National University of Computer and Emerging Sciences

School of Computing **Fall 2018** **Islamabad Campus**

MT-119Calculus & Analytical Geometry

Serial No:

Midterm Exam

Total Time: 2 Hours

Total Marks: 100

Tuesday, October 23, 2018

Course Instructor(s)

Ms. Hamda Khan, Mr. Usman Ashraf

Dr. Raheel

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Signature of Invigilator

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Student NameRoll No Section Signature

DO NOT OPEN THE QUESTION BOOK OR START UNTIL INSTRUCTED.

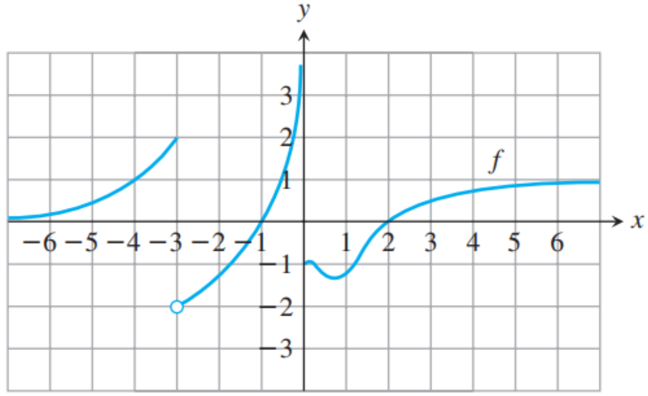
Instructions:

1. Attempt on question paper. Attempt all of them. Read the question carefully, understand the question, and then attempt it.
2. No additional sheet will be provided for rough work. Use the back of the last page for rough work.
3. If you need more space write on the back side of the paper and clearly mark question and part number etc.
4. After asked to commence the exam, please verify that you have fourteen (14) different printed pages including this title page. There are a total of 10 questions.
5. Calculator sharing is strictly prohibited.
6. Use permanent ink pens only. Any part done using soft pencil will not be marked and cannot be claimed for rechecking.

	Q-1	Q-2	Q-3	Q-4	Q-5	Q-6	Q-7	Q-8	Q-9	Q-10	Total
Marks Obtained											
Total Marks	10	10	10	10	10	5	15	10	10	10	100

Vetted By: ______________________________Vetter Signature: ______________________

Question 1 [10Marks]



For the function f whose graph is given above, determine the following:

a) $\lim_{x\to 2} f(x)$ b) $\lim_{x\to -3^+} f(x)$ c) $\lim_{x\to -3^-} f(x)$ d) $\lim_{x\to -3} f(x)$ e) $\lim_{x\to 0^+} f(x)$

f) $\lim_{x\to 0^-} f(x)$ g) $\lim_{x\to 0} f(x)$ h) $\lim_{x\to \infty} f(x)$

(a) $\lim_{x\to 2} f(x) = 0$

(b) $\lim_{x\to -3^+} f(x) = -2$

(c) $\lim_{x\to -3^-} f(x) = 2$

(d) $\lim_{x\to 3} f(x) = \text{does not exist}$

(e) $\lim_{x\to 0^+} f(x) = -1$

(f) $\lim_{x\to 0^-} f(x) = +\infty$

(g) $\lim_{x\to 0} f(x) = \text{does not exist}$

(h) $\lim_{x\to \infty} f(x) = 1$

Question 2 [10Marks]

A 24-in. by 36-in. sheet of cardboard is folded in half to form a 24-in. by 18-in. rectangle as shown in the accompanying figure. Then four congruent squares of side length x are cut from the corners of the folded rectangle. The sheet is unfolded, and the six tabs are folded up to form a box with sides and a lid.

a) Write a formula V(x) for the volume of the box.

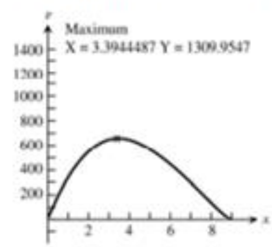
b) Find the domain of V for the problem situation and graph V over this domain.

c) Find the maximum volume and the value of corresponding x analytically.

d) Find a value of x that yields a volume of 1120 in^3.

17. (a) The "sides" of the suitcase will measure $24-2x$ in. by $18-2x$ in. and will be $2x$ in. apart, so the volume formula is $V(x) = 2x(24-2x)(18-2x) = 8x^3 - 168x^2 + 862x$.

(b) We require $x > 0$, $2x < 18$, and $2x < 12$. Combining these requirements, the domain is the interval $(0, 9)$.



(c) The maximum volume is approximately 1309.95 in.^{3} when $x \approx 3.39$ in.

(d) $V'(x) = 24x^2 - 336x + 864 = 24(x^2 - 14x + 36)$. The critical point is at $x = \frac{14 \pm \sqrt{(-14)^2 - 4(1)(36)}}{2(1)} = \frac{14 \pm \sqrt{52}}{2}$ $= 7 \pm \sqrt{13}$, that is, $x \approx 3.39$ or $x \approx 10.61$. We discard the larger value because it is not in the domain. Since $V''(x) = 24(2x - 14)$ which is negative when $x \approx 3.39$, the critical point corresponds to the maximum volume. The maximum value occurs at $x = 7 - \sqrt{13} \approx 3.39$, which confirms the results in (c).

(e) $8x^3 - 168x^2 + 862x = 1120 \Rightarrow 8(x^3 - 21x^2 + 108x - 140) = 0 \Rightarrow 8(x-2)(x-5)(x-14) = 0$. Since 14 is not in the domain, the possible values of x are $x = 2$ in. or $x = 5$ in.

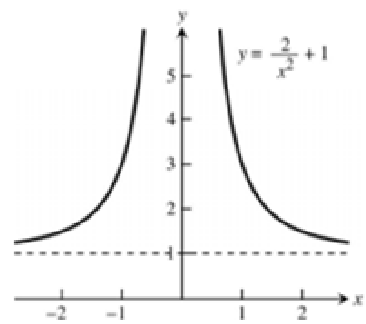
Question 3A [5Marks]

Graph the function by starting with the graph of one of the standard functions and then applying an appropriate transformation.

$$y = \frac{2}{x^2} + 1$$

Let $f(x) = \frac{1}{x^2}$ and $g(x) = \frac{2}{x^2} + 1 = \frac{1}{\left(\frac{x^2}{2}\right)} + 1$

$= \frac{1}{\left(x/\sqrt{2}\right)^2} + 1 = \frac{1}{\left[\left(1/\sqrt{2}\right)x\right]^2} + 1$. Since $\sqrt{2} \approx 1.4$, we see that the graph of $f(x)$ stretched horizontally by a factor of 1.4 and shifted up 1 unit is the graph of $g(x)$.



Question 3B [5Marks]

Use the limits to find all the asymptotes.

$$y = \sqrt{\frac{x^2 + 9}{9x^2 + 1}}$$

thus $y = 1$ and $y = -1$ are horizontal asymptotes.

d) $y = \sqrt{\frac{x^2+9}{9x^2+1}}$: $\lim\limits_{x\to\infty} \sqrt{\frac{x^2+9}{9x^2+1}} = \lim\limits_{x\to\infty} \sqrt{\frac{1+\frac{9}{x^2}}{9+\frac{1}{x^2}}} = \sqrt{\frac{1+0}{9+0}} = \frac{1}{3}$ and $\lim\limits_{x\to-\infty} \sqrt{\frac{x^2+9}{9x^2+1}} = \lim\limits_{x\to-\infty} \sqrt{\frac{1+\frac{9}{x^2}}{9+\frac{1}{x^2}}} = \sqrt{\frac{1+0}{9+0}} = \frac{1}{3}$,

thus $y = \frac{1}{3}$ is a horizontal asymptote.

Find vertical asymptote too

Question 4A [5Marks]

Check whether the given problem qualifies to be solved using L'Hospital's rule. If yes, then solve:

$$\lim_{x\to\infty}\frac{3x+2^x}{2x+3^x}$$

$$\lim_{x\to\infty}\frac{3x+2^x}{2x+3^x}=\frac{\text{“ }3(\infty)+2^\infty\text{ ”}}{2(\infty)+3^\infty}=\frac{\text{“ }\infty+\infty\text{ ”}}{\infty+\infty}=\frac{\text{“ }\infty\text{ ”}}{\infty}$$

call that $D\{a^{x)}\}=a^x\cdot\ln a$.)

$$=\lim_{x\to\infty}\frac{3+2^x\ln 2}{2+3^x\ln 3}=\frac{\text{“ }3+2^\infty\ln 2\text{ ”}}{2+3^\infty\ln 3}=\frac{\text{“ }3+\infty\text{ ”}}{2+\infty}=\frac{\text{“ }\infty\text{ ”}}{\infty}$$

$$=\lim_{x\to\infty}\frac{0+2^x\ln 2\cdot\ln 2}{0+3^x\ln 3\cdot\ln 3}$$

$$=\lim_{x\to\infty}\frac{2^x(\ln 2)^2}{3^x(\ln 3)^2}=\frac{2^\infty(\ln 2)^2}{3^\infty(\ln 3)^2}=\frac{(\infty)(\ln 2)^2}{(\infty)(\ln 3)^2}=\frac{\text{“ }\infty\text{ ”}}{\infty}$$

$$=\lim_{x\to\infty}\frac{2^x(\ln 2)^2}{3^x(\ln 3)^2}$$

$$=\lim_{x\to\infty}\frac{2^x(\ln 2)^2}{3^x(\ln 3)^2}$$

$$=\lim_{x\to\infty}\frac{2^x}{3^x}\cdot\frac{(\ln 2)^2}{(\ln 3)^2}$$

$$=\lim_{x\to\infty}\left(\frac{2}{3}\right)^x\cdot\frac{(\ln 2)^2}{(\ln 3)^2}$$

$$=\left(\frac{2}{3}\right)^\infty\cdot\frac{(\ln 2)^2}{(\ln 3)^2}$$

$$=(0)\frac{(\ln 2)^2}{(\ln 3)^2}$$

$$=0$$

Question 4B [5Marks]

Find the limit of the given function without using L'Hospital's Rule.

$$\lim_{\theta \to 0} \frac{\cos(\theta) - 1}{\theta}$$

Question 5 [10Marks]

For what value or values of the constant m, if any, is

$$f(x) = \begin{cases} \sin 2x, & x \le 0 \\ mx, & x > 0 \end{cases}$$

a) Continuous at $x = 0$?

b) Differentiable at $x = 0$?

98. (a) $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \sin 2x = 0$ and $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} mx = 0 \Rightarrow \lim_{x\to 0} f(x) = 0$, independent of m; since $f(0) = 0 = \lim_{x\to 0} f(x)$ it follows that f is continuous at $x = 0$ for all values of m.

(b) $\lim_{x\to 0^-} f'(x) = \lim_{x\to 0^-} (\sin 2x)' = \lim_{x\to 0^-} 2\cos 2x = 2$ and $\lim_{x\to 0^+} f'(x) = \lim_{x\to 0^+} (mx)' = \lim_{x\to 0^+} m = m \Rightarrow f$ is differentiable at $x = 0$ provided that $\lim_{x\to 0^-} f'(x) = \lim_{x\to 0^+} f'(x) \Rightarrow m = 2$.

Question 6 [5Marks]

Find the derivatives of the function

$$y = 2\sqrt{x-1}\sec^{-1}\sqrt{x}$$

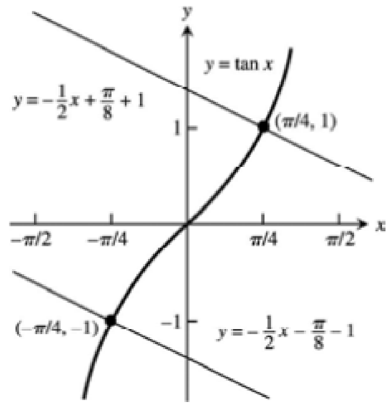
$$y = 2\sqrt{x-1}\sec^{-1}\sqrt{x} = 2(x-1)^{1/2}\sec^{-1}(x^{1/2})$$

$$\Rightarrow \frac{dy}{dx} = 2\left[\left(\tfrac{1}{2}\right)(x-1)^{-1/2}\sec^{-1}(x^{1/2}) + (x-1)^{1/2}\left(\frac{\left(\tfrac{1}{2}\right)x^{-1/2}}{\sqrt{x}\sqrt{x-1}}\right)\right] = 2\left(\frac{\sec^{-1}\sqrt{x}}{2\sqrt{x-1}} + \frac{1}{x}\frac{1}{2}\right) = \frac{\sec^{-1}\sqrt{x}}{\sqrt{x-1}} + \frac{1}{x}$$

Question 7A [10Marks]

Find the points on the curve $y = tanx, -\frac{\pi}{2} < x < \frac{\pi}{2}$, where the normal is parallel to the line $y = -x/2$. Sketch the curve and normal together, labeling each with its equation.

105. $y = \tan x, -\frac{\pi}{2} < x < \frac{\pi}{2} \Rightarrow \frac{dy}{dx} = \sec^2 x$; now the slope of $y = -\frac{x}{2}$ is $-\frac{1}{2} \Rightarrow$ the normal line is parallel to $y = -\frac{x}{2}$ when $\frac{dy}{dx} = 2$. Thus, $\sec^2 x = 2 \Rightarrow \frac{1}{\cos^2 x} = 2 \Rightarrow \cos^2 x = \frac{1}{2} \Rightarrow \cos x = \frac{\pm 1}{\sqrt{2}} \Rightarrow x = -\frac{\pi}{4}$ and $x = \frac{\pi}{4}$ for $-\frac{\pi}{2} < x < \frac{\pi}{2} \Rightarrow \left(-\frac{\pi}{4}, -1\right)$ and $\left(\frac{\pi}{4}, 1\right)$ are points where the normal is parallel to $y = -\frac{x}{2}$.



Question 7B [5Marks]

Use logarithmic differentiation to find the derivative of y with respect to appropriate variable.

$$y = (lnx)^{\frac{1}{lnx}}$$

$$y = (\ln x)^{1/\ln x} \Rightarrow \ln y = \left(\frac{1}{\ln x}\right)\ln(\ln x) \Rightarrow \frac{y'}{y} = \left(\frac{1}{\ln x}\right)\left(\frac{1}{\ln x}\right)\left(\frac{1}{x}\right) + \ln(\ln x)\left[\frac{-1}{(\ln x)^2}\right]\left(\frac{1}{x}\right) \Rightarrow y' = (\ln x)^{1/\ln x}\left[\frac{1-\ln(\ln x)}{x(\ln x)^2}\right]$$

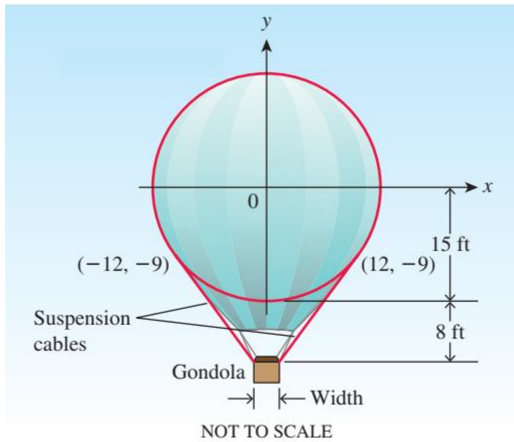
Question 8 [10Marks]

A particle moves along the curve $y = x^{\frac{3}{2}}$ in the first quadrant in such a way that its distance from the origin increases at the rate of 11 units per second. Find horizontal rate of change when x=3.

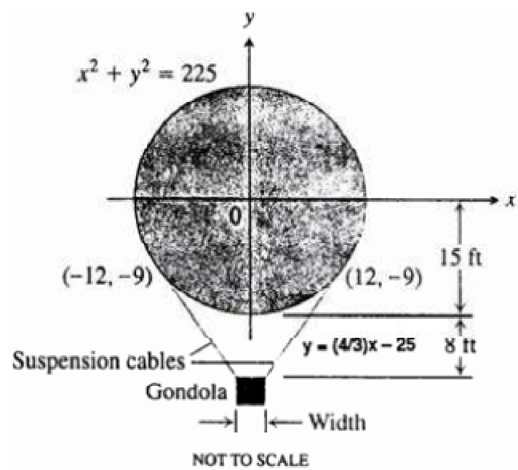
Let D be the distance from the origin. We are given that $\frac{dD}{dt} = 11$ units/sec. Then $D^2 = x^2 + y^2 = x^2 + (x^{3/2})^2$ $= x^2 + x^3 \Rightarrow 2D\frac{dD}{dt} = 2x\frac{dx}{dt} + 3x^2\frac{dx}{dt} = x(2+3x)\frac{dx}{dt}$; $x = 3 \Rightarrow D = \sqrt{3^2 + 3^3} = 6$ and substitution in the derivative equation gives $(2)(6)(11) = (3)(2+9)\frac{dx}{dt} \Rightarrow \frac{dx}{dt} = 4$ units/sec.

Question 9 [10Marks]

The designer of a 30-ft-diameter spherical hot air balloon wants to suspend the gondola 8 ft below the bottom of the balloon with cables tangent to the surface of the balloon, as shown. Two of the cables are shown running from the top edges of the gondola to their points of tangency, $(-12,-9)$ *and* $(12,-9)$. How wide should the gondola be when the radius of the balloon is 15ft?



When $x^2+y^2=225$, then $y'=-\frac{x}{y}$. The tangent line to the balloon at $(12,-9)$ is $y+9=\frac{4}{3}(x-12)$ $\Rightarrow y=\frac{4}{3}x-25$. The top of the gondola is $15+8=23$ ft below the center of the balloon. The intersection of $y=-23$ and $y=\frac{4}{3}x-25$ is at the far right edge of the gondola $\Rightarrow -23=\frac{4}{3}x-25 \Rightarrow x=\frac{3}{2}$. Thus the gondola is $2x=3$ ft wide.



Question 10 [10Marks]

Calculate derivative by definition by applying on any one case:

$$f(x) = |x^2 - 3x|$$

$f(x) = | x^2 - 3x | = | x(x-3) |$

$$= \begin{cases} x(x-3), & \text{if } x \le 0 \\ -x(x-3), & \text{if } 0 < x < 3 \\ x(x-3), & \text{if } x \ge 3 \end{cases}$$

$$= \begin{cases} x^2 - 3x, & \text{if } x \le 0 \\ 3x - x^2, & \text{if } 0 < x < 3 \\ x^2 - 3x, & \text{if } x \ge 3\ . \end{cases}$$

Assume that $x < 0$. Then

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{(x+\Delta x)^2 - 3(x+\Delta x) - (x^2 - 3x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{x^2 + 2x\Delta x + (\Delta x)^2 - 3x - 3\Delta x - x^2 + 3x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{2x\Delta x + (\Delta x)^2 - 3\Delta x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\Delta x(2x + \Delta x - 3)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} (2x + \Delta x - 3)$$

$$= 2x - 3\ .$$

Assume that $x>3$. Then it is also true that $f(x) = 2x - 3$. Assume that $0 < x < 3$. Then

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{3(x+\Delta x) - (x+\Delta x)^2 - (3x - x^2)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{3x + 3\Delta x - (x^2 + 2x\Delta x + (\Delta x)^2) - 3x + x^2}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{3x + 3\Delta x - x^2 - 2x\Delta x - (\Delta x)^2 - 3x + x^2}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{3\Delta x - 2x\Delta x - (\Delta x)^2}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\Delta x(3 - 2x - \Delta x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} (3 - 2x - \Delta x)$$

$$= 3 - 2x\ .$$

Now check for differentiability at x=0, i.e., compute $f'(0)$. Then

$$f'(0) = \lim_{\Delta x \to 0} \frac{f(0+\Delta x) - f(0)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{f(\Delta x) - 0}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{f(\Delta x)}{\Delta x} .$$

If $\Delta x > 0$, then

$$\lim_{\Delta x \to 0^+} \frac{f(\Delta x)}{\Delta x} = \lim_{\Delta x \to 0^+} \frac{3(\Delta x) - (\Delta x)^2}{\Delta x}$$

$$= \lim_{\Delta x \to 0^+} \frac{\Delta x(3 - \Delta x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0^+} (3 - \Delta x)$$

$$= 3.$$

If $\Delta x < 0$, then

$$\lim_{\Delta x \to 0^-} \frac{f(\Delta x)}{\Delta x} = \lim_{\Delta x \to 0^-} \frac{(\Delta x)^2 - 3(\Delta x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0^-} \frac{\Delta x(\Delta x - 3)}{\Delta x}$$

$$= \lim_{\Delta x \to 0^-} (\Delta x - 3)$$

$$= - 3.$$

Since the one-sided limits exist but are NOT EQUAL, $f'(0) = \lim_{\Delta x \to 0} \frac{f(\Delta x)}{\Delta x}$ does not exist, and f is not differentiable at $x = 0$. Now check for differentiability at x=3, i.e., compute $f'(3)$. Then

$$f'(3) = \lim_{\Delta x \to 0} \frac{f(3+\Delta x) - f(3)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{f(3+\Delta x) - 0}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{f(3+\Delta x)}{\Delta x} .$$

If $\Delta x > 0$, then $3 + \Delta x > 3$ so that

$$\lim_{\Delta x \to 0^+} \frac{f(3+\Delta x)}{\Delta x} = \lim_{\Delta x \to 0^+} \frac{(3+\Delta x)^2 - 3(3+\Delta x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0^+} \frac{9 + 6\Delta x + (\Delta x)^2 - 9 - 3\Delta x}{\Delta x}$$

$$= \lim_{\Delta x \to 0^+} \frac{3\Delta x + (\Delta x)^2}{\Delta x}$$

$$= \lim_{\Delta x \to 0^+} \frac{\Delta x(3 + \Delta x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0^+} (3 + \Delta x)$$

$$= 3.$$

If $\Delta x < 0$, then $3 + \Delta x < 3$ so that

$$\lim_{\Delta x \to 0^-} \frac{f(3+\Delta x)}{\Delta x} = \lim_{\Delta x \to 0^-} \frac{3(3+\Delta x) - (3+\Delta x)^2}{\Delta x}$$

$$= \lim_{\Delta x \to 0^-} \frac{9 + 3\Delta x - (9 + 6\Delta x + (\Delta x)^2)}{\Delta x}$$

$$= \lim_{\Delta x \to 0^-} \frac{9 + 3\Delta x - 9 - 6\Delta x - (\Delta x)^2}{\Delta x}$$

$$= \lim_{\Delta x \to 0^-} \frac{-3\Delta x - (\Delta x)^2}{\Delta x}$$

$$= \lim_{\Delta x \to 0^-} \frac{\Delta x(-3 - \Delta x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0^-} (-3 - \Delta x)$$

$$= -3.$$

Since the one-sided limits exist but are NOT EQUAL, $f'(3) = \lim_{\Delta x \to 0} \frac{f(3+\Delta x)}{\Delta x}$ does not exist, and *f* is not differentiable at $x = 3$. Summarizing, the derivative of *f* is

$$f'(x) = \begin{cases} 2x - 3, & \text{if } x < 0 \\ 3 - 2x, & \text{if } 0 < x < 3 \\ 2x - 3, & \text{if } x > 3 \ . \end{cases}$$

Function *f* is not differentiable at x=0 or x=3.

Click HERE to return to the list of problems.

SOLUTION 12 : First, determine if *f* is continuous at x=2, i.e., determine if $\lim_{x \to 2} f(x) = f(2) = 0$. For $x > 2$

$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} \left\{ \frac{1}{4}x^3 - \frac{1}{2}x^2 \right\}$$

$$= \frac{1}{4}(2)^3 - \frac{1}{2}(2)^2$$

$$= 0.$$

For $x < 2$

For $x < 2$

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} \frac{-6x-6}{x^2+2}$$

$$= \frac{-6(2)-6}{(2)^2+2}$$

$$= -3.$$

Thus, the one-sided limits exist but are NOT EQUAL, so that $\lim_{x \to 2} f(x)$ does not exist and function f is NOT CONTINUOUS AT x=0 . Since function f is NOT CONTINUOUS AT x=0 , function f is NOT DIFFERENTIABLE at x=2 .

REMARK 1 : Use of the limit definition of the derivative of f at x=2 also leads to a correct solution to this problem.

REMARK 2 : What follows is a common INCORRECT attempt to solve this problem using another method. For x>2

$$f'(x) = \frac{3}{4}x^2 - x.$$

For x<2

$$f'(x) = \frac{(x^2+2)(-6) - (-6x-6)(2x)}{(x^2+2)^2}$$

$$= \frac{6x^2+12x-12}{(x^2+2)^2}.$$

Then

$$\lim_{x \to 2^+} f'(x) = \lim_{x \to 2^+} \left\{ \frac{3}{4}x^2 - x \right\}$$

$$= \frac{3}{4}(2)^2 - 2$$

$$= 1,$$

and

$$\lim_{x \to 2^-} f'(x) = \lim_{x \to 2^-} \frac{6x^2+12x-12}{(x^2+2)^2}$$

$$= \frac{6(2)^2+12(2)-12}{((2)^2+2)^2}$$

$$= 1.$$

An INCORRECT conclusion would be that $f(2) = 1$. If f were continuous at x=2 , this would be a valid method to compute $f(2)$.