Inverse Options in a Black-Scholes World

Carol Alexander* and Arben Imeraj†

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Abstract

Most trading in cryptocurrency options is on inverse products, so called because the contract size is denominated in US dollars and they are margined and settled in crypto, typically bitcoin or ether. Their popularity stems from allowing professional traders in bitcoin or ether options to avoid transferring fiat currency to and from the exchanges. We derive new analytic pricing and hedging formulae for inverse options under the assumption that the underlying follows a geometric Brownian motion. The boundary conditions and hedge ratios exhibit relatively complex but very important new features which warrant further analysis and explanation. We also illustrate some inconsistencies, exhibited in time series of Deribit bitcoin option implied volatilities, which indicate that traders may be applying direct option hedging and valuation methods erroneously. This could be because they are unaware of the correct, inverse option characteristics which are derived in this paper.

Keywords: Bitcoin, Delta, Deribit, Ether, Gamma, Implied Volatility, Pricing Formula, Vega

JEL Classification: C02, G12, G23

*University of Sussex Business School. Email: c.alexander@sussex.ac.uk
†University of Sussex Business School. Email: a.imeraj@sussex.ac.uk

1 Introduction

During the last few years a number of exchanges have started to offer European options on bitcoin and ether against the US dollar. Many platforms that offer crypto options are regulated (or, at least semi-regulated) including the CME, LedgerX, FTX and IQ Option. However, the Deribit exchange, which is registered in Panama at the time of writing, is not regulated at all. Yet, almost two-thirds of the trading volume on crypto options is currently on that exchange. The price bubble in crypto assets at the end of 2020 precipitated a rapid growth in the popularity of these products and by May 2021 the daily trading volume of bitcoin options on Deribit alone exceeded $3 billion.

Apart from regulatory oversight, the main factor that differentiates these platforms is the type of options that they offer. For example, CME and LedgerX run order books in standard European put and call options with a contract size in bitcoin or ether and margined and settled in USD.[1] But $\sim 90\%$ of open interest and trading volume on crypto options has always been on Deribit, which only runs order books in so-called *inverse* options. These have a contract size in USD with the margining and settlement in bitcoin or ether. The reason why so much volume goes to Deribit is that professional traders do not need to use fiat currency for on-boarding and off-boarding the exchange. In fact, Derbit only allows deposits in bitcoin, which can then be swapped to ether if required, specifically for margining and settlement of their inverse options. Other large but unregulated exchanges like Binance allow transfers of stable coins such as tether, because they offer *direct* options exactly like the CME and LedgerX products, except that margining and settlement is in tether or another stable USD coin.

Several previous papers examine bitcoin option valuation, including Li et al. (2019); Cretarola et al. (2020); Hou et al. (2020); Siu and Elliott (2021) and Cao and Celik (2021) but all of them propose valuation models for *direct* options. The papers are differentiated by the stochastic process that is assumed for modelling the bitcoin price dynamics, some of which are highly complex.

Despite their overwhelming dominance in terms of volume and open interest, this is the first study of inverse options. The usual Black-Scholes (BS) formulae for prices and hedge ratios do not apply because the pay-offs and boundary conditions are different. Amongst many other results, we prove that inverse puts and calls have different gammas and different vegas, and that an in-the-money (ITM) inverse call may have a negative gamma and a negative vega. Our work is very important for professional traders because the results derived here are critical for achieving a balanced book of options and perpetuals which is hedged against both price and volatility risks.

Clearly, there is a very significant gap in the literature which this paper fills. In the following: Section 2 describes inverse derivatives products, in some detail because they are unique to crypto markets and will be unfamiliar to many readers; Section 3 derives a formula for the price of a standard European inverse option under the geometric Brownian motion assumption; Section 4 derives the BS-type hedge ratios for inverse options and describes their behaviour; Section 5 uses historical Deribit option prices to depict time series of implied volatilities; and Section 6 concludes.

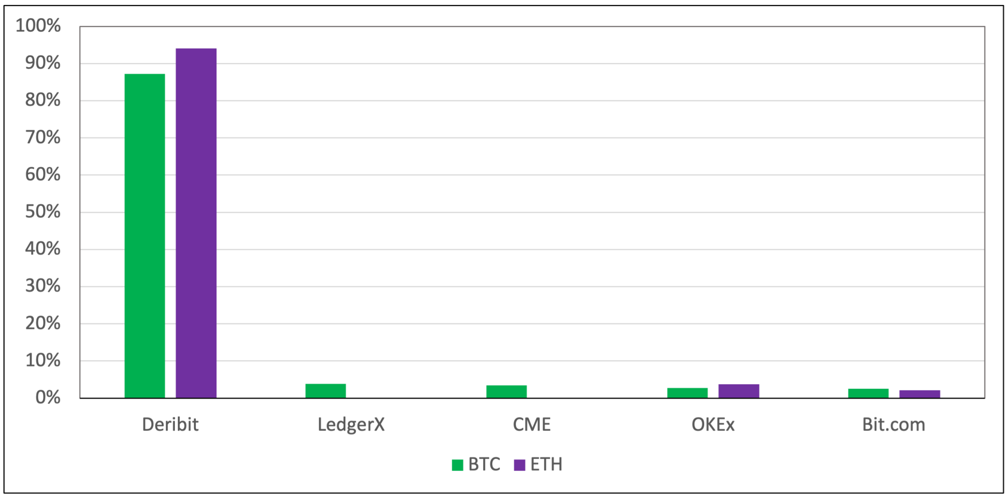
[1] See, for instance, LedgerX Contract Specifications. The CME lists two types of option contracts: standard and micro, i.e. an option on one or one tenth of a bitcoin. FTX offers a wide range of crypto options through a Request for Quote system and IQ Option offers several products, including binary options, but only to professional traders.

2 Inverse Options and Perpetual Futures

Figure 1 substantiates our claims about the dominance of Deribit for crypto options trading. It exhibits the share of open interest on bitcoin and ether options in the order books of different crypto options exchanges, as of 21 July 2020. Data are from Bybt.com, which also provides historical data showing that Deribit inverse options have always been this dominant in the crypto options market.

Figure 1: Percentage of Total Open Interest on Bitcoin and Ether Options by Exchange

The percentage of direct and inverse bitcoin (green) and ether (purple) options open interest on each centralised exchange that runs an options order book, as of 21 July 2020. The Deribit, OKEx and Bit.com options are inverse type, and LedgerX and CME are standard, direct options. Data obtained from Bybt



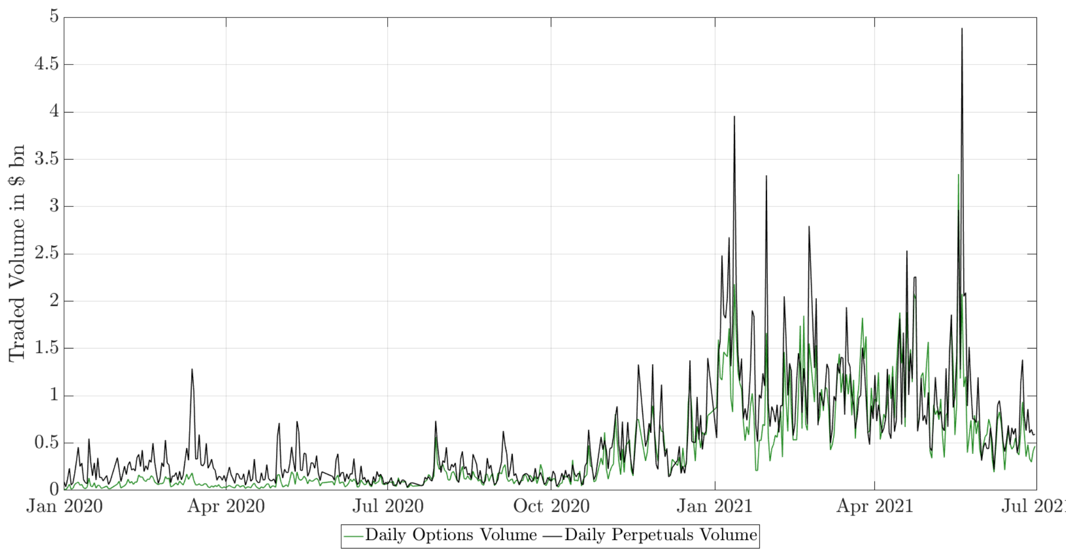
At the time of writing, Deribit lists seven standard inverse futures on a regular issuing schedule with fixed expiry dates. There are more option maturities than futures maturities but the vast majority of futures trading is on its perpetual products and it is these, rather than standard futures, which professional traders use for hedging.[2] As their name suggests, perpetual futures do not expire – but their price is tied much more closely to the spot price than fixed-expiry futures prices. This is achieved via a regular funding payment, every 8 hours. Like inverse products, perpetuals are unique to cryptocurrency markets, and they are now attracting much academic interest. We shall not dwell on them here because many other papers have already described these products in considerable detail – see, for instance, Alexander and Heck (2020) and references therein. The growth in Deribit's perpetual stems from its use for delta hedging Deribit options – these being by far the most popular and liquid of all bitcoin options, as already mentioned above. Around 85% of futures trading on Deribit is on the perpetual contract with only 15% on fixed-expiry futures, as measured by the notional amount traded.

Deribit actually started listing bitcoin inverse perpetuals and options in 2017, but bitcoin options trading was relatively thin generally, until 2020. To see this, Figure 2 depicts the notional amount traded each day on these perpetuals, in black, as well as the notional traded on bitcoin inverse options of all strikes and maturities, in green.

[2] See CryptoCompare Monthly Review

Figure 2: Trading Volumes on Deribit Bitcoin Options and Perpetuals

Trading volume in $ billion on Deribit bitcoin options (green) and perpetuals (black) from 1 January 2020 to 1 July 2021, measured by the notional amount traded per day.



The upper boundaries for inverse option pay-offs differ from those for direct options. To see this, Table 1 compares the pay-offs to each type of option under different scenarios for the underlying S, where all four options (direct and inverse, calls and puts) have the same strike $K = 100$. The direct option pay-offs are the usual: $\max[S - K, 0]$ for a call and $\max[K - S, 0]$ for a put; but for the inverse options we divide the corresponding direct pay-off by S. This way, while the direct and inverse options share the same lower boundary of 0, the pay-off to an inverse call $\to 1$ as $S \to \infty$ and the inverse put pay-off $\to \infty$ as $S \to \infty$.

Table 1: Example Comparing Direct and Inverse Option Pay-Offs

The pay-off to direct and inverse options of strike 100 under various scenarios for the underlying price at maturity.

K = 100	Underlying Price	Direct Option Payoff	Inverse Option Payoff
Call	100	0	0
	200	100	0.5
	500	400	0.8
	1000	900	0.9
Put	0.1	99.9	999
	1	99	99
	10	90	9
	50	50	1
	100	0	0

These innovative option products, which are unique to cryptocurrency markets, lie somewhere between currency options and quanto options. Intuitively, one could argue that inverse options are like currency options, but options on currencies are usually denominated in the same currency, e.g. a USD/EUR option denominated in USD has both strike and premium measured in USD. By contrast, a bitcoin inverse option is denominated in BTC – the margin account and settlement are in BTC – but the contract size and strikes are both in USD. The interpretation of a quanto option is also not accurate, because the underlying of an inverse option is not just in a different currency, the underlying *is* a different exchange rate.

The crypto market is highly fragmented, with trading split between many different centralised and decentralised, regulated and unregulated exchanges. For this reason, all crypto derivatives exchanges link settlements to a price index rather than a single traded price. For example, the bitcoin inverse options and the corresponding perpetuals listed on Deribit use the 'Deribit Bitcoin Index' as the underlying. This is the arithmetic average of bitcoin spot prices on several major centralised exchanges. See Deribit Index for a more detailed explanation.

We denote the USD value of a bitcoin spot price index at time t as S_t. A direct European call option to buy 1 bitcoin at a strike of K USD at time T has pay-off $\Psi^c(S_t) = \max[S_t - K, 0]$. But for an inverse option, even though the strike K is measured in USD per bitcoin, the pay-off is converted into BTC. That is, setting $\widehat{K} = K^{-1}$ and $\widehat{S}_t = S_t^{-1}$ the inverse call pay-off is given by:

$$\widehat{\Psi}^c(S_t) = \widehat{S}_T\Psi^c(S_t) = \max\left[1 - \widehat{S}_T K, 0\right] = K\max\left[\widehat{K} - \widehat{S}_T, 0\right]. \tag{1}$$

This shows that an inverse call with USD strike K has pay-off equal to K times the pay-off to a direct put with strike $\widehat{K}$ bitcoin on the underlying $\widehat{S}_t$, i.e. the value of USD in bitcoin. It also shows that the maximum pay-off to an inverse call is 1 bitcoin, irrespective of the USD value of its strike.

Similarly, denoting the direct put pay-off by $\Psi^p(S_t) = \max[K - S_t, 0]$, an inverse put has pay-off given by:

$$\widehat{\Psi}^p(S_t) = \widehat{S}_T\Psi^p(S_t) = \max\left[\widehat{S}_T K - 1, 0\right] = K\max\left[\widehat{S}_T - \widehat{K}, 0\right]. \tag{2}$$

That is, the inverse put with strike K USD is K times the pay-off to a direct call with strike $\widehat{K}$ bitcoin on the underlying $\widehat{S}_t$. By contrast with the call, the pay-off to an inverse put is unbounded above. That the boundary conditions are different for inverse options is also evident from Figure 3 in Section 3, where the bold lines indicate the pay-offs using blue for the direct option and red for the inverse. The figure clearly illustrates the difference in boundary pay-off conditions for inverse calls and puts which we have already explained using the example in Table 1.

3 Derivation of Inverse Option Price under Geometric Brownian Motion

The settlement price of Deribit inverse bitcoin options is not the price of a tradable asset, so we price it using its hedging instrument – the bitcoin perpetual futures contract whose price, in USD per bitcoin, we denote by F_t, with t denoting running time between issue at time 0 and the option's maturity at time T. As inBlack and Scholes (1973) and Merton (1973) we assume that F_t follows a geometric Brownian motion (GBM) with volatility σ, for now assuming both dividend yield and discount rates are zero, so that:

$$\frac{dF_t}{F_t} = \sigma dW_t.$$

Then $\widehat{F}_t = F_t^{-1}$ is also a geometric Brownian motion, indeed Itô's lemma yields:

$$d \ln \widehat{F}_t = \frac{1}{2}\sigma^2 dt + \sigma dZ_t,$$

where $Z_t = -W_t$. Thus, at any time $t \in [0, T]$ we can take the inverse call option's expected pay-off under the risk-neutral measure $\mathcal{Q}$ to obtain the time t value C_t of a call option with pay-off $\widehat{\Psi}^c(F_T)$ given by (1). Sine we assume the discount rate r is zero, this is:

$$C_t = \mathbb{E}^{\tilde{\mathcal{Q}}}\left[\widehat{\Psi}\left(F_T\right)\right] = K\int_{-\infty}^{\infty}\left[\widehat{K} - \widehat{F_T}(z)\right]^+ \phi(z)dz,$$

where $\tau = T - t$, $\phi(\cdot)$ is the standard normal density function and $\widehat{F_T}(z) = \widehat{F}_t \exp\left\{\frac{1}{2}\sigma^2\tau + \sigma\sqrt{\tau}z\right\}$ where z is drawn from a standard normal distribution. Note that

$$\left[\widehat{K} - \widehat{F_T}(z)\right]^+ = 0 \Leftrightarrow z \geq \frac{\ln\left(\frac{F_t}{K}\right)}{\sigma\sqrt{\tau}} - \frac{1}{2}\sigma\sqrt{\tau} = d_2,$$

where above we use the standard BS notation for d_2. Thus, using $\Phi(\cdot)$ to denote the standard normal distribution function:

$$C_t = K\int_{-\infty}^{d_2}\left(\widehat{K} - \widehat{F_T}(z)\right)\phi(z)dz = \Phi(d_2) - F_t^{-1}K\int_{-\infty}^{d_2}\exp\left\{\frac{1}{2}\sigma^2\tau + \sigma\sqrt{\tau}z\right\}\phi(z)dz.$$

Evaluating the integral yields the GBM price of the inverse call option with strike K as:

$$C_t = \Phi(d_2) - \exp\left\{\sigma^2\tau\right\}F_t^{-1}K\Phi(d_2 - \sigma\sqrt{\tau}). \tag{3}$$

A similar argument yields the time t GBM price of an inverse put option with stike K USD and maturity T as:

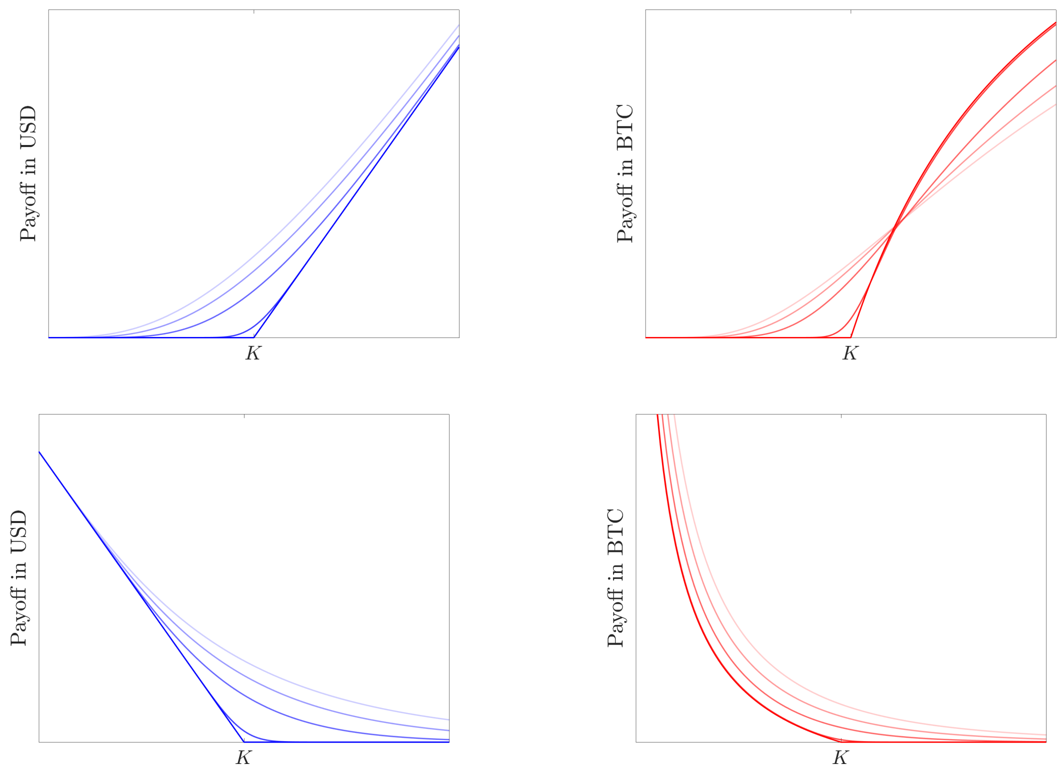
$$P_t = \exp\left\{\sigma^2\tau\right\}F_t^{-1}K\Phi(\sigma\sqrt{\tau} - d_2) - \Phi(-d_2). \tag{4}$$

So as not to complicate the derivation above, we have assumed the dividend yield y and risk-free rate r are zero, but for formulae with non-zero y and r see Table 2 in Section 4. Note that y can be positive or negative. The perpetuals price is tied to the spot via a funding payment mechanism which is equivalent to the dividend yield. Each exchange has their own funding rate calculations, and these are typically positive if the perpetuals price is above the spot bitcoin rate, and negative otherwise. The Deribit funding rate mechanism is explained here.

Figure 3 compares the standard, direct BS prices and pay-offs with the prices and pay-offs for inverse options, derived above, assuming that the underling follows a GBM. The left hand side shows the familiar convex structure of the BS pricing function with an increasing gamma and a positive vega and the price approaches the pay-off as the option approaches expiry. But the inverse option GBM pricing functions behave very differently. A deep ITM inverse call option could decrease in value when the underlying price increases, and being capped at 1 bitcoin it has very little upward potential. Indeed, very deep ITM inverse call option prices could be much lower than one would

Figure 3: Pay-Offs for Direct and Inverse Call and Put Options and Prices under GBM

Option pay-offs (in bold) and prices obtained using the Black-Scholes formula for the direct options and our formula (**??**) for the inverse options. Prices are depicted as a function of the underlying price with a thicker line as the option approaches expiry. Time to maturities of 10 days, 3 months, 6 months and 1 year are shown. In the array of plots, the first row depicts calls and the second depicts puts; the left column depicts the direct option prices and pay-offs in blue, and the right shows the inverse option prices in red. All four plots are calculated using the same K, r and σ but with different contingent claims Ψ and $\hat{\Psi}$, respectively.



think from looking at ATM option prices.[3] By contrast, deep ITM inverse put option prices can become very high as the price of the underlying falls. While the BS delta for a direct put option is always finite, the delta of an inverse put option tends to infinity as the underlying price decreases towards zero.

The convex behaviour of the inverse call option pricing function reaches a turning point as the underlying increases, and thereafter it becomes a concave function of the underlying price. Hence, the delta has an interior point maximum which in turn affects the sign of the gamma: before the underlying price reaches the point of delta inflection the gamma is positive, but as the price moves above the inflection point level the gamma becomes negative.

[3] The area around ATM and slightly ITM inverse call options is extremely sensitive. At maturity, if the underlying price is 10% higher than the strike, the pay-off would be roughly 0.091 bitcoin; if the underlying price is twice the strike the pay-off will be 0.5 bitcoin; and if the underlying price was 10 times the strike the strike, the pay-off would be 0.9 bitcoin.

4 Hedge Ratios

Table 2 summarizes the formulae for the prices of both direct and inverse European call and put options and their price and volatility sensitivities, under the GBM assumption. The direct option formulae are well known, and see the Appendix for the derivation of the delta, gamma and vega of the inverse call option – the derivation for the put is omitted because it is very similar. The next version of this discussion paper will contain all the inverse Greeks, also with non-zero r and y.

Table 2: Direct and and Inverse Option Prices and Greeks under GBM

Name			BS Formula	Inverse Formula
Price	f		$\omega\left[e^{-y\tau}F\Phi(\omega d_1) - e^{-r\tau}K\phi(\omega d_2)\right]$	$\omega\left[e^{-r\tau}\Phi(\omega d_2) - e^{(y-r+\sigma^2)\tau}F^{-1}K\Phi(\omega d_3)\right]$
Delta	δ	$\frac{\partial f}{\partial F}$	$\omega e^{-y\tau}\Phi(\omega d_1)$	$\omega e^{\sigma^2\tau}F^{-2}K\Phi(\omega d_2)$
Gamma	γ	$\frac{\partial^2 f}{\partial F^2}$	$e^{-y\tau}\phi(d_1)\left(F\sigma\sqrt{\tau}\right)^{-1}$	$e^{\sigma^2\tau}F^{-3}K\left[\phi(\omega d_3)\left(\sigma\sqrt{\tau}\right)^{-1} - 2\omega\Phi(\omega d_3)\right]$
Vega	ν	$\frac{\partial f}{\partial \sigma}$	$e^{-y\tau}F\phi(d_1)\sqrt{\tau}$	$\phi(d_2)\sqrt{\tau} - 2\omega e^{\sigma^2\tau}\sigma\tau F^{-1}K\Phi(\omega d_3)$

We assume the bitcoin perpetuals price F follows a GBM with volatility σ, where the drift depends on the discount rate r and dividend yield y. The direct or inverse call or put option has strike K and residual time to maturity τ and we use the notation $\omega = \pm 1$ according as the option is a call or a put. Suppressing all time subscripts for simplicity we set $d_1 = \ln\left(\frac{F}{K}\right)\left[\sigma\sqrt{\tau}\right]^{-1} + (r - y + \frac{1}{2}\sigma)\sqrt{\tau}$, $d_2 = d_1 - \sigma\sqrt{\tau}$ and $d_3 = d_2 - \sigma\sqrt{\tau}$. Inverse Greek formulae assume $r = y = 0$ because non-zero values rapidly complicate the calculations.

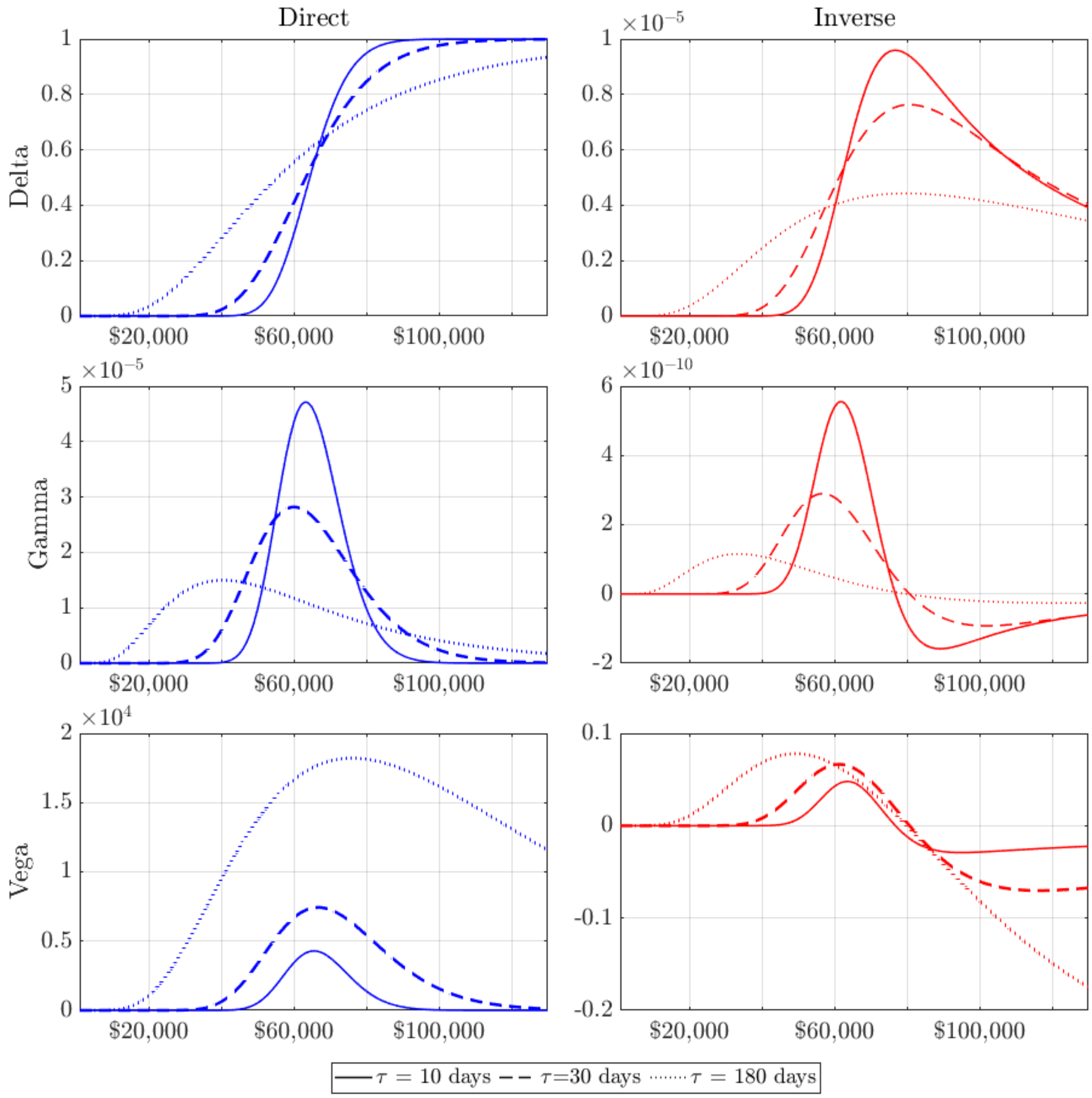
Note that the inverse Greeks represent the sensitivity of the price of the option – denominated in BTC – to a unit change in bitcoin's USD price or its volatility. Consider, for example, the delta, δ which is always between 0 and 1 for a long position on a direct call option,[4] and it represents the approximate change in the option price for a unit change in the underlying price, where both prices are measured in USD. For a direct call $\delta \to 1$ as $F \to \infty$. In contrast, the inverse delta represents a change in the BTC price of the option for a unit change in the USD price of the underlying. And, because of the term $F^{-2}K$ in the inverse delta formula, we have that $\delta \to 0$ as $F \to \infty$. Also note that, while BS gamma and vega for direct options are the same, the gamma is different for an inverse call and inverse put. By the same token, the vega of an inverse call is different from the vega of an inverse put.

Figure 4 depicts the direct and inverse price and volatility sensitivities for European call options of strike $65,000$ but with different maturities. We set the underlying volatility at 80% and examine the behaviour of the hedge ratios as the underlying price F varies. In each case, we compare the direct option in blue with the inverse option in red, and draw the hedge ratios as a function of the underlying in USD. The direct option sensitivities exhibit well-known features, with delta monotonically increasing – following the normal distribution function – and the gamma and vega

[4] This is why the common practical interpretation of the delta is the probability of an option ending ITM.

Figure 4: Direct and Inverse Greeks

Direct and inverse Greeks as a function of the underlying F with fixed strike K=\$65,000, volatility σ=80% and different times to maturity: from 10 days (bold), 3 months (dashed) and 6 months (dotted). The left, blue column represents the direct Greeks, while the right, red column depicts the inverse Greeks.



being positive – following the standard normal density. The delta becomes steeper, the gamma increases and the vega degreases as the option approaches expiry.

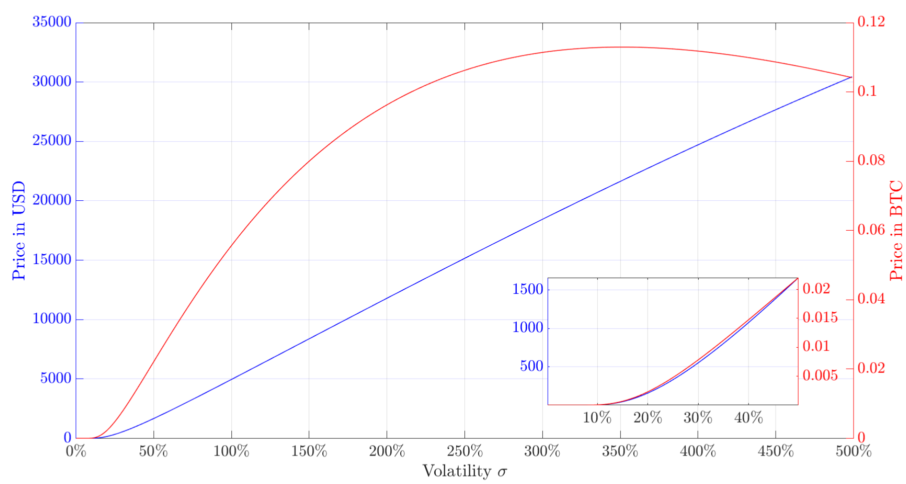
By contrast the inverse delta is not monotonously increasing, it reaches a maximum when the underlying price just exceeds the strike and declines thereafter. This point of delta inflection is associated with the shift from the convexity to concavity of the pricing function, as previously noted. The inverse option delta is still positive, but it is no longer bounded above by 1. Its maximum value is always achieved just after the call becomes ITM. Up to this turning point (the exact value of which depends on the option characteristics) the inverse delta behaves similarly to the direct option delta. The same remark applies to the gamma, and the vega. However, ITM inverse calls have a delta, gamma and vega that can be quite different from the direct option equivalents. The divergences between the direct and inverse call option hedge ratios becomes more and more evident as the option moves deeper ITM, and the inverse option gamma becomes negative as the delta tends towards zero. The call option vega also becomes negative because, when volatility increases, a deep ITM option which is capped at 1 has greater change to loose value than gain it.

5 Implications for Volatility

Figure 5 depicts a call option's price as a function of volatility, comparing the behaviour of the direct option BS price in blue with the inverse option pricing formula (1) in red. As is well known, the BS direct option price is almost linear in volatility. By contrast, our indirect option price is convex at low volatility levels and behaves similarly to the direct option (see insert) but it as volatility increases it becomes concave. This is because the vega of an inverse option can be negative. Unlike the direct option, whose price always increases with volatility, the inverse call option price reaches maximum price at some level of volatility, after which its price decreases. Again, this maximum price depends on the level of the strike and the underlying price, as well as the option maturity.

Figure 5: Direct and Inverse Option Prices as a Function of Implied Volatility

Comparison of the direct (blue, left-hand scale) and inverse (red, right-hand scale) call option prices with fixed $F = \$60,000, K = \$65,000, \tau = 30$ days and zero interest rates, but with variable σ. The small inset box shows their similar shapes but only on the range from 0%-50% volatility.



Now we examine the implied volatilities backed out from traded market prices of Deribit inverse options using our new pricing formula. To exhibit a time series over a large sample, we construct synthetic, 30-day constant maturity options with fixed moneyness $K/F = 1.1$. We choose a moneyness of 1.1 since these call options have the greatest trading volume for maturities between 2 weeks and 2 months. This construction entails using 4 options. The strikes are selected to be immediately adjacent and either side of the strike with moneyness 1.1 and the maturities are the two that are traded, again immediately either side of 30 days. We interpolate between the prices of each pair of options of the same maturity but with different strikes, using a piecewise cubic Hermite interpolating polynomial. The interpolation yields a synthetic price for an option of moneyness 1.1. Then we find the synthetic futures price using put-call parity (PCP).[5] Then we use this PCP futures price to back out the implied volatilities using the relevant direct or indirect pricing formula. Finally we use linear interpolation between the synthetic option's implied variances to obtain a 30-day implied variance. Then square rooting gives the fixed-moneyness, fixed-maturity synthetic option implied volatility shown in Figure 6.

Figure 6: Comparison of Implied Volatilities

Implied volatilities backed out from Deribit's inverse bitcoin options at 00:00 UTC from 15 March 2019 to 15 Feb 2021. We do this using the standard BS formula for the option price converted into USD using the current USD/BTC rate (blue) and by apply the correct invers option BS price formula to the actual BTC option price (in red).

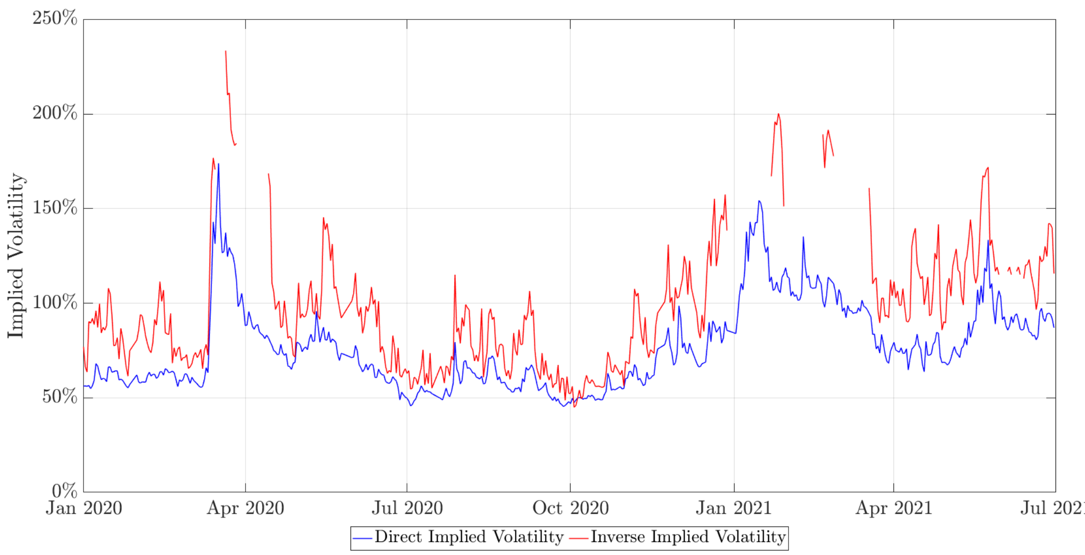


Figure 6 compares daily time series for the two implied volatilities over a 18-month period starting from January 2020 and using Deribit option prices at 00:00 UTC. We see that the correct inverse implied volatility almost always exceeds the implied volatility that is backed out using the standard – but incorrect – BS formula. During tranquil periods, both implied volatilities are relatively close to each other. Small difference are expected because of the relatively large tick size of 0.0005BTC for bitcoin options. However, the inverse option implied volatility is highly sensitive to price shocks and we observe differences of up to 40%, e.g. on 12 March 2020. The gaps in the time series of inverse implied volatility are because the traded option's price exceeded the maximum price of the GBM inverse option pricing formula. That is, at least one of the 4 options has a price so far above the upper bound for the GBM price that the implied volatility does not exist. Figure 6 also indicates that crypto option traders are indeed converting the BTC price of the option into

[5] The PCP relations changes for inverse options. For direct options is it $C - P = F - e^{r\tau}K$ but for inverse options it is $F(C-P) = F - e^{-r\tau}K$.

USD, then using the standard BS formula to back out implied volatility – even though this is not correct.

The implied volatility smile or skew that we observe in equity index options – and in standard, direct call and put options on commodities, interest rates and exchange rates – is evidence that traders do not believe in the GBM assumption. By the same token, the implied volatilities backed out using our inverse option pricing formula do not always exist, and this is clear evidence that bitcoin options professional traders are not assuming a GBM price process when they make a market in these options. The steepness and symmetry of implied volatility smiles in bitcoin options shows that traders assume the chance of large upward or downward bitcoin price moves is very much more likely than the GBM assumption would allow. They believe this, because there is clear empirical evidence that log returns on bitcoin are not normally distributed – both upper and lower tails of the distribution are much heavier than normal tails and the distribution also has a pronounced skew.

6 Conclusion

We derive new formulae for pricing and hedging inverse call and put options based on the assumption that the underlying price follows a GBM. At the time of writing all inverse options are for trading bitcoin or ether against the US dollar, so the GBM assumption is certainly not realistic. However, a similar comment applies to *every* financial options market, from equities to commodities, and this does not mean the BS formula is irrelevant. Far from it, we still use the BS formula to back out implied volatilities and numerous professional traders use the BS hedge ratios to balance their options books for delta-gamma-vega neutrality.

Therefore, in deriving BS-type formulae for inverse options, this paper fills a fundamental gap in the literature on crypto derivatives. The importance of our work includes our finding that the correct hedge ratios for inverse options behave very differently from the standard BS Greeks. In particular:

- The delta of an ITM inverse call starts to decrease towards zero as the call moves deeper ITM
- ITM inverse call options may have negative gamma and negative vega
- The delta of an ITM inverse put rapidly increases as the put moves deeper ITM
- The vega of a deep ITM inverse put is much larger than the vega of a deep ITM direct put
- Inverse call and puts do not have the same gamma or the same vega

Several previous papers examine the pricing of bitcoin options under stochastic processes that are far more complex than GBM, but none of them examine inverse options. Inverse options constitute 90% of the crypto options market, and almost all are traded on Deribit. The importance of our work is that professional traders in Deribit bitcoin and ether options now have the *correct* BS-type prices and hedge ratios to use for balancing their options books.

Appendix: Derivation of the Greeks

$$
\begin{aligned}
\widehat{\delta} := \frac{\partial \widehat{f}}{\partial F} &= \frac{\partial}{\partial F}\left[\Phi(d_2) - e^{\sigma^2\tau}\frac{K}{F}\Phi(d_3)\right] \\
&= \frac{\partial \Phi(d_2)}{\partial d_2}\frac{\partial d_2}{\partial F} - e^{\sigma^2\tau}\left[-\frac{K}{F^2}\Phi(d_3) + \frac{K}{F}\frac{\partial \Phi(d_3)}{\partial d_3}\frac{\partial d_3}{\partial F}\right] \\
&= \frac{\phi(d_2)}{F\sigma\sqrt{\tau}} - e^{\sigma^2\tau}\left[-\frac{K}{F^2}\Phi(d_3) + \frac{K}{F^2\sigma\sqrt{\tau}}\phi(d_3)\right] \\
&= \frac{\phi(d_2)}{F\sigma\sqrt{\tau}} + e^{\sigma^2\tau}\frac{K}{F^2}\Phi(d_3) - e^{\sigma^2\tau}\frac{K}{F^2\sigma\sqrt{\tau}}\phi(d_2 - \sigma\sqrt{\tau}) \\
&= e^{\sigma^2\tau}F^{-2}K\Phi(d_3)
\end{aligned}
$$

$$
\widehat{\gamma} := \frac{\partial \widehat{\delta}}{\partial F} = \frac{\partial}{\partial F}\left[e^{\sigma^2\tau}\frac{K}{F^2}\Phi(d_3)\right] = e^{\sigma^2\tau}F^{-3}K\left[\frac{\phi(d_3)}{\sigma\sqrt{\tau}} - 2\Phi(d_3)\right]
$$

$$
\begin{aligned}
\widehat{\nu} := \frac{\partial \widehat{f}}{\partial \sigma} &= \frac{\partial}{\partial \sigma}\left[\Phi(d_2) - \exp\left\{\sigma^2\tau\right\}\frac{K}{F}\Phi(d_3)\right] \\
&= \frac{\partial \Phi(d_2)}{\partial d_2}\frac{\partial d_2}{\partial \sigma} - \frac{K}{F}\left[\frac{\partial e^{\sigma^2\tau}}{\partial \sigma}\Phi(d_3) + e^{\sigma^2\tau}\frac{\partial \Phi(d_3)}{\partial d_3}\frac{\partial d_3}{\partial \sigma}\right] \\
&= \phi(d_2)\frac{\partial d_2}{\partial \sigma} - e^{\sigma^2\tau}\frac{K}{F}\left[2\tau\sigma\Phi(d_3) + \phi(d_2 - \sigma\sqrt{\tau})\left(\frac{\partial d_2}{\partial \sigma} - \sqrt{\tau}\right)\right] \\
&= \phi(d_2)\sqrt{\tau} - 2e^{\sigma^2\tau}F^{-1}K\tau\sigma\Phi(d_3)
\end{aligned}
$$

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