Risk Factor Models

Why do we need a factor model?

Risktakers need to know their portfolio's factor exposure so that they can take a certain factor tilt depending on the market condition.

Risk managers need to know factor exposure so that they can estimate portfolio risk.
Note: Risk = Standard Deviation

Following two inputs are required to calculate standard deviation:
1. Asset Weights
2. Asset returns covariance matrix

Asset weights are easy to obtain:

$$\text{Weight}(w) = \text{Price}(p) \times \text{Quantity}(q) \tag{1}$$

Calculating covariance matrix is not easy:
The obvious solution of calculating variances and covariances using history of asset returns is inaccurate. A universe of N asset would require $O(N^2)$ data points.
Covariance matrix grows parabolic with respect to number of securities N.

Risk Calculations

Assuming Q to be a $n \times n$ covariance matrix of asset returns:
h to be the $n \times 1$ portfolio weight matrix:

$$Q = \begin{pmatrix} \sigma_1^2 & \cdots & \rho_{1n}\sigma_1\sigma_n \\ \vdots & \ddots & \vdots \\ \rho_{n1}\sigma_n\sigma_1 & \cdots & \sigma_n^2 \end{pmatrix} \tag{2}$$

$$h = \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix} \tag{3}$$

Where, ρ_{ij} is the correlation between asset i and asset j,
σ_i is the standard deviation of asset i,
Where, w_i, $i = 1, \ldots, n$ are the weights of each asset.

The risk of the portfolio is simply:

$$\sigma_h = \sqrt{h^T Q h} \tag{4}$$

Asset returns can be decomposed into a portion driven by common factors (systematic part) and residual component (specific/idiosyncratic part).

$$\begin{pmatrix} r_1 \\ \vdots \\ r_n \end{pmatrix} = \begin{pmatrix} b_{11} & \cdots & b_{1m} \\ \vdots & \ddots & \vdots \\ b_{n1} & \cdots & b_{nm} \end{pmatrix}_{n \times m} \begin{pmatrix} f_1 \\ \vdots \\ f_m \end{pmatrix} + \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} \tag{5}$$

Or, in matrix form:

$$r = Bf + u \tag{6}$$

Assuming specific returns are uncorrelated amongst themselves and with factor returns, the asset return covariance matrix becomes:

$$Q = B \Sigma B^T + \Delta^2 \tag{7}$$

Where,
Σ is the $m \times m$ factor covariance matrix of factor returns,
Δ^2 is the diagonal matrix of specific variances.

Estimation Universe

There are two sets of assets:

- The model universe, i.e., all of the stocks contained in a particular model
- The estimation universe, which is a subset of model universe and is used to estimating factor returns

The estimation universe must be:

- **Representative**: It should reflect the full breadth of investment opportunities
- **Liquid**: The assets must have reliable and regular prices.
- **Stable**: This is to ensure factor exposures are well behaved across time.

Factor Exposures/Betas

Factors are broadly classified into:

- Market and country factors
- Industry factors
- Style factors

Market and country factors: These factors define the broad market and country behaviors.

Industry Factors

Barra primarily uses GICS classification for industry membership. It assigns multiple industry exposures to stocks based on the firm's business segment reporting. Specifically, Barra regresses the market cap of the stocks against their reported assets within each industry.

$$M_n = \sum_k A_{nk}\beta_k^A + \epsilon_n \tag{8}$$

Where, A_{nk} is the asset of stock within industry k, β_k^A is the industry beta. The industry exposures using assets are given by:

$$X_{nk}^A = \frac{A_{nk}\beta_k^A}{\sum_i A_{ni}\beta_i^A} \tag{9}$$

BARRA also uses Sales as an explanatory variable to estimate industry exposures:

$$X_{nk} = 0.75X_{nk}^A + 0.25X_{nk}^S \tag{10}$$

Factor Returns

Once factor betas are calculated, Ordinary Least Square (OLS) regression solution may be used to estimate factor returns.

The solution of the equation is:

$$f_{ols} = \arg\min_f \sum_{i=1}^{n} u_i^2 \tag{11}$$

$$f_{ols} = (B^T B)^{-1} B^T r \tag{12}$$

To tackle this, both Barra and Axioma use weighted least-square regression, assuming that the variance of specific returns is inversely proportional to the square root of total market capitalization. The weighting scheme also tunes the regression estimates in favor of larger assets.

$$f_{wls} = (B^T W B)^{-1} B^T W r \tag{13}$$

Where,

$$W = \begin{pmatrix} \sigma_1^{-2} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \sigma_n^{-2} \end{pmatrix} \tag{14}$$

and

$$\sigma_i^2 \propto \frac{1}{\sqrt{M_i}} \tag{15}$$

M_i is the market cap of asset i.

Factor Covariance Matrix

Both Barra and Axioma estimate factor volatilities and correlations separately. The covariance matrix is calculated simply as,

$$\Sigma = \text{Sigma} \times \text{Correl} \times \text{Sigma} \tag{16}$$

Where,

$$\text{Sigma} = \begin{pmatrix} \sigma_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \sigma_m \end{pmatrix} \tag{17}$$

$$\text{Correl} = \begin{pmatrix} \rho_{11} & \cdots & \rho_{1m} \\ \vdots & \ddots & \vdots \\ \rho_{m1} & \cdots & \rho_{mm} \end{pmatrix} \tag{18}$$

Specific Risk Calculation

Both Barra and Axioma estimate asset level specific risk directly from the time series of specific returns,

$$\sigma_i^2 = \frac{1}{T-1} \sum_{t=1}^{T} w_t \left(u_{i,t} - u_{\text{mean},i}\right)^2 \tag{19}$$

Barra applies Bayesian shrinkage that shrink the specific volatility estimates toward the cap-weighted mean specific volatility for the size decile S_n to which the stock belongs.

$$\sigma_i^{SH} = v_n \sigma_{\text{mean},S_n} + (1 - v_n)\sigma_i \tag{20}$$

Where,

$$\sigma_{\text{mean},S_n} = \sum_{k \in S_n} w_k \sigma_k \tag{21}$$

w_k is the market cap of stock k and v_n is the shrinkage intensity.

Back To Risk Calculations

The risk of the portfolio is simply:

$$\sigma_h = \sqrt{h^T Q h} \tag{22}$$

Where $Q = B \Sigma B^T + \Delta^2$ and h is an $n \times 1$ vector of portfolio holdings.
Let $X_P = B^T h$, then:

$$\sigma_h^2 = X_P^T \Sigma X_P + h^T \Delta^2 h \tag{23}$$

Let h_B be the benchmark holding vector, then we can define:

$$h_{PA} = h - h_B \tag{24}$$

$$X_{PA} = B^T h_{PA} \tag{25}$$

$$X_B = B^T h_B \tag{26}$$

Passive or beta risk is given by:

$$\psi_P^2 = X_B^T \Sigma X_B + h_B^T \Delta^2 h_B \tag{27}$$

Active risk or tracking error is given by:

$$\psi_A^2 = X_{PA}^T \Sigma X_{PA} + h_{PA}^T \Delta^2 h_{PA} \tag{28}$$