2 (a) $y_{ij}|u_i \sim N(\beta_0+\beta_1 x_{ij}+u_i,\ \sigma_\varepsilon^2)$, to integrate out u_i we need to get its posterior distribution.

$$f(u_i|\theta^{old},\ \boldsymbol{y_i}) \propto f(y_{i1}|u_i)f(y_{i2}|u_i)\cdots f(y_{iJ}|u_i)f(u_i|\theta^{old})$$

$$\propto \exp\left(-\frac{1}{2\sigma_\varepsilon^2}\sum_{j=1}^{J}(y_{ij}-\beta_0-\beta_1 x_{ij}-u_i)^2 - \frac{u_i^2}{2\sigma_u^2}\right)$$

$$\propto \exp\left(-\frac{1}{2\frac{\sigma_u^2\sigma_\epsilon^2/J}{\sigma_u^2+\sigma_\varepsilon^2/J}}\left(u_i-\frac{\sigma_u^2(\overline{y}_i-\beta_0-\beta_1\overline{x}_i)}{\sigma_u^2+\sigma_\varepsilon^2/J}\right)^2\right)$$

$$u_i|\theta^{old},\boldsymbol{y_i} \sim N(E_i,\ \sigma_{old}^2)\quad where\ E_i=\frac{\sigma_u^2(\overline{y}_i-\beta_0-\beta_1\overline{x}_i)}{\sigma_u^2+\sigma_\varepsilon^2/J},\ \sigma_{old}^2=\frac{\sigma_u^2\sigma_\epsilon^2/J}{\sigma_u^2+\sigma_\varepsilon^2/J}$$

$(\boldsymbol{y},\boldsymbol{u})$ is complete data, we need to integrate out u_i on θ^{old}, $\theta^{old}=(E_i,\ \sigma_{old}^2)$.

Utilize $E(u_i)=E_i$, $E(u_i^2)=\sigma_{old}^2+E_i^2$, $E(u_i-\lambda)^2=\sigma_{old}^2+E_i^2-2\lambda E_i+\lambda^2=\sigma_{old}^2+(E_i-\lambda)^2$

$$Q(\theta|\theta^{old})=E_{\boldsymbol{u}|\theta^{old}}[\log P(\boldsymbol{y},\boldsymbol{u}|\theta)]=\int_{-\infty}^{\infty}(\log P(\boldsymbol{y},\boldsymbol{u}|\theta))f(\boldsymbol{u}|\theta^{old})d\boldsymbol{u}$$

$$=\sum_{i=1}^{I}E_{u|\theta^{old}}[\log[f(y_{i1}|u_i)f(y_{i2}|u_i)\cdots f(y_{iJ}|u_i)f(u_i|\theta)]]$$

$$=\sum_{i=1}^{I}E_{u|\theta^{old}}\left[\sum_{j=1}^{J}\left(-\frac{1}{2}\log\sigma_\varepsilon^2-\frac{1}{2\sigma_\varepsilon^2}(y_{ij}-\beta_0-\beta_1x_{ij}-u_i)^2\right)-\frac{1}{2}\log\sigma_u^2-\frac{u_i^2}{2\sigma_u^2}\right]$$

$$=\sum_{i=1}^{I}\left[-\frac{J}{2}\log\sigma_\varepsilon^2-\frac{1}{2\sigma_\varepsilon^2}\sum_{j=1}^{J}(\sigma_{old}^2+(y_{ij}-\beta_0-\beta_1x_{ij}-E_i)^2)-\frac{1}{2}\log\sigma_u^2-\frac{\sigma_{old}^2+E_i^2}{2\sigma_u^2}\right]$$

$$=-\frac{IJ}{2}\log\sigma_\varepsilon^2-\frac{IJ\sigma_{old}^2}{2\sigma_\varepsilon^2}-\frac{1}{2\sigma_\varepsilon^2}\sum_{i=1}^{I}\sum_{j=1}^{J}(y_{ij}-\beta_0-\beta_1x_{ij}-E_i)^2-\frac{I}{2}\log\sigma_u^2-\frac{I\sigma_{old}^2}{2\sigma_u^2}-\frac{1}{2\sigma_u^2}\sum_{i=1}^{I}E_i^2$$

Notice $\frac{\partial\theta^{old}}{\partial\sigma_\varepsilon^2}=\frac{\partial\theta^{old}}{\partial\sigma_u^2}=\frac{\partial\theta^{old}}{\partial\beta_0}=\frac{\partial\theta^{old}}{\partial\beta_1}=0$, we have M step as following:

$$\begin{cases}\dfrac{\partial Q}{\partial\beta_0}=\dfrac{1}{\sigma_\varepsilon^2}\displaystyle\sum_{i=1}^{I}\sum_{j=1}^{J}(y_{ij}-\beta_0-\beta_1x_{ij}-E_i)=0\\[2ex]\dfrac{\partial Q}{\partial\beta_1}=\dfrac{1}{\sigma_\varepsilon^2}\displaystyle\sum_{i=1}^{I}\sum_{j=1}^{J}x_{ij}(y_{ij}-\beta_0-\beta_1x_{ij}-E_i)=0\end{cases}$$

Solve this equation group, we have

$$\beta_0=\left(\frac{\sum_{i=1}^{I}\sum_{j=1}^{J}x_{ij}(y_{ij}-E_i)}{\sum_{i=1}^{I}\sum_{j=1}^{J}x_{ij}^2}-\frac{\sum_{i=1}^{I}\sum_{j=1}^{J}(y_{ij}-E_i)}{\sum_{i=1}^{I}\sum_{j=1}^{J}x_{ij}}\right)\Bigg/\left(\frac{\sum_{i=1}^{I}\sum_{j=1}^{J}x_{ij}}{\sum_{i=1}^{I}\sum_{j=1}^{J}x_{ij}^2}-\frac{IJ}{\sum_{i=1}^{I}\sum_{j=1}^{J}x_{ij}}\right)$$

$$\beta_1=\sum_{i=1}^{I}\sum_{j=1}^{J}x_{ij}(y_{ij}-\beta_0-E_i)\Big/\sum_{i=1}^{I}\sum_{j=1}^{J}x_{ij}^2$$

where β_0 is as above.

$$\frac{\partial Q}{\partial \sigma_\varepsilon^2} = - \frac{IJ}{2\sigma_\varepsilon^2} + \frac{IJ\sigma_{old}^2}{2\sigma_\varepsilon^4} + \frac{1}{2\sigma_\varepsilon^4}\sum_{i=1}^{I}\sum_{j=1}^{J}(y_{ij} - \beta_0 - \beta_1 x_{ij} - E_i)^2 = 0$$

$$\Longrightarrow \ \sigma_\varepsilon^2 = \sigma_{old}^2 + \frac{1}{IJ}\sum_{i=1}^{I}\sum_{j=1}^{J}(y_{ij} - \beta_0 - \beta_1 x_{ij} - E_i)^2$$

where β_0, β_1 are as above.

$$\frac{\partial Q}{\partial \sigma_u^2} = - \frac{I}{2\sigma_u^2} + \frac{I\sigma_{old}^2}{2\sigma_u^4} + \frac{1}{2\sigma_u^4}\sum_{i=1}^{I} E_i^2 = 0$$

$$\Longrightarrow \ \sigma_u^2 = \sigma_{old}^2 + \frac{1}{I}\sum_{i=1}^{I} E_i^2$$

In above 4 formulas,

$$E_i = \frac{\sigma_u^2(\overline{y}_i - \beta_0 - \beta_1 \overline{x}_i)}{\sigma_u^2 + \sigma_\varepsilon^2/J},\ \sigma_{old}^2 = \frac{\sigma_u^2\sigma_\epsilon^2/J}{\sigma_u^2 + \sigma_\varepsilon^2/J},$$ where $\beta_0, \beta_1, \sigma_u^2, \sigma_\varepsilon^2$ are from last step (old variables).

3 (a) Complete-data likelihood

$$g(y_i, u_i|\boldsymbol{\omega}) = \prod_{j=1}^{k}[\omega_j f_j(y_i)]^{I(u_i=j)} = \sum_{j=1}^{k}\omega_j f_j(y_i) I(u_i = j)$$

3 (b) Marginal distribution

$$g(y_i|\boldsymbol{\omega}) = \sum_{u_i=1}^{k} g(y_i, u_i|\boldsymbol{\omega}) = \sum_{j=1}^{k}\omega_j f_j(y_i)$$

3 (d) The probability of $u_i = j$ is

$$P(u_i = j|y_i, \mu_j^{(0)}) = \frac{P(y_i|u_i = j)\omega_j}{P(y_i)} = \frac{P(y_i|u_i = j)\omega_j}{\sum_{j=1}^{k} P(y_i|u_i = j)\omega_j} = \frac{\omega_j \exp\left(-\frac{1}{2}(y_i - \mu_j^{(0)})^2\right)}{\sum_{j=1}^{k}\omega_j \exp\left(-\frac{1}{2}(y_i - \mu_j^{(0)})^2\right)}$$

E-step:

$$Q(\boldsymbol{\mu}|\boldsymbol{\mu}^{(0)}) = \sum_{i=1}^{n} E_{\boldsymbol{u}|\boldsymbol{\mu}^{(0)}}[\log g(y_i, u_i|\boldsymbol{\mu})] = \sum_{i=1}^{n}\sum_{j=1}^{k}[\log\omega_j + \log f_j(y_i)] P(u_i = j|y_i)$$

$$\propto \sum_{i=1}^{n}\sum_{j=1}^{k} - \frac{(y_i - \mu_j)^2}{2} P(u_i = j|y_i)$$

M-step:

$$\frac{\partial Q}{\partial \mu_j} = \sum_{i=1}^{n}\frac{1}{\sigma_j^2}(y_i - \mu_j) P(u_i = j|y_i) = 0 \quad \Longrightarrow \quad \mu_j^{(1)} = \frac{\sum_{i=1}^{n} y_i P(u_i = j|y_i)}{\sum_{i=1}^{n} P(u_i = j|y_i)}$$

3 (e) E-step

$$Q(\boldsymbol{\omega},\boldsymbol{\mu}|\boldsymbol{\omega}^{(0)},\boldsymbol{\mu}^{(0)})=\sum_{i=1}^{n}E_{\boldsymbol{u}|\boldsymbol{\omega}^{(0)},\boldsymbol{\mu}^{(0)}}[\log g(y_i,u_i|\boldsymbol{\omega},\boldsymbol{\mu})]=\sum_{i=1}^{n}\sum_{j=1}^{k}[\log\omega_j+\log f_j(y_i)]P(u_i=j|y_i)$$

$$\propto\sum_{i=1}^{n}\sum_{j=1}^{k}\left[\log\omega_j-\frac{1}{2}(y_i-\mu_j)^2\right]P(u_i=j|y_i)$$

To find the expression for ω_j to maximize Q, we use Lagrange multiplier λ under the constraint that $\sum_{j=1}^{k}\omega_j=1$ and $\sum_{j=1}^{k}P(u_i=j)=1$, solve the equation $\frac{\partial}{\partial\omega_j}\left(Q+\lambda\left(1-\sum_{j=1}^{k}\omega_j\right)\right)=0$

We have $\frac{1}{\omega_j}\sum_{i=1}^{n}P(u_i=j)-\lambda=0,\ for\ 1\leq j\leq k$, so

$$\begin{cases}\omega_1\lambda=\sum_{i=1}^{n}P(u_i=1)\\ \omega_2\lambda=\sum_{i=1}^{n}P(u_i=2)\\ \omega_3\lambda=\sum_{i=1}^{n}P(u_i=3)\end{cases}$$

Sum it, we have $\lambda=\sum_{i=1}^{n}1=n$

M-step, so

$$\omega_j^{(1)}=\frac{1}{n}\sum_{i=1}^{n}P(u_i=j|\omega_j^{(0)},\mu_j^{(0)})=\frac{1}{n}\sum_{i=1}^{n}\frac{\omega_j^{(0)}\exp\left(-\frac{1}{2}(y_i-\mu_j^{(0)})^2\right)}{\sum_{j=1}^{k}\omega_j^{(0)}\exp\left(-\frac{1}{2}(y_i-\mu_j^{(0)})^2\right)}$$

$$\frac{\partial Q}{\partial\mu_j}=\sum_{i=1}^{n}\frac{1}{\sigma_j^2}(y_i-\mu_j)P(u_i=j)=0\implies\mu_j^{(1)}=\frac{\sum_{i=1}^{n}y_iP(u_i=j|y_i)}{\sum_{i=1}^{n}P(u_i=j|y_i)},\quad 1\leqslant j\leqslant 3$$