{aligned} x^2y - V_0 &= 0 \\ xy^2 - V_0 &= 0 \end{aligned}$$
So, $x = \sqrt[3]{V_0}$, $y = \sqrt[3]{V_0}$, and $z = \sqrt[3]{V_0}$.

15. $x_1 = 3;\ x_2 = 6$ **17.** Proof

19. $x = \sqrt{2}/2 \approx 0.707$ km
$y = (3\sqrt{2} + 2\sqrt{3})/6 \approx 1.284$ km

21. (a) $S = \sqrt{x^2 + y^2} + \sqrt{(x + 2)^2 + (y - 2)^2} + \sqrt{(x - 4)^2 + (y - 2)^2}$



The surface has a minimum.

(b) $S_x = \dfrac{x}{\sqrt{x^2 + y^2}} + \dfrac{x + 2}{\sqrt{(x + 2)^2 + (y - 2)^2}} + \dfrac{x - 4}{\sqrt{(x - 4)^2 + (y - 2)^2}}$

$S_y = \dfrac{y}{\sqrt{x^2 + y^2}} + \dfrac{y - 2}{\sqrt{(x + 2)^2 + (y - 2)^2}} + \dfrac{y - 2}{\sqrt{(x - 4)^2 + (y - 2)^2}}$

(c) $-\dfrac{1}{\sqrt{2}}\mathbf{i} - \left(\dfrac{1}{\sqrt{2}} - \dfrac{2}{\sqrt{10}}\right)\mathbf{j}$

$\theta \approx 186.0°$

(d) $t = 1.344$; $(x_2, y_2) \approx (0.05, 0.90)$

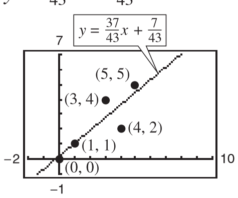
(e) $(x_4, y_4) \approx (0.06, 0.45)$; $S = 7.266$

(f) $-\nabla S(x, y)$ gives the direction of greatest rate of decrease of S. Use $\nabla S(x, y)$ when finding a maximum.

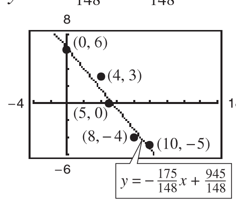
23. Write the equation to be maximized or minimized as a function of two variables. Take the partial derivatives and set them equal to zero or undefined to obtain the critical points. Use the Second Partials Test to test for relative extrema using the critical points. Check the boundary points.

25. (a) $y = \frac{3}{4}x + \frac{4}{3}$ (b) $\frac{1}{6}$ **27.** (a) $y = -2x + 4$ (b) 2

29. $y = \frac{37}{43}x + \frac{7}{43}$

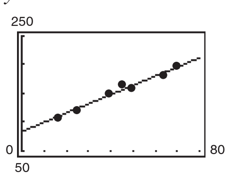


31. $y = -\frac{175}{148}x + \frac{945}{148}$



33. (a) $y = 1.6x + 84$

(b)



(c) 1.6

35. $y = 14x + 19$

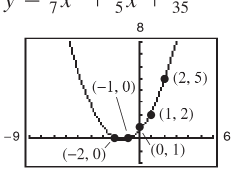
41.4 bushels per acre

37. $a\sum_{i=1}^{n} x_i^4 + b\sum_{i=1}^{n} x_i^3 + c\sum_{i=1}^{n} x_i^2 = \sum_{i=1}^{n} x_i^2 y_i$

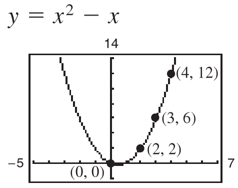
$a\sum_{i=1}^{n} x_i^3 + b\sum_{i=1}^{n} x_i^2 + c\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} x_i y_i$

$a\sum_{i=1}^{n} x_i^2 + b\sum_{i=1}^{n} x_i + cn = \sum_{i=1}^{n} y_i$

39. $y = \frac{3}{7}x^2 + \frac{6}{5}x + \frac{26}{35}$

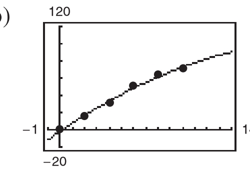


41. $y = x^2 - x$



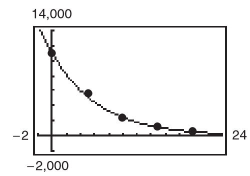
43. (a) $y = -0.22x^2 + 9.66x - 1.79$

(b)



45. (a) $\ln P = -0.1499h + 9.3018$ (b) $P = 10{,}957.7e^{-0.1499h}$

(c)

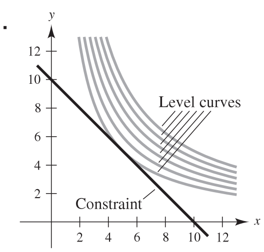


(d) Proof

47. Proof

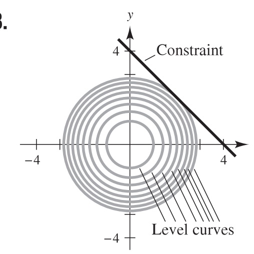
Section 13.10 (page 976)

1.



$f(5, 5) = 25$

3.



$f(2, 2) = 8$

5. $f(1, 2) = 5$ **7.** $f(25, 50) = 2600$

9. $f(1, 1) = 2$ **11.** $f(3, 3, 3) = 27$ **13.** $f\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) = \frac{1}{3}$

15. Maxima: $f\left(\sqrt{2}/2, \sqrt{2}/2\right) = \frac{5}{2}$

$f\left(-\sqrt{2}/2, -\sqrt{2}/2\right) = \frac{5}{2}$

Minima: $f\left(-\sqrt{2}/2, \sqrt{2}/2\right) = -\frac{1}{2}$

$f\left(\sqrt{2}/2, -\sqrt{2}/2\right) = -\frac{1}{2}$

17. $f(8, 16, 8) = 1024$ **19.** $\sqrt{2}/2$ **21.** $3\sqrt{2}$ **23.** $\sqrt{11}/2$

25. 0.188 **27.** $\sqrt{3}$ **29.** $(-4, 0, 4)$

31. Optimization problems that have restrictions or constraints on the values that can be used to produce the optimal solutions are called constrained optimization problems.

33. $\sqrt{3}$ **35.** $x = y = z = 3$

37. 9 ft × 9 ft × 8.25 ft; \$26.73 **39.** $a = b = c = k/3$

41. Proof **43.** $2\sqrt{3}a/3 \times 2\sqrt{3}b/3 \times 2\sqrt{3}c/3$

45. $\sqrt[3]{360} \times \sqrt[3]{360} \times \frac{4}{3}\sqrt[3]{360}$ ft

47. $r = \sqrt[3]{\dfrac{v_0}{2\pi}}$ and $h = 2\sqrt[3]{\dfrac{v_0}{2\pi}}$ **49.** Proof

51. $P(15{,}625/18, 3125) \approx 226{,}869$

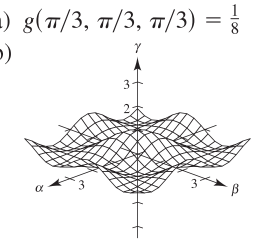
53. $x \approx 191.3$

$y \approx 688.7$

Cost ≈ \$55,095.60

55. (a) $g(\pi/3, \pi/3, \pi/3) = \frac{1}{8}$

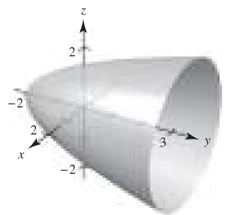
(b)



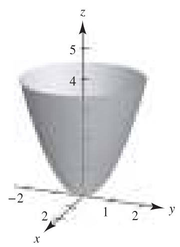
Maximum values occur when $\alpha = \beta$.

Review Exercises for Chapter 13 (page 978)

1.



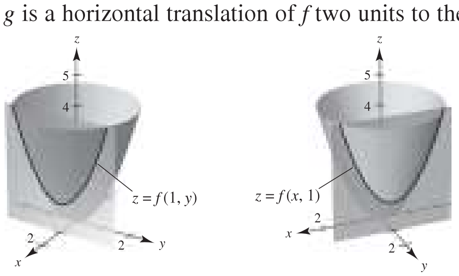
3. (a)



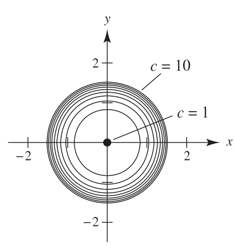
(b) g is a vertical translation of f two units upward.

(c) g is a horizontal translation of f two units to the right.

(d)



5.



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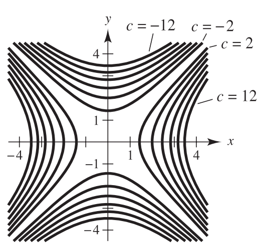
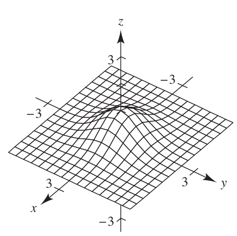
7.



Generated by Mathematica

9.



11. Limit: $\frac{1}{2}$
Continuous except at $(0, 0)$

13. Limit: 0
Continuous

15. $f_x(x, y) = e^x \cos y$
$f_y(x, y) = -e^x \sin y$

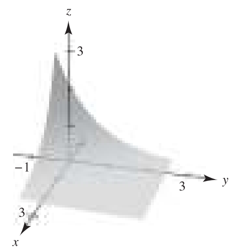
17. $\partial z/\partial x = -e^{-x}$
$\partial z/\partial y = -e^{-y}$

19. $g_x(x, y) = [y(y^2 - x^2)]/(x^2 + y^2)^2$
$g_y(x, y) = [x(x^2 - y^2)]/(x^2 + y^2)^2$

21. $f_x(x, y, z) = -yz/(x^2 + y^2)$
$f_y(x, y, z) = xz/(x^2 + y^2)$
$f_z(x, y, z) = \arctan y/x$

23. $u_x(x, t) = cne^{-n^2t} \cos nx$
$u_t(x, t) = -cn^2e^{-n^2t} \sin nx$

25. Answers will vary. Example:



27. $f_{xx}(x, y) = 6$
$f_{yy}(x, y) = 12y$
$f_{xy}(x, y) = f_{yx}(x, y) = -1$

29. $h_{xx}(x, y) = -y \cos x$
$h_{yy}(x, y) = -x \sin y$
$h_{xy}(x, y) = h_{yx}(x, y) = \cos y - \sin x$

31. $\partial^2 z/\partial x^2 + \partial^2 z/\partial y^2 = 2 + (-2) = 0$

33. $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2} = \dfrac{6x^2y - 2y^3}{(x^2 + y^2)^3} + \dfrac{-6x^2y + 2y^3}{(x^2 + y^2)^3} = 0$

35. $(xy \cos xy + \sin xy)\, dx + (x^2 \cos xy)\, dy$

37. 0.6538 cm, 5.03% **39.** $\pm\pi$ in.^{3}

41. $dw/dt = (8t - 1)/(4t^2 - t + 4)$

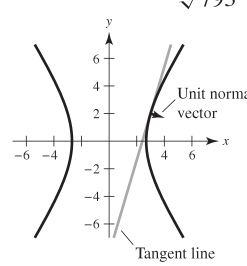
43. $\partial w/\partial r = (4r^2t - 4rt^2 - t^3)/(2r - t)^2$
$\partial w/\partial t = (4r^2t - rt^2 + 4r^3)/(2r - t)^2$

45. $\partial z/\partial x = (-2x - y)/(y + 2z)$
$\partial z/\partial y = (-x - 2y - z)/(y + 2z)$

47. -50 **49.** $\frac{2}{3}$ **51.** $\langle 4, 4\rangle, 4\sqrt{2}$ **53.** $\left\langle -\frac{1}{2}, 0\right\rangle, \frac{1}{2}$

55. (a) $54\mathbf{i} - 16\mathbf{j}$ (b) $\dfrac{27}{\sqrt{793}}\mathbf{i} - \dfrac{8}{\sqrt{793}}\mathbf{j}$ (c) $y = \dfrac{27}{8}x - \dfrac{65}{8}$

(d)

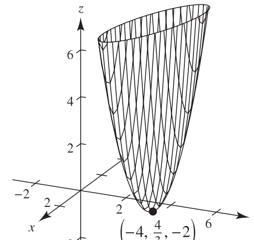


57. Tangent plane: $4x + 4y - z = 8$
Normal line: $x = 2 + 4t$, $y = 1 + 4t$, $z = 4 - t$

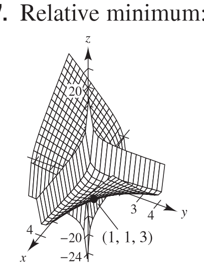
59. Tangent plane: $z = 4$
Normal line: $x = 2$, $y = -3$, $z = 4 + t$

61. $(x - 2)/1 = (y - 2)/1 = (z - 5)/(-4)$ **63.** $\theta \approx 36.7°$

65. Relative minimum: $\left(-4, \frac{4}{3}, -2\right)$



67. Relative minimum: $(1, 1, 3)$



69. The level curves are hyperbolas. The critical point $(0, 0)$ may be a saddle point or an extremum.

71. $x_1 = 94, x_2 = 157$ **73.** $f(49.4, 253) = 13{,}201.8$

75. (a) $y = 0.004x^2 + 0.07x + 19.4$ (b) 50.6 kg

77. Maximum: $f\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) = \frac{1}{3}$

79. $x = \sqrt{2}/2 \approx 0.707$ km; $y = \sqrt{3}/3 \approx 0.577$ km; $z = \left(60 - 3\sqrt{2} - 2\sqrt{3}\right)/6 \approx 8.716$ km

P.S. Problem Solving (page 981)

1. (a) 12 square units (b) Proof (c) Proof

3. (a) $y_0z_0(x - x_0) + x_0z_0(y - y_0) + x_0y_0(z - z_0) = 0$

(b) $x_0y_0z_0 = 1 \implies z_0 = 1/x_0y_0$

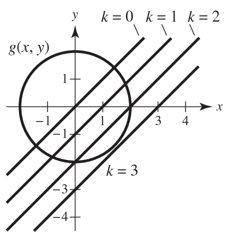
Then the tangent plane is

$$y_0\left(\frac{1}{x_0y_0}\right)(x - x_0) + x_0\left(\frac{1}{x_0y_0}\right)(y - y_0) + x_0y_0\left(z - \frac{1}{x_0y_0}\right) = 0.$$

Intercepts: $(3x_0, 0, 0), (0, 3y_0, 0), \left(0, 0, \dfrac{3}{x_0y_0}\right)$

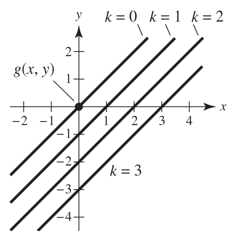
$V = \frac{1}{3}bh = \frac{9}{2}$

5. (a)



Maximum value: $2\sqrt{2}$

(b)



Maximum and minimum value: 0

The method of Lagrange multipliers does not work because $\nabla g(x_0, y_0) = \mathbf{0}$.

7. $2\sqrt[3]{150} \times 2\sqrt[3]{150} \times 5\sqrt[3]{150}/3$

9. (a)
$$\begin{aligned} x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} &= xCy^{1-a}ax^{a-1} + yCx^a(1-a)y^{1-a-1} \\ &= ax^aCy^{1-a} + (1-a)x^aC(y^{1-a}) \\ &= Cx^ay^{1-a}[a + (1-a)] \\ &= Cx^ay^{1-a} \\ &= f(x, y) \end{aligned}$$

(b)
$$\begin{aligned} f(tx, ty) &= C(tx)^a(ty)^{1-a} \\ &= Ctx^ay^{1-a} \\ &= tCx^ay^{1-a} \\ &= tf(x, y) \end{aligned}$$

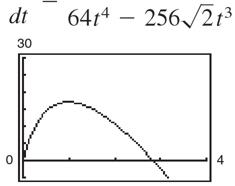
11. (a) $x = 32\sqrt{2}t$

$y = 32\sqrt{2}t - 16t^2$

(b) $\alpha = \arctan\left(\dfrac{y}{x + 50}\right) = \arctan\left(\dfrac{32\sqrt{2}t - 16t^2}{32\sqrt{2}t + 50}\right)$

(c) $\dfrac{d\alpha}{dt} = \dfrac{-16\left(8\sqrt{2}t^2 + 25t - 25\sqrt{2}\right)}{64t^4 - 256\sqrt{2}t^3 + 1024t^2 + 800\sqrt{2}t + 625}$

(d)

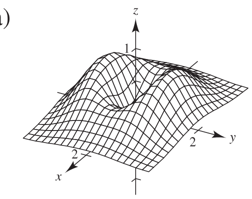


No; The rate of change of α is greatest when the projectile is closest to the camera.

(e) α is maximum when $t = 0.98$ second.

No; the projectile is at its maximum height when $t = \sqrt{2} \approx 1.41$ seconds.

13. (a)

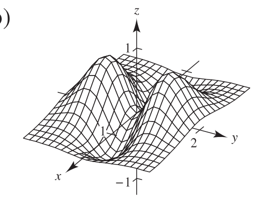


Minimum: $(0, 0, 0)$

Maxima: $(0, \pm 1, 2e^{-1})$

Saddle points: $(\pm 1, 0, e^{-1})$

(b)



Minima: $(\pm 1, 0, -e^{-1})$

Maxima: $(0, \pm 1, 2e^{-1})$

Saddle point: $(0, 0, 0)$

(c) $\alpha > 0$

Minimum: $(0, 0, 0)$

Maxima: $(0, \pm 1, \beta e^{-1})$

Saddle points: $(\pm 1, 0, \alpha e^{-1})$

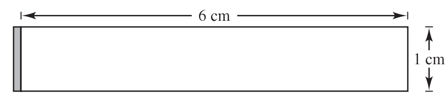
$\alpha < 0$

Minima: $(\pm 1, 0, \alpha e^{-1})$

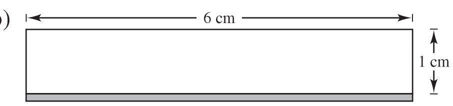
Maxima: $(0, \pm 1, \beta e^{-1})$

Saddle point: $(0, 0, 0)$

15. (a)



(b)



(c) Height

(d) $dl = 0.01, dh = 0$: $dA = 0.01$

$dl = 0, dh = 0.01$: $dA = 0.06$

17–19. Proofs

Chapter 14

Section 14.1 (page 990)

1. $2x^2$ **3.** $y\ln(2y)$ **5.** $(4x^2 - x^4)/2$

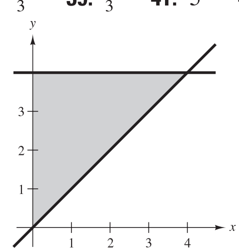
7. $(y/2)[(\ln y)^2 - y^2]$ **9.** $x^2(1 - e^{-x^2} - x^2e^{-x^2})$ **11.** 3

13. $\frac{8}{3}$ **15.** $\frac{1}{2}$ **17.** 2 **19.** $\frac{1}{3}$ **21.** 1629 **23.** $\frac{2}{3}$ **25.** 4

27. $\pi/2$ **29.** $\pi^2/32 + \frac{1}{8}$ **31.** $\frac{1}{2}$ **33.** Diverges **35.** 24

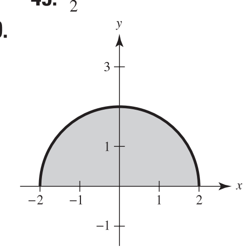
37. $\frac{16}{3}$ **39.** $\frac{8}{3}$ **41.** 5 **43.** πab **45.** $\frac{9}{2}$

47.



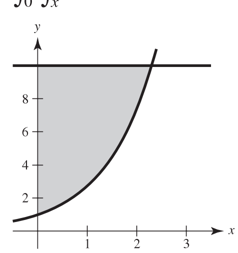
$$\int_0^4 \int_x^4 f(x, y)\,dy\,dx$$

49.



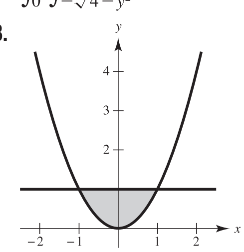
$$\int_0^2 \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} f(x, y)\,dx\,dy$$

51.



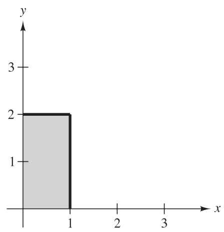
$$\int_0^{\ln 10} \int_{e^x}^{10} f(x, y)\,dy\,dx$$

53.



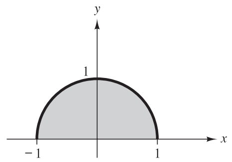
$$\int_0^1 \int_{-\sqrt{y}}^{\sqrt{y}} f(x, y)\,dx\,dy$$

55.



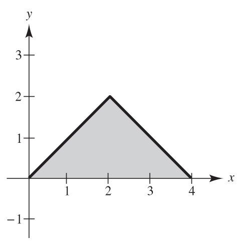
$$\int_0^1\int_0^2 dy\,dx = \int_0^2\int_0^1 dx\,dy = 2$$

57.



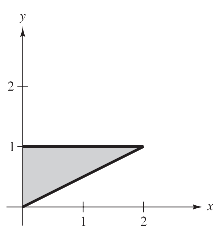
$$\int_0^1\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} dx\,dy = \int_{-1}^1\int_0^{\sqrt{1-x^2}} dy\,dx = \frac{\pi}{2}$$

59.



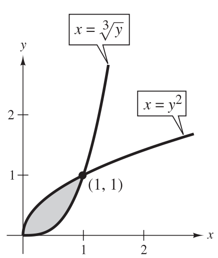
$$\int_0^2\int_0^x dy\,dx + \int_2^4\int_0^{4-x} dy\,dx = \int_0^2\int_y^{4-y} dx\,dy = 4$$

61.



$$\int_0^2\int_{x/2}^1 dy\,dx = \int_0^1\int_0^{2y} dx\,dy = 1$$

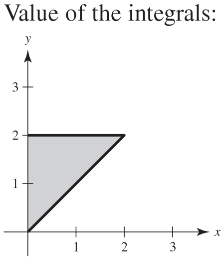
63.



$$\int_0^1\int_{y^2}^{\sqrt[3]{y}} dx\,dy = \int_0^1\int_{x^3}^{\sqrt{x}} dy\,dx = \frac{5}{12}$$

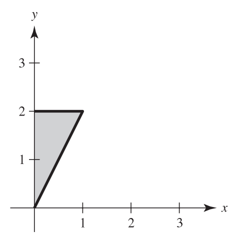
65. The first integral arises using vertical representative rectangles. The second two integrals arise using horizontal representative rectangles.
Value of the integrals: $15{,}625\pi/24$

67.



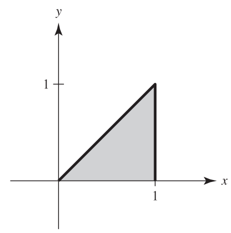
$$\int_0^2\int_x^2 x\sqrt{1+y^3}\,dy\,dx = \frac{26}{9}$$

69.



$$\int_0^1\int_{2x}^2 4e^{y^2}\,dy\,dx = e^4 - 1 \approx 53.598$$

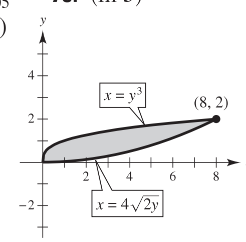
71.



$$\int_0^1\int_y^1 \sin x^2\,dx\,dy = \frac{1}{2}(1-\cos 1) \approx 0.230$$

73. $\frac{1664}{105}$ **75.** $(\ln 5)^2$

77. (a)



(b) $\displaystyle\int_0^8\int_{x^2/32}^{\sqrt[3]{x}} (x^2y - xy^2)\,dy\,dx$ (c) $67{,}520/693$

79. 20.5648 **81.** $15\pi/2$

83. An iterated integral is an integral of a function of several variables. Integrate with respect to one variable while holding the other variables constant.

85. If all four limits of integration are constant, the region of integration is rectangular.

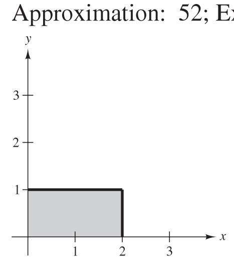
87. True

Section 14.2 (page 1000)

1. 24 (approximation is exact)

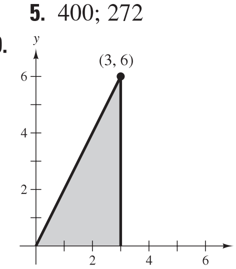
3. Approximation: 52; Exact: $\frac{160}{3}$ **5.** 400; 272

7.



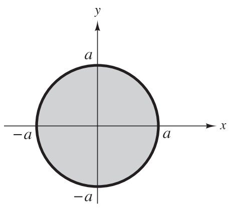
8

9.



36

11.



0

13. $\displaystyle\int_0^3\int_0^5 xy\,dy\,dx = \frac{225}{4}$

$\displaystyle\int_0^5\int_0^3 xy\,dx\,dy = \frac{225}{4}$

15. $\displaystyle\int_1^2\int_x^{2x} \frac{y}{x^2+y^2}\,dy\,dx = \frac{1}{2}\ln\frac{5}{2}$

$\displaystyle\int_1^2\int_1^y \frac{y}{x^2+y^2}\,dx\,dy + \int_2^4\int_{y/2}^2 \frac{y}{x^2+y^2}\,dx\,dy = \frac{1}{2}\ln\frac{5}{2}$

17. $\displaystyle\int_0^1\int_{4-x}^{4-x^2} -2y\,dy\,dx = -\frac{6}{5}$

$\displaystyle\int_3^4\int_{4-y}^{\sqrt{4-y}} -2y\,dx\,dy = -\frac{6}{5}$

19. $\displaystyle\int_0^3\int_{4y/3}^{\sqrt{25-y^2}} x\,dx\,dy = 25$

$\displaystyle\int_0^4\int_0^{3x/4} x\,dy\,dx + \int_4^5\int_0^{\sqrt{25-x^2}} x\,dy\,dx = 25$

21. 4 **23.** 4 **25.** 12 **27.** $\frac{3}{8}$ **29.** 1 **31.** $32\sqrt{2}\,\pi/3$

33. $\displaystyle\int_0^1\int_0^x xy\,dy\,dx = \frac{1}{8}$ **35.** $\displaystyle\int_0^2\int_0^4 x^2\,dy\,dx = \frac{32}{3}$

37. $\displaystyle 2\int_0^1\int_0^x \sqrt{1-x^2}\,dy\,dx = \frac{2}{3}$

39. $\displaystyle\int_0^2\int_0^{\sqrt{4-x^2}} (x+y)\,dy\,dx = \frac{16}{3}$

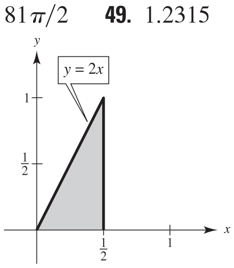
41. $\displaystyle 2\int_0^2\int_0^{\sqrt{1-(x-1)^2}} (2x-x^2-y^2)\,dy\,dx$

43. $\displaystyle 4\int_0^2\int_0^{\sqrt{4-x^2}} (x^2+y^2)\,dy\,dx$

45. $\displaystyle\int_0^2\int_{-\sqrt{2-2(y-1)^2}}^{\sqrt{2-2(y-1)^2}} (4y-x^2-2y^2)\,dx\,dy$

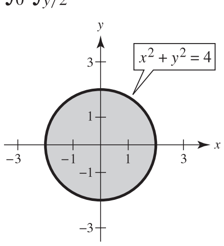
47. $81\pi/2$ **49.** 1.2315 **51.** Proof

53.



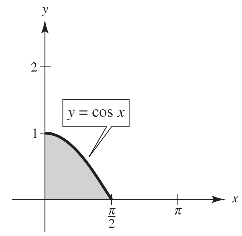
$\displaystyle\int_0^1\int_{y/2}^{1/2} e^{-x^2}\,dx\,dy = 1 - e^{-1/4} \approx 0.221$

55.



$\displaystyle\int_{-2}^2\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \sqrt{4-y^2}\,dy\,dx = \frac{64}{3}$

57.



$\displaystyle\int_0^1\int_0^{\arccos y} \sin x\sqrt{1+\sin^2 x}\,dx\,dy = \frac{1}{3}\left(2\sqrt{2}-1\right)$

59. 2 **61.** $\frac{8}{3}$ **63.** $(e-1)^2$ **65.** 25,645.24

67. See "Definition of Double Integral" on page 994. The double integral of a function $f(x, y) \ge 0$ over the region of integration yields the volume of that region.

69. (a) The total snowfall in county R
(b) The average snowfall in county R

71. No; 6π is the greatest possible value. **73.** Proof; $\frac{1}{5}$

75. Proof; $\frac{7}{27}$ **77.** 2500 m^3 **79.** (a) 1.784 (b) 1.788

81. (a) 11.057 (b) 11.041 **83.** d

85. False. $\displaystyle V = 8\int_0^1\int_0^{\sqrt{1-y^2}} \sqrt{1-x^2-y^2}\,dx\,dy.$

87. $\frac{1}{2}(1-e)$ **89.** R: $x^2+y^2 \le 9$ **91.** About 0.82736

93. Putnam Problem A2, 1989

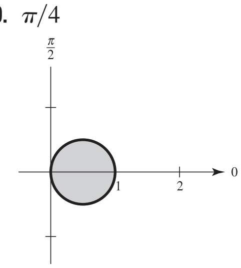
Section 14.3 (page 1009)

1. Rectangular **3.** Polar

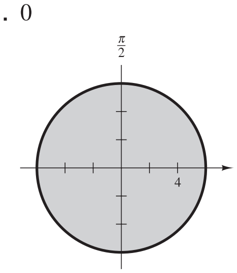
5. The region R is a half-circle of radius 8. It can be described in polar coordinates as
$R = \{(r, \theta): 0 \le r \le 8, 0 \le \theta \le \pi\}$.

7. The region R is a cardioid with $a = b = 3$. It can be described in polar coordinates as
$R = \{(r, \theta): 0 \le r \le 3 + 3\sin\theta, 0 \le \theta \le 2\pi\}$.

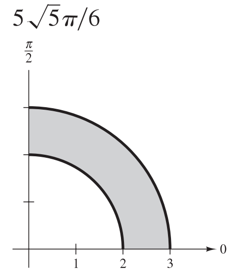
9. $\pi/4$



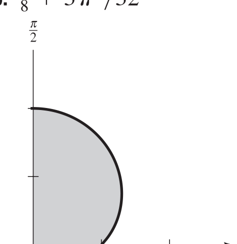
11. 0



13. $5\sqrt{5}\pi/6$



15. $\frac{9}{8} + 3\pi^2/32$



17. $a^3/3$ **19.** 4π **21.** $243\pi/10$ **23.** $\frac{2}{3}$ **25.** $(\pi/2)\sin 1$

27. $\displaystyle\int_0^{\pi/4}\int_0^{2\sqrt{2}} r^2\,dr\,d\theta = \frac{4\sqrt{2}\pi}{3}$

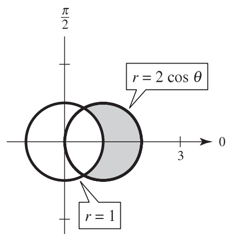
29. $\displaystyle\int_0^{\pi/2}\int_0^2 r^2(\cos\theta + \sin\theta)\,dr\,d\theta = \frac{16}{3}$

31. $\displaystyle\int_0^{\pi/4}\int_1^2 r\theta\,dr\,d\theta = \frac{3\pi^2}{64}$ **33.** $\frac{1}{8}$ **35.** $\frac{250\pi}{3}$

37. $\frac{64}{9}(3\pi - 4)$ **39.** $2\sqrt{4 - 2\sqrt[3]{2}}$ **41.** 1.2858

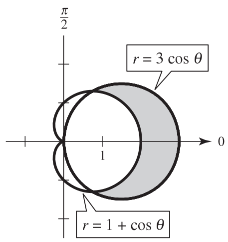
43. 9π **45.** $3\pi/2$ **47.** π

49.



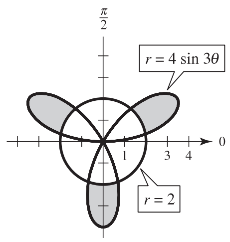
$\frac{\pi}{3} + \frac{\sqrt{3}}{2}$

51.



π

53.



$\frac{4\pi}{3} + 2\sqrt{3}$

55. Let R be a region bounded by the graphs of $r = g_1(\theta)$ and $r = g_2(\theta)$ and the lines $\theta = a$ and $\theta = b$. When using polar coordinates to evaluate a double integral over R, R can be partitioned into small polar sectors.

57. r-simple regions have fixed bounds for θ and variable bounds for r.
θ-simple regions have variable bounds for θ and fixed bounds for r.

59. (a) $\displaystyle\int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} f(x, y)\,dy\,dx$

(b) $\displaystyle\int_0^{2\pi}\int_0^3 f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$

(c) Choose the integral in part (b) because the limits of integration are less complicated.

61. Insert a factor of r; Sector of a circle **63.** 56.051 **65.** c

67. False. Let $f(r, \theta) = r - 1$ and let R be a sector where $0 \le r \le 6$ and $0 \le \theta \le \pi$.

69. (a) 2π (b) $\sqrt{2\pi}$ **71.** 486,788

73. (a) $\displaystyle\int_2^4\int_{y/\sqrt{3}}^{y} f\,dx\,dy$

(b) $\displaystyle\int_{2/\sqrt{3}}^{2}\int_2^{\sqrt{3}x} f\,dy\,dx + \int_2^{4/\sqrt{3}}\int_x^{\sqrt{3}x} f\,dy\,dx + \int_{4/\sqrt{3}}^{4}\int_x^4 f\,dy\,dx$

(c) $\displaystyle\int_{\pi/4}^{\pi/3}\int_{2\csc\theta}^{4\csc\theta} f\,r\,dr\,d\theta$

75. $A = \dfrac{\Delta\theta r_2^2}{2} - \dfrac{\Delta\theta r_1^2}{2} = \Delta\theta\left(\dfrac{r_1 + r_2}{2}\right)(r_2 - r_1) = r\,\Delta r\,\Delta\theta$

Section 14.4 (page 1018)

1. $m = 4$ **3.** $m = \frac{1}{8}$

5. (a) $m = ka^2$, $(a/2, a/2)$ (b) $m = ka^3/2$, $(a/2, 2a/3)$
(c) $m = ka^3/2$, $(2a/3, a/2)$

7. (a) $m = ka^2/2$, $(a/3, 2a/3)$ (b) $m = ka^3/3$, $(3a/8, 3a/4)$
(c) $m = ka^3/6$, $(a/2, 3a/4)$

9. (a) $\left(\dfrac{a}{2} + 5, \dfrac{a}{2}\right)$ (b) $\left(\dfrac{a}{2} + 5, \dfrac{2a}{3}\right)$
(c) $\left(\dfrac{2(a^2 + 15a + 75)}{3(a + 10)}, \dfrac{a}{2}\right)$

11. $m = k/4$, $(2/3, 8/15)$ **13.** $m = 30k$, $(14/5, 4/5)$

15. (a) $m = k(e - 1)$, $\left(\dfrac{1}{e - 1}, \dfrac{e + 1}{4}\right)$
(b) $m = \dfrac{k}{4}(e^2 - 1)$, $\left(\dfrac{e^2 + 1}{2(e^2 - 1)}, \dfrac{4(e^3 - 1)}{9(e^2 - 1)}\right)$

17. $m = 256k/15$, $(0, 16/7)$ **19.** $m = \dfrac{2kL}{\pi}$, $\left(\dfrac{L}{2}, \dfrac{\pi}{8}\right)$

21. $m = \dfrac{k\pi a^2}{8}$, $\left(\dfrac{4\sqrt{2}a}{3\pi}, \dfrac{4a(2 - \sqrt{2})}{3\pi}\right)$

23. $m = \dfrac{k}{8}(1 - 5e^{-4})$, $\left(\dfrac{e^4 - 13}{e^4 - 5}, \dfrac{8}{27}\left[\dfrac{e^6 - 7}{e^6 - 5e^2}\right]\right)$

25. $m = k\pi/3$, $\left(81\sqrt{3}/(40\pi), 0\right)$

27. $\bar{\bar{x}} = \sqrt{3}b/3$
$\bar{\bar{y}} = \sqrt{3}h/3$

29. $\bar{\bar{x}} = a/2$
$\bar{\bar{y}} = a/2$

31. $\bar{\bar{x}} = a/2$
$\bar{\bar{y}} = a/2$

33. $I_x = kab^4/4$
$I_y = kb^2a^3/6$
$I_0 = (3kab^4 + 2ka^3b^2)/12$
$\bar{\bar{x}} = \sqrt{3}a/3$
$\bar{\bar{y}} = \sqrt{2}b/2$

35. $I_x = 32k/3$
$I_y = 16k/3$
$I_0 = 16k$
$\bar{\bar{x}} = 2\sqrt{3}/3$
$\bar{\bar{y}} = 2\sqrt{6}/3$

37. $I_x = 16k$
$I_y = 512k/5$
$I_0 = 592k/5$
$\bar{\bar{x}} = 4\sqrt{15}/5$
$\bar{\bar{y}} = \sqrt{6}/2$

39. $I_x = 3k/56$
$I_y = k/18$
$I_0 = 55k/504$
$\bar{\bar{x}} = \sqrt{30}/9$
$\bar{\bar{y}} = \sqrt{70}/14$

41. $\displaystyle 2k\int_{-b}^{b}\int_0^{\sqrt{b^2-x^2}} (x - a)^2\,dy\,dx = \frac{k\pi b^2}{4}(b^2 + 4a^2)$

43. $\displaystyle\int_0^4\int_0^{\sqrt{x}} kx(x - 6)^2\,dy\,dx = \frac{42{,}752k}{315}$

45. $\displaystyle\int_0^a\int_0^{\sqrt{a^2-x^2}} k(a - y)(y - a)^2\,dy\,dx = ka^5\left(\frac{7\pi}{16} - \frac{17}{15}\right)$

47. See definitions on page 1014. **49.** Answers will vary.

51. $L/3$ **53.** $L/2$ **55.** Proof

Section 14.5 (page 1025)

1. 24 **3.** 12π **5.** $\frac{1}{2}\left[4\sqrt{17} + \ln\left(4 + \sqrt{17}\right)\right]$

7. $\frac{4}{27}\left(31\sqrt{31} - 8\right)$ **9.** $\sqrt{2} - 1$ **11.** $\sqrt{2}\pi$

13. $2\pi a\left(a - \sqrt{a^2 - b^2}\right)$ **15.** $48\sqrt{14}$ **17.** 20π

19. $\displaystyle\int_0^1\int_0^x \sqrt{5 + 4x^2}\,dy\,dx = \frac{27 - 5\sqrt{5}}{12} \approx 1.3183$

21. $\displaystyle\int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \sqrt{1 + 4x^2 + 4y^2}\,dy\,dx$
$\displaystyle = \frac{\pi}{6}\left(37\sqrt{37} - 1\right) \approx 117.3187$

23. $\displaystyle\int_0^1\int_0^1 \sqrt{1 + 4x^2 + 4y^2}\,dy\,dx \approx 1.8616$ **25.** e

27. 2.0035 **29.** $\int_{-1}^{1}\int_{-1}^{1} \sqrt{1 + 9(x^2 - y)^2 + 9(y^2 - x)^2}\, dy\, dx$

31. $\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \sqrt{1 + e^{-2x}}\, dy\, dx$

33. $\int_{0}^{4}\int_{0}^{10} \sqrt{1 + e^{2xy}(x^2 + y^2)}\, dy\, dx$

35. If f and its first partial derivatives are continuous on the closed region R in the xy-plane, then the area of the surface S given by $z = f(x, y)$ over R is

$$\int_R\int \sqrt{1 + [f_x(x, y)]^2 + [f_y(x, y)]^2}\, dA.$$

37. No. The size and shape of the graph stay the same, just the position is changed. So, the surface area does not increase.

39. 16 **41.** (a) $812\pi\sqrt{609}$ cm^3 (b) $100\pi\sqrt{609}$ cm^2

Section 14.6 (page 1035)

1. 18 **3.** $\frac{1}{10}$ **5.** $\frac{15}{2}(1 - 1/e)$ **7.** $-\frac{40}{3}$ **9.** $\frac{324}{5}$

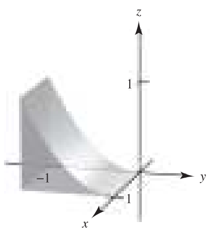
11. 2.44167 **13.** $V = \int_{0}^{5}\int_{0}^{5-x}\int_{0}^{5-x-y} dz\, dy\, dx$

15. $V = \int_{-\sqrt{6}}^{\sqrt{6}}\int_{-\sqrt{6-y^2}}^{\sqrt{6-y^2}}\int_{0}^{6-x^2-y^2} dz\, dx\, dy$

17. $V = \int_{-4}^{4}\int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}}\int_{(x^2+y^2)/2}^{\sqrt{80-x^2-y^2}} dz\, dy\, dx$

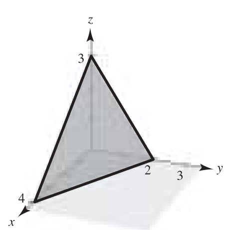
19. $\frac{256}{15}$ **21.** $4\pi a^3/3$ **23.** $\frac{256}{15}$ **25.** 10

27.



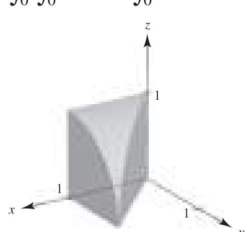
$\int_{0}^{1}\int_{0}^{1}\int_{-1}^{-\sqrt{z}} dy\, dz\, dx$

29.



$\int_{0}^{3}\int_{0}^{(12-4z)/3}\int_{0}^{(12-4z-3x)/6} dy\, dx\, dz$

31.



$\int_{0}^{1}\int_{0}^{x}\int_{0}^{\sqrt{1-y^2}} dz\, dy\, dx$

33. $\int_{0}^{1}\int_{0}^{x}\int_{0}^{3} xyz\, dz\, dy\, dx,\ \int_{0}^{1}\int_{y}^{1}\int_{0}^{3} xyz\, dz\, dx\, dy,$

$\int_{0}^{1}\int_{0}^{3}\int_{0}^{x} xyz\, dy\, dz\, dx,\ \int_{0}^{3}\int_{0}^{1}\int_{0}^{x} xyz\, dy\, dx\, dz,$

$\int_{0}^{3}\int_{0}^{1}\int_{y}^{1} xyz\, dx\, dy\, dz,\ \int_{0}^{1}\int_{0}^{3}\int_{y}^{1} xyz\, dx\, dz\, dy$

35. $\int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_{0}^{4} xyz\, dz\, dy\, dx,\ \int_{-3}^{3}\int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}}\int_{0}^{4} xyz\, dz\, dx\, dy,$

$\int_{-3}^{3}\int_{0}^{4}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} xyz\, dy\, dz\, dx,\ \int_{0}^{4}\int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} xyz\, dy\, dx\, dz,$

$\int_{0}^{4}\int_{-3}^{3}\int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} xyz\, dx\, dy\, dz,\ \int_{-3}^{3}\int_{0}^{4}\int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} xyz\, dx\, dz\, dy$

37. $\int_{0}^{1}\int_{0}^{1-z}\int_{0}^{1-y^2} dx\, dy\, dz,\ \int_{0}^{1}\int_{0}^{1-y}\int_{0}^{1-y^2} dx\, dz\, dy,$

$\int_{0}^{1}\int_{0}^{2z-z^2}\int_{0}^{1-z} 1\, dy\, dx\, dz + \int_{0}^{1}\int_{2z-z^2}^{1}\int_{0}^{\sqrt{1-x}} 1\, dy\, dx\, dz,$

$\int_{0}^{1}\int_{1-\sqrt{1-x}}^{1}\int_{0}^{1-z} 1\, dy\, dz\, dx + \int_{0}^{1}\int_{0}^{1-\sqrt{1-x}}\int_{0}^{\sqrt{1-x}} 1\, dy\, dz\, dx,$

$\int_{0}^{1}\int_{0}^{\sqrt{1-x}}\int_{0}^{1-y} dz\, dy\, dx$

39. $m = 8k$ **41.** $m = 128k/3$

$\bar{x} = \frac{3}{2}$ $\bar{z} = 1$

43. $m = k\int_{0}^{b}\int_{0}^{b}\int_{0}^{b} xy\, dz\, dy\, dx$

$M_{yz} = k\int_{0}^{b}\int_{0}^{b}\int_{0}^{b} x^2y\, dz\, dy\, dx$

$M_{xz} = k\int_{0}^{b}\int_{0}^{b}\int_{0}^{b} xy^2\, dz\, dy\, dx$

$M_{xy} = k\int_{0}^{b}\int_{0}^{b}\int_{0}^{b} xyz\, dz\, dy\, dx$

45. $\bar{x}$ will be greater than 2, and $\bar{y}$ and $\bar{z}$ will be unchanged.

47. $\bar{x}$ and $\bar{z}$ will be unchanged, and $\bar{y}$ will be greater than 0.

49. $(0, 0, 3h/4)$ **51.** $\left(0, 0, \frac{3}{2}\right)$ **53.** $\left(5, 6, \frac{5}{4}\right)$

55. (a) $I_x = 2ka^5/3$ (b) $I_x = ka^8/8$
$I_y = 2ka^5/3$ $I_y = ka^8/8$
$I_z = 2ka^5/3$ $I_z = ka^8/8$

57. (a) $I_x = 256k$ (b) $I_x = 2048k/3$
$I_y = 512k/3$ $I_y = 1024k/3$
$I_z = 256k$ $I_z = 2048k/3$

59. Proof **61.** $\int_{-1}^{1}\int_{-1}^{1}\int_{0}^{1-x} (x^2 + y^2)\sqrt{x^2 + y^2 + z^2}\, dz\, dy\, dx$

63. (a) $m = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{0}^{4-x^2-y^2} kz\, dz\, dy\, dx$

(b) $\bar{x} = \bar{y} = 0$, by symmetry.

$\bar{z} = \frac{1}{m}\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{0}^{4-x^2-y^2} kz^2\, dz\, dy\, dx$

(c) $I_z = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{0}^{4-x^2-y^2} kz(x^2 + y^2)\, dz\, dy\, dx$

65. See "Definition of Triple Integral" on page 1027 and Theorem 14.4, "Evaluation by Iterated Integrals," on page 1028.

67. (a) Solid B

(b) Solid B has the greater moment of inertia because it is more dense.

(c) Solid A will reach the bottom first. Because Solid B has a greater moment of inertia, it has a greater resistance to rotational motion.

69. $\frac{13}{3}$ **71.** $\frac{3}{2}$

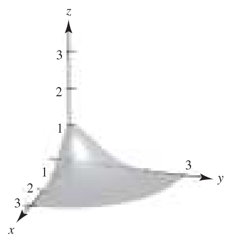
73. Q: $3z^2 + y^2 + 2x^2 \le 1$; $4\sqrt{6}\pi/45 \approx 0.684$

75. $a = 2, \frac{16}{3}$ **77.** Putnam Problem B1, 1965

Section 14.7 (page 1043)

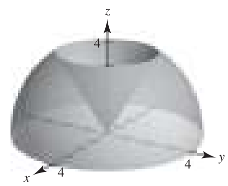
1. 27 **3.** $\frac{52}{45}$ **5.** $\pi/8$ **7.** $\pi(e^4 + 3)$

9.



$(1 - e^{-9})\pi/4$

11.



$64\sqrt{3}\pi/3$

13. Cylindrical: $\displaystyle\int_0^{2\pi}\int_0^2\int_{r^2}^4 r^2\cos\theta\,dz\,dr\,d\theta = 0$

Spherical: $\displaystyle\int_0^{2\pi}\int_0^{\arctan(1/2)}\int_0^{4\sec\phi}\rho^3\sin^2\phi\cos\theta\,d\rho\,d\phi\,d\theta$

$\displaystyle + \int_0^{2\pi}\int_{\arctan(1/2)}^{\pi/2}\int_0^{\cot\phi\csc\phi}\rho^3\sin^2\phi\cos\phi\,d\rho\,d\phi\,d\theta = 0$

15. Cylindrical: $\displaystyle\int_0^{2\pi}\int_0^a\int_a^{a+\sqrt{a^2-r^2}} r^2\cos\theta\,dz\,dr\,d\theta = 0$

Spherical: $\displaystyle\int_0^{\pi/4}\int_0^{2\pi}\int_{a\sec\phi}^{2a\cos\phi}\rho^3\sin^2\phi\cos\theta\,d\rho\,d\theta\,d\phi = 0$

17. $(2a^3/9)(3\pi - 4)$ **19.** $\pi/16$ **21.** $(2a^3/9)(3\pi - 4)$

23. $48k\pi$ **25.** $\pi r_0^2 h/3$ **27.** $(0, 0, h/5)$

29. $\displaystyle I_z = 4k\int_0^{\pi/2}\int_0^{r_0}\int_0^{h(r_0-r)/r_0} r^3\,dz\,dr\,d\theta = 3mr_0^2/10$

31. Proof **33.** $9\pi\sqrt{2}$ **35.** $16\pi^2$

37. $k\pi a^4$ **39.** $(0, 0, 3r/8)$ **41.** $k\pi/192$

43. Rectangular to cylindrical: $r^2 = x^2 + y^2$
$\tan\theta = y/x$
$z = z$

Cylindrical to rectangular: $x = r\cos\theta$
$y = r\sin\theta$
$z = z$

45. $\displaystyle\int_{\theta_1}^{\theta_2}\int_{g_1(\theta)}^{g_2(\theta)}\int_{h_1(r\cos\theta,\, r\sin\theta)}^{h_2(r\cos\theta,\, r\sin\theta)} f(r\cos\theta, r\sin\theta, z)r\,dz\,dr\,d\theta$

47. (a) r constant: right circular cylinder about z-axis
θ constant: plane parallel to z-axis
z constant: plane parallel to xy-plane

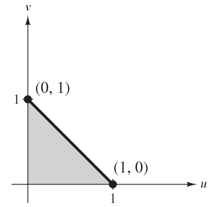
(b) ρ constant: sphere
θ constant: plane parallel to z-axis
ϕ constant: cone

49. $\frac{1}{2}\pi^2 a^4$ **51.** Putnam Problem A1, 2006

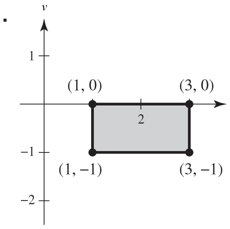
Section 14.8 (page 1050)

1. $-\frac{1}{2}$ **3.** $1 + 2v$ **5.** 1 **7.** $-e^{2u}$

9.



11.



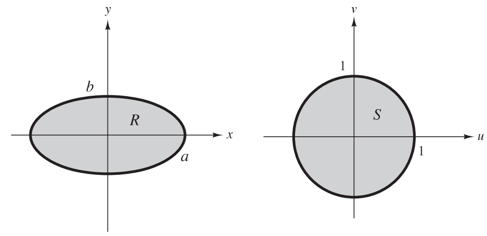
13. $\displaystyle\int_R\int 3xy\,dA = \int_{-2/3}^{2/3}\int_{1-x}^{(1/2)x+2} 3xy\,dy\,dx$

$\displaystyle + \int_{2/3}^{4/3}\int_{(1/2)x}^{(1/2)x+2} 3xy\,dy\,dx + \int_{4/3}^{8/3}\int_{(1/2)x}^{4-x} 3xy\,dy\,dx = \frac{164}{9}$

15. $\frac{8}{3}$ **17.** 36 **19.** $(e^{-1/2} - e^{-2})\ln 8 \approx 0.9798$ **21.** 96

23. $12(e^4 - 1)$ **25.** $\frac{100}{9}$ **27.** $\frac{2}{5}a^{5/2}$ **29.** One

31. (a)



(b) ab (c) πab

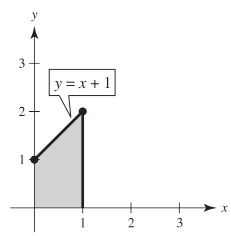
33. See "Definition of the Jacobian" on page 1045. **35.** $u^2 v$

37. $-uv$ **39.** $-\rho^2\sin\phi$ **41.** Putnam Problem A2, 1994

Review Exercises for Chapter 14 (page 1052)

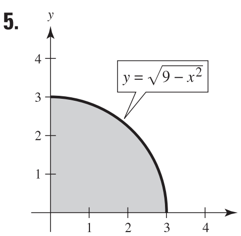
1. $x - x^3 + x^3\ln x^2$

3.



$\frac{29}{6}$

5.



36

7. $\displaystyle\int_0^3\int_0^{(3-x)/3} dy\,dx = \int_0^1\int_0^{3-3y} dx\,dy = \frac{3}{2}$

9. $\displaystyle\int_{-5}^{3}\int_{-\sqrt{25-x^2}}^{\sqrt{25-x^2}} dy\,dx$

$\displaystyle = \int_{-5}^{-4}\int_{-\sqrt{25-y^2}}^{\sqrt{25-y^2}} dx\,dy + \int_{-4}^{4}\int_{-\sqrt{25-y^2}}^{3} dx\,dy$

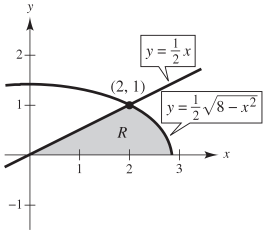
$\displaystyle + \int_4^5\int_{-\sqrt{25-y^2}}^{\sqrt{25-y^2}} dx\,dy$

$= 25\pi/2 + 12 + 25\arcsin\frac{3}{5} \approx 67.36$

11. $\displaystyle 4\int_0^1\int_0^{x\sqrt{1-x^2}} dy\,dx = 4\int_0^{1/2}\int_{\sqrt{(1-\sqrt{1-4y^2})/2}}^{\sqrt{(1+\sqrt{1-4y^2})/2}} dx\,dy = \frac{4}{3}$

13. $\int_2^5 \int_{x-3}^{\sqrt{x-1}} dy\,dx + 2\int_1^2 \int_0^{\sqrt{x-1}} dy\,dx = \int_{-1}^2 \int_{y^2+1}^{y+3} dx\,dy = \frac{9}{2}$

15. Both integrations are over the common region R, as shown in the figure. Both integrals yield $\frac{4}{3} + \frac{4}{3}\sqrt{2}$.

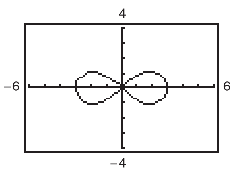


17. $\frac{3296}{15}$ **19.** $\frac{40}{3}$ **21.** 13.67°C **23.** $k = 1, 0.070$ **25.** c

27. True **29.** True **31.** $(h^3/6)\left[\ln\left(\sqrt{2} + 1\right) + \sqrt{2}\right]$

33. $9\pi/2$ **35.** $\pi h^3/3$

37. (a) $r = 3\sqrt{\cos 2\theta}$ (b) 9 (c) $3\left(3\pi - 16\sqrt{2} + 20\right) \approx 20.392$



(b) 9 (c) $3\left(3\pi - 16\sqrt{2} + 20\right) \approx 20.392$

39. (a) $m = k/4, \left(\frac{32}{45}, \frac{64}{55}\right)$ (b) $m = 17k/30, \left(\frac{936}{1309}, \frac{784}{663}\right)$

41. $I_x = ka^2b^3/6$
$I_y = ka^4b/4$
$I_0 = (2ka^2b^3 + 3ka^4b)/12$
$\bar{\bar{x}} = a/\sqrt{2}$
$\bar{\bar{y}} = b/\sqrt{3}$

43. $\dfrac{\left(101\sqrt{101} - 1\right)\pi}{6}$ **45.** $\frac{1}{6}\left(37\sqrt{37} - 1\right)$

47. (a) 30,415.74 ft^3 (b) 2081.53 ft^2 **49.** $324\pi/5$

51. $(abc/3)(a^2 + b^2 + c^2)$ **53.** $8\pi/15$ **55.** $\frac{32}{3}\left(\pi/2 - \frac{2}{3}\right)$

57. $\left(0, 0, \frac{1}{4}\right)$ **59.** $(3a/8, 3a/8, 3a/8)$ **61.** $833k\pi/3$

63. (a) $\dfrac{1}{3}\pi h^2(3a - h)$ (b) $\left(0, 0, \dfrac{3(2a - h)^2}{4(3a - h)}\right)$ (c) $\left(0, 0, \dfrac{3}{8}a\right)$

(d) a (e) $(\pi/30)h^3(20a^2 - 15ah + 3h^2)$ (f) $4\pi a^5/15$

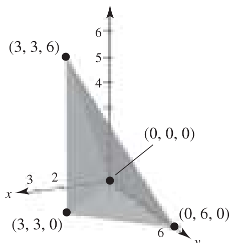
65. Volume of a torus formed by a circle of radius 3, centered at (0, 3, 0) and revolved about the z-axis

67. -9 **69.** $5 \ln 5 - 3 \ln 3 - 2 \approx 2.751$

P.S. Problem Solving (page 1055)

1. $8\left(2 - \sqrt{2}\right)$ **3.** (a)–(g) Proofs **5.** $\frac{1}{3}$

7.



$$\int_0^3 \int_0^{2x} \int_x^{6-x} dy\,dz\,dx = 18$$

9. $\sqrt{\pi}/4$

11. If $a, k > 0$, then $1 = ka^2$ or $a = 1/\sqrt{k}$.

13. Answers will vary.

15. The greater the angle between the given plane and the xy-plane, the greater the surface area. So, $z_2 < z_1 < z_4 < z_3$.

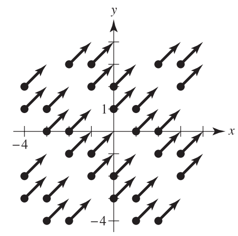
17. The results are not the same. Fubini's Theorem is not valid because f is not continuous on the region $0 \le x \le 1$, $0 \le y \le 1$.

Chapter 15

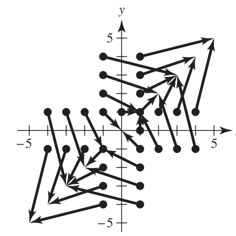
Section 15.1 (page 1067)

1. d **2.** c **3.** e **4.** b **5.** a **6.** f

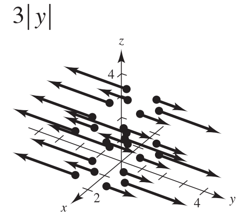
7. $\sqrt{2}$



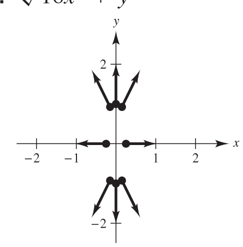
9. $\sqrt{x^2 + y^2}$



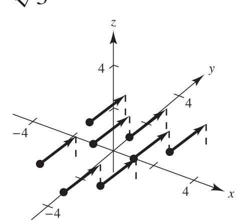
11. $3|y|$



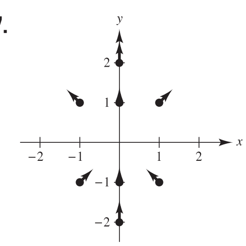
13. $\sqrt{16x^2 + y^2}$



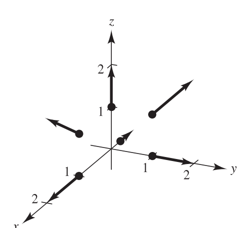
15. $\sqrt{3}$



17.



19.



21. $2x\mathbf{i} + 4y\mathbf{j}$

23. $(10x + 3y)\mathbf{i} + (3x + 2y)\mathbf{j}$ **25.** $6yz\mathbf{i} + 6xz\mathbf{j} + 6xy\mathbf{k}$

27. $2xye^{x^2}\mathbf{i} + e^{x^2}\mathbf{j} + \mathbf{k}$

29. $[xy/(x + y) + y \ln(x + y)]\mathbf{i} + [xy/(x + y) + x \ln(x + y)]\mathbf{j}$

31–33. Proofs **35.** Conservative because $\partial N/\partial x = \partial M/\partial y$.

37. Not conservative because $\partial N/\partial x \ne \partial M/\partial y$.

39. Conservative: $f(x, y) = xy + K$

41. Conservative: $f(x, y) = x^2y + K$

43. Not conservative **45.** Not conservative
47. Conservative: $f(x, y) = e^x \cos y + K$ **49.** $4\mathbf{i} - \mathbf{j} - 3\mathbf{k}$
51. $-2\mathbf{k}$ **53.** $2x/(x^2 + y^2)\mathbf{k}$
55. $\cos(y - z)\mathbf{i} + \cos(z - x)\mathbf{j} + \cos(x - y)\mathbf{k}$
57. Conservative: $f(x, y, z) = \frac{1}{2}(x^2y^2z^2) + K$
59. Not conservative **61.** Conservative: $f(x, y, z) = xz/y + K$
63. $2x + 4y$ **65.** $\cos x - \sin y + 2z$ **67.** 4 **69.** 0
71. See "Definition of Vector Field" on page 1058. Some physical examples of vector fields include velocity fields, gravitational fields, and electric force fields.
73. See "Definition of Curl of a Vector Field" on page 1064.
75. $9x\mathbf{j} - 2y\mathbf{k}$ **77.** $z\mathbf{j} + y\mathbf{k}$ **79.** $3z + 2x$ **81.** 0
83–89. Proofs
91. $f(x, y, z) = \|\mathbf{F}(x, y, z)\| = \sqrt{x^2 + y^2 + z^2}$

$$\ln f = \frac{1}{2}\ln(x^2 + y^2 + z^2)$$
$$\nabla \ln f = \frac{x}{x^2 + y^2 + z^2}\mathbf{i} + \frac{y}{x^2 + y^2 + z^2}\mathbf{j} + \frac{z}{x^2 + y^2 + z^2}\mathbf{k} = \frac{\mathbf{F}}{f^2}$$

93. $f^n = \|\mathbf{F}(x, y, z)\|^n = \left(\sqrt{x^2 + y^2 + z^2}\right)^n$

$$\nabla f^n = n\left(\sqrt{x^2 + y^2 + z^2}\right)^{n-1}\left(\frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{\sqrt{x^2 + y^2 + z^2}}\right) = nf^{n-2}\mathbf{F}$$

95. True
97. False. Curl f is meaningful only for vector fields, when direction is involved.

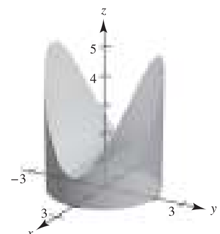
Section 15.2 (page 1079)

1. $\mathbf{r}(t) = \begin{cases} t\mathbf{i} + t\mathbf{j}, & 0 \le t \le 1 \\ (2 - t)\mathbf{i} + \sqrt{2 - t}\,\mathbf{j}, & 1 \le t \le 2 \end{cases}$

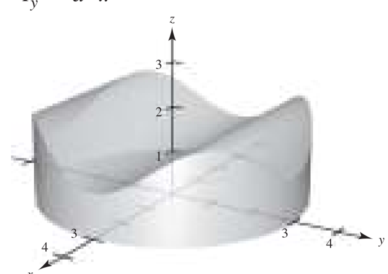
3. $\mathbf{r}(t) = \begin{cases} t\mathbf{i}, & 0 \le t \le 3 \\ 3\mathbf{i} + (t - 3)\mathbf{j}, & 3 \le t \le 6 \\ (9 - t)\mathbf{i} + 3\mathbf{j}, & 6 \le t \le 9 \\ (12 - t)\mathbf{j}, & 9 \le t \le 12 \end{cases}$

5. $\mathbf{r}(t) = 3\cos t\mathbf{i} + 3\sin t\mathbf{j}, \quad 0 \le t \le 2\pi$ **7.** 20 **9.** $5\pi/2$
11. (a) C: $\mathbf{r}(t) = t\mathbf{i} + t\mathbf{j}$, $0 \le t \le 1$ (b) $2\sqrt{2}/3$
13. (a) C: $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j}$, $0 \le t \le \pi/2$ (b) $\pi/2$
15. (a) C: $\mathbf{r}(t) = t\mathbf{i}$, $0 \le t \le 1$ (b) $1/2$
17. (a) C: $\mathbf{r}(t) = \begin{cases} t\mathbf{i}, & 0 \le t \le 1 \\ (2 - t)\mathbf{i} + (t - 1)\mathbf{j}, & 1 \le t \le 2 \\ (3 - t)\mathbf{j}, & 2 \le t \le 3 \end{cases}$
(b) $\frac{19}{6}\left(1 + \sqrt{2}\right)$
19. (a) C: $\mathbf{r}(t) = \begin{cases} t\mathbf{i}, & 0 \le t \le 1 \\ \mathbf{i} + t\mathbf{k}, & 0 \le t \le 1 \\ \mathbf{i} + t\mathbf{j} + \mathbf{k}, & 0 \le t \le 1 \end{cases}$ (b) $\frac{23}{6}$
21. $8\sqrt{5}\pi(1 + 4\pi^2/3) \approx 795.7$ **23.** 2
25. $(k/12)\left(41\sqrt{41} - 27\right)$ **27.** 1 **29.** $\frac{1}{2}$ **31.** $\frac{9}{4}$
33. About 249.49 **35.** 66 **37.** 0 **39.** $-10\pi^2$
41. Positive **43.** Zero
45. (a) $\frac{236}{3}$; Orientation is from left to right, so the value is positive.
(b) $-\frac{236}{3}$; Orientation is from right to left, so the value is negative.

47. $\mathbf{F}(t) = -2t\mathbf{i} - t\mathbf{j}$
$\mathbf{r}'(t) = \mathbf{i} - 2\mathbf{j}$
$\mathbf{F}(t) \cdot \mathbf{r}'(t) = -2t + 2t = 0$
$\int_C \mathbf{F} \cdot d\mathbf{r} = 0$
49. $\mathbf{F}(t) = (t^3 - 2t^2)\mathbf{i} + (t - t^2/2)\mathbf{j}$
$\mathbf{r}'(t) = \mathbf{i} + 2t\mathbf{j}$
$\mathbf{F}(t) \cdot \mathbf{r}'(t) = t^3 - 2t^2 + 2t^2 - t^3 = 0$
$\int_C \mathbf{F} \cdot d\mathbf{r} = 0$
51. 1010 **53.** $\frac{190}{3}$ **55.** 25 **57.** $\frac{63}{2}$ **59.** $-\frac{11}{6}$ **61.** $\frac{316}{3}$
63. $5h$ **65.** $\frac{1}{2}$ **67.** $(h/4)\left[2\sqrt{5} + \ln\left(2 + \sqrt{5}\right)\right]$
69. $\frac{1}{120}\left(25\sqrt{5} - 11\right)$
71. (a) $12\pi \approx 37.70$ cm^2 (b) $12\pi/5 \approx 7.54$ cm^3
(c)



73. $I_x = I_y = a^3\pi$
75. (a)



(b) 9π cm^2 ≈ 28.274 cm^2
(c) $\text{Volume} = 2\int_0^3 2\sqrt{9 - y^2}\left[1 + 4\frac{y^2}{9}\left(1 - \frac{y^2}{9}\right)\right] dy = 27\pi/2 \approx 42.412$ cm^3
77. 1750 ft-lb
79. See "Definition of Line Integral" on page 1070 and Theorem 15.4, "Evaluation of a Line Integral as a Definite Integral" on page 1071.
81. z_3, z_1, z_2, z_4; The greater the height of the surface over the curve $y = \sqrt{x}$, the greater the lateral surface area.
83. False: $\int_C xy\, ds = \sqrt{2}\int_0^1 t^2\, dt.$
85. False: the orientations are different. **87.** -12

Section 15.3 (page 1090)

1. (a) $\int_0^1 (t^2 + 2t^4)\, dt = \frac{11}{15}$
(b) $\int_0^{\pi/2} (\sin^2\theta\cos\theta + 2\sin^4\theta\cos\theta)\, d\theta = \frac{11}{15}$
3. (a) $\int_0^{\pi/3} (\sec\theta\tan^2\theta - \sec^3\theta)\, d\theta \approx -1.317$
(b) $\int_0^3 \left[\frac{\sqrt{t}}{2\sqrt{t + 1}} - \frac{\sqrt{t + 1}}{2\sqrt{t}}\right] dt \approx -1.317$

5. Conservative **7.** Not conservative **9.** Conservative

11. (a) 1 (b) 1 **13.** (a) 0 (b) $-\frac{1}{3}$ (c) $-\frac{1}{2}$

15. (a) 64 (b) 0 (c) 0 (d) 0 **17.** (a) $\frac{64}{3}$ (b) $\frac{64}{3}$

19. (a) 32 (b) 32 **21.** (a) $\frac{2}{3}$ (b) $\frac{17}{6}$ **23.** (a) 0 (b) 0

25. 72 **27.** -1 **29.** 0 **31.** (a) 2 (b) 2 (c) 2

33. 11 **35.** 30,366 **37.** 0

39. (a) $d\mathbf{r} = (\mathbf{i} - \mathbf{j})\,dt \Rightarrow \int_0^{50} 175\,dt = 8750$ ft-lb

(b) $d\mathbf{r} = \left(\mathbf{i} - \frac{1}{25}(50 - t)\mathbf{j}\right) dt \Rightarrow 7\int_0^{50} (50 - t)\,dt$

$= 8750$ ft-lb

41. See Theorem 15.5, "Fundamental Theorem of Line Integrals," on page 1084.

43. (a) 2π (b) 2π (c) -2π (d) 0

45. Yes, because the work required to get from point to point is independent of the path taken.

47. False. It would be true if $\mathbf{F}$ were conservative.

49. True **51.** Proof

53. (a) Proof (b) $-\pi$ (c) π

(d) -2π; does not contradict Theorem 15.7 because $\mathbf{F}$ is not continuous at $(0, 0)$ in R enclosed by C.

(e) $\nabla\left(\arctan\frac{x}{y}\right) = \dfrac{1/y}{1 + (x/y)^2}\,\mathbf{i} + \dfrac{-x/y^2}{1 + (x/y)^2}\,\mathbf{j}$

Section 15.4 (page 1099)

1. $\frac{1}{30}$ **3.** 0 **5.** About 19.99 **7.** $\frac{9}{2}$ **9.** 56 **11.** $\frac{4}{3}$ **13.** 0

15. 0 **17.** $\frac{1}{12}$ **19.** 32π **21.** π **23.** $\frac{225}{2}$ **25.** πa^2 **27.** $\frac{9}{2}$

29. See Theorem 15.8 on page 1093. **31.** Proof **33.** $\left(0, \frac{8}{5}\right)$

35. $\left(\frac{8}{15}, \frac{8}{21}\right)$ **37.** $3\pi a^2/2$ **39.** $\pi - 3\sqrt{3}/2$

41. (a) $51\pi/2$ (b) $243\pi/2$

43. $\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C M\,dx + N\,dy = \int\!\!\int_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA = 0;$

$I = -2\pi$ when C is a circle that contains the origin.

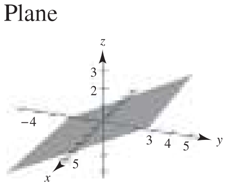
45. $\frac{19}{2}$ **47–49.** Proofs

Section 15.5 (page 1109)

1. e **2.** f **3.** b **4.** a **5.** d **6.** c

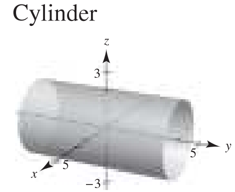
7. $y - 2z = 0$

Plane

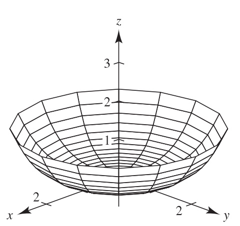


9. $x^2 + z^2 = 4$

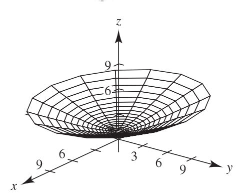
Cylinder



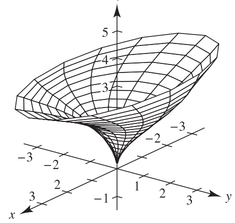
11.



13.



15.



17. The paraboloid is reflected (inverted) through the xy-plane.

19. The height of the paraboloid is increased from 4 to 9.

21. $\mathbf{r}(u, v) = u\mathbf{i} + v\mathbf{j} + v\mathbf{k}$

23. $\mathbf{r}(u, v) = \frac{1}{2}u\cos v\mathbf{i} + u\mathbf{j} + \frac{1}{3}u\sin v\mathbf{k},\ u \ge 0,\ 0 \le v \le 2\pi$ *or* $\mathbf{r}(x, y) = x\mathbf{i} + \sqrt{4x^2 + 9y^2}\mathbf{j} + z\mathbf{k}$

25. $\mathbf{r}(u, v) = 5\cos u\mathbf{i} + 5\sin u\mathbf{j} + v\mathbf{k}$

27. $\mathbf{r}(u, v) = u\mathbf{i} + v\mathbf{j} + u^2\mathbf{k}$

29. $\mathbf{r}(u, v) = v\cos u\mathbf{i} + v\sin u\mathbf{j} + 4\mathbf{k},\quad 0 \le v \le 3$

31. $x = u,\ y = \dfrac{u}{2}\cos v,\ z = \dfrac{u}{2}\sin v,\quad 0 \le u \le 6,\ 0 \le v \le 2\pi$

33. $x = \sin u\cos v,\ y = \sin u\sin v,\ z = u$
$0 \le u \le \pi,\ 0 \le v \le 2\pi$

35. $x - y - 2z = 0$ **37.** $4y - 3z = 12$ **39.** $8\sqrt{2}$

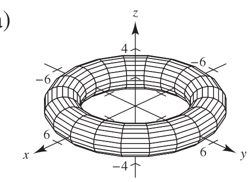
41. $2\pi ab$ **43.** $\pi ab^2\sqrt{a^2 + 1}$

45. $(\pi/6)\left(17\sqrt{17} - 1\right) \approx 36.177$

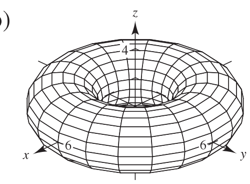
47. See "Definition of Parametric Surface" on page 1102.

49–51. Proofs

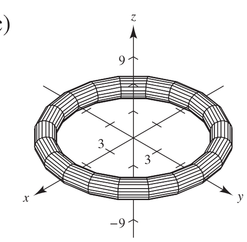
53. (a)



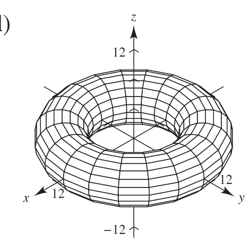
(b)



(c)



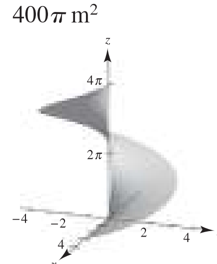
(d)



The radius of the generating circle that is revolved about the z-axis is b, and its center is a units from the axis of revolution.

55. 400π m^2

57.

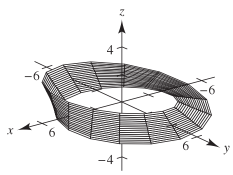


$2\pi\left[\frac{3}{2}\sqrt{13} + 2\ln\left(3 + \sqrt{13}\right) - 2\ln 2\right]$

59. Answers will vary. Sample answer: Let
$x = (2 - u)(5 + \cos v)\cos 3\pi u$
$y = (2 - u)(5 + \cos v)\sin 3\pi u$
$z = 5u + (2 - u)\sin v$
where $-\pi \le u \le \pi$ and $-\pi \le v \le \pi$.

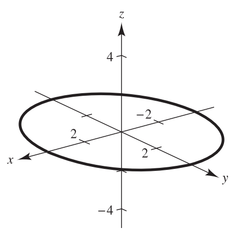
Section 15.6 (page 1122)

1. $12\sqrt{2}$ **3.** 2π **5.** $27\sqrt{3}/8$ **7.** $(391\sqrt{17} + 1)/240$
9. About -11.47 **11.** $\frac{364}{3}$ **13.** $12\sqrt{5}$ **15.** 8 **17.** $\sqrt{3}\pi$
19. $32\pi/3$ **21.** 486π **23.** $-\frac{4}{3}$ **25.** $3\pi/2$ **27.** 20π
29. 384π **31.** 0 **33.** Proof **35.** $2\pi a^3 h$ **37.** $64\pi\rho$
39. See Theorem 15.10, "Evaluating a Surface Integral," on page 1112.
41. See "Definition of Flux Integral," on page 1118; see Theorem 15.11, "Evaluating a Flux Integral," on page 1118.
43. (a)



(b) If a normal vector at a point P on the surface is moved around the Möbius strip once, it will point in the opposite direction.

(c)



Circle

(d) Construction
(e) A strip with a double twist that is twice as long as the Möbius strip.

Section 15.7 (page 1130)

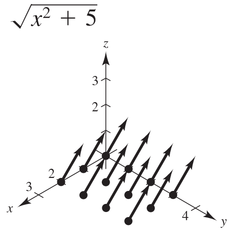
1. a^4 **3.** 18 **5.** π **7.** $3a^4$ **9.** 0 **11.** 108π
13. 0 **15.** 2304 **17.** $1024\pi/3$ **19.** 0
21. See Theorem 15.12, "The Divergence Theorem," on page 1124.
23–29. Proofs

Section 15.8 (page 1137)

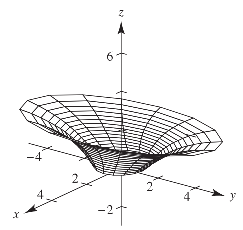
1. $(xz - e^z)\mathbf{i} - (yz + 1)\mathbf{j} - 2\mathbf{k}$ **3.** $[2 - 1/(1 + x^2)]\mathbf{j} - 8x\mathbf{k}$
5. $z(x - 2e^{y^2+z^2})\mathbf{i} - yz\mathbf{j} - 2ye^{x^2+y^2}\mathbf{k}$ **7.** 18π **9.** 0
11. -12 **13.** 2π **15.** 0 **17.** $\frac{8}{3}$ **19.** $a^5/4$ **21.** 0
23. See Theorem 15.13, "Stokes's Theorem," on page 1132.
25–27. Proofs **29.** Putnam Problem A5, 1987

Review Exercises for Chapter 15 (page 1138)

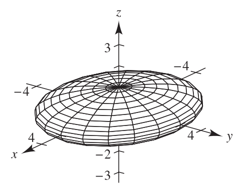
1. $\sqrt{x^2 + 5}$



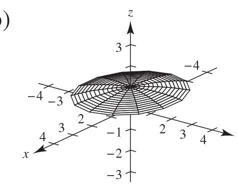
3. $(4x + y)\mathbf{i} + x\mathbf{j} + 2z\mathbf{k}$
5. Conservative: $f(x, y) = y/x + K$
7. Conservative: $f(x, y) = \frac{1}{2}x^2y^2 - \frac{1}{3}x^3 + \frac{1}{3}y^3 + K$
9. Not conservative **11.** Conservative: $f(x, y, z) = x/(yz) + K$
13. (a) $\operatorname{div} \mathbf{F} = 2x + 2xy + x^2$ (b) $\operatorname{curl} \mathbf{F} = -2xz\mathbf{j} + y^2\mathbf{k}$
15. (a) $\operatorname{div} \mathbf{F} = -y\sin x - x\cos y + xy$
(b) $\operatorname{curl} \mathbf{F} = xz\mathbf{i} - yz\mathbf{j}$
17. (a) $\operatorname{div} \mathbf{F} = 1/\sqrt{1 - x^2} + 2xy + 2yz$
(b) $\operatorname{curl} \mathbf{F} = z^2\mathbf{i} + y^2\mathbf{k}$
19. (a) $\operatorname{div} \mathbf{F} = \dfrac{2x + 2y}{x^2 + y^2} + 1$
(b) $\operatorname{curl} \mathbf{F} = \dfrac{2x - 2y}{x^2 + y^2}\mathbf{k}$
21. (a) $\frac{125}{3}$ (b) 2π **23.** 6π **25.** (a) 18 (b) 18π
27. $9a^2/5$ **29.** $(\sqrt{5}/3)(19 - \cos 6) \approx 13.446$ **31.** 1
33. $2\pi^2$ **35.** 36 **37.** $\frac{4}{3}$ **39.** $\frac{8}{3}(3 - 4\sqrt{2}) \approx -7.085$
41. 6 **43.** (a) 15 (b) 15 (c) 15
45. 1 **47.** 0 **49.** 0
51.



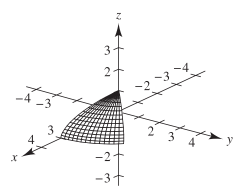
53. (a)



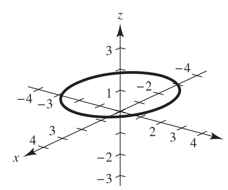
(b)



(c)



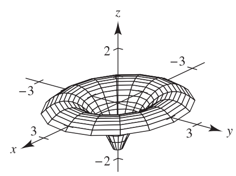
(d)



Circle

(e) About 14.436 (f) About 4.269

55.



0

57. 66 **59.** $2a^6/5$ **61.** Proof

P.S. Problem Solving (page 1141)

1. (a) $(25\sqrt{2}/6)k\pi$ (b) $(25\sqrt{2}/6)k\pi$
3. $I_x = (\sqrt{13}\pi/3)(27 + 32\pi^2)$; $I_y = (\sqrt{13}\pi/3)(27 + 32\pi^2)$;
$I_z = 18\sqrt{13}\pi$
5. Proof **7.** $3a^2\pi$ **9.** (a) 1 (b) $\frac{13}{15}$ (c) $\frac{5}{2}$ **11.** Proof
13. $M = 3mxy(x^2 + y^2)^{-5/2}$
$\partial M/\partial y = 3mx(x^2 - 4y^2)/(x^2 + y^2)^{7/2}$
$N = m(2y^2 - x^2)(x^2 + y^2)^{-5/2}$
$\partial N/\partial x = 3mx(x^2 - 4y^2)/(x^2 + y^2)^{7/2}$
Therefore, $\partial N/\partial x = \partial M/\partial y$ and $\mathbf{F}$ is conservative.