Differential Equations

PPLATO PROMOTING PHYSICS LEARNING & TEACHING OPPORTUNITIES

SEPARATION OF VARIABLES

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A Tutorial Module for learning the technique of separation of variables

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Full worked solutions

1. Theory

If one can re-arrange an ordinary differential equation into the following standard form:

$$\frac{dy}{dx} = f(x)g(y),$$

then the solution may be found by the technique of **SEPARATION OF VARIABLES**:

$$\int \frac{dy}{g(y)} = \int f(x)\, dx\,.$$

This result is obtained by dividing the standard form by $g(y)$, and then integrating both sides with respect to x.

2. Exercises

Click on Exercise links for full worked solutions (there are 16 exercises in total)

Exercise 1.
Find the general solution of $\frac{dy}{dx} = 3x^2 e^{-y}$ and the particular solution that satisfies the condition $y(0) = 1$

Exercise 2.
Find the general solution of $\frac{dy}{dx} = \frac{y}{x}$

Exercise 3.
Solve the equation $\frac{dy}{dx} = \frac{y+1}{x-1}$ given the boundary condition: $y = 1$ at $x = 0$

Exercise 4.

Solve $y^2\dfrac{dy}{dx} = x$ and find the particular solution when $y(0) = 1$

Exercise 5.

Find the solution of $\dfrac{dy}{dx} = e^{2x+y}$ that has $y = 0$ when $x = 0$

Exercise 6.

Find the general solution of $\dfrac{xy}{x+1} = \dfrac{dy}{dx}$

Exercise 7.

Find the general solution of $x\sin^2 y.\dfrac{dy}{dx} = (x+1)^2$

Exercise 8.

Solve $\dfrac{dy}{dx} = -2x\tan y$ subject to the condition: $y = \frac{\pi}{2}$ when $x = 0$

Exercise 9.

Solve $(1 + x^2)\dfrac{dy}{dx} + xy = 0$
and find the particular solution when $y(0) = 2$

Exercise 10.

Solve $x\dfrac{dy}{dx} = y^2 + 1$ and find the particular solution when $y(1) = 1$

Exercise 11.

Find the general solution of $x\dfrac{dy}{dx} = y^2 - 1$

Exercise 12.

Find the general solution of $\dfrac{1}{y}\dfrac{dy}{dx} = \dfrac{x}{x^2+1}$

Exercise 13.

Solve $\dfrac{dy}{dx} = \dfrac{y}{x(x+1)}$ and find the particular solution when $y(1) = 3$

Exercise 14.

Find the general solution of $\sec x \cdot \dfrac{dy}{dx} = \sec^2 y$

Exercise 15.

Find the general solution of $\operatorname{cosec}^3 x\, \dfrac{dy}{dx} = \cos^2 y$

Exercise 16.

Find the general solution of $(1 - x^2)\,\dfrac{dy}{dx} + x(y - a) = 0$, where a is a constant

3. Answers

1. General solution is $y = \ln(x^3 + A)$, and particular solution is $y = \ln(x^3 + e)$,

2. General solution is $y = kx$,

3. General solution is $y + 1 = k(x - 1)$, and particular solution is $y = -2x + 1$,

4. General solution is $\frac{y^3}{3} = \frac{x^2}{2} + C$, and particular solution is $y^3 = \frac{3x^2}{2} + 1$,

5. General solution is $y = -\ln\left(-\frac{1}{2}\, e^{2x} - C\right)$, and particular solution is $y = -\ln\left(\frac{3-e^{2x}}{2}\right)$,

6. General solution is $e^x = ky(x + 1)$,

7. General solution is $\frac{y}{2} - \frac{1}{4}\sin 2y = \frac{x^2}{2} + 2x + \ln\, x + C$,

8. General solution is $\sin y = e^{-x^2+A}$, and particular solution is $\sin y = e^{-x^2}$,

9. General solution is $y(1+x^2)^{\frac{1}{2}} = k$, and particular solution is $y(1+x^2)^{\frac{1}{2}} = 2$,

10. General solution is $\tan^{-1} y = \ln\, x + C$, and particular solution is $\tan^{-1} y = \ln\, x + \frac{\pi}{4}$,

11. General solution is $y - 1 = kx^2(y+1)$,

12. General solution is $y^2 = k(x^2+1)$,

13. General solution is $y = \frac{kx}{x+1}$, and particular solution is $y = \frac{6x}{x+1}$,

14. General solution is $2y + \sin 2y = 4\sin x + C$,

15. General solution is $\tan y = -\cos x + \frac{1}{3}\cos^3 x + C$,

16. General solution is $y - a = k(1 - x^2)^{\frac{1}{2}}$.

4. Standard integrals

$f(x)$	$\int f(x)dx$	$f(x)$	$\int f(x)dx$
x^n	$\frac{x^{n+1}}{n+1} \quad (n \neq -1)$	$[g(x)]^n g'(x)$	$\frac{[g(x)]^{n+1}}{n+1} \quad (n \neq -1)$
$\frac{1}{x}$	$\ln \lvert x \rvert$	$\frac{g'(x)}{g(x)}$	$\ln \lvert g(x) \rvert$
e^x	e^x	a^x	$\frac{a^x}{\ln a} \quad (a > 0)$
$\sin x$	$-\cos x$	$\sinh x$	$\cosh x$
$\cos x$	$\sin x$	$\cosh x$	$\sinh x$
$\tan x$	$-\ln \lvert \cos x \rvert$	$\tanh x$	$\ln \cosh x$
$\operatorname{cosec} x$	$\ln \left\lvert \tan \frac{x}{2} \right\rvert$	$\operatorname{cosech} x$	$\ln \left\lvert \tanh \frac{x}{2} \right\rvert$
$\sec x$	$\ln \lvert \sec x + \tan x \rvert$	$\operatorname{sech} x$	$2 \tan^{-1} e^x$
$\sec^2 x$	$\tan x$	$\operatorname{sech}^2 x$	$\tanh x$
$\cot x$	$\ln \lvert \sin x \rvert$	$\coth x$	$\ln \lvert \sinh x \rvert$
$\sin^2 x$	$\frac{x}{2} - \frac{\sin 2x}{4}$	$\sinh^2 x$	$\frac{\sinh 2x}{4} - \frac{x}{2}$
$\cos^2 x$	$\frac{x}{2} + \frac{\sin 2x}{4}$	$\cosh^2 x$	$\frac{\sinh 2x}{4} + \frac{x}{2}$

$f(x)$	$\int f(x)\,dx$	$f(x)$	$\int f(x)\,dx$
$\frac{1}{a^2+x^2}$	$\frac{1}{a}\tan^{-1}\frac{x}{a}$	$\frac{1}{a^2-x^2}$	$\frac{1}{2a}\ln\left\lvert\frac{a+x}{a-x}\right\rvert \quad (0<\lvert x\rvert<a)$
	$(a>0)$	$\frac{1}{x^2-a^2}$	$\frac{1}{2a}\ln\left\lvert\frac{x-a}{x+a}\right\rvert \quad (\lvert x\rvert>a>0)$
$\frac{1}{\sqrt{a^2-x^2}}$	$\sin^{-1}\frac{x}{a}$	$\frac{1}{\sqrt{a^2+x^2}}$	$\ln\left\lvert\frac{x+\sqrt{a^2+x^2}}{a}\right\rvert \quad (a>0)$
	$(-a<x<a)$	$\frac{1}{\sqrt{x^2-a^2}}$	$\ln\left\lvert\frac{x+\sqrt{x^2-a^2}}{a}\right\rvert \quad (x>a>0)$
$\sqrt{a^2-x^2}$	$\frac{a^2}{2}\left[\sin^{-1}\left(\frac{x}{a}\right)\right.$	$\sqrt{a^2+x^2}$	$\frac{a^2}{2}\left[\sinh^{-1}\left(\frac{x}{a}\right)+\frac{x\sqrt{a^2+x^2}}{a^2}\right]$
	$\left.+\frac{x\sqrt{a^2-x^2}}{a^2}\right]$	$\sqrt{x^2-a^2}$	$\frac{a^2}{2}\left[-\cosh^{-1}\left(\frac{x}{a}\right)+\frac{x\sqrt{x^2-a^2}}{a^2}\right]$

5. Tips on using solutions

- When looking at the THEORY, ANSWERS, INTEGRALS, or TIPS pages, use the Back button (at the bottom of the page) to return to the exercises.

- Use the solutions intelligently. For example, they can help you get started on an exercise, or they can allow you to check whether your intermediate results are correct.

- Try to make less use of the full solutions as you work your way through the Tutorial.

Full worked solutions

Exercise 1.

This is of the form $\boxed{\dfrac{dy}{dx} = f(x)g(y)}$, where $f(x) = 3x^2$ and $g(y) = e^{-y}$, so we can separate the variables and then integrate,

$$\text{i.e.} \quad \int e^y dy = \int 3x^2 dx \quad \text{i.e. } e^y = x^3 + A$$

(where A = arbitrary constant).

i.e. $y = \ln(x^3 + A)$: General solution

Particular solution: $y(x) = 1$ when $x = 0$ i.e. $e^1 = 0^3 + A$

$$\text{i.e. } A = e \quad \text{and} \quad y = \ln(x^3 + e)\,.$$

Return to Exercise 1

Exercise 2.

This is of the form $\boxed{\dfrac{dy}{dx} = f(x)g(y)}$, where $f(x) = \frac{1}{x}$ and $g(y) = y$, so we can separate the variables and then integrate,

$$\int \frac{dy}{y} = \int \frac{dx}{x}$$

$$\begin{aligned} \text{i.e.} \quad \ln\ y &= \ln\ x + C \\ &= \ln\ x + \ln\ k \quad (\ln\ k = C = \text{constant}) \\ \text{i.e.} \quad \ln\ y - \ln\ x &= \ln\ k \\ \text{i.e.} \quad \ln(y/x) &= \ln\ k \\ \text{i.e.} \quad y &= kx\,. \end{aligned}$$

Return to Exercise 2

Exercise 3.

Find the general solution first. Then apply the boundary condition to get the particular solution.

Equation is of the form: $\dfrac{dy}{dx} = f(x)g(y)$, where $\begin{array}{l} f(x) = \frac{1}{x-1} \\ g(y) = y+1 \end{array}$

so separate variables and integrate.

$$\text{i.e. } \int \frac{dy}{y+1} = \int \frac{dx}{x-1}$$

$$\begin{aligned} \text{i.e. } \ln(y+1) &= \ln(x-1) + C \\ &= \ln(x-1) + \ln k \qquad (k = \text{arbitrary constant}) \end{aligned}$$

$$\text{i.e. } \ln(y+1) - \ln(x-1) = \ln k$$

$$\text{i.e. } \ln\left(\frac{y+1}{x-1}\right) = \ln k$$

i.e. $\dfrac{y+1}{x-1} = k$

i.e. $y + 1 = k(x-1)$ (<u>general solution</u>)

Now determine k for particular solution with $y(0) = 1$.

$$\begin{array}{lll} \begin{array}{l} x = 0 \\ y = 1 \end{array} & \text{gives} & 1 + 1 = k(0-1) \\ & \text{i.e.} & 2 = -k \\ & \text{i.e.} & k = -2 \end{array}$$

<u>Particular solution</u>: $y + 1 = -2(x-1)$ i.e. $y = -2x + 1$.

Return to Exercise 3

Exercise 4.

Use separation of variables to find the general solution first.

$$\int y^2 dy = \int x\,dx \quad \text{i.e. } \frac{y^3}{3} = \frac{x^2}{2} + C$$

(<u>general solution</u>)

<u>Particular solution</u> with $y = 1, x = 0$: $\frac{1}{3} = 0 + C$ i.e. $C = \frac{1}{3}$

i.e. $y^3 = \frac{3x^2}{2} + 1$.

Return to Exercise 4

Exercise 5.

General solution first then find particular solution.

Write equation as: $$\frac{dy}{dx} = e^{2x}e^{y}\,(\equiv f(x)g(y))$$

Separate variables and integrate: $$\int \frac{dy}{e^{y}} = \int e^{2x}dx$$

$$\text{i.e.}\quad -e^{-y} = \tfrac{1}{2}e^{2x} + C$$

$$\text{i.e.}\quad e^{-y} = -\tfrac{1}{2}e^{2x} - C$$

$$\text{i.e.}\quad -y = \ln\left(-\tfrac{1}{2}e^{2x} - C\right)$$

$$\text{i.e.}\quad y = -\ln\left(-\tfrac{1}{2}e^{2x} - C\right).$$

<u>Particular solution</u>: $\begin{matrix} x=0 \\ y=0 \end{matrix}$ gives $0 = -\ln\left(-\frac{1}{2} - C\right)$

i.e. $-\frac{1}{2} - C = 1$

i.e. $C = -\frac{3}{2}$

$\therefore$ $y = -\ln\left(\frac{3-e^{2x}}{2}\right).$

Return to Exercise 5

Exercise 6.

Separate variables and integrate:

$$\int \frac{x}{x+1}\,dx = \int \frac{dy}{y}$$

$$\nearrow$$

Numerator and denominator of same degree in x: reduce degree of numerator using long division.

i.e. $\frac{x}{x+1} = \frac{x+1-1}{x+1} = \frac{x+1}{x+1} - \frac{1}{x+1} = 1 - \frac{1}{x+1}$

i.e. $\int \left(1 - \frac{1}{x+1}\right) dx = \int \frac{dy}{y}$

i.e. $x - \ln(x+1) = \ln\, y + \ln\, k$ (ln k = constant of integration)

i.e. $x = \ln(x+1) + \ln\, y + \ln\, k$

$= \ln[ky(x+1)]$

i.e. $e^x = ky(x+1)\,.$ General solution.

Return to Exercise 6

Exercise 7.

Separate variables and integrate:

$$
\begin{aligned}
\text{i.e.} \quad \int \sin^2 y dy &= \int \frac{(x+1)^2}{x} dx \\
\text{i.e.} \quad \int \frac{1}{2}(1-\cos 2y) dy &= \int \frac{x^2+2x+1}{x} dx \\
\text{i.e.} \quad \frac{1}{2}\int dy - \frac{1}{2}\int \cos 2y dy &= \int \left(x+2+\frac{1}{x}\right) dx \\
\text{i.e.} \quad \frac{1}{2}\, y - \frac{1}{2}\cdot\frac{1}{2}\sin 2y &= \frac{1}{2}x^2 + 2x + \ln\, x + C\,.
\end{aligned}
$$

Return to Exercise 7

Exercise 8.
General solution first.

Separate variables: i.e. $\dfrac{dy}{\tan y} = -2x\,dx$

Integrate: i.e. $\displaystyle\int \cot y\,dy = -2\int x dx$

i.e. $\ln(\sin y) = -2\cdot\frac{x^2}{2} + A$

i.e. $\ln(\sin y) = -x^2 + A$

i.e. $\sin y = e^{-x^2+A}$

$$\left\{\text{Note: } \int \frac{\cos y}{\sin y} dy \ \text{ is of form } \int \frac{f'(y)}{f(y)}\,dy = \ln[f(y)] + C\right\}$$

Particular solution: $y = \frac{\pi}{2}$ when $x = 0$

$$\begin{aligned} \text{gives} \quad & \sin\tfrac{\pi}{2} = e^{A} \\ \text{i.e.} \quad & 1 = e^{A} \\ \text{i.e.} \quad & A = 0 \\ \therefore \quad & \text{Required solution is } \sin y = e^{-x^2}\,. \end{aligned}$$

Return to Exercise 8

Exercise 9.

Separate variables and integrate:

$$\begin{aligned}(1+x^2)\frac{dy}{dx} &= -xy \\ \text{i.e.} \qquad \int \frac{dy}{y} &= -\int \frac{x}{1+x^2}\,dx \\ \text{i.e.} \qquad \int \frac{dy}{y} &= -\frac{1}{2}\int \frac{2x}{1+x^2}\,dx\end{aligned}$$

$$\left[\text{compare with } \int \frac{f'(x)}{f(x)}\,dx\right]$$

i.e. $\ln y = -\frac{1}{2}\ln(1+x^2) + \ln k \qquad (\ln k = \text{ constant})$

i.e. $\ln y + \ln(1+x^2)^{\frac{1}{2}} = \ln k$

i.e. $\ln\left[y(1+x^2)^{\frac{1}{2}}\right] = \ln k$

i.e. $y(1+x^2)^{\frac{1}{2}} = k, \qquad$ (general solution).

Particular solution

$$\begin{aligned} y(0) = 2, \quad & \text{i.e. } y(x) = 2 \text{ when } x = 0 \\ & \text{i.e. } 2(1+0)^{\frac{1}{2}} = k \\ & \text{i.e. } k = 2 \\ & \text{i.e. } y(1+x^2)^{\frac{1}{2}} = 2\,. \end{aligned}$$

Return to Exercise 9

Exercise 10.

$$\int \frac{dy}{y^2+1} = \int \frac{dx}{x}$$

$$\left\{\text{Standard integral: } \int \frac{dy}{1+y^2} = \tan^{-1} y + C\right\}$$

i.e. $\tan^{-1} y = \ln\, x + C$. <u>General solution</u>.

Particular solution with $y = 1$ when $x = 1$:

$\tan \frac{\pi}{4} = 1 \;\therefore \tan^{-1}(1) = \frac{\pi}{4}$, while $\ln\, 1 = 0$ (i.e. $1 = e^0$)

$\therefore\ \frac{\pi}{4} = 0 + C$ i.e. $C = \frac{\pi}{4}$

<u>Particular solution</u> is: $\tan^{-1} y = \ln\, x + \frac{\pi}{4}$.

Return to Exercise 10

Exercise 11.

$$\begin{aligned}\int \frac{dy}{y^2-1} &= \int \frac{dx}{x}\\ \underline{\text{Partial fractions}}:\ \frac{1}{y^2-1} = \frac{A}{y-1}+\frac{B}{y+1} &= \frac{A(y+1)+B(y-1)}{(y-1)(y+1)}\\ &= \frac{(A+B)y+(A-B)}{y^2-1}\end{aligned}$$

Compare numerators: $1 = (A+B)y+(A-B)$ [true for all y]

$\therefore$

$$\begin{array}{r} A+B=0\\ A-B=1\\ \hline 2A=1\\ \hline \end{array}$$

$$\therefore\ A=\tfrac{1}{2},\ B=-\tfrac{1}{2}\ .$$

i.e. $\int \frac{A}{y-1} + \frac{B}{y+1}\, dy = \int \frac{dx}{x}$

i.e. $\frac{1}{2} \int \frac{1}{y-1} - \frac{1}{y+1}\, dy = \int \frac{dx}{x}$

i.e. $\frac{1}{2} \left[\ln(y-1) - \ln(y+1)\right] = \ln\, x + \ln\, k$

i.e. $\ln(y-1) - \ln(y+1) - 2\, \ln\, x = 2\, \ln\, k$

i.e. $\ln\left[\frac{y-1}{(y+1)x^2}\right] = 2\, \ln\, k$

i.e. $y - 1 = k' x^2 (y+1), \qquad (k' = k^2 = \text{constant})\,.$

Return to Exercise 11

Exercise 12.

$$\int \frac{dy}{y} = \int \frac{x}{x^2+1}\,dx = \frac{1}{2}\int \frac{2x}{x^2+1}\,dx$$

$$\left\{\underline{\text{Note}} : \int \frac{f'(x)}{f(x)}\,dx = \ln[f(x)] + A\right\}$$

$$\begin{aligned}
\text{i.e.} \quad \ln\, y &= \frac{1}{2}\,\ln\left(x^2+1\right) + C \\
\text{i.e.} \quad \frac{1}{2}\ln\, y^2 &= \frac{1}{2}\,\ln\left(x^2+1\right) + C \ \text{\{get same coefficients to} \\
& \qquad\qquad \text{allow log manipulations\}} \\
\text{i.e.} \quad \frac{1}{2}\,\ln\left[\frac{y^2}{x^2+1}\right] &= C \\
\text{i.e.} \quad \frac{y^2}{x^2+1} &= e^{2C} \\
\text{i.e.} \quad y^2 &= k\left(x^2+1\right), \quad (\text{where } k = e^{2C} = \text{ constant})\,.
\end{aligned}$$

Return to Exercise 12

Exercise 13.

$$\int \frac{dy}{y} = \int \frac{dx}{x(x+1)}$$

Use partial fractions:

$$\begin{aligned} \frac{1}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1} &= \frac{A(x+1)+Bx}{x(x+1)} \\ &= \frac{(A+B)x+A}{x(x+1)} \end{aligned}$$

Compare numerators: $1 = (A+B)x + A \quad$ (true for all x)

i.e. $A + B = 0$ and $A = 1, \quad \therefore B = -1$

i.e. $\displaystyle\int \frac{dy}{y} = \int \left(\frac{1}{x} - \frac{1}{x+1}\right) dx$

i.e. $\ln y = \ln x - \ln(x+1) + C$

i.e. $\ln\, y - \ln\, x + \ln\,(x+1) = \ln\, k \qquad (\ln\, k = C = \text{constant})$

i.e. $\ln\left[\dfrac{y(x+1)}{x}\right] = \ln\, k$

i.e. $\dfrac{y(x+1)}{x} = k$

i.e. $y = \dfrac{kx}{x+1}$. General solution.

Particular solution with $y(1) = 3$:

$$x = 1, \quad y = 3 \quad \text{gives} \quad 3 = \tfrac{k}{1+1}$$

$$\text{i.e. } k = 6$$

$$\text{i.e. } y = \tfrac{6x}{x+1} \quad .$$

Return to Exercise 13

Exercise 14.

$$\int \frac{dy}{\sec^2 y} = \int \frac{dx}{\sec x}$$

$$\begin{array}{lrcl}
\text{i.e.} & \int \cos^2 y\, dy & = & \int \cos x\, dx \\
\text{i.e.} & \int \frac{1+\cos 2y}{2}\, dy & = & \int \cos x\, dx \\
\text{i.e.} & \frac{y}{2} + \frac{1}{2}\cdot\frac{1}{2}\sin 2y & = & \sin x + C \\
\text{i.e.} & 2y + \sin 2y & = & 4\sin x + C' \\
 & & & \\
 & (\text{where } C' & = & 4C = \text{constant})\,.
\end{array}$$

Return to Exercise 14

Exercise 15.

$$\begin{aligned}
\text{i.e.} \int \frac{dy}{\cos^2 y} &= \int \frac{dx}{\text{cosec}^3 x} \\
&= \int \sin^3 x \, dx \\
&= \int \sin^2 x \cdot \sin x \, dx \\
&= \int (1 - \cos^2 x) \cdot \sin x \, dx \\
&= \int \sin x \, dx \underbrace{- \int \cos^2 x \cdot \sin x \, dx}_{\substack{\text{set } u = \cos x \text{ , so } \frac{du}{dx} = -\sin x \\ \text{and } \cos^2 x \cdot \sin x \, dx = -u^2 du}}
\end{aligned}$$

LHS is standard integral

$$\int \sec^2 y\, dy = \tan y + A\,.$$

This gives, $$\tan y = -\cos x - \left(-\frac{\cos^3 x}{3}\right) + C$$

$$\text{i.e.}\quad \tan y = -\cos x + \frac{\cos^3 x}{3} + C\,.$$

Return to Exercise 15

Exercise 16.

$$
\begin{array}{ll}
\text{i.e.} & (1-x^2)\frac{dy}{dx} = -x(y-a) \\[2ex]
\text{i.e.} & \int \frac{dy}{y-a} = -\int \frac{x}{1-x^2}dx \\[2ex]
\text{i.e.} & \int \frac{dy}{y-a} = +\frac{1}{2}\int \frac{-2x}{1-x^2}dx \quad \text{[compare RHS integral with } \int \frac{f'(x)}{f(x)}\,dx\text{]} \\[2ex]
\text{i.e.} & \ln(y-a) = \frac{1}{2}\ \ln(1-x^2) + \ln\ k \\[2ex]
\text{i.e.} & \ln(y-a) - \ln(1-x^2)^{\frac{1}{2}} = \ln\ k \\[2ex]
\text{i.e.} & \ln\left[\frac{y-a}{(1-x^2)^{\frac{1}{2}}}\right] = \ln\ k \\[2ex]
\therefore & y-a = k(1-x^2)^{\frac{1}{2}}\ .
\end{array}
$$

Return to Exercise 16