Conditional Probability:

Date: 1-Oct 2021

$$P(B|A) = \frac{P(A \cap B)}{P(A)} \quad \text{if } P(A) > 0$$

OR

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \quad \text{if } P(B) > 0$$

For Independent Event

$$P(B/A) = P(B)$$

$$P(A|B) = P(A)$$

PROBLEM: 1 Suppose that we have portable battery box of a robot containing 20 batteries, of which 5 are defective. If 2 batteries are selected at random and removed from the box in succession without replacing the 1st, What is the prob that both batteries are defective?

Date:

Solution: Let A be the event that the first battery is defective and B be the event that the second battery is defective. Hence

$$P(A \cap B) = P(A)\,P(B) = \left(\frac{5}{20}\right)\left(\frac{4}{19}\right)$$

$$= 1/_{19} \text{ Ans}$$

PROBLEM 2: A ROBOT is working in a cognitive Environment where there are two bags. Bag 1 contains 4 White Markers and 3 Black Markers. Bag 2 contains 3 White Markers and 5 Black Markers. One marker is drawn from the 1st bag and placed in the second bag by the Robot (Unseen). What is the Prob that a marker now drawn from the second bag is black?

Solution:

Solution :-   Date: ______

Let B_1, B_2 and W_1 represents resp.

B_1 the drawing of a black marker from Bag 1

B_2 " " " " Bag 2

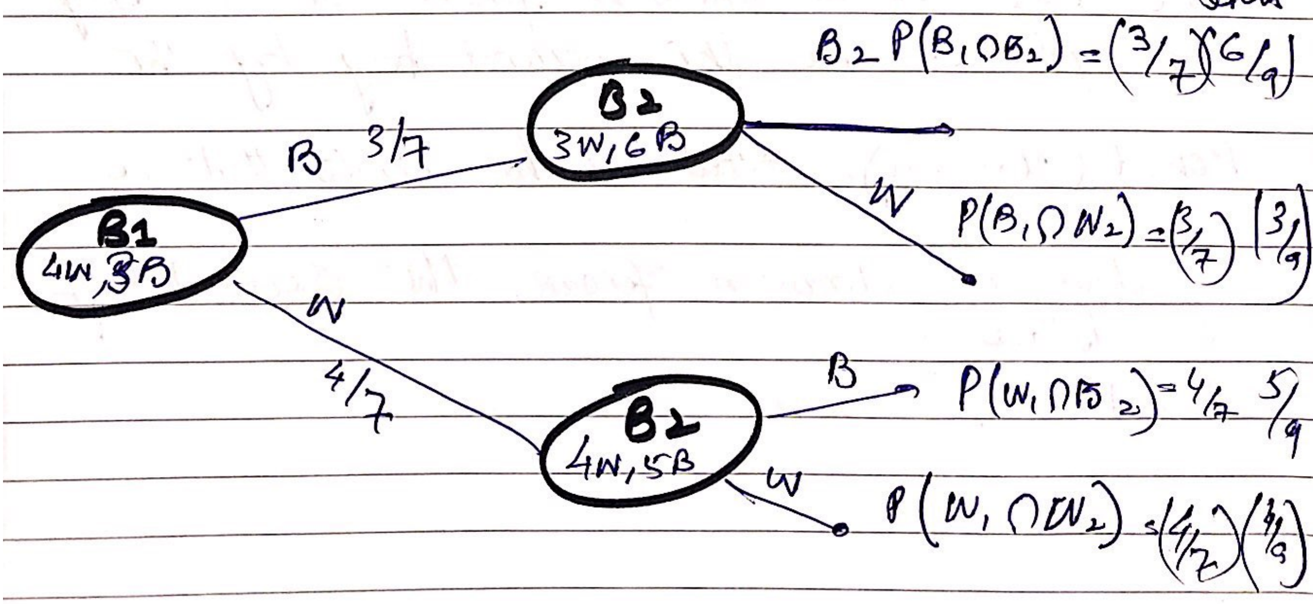
W_1 " " white " Bag 1

$$P\left[(B_1 \cap B_2) \text{ or } (W_1 \cap B_2)\right] = P(B_1 \cap B_2) + P(W_1 \cap B_2)$$

$$= P(B_1)\,P(B_2/B_1) + P(W_1)\,P(B_2/W_1)$$

$$= \left(\frac{3}{7}\right)\left(\frac{6}{9}\right) + \left(\frac{4}{7}\right)\left(\frac{5}{9}\right) = \frac{38}{63}$$

Ans



Baye's Theorem :

We have,

$$P(A/B) = \frac{P(A \cap B)}{P(B)} \longrightarrow (A)$$

$$P(B/A) = \frac{P(A \cap B)}{P(A)} \longrightarrow (B)$$

$$P(A/B)\,P(B) = P(B/A)\,P(A)$$

$$\underbrace{P(A/B)}_{\text{Posterior}} = \frac{\underbrace{P(B/A)}_{\text{Likelihood}}\;\overbrace{P(A)}^{\text{Prior}}}{\underbrace{P(B)}_{\text{Marginal}}}$$

Random Variables and Probability Distributions

Date:

$S = \{00 \quad 01 \quad 10 \quad 11\}$

X is a random variable that is associated with the no of 1 occuring

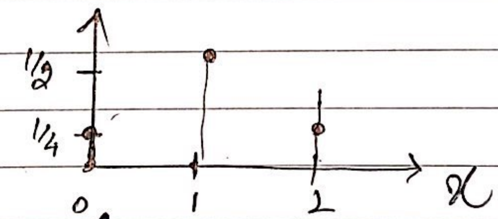
$X = 0, 1, 2$

$f(x) = f(0) = 1/4$

$f(1) = 2/4 = 1/2$

$f(2) = 1/4$

x	0	1	2
$f(x)$	$1/4$	$1/2$	$1/4$



So we define new $(x, f(x))$

① $f(x) \geq 0$

② $\sum_{x} f(x) = 1$

③ $P(X = x) = f(x)$

PROBLEM:

$S = \{000, 010, 110, 100, 001, 011, 111, 101\}$

Let X be a r.v that denote the no of ones in a QAM symbol. Find the pdf.

Sol: $X = 0, 1, 2, 3$

Hence

$$f(0) = P(X=0) = 1/8 = P(\{000\})$$

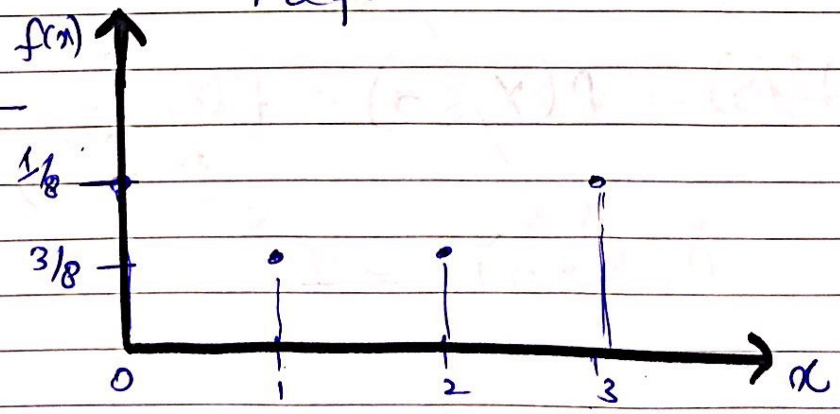
$$f(1) = P(X=1) = 1/8 + 1/8 + 1/8 = 3/8$$

$\vdots$

x	0	1	2	3
$f(x)$	$1/8$	$3/8$	$3/8$	$1/8$

$\underbrace{\qquad\qquad\qquad\qquad}_{\text{pdf}}$

Now



Date:

$F(x) = P(X \leq x) = \sum_{t \leq x} f(t)$

$x < 0 \quad = P(X < 0) = 0$

$F(0) = P(X \leq 0) = 1/8$

$F(1) = P(X \leq 1) = f(0) + f(1)$

$= 1/8 + 3/8$

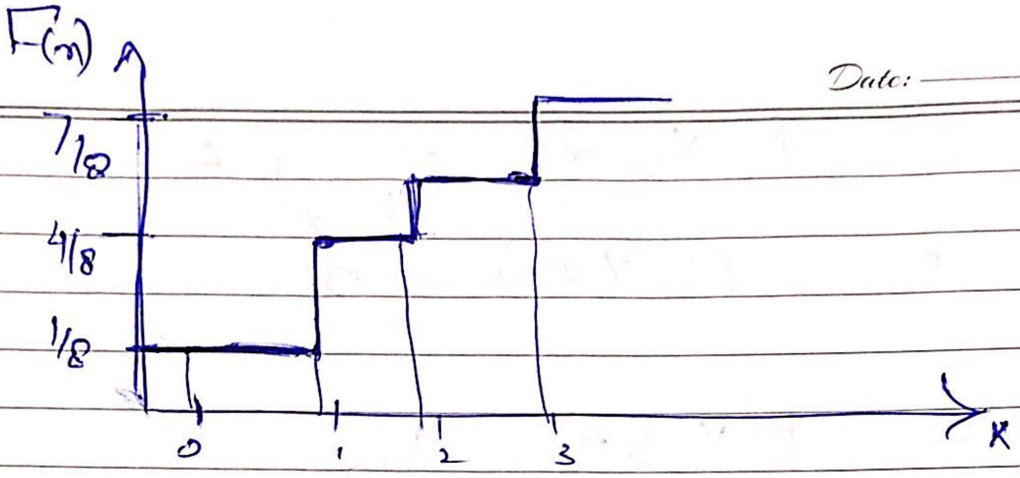
$F(1) \quad = 4/8$

$F(2) = P(X \leq 2) = 1/8 + 3/8 + 3/8 = 7/8$

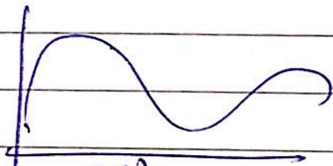
$F(3) = P(X \leq 3) = 1/8 + 3/8 + 3/8 + 1/8 = 1$

$$F(x) = \begin{cases} 0 & x < 0 \\ 1/8 & 0 \leq x < 1 \\ 4/8 & 1 \leq x < 2 \\ 7/8 & 2 \leq x < 3 \\ 1 & x \geq 3 \end{cases}$$

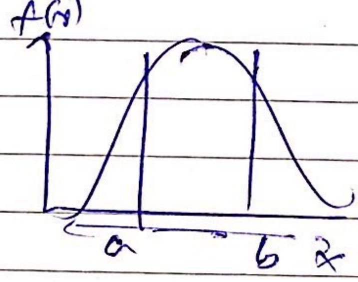
Date:

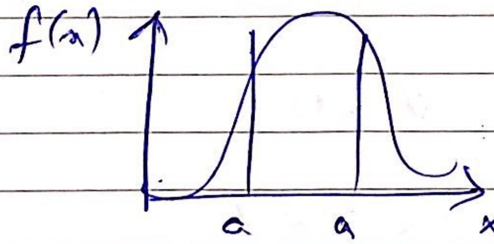


Continous P. Distribution



$\propto \quad P(a < x < b) = \int_a^b f(x)\,dx$





(1) $f(x) \geq 0$ (2) $\int_{-\infty}^{\infty} f(x)\,dx = 1$

(3) $P(a < x < b) = \int_a^b f(x)\,dx$

Cumulative Distribution

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(t)\,dt \qquad -\infty < x < \infty$$

Problem:

$$f(x) = \begin{cases} x^2/3 & -1 < x < 2 \\ 0 & \text{else where} \end{cases}$$

① Verify c2.

$$= \int_{-\infty}^{\infty} \frac{x^2}{3}\,dx = \int_{-1}^{2} \frac{x^2}{3}\,dx$$

$$= \frac{x^3}{9}\Big|_{-1}^{2} = \frac{1}{9}\left[8+1\right] = 1$$

② Find $P(0 < X \leq 1)$

$$= \int_{0}^{1} x^2/3\,dx = \frac{x^3}{9}\Big|_{0}^{1} = \frac{1}{9}\left[1+0\right]$$

$$= 1/9 \text{ Ans}$$

* Find $F(x)$ of the Previous example:-

$$F(x) = P(X \le x) = \int_{-\infty}^{x} f(t)\,dt$$

$$F(x) = \int_{-\infty}^{x} \frac{t^2}{3}\,dt = \frac{x^3+1}{9}$$

$$F(x) = \begin{cases} 0 & x < -1 \\ \frac{x^3+1}{9} & -1 < x < 2 \\ 1 & x \ge 2 \end{cases}$$

Find $P(0 < X \le 1) = F(1) - F(0)$

$$= \frac{2}{9} - \frac{1}{9} = \frac{1}{9}$$

Joint P. D.:-

Discrete:-

① $f(x, y) \geq 0$

② $\sum_x \sum_y f(x, y) = 1$

③ $P(X = x, Y = y) = f(x, y)$

Continuous:-

① $f(x, y) \geq 0$

② $\int_x \int_y f(x, y)\, dx\, dy = 1$

③ $P(X = x, Y = y) = f(x, y)$

PROBLEM, 16

Date:

X = no. of ① HL Lines and

Y = no of O_{SB} Line. Find JPD

$$f(x,y) = \frac{\binom{3}{x}\binom{2}{y}\binom{3}{2-x-y}}{\binom{8}{2}}$$

If $x=0$ and $y=0$

$$= \frac{\binom{3}{0}\binom{2}{0}\binom{3}{2}}{\binom{8}{2}} = \frac{\frac{3!}{3!}\times\frac{2!}{2!}\times\frac{3!}{1!}}{\frac{8!}{6!}}$$

$$= \frac{1\times1\times3\times2\times\cancel{6\times5\times4\times3\times2\times1}}{\cancel{8}^{4}\times7\times\cancel{6\times5\times4\times3\times2}}$$

$$= \frac{3}{28}$$

$f(x,y)$		x: 0	1	2	Total
y	0	3/28	9/28	3/28	15/28
	1	3/14	3/14	0	3/7
	2	1/28	0	0	1/28
Total		5/14	15/28	3/28	1

Problem:

$$f(x,y) = \begin{cases} \frac{2}{5}(2x+3y) & 0 \le x \le 1,\ 0 \le y \le 1 \\ 0 & \text{elsewhere} \end{cases}$$

Proof

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$$

$$= \cancel{\frac{2}{5}\iint(2x+3y)\,dx\,dy}$$

$$= \cancel{\frac{4x}{5}\,dx\,dy + \frac{6}{5}y \int \frac{4x}{5}\,dx} + \int 3y\,dy$$

$$= \frac{2}{5}\iint (2x+3y)\,dx\,dy$$

$$= \iint \left(\frac{4x}{5} + \frac{6y}{5}\right)dx\,dy$$

$$= \cancel{\int \frac{2}{5}x^2 + \frac{3}{5}y^2}$$

$$= \int_0^1 \left(\frac{2x^2}{5}\Big|_0^1 + \frac{6xy}{5}\Big|_0^1\right) dy$$

$$= \int_0^1 \left(\frac{2}{5} + \frac{6y}{5}\right) dy$$

$$= \frac{2}{5}\,y\Big|_0^1 + \frac{3}{5}\,y^2\Big|_0^1$$

$$= \frac{2}{5} + \frac{3}{5} = 1$$

Date:

Marginal Distributions:

$$g(x) = \sum_{y} f(x, y)$$

$$h(y) = \sum_{x} f(x, y)$$

$$g(0) = \sum_{y} f(x, y)$$

$$= \sum_{0}^{2} f(x, y) = \sum_{0}^{2} f(0, y)$$

$$g(0) = f(0,0) + f(0,1) + f(0,2)$$

$g(1) =$

$g(2) =$

x	0	1	2
$g(x)$			

y	0	1	2
$h(y)$			

$$g(x) = \int_{-\infty}^{\infty} f(x, y)\, dy = \int_{0}^{1} \frac{2}{5}(2x + 3y)\, dy$$

$$= \frac{4}{5} xy \Big|_0^1 + \frac{\cancel{6}}{5} \frac{y^2}{2} \Big|_0^1$$

$$= \frac{4x}{5} + 3/5 = \frac{4x+3}{5}$$

Scanned with CamScanner

Let X and Y be two R.V.

The condition P. distribution is given as.

$$f(y/x) = \frac{f(x,y)}{g(x)} \qquad g(x) > 0$$

$$f(x/y) = \frac{f(x,y)}{h(y)} \qquad h(y) > 0$$

$f(x/y=1)$?) $f(x/y) = \frac{f(x,y)}{h(y)}$

$$f(x/y=1) = \frac{f(x,1)}{h(1)} = \frac{f(x,1)}{3/7}$$

$$f(0/1) = f(0,1)/3/7 = 7/3 \times 3/28$$

$$f(1/1) = f(1,1)/3/7$$

$$f(2/1) = f(2,1)/3/7$$

Mean and R.V

Date:

X r.v with p.d f(x). μ as expected value of X is

$\rightarrow \mu = E(x) = \sum_{x} x f(x)$

$\rightarrow \mu = E(x) = \int_{-\infty}^{\infty} x f(x)$

$\Rightarrow f(x) = \begin{cases} \frac{20,000}{x^3} & x > 100 \\ 0 & \text{elsewhere} \end{cases}$

Find μ

$f(x) = \int_{100}^{\infty} \frac{20,000}{x^3} dx$

$= 20,000 \int x^{-3} dx \quad 20,000 \left. \frac{x^{-3+1}}{-3+1} \right|_{100}^{\infty}$

$= \frac{20,000}{-2} \left[\frac{1}{x^2} \right]$

$= \frac{\cancel{20,000}^{10000}}{-\cancel{2}} \left[\frac{1}{\infty} - \frac{1}{100^2} \right]$

$= 1000 \times \frac{1}{100 \times 100}$

$\frac{1}{x^3}$ x^{-3} x^{-3}

$\frac{x^{-3+1}}{-3+1} = \frac{x^{-2}}{-2}$

$$= \int_{100}^{\infty} \frac{20,000}{x^2} \, dx$$

$$= -20,000 \left[+\frac{1}{x^2} \Big|_{100}^{\infty} \right]$$

$$= 20,000 \left[\frac{1}{\infty} + \frac{1}{100 \times 100} \right]$$

$$= \frac{20,000}{100 \times 100} = 2000$$

$$= \int x \frac{20,000}{x^3} \quad \int \frac{20,000}{x^2} \, dx$$

$$= \frac{x^{-2+1}}{-2+1} \Rightarrow -20,000 \left[\frac{1}{x^2} \Big|_{100}^{\infty} \right]$$

$$= -20,000 \left[\frac{-1}{100 \times 100} \right]$$

$$= 200$$

Let X be R.V $f(x)$ Date:

μ of $g(X)$ $\mu g(x) = E[g(X)]$

$= \sum g(x) f(x)$

$\mu g(x) = E[g(x)] = \int_{\pm\infty} g(x) f(x) dx$

x	4	5	6	7	8	9
$P(X=x)$	$1/12$	$1/12$	$1/2$			

$g(x) = 2x - 1$. Find $\mu g(x)$

$\mu g(x) = E[g(x)] = E(2x-1)$

$= \sum_{4}^{9} (2x-1) f(x)$

$= (2 \times 2 - 1) 1/12 + 9 \; 1/12$

X r.v $f(x)$ Date:

Variance of X is

$$\text{Vari} - \sigma^2 = E[(X-\mu)^2] = \sum_x (x-\mu)^2 f(x)$$

$$\sigma^2 = E[(x-\mu)^2] = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\,dx$$

σ = STd

x	1	2	3
$f(x)$	0.3	0.4	0.3

Find σ^2

$$\mu = \sum x f(x) = 1\times 0.3 + 2\times 0.4 + 3\times 0.3$$

$$\mu = 2.0$$

$$\sigma^2 = \sum (x-\mu)^2 f(x) = (1-2)^2\, 0.3 + (2-2)^2\, 0.4 + (3-2)^2 (0.3)$$

$$\sigma^2 = 0.6$$

$$\boxed{\sigma^2 = E(X^2) - \mu^2}$$

$$\sigma^2 = E(X^2) - \mu^2$$

$$\mu = \sum x f(x) = E(x)$$

Date:

$$\sigma^2 = \sum_x (x-\mu)^2 f(x)$$

$$= \sum \left(x^2 - 2x\mu + \mu^2\right) f(x)$$

$$= \sum_x x^2 f(x) - 2\sum_x x\mu f(x) + \sum_x \mu^2 f(x)$$

$$= \sum_x x^2 f(x) - 2\mu^2 + \mu^2$$

$$= \boxed{E(x^2) - \mu^2}$$

Variance:

$$\boxed{\sigma^2 = \sum_x (X-\mu)^2 f(x)}$$

$$\sigma^2 = E[(x-\mu)^2]$$

Vari $\rightarrow$ $\boxed{\sigma^2 = E[x^2] - \mu^2}$

Std $\rightarrow \sigma$

Date:

Normal Distribution

$P(a < X < b)$

C.R.V

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

Std

$$= \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left[\frac{(x-\mu)}{\sigma}\right]^2}$$

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

μ

σ

$X \sim N(\mu, \sigma^2)$

$-\infty < x < \infty$

$-\infty < \mu < \infty$

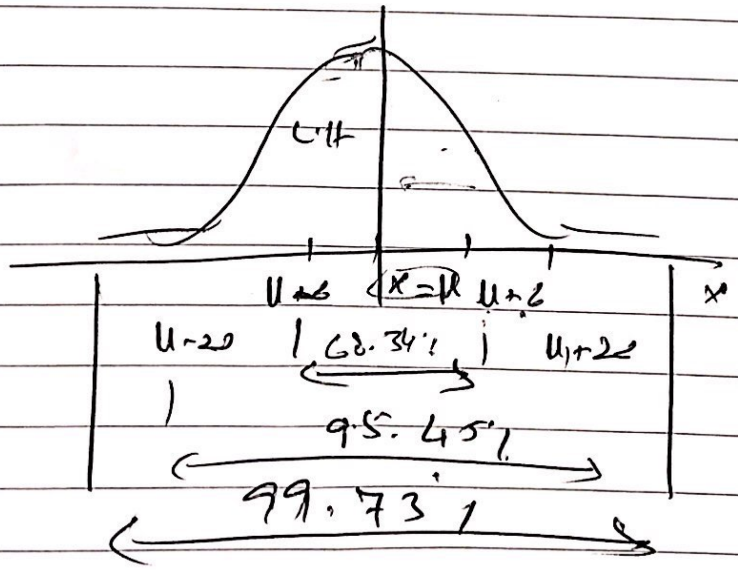
$\Leftarrow \sigma^2 \geq 0$

≥ 0

μ

σ^2

$\sigma^2 \geq 0$



Date:

Standard Normal Distribution;

Let Z R.V standard N.D

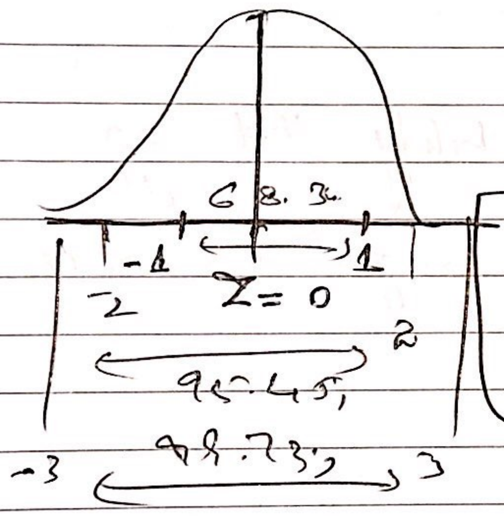
$$Z = \frac{x-\mu}{\sigma} \qquad Z \sim SN(0, 1)$$

$$-\infty < Z < \infty$$

$\mu = 0$ $\qquad x = \mu + \sigma$

$\qquad\qquad\; x = \mu - \sigma$

$\sigma^2 = 1$ $\qquad 0 - 1$



$$\boxed{f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} z^2}}$$

$P(a < x < b)$

$X \sim N(\mu, \sigma^2) \longrightarrow P(z_1 < Z < z_2)$

$Z \sim SN(0, 1)$

$$z_1 = \frac{a - \mu}{\sigma}$$

$$z_2 = \frac{b - \mu}{\sigma}$$

PROBLEM

Date:

Sensor

(SA)

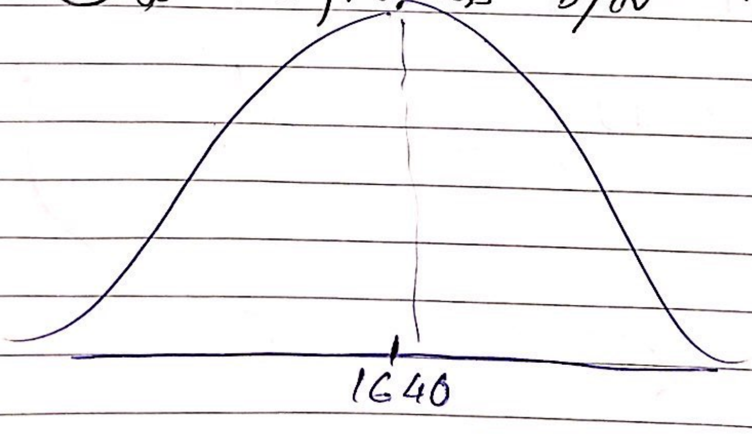
A 100-watt light bulb has an average brightness of 1640 lumens with a standard deviation of 62 lumens.

Find

(a) What is the probability that 100-watt light bulb will have a brightness more than 1800 lumens.

(b) " " " " brightness less than 1550 lumens

(c) What is the probability bulb will have a brightness b/w 1600 + 1700



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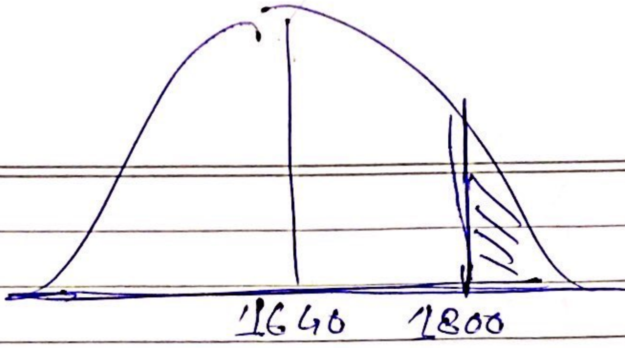


$$z_s = \frac{X-\mu}{\sigma} = \frac{1800-1640}{62}$$

$$Z = 2.58$$

$P(X \geq 1800)$

$P(Z \geq 2.5) = ??$ $\cancel{Z < 2.5}$

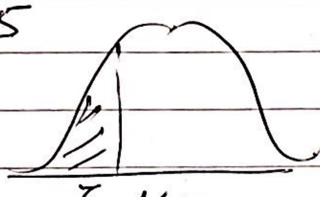
$= 0.9938$ $\cancel{0.9951}$

$= \cancel{1 - 0.9938}$

$= 0.0049 = \boxed{0.0049}$ A

(b) $P(X < 1550)$ $P(Z < -1.45)$

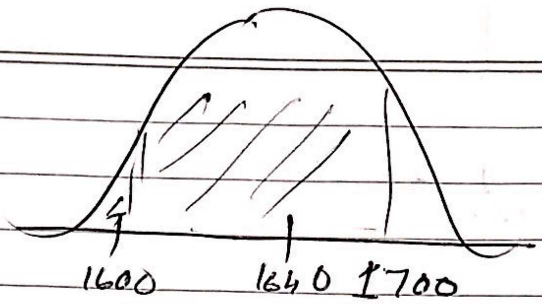
$$Z = \frac{1550-1640}{62} = -1.45$$



$P(Z < -1.45) = 0.0735$

Date:

(a)



$$Z_2 = \frac{1700 - 1640}{62} = 0.96 = 0.8315$$

$$Z_1 = \frac{1600 - 1640}{62} = -0.645 = 0.2611$$

$$P(Z_1 < Z < Z_2) = 0.8315 - 0.2611$$

$$= 0.5704$$

Date: 8th oct 2021

Normal Distribution

$a < x < b$ → continou

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

μ

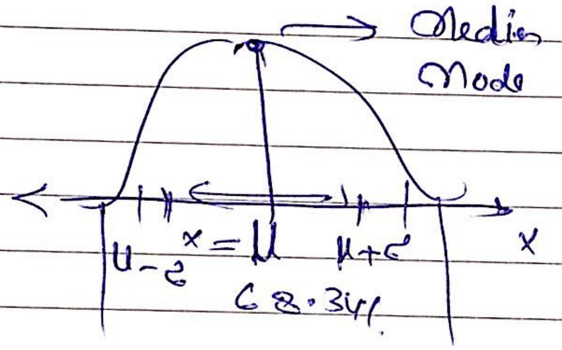
σ

$X \sim N(\mu, \sigma^2)$

$-\infty < \mu < \infty$

$-\infty < x < \infty$

$\sigma^2 > 0$



σ^2

$\mu + 1\sigma$

$= 0 + 1$

95.45%

$\mu - 3\sigma$ 99.73 $\mu + 3\sigma$

To solve a problem we do normalization

Let Z be a R.V SRID $Z \sim N(0, 1)$

Stand → Normal variable

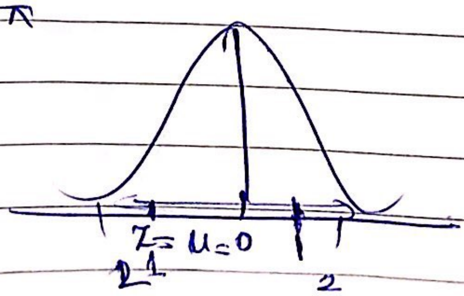
Hence $Z = \frac{x - \mu}{\sigma}$ → Mean, → Vari

$$f(z) = \frac{1}{1\sqrt{2\pi}} e^{-\frac{1}{2}(Z)^2}$$

$\infty < Z < \infty$

$\mu = 0$

$\sigma^2 = 1$



Date: 9th Oct 2021

Normal Distribution

$a < x < b$ → continou

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

μ

σ

$X \sim N(\mu, \sigma^2)$

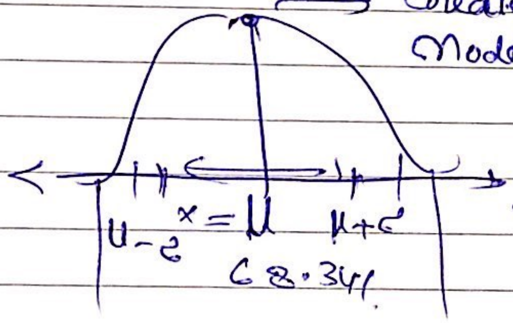
$-\infty < \mu < \infty$

$-\infty < x < \infty$

$\sigma^2 > 0$

→ Median

Mode



σ^2

$\mu + 1\sigma$

$= 0 + 1$

95.45%

$\mu - 3\sigma$ 99.73 $\mu + 3\sigma$

To solve a problem we do normalization

Let Z be a R.V SRND $Z \sim N(0, 1)$

Stand → Normal variable

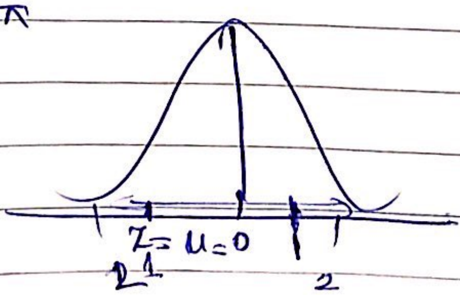
Hence $Z = \frac{x - \mu}{\sigma}$ → Mean, → vari

$$f(z) = \frac{1}{1\sqrt{2\pi}} e^{-\frac{1}{2}(Z)^2}$$

$\infty < Z < \infty$

$\mu = 0$

$\sigma^2 = 1$



1th Oct 2021

Date:

Step to solve question

(1) convert x into Z $\quad Z = \frac{x-\mu}{\sigma}$

$x = a \Rightarrow \frac{a-\mu}{\sigma}$

$x = b \Rightarrow \frac{b-\mu}{\sigma}$

$P(a < x < b)$

$P\left(\frac{a-\mu}{\sigma} < z < \frac{b-\mu}{\sigma}\right)$

tinou

$< \infty$

$< \infty$

> 0

σ

$N(0, 1)$

le

$< \infty$

$= 0$

$= 1$

$\cancel{X} = \{ x_1 \; x_2 \; x_3 - x_3 \}_1$

X, Y, Z

Date:

$f(x) \quad X = 0, 1, 2, 3$

Stochastic $\quad f(x) = \frac{1}{2} \; \frac{1}{4} \; \frac{1}{2} \; \frac{1}{1}$

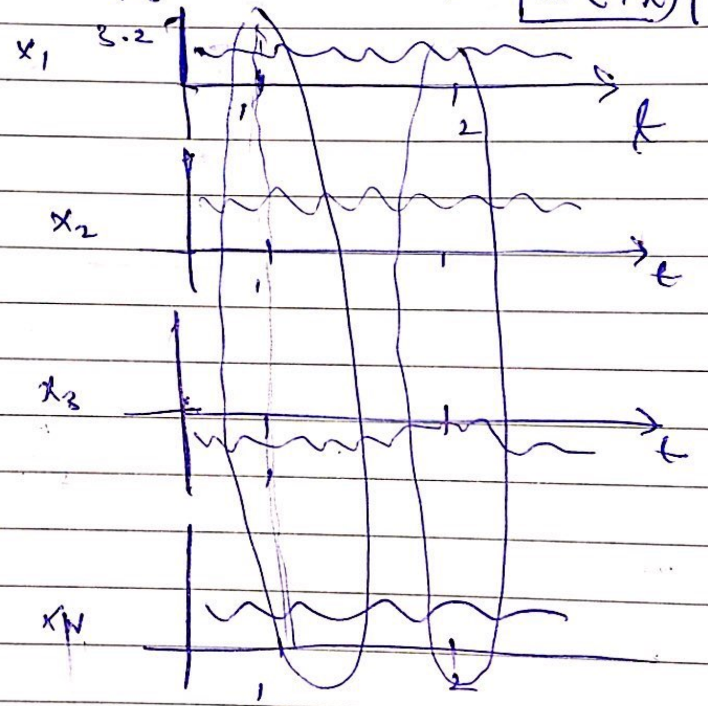
Voltmeter $\quad$ Sample Osculloscope $\boxed{X(x, t)}$

1 Sensor $= 3.2$

2 $= 4.2$

3 $= -1.5$

$= \int x f(x) dx$



$X(1) \qquad X(2)$

$X(t_1 \qquad X(t_2)$

X – R.V

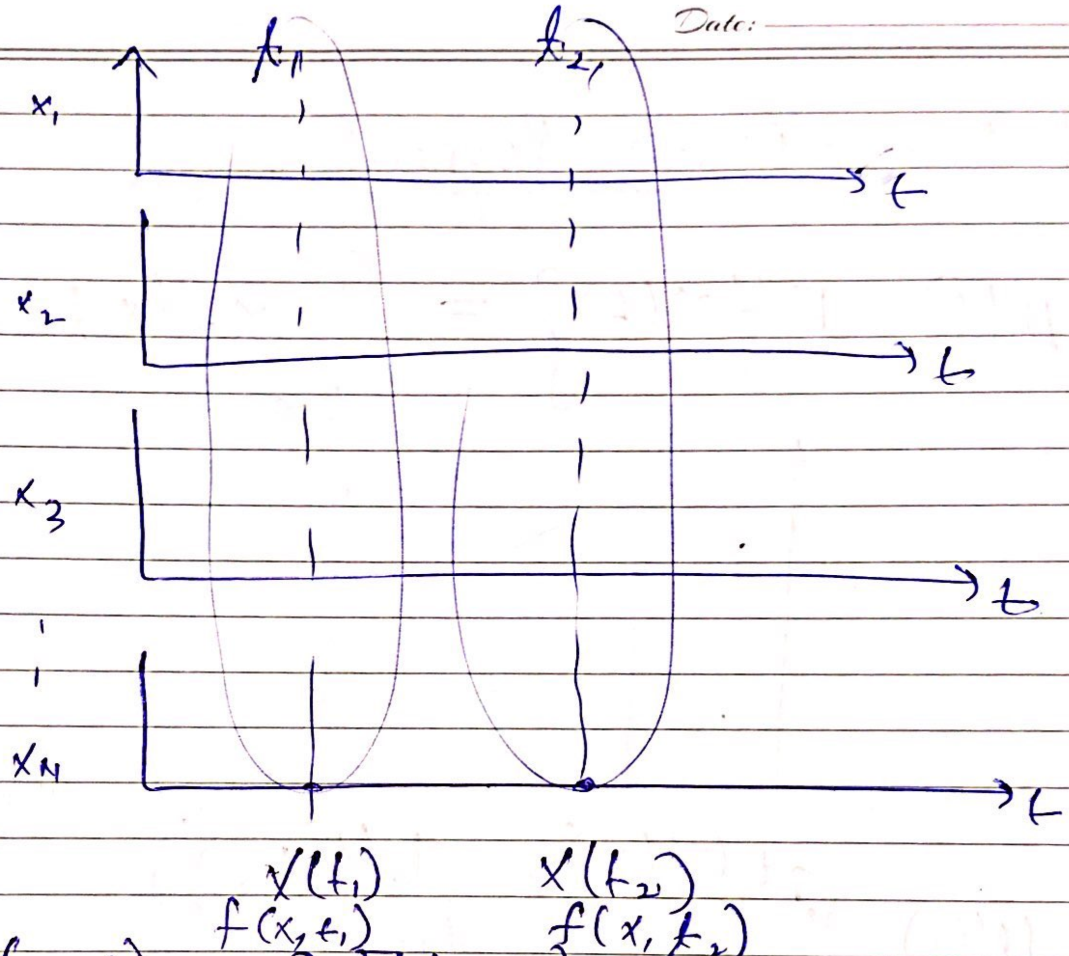
$X = \{ x_1 \; x_2 \; x_3 \; x_n - x_N \}_t$

$X_t \Rightarrow f(x)$

$f(x, t)$

$X_t \Rightarrow f(x, t)$

Ensemble function — All possible time f



$$f(x,t) = \frac{\partial F(x,t)}{\partial x}$$

Joint PDF

$$f(x_1,x_2;\, t_1,t_2) = \frac{\partial F(x_1,x_2,t_1,t_2)}{\partial x_1\,\partial x_2}$$

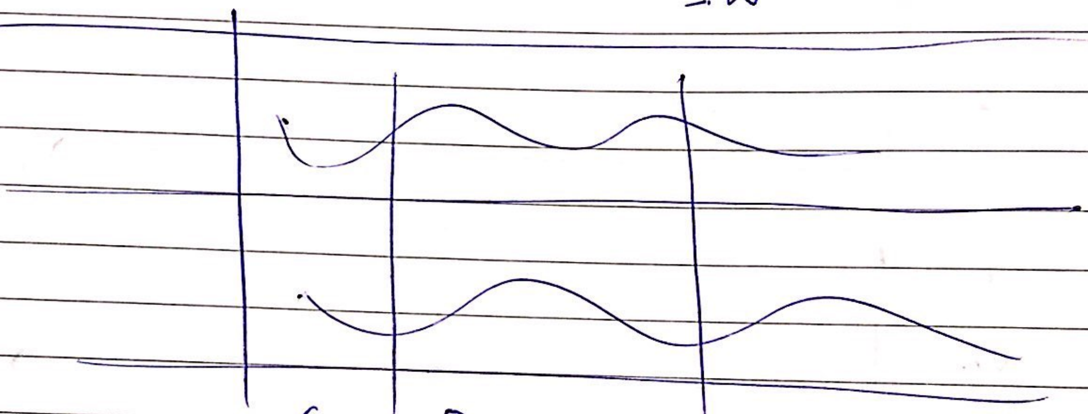
$x(t) = \cos(w_0 t + \Phi)$ $\quad \Phi \quad -\pi \rightarrow \pi$

$x(t) = e^{-\gamma t}$

Date:

First ORDER Moment:

$$\boxed{\mu(t) = E\{X(t)\} = \int_{\pm\infty} x f(x,t)\, dx}$$



$\mu(t_1)$ $E(X(t_1))$ $E(X(t_2))$

$\mu(t_2)$

2ND Order Moment:

$X(t_1)$ and $X(t_2)$

$$R(t_1, t_2) = E\{X(t_1)\, X(t_2)\}$$

$$= \int_{\pm\infty}\int_{\pm\infty} x_1 x_2\, f(x_1, x_2; t_1, t_2)\, dx_1\, dx_2$$

Date:

Wide Sense Stationary

$$\mu(t) = \mu \longrightarrow \text{constant}$$

$$\rightarrow R_{xx}(t_1, t_2) = R_{xx}(\tau)$$

x —— x

Date:

Random Process: $X \leq x(s,t)$ $\quad X_1 = X(t_1)$

$$f(x_1;t_1) = \frac{\partial F_X(x_1;t_1)}{\partial x_1}$$

$$f(x_1,x_2 - x_n; t_1,t_2 - t_n) = f_X(x_1;t_1)\, f_X(x_2,t_2) \ldots$$

Mean: $X(t)$ $\quad \bar{X}(t) = E[X(t)] = \int_{\pm} x f_X(x;t)\,dt$

Autocorrelation $X(t)$ $\quad X_1$ & X_2 two R.V

t_1 t_2 $\quad f(x_1\, x_2; t_2\, t_1)$

$$E[x_1 x_2] = \cancel{E X(x_1)}\; E[X(t_1)X(t_2)]$$

$$R_{XX}(t_1,t_2) = E[X_1 X_2] = E[X(t_1)\, X(t_2)]$$

$$R_{XX}(t_1,t_2) = \iint x_1 x_2\, f_X(x_1,x_2; t_1, t_2)\, dx_1\, dx_2$$

Cross: $X(t)$ $\quad Y(t)$

$$R_{XY}(t_1,t_2) = \iint xy\, f_{XY}(x,y;t_1,t_2)\,dx\,dy$$

Stationary $\quad f_X(x_1;t_1) = f_X(x_1; t_1 + \Delta t)$

E'($\quad E[X(t)] = \bar{X} = \text{constant}$

$$f_X(x_1,x_2;t_1,t_2) = f_X(x_1,x_2; t_1+\Delta t, t_2+\Delta t)$$

$$R_{XX}(t_1,t_2) = E[X(t_1)X(t_2)] \qquad \tau = t_2 - t_1$$

Date:

$R_{xx}(t_1, t_2) = E[X(t_1)X(t_2)]$

$R_{xx}(t_1, t_1+\tau) = E[X(t_1)\,X(t_1+\tau)]$

$= R_{xx}(\tau) \cdot R_{xx}(\tau)$

$E[X(t)] = \bar{X} = \text{constant}$ $E[X(t)\,X(t+\tau)]$
$= R_{xx}(\tau)$

$f(x_1, x_2 — x_N; t_1, t_2 — t_N) = f(x_1, x_2 — x_N; t_1+\tau$

SSS is WSS

$X(t)$ $\tilde{x} = A[x(t)] = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} x(t)\,dt$

$x(t)$ $\tau = t_2 - t_1$ $x(t)$

① $X(t) = E[X^2(t)] = R_{xx}(0)$

$X(t)$, $R_{xx}(\tau) = E[X(t)\,X(t+\tau)]$ $\tau = 0$,
$R_{xx}(0) = E[X(t)\,X(t)] = E[X^2(t)]$

② $|R_{xx}(\tau)| \le R_{xx}(0)$

$R_{xx}(-\tau) = R_{xx}(\tau)$ $E[X(t)\,X(t+\tau)]$
$\tau = -\tau$

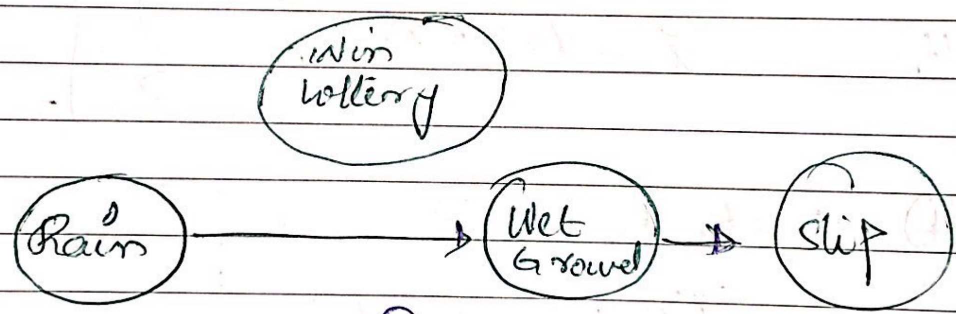
Bayesian N/Ws..

Date:

Probabilistic Graphical Models..

→ (PGMs) represent probability distributions
→ They encode conditional independence structure with graphs
→ They enable graph algorithms for Inference and Learning.

Bayesian N/Ws



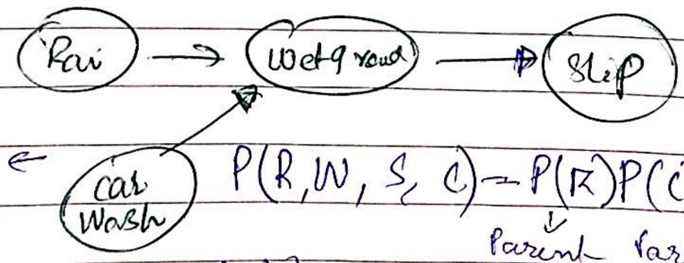
Joint Prob distribution $P(L, R, W)$

$$P(L, R, W) = P(L)P(R)P(W|R)$$

$$P(L, R, W, S) = P(L)P(R)P(W|R)\,P(S|W)$$

or $$= P(L)\,P(R)\,P(W|R)\,P(S|W, R)$$

Parent ←



Parent ←

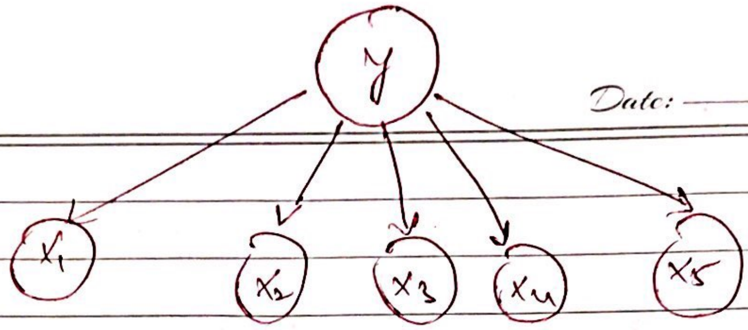
$$P(R, W, S, C) = P(R)P(C)P(W|C, R)P(S|W)$$

Parent Parents

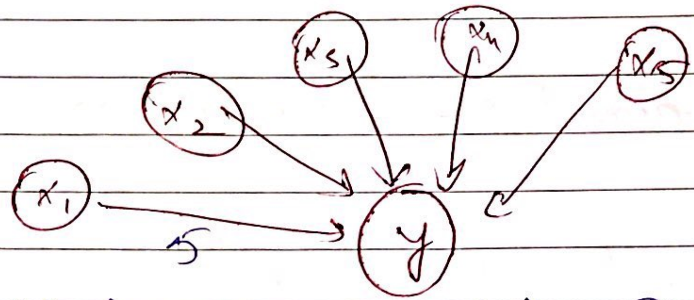
$$P(X \mid \text{Parent}(X))$$

Date:

(A)



(B)



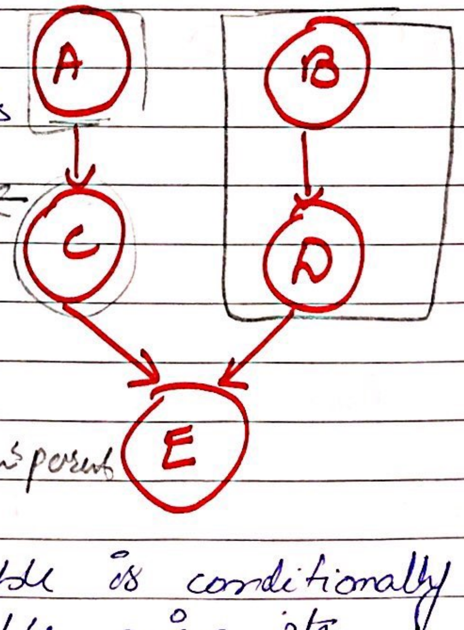
(A) $\rightarrow P(Y) \prod_{i=1} P(X_i / Y)$

(B) $P(Y / x_1, x_2, x_3, x_4, x_5) \prod_{i=1}^{5} P(x_i)$

↳ Logistic Regression (with i/p Likelihood)

Independence in Bayes Nets:

Observe A

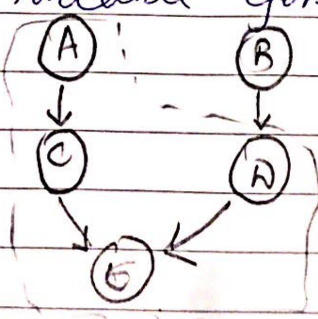


Each variable is conditionally independent of its non-descendants given its parents. consider.

E depends on C B and D are independent of C

Parents, children and children's parents

→ Markov Blanket → Each variable is conditionally independent of any other variable given its M. Blanket.



Node C has a Markov Blanket in which B is independent of C.

Inference:

Date:

JPD Given a Bayesian Ntw describing
→ $P(X, Y, Z)$. What is $P(Y)$?

(1) First Approach:- Enumeration

$$P(R, W, S, C) = P(R)P(C)\,P(W|C, R)P(S|W)$$

Let says $P(R|S) = ??$

$$P(r|s) = \sum_{w}\sum_{c} P(r, w, s, c) \Big/ P(s)$$

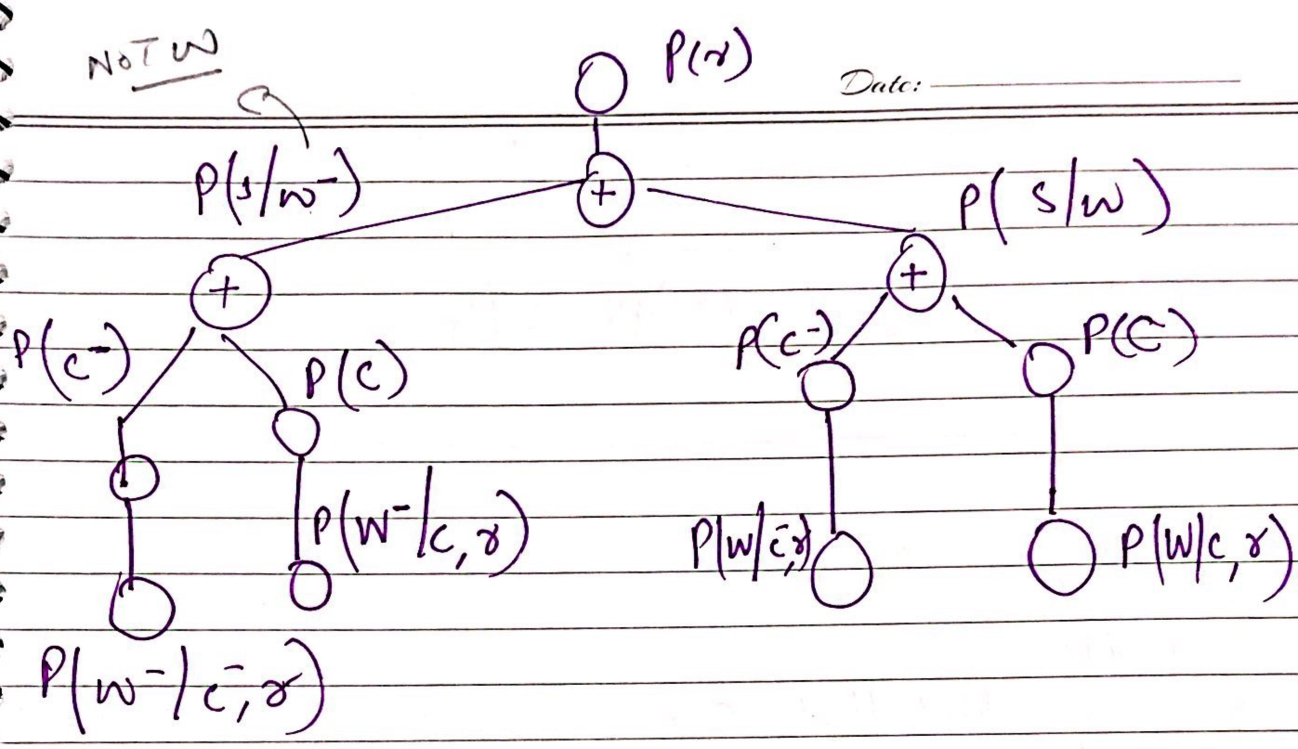
$$\boxed{P(r|s) \propto \sum_{w}\sum_{c} P(r)P(c)\,P(w|c, r)P(s|w)} \quad \text{-(A)}$$

This is expensive, we have to enumerate overall values of $P(W)\ P(C)$. We can use BSW to solve this problem.

From eq (A), $P(R)\ P(C)$ can be taken out,

$$P(r|s) \propto P(r)\sum_{w} P(s|w) \sum_{c} P(c)P(w|c, r)$$

Still expensive $O(2^n)$



2) Second Approach :- Variable Elimination

$$P(r/s) \propto \sum_{w} \sum_{c} P(r) \boxed{P(c)\, P(w/c, r)}\, P(s/w)$$

Let $\quad f_c(w) = \sum_{c} P(c)\, P(w/c, r)$

$$\boxed{P(r/s) \propto \sum_{W} P(r)\, P(s/w)\, f_c(w)}$$

Lets,

$$P(W, X, Y, Z) = P(w)\, P(x/w)\, P(Y/x)\, P(Z/Y)$$

$P(Y) = ??$

$$P(Y) = \sum_{W} \sum_{x} \sum_{Z} P(w)\, P(x/W)\, P(Y/x)\, P(Z/Y)$$

Let

$$f_w(x) = \sum_{W} P(w)\, P(x/w) \quad \text{Eliminated } W \text{ Variable}$$

$$\boxed{P(Y) = \sum_{x} \sum_{Z} f_w(x)\, P(Y/x)\, P(Z/Y)}$$

Date:

$$f_X(Y) = \sum_X f_W(X) P(Y|X)$$

$$\boxed{P(Y) = \sum_Z f_X(Y) P(Z|Y)}$$

— x x x —

Examples:

Installed a new Burglar alarm installed at Home.

→ It is fairly reliable at detecting burglary but also sometimes responds to minor earthquakes

→ You have two neighbors, John and Merry, who promised to call you at work when they hear the alarm.

→ John always calls when he hears the alarm but sometimes confuses telephone ringing with the alarm and calls too.

→ Merry likes loud music and sometimes misses the alarm.

Given the Evidence of who has or has not called, we'd like to estimate the prob of a burglary.

→ No Burglary $1 - .001 =$

Burglary

P(B)
.001

Date:

Earthquake

P(E)
.002

$= 1 - .002$

Alarm

B E	P(A/B,E)
T T	.95
T F	.94
F T	.29
F F	.001

Johncalls

Marycalls

A	P(J/A)	P(J'/A)
T	.90	1 − .90
F	.05	P(J'/A') = 1 − .05

A	P(M/A)	P(M'/A)
T	.70	1 − .70
F	.01	P(M'/A')
		1 − .01

① What is the Prob that the alarm has sounded but neither a Burglary nor an Earthquake has occured and both John and Merry call?

Solution

$$P(\underbrace{J \cap M}_{\text{depends on alarm}} \cap \underbrace{A}_{\text{depends on B and E}} \cap \underbrace{B' \cap E'}_{\text{Independent}})$$

$$= \boxed{P(j/a)\, P(m/a)\, P(a/b^-, e^-)\, P(b^-)\, P(e^-)}$$

$$= (0.90) \times (0.70)\,(0.001) \times (0.999)(0.998)$$

$$= \boxed{0.00062}$$

② What is the probability that John call?

$$P(J) = P(J/a)\,P(a) + P(J/a^-)\,P(a^-) \rightarrow P(b)P(E)$$

$$= P(J/a)\{\underline{P(a/b,e)}^{0.95} * P(b,e) + \underline{P(a/b^-,e)} * P(b^-,e)$$
$$+ \underline{P(a/b,e^-)} * P(b,e^-) + P(a/b^-,e^-) * P(b^-,e^-)\}$$

$+$

Date:

$$+ P(j/a^-)\{P(a^-/b,e) * P(b,e) + P(a^-/b^-,e) * P(b^-,e) + P(a^-/b,e^-) * P(b,e^-) + P(a^-/b^-,e^-) * P(b^-,e^-)\}$$

$$= 0.90 * 0.00252 + 0.05 * 0.9974$$

$$= \boxed{0.0521}$$

Keypoints:

→ Bayesian Ntws are directed acyclic graphs that represent dependencies b/w variables in a Prob Models.

→ Many time series models such HMM and KF are dynamic B.N.

→ BNs are used for solving such problems with uncertainty.

Dynamic Bayesian Networks

Date:

$$Y_1 \longrightarrow Y_2 \longrightarrow Y_3 \longrightarrow \cdots \longrightarrow Y_T$$

Fig: A BN represent a 1st order Markov Process

In time series modeling, we observe the values of certain variables at different point in time.

→ The assumption that an event can cause another event in the future but not vice-versa, simplifies the design of BN for time series.

→ Assigning a time index t to each variable, one of the simplest causal models for a sequence of data $\{Y_1, \dots, Y_T\}$ is a first order Markov Model in which each variable is directly influenced only by the previous variable

$$P(Y_1, Y_2 \dots Y_T) = P(Y_1)\,P(Y_2|Y_1) \dots (Y_T | Y_{T-1})$$

→ These models don't directly represent dependencies b/w observables over more than one time step. Having observed $\{Y_1 \dots Y_t\}$, the model will only make use of Y_t to predict Y_{t+1}.

→ One simple way of extending Markov models is to allow higher order interactions b/w variables. e.g. τ^{th}-order Markov model allows arcs from $\{Y_{t-\tau}, \dots Y_{t-1}\}$ to Y_t.

→ Another way to extend Markov models is to

Date:

posit that the observations are dependent on a hidden variable, which we call the state, and that the sequence of states is a Markov process.

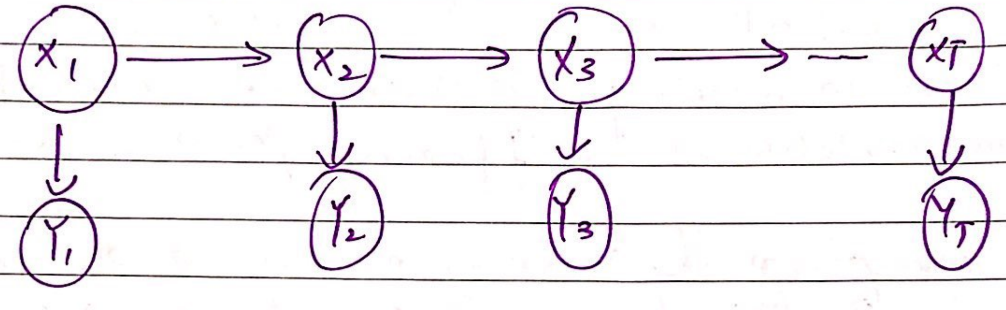


Fig:- A Bayesian NTW specifying conditional independence relations for a state-space model.

* Example 1: State-Space Models:

→ In a state-space Models, a sequence of D-dimensional real-valued observation vectors $\{Y_1 \ldots Y_T\}$, is modeled by assuming that at each time step Y_t was generated from a K-dimensional real-valued hidden state variable X_t, and that the sequence of X's define a first order Markov process.

→ Using $\{Y_t\}$ to denote sequences from $t=1$ to $t=T$,

$$P(\{X_t, Y_t\}) = P(X_1)P(Y_1|X_1)\prod_{t=2}^{T} P(X_t|X_{t-1})P(Y_t|X_t) \rightarrow (1)$$

Date:

$P(X_t / X_{t-1})$ State Transition Probability

and can be written as:

$$X_t = f_t(X_{t-1}) + w_t$$

f_t is the deterministic transition function determining the mean of X_t given X_{t-1} and w_t is a zero-mean random noise vector.

$P(Y_t / X_t)$ Observation Prob. and can be written as,

$$Y_t = g_t(X_t) + v_t$$

If both transition and observation functions are linear and time-invariant and the w_k and v_k is zero-mean random means Gaussian, the model becomes a Linear-Gaussian State-space Model:

$$X_t = A X_{t-1} + W_t \longrightarrow (2)$$

$$Y_t = C X_t + v_t \longrightarrow (3)$$

A is the state T. Matrix and C is the observation Matrix.

- After the observation can be divided into a set of input (Predictor) variables and o/p (Responses) variables. Again, Assume Linearity and Gaussian noise, we can write (2) as

$$X_t = A X_{t-1} + B U_t + W_t \rightarrow (4)$$

Date:

→ U_k is the i/p observation vector and B is the i/p matrix.

– The B. N ~~corresponding to this model.~~

* Learning B.N:

→ It is often the case we don't know c.d throughout the NTW

→ To solve this problem, we deploy some kind of learning algorithms.

→ The role of learning is to adjust the parameters of the B.N so that pdf ~~the of~~ defined by the NTW sufficiently describes statistical behaviour of the observed data.

Let M be a parametric BN model with parameter Θ of the Prob distribution defined by the model. Let $Pr(M)$ and $Pr(\Theta/M)$ be the prior distributions over the set of models and the space of parameters in those models respectively.

→ Given some observed data assumed to have been generated by the model, as defined by the goal of learning in B.N. We have to estimate the model parameters Θ, such that the posterior prob of the model given data Z_L becomes maximized.

Date:

Hence,

$$Pr(M|Z_L) = \frac{Pr(M)}{Pr(Z_L)} \int_{\theta} Pr(Z_L|\theta, M)\, Pr(\theta|M)\, d\theta \rightarrow (A)$$

We assumed that $Pr(\theta/M)$ is peaked around maximum likelihood (ML) estimates of those parameters. We can transform eq (A) into:

$$Pr(M|Z_L) \simeq \frac{Pr(M)\, Pr(Z_L|\theta_{ML}, M)\, Pr(\theta_{ML}|M)}{Pr(Z_L)} \rightarrow (B)$$

where the maximum likelihood estimate θ_{ML} for a given model M is obtained from max term:

$$\theta_{ML} = \arg\max_{\theta} \log Pr(Z_L|\theta) \rightarrow (C)$$

→ We can even consider a case where not all of the variables Z in the model of a BN M are observed (represented by X). We can describe the goal of learning in such problem as:

$$\hat{\theta} = \arg\max_{\theta} \log \sum_{X} P(Y, X|\theta) \rightarrow (D)$$

P denotes joint pdfs defined by the NW. We can minimize the cost fn as;

$$J(\theta) = -\log \sum_{X} P(Y, X|\theta) \rightarrow (E)$$

Date:

To min the cost we can use an optimisation technique that rely on the gradient of the cost fn or we can use an iterative procedure for the optimisation called Expectation-Maximization (EM) algorithm or variance of that procedure known as Generalized (EM) etc.

Dynamic B.N:

→ DBN special case of singly connected BN specially aimed at time series modeling.

→ Variables in DBN exhibits temporal dimension.

→ The states of DBN satisfy Markovian condition

⊛ The state of a system at time t depends only on its immediate past t-1.

→ We can describe DBN consisting of P.d on the sequence of T-hidden-state variables

$X = \{x_0 \ldots x_{T-1}\}$ and the sequence of T observable variable $Y = \{y_0, \ldots y_{T-1}\}$

where T is the Time boundary for the given event we're investigating. We can express:

$$Pr(X, Y) = \prod_{t=1}^{T-1} Pr(x_t / x_{t-1}) \prod_{t=0}^{T-1} Pr(y_t / x_t) Pr(x_0)$$

→ (1)

To complety Defining DBN Date:
we need three parameters:

(1) State Transition $Pr(x_t | x_{t-1})$ (2) Observation $Pr(y_t | x_t)$

(3) Initial state distribution $Pr(x_0)$: {Initial Prob dis in the beginning of the process.