Ex 4.1 Laplace Transform

$$\mathcal{L}[f(t)] = \int_0^\infty e^{-st} f(t)\, dt$$

i) $\mathcal{L}(t^n) = \dfrac{n!}{s^{n+1}}$

ii) $\mathcal{L}(1) = \dfrac{1}{s}$

iii) $\mathcal{L}(e^{at}) = \dfrac{1}{s-a}$

iv) $\mathcal{L}(\sin kt) = \dfrac{k}{s^2+k^2}$

v) $\mathcal{L}(\cos kt) = \dfrac{s}{s^2+k^2}$

vi) $\mathcal{L}(\sinh kt) = \dfrac{k}{s^2-k^2}$

vii) $\mathcal{L}(\cosh kt) = \dfrac{s}{s^2-k^2}$

Ex 4.2 (Laplace of a Derivative)

~~$\mathcal{L}\{f(t)\}$~~

$$\mathcal{L}[f'(t)] = \int_0^\infty e^{-st} f'(t)\,dt$$

Integrating by parts

$$\int U.V = V\int U - \int \int U.V'$$

$$= e^{-st}\int_0^\infty f'(t) - \int_0^\infty \int f'(t) . \frac{d}{dt} e^{-st}\,dt$$

~~$= e^{-st}[f(t)]_0^\infty - \int_0$~~

$$= \left[e^{-st} f(t)\right]_0^\infty - \int_0^\infty f(t) . e^{-st}(-s)\,dt$$

$$= \left[e^{-st} f(t)\right]_0^\infty + s\int_0^\infty f(t) e^{-st}\,dt$$

$$\because \int_0^\infty f(t) e^{-st} = \mathcal{L}(f(t)) = F(s)$$

$$= \left(e^{-\infty} f(\infty) - e^0 f(0)\right) + s F(s)$$

$$= 0 - f(0) + s F(s)$$

$$\boxed{\mathcal{L}[f'(t)] = s F(s) - f(0)}$$

$$\boxed{\mathcal{L}[f''(t)] = s^2 F(s) - s f(0) - f'(0)}$$

$$\boxed{\mathcal{L}[f'''(t)] = s^3 F(s) - s^2 f(0) - s f'(0) - f''(0)}$$

Simplification of L'Hospital

$$\lim_{t\to\infty} e^{-st} f(t) = \frac{f(t)}{e^{st}} \quad \text{or} \quad \frac{e^{-st}}{1/f(t)}$$

<u>Example question:-</u>

$$\frac{dy}{dt} + 3y = 13 \sin 2t \quad ; \quad f(0) = 6$$

$$\mathcal{L}(y') + 3\mathcal{L}(y) = 13\,\mathcal{L}(\sin 2t)$$

$$sF(s) - f(0) + 3F(s) = 13\left(\frac{2}{s^2+4}\right)$$

$$sF(s) - 6 + 3F(s) = \frac{26}{s^2+4}$$

$$F(s)(s+3) = \frac{26}{s^2+4} + 6$$

$$F(s) = \frac{26 + 6(s^2+4)}{(s+3)(s^2+4)}$$

$$= \frac{26 + 6s^2 + 24}{(s+3)(s^2+4)}$$

$$F(s) = \frac{6s^2+50}{(s+3)(s^2+4)}$$

Now,

$$\frac{6s^2+50}{(s+3)(s^2+4)} = \frac{A}{s+3} + \frac{Bs+C}{(s^2+4)}$$

$$6s^2+50 = A(s^2+4) + Bs+C(s+3)$$

$$6s^2+50 = As^2+4A+Bs^2+3Bs+Cs+3C$$

$$6s^2+50 = s^2(A+B) + s(3B+C) + 4A+3C$$

$6 = A+B$ | $3B+C = 0$ or $C = -3B$ | $4A+3C = 50$

Solving by calculator simultaneously

$\boxed{A=8}$, $\boxed{B=-2}$; $\boxed{C=6}$

$$= \frac{8}{s+3} - \frac{2s+6}{s^2+4}$$

$$= \frac{8}{s+3} - 2\frac{s}{s^2+4} + \frac{6}{s^2+4}$$

$$= \mathcal{L}^{-1}\left(\frac{1}{s-(-3)}\right) - 2\mathcal{L}^{-1}\left(\frac{s}{s^2+2^2}\right) + 3\mathcal{L}^{-1}\left(\frac{2}{s^2+2^2}\right)$$

$$\boxed{y(t) = e^{-3t} - 2\cos 2t + 3\sin 2t}$$

Ex 4.3:-

TRANSLATION THEOREM:

$f(t) = t^2 e^t$

$\mathcal{L}[f(t)] = \int_0^\infty e^{-st}(t^2 e^t)\,dt$

$= \int_0^\infty e^{-t(s-1)} t^2\,dt$

↳ very long by using by part

First Translation THEOREM/Shift Theorem:-

~~$\mathcal{L}[f(t)] = F(s-a)$~~

$\mathcal{L}[f(t)\,\boxed{e^{at}}] = F(s-a)$

For the application of this theorem exp must be present. the other one can be polynomial/ trig/hyperbolic (Sin/Cos) (Sinh/Cosh)

$\mathcal{L}[e^{3t}.t^2] = \left.\frac{2!}{s^3}\right|_{s\to s-a}$

$= \frac{2!}{(s-3)^3}$

This theorem is also called translation on s-axis.

In the case of inverse of First-Shift theorem, the power in the denominator will be <1,

For eg.:

$$\mathcal{L}^{-1}(F(s)) = \mathcal{L}^{-1}\left[\frac{2s+5}{(s-3)^2}\right]$$

$$\frac{2s+5}{(s-3)^2} = \frac{A}{s-3} + \frac{B}{(s-3)^2}$$

$$2s+5 = A(s-3) + B$$

$$2s+5 = As - 3A + B$$

$$\boxed{A = 2}$$

$$-3A + B = 5$$

$$-6 + B = 5$$

$$\boxed{B = 11}$$

$$y = 2\mathcal{L}^{-1}\left(\frac{1}{s-3}\right) + 11\,\mathcal{L}^{-1}\left\{\frac{1}{(s-3)^2}\right\}$$

$$= 2e^{3} + 11\,\mathcal{L}^{-1}\left(\frac{1}{s^2}\bigg|_{s \to s-3}\right)$$

$$\boxed{y = 2e^{3t} + 11\,te^{3t}}$$

Now, Generalised form.

$$\mathcal{L}(e^{at} t^n) = e^{at} \frac{n!}{s^{n+1}}$$

$$\boxed{\mathcal{L}(e^{at} t^n) = \frac{n!}{(s-a)^{n+1}}}$$

$$\mathcal{L}(e^{at} \cos nt) = \left[\frac{s}{s^2+k^2}\right]$$

Put $s = s-a$

$$\boxed{\mathcal{L}(e^{at} \cos nt) = \frac{s-a}{(s-a)^2+k^2}}$$