

Notes and Solved Problems for Common Exam 2

3. GAUSS LAW

Key concepts: Gaussian Surface, Flux, Enclosed Charge

Gauss Law is equivalent to the Coulomb law but sometimes more useful. Gauss law considers a flux of an electric field thru some hypothetical surface (called a Gaussian surface) The flux of the field E thru the surface S is formally an integral

$$\Phi = \int \vec{E} d\vec{S}$$

The Gauss law relates the flux of the electric field thru the Gaussian surface with the total charge enclosed by this surface

$$\Phi = \int \vec{E} d\vec{S} = \frac{q_{total}}{\epsilon_0}$$

The usefulness of the Gauss law is seen when calculating electric fields near symmetrical objects. For these situations, the electric field can for example be a constant on the surface of the integration and can be taken out of the integral defined above. The integral can then often be done easily (it is just the area of the Gaussian surface) and one can immediately find an expression for the electric field on the surface.

The units of electric flux: Nm^2 / C

Gauss's Law helps us understand the behavior of electric fields inside the conductors. Since the conductors are the objects where the electrons move freely the electric field must be zero everywhere inside the conductor. The reason: if a conducting object is placed in a non-zero external electric field the charge inside the conductor will move towards its surface until the electric field created by this displaced charge completely compensates the external electric field. The total electric field inside the conductor is therefore zero. If that were not so, free charges would move due to the field and make it so.

The Gauss law also helps us understand the distribution of electric charge placed into a conductor. Since the electric field inside every conductor is always zero, there cannot be extra electric charge located inside of one. Therefore, any extra charge placed into the conductor will move until it is distributed only on the surface of the conductor. This can be proven by considering any Gaussian surface lying completely inside the conductor, since electric field inside the conductor is zero, there is no charge which is enclosed inside this Gaussian surface.

Typical problems related to Gauss Law:

Problem 12. Find the flux through a spherical Gaussian surface of radius $a = 1$ m surrounding a charge of 8.85 pC.

- A) $1 \times 10^{-16} \text{ N m}^2/\text{C}$
- B) $1 \times 10^{-12} \text{ N m}^2/\text{C}$
- C) $1 \times 10^{-8} \text{ N m}^2/\text{C}$
- D) $1 \times 10^{-4} \text{ N m}^2/\text{C}$
- E) $1 \text{ N m}^2/\text{C}$



Solution. The flux thru the Gaussian surface is the charge located inside the surface. Therefore: $\Phi = q / \epsilon_0 = 8.85 \times 10^{-12} / 8.85 \times 10^{-12} = 1 \text{ Nm}^2 / \text{C}$
 Answer E

Problem 13. A positive charge $Q = 8 \text{ mC}$ is placed inside the cavity of a neutral spherical conducting shell with an inner radius a and an outer radius b . Find the charges induced at the inner and outer surfaces of the shell.

- A) Inner charge = -8 mC , Outer charge = $+8 \text{ mC}$
- B) Inner charge = $+8 \text{ mC}$, Outer charge = -8 mC
- C) Inner charge = 0 mC , Outer charge = $+8 \text{ mC}$
- D) Inner charge = -8 mC , Outer charge = 0 mC
- E) Inner charge = 0 mC , Outer charge = 0 mC



Solution. Inside any conductor the electric field must equal zero. Therefore, the electric field created by the external charge inside the shell is cancelled by electric fields created by the induced charged on the inner and outer surfaces of the shell. Choosing the Gaussian surface to enclose the external charge and the inner surface of the shell and applying the Gauss law: $\Phi = \int E dS = 0 = Q + q_{in}$

since the electric field inside the conductor is zero. This gives us $q_{in} = -Q = -8 \text{ mC}$
 Since the conducting shell is electrically neutral, the charge induced on the inner part of the shell has been taken from the outer surface of the shell which implies $q_{in} + q_{out} = 0 \Rightarrow q_{out} = 8 \text{ mC}$
 Answer A.

Problem 14. A positive charge $Q = 8 \text{ mC}$ is placed inside a spherical conducting shell with inner radius a and outer radius b which has an extra charge of 4 mC placed somewhere on it. When all motion of charges ends (after 10^{-15} sec), find the charges on the inner and outer surfaces of the shell.

- A) Inner charge = -8 mC , Outer charge = $+8 \text{ mC}$
- B) Inner charge = $+8 \text{ mC}$, Outer charge = -8 mC
- C) Inner charge = $+8 \text{ mC}$, Outer charge = -12 mC
- D) Inner charge = -8 mC , Outer charge = 12 mC
- E) Inner charge = -4 mC , Outer charge = $+4 \text{ mC}$



Solution. Inside any conductor the electric field equals zero. Therefore, the electric field created by the external charge inside the shell is compensated by

the electric fields created by the induced charges on the inner and outer surfaces of the shell. Choosing the Gaussian surface to enclose the external charge and

the inner surface of the shell and applying the Gauss law: $\Phi = \int E dS = 0 = Q + q_{in}$

since the electric field inside the conductor is zero. This gives us $Q = -q_{in} = -8mC$

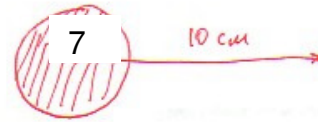
The conducting shell here is not electrically neutral, but has some extra charge 4 mC. Since net charge has to remain constant (none of it can escape), the charge on the outer surface of the shell equals the net charge plus a charge equal in magnitude to that on the inner surface but of the opposite sign. The

condition is $q_{in} + q_{out} = 4mC \Rightarrow q_{out} = 12mC$

Answer D.

Problem 15. Find the value of the electric field at a distance $r = 10$ cm from the center of a non conducting sphere of radius $R = 1$ cm which has an extra positive charge equal to 7 C uniformly distributed within the volume of the sphere.

- A) 6.3×10^{12} N/C
- B) 7.5×10^{-6} N/C
- C) 1.2×10^0 N/C
- D) 9.1×10^{-3} N/C
- E) 8.5×10^2 N/C



Solution. If the charge at the sphere is distributed uniformly, for $r > R$ the electric field created by it is the same as the electric field of the same amount of point charge located at the center of the sphere (Shell Theorem)..

$$E = k \frac{q}{r^2} = 8.99 \times 10^9 \frac{7}{(10 \times 10^{-2})^2} = 6.3 \times 10^{12} \text{ N/C}$$

Answer A.

Problem 16. A positive charge is placed inside a spherical metallic shell with inner radius a and outer radius b . The charge is placed at shifted position relative to the center of the shell. Describe the charge distribution induced at the shell surfaces.

A) A negative charge with uniform surface density will be induced on the inner surface, a positive charge will be induced on the outer surface.

B) A negative charge with non-uniform surface density will be induced on the inner surface, a positive charge will be induced on the outer surface.

C) A positive charge with uniform surface density will be induced on the inner surface, a negative charge will be induced on the outer surface.

D) A positive charge with non-uniform surface density will be induced on the inner surface, a negative charge will be induced at the outer surface.



E) A negative charge with uniform surface density will be induced on the inner surface, a positive charge will be induced on the outer surface.

Solution. First, since the external charge placed inside has a positive sign, the inner surface will be charged negatively, and the outer surface will be charged positively. There is no charge induced in the bulk of the metallic shell. Second since the external charge is not precisely at the center of the shell, the induced charge density on the inner surface will not be uniform.

Answer B.

4. ELECTRIC POTENTIAL

Key concepts:

Just as we visualize charged objects creating an electric field around them, they also create a potential field. The potential field or equivalently electric potential is directly related to the electric field:

$$V(x) = -\int E(x)dx$$

$$E(x) = -\frac{dV}{dx}$$

Therefore the electric field can be represented either by specifying the electric field itself as a function of r , or by specifying the electric potential at each point r .

In some problems the concept of electric potential may be more useful than the concept of electric field since the potential is a scalar function (scalar defined for each point of the space) while the electric field is a vector function (vector defined for each point of the space)

For example, the electric potential of a point charge located at the origin ($r=0$) as a function of r is given by

$$V(r) = k \frac{q}{r}$$

It is easy to see that the derivative of this function with respect to r and multiplied by -1 gives the electric field of the point charge

$$E(r) = k \frac{q}{r^2}$$

Note that the potential is defined with respect to some arbitrary constant since adding a constant to the potential does not influence its derivative with respect to r . We can for example write $V(r) = k \frac{q}{r} + V_0$ which will produce the same electric field. Usually the constant is chosen in such a way that the electric potential is equal zero when r goes to infinity. For the point charge this will produce $V_0 = 0$ in the example above.

Another important example is the potential created by an infinite charged plate. The electric field created by the plate having surface density σ is given by $E = \sigma / 2\epsilon_0$, i.e. it remains constant and does not depend on the distance. The electric potential of this plate is an integral with respect to x , and therefore: $V(x) = -Ex + \text{const.}$ where x is the distance from the plate to the point where we measure the potential. Note that the potential increases as x goes to infinity, therefore in this example it is not possible to fix *const.* to a value that makes the potential at infinity equal to zero. However if we are interested in the potential **difference** between two points, say x_1 and x_2 , the constant will drop out:

$$V(x_1) - V(x_2) = -E(x_1 - x_2) = -\frac{\sigma}{2\epsilon_0}(x_1 - x_2)$$

The potential can be visualized by drawing so called equipotential surfaces (surfaces of the same potential). For example, for the point charge, these surfaces are the spheres centered at $r=0$ since at all points of the surface, r is

the same, therefore $V(r) = k \frac{q}{r}$ is the same, which produces an equipotential

surface. For the potential created by the infinite plate, the equipotential surfaces are the planes parallel to the charged plate since the potential is

$V(x) = -Ex + \text{const.}$ and all points separated by the distance x from the plate form a plane surface.

Note that the equipotential surfaces are always **perpendicular** to the electric field lines. For example, the electric field lines for a point charge are lines originating radially from the origin, while equipotential surfaces are spheres centered at the origin. For an infinite charged plate, the electric field lines are the lines perpendicular to the plate while equipotential surfaces are the planes parallel to the plate.

Note also that it costs **zero energy** to transfer charge between any two points on an equipotential surface: Since the work done by the electric field described by the potential $V(r)$ to move the charge Q from the point \mathbf{r}_1 to the point \mathbf{r}_2 is given by:

$W = Q * [V(r_1) - V(r_2)]$, this work is always equal zero as long as $V(r_1) = V(r_2)$ which is the formal definition of an equipotential surface.

A comment about the test charge placed into the electric potential: we learned that a positive charge placed into the electric field would move along the field direction while a negative charge placed into the electric field would move in the direction opposite to the field direction. If we follow the field direction, the

potential decreases. This simply follows from the fact that $E(x) = -\frac{dV}{dx}$, or, in

other words, if the potential decreases, its derivative is negative, and taking the minus sign into account produces a positive $E(x)$.

Therefore, the positively charged particle will always move from the region of the higher potential to the region of the lower potential. Likewise, a ball placed on top of a hill will move down the hill since the potential energy of the ball on top of the hill is larger than its potential energy lower down, and the ball “wants” to minimize its potential energy. (Here we should recall the relationship for the gravitational potential $V(h) = mgh$, where m is the mass of the ball, g is free fall acceleration and h is the height.)

For negatively charged particles the situation is reversed: negatively charged particles would always move from the region of the lower potential to the region of the higher potential, although they are still moving spontaneously from higher

to lower potential **energy**. The example of the ball does not represent this: a ball placed near the bottom of a hill will not spontaneously move upward. This does not happen of course since there is no such thing as negative masses in mechanics.

Typical problems related to electric potential:

Problem 1. Calculate the difference between the potential at $r = 5 \text{ cm}$ and $r = 10 \text{ cm}$ for a single point charge of 1 C located at the origin.

- A. $9 \cdot 10^{-10} \text{ V}$
- B. $9 \cdot 10^{-5} \text{ V}$
- C. $9 \cdot 10^0 \text{ V}$
- D. $9 \cdot 10^{+5} \text{ V}$
- E. $9 \cdot 10^{+10} \text{ V}$

Solution. The potential created by a single point charge is given by $V(r) = k \frac{q}{r}$.

Therefore

$$V(r_1) - V(r_2) = kq(1/r_1 - 1/r_2) = 9 \cdot 10^9 \cdot 1 \cdot (1/(5 \cdot 10^{-2}) - 1/(10 \cdot 10^{-2})) = 9 \cdot 10^{10} \text{ V}$$

Answer E.

Problem 2. What are the equipotential surfaces of the uniformly charged sphere?

- A. Equipotential surfaces are the planes crossing the sphere and perpendicular to z axis
- B. Equipotential surfaces are the cylinders with the common axis along z direction
- C. Equipotential surfaces are the spheres centered at the center of the charged sphere
- D. Equipotential surfaces are elliptical and stretched along z direction.
- E. Insufficient information.

Solution. For the uniformly charged sphere the electric potential at r larger than the sphere radius is the same as for the point charge concentrated at the center of the sphere. This is the same as for the electric field created by the uniformly charged sphere, which can be found as from the same point charge placed at the sphere center. Therefore the equipotential surfaces are spheres centered at the origin.

Answer C.

Problem 3. Find the electric potential at the center of the equilateral triangle whose side is 1 m if there are three positive charges of 1 C , 2 C and 3 C in its corners. (Assume that $V(r)=0$ when r goes to infinity)

- A. $9.07 \cdot 10^{-10} \text{ V}$

- B. $3.21 \cdot 10^{-5} \text{ V}$
- C. $7.09 \cdot 10^0 \text{ V}$
- D. $9.23 \cdot 10^{+10} \text{ V}$
- E. $2.31 \cdot 10^{+14} \text{ V}$

Solution. The electric potential at the center of the triangle is the sum of the electric potential from each charge: $V = V_1(r) + V_2(r) + V_3(r)$ where r is the distance between the charge and the center of the triangle which is the same for all three charges (as long as the triangle is equilateral). If the side is 1 m, the distance to the center is $0.5/\cos(30)=1/\sqrt{3}$. We finally have

$$V = kq_1/r + kq_2/r + kq_3/r = 9 \cdot 10^9 \cdot \sqrt{3} \cdot (1 + 2 + 3) = 9.23 \cdot 10^{10} \text{ V}$$

Answer D.

Problem 4. If the potential has a form of stairs at which points the electric field is the largest?

- A. At the bottom of the stairs.
- B. At the top of the stairs
- C. At flat area of each step of the stairs
- D. At transition points between the steps
- E. The electric field is constant across the stairs.

Solution. The electric field is the derivative of the potential with respect to x . Therefore the electric field is the largest at the points where the derivative of the potential with respect to x is largest. If the potential has a step like form, its derivative is zero at all flat areas of the steps while the derivative goes to infinity at transition points between each step (since the derivative is simply given by the slope, i.e. $\tan(0)=\text{infinty}$). Therefore electric field is infinite at transition points and zero everywhere else.

Answer D.

Problem 5. Find the value of the surface charge density for the infinitely charged plate if it is known that each two equipotential surfaces separated by 1 m have potential difference of 5 V.

- A. $8.85 \cdot 10^{-11} \text{ V}$
- B. $8.85 \cdot 10^{-9} \text{ V}$
- C. $8.85 \cdot 10^{-7} \text{ V}$
- D. $8.85 \cdot 10^{-5} \text{ V}$
- E. $8.85 \cdot 10^{-3} \text{ V}$

Solution. For infinitely charged plate, the electric field is constant and it is given by $E = \frac{\sigma}{2\epsilon_0}$. The potential corresponding to the constant field is given by:

$V(x) = -Ex + \text{const.}$ The equipotential surfaces between any two points x_1 and x_2

have the potential difference: $V(x_1) - V(x_2) = -E(x_1 - x_2)$. From here we conclude that the absolute value of the electric field

$$E = (V(x_1) - V(x_2)) / (x_1 - x_2) = 5 / 1 = 5 \text{ V} / \text{m} = 5 \text{ N} / \text{C}$$

It follows that $\sigma = 2E\epsilon_0 = 2 * 5 * 8.85 * 10^{-12} = 8.85 * 10^{-11} \text{ C} / \text{m}^2$

Answer A.

5. CAPACITANCE

Key concepts:

A typical capacitor consists of metallic plates separated by some distance.

If a capacitor is charged (the plates have charge $+Q$ and $-Q$), it maintains a potential difference between the plates.

If the plates have a charge $+Q$ and $-Q$, the charge is related to the potential difference between the plates by

$$Q = C \cdot V$$

where C is called the capacitance of the capacitor. If a battery with potential difference V is attached to capacitor, the charge after a long time will be $Q = C \cdot V$

Units of C are "Farads": $1 \text{ F} = 1 \text{ Coulomb/1 Volt}$.

Frequently used units also are $\mu\text{F} = 10^{-6} \text{ F}$ and $\text{pF} = 10^{-12} \text{ F}$.

Formulas for capacitances:

Parallel plate capacitor: $C = \epsilon_0 \cdot A / d$, where A is an area of the plate, and d is the distance between the plates.

Spherical capacitor: $C = 4\pi\epsilon_0 R$, where R is the radius of the sphere.

If a capacitor is filled with a dielectric material, its capacitance is increased by κ times where κ is the dielectric constant of the material. The dielectric constant is always > 1 .

In other words, if you know the capacitance of the vacuum capacitor (no dielectric between the plates), the capacitance of the capacitor with the dielectric is $C_{\text{diel}} = \kappa C_{\text{air}}$

If capacitors are connected in parallel, the sum of capacitances gives the equivalent capacitance: $C_{\text{eq}} = C_1 + C_2 + C_3 + \dots$

If capacitors are connected in series, the sum of reciprocal capacitances gives the **reciprocal** (!!!) equivalent capacitance

$$C_{\text{eq}}^{-1} = C_1^{-1} + C_2^{-1} + C_3^{-1} + \dots$$

Typical problems involving capacitors:

Problem 6. Find the capacitance of a parallel plate capacitor with plates of area 1 cm^2 , if the distance between them is equal to 1 mm and it is filled with a dielectric solid having dielectric constant $\kappa = 10$.

- A. 885 F
- B. 8.85 kF
- C. 8.85 pF
- D. 885 mF
- E. $885 \mu\text{F}$

Solution.

$$C = \epsilon_0 \cdot \kappa \cdot A / d = 8.85 \cdot 10^{-12} \cdot 10 \cdot (0.0001) / 0.001 = 8.85 \cdot 10^{-10} \text{ F} = 885 \text{ pF}.$$

Answer C

Problem 7. What is the charge that appears on the plates of the 10 pF capacitor if it is connected to a battery of 9 V :

- A. 90 pC
- B. 90 mC
- C. 90 kC
- D. 90 C
- E. $90 \mu\text{C}$

Solution.

$$Q = C \cdot V = 10 \cdot 10^{-12} \cdot 9 = 90 \cdot 10^{-12} \text{ C} = 90 \text{ pC}.$$

Answer A.

Problem 8. What is the voltage that should be applied to the $5 \mu\text{F}$ capacitor to accumulate $1 \mu\text{C}$ charge on its plates?

- A. 0.2 V
- B. 0.2 mV
- C. 0.2 kV
- D. 0.2 nV
- E. $0.2 \mu\text{V}$

Solution.

$$V = Q / C = 10^{-6} / (5 \cdot 10^{-6}) = 0.2 \text{ V}$$

Answer A.

Problem 9. Find the value of the electric field between the plates of the parallel-plate capacitor if the voltage is 120 V and the distance between the plates is 1 mm .

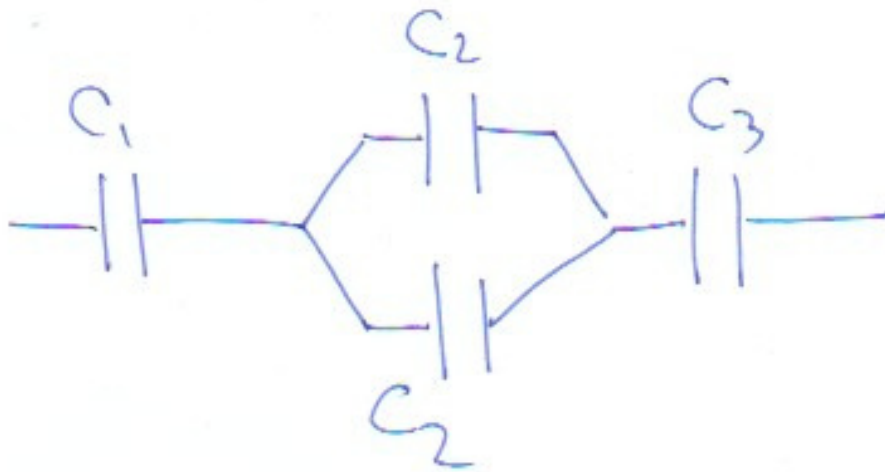
- A. 120 kN/C
- B. 120 N/C
- C. 120 mN/C
- D. 120 nN/C
- F. 120 μ N/C

Solution.

$$E = V/d = 120/(1 \times 10^{-3}) = 120000 \text{ N/C} = 120 \text{ kN/C}$$

Answer A.

Problem 10. Find the equivalent capacitance for the following circuit if $C_1=1 \text{ pF}$, $C_2=0.5 \text{ pF}$, $C_3=1 \text{ pF}$



- A. 1 pF
- B. 1/2 pF
- C. 1/3 pF
- D. 1/4 pF
- E. 1/5 pF

Solution.

Two capacitors C_2 are in parallel, therefore equivalent capacitance is $0.5+0.5=1 \text{ pF}$. Now all capacitors are in serial, therefore the sum of reciprocal values is $1/1+1/1+1/1=3$. The reciprocal of 3 is $1/3 \text{ pF}$ which is the answer.

Answer C.

Problem 11. Find the value of the electric field exactly in the middle between the plates of a cylindrical capacitor if its inner radius 1 mm, its outer radius is 2 mm, its height is 1 cm, and its charge is 1 mC.

- A. $0.5 \cdot 10^{-4}$ N/C
- B. $6.2 \cdot 10^{-21}$ N/C
- C. $1.2 \cdot 10^{+12}$ N/C
- D. $0.1 \cdot 10^{-8}$ N/C
- E. $7.9 \cdot 10^{+3}$ N/C

Solution. First recall the expression for the electric field created by a single charged cylinder of length h at the distance r (larger than the radius of the cylinder). Applying the Gauss theorem for the surface surrounding the charged cylinder gives:

$$E(r) = \frac{q}{2\pi r h \epsilon_0}$$

For a cylindrical capacitor, there are inner and outer cylinders; the electric field between the cylinders is created only by the inner cylinder since for the outer cylinder r is less than its radius and application of the Gauss theorem produces zero electric field (choosing any cylindrical Gaussian surface with the radius less than the radius of the charged cylinder will not enclose any charge). Therefore, the electric field behaves as a function of r for the cylindrical capacitor as follows

$$E(r) = \frac{q}{2\pi r h \epsilon_0}, r_{in} < r < r_{out}$$

Finally, the electric field has to be evaluated at the middle point, i.e. at

$$r = (r_{in} + r_{out}) / 2$$

And it is given by

$$E(r) = \frac{q}{2\pi r_{mid} h \epsilon_0} = \frac{10^{-3}}{2 * 3.14 * 1.5 * 10^{-3} * 10^{-2} * 8.85 * 10^{-12}} = 1.2 * 10^{12} \text{ N / C}$$

6. CURRENT AND RESISTANCE

Key concepts:

Electric current is the amount of charge passed through the conductor per unit time:

$$i = dQ/dt, \quad \text{or} \quad dQ = i dt$$

For a constant (in time) charge flow: $i = Q/t$, therefore the total charge passed through the conductor for a time interval t can be found: $Q = i \cdot t$.

The direction of the current is assumed to be the direction of the positive charge flow.

Units of current: Ampere. $1 \text{ A} = 1 \text{ C} / 1 \text{ s}$

Concept of the current density: if the current flows thru a cross-sectional area A of the wire, the current density J (current per unit area) is defined as the ratio

$$J = i / A$$

If the constant current i flows thru the wire of different cross-sections A_1, A_2, A_3, \dots , the current itself remains constant for any cross section, but the current density for different cross-sections will be different: $i = J_1 A_1 = J_2 A_2 = J_3 A_3 = \dots$

Electrons travel thru the wire with some velocity. Since the electrons are continuously accelerated by the electric field created by the battery, and slow down due to imperfectness of the lattice (impurities, vacancies, lattice vibrations, etc), the electrons velocity remains constant on the average and it is called *drift velocity*. The drift velocity v_d is connected to the current density J as follows:

$J = nev_d$ where n is the number of electrons per unit volume (electrons density), e is the charge of single electron, which is $1.6 \cdot 10^{-19} \text{ C}$. Typical drift velocities of the electrons are very small (of the order $10^{-6} - 10^{-8} \text{ m/s}$)

If the potential difference is applied across a piece of wire, a current exists in the wire. The current i is related to the applied potential difference V via: $i = V/R$ which is known to be Ohm's law. The coefficient R is called resistance and is directly related to the fact that the lattice resists to electrons flow via its imperfectness.

Units of R are Ohms. (Ω): $1 \Omega = 1 \text{ V} / 1 \text{ A}$.

The resistance of the wire depends on its length L and the cross-sectional area A and the characteristics of the material itself which is called resistivity ρ . The relationship between the resistance R and the resistivity ρ for the wire is: $R = \rho \cdot L / A$

Power (energy spent/gained per unit time) consumed in the circuit is the product of the current times the voltage: $P = i * V$

Assuming that the current is connected to the voltage via the Ohm law: $i = V / R$,

The following formulae are also valid: $P = i^2 R = V^2 / R$

Typical problems related to current and resistance:

Problem 12. A current of 5 A exists in the wire for 3 min. What is the total charge passed through any cross-sectional area of the wire?

- A. 0.9 kC
- B. 0.9 C
- C. 0.9 mC
- D. 0.9 nC
- E. 0.9 μ C

Solution. $Q = i * t = 5 * 3 * 60 = 900 \text{ C} = 0.9 \text{ kC}$.

Answer A.

Problem 13. A metallic wire has resistivity $10 \mu\Omega \cdot \text{cm}$, length L equal to 50 cm and the diameter d equal to 2 mm. What is its resistance?

- A. 16 m Ω
- B. 16 Ω
- C. 16 k Ω
- D. 16 n Ω
- E. 16 $\mu\Omega$

Solution.

$$R = \rho * L / A = \rho * L / (\pi R^2) = 4 * \rho * L / (\pi d^2) = \\ 4 * 10 * 10^{-6} * 10^{-2} * 0.5 / (3.14 * 4 * 10^{-6}) = 0.016 \Omega = 16 \text{ m}\Omega$$

Answer A.

Problem 14. If a piece of wire is stretched by n times what happens to the resistance?

- A. Increases as n
- B. Increases as n^2
- C. Decreases as n
- D. Decreases as n^2
- E. Does not change

Solution. Assume that the resistivity, density and consequently the volume do not change during the stretching process. Then, if the wire in the old state has the length L_{old} and the cross-sectional area A_{old} while the wire in the new stretched state has the length L_{new} and the cross-sectional area A_{new} , the old volume $L_{old} * A_{old}$ is equal to the new volume $L_{new} * A_{new}$.

It is therefore, $A_{new} / A_{old} = L_{old} / L_{new}$

$$\text{New resistance } R_{new} = \rho * L_{new} / A_{new} = \rho * L_{new} / A_{old} * L_{new} / L_{old} = \rho * L_{old} / A_{old} * (L_{new} / L_{old})^2 = R_{old} * (L_{new} / L_{old})^2$$

Therefore, new resistance $R_{new} = R_{old} * n^2$ since we assume that $L_{new} / L_{old} = n$

Answer B.

Problem 15. What is the drift velocity of the electrons for the current of 1.6 A passing thru the wire which has a cross-sectional area of 1 mm². (Assume that typical density of the electrons is 10³⁰ el/m³)

- A. 10⁻¹⁵ m/s
- B. 10⁻¹⁰ m/s
- C. 10⁻⁵ m/s
- D. 10⁺⁵ m/s
- E. 10⁺¹⁵ m/s

Solution. A drift velocity is connected to the current density as follows

$J = nev_d$. For the current itself, this relationship reads as follows: $i = nev_d A$, where A is the cross-section of the wire. Evaluating the drift velocity gives

$$v_d = \frac{i}{neA} = \frac{1.6}{10^{30} * 1.6 * 10^{-19} * 10^{-6}} = 10^{-5} \text{ m/s}$$

Problem 16. How much money you will spend per day for having a bulb working in the house circuit with the voltage of 110V and the current of 1 A. (Assume that 1 kW*hour costs 10 cents)

- A. 0 \$
- B. 0.26 \$
- C. 2.6 \$
- D. 26.0 \$
- E. 260.0 \$

Solution. Let's calculate the power consumed by our bulb:

$$P = i * V = 1 * 110 = 110 \text{ Watt},$$

The energy, which will be required to keep the bulb lighting for a whole day (24 hours=24*3600 s) is given by: $E = P * t = 110 * 24 * 3600 = 9.5 * 10^6 \text{ Joules}$.

Alternatively, since we know the price in kW*hour, we can calculate the energy in

more appropriate units: $E = 0.11(kW) * 24(hours) = 2.64kW * hour$. If 1 kW*hour costs 10 cents, 2.64 kW*hour will cost 26 cents.

Answer B.

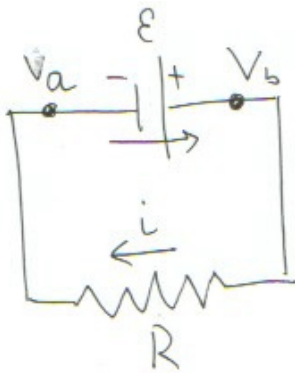
7. SINGLE LOOP CIRCUITS

Key concepts:

A Battery is a device that maintains a potential difference between its two poles. The poles are denoted “-“ and “+”. The potential at the “-“ pole is smaller than the potential at the “+” pole.

The important characteristic of the battery is its electromotive force (EMF) denoted as \mathcal{E} . If the battery is ideal, the potential difference V that it maintains is equal to \mathcal{E} . For real batteries, there is an internal resistance usually denoted “ r ” that acts in series with the battery.

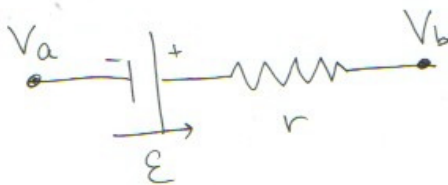
The simplest circuit consists of a battery and resistor in a single loop circuit:



The current within the circuit flows from the region of higher potential to the region of lower potential (see picture above), i.e. from V_b to V_a .

Due to chemical reactions, the current inside the battery flows from “-“ to “+”, i.e. from the region of lower potential (V_a) to the region of the higher potential (V_b). The electromotive force is directed from “-“ to “+” i.e. it “pushes

the positive charge inside the battery in the direction of lower potential to the direction of higher potential.



Real batteries have some internal resistance, therefore the potential difference they maintain is different from \mathcal{E} : $V = \mathcal{E} - i \cdot r$, where r is the internal resistance of the battery and i is the current through the battery.

Real batteries can be represented by two pieces: the ideal battery which maintains $V = \mathcal{E}$ and the resistor r . The potential difference V that is available is less than the EMF by $i \cdot r$, where i is the current through the battery.

If resistors are connected in serial, the sum of resistances gives the equivalent resistance:

$$R_{eq} = R_1 + R_2 + R_3 + \dots$$

If resistors are connected in parallel the sum of reciprocal resistances gives the reciprocal equivalent resistance:

$$R_{eq}^{-1} = R_1^{-1} + R_2^{-1} + R_3^{-1} + \dots$$

These relations are reversed from those describing capacitors in series and parallel.

The current through a single loop circuit is the same everywhere and can be found by applying the **Kirchoff loop rule**:

The sum of all potential differences in a complete loop through the circuit should be equal zero.

The power P is the energy (spent or gained) per unit time. In the problems related to the circuits the energy can be supplied by the batteries, the energy can dissipate in resistors and bulbs, etc.

Power P can be found as a product of the current times voltage.

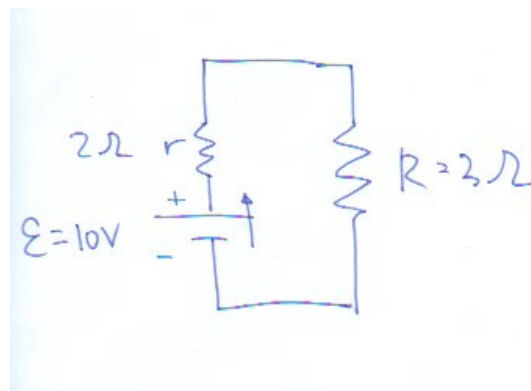
For example the energy rate (power) dissipated in the resistor is the product of the current inside the resistor times the voltage across the resistor: $P = i * V = i^2 * R = V^2 / R$, where we have used the relationship $i = V / R$.

For ideal batteries, the battery supplies the power to the circuit $P = i * E$.

Typical problems related to single loop circuits:

Problem 1. A circuit consists of the real battery ($E = 10\text{V}$, $r = 2\ \Omega$) and the resistor $R = 3\ \Omega$. Calculate current and the total power released in the circuit.

- A. 1 A, 10 W
- B. 0.5 A, 5 W
- C. 2 A, 20 W
- D. 3 A, 15 W
- E. 5 A, 3 W



Solution. First, walk through the loop and write the Kirchoff equation. This will give us the equation for the current. Since the battery arrow (from “-“ to “+“) is pointed up, and there is only one battery, the direction of the current is clockwise. Walking clockwise:

$$E - i * r - i * R = 0$$

$$i = E / (r + R) = 2\text{A}.$$

Now, the total power released in the circuit is the sum of the power lost in the resistor r and R . Since the current is known, the useful formula for power $P=i*V=i^2*R$.

Power released in r : $P_r=i^2*r=2*2*2=8\text{ W}$

Power released in R : $P_R=i^2*R=2*2*3=12\text{ W}$

The total power released in the circuit is therefore $12+8=20\text{W}$.

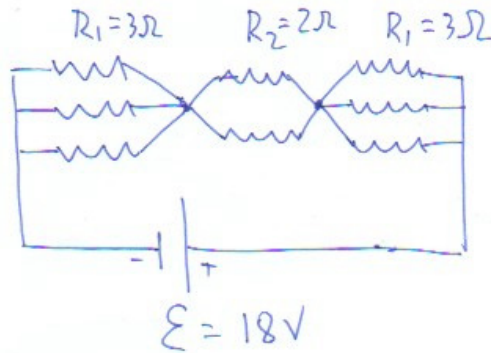
Note that the same answer can be found quicker: this is the power which ideal piece of the battery supplies to the circuit:

$$P=i*E=2*10=20\text{ W}$$

Answer C.

Problem 2. A circuit consists of the ideal battery ($E=18\text{ V}$) and two kinds of resistors $R_1=3\Omega$, $R_2=2\Omega$. Calculate current through the battery and the total power released in the circuit.

- A. 3 A, 64W
- B. 6 A, 108 W
- C. 12 A, 216 W
- D. 1 A, 32 W
- E. 0.5 A, 108 W



Solution. While formally it is a multi loop circuit, the problem is simple since we need to find the current through the battery.

Therefore, first, we simplify the circuit by calculating equivalent resistance. We have first three resistors in parallel, its resistance is $1/(1/3+1/3+1/3)=1\Omega$. Second, two resistors are in parallel, therefore the equivalent resistance is $1(1/2+1/2)=1\Omega$. The third three resistors are in parallel, their resistance is $1/(1/3+1/3+1/3)=1\Omega$.

After, we have three serial connections with the total resistance $1+1+1=3\Omega$.

The second step is to find the current through the battery since we have a simple single loop circuit with the battery and one equivalent resistor of 3Ω .

$$E-i*R_{eq}=0$$

$$i=E/R_{eq}=18/3=6\text{ A.}$$

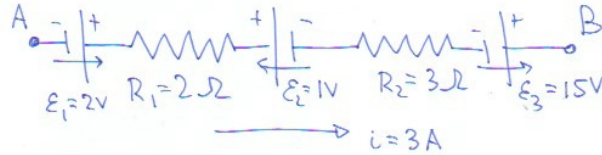
The total power released in the circuit is the sum of all powers released in all resistors. This is the same as the power which battery supplies to the circuit.

$$\text{Therefore } P=i*V=6*18=108\text{ W}$$

Answer B.

Problem 3. A fragment of the circuit is shown on the Figure. What is the potential difference between the points B and A?

- A. 1 V
- B. 2 V
- C. -5 V
- D. 4 V
- E. -3 V



Solution. It is easy to walk in the direction of the current from the point A to the point B. Writing the Kirchhoff equation:

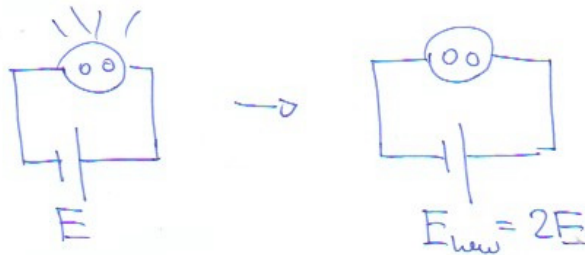
$$V_A + \varepsilon_1 - R_1 \cdot i - \varepsilon_2 - R_2 \cdot i + \varepsilon_3 = V_B$$

$$\varepsilon_1 - R_1 \cdot i - \varepsilon_2 - R_2 \cdot i + \varepsilon_3 = V_B - V_A = 2 - 2 \cdot 3 - 1 - 3 \cdot 3 + 15 = 1 \text{ V}$$

Answer A.

Problem 4. A bulb is connected to the battery of some EMF equal to E. Will the bulb be brighter or dimmer if it will be reconnected to a new battery of $E_{\text{new}} = 2E$. By how much?

- A. Twice brighter
- B. Twice dimmer
- C. The same
- D. Four times brighter
- E. Four times dimmer



Solution. The brightness of the bulb is proportional to its power. Therefore we have to figure out what happens to the power released in the bulb if we change the voltage across it. If the bulb has some resistance R, and it is connected to some ideal battery E, its power $P = i \cdot V = i \cdot E$ since the voltage across the bulb is equal to the EMF of the battery. The current in the circuit $i = E/R$. Therefore we can express the power of the bulb via E and R as follows:
 $P = i \cdot V = i \cdot E = E \cdot E/R = E^2/R$.

This formula tells us if we increase the applied voltage by 2, the power released in the bulb will increase quadratically, i.e. by 4.

$$P_{\text{new}} = E_{\text{new}}^2/R = (2 \cdot E)^2/R = 4P_{\text{old}}$$

Answer D.