National University of Computer & Emerging Sciences, Karachi
Fall-2020 Department of Computer Science
Mid Term-2
24th November 2020, 10:30 AM – 12:00 PM

Course Code: CS302	**Course Name:** Design and Analysis of Algorithm
Instructor Name / Names: Dr. Muhammad Atif Tahir, Dr. Fahad Sherwani, Zeshan Khan, Waqas Sheikh, Sohail Afzal	
Student Roll No:	**Section:**

Instructions:

- Return the question paper.
- Read each question completely before answering it. There are **6 questions** on **3 pages.**
- In case of any ambiguity, you may make assumption. But your assumption should not contradict any statement in the question paper.

Time: 90 minutes. **Max Marks: 17.5**

Question # 1 **[3 marks]**

Consider the following instance of the 0/1 knapsack problem

items	1	2	3	4
values	3	4	5	6
weights	2	3	4	5

The maximum allowable total weight in the knapsack is W = 5.

Find an optimal solution for the above problem with the weights and values given above using Dynamic Programming. Be sure to state both the maximum value as well as the item(s) that will make maximum value in the given capacity W. Show all steps. [see appendix]

Solution

Step-01:

- Draw a table say 'T' with (n+1) = 4 + 1 = 5 number of rows and (w+1) = 5 + 1 = 6 number of columns.
- Fill all the boxes of 0^{th} row and 0^{th} column with 0.

	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0					
2	0					
3	0					
4	0					

Step-02:

Start filling the table row wise top to bottom from left to right using the formula-

$$T (i , j) = \max \{ T (i-1 , j) , value_i + T(i-1 , j – weight_i) \}$$

Finding T(1,1)-

We have,

- $i = 1$
- $j = 1$
- $(value)_i = (value)_1 = 3$
- $(weight)_i = (weight)_1 = 2$

Substituting the values, we get-

$T(1,1) = \max \{ T(1-1 , 1) , 3 + T(1-1 , 1-2) \}$

$T(1,1) = \max \{ T(0,1) , 3 + T(0,-1) \}$

$T(1,1) = T(0,1)$ { Ignore $T(0,-1)$ }

$T(1,1) = 0$

Finding T(1,2)-

We have,

- $i = 1$
- $j = 2$
- $(value)_i = (value)_1 = 3$
- $(weight)_i = (weight)_1 = 2$

Substituting the values, we get-

T(1,2) = max { T(1-1 , 2) , 3 + T(1-1 , 2-2) }

T(1,2) = max { T(0,2) , 3 + T(0,0) }

T(1,2) = max {0 , 3+0}

T(1,2) = 3

Finding T(1,3)-

We have,

- $i = 1$
- $j = 3$
- $(value)_i = (value)_1 = 3$
- $(weight)_i = (weight)_1 = 2$

Substituting the values, we get-

T(1,3) = max { T(1-1 , 3) , 3 + T(1-1 , 3-2) }

T(1,3) = max { T(0,3) , 3 + T(0,1) }

T(1,3) = max {0 , 3+0}

T(1,3) = 3

Finding T(1,4)-

We have,

- i = 1
- j = 4
- $(value)_i = (value)_1 = 3$
- $(weight)_i = (weight)_1 = 2$

Substituting the values, we get-

T(1,4) = max { T(1-1 , 4) , 3 + T(1-1 , 4-2) }

T(1,4) = max { T(0,4) , 3 + T(0,2) }

T(1,4) = max {0 , 3+0}

T(1,4) = 3

Finding T(1,5)-

We have,

- i = 1
- j = 5
- $(value)_i = (value)_1 = 3$
- $(weight)_i = (weight)_1 = 2$

Substituting the values, we get-

T(1,5) = max { T(1-1 , 5) , 3 + T(1-1 , 5-2) }

T(1,5) = max { T(0,5) , 3 + T(0,3) }

T(1,5) = max {0 , 3+0}

T(1,5) = 3

Finding T(2,1)-

We have,

- i = 2
- j = 1
- $(\text{value})_i = (\text{value})_2 = 4$
- $(\text{weight})_i = (\text{weight})_2 = 3$

Substituting the values, we get-

T(2,1) = max { T(2-1 , 1) , 4 + T(2-1 , 1-3) }

T(2,1) = max { T(1,1) , 4 + T(1,-2) }

T(2,1) = T(1,1)   { Ignore T(1,-2) }

T(2,1) = 0

Finding T(2,2)-

We have,

- i = 2
- j = 2
- $(\text{value})_i = (\text{value})_2 = 4$
- $(\text{weight})_i = (\text{weight})_2 = 3$

Substituting the values, we get-

T(2,2) = max { T(2-1 , 2) , 4 + T(2-1 , 2-3) }

T(2,2) = max { T(1,2) , 4 + T(1,-1) }

T(2,2) = T(1,2)   { Ignore T(1,-1) }

T(2,2) = 3

Finding T(2,3)-

We have,

- $i = 2$
- $j = 3$
- $(value)_i = (value)_2 = 4$
- $(weight)_i = (weight)_2 = 3$

Substituting the values, we get-

T(2,3) = max { T(2-1 , 3) , 4 + T(2-1 , 3-3) }

T(2,3) = max { T(1,3) , 4 + T(1,0) }

T(2,3) = max { 3 , 4+0 }

T(2,3) = 4

Finding T(2,4)-

We have,

- $i = 2$
- $j = 4$
- $(value)_i = (value)_2 = 4$
- $(weight)_i = (weight)_2 = 3$

Substituting the values, we get-

T(2,4) = max { T(2-1 , 4) , 4 + T(2-1 , 4-3) }

T(2,4) = max { T(1,4) , 4 + T(1,1) }

T(2,4) = max { 3 , 4+0 }

T(2,4) = 4

Finding T(2,5)-

We have,

- i = 2
- j = 5
- $(value)_i = (value)_2 = 4$
- $(weight)_i = (weight)_2 = 3$

Substituting the values, we get-

T(2,5) = max { T(2-1 , 5) , 4 + T(2-1 , 5-3) }

T(2,5) = max { T(1,5) , 4 + T(1,2) }

T(2,5) = max { 3 , 4+3 }

T(2,5) = 7

Similarly, compute all the entries.

After all the entries are computed and filled in the table, we get the following table-

	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	0	3	3	3	3
2	0	0	3	4	4	7
3	0	0	3	4	5	7
4	0	0	3	4	5	7

T-Table

- The last entry represents the maximum possible value that can be put into the knapsack.
- So, maximum possible value that can be put into the knapsack = 7.

Identifying Items To Be Put Into Knapsack-

Following Step-04,

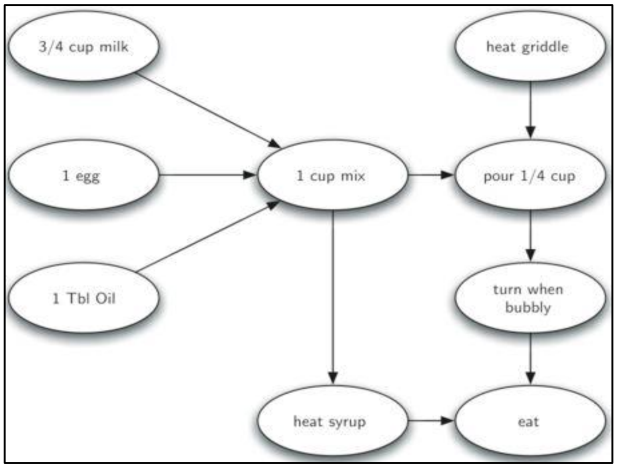
- We mark the rows labelled “1” and “2”.
- Thus, items that must be put into the knapsack to obtain the maximum value 7 are-

Item-1 and Item-2

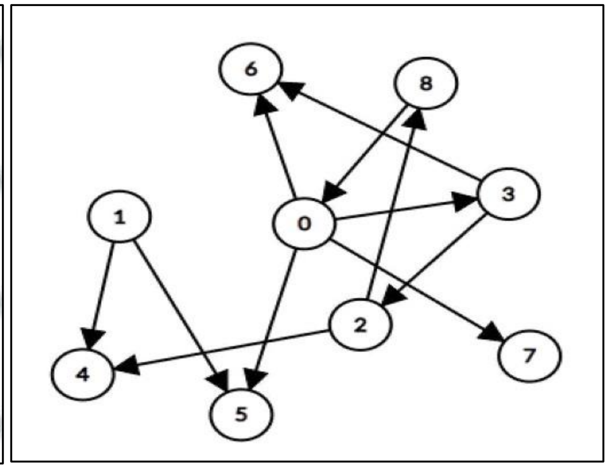
<u>Question # 2</u> **[1+0.5 = 1.5 marks]**

Given two directed graphs in part (a) and part (b), you need to write one possible topological ordering of both graphs.

Part (a)

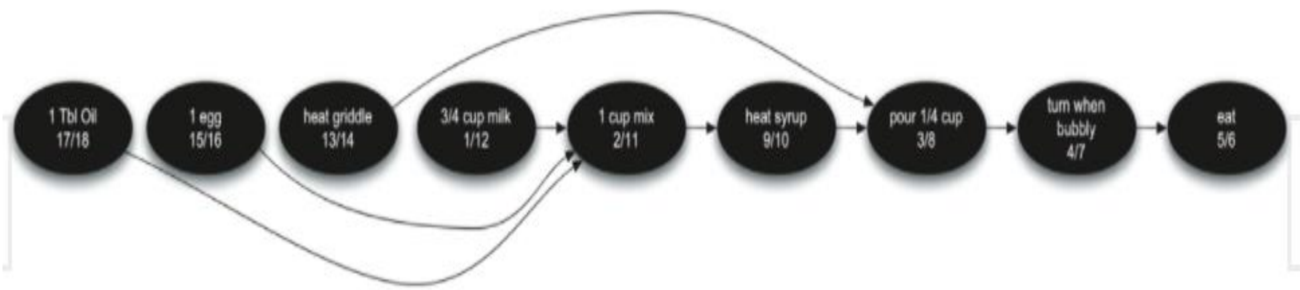


Part (b)



<u>Solution</u>

Part (a):



Part (b):

Not possible as there is cycle among nodes (8,0,3,2)

Question # 3 **[1 + 1 = 2 marks]**

Compute the time complexity for both below mentioned algorithms. Show all steps.

$$
\begin{array}{l}
void\ Algorithm1(A, int\ n)\{ \\
\quad for(int\ i = 0; i < \sqrt{n}; i++)\{ \\
\quad\quad for\ (int\ j = 0;\ j < \sqrt{n}; j++)\{ \\
\quad\quad\quad int\ temp = A[j][i]/A[i][i] \\
\quad\quad\quad for\ (int\ k = i+1; k \le n+i; k++)\{ \\
\quad\quad\quad\quad A[j][k-i] -= temp * A[i][k-i] \\
\quad\quad\quad \} \\
\quad\quad \} \\
\quad \} \\
\}
\end{array}
$$

$$
\begin{array}{l}
void\ Algorithm2(A, int\ n)\{ \\
\quad for(int\ i = 0; i < n/2; i++)\{ \\
\quad\quad for\ (int\ j = 1;\ j \le n/2; j++)\{ \\
\quad\quad\quad for\ (int\ k = i+1; k \le i+5; k++)\{ \\
\quad\quad\quad\quad for\ (int\ l = 1; l \le n; l++)\{ \\
\quad\quad\quad\quad\quad sum += A[i+j][l] \\
\quad\quad\quad\quad \} \\
\quad\quad\quad \} \\
\quad\quad \} \\
\quad \} \\
\}
\end{array}
$$

Solution

Algorithm 1:

$$T(n) = \sum_{i=0}^{\sqrt{n}} \left(2 + \left(\sum_{j=0}^{\sqrt{n}} 2 + \left(\sum_{k=i+1}^{n+i} (3) \right) \right) \right)$$

$$T(n) = \sum_{i=0}^{\sqrt{n}} \left(2 + \left(\sum_{j=0}^{\sqrt{n}} 2 + 3n \right) \right)$$

$$T(n) = \sum_{i=0}^{\sqrt{n}} \left(2 + 2\sqrt{n} + 3n\sqrt{n} + 2 + 3n \right)$$

$$T(n) = \left(4 + 2\sqrt{n} + 3n\sqrt{n} + 3n \right)(\sqrt{n} + 1)$$

$$T(n) = \left(4\sqrt{n} + 2n + 3n^2 + 3n\sqrt{n} + 4 + 2\sqrt{n} + 3n\sqrt{n} + 3n \right)$$

$$T(n) = 3n^2 + 6n\sqrt{n} + 5n + 6\sqrt{n} + 4$$

$$T(n) = \theta(n^2)$$

Algorithms 2:

$$T(n) = \sum_{i=0}^{\frac{n}{2}} \left(2 + \left(\sum_{j=1}^{\frac{n}{2}} 2 + \left(\sum_{k=i+1}^{i+5} 2 + \left(\sum_{l=1}^{n} 3 \right) \right) \right) \right)$$

$$T(n) = \sum_{i=0}^{\frac{n}{2}} \left(2 + \left(\sum_{j=1}^{\frac{n}{2}} 2 + \left(\sum_{k=i+1}^{i+5} 2 + 3n \right) \right) \right)$$

$$T(n) = \sum_{i=0}^{\frac{n}{2}} \left(2 + \left(\sum_{j=1}^{\frac{n}{2}} 2 + 10 + 6n \right) \right)$$

$$T(n) = \sum_{i=0}^{\frac{n}{2}} \left(2 + \left(\sum_{j=1}^{\frac{n}{2}} 12 + 6n \right) \right)$$

$$T(n) = \sum_{i=0}^{\frac{n}{2}} (2 + 6n + 3n^2)$$

$$T(n) = (n + 3n^2 + \frac{3}{2}n^3 + 2 + 6n + 3n^2)$$

$$T(n) = (\frac{3}{2}n^3 + 6n^2 + 7n + 2)$$

$$T(n) = \theta(n^3)$$

Question # 4 **[3 marks]**

Let A and B be two sequences of *"n"* integers each, in the range $[1, n^2]$, from square series. Given an integer x, design an O(n)-time algorithm for determining if there is an integer "a" in A and an integer "b" in B such that $x = a + b$. (You may write algorithm in plain text)

Solution

Sort both a and b by count sort altered version

```
countingSort(array):

  size = len(array)

  output = [0] * size

  count = [0] * 10

  for i in range(0, size):

          count[sqrt(array[i])] += 1//Modification

  for i in range(1, 10):

          count[i] += count[i - 1]

  i = size – 1

  while i >= 0:

          output[count[array[i]] - 1] = array[i]

          count[array[i]] -= 1

          i -= 1

  for i in range(0, size):

          array[i] = (output[i]* output[i]) //Modification
```

Then apply the following linear algorithm

While(a<A.size and b>0):

```
        if(A[a] + B[b] < x) -> update index a to be a + 1
        if(A[a] + B[b] > x) -> update index b to be b - 1
        if(A[a] + B[b] = x) -> success
```

Question # 5 **[4 marks]**

Given a graph *G* and a minimum spanning tree *T*, suppose that we decrease the weight of one of the edges not in *T*. Give an algorithm for finding the minimum spanning tree in the modified graph. (You may write algorithm in plain text)

Solution

If we were to add in this newly decreased edge to the given tree, we would be creating a cycle. Then, if we were to remove any one of the edges along this cycle, we would still have a spanning tree. This means that we look at all the weights along this cycle formed by adding in the decreased edge, and remove the edge in the cycle of maximum weight. This does exactly what we want since we could only possibly want to add in the single decreased edge, and then, from there we change the graph back to a tree in the way that makes its total weight minimized.

Question # 6 **[0.5 + 0.5 + 3 = 4 marks]**

Answer the following.

a) How do we decide to split the matrix-chain and parenthesize for the optimization of multiplications operations? (0.5 marks)

Solution (a)

The matrix-chain problem can be solved recursively to determine the best value of k, we will consider all possible values of k, and pick the best of them to split the matrix-chain accordingly, providing optimal multiplication operations for each subproblem. Parenthesizing can be done in the form of (A[1…k])+(A[k+1…..n]).

b) Which of the following is the recurrence relation for the matrix-chain multiplication problem where mat[i-1] * mat[i] gives the dimension of the i^{th} matrix? (0.5 marks)

1) M[i, j] = 1 if i=j

   M[i, j] = min{ M[i, k] + M[k+1, j]}

2) M [i, j] = 0 if i=j

M [i, j] = min{M[i, k] + M[k+1, j]}

3) M [i, j] = 0 if i=j

M [i, j] = max{M[i, k] + M[k+1, j]} + mat[i-1]*mat[k]*mat[j]

4) M [i, j] = 0 if i=j

M [i, j] = min{M[i, k] + M[k+1, j]} + mat[i-1]*mat[k]*mat[j]

Solution (b)

4) M[i,j] = 0 if i=j

M[i,j] = min{M[i, k] + M[k+1, j]} + mat[i-1]*mat[k]*mat[j]

c) Design a **recursive algorithm** to compute the minimum number of scalar multiplications for the chain matrix product $A_{i...j}$ in a **top-down manner**. You can also write the algorithm in English. (3 marks)

Solution (c)

```
Rec-Matrix-Chain(array p, int i, int j) {
    if (i = = j) m[i, i] = 0;                // basic case
    else {
        m[i, j] = infinity;                  // initialize
        for k = i to j − 1 do {              // try all possible splits
            cost=Rec-Matrix-Chain(p, i, k) + Rec-Matrix-Chain(p, k + 1, j) + p[i − 1]*p[k]*p[j];
            if (cost<m[i, j]) then
                m[i, j]= cost;
            }
        }                                    // update if better
    return m[i,j];                           // return final cost
}
```

BEST OF LUCK