Quiz 6

Problem 1. Locate all relative maxima, relative minima, and saddle points for $f(x,y) = x^3 - 9xy + y^3$. (Use the second derivatives test to identify the type of each critical point.)

Solution. We first find critical points solving the equations $f_x = 3x^2 - 9y = 0$ and $f_y = -9x + 3y^2 = 0$. We obtain from the first equation that $y = \frac{x^2}{3}$, and then substitute this y to the second equation: $-9x + \frac{x^4}{3} = 0$, or equivalently, $-27x + x^4 = 0$, i.e., $x(x^3 - 27) = 0$. We obtain from here that either $x = 0$ or $x = 3$, and $y = 0$ or $y = 3$ respectively. Thus $f(x,y)$ has two critical points: $(0,0)$ and $(3,3)$. Now compute the second derivatives: $f_{xx} = 6x$, $f_{xy} = f_{yx} = -9$, $f_{yy} = 6y$. We also have $D(x,y) = f_{xx}f_{yy} - f_{xy}^2 = 36xy - 81$. Next, $D(0,0) = -81 < 0$, therefore $(0,0)$ is a saddle point. $f_{xx}(3,3) = 18 > 0$ and $D(3,3) = 243 > 0$, therefore $(3,3)$ is a point of relative minimum.

Problem 2. Find the absolute maximum and the absolute minimum of $f(x,y) = x^2 + 2y^2 - x$ on the closed disk $x^2 + y^2 \leq 4$.

Solution. We start with finding critical points of $f(x,y)$ inside the disk. We have $f_x = 2x - 1 = 0$ and $f_y = 4y = 0$ for $x = \frac{1}{2}$ and $y = 0$. Thus, there is only one critical point inside the disk: $P_1(\frac{1}{2}, 0)$. Next, consider the function $f(x,y)$ restricted to the boundary of the disk, the circle $x^2 + y^2 = 4$. We plug in $y^2 = 4 - x^2$ into the expression for $f(x,y)$ and obtain the function

$$\phi(x) = x^2 + 2(4 - x^2) - x = x^2 + 8 - 2x^2 - x = -x^2 - x + 8.$$

This function is defined on the closed interval $[-2, 2]$. We first find the critical points of $\phi(x)$ inside the interval: $\phi'(x) = -2x - 1 = 0$ at $x = -\frac{1}{2}$. We have that the corresponding y takes the values

$$y = \pm\sqrt{4 - \left(-\frac{1}{2}\right)^2} = \pm\frac{\sqrt{15}}{2}.$$

Therefore, there are two more points of the closed disk where the absolute extrema may occure: $P_2(-\frac{1}{2}, \frac{\sqrt{15}}{2})$ and $P_3(-\frac{1}{2}, -\frac{\sqrt{15}}{2})$. In addition, $\phi(x)$ may take the extremal values at $x = -2$ and $x = 2$. The corresponding value of y is $y = \sqrt{4 - (\pm 2)^2} = 0$ for both points. Thus we add two more suspicious points, $P_4(-2, 0)$ and $P_5(2, 0)$. In order to find the absolute extrema, we evaluate $f(x,y)$ at the points P_1, P_2, P_3, P_4, and P_5, and find the largest and smallest values. We have $f(\frac{1}{2}, 0) = -\frac{1}{4}$, $f(-\frac{1}{2}, \pm\frac{\sqrt{15}}{2}) = \frac{33}{4}$, $f(-2, 0) = 6$, $f(2, 0) = 2$. Therefore, $f(x,y)$ has the absolute minimum at $P_1(\frac{1}{2}, 0)$ which is equal to $-\frac{1}{4}$, and the absolute maximum at $P_2(-\frac{1}{2}, \frac{\sqrt{15}}{2})$ and $P_3(-\frac{1}{2}, -\frac{\sqrt{15}}{2})$ which is equal to $\frac{33}{4}$.