**2 (a)**  $y_{ij}|u_i \sim N(\beta_0 + \beta_1 x_{ij} + u_i, \sigma_{\varepsilon}^2)$ , to integrate out  $u_i$  we need to get its posterior distribution.

$$egin{aligned} f(u_i| heta^{old},\,m{y_i}) &\propto f(y_{i1}|u_i)f(y_{i2}|u_i) \cdot \cdot \cdot \cdot f(y_{iJ}|u_i)f(u_i| heta^{old}) \ &\propto \exp\left(-rac{1}{2\sigma_arepsilon^2} \sum_{j=1}^J (y_{ij}-eta_0-eta_1 x_{ij}-u_i)^2 - rac{u_i^2}{2\sigma_u^2}
ight) \ &\propto \exp\left(-rac{1}{2rac{\sigma_u^2\sigma_arepsilon^2/J}{\sigma_u^2+\sigma_arepsilon^2/J}} \left(u_i - rac{\sigma_u^2ig(ar{y}_i-eta_0-eta_1ar{x}_iig)}{\sigma_u^2+\sigma_arepsilon^2/J}
ight)^2
ight) \ &u_i| heta^{old}, m{y_i} \sim N(E_i,\,\sigma_{old}^2) \ \ \, where\,\, E_i = rac{\sigma_u^2ig(ar{y}_i-eta_0-eta_1ar{x}_iig)}{\sigma_u^2+\sigma_arepsilon^2/J},\, \sigma_{old}^2 = rac{\sigma_u^2\sigma_arepsilon^2/J}{\sigma_u^2+\sigma_arepsilon^2/J} \end{aligned}$$

(y, u) is complete data, we need to integrate out  $u_i$  on  $\theta^{old}$ ,  $\theta^{old} = (E_i, \sigma_{old}^2)$ .

$$\text{Utilize } E(u_i) = E_i \;\;, \;\; E(u_i^2) = \sigma_{old}^2 \;\; + \;\; E_i^2 \;\;, \;\; E(u_i - \lambda)^2 = \sigma_{old}^2 + E_i^2 - 2\lambda E_i + \lambda^2 = \sigma_{old}^2 + (E_i - \lambda)^2$$

$$Q( heta| heta^{old}) = E_{oldsymbol{u}| heta^{old}}[\log P(oldsymbol{y},oldsymbol{u}| heta)] = \int_{-\infty}^{\infty} (\log P(oldsymbol{y},oldsymbol{u}| heta))f(oldsymbol{u}| heta^{old})doldsymbol{u}$$

$$=\sum_{i=1}^{r}E_{u| heta^{old}}igll[\logigl[f(y_{i1}|u_i)f(y_{i2}|u_i)\cdot\cdot\cdot\cdot f(y_{iJ}|u_i)f(u_i| heta)igr]igr]$$

$$= \sum_{i=1}^{I} E_{u|\theta^{old}} \Bigg[ \sum_{j=1}^{J} \left( -\frac{1}{2} \log \sigma_{\varepsilon}^2 - \frac{1}{2\sigma_{\varepsilon}^2} \left(y_{ij} - \beta_0 - \beta_1 x_{ij} - u_i \right)^2 \right) \ - \ \frac{1}{2} \log \sigma_u^2 - \frac{u_i^2}{2\sigma_u^2} \Bigg]$$

$$I = \sum_{i=1}^{I} \left[ -rac{J}{2} \log \sigma_arepsilon^2 - rac{1}{2\sigma_arepsilon^2} \sum_{i=1}^{J} \left(\sigma_{old}^2 + (y_{ij} - eta_0 - eta_1 x_{ij} - E_i)^2
ight) - rac{1}{2} \log \sigma_u^2 - rac{\sigma_{old}^2 + E_i^2}{2\sigma_u^2} 
ight]$$

$$=-\frac{IJ}{2}\log\sigma_{\varepsilon}^2-\frac{IJ\sigma_{old}^2}{2\sigma_{\varepsilon}^2}-\frac{1}{2\sigma_{\varepsilon}^2}\sum_{i=1}^{I}\sum_{j=1}^{J}(y_{ij}-\beta_0-\beta_1x_{ij}-E_i)^2-\frac{I}{2}\log\sigma_u^2-\frac{I\sigma_{old}^2}{2\sigma_u^2}-\frac{1}{2\sigma_u^2}\sum_{i=1}^{I}E_i^2$$

Notice  $\frac{\partial \theta^{old}}{\partial \sigma_{s}^{2}} = \frac{\partial \theta^{old}}{\partial \sigma_{u}^{2}} = \frac{\partial \theta^{old}}{\partial \beta_{0}} = \frac{\partial \theta^{old}}{\partial \beta_{1}} = 0$ , we have M step as following:

$$egin{aligned} rac{\partial Q}{\partial eta_0} &= rac{1}{\sigma_arepsilon^2} \sum_{i=1}^I \sum_{j=1}^J (y_{ij} - eta_0 - eta_1 x_{ij} - E_i) = 0 \ rac{\partial Q}{\partial eta_1} &= rac{1}{\sigma_arepsilon^2} \sum_{i=1}^I \sum_{j=1}^J x_{ij} (y_{ij} - eta_0 - eta_1 x_{ij} - E_i) = 0 \end{aligned}$$

Solve this equation group, we have

$$eta_0 \! = \! \left( rac{\sum\limits_{i=1}^{I} \sum\limits_{j=1}^{J} x_{ij} (y_{ij} \! - \! E_i)}{\sum\limits_{i=1}^{I} \sum\limits_{j=1}^{J} x_{ij}^2} \, - rac{\sum\limits_{i=1}^{I} \sum\limits_{j=1}^{J} (y_{ij} \! - \! E_i)}{\sum\limits_{i=1}^{I} \sum\limits_{j=1}^{J} x_{ij}} 
ight) / \left( rac{\sum\limits_{i=1}^{I} \sum\limits_{j=1}^{J} x_{ij}}{\sum\limits_{i=1}^{J} \sum\limits_{j=1}^{J} x_{ij}^2} - rac{IJ}{\sum\limits_{i=1}^{I} \sum\limits_{j=1}^{J} x_{ij}} 
ight)$$

$$eta_1 = \sum_{i=1}^{I} \sum_{j=1}^{J} x_{ij} (y_{ij} - eta_0 - E_i) \; / \; \sum_{i=1}^{I} \sum_{j=1}^{J} x_{ij}^2$$

where  $\beta_0$  is as above.

$$rac{\partial Q}{\partial \sigma_arepsilon^2} = -rac{IJ}{2\sigma_arepsilon^2} + rac{IJ\sigma_{old}^2}{2\sigma_arepsilon^4} + rac{1}{2\sigma_arepsilon^4} \sum_{i=1}^I \sum_{j=1}^J (y_{ij} - eta_0 - eta_1 x_{ij} - E_i)^{\,2} = 0$$

$$\implies \sigma_{arepsilon}^{\,2} = \sigma_{old}^{\,2} + rac{1}{IJ} \sum_{i=1}^{I} \sum_{j=1}^{J} (y_{ij} - eta_0 - eta_1 x_{ij} - E_i)^{\,2}$$

where  $\beta_0$ ,  $\beta_1$  are as above.

$$rac{\partial Q}{\partial \sigma_u^2} = -rac{I}{2\sigma_u^2} + rac{I\sigma_{old}^2}{2\sigma_u^4} + rac{1}{2\sigma_u^4}\sum_{i=1}^I E_i^2 = 0$$

$$\implies \sigma_u^2 = \sigma_{old}^2 + rac{1}{I} \sum_{i=1}^I E_i^2$$

In above 4 formulas,

$$E_i = \frac{\sigma_u^2 \left(\overline{y}_i - \beta_0 - \beta_1 \overline{x}_i\right)}{\sigma_u^2 + \sigma_\varepsilon^2 / J}, \ \sigma_{old}^2 = \frac{\sigma_u^2 \sigma_\epsilon^2 / J}{\sigma_u^2 + \sigma_\varepsilon^2 / J}, \ \text{where} \ \beta_0, \beta_1, \sigma_u^2, \sigma_\varepsilon^2 \ \text{are from last step (old variables)}.$$

3 (a) Complete-data likelihood

$$g(y_i, u_i | oldsymbol{\omega}) = \prod_{j=1}^k ig[\omega_j f_j(y_i)ig]^{I(u_i=j)} = \sum_{j=1}^k \omega_j f_j(y_i) I(u_i\!=\!j)$$

3 (b) Marginal distribution

$$g(y_i|oldsymbol{\omega}) = \sum_{u_i=1}^k g(y_i, u_i|oldsymbol{\omega}) = \sum_{j=1}^k \omega_j f_j(y_i)$$

**3 (d)** The probability of  $u_i = j$  is

$$P(u_i = j | y_i, \mu_j^{(0)}) = rac{P(y_i | u_i = j) \omega_j}{P(y_i)} = rac{P(y_i | u_i = j) \omega_j}{\displaystyle\sum_{j=1}^k P(y_i | u_i = j) \omega_j} = rac{\omega_j \exp\left(-rac{1}{2} \left(y_i - \mu_j^{(0)}
ight)^2
ight)}{\displaystyle\sum_{j=1}^k \omega_j \exp\left(-rac{1}{2} \left(y_i - \mu_j^{(0)}
ight)^2
ight)}$$

E-step:

$$egin{aligned} Q(oldsymbol{\mu}|oldsymbol{\mu}^{(0)}) &= \sum_{i=1}^n E_{oldsymbol{u}|oldsymbol{\mu}^{(0)}} igl[\log g(y_i,u_i|oldsymbol{\mu})igr] = \sum_{i=1}^n \sum_{j=1}^k igl[\log \omega_j + \log f_j(y_i)igr] P(u_i = j|y_i) \ &\propto \sum_{i=1}^n \sum_{j=1}^k -rac{(y_i - \mu_j)^2}{2} P(u_i = j|y_i) \end{aligned}$$

M-step:

$$rac{\partial Q}{\partial \mu_j} = \sum_{i=1}^n rac{1}{\sigma_j^2} (y_i - \mu_j) P(u_i = j | y_i) = 0 \quad \Longrightarrow \quad \mu_j^{(1)} = rac{\sum_{i=1}^n y_i P(u_i = j | y_i)}{\sum_{i=1}^n P(u_i = j | y_i)}$$

## 3 (e) E-step

$$egin{aligned} Q(oldsymbol{\omega},oldsymbol{\mu}^{(0)},oldsymbol{\mu}^{(0)}) &= \sum_{i=1}^n E_{oldsymbol{u}|oldsymbol{\omega}^{(0)},oldsymbol{\mu}^{(0)}} igl[\log g(y_i,u_i|oldsymbol{\omega},oldsymbol{\mu})igr] &= \sum_{i=1}^n \sum_{j=1}^k igl[\log \omega_j + \log f_j(y_i)igr] P(u_i=j|y_i) \ &\propto \sum_{i=1}^n \sum_{j=1}^k igl[\log \omega_j - rac{1}{2} \left(y_i - \mu_j
ight)^2igr] P(u_i=j|y_i) \end{aligned}$$

To find the expression for  $\omega_j$  to maximize Q, we use Lagrange multiplier  $\lambda$  under the constraint that  $\sum_{j=1}^k \omega_j = 1$  and  $\sum_{j=1}^k P(u_i = j) = 1$ , solve the equation  $\frac{\partial}{\partial \omega_j} \left( Q + \lambda \left( 1 - \sum_{j=1}^k \omega_j \right) \right) = 0$  We have  $\frac{1}{\omega_j} \sum_{i=1}^n P(u_i = j) - \lambda = 0$ , for  $1 \le j \le k$ , so  $\left( \omega_1 \lambda = \sum_{i=1}^n P(u_i = 1) \right)$ 

$$\begin{cases} \omega_1 \lambda = \sum_{i=1}^n P(u_i = 1) \\ \omega_2 \lambda = \sum_{i=1}^n P(u_i = 2) \\ \omega_3 \lambda = \sum_{i=1}^n P(u_i = 3) \end{cases}$$

Sum it, we have  $\lambda = \sum_{i=1}^{n} 1 = n$ 

M-step, so

$$P(\omega_j^{(1)} = rac{1}{n} \sum_{i=1}^n P(u_i = j | \omega_j^{(0)}, \mu_j^{(0)}) = rac{1}{n} \sum_{i=1}^n rac{\omega_j^{(0)} \exp\left(-rac{1}{2} \left(y_i - \mu_j^{(0)}
ight)^2
ight)}{\sum_{j=1}^k \omega_j^{(0)} \exp\left(-rac{1}{2} \left(y_i - \mu_j^{(0)}
ight)^2
ight)}$$

$$rac{\partial Q}{\partial \mu_j} = \sum_{i=1}^n rac{1}{\sigma_j^2} (y_i - \mu_j) P(u_i = j) = 0 \;\; \implies \; \mu_j^{(1)} = rac{\sum_{i=1}^n y_i P(u_i = j | y_i)}{\sum_{i=1}^n P(u_i = j | y_i)}, \;\;\; 1 \leqslant j \leqslant 3$$