Chapter 9

9.2 Let a_{ij} be element in ith row and jth column. Since saddle point is minimum of row and maximum of column

	j	l
i	a_{ij}	a_{il}
k	a_{kj}	a_{kl}

$a_{ij} \ge a_{kj} \ge a_{kl} \ge a_{il} \ge a_{ij}$

$\therefore$ must all be equal signs

$a_{ij} = a_{kj} = a_{kl} = a_{il}$ and both parts are proved

9.3 If we let $x = 0$ for n heads, $x = 1$ at least one tail
Only changes in risk functions are that
$R(d_1,\ \theta_2) = \dfrac{1}{2^n}$ and $R(d_4,\ \theta_2) = 1 - \dfrac{1}{2^n}$

dominance same as before

resulting risk functions given by

	d_1	d_2
θ_1	0	1
θ_2	$1/2^n$	0

9.4 $R(d_1,\ \theta) = \int_0^\theta c(kx-\theta)^2 1 \frac{1}{\theta} d\theta$

$$= \frac{c}{\theta}\left[\frac{(kx-\theta)^3}{3k}\right]\Bigg|_0^c$$

$$= \frac{c}{\theta}\left[\frac{(k\theta-\theta)^3}{3k} - \frac{\theta^3}{3k}\right] = \frac{c\theta^2}{3}(k^3 - 3k + 3)$$

9.5 $p(x < k) = \int_0^k \frac{2x}{\theta^2} dx = \frac{k^2}{\theta_2}$

	θ_1	θ_2
θ_1	0	C
θ_2	C	0

Probability	
$\dfrac{k^2}{\theta_1^2}$	$1 - \dfrac{k^2}{\theta_1^2}$
$\dfrac{k^2}{\theta_2^2}$	$1 - \dfrac{k^2}{\theta_2^2}$

$R(d,\ \theta_1) = C\left(1 - \dfrac{k^2}{\theta_1^2}\right),\ R(d,\ \theta_2) = C \cdot \dfrac{k^2}{\theta_2^2}$

For mimimax solution $C\left(1 - \dfrac{k^2}{\theta_1^2}\right) = C \cdot \dfrac{k^2}{\theta_2^2} \qquad k = \dfrac{\theta_1\theta_2}{\sqrt{\theta_1^2 + \theta_2^2}}$

9.6 Maximizing $R(d,\ \theta)$ with respect to θ yields

$$\theta = \frac{2ab-n}{2(b^2-n)}$$

Substituting this value into $R(d,\ \theta)$ and differentiating partially with respect to a and b yields $a = \frac{1}{2}\sqrt{n}$ and $b = \sqrt{n}$.

9.7 $E(\Theta) = \int_0^1 x\ dx = \frac{1}{2},\ E(\Theta^2) = \int_0^1 x^2\ dx = \frac{1}{3}$

Substituting into $R(d,\ \Theta)$ yields

Bayes Risk = $\frac{c}{(n+b)^2}\left[\frac{1}{3}(b^2-n) + \frac{1}{2}(n-2ab) + a^2\right]$

Differentiating partially with respect to a and equating to 0 yields $a = \frac{b}{2}$. Substituting $a = \frac{b}{2}$ into Bayes risk and differentiating with respect to b yields b= 2. So a = 1 and $d(x) = \frac{x+1}{n+2}$.

9.8 $g(x) = \int_x^{\infty} e^{-\theta} d\,\theta = e^{-x}$ for $x > 0$

$g(x) = 0$ elsewhere

$\phi(\theta|x) = \frac{f(x,\ \theta)}{g(x)} = \frac{e^{-\theta}}{e^{-x}} = e^{x-\theta}$ for $\theta > x$

$\phi(\theta|x) = 0$ elsewhere

9.9 **(a)** $g(x,\ \theta) = \theta(1-\theta)^{x-1}$ $x = 1,\ 2,\ 3,\ ...$

$f(x,\ \theta) = \theta(1-\theta)^{x-1}\cdot 1$ $x = 1,\ 2,\ 3,\ ...$ $0 < \theta < 1$

Beta distribution with $a = 2,\ \beta = x$

$$g(x) = \int_0^1 \theta(1-\theta)^{x-1} d\theta = \frac{\Gamma(2)\Gamma(x)}{\Gamma(x+2)} = \frac{1}{x(x+1)}$$

$\phi(\theta|x) = \frac{\theta(1-\theta)^{x-1}}{1/x(x+1)} = x(x+1)\theta(1-\theta)^{x-1}$ $0 < \theta < 1$

$\phi(\theta|x) = 0$ elsewhere

(b) $$\sum_{x=1}^{\infty}\left\{\int_0^1 c\left[d(x)-\theta\right]^2\theta(1-\theta)^{x-1}x(x+1)\,d\theta\right\}$$

$$c\int_0^1 2\left[d(x)-\theta\right]\theta(1-\theta)^{x-1}x(x+1)d\theta$$

$$2cx(x+1)\int_0^1\left[d(x)-\theta\right]\theta(1-\theta)^{x-1}d\theta=0$$

$$d(x)\int_0^1\theta(1-\theta)^{x-1}d\theta=\int_0^1\theta^2(1-\theta)^{x-1}d\theta$$

$$d(x)\cdot\frac{1}{x(x+1)}=\frac{\Gamma(3)\Gamma(x)}{\Gamma(x+3)}=\frac{2(x+1)!}{(x+2)!}=\frac{2}{(x+2)(x+1)x}$$

$$d(x)=\frac{2}{x+2}$$

9.10

	expand	wait		
Good times	−164,000	−80,000	0.4	4/11
Recession	40,000	−8,000	0.6	7/11

(a) $E=(0.4)(-164{,}000)+(0.6)(40{,}000)=-41{,}600$

$E=(0.4)(-80{,}000)+(0.6)(-8{,}000)=-36{,}800$

Manufacturer should expand now.

(b) $E=\frac{4}{11}(-164{,}000)+\frac{7}{11}(40{,}000)=-34{,}182$

$E=\frac{4}{11}(-80{,}000)+\frac{7}{11}(-8{,}000)=-34{,}182$

Does not matter.

9.11 **(a)**

	expand	wait	
Good times	−200,000	−80,000	1/3
Recession	40,000	−8,000	2/3

$E=\frac{1}{3}(-200{,}000)+\frac{2}{3}(40{,}000)=-40{,}000$

$E=\frac{1}{3}(-80{,}000)+\frac{2}{3}(-8{,}000)=-32{,}000$

Manufacturer should expand now. Decision reversed.

(b)

	expand	wait	
good times	−164,000	−80,000	2/5
recession	60,000	−8,000	3/5

$$E = \frac{2}{5}(-164{,}000) + \frac{3}{5}(60{,}000) = -29{,}600$$
$$E = \frac{2}{5}(-80{,}000) + \frac{3}{5}(-8{,}000) = -36{,}800$$

Manufacturer should expand now. Decision reversed.

9.12 Reservation at

	x	Y	(a)	(b)
x	65	68.40	3/4	5/6
Y	72	62.40	1/4	1/6

(a) $EC = \frac{3}{4}(66) + \frac{1}{4}(72) = 67.50$

$EC = \frac{3}{4}(68.40) + \frac{1}{4}(62.40) = 66.90$ Make reservation at Hotel Y.

(b) $EC = \frac{5}{6}(66) + \frac{1}{6}(72) = 67$

$EC = \frac{5}{6}(68.40) + \frac{1}{6}(62.40) = 67.40$ Make reservation at Hotel x

9.13 go to

	27	33	(a)	(b)	(c)
should go to 27	27	45	1/6	1/3	1/4
33	39	33	5/6	2/3	3/4

(a) $ED = \frac{1}{6}(27) + \frac{5}{6}(39) = 37$

$ED = \frac{1}{6}(45) + \frac{5}{6}(33) = 35$ Should go to site 33 miles from lumberyard.

(b) $ED = \frac{1}{3}(27) + \frac{2}{3}(39) = 35$

$ED = \frac{1}{3}(45) + \frac{2}{3}(33) = 37$ Should go to site 27 miles from lumberyard.

(c) $ED = \frac{1}{4}(27) + \frac{3}{4}(39) = 36$ Does not matter.

$ED = \frac{1}{4}(45) + \frac{3}{4}(33) = 36$

9.14 **(a)** If he goes to x worst cost is 72.00; if he goes to Y worst cost is 68.40. Worst cost is minimized if he chooses Y.

(b) If he goes to (27) worst distance is 39; if he goes to (33) worst distance is 45; worst distance is least if he goes to site 27 miles from lumberyard.

9.15 **(a)** If he expands now, maximum gain is 164,000; if he waits maximum gain is 80,000. Maximum gain is maximized if he expands now.

(b) If she chooses x, minimum cost is 66; if she chooses Y minimum cost is 62.40; minimum cost is minimized if she chooses Y.

(c) If he goes to (27), minimum distance is 27; if he goes to (33) minimum distance is 33; minimum distance is minimized if he goes to site 27 miles from lumberyard.

9.16 **(a)** opportunity losses are

0	84,000
48,000	0

(b) Maximum opportunity losses are 48,000 and 84,000; these are minimized if he expands now.

9.17 **(a)** opportunity losses are

0	2.40
9.60	0

Maximum opportunity losses are 9.60 and 2.40; they are minimized if she chooses Hotel Y.

(b) opportunity losses are

0	18
6	0

Maximum opportunity losses are 6 and 18; they are minimized if he chooses to go to site 27 miles from lumberyard.

9.18 Expected losses with perfect information = $\frac{1}{3}(-164{,}000)+\frac{2}{3}(-8{,}000)=-60{,}000$

60,000 exceeds 28,000 and 32,000 by more than 15,000
Hiring the forecaster is worthwhile.

9.19 **(a)** Cross out first row, cross out second column, optimum strategies I and 2; value = 5
(b) Cross out first column, cross out second row, optimum strategies II and 1; value = 11
(c) Cross out third column, cross out second row, cross out second column, cross out third row, optimum strategies I and 1; value = -5.
(d) Cross out third column, cross out third row, cross out second column, cross out first row, optimum strategies I and 2; value = 8.

9.20 **(a)** Mimima of rows are $-2,\ 0,\ -4$; only second is largest of its column. Saddle point corresponds to I and 2; value = 0.

(b) Mimima of rows are 2, 3, 5, and 5; first two are not maxima of their columns; others are. Saddle point corresponds to I and 3; I and 4, III and 3, III and 4; value = 5 in each case.

9.21 (a)

	no glasses	glasses
no knives	0	−6
knives	8	3

(b) Minimum of second row is maximum of second column saddle point. Optimum strategies are for Station A to give away glasses and Station B to give away knives.

9.22

p	8	−5
$1-p$	2	6

$$8p+2(1-p)=-5p+6(1-p)$$
$$8p+2-2p=-5p+6-6p$$
$$17p=4 \qquad p=\frac{4}{17}$$

probabilities are $\frac{4}{17}$ and $\frac{13}{17}$

9.23

	x	$1-x$
y	3	−4
$1-y$	−3	1

(a) $3x-4(1-x)=-3x+(1-x)$

$11x=5 \qquad x=\frac{5}{11}$

probabilities are $\frac{5}{11}$ and $\frac{6}{11}$

(b) $3y-3(1-y)=-4y+(1-y)$

$11y-4 \qquad$ probabilities are $\frac{4}{11}$ and $\frac{7}{11}$

(c) $3\cdot\frac{4}{11}-3\cdot\frac{7}{11}=-\frac{9}{11}$

9.24

x	$1-x$
66	68.40
72	62.40

$$66x+68.40(1-x)=72x+62.40(1-x)$$
$$6(1-x)=6x \qquad 1-x=x \qquad x=\frac{1}{2}$$

probabilities are $\frac{1}{2}$ and $\frac{1}{2}$

9.25

enemy attacks

		y 2	$1-y$ 10
country defends	x 2	12	2
	$1-x$ 10	10	12

$12x+10(1-x)=2x+12(1-x)$

$12x=2 \qquad x=\frac{1}{6}$

for defends $\frac{1}{6}$ and $\frac{5}{6}$

$12y+2(1-y)=10y+12(1-y)$

$12y=10 \qquad y=\frac{5}{6} \qquad$ for enemy $\frac{5}{6}$ and $\frac{1}{6}$

value is $12\cdot\frac{5}{6}+2\cdot\frac{1}{6}=10\frac{1}{3}$ which is \$10,333,333

9.26 **(a)**

first person

		1	4
second	0	−1	2
	3	2	−7

(b) $-x+2(1-x)=2x-7(1-x)$

$12x=9 \qquad x=\frac{3}{4} \qquad \frac{3}{4}$ and $\frac{1}{4}$

(c) $-y+2(1-y)=2y-7(1-y)$

$12y=9 \qquad y=\frac{3}{4} \qquad \frac{3}{4}$ and $\frac{1}{4}$

9.27

first

		lowers	not
second	lowers	\$80	\$70
	not	\$140	\$100

(a) Minima are \$80 and \$70. Maximized if he lowers prices.

(b) They might lower prices on alternate days.

$\frac{140+70}{2}=105$

9.28 **(a)**

	0	1/2	1
0	0	50	100
1/2	50	0	50
1	100	50	0

(b) $d_1(0)=0,\ d_1(1)=0;\ d_2(0)=0,\ d_2(1)=\frac{1}{2};\ d_3(0)=0,\ d_3(1)=1;$

$d_4(0)=\frac{1}{2},\ d_4(1)=0;\ d_5(0)=\frac{1}{2},\ d_5(1)=\frac{1}{2};\ d_6(0)=\frac{1}{2},\ d_6(1)=1;$

$d_7(0)=1,\ d_7(1)=0;\ d_8(0)=1,\ d_8(1)=\frac{1}{2};\ d_9(0)=1,\ d_9(1)=1$

(c) The risk functions are

	d_1	d_2	d_3	d_4	d_5	d_6	d_7	d_8	d_9
0	0	0	0	50	50	50	100	100	100
1/2	50	25	50	25	0	25	50	25	50
1	100	50	0	100	50	0	100	50	0

d_1, d_4, d_7, and d_8 are eliminated by dominances; only d_2, d_3, d_5, d_6 are admissible and by inspection the maximum is 50 in each case. Accordingly by minimax criterion they area all equally good.

(d) Bayes risks are

$$d_2 \quad 0\cdot\frac{1}{3}+25\cdot\frac{1}{3}+50\cdot\frac{1}{3}=25$$

$$d_3 \quad 0\cdot\frac{1}{3}+50\cdot\frac{1}{3}+0\cdot\frac{1}{3}=16\frac{2}{3}$$

$$d_5 \quad 50\cdot\frac{1}{3}+0\cdot\frac{1}{3}+50\cdot\frac{1}{3}=33\frac{1}{3}$$

$$d_6 \quad 50\cdot\frac{1}{3}+25\cdot\frac{1}{3}+0\cdot\frac{1}{3}=25$$

Bayes risk is minimum for d_3.

9.29 **(a)**

	1/4	1/2
1/4	0	160
1/2	160	0

(b)

$$d_1(0)=\frac{1}{4},\ d_1(1)=\frac{1}{4};\ d_1(2)=\frac{1}{4},\ d_2(0)=\frac{1}{4};\ d_2(1)=\frac{1}{4},\ d_2(2)=\frac{1}{2};$$

$$d_3(0)=\frac{1}{4},\ d_3(1)=\frac{1}{2};\ d_3(2)=\frac{1}{4},\ d_4(0)=\frac{1}{4};\ d_4(1)=\frac{1}{2},\ d_4(2)=\frac{1}{2};$$

$$d_5(0)=\frac{1}{2},\ d_5(1)=\frac{1}{4};\ d_5(2)=\frac{1}{4},\ d_6(0)=\frac{1}{2};\ d_6(1)=\frac{1}{4},\ d_6(2)=\frac{1}{2};$$

$$d_7(0)=\frac{1}{2},\ d_7(1)=\frac{1}{2};\ d_7(2)=\frac{1}{4},\ d_8(0)=\frac{1}{2};\ d_8(1)=\frac{1}{2},\ d_8(2)=\frac{1}{2}$$

(c) The risk functions are

	d_1	d_2	d_3	d_4	d_5	d_6	d_7	d_8
1/4	0	10	60	70	60	100	150	160
1/2	160	120	80	40	120	80	40	0

probabilities for $\theta=\frac{1}{4}$ are $\frac{9}{16},\ \frac{6}{16},\ \frac{1}{16}$

$\theta=\frac{1}{2}$ are $\frac{1}{4},\ \frac{1}{2},\ \frac{1}{4}$

d_7 is dominated by d_4, d_6 is dominated by d_3, d_5 is dominated by d_2 and d_3.

(d) The maxima corresponds to d_1, d_2, d_3, d_4, and d_8 are 160, 120, 80, 70, and 160. So the minimax criterion yields d_4.

(e) The five Bayes risks are $\frac{160}{3}$, $\frac{140}{3}$, $\frac{200}{3}$, $\frac{180}{3}$, $\frac{320}{3}$ so that d_2 is best.

9.30 **(a)**

		no inspection	inspect 1	inspect 2
	0	0	β	2β
repeat	1	$\alpha+2\beta+\phi$	$\frac{\alpha}{2}+2\beta+\phi$	$2\beta+\phi$
	2	$\alpha+2\beta+2\phi$	$2\beta+2\phi$	$2\beta+2\phi$

(b)

0	β	2β
$35+2\beta$	$22.50+2\beta$	$10+2\beta$
$45+2\beta$	$20+2\beta$	$20+2\beta$

maxima

$45+2\beta$ $\quad$ $22.50+2\beta$ $\quad$ $20+2\beta$ $\quad$ should inspect both

$\uparrow$

min

(c)

0	12	24
64	59	54
94	84	85

Bayes risks are

$$0(0.70)+64(0.20)+94(0.10)=22.2 \quad \leftarrow$$
$$12(0.70)+59(0.20)+84(0.10)=28.6$$
$$24(0.70)+54(0.20)+84(0.10)=36.0$$

Minimized if shipped without inspection

9.31 $\delta(\theta)=R(d_1,\theta)-R(d_2,\theta)=(1{,}000\theta-2{,}000)[B(1;10,\theta)-B(0;10,\theta)]$
As in the example, the first term always negative, and the second term is always positive; thus, $\delta(\theta)$ is always negative. As before, d_1 dominates d_2 and it is preferred.

9.32 $\delta(\theta)=(C_w\cdot n\theta-C_d)[B(2;n,\theta)-B(1;n;\theta)]$.
Since the second term of this product is always positive, d_2 will dominate d_1 when the first term is positive, that is, when $C_w n\theta > C_d$, as long as there is a value of $\theta \le 1$ that satisfies this inequality. Thus, strategy 2 will be preferable to strategy 1 whenever $\frac{C_4}{nC_w} < \theta \le 1$