(a) For $Y \sim \text{Bin}(n,\theta)$, prove that

$$\begin{aligned}\sum_{k=y}^{n}\binom{n}{k}\theta^k(1-\theta)^{n-k} &= \frac{\Gamma(n+1)}{\Gamma(n-y+1)\Gamma(y)}\int_0^\theta t^{y-1}(1-t)^{n-y}dt \\ &= I_\theta(y, n-y+1)\end{aligned}$$

Proof.

We use the mathematical induction method to prove the above result.

(i) Note that if $k=n$,

$$\binom{n}{n}\theta^n(1-\theta)^{n-n} = \theta^n = \frac{\Gamma(n+1)}{\Gamma(1)\Gamma(n)}\int_0^\theta t^{n-1}dt$$

(ii) If it holds that for $n-y-1 \geq l \geq 0$

$$\sum_{k=y+l+1}^{n}\binom{n}{k}\theta^k(1-\theta)^{n-k} = \frac{\Gamma(n+1)}{\Gamma(n-y-l)\Gamma(y+l+1)}\int_0^\theta t^{y+l}(1-t)^{n-y-l-1}dt. \quad (1)$$

We want to obtain

$$\sum_{k=y+l}^{n}\binom{n}{k}\theta^k(1-\theta)^{n-k} = \frac{\Gamma(n+1)}{\Gamma(n-y-l+1)\Gamma(y+l)}\int_0^\theta t^{y+l-1}(1-t)^{n-y-l}dt. \quad (2)$$

Using the methods of integration by parts,

$$\begin{aligned}\text{RHS of (2)} &= \frac{n!}{(y+l-1)!(n-y-l)!}\left[\frac{1}{y+l}t^{y+l}(1-t)^{n-y-l}\Big|_0^\theta + \int_0^\theta \frac{n-y-l}{y+l}t^{y-l}(1-t)^{n-y-l-1}dt\right] \\ &= \binom{n}{y+l}\theta^{y+l}(1-\theta)^{n-y-l} + \frac{n!}{(y+l)!(n-y-l-1)!}\int_0^\theta t^{y+l}(1-t)^{n-y-l-1}dt \\ &= \binom{n}{y+l}\theta^{y+l}(1-\theta)^{n-y-l} + \frac{\Gamma(n+1)}{\Gamma(n-y-l)\Gamma(y+l+1)}\int_0^\theta t^{y+l}(1-t)^{n-y-l-1}dt \\ &= \binom{n}{y+l}\theta^{y+l}(1-\theta)^{n-y-l} + \sum_{k=y+l+1}^{n}\binom{n}{k}\theta^k(1-\theta)^{n-k} = \text{LHS of (2)}\end{aligned}$$

Thus, with (i) and (ii), we can induce that (2) holds when $l=0$.

We can also obtain the result by utilizing order statistics for a uniform distribution. Recall how we can obtain binomial samples $Y \sim \text{Bin}(n,p)$ from $\text{Unif}(0,1)$:

$$y_i = \begin{cases} 0, & u_i > p, \\ 1, & u_i \leq p, \end{cases}$$

where $\{u_i\}_{i=1}^n$ are samples drawn from $\text{Unif}(0,1)$ and $Y = \sum_{i=1}^n y_i$. Given an observation y, the event $\{Y \geq y\}$ indicates that at least y elements in the sequence $\{u_i\}_{i=1}^n$ are smaller than p, i.e., the y-th order statistic of $(u_1,\ldots,u_n)$, $u_i \overset{i.i.d.}{\sim} \text{Unif}(0,1)$ denoted by $u_{(y)}$ is smaller than p. The probability density function of $u_{(y)}$ is

$$f_{u_{(y)}}(u) = \frac{n!}{(y-1)!(n-y)!}u^{y-1}(1-u)^{n-y},$$

Thus, it holds that

$$P(Y \geq y) = P(u_{(y)} \leq p) = \int_0^p f_{u_{(y)}}(u)du = \int_0^p \frac{\Gamma(n+1)}{\Gamma(n-y+1)\Gamma(y)}u^{y-1}(1-u)^{n-y}du$$

$\square$

(b) For $Y \sim \mathrm{NB}(r, \theta)$, prove that

$$\begin{aligned}\sum_{k=y}^{\infty} \binom{k+r-1}{k} \theta^k (1-\theta)^r &= \frac{\Gamma(y+r)}{\Gamma(r)\Gamma(y)} \int_0^{\theta} t^{y-1}(1-t)^{r-1} dt \\ &= I_\theta(y, r)\end{aligned}$$

Proof.
It is equivalent to prove

$$\Pr(Y \leq y) = \sum_{k=0}^{y} \binom{k+r-1}{k} \theta^k (1-\theta)^r = 1 - I_\theta(y+1, r) = I_{1-\theta}(r, y+1) \tag{3}$$

$$\begin{aligned}\Pr(Y \leq y) &= \Pr\{y \text{ or fewer successes to get } r \text{ failures}\} \\ &= \Pr\{y+r \text{ or fewer trials to get } r \text{ failures}\} \\ &= \Pr\{r\text{-th failure in } (y+r)\text{-th trial or before}\} \\ &= \Pr\{\text{at most } y \text{ successes in } (y+r) \text{ trails}\} \\ &= \Pr(Z \leq y | Z \sim \mathrm{Bin}(y+r, \theta)) \qquad (4)\end{aligned}$$

According to the results in (a),

$$(4) = 1 - I_\theta(y+1, y+r-(y+1)+1) = 1 - I_\theta(y+1, r) = I_{1-\theta}(r, y+1)$$

$\square$