Final Term Exam Math Stats and Linear Algebra December 15, 2022

ERP No.	Name

Note: Please solve all FIVE problems. Each problem has a maximum credit of 8 marks. To get the full credit you should provide the reasoning and show all the work.

Question 1

a) Consider the following homogeneous system of three linear equations in three unknowns

$$x_1 - x_2 + 3x_3 = 0$$
$$x_1 + x_2 - hx_3 = 0$$
$$x_1 - x_2 + hx_3 = 0$$

- For what value(s) of h the system has only trivial solution?
- For what value(s) of h the system has nontrivial solutions? Also find all such solutions.

b) Consider the following non-homogeneous system of three linear equations in three unknowns

$$x_1 - x_2 + 3x_3 = 0$$
$$x_1 + x_2 - hx_3 = 2$$
$$x_1 - x_2 + hx_3 = 0$$

- Show that the system is consistent for all values of h.
- For what value(s) of h the system has a unique solution? Also find the unique solution.
- For what value(s) of h the system has infinitely many solutions? Also find all such solutions.

Question 2

a) Find the standard matrix of a linear transformation T: $R^2 \to R^2$ which rotates a point about the origin through an angle of $3\pi/4$ radians (in clockwise direction).

b) For what value(s) of k will the vector $\begin{bmatrix} 1 \\ 2 \\ k \end{bmatrix}$ in R^3 be a linear combination of the vectors $\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$? Also find the corresponding weights.

c) The columns of $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ are $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and $\mathbf{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. Suppose T is a linear transformation from R^3 to R^4 where $T(\mathbf{e}_1) = \begin{bmatrix} 1 \\ 2 \\ 4 \\ 6 \end{bmatrix}$, $T(\mathbf{e}_2) = \begin{bmatrix} 6 \\ 4 \\ 2 \\ 1 \end{bmatrix}$ and $T(\mathbf{e}_3) = \begin{bmatrix} 7 \\ 6 \\ 6 \\ 7 \end{bmatrix}$. Compute $T(\mathbf{x})$ for any $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$.

Question 3

a) What is meant by an **elementary** matrix? Identify the **elementary** matrices among the following. Also find the inverses of those matrices which are **elementary**. (In each case your answer must be supported by valid reasons.)

i. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ ii. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ iii. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ iv. $\begin{bmatrix} -10 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

b) By using elementary row operations find the inverse of

$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix}$ and use this inverse to solve the system $A\underline{x} = \begin{bmatrix} 14 \\ 8 \\ 13 \end{bmatrix}$.

c) By examining the elementary row operations which you have performed in part **b)** above, find the determinant of A.

Question 4

a) If the probability density of X is given by $f(x) = \begin{cases} \frac{x}{4.5} & for\ 0 \le x \le 3 \\ 0 & otherswise \end{cases}$

find the probability density of $Y = X^3$.

b) If the joint density of X and Y is given by

$$f(x, y) = \begin{cases} e^{-(x+y)} & for\ x > 0\ and\ y > 0 \\ 0 & otherwise \end{cases}$$

find the probability density of $Z = \frac{X + Y}{2}$.

Question 5

a) Let X and Y be two jointly continuous random variables with joint PDF

$$f(x,y) = \begin{cases} x + cy^2 \; for \; 0 \le x \le 1 \; and \; 0 \le y \le 1 \\ 0 \qquad\qquad\qquad\qquad\qquad otherwise \end{cases}$$

i) Find the constant c.

ii) Find $P(0 \le X \le \frac{1}{2}, 0 \le Y \le \frac{1}{2})$.

b) Let X and Y be two jointly continuous random variables with joint PDF

$$f(x,y) = \begin{cases} cx^2y \quad for \; 0 \le y \le x \le 1 \\ 0 \qquad\qquad\qquad otherwise \end{cases}$$

i) Find the constant c.
ii) Find the marginal densities of X and Y.
iii) Find $P(X \ge 2Y)$.
iv) Find $P(X \ge 4Y \mid X \ge 2Y)$.

c) Let X and Y be two jointly continuous random variables with joint PDF

$$f(x,y) = \begin{cases} \frac{x^2}{4} + \frac{y^2}{4} + \frac{xy}{6} \quad for \; 0 \le x \le 1, 0 \le y \le 2 \\ 0 \qquad\qquad\qquad\qquad otherwise \end{cases}$$

i) Find the marginal densities of X and Y.
ii) For $0 \le y \le 2$, find the conditional density of X given $Y = y$.
iii) $P(X < \frac{1}{2} | Y = y)$.