Math stats & Linear Algebra L14 Final Term Exam Dec 15, 2022

Q1- a) $\begin{bmatrix}1 & -1 & 3\\ 1 & 1 & -h\\ 1 & -1 & h\end{bmatrix} \simeq \begin{bmatrix}1 & -1 & 3\\ 0 & 2 & -3-h\\ 0 & 0 & -3+h\end{bmatrix}$ • For $\boxed{h\neq 3}$ all columns are pivot columns so in this case we have $\boxed{\text{trivial Sols}}$ only. $\underline{x}=\underline{0}$

•• For $\boxed{h=3}$ last column is a pivot column, so x_3 is a free variable. So system has nontrivial solutions. We have in this case $\begin{bmatrix}1 & -1 & 3\\ 0 & 2 & -6\\ 0 & 0 & 0\end{bmatrix} \simeq \begin{bmatrix}1 & -1 & 3\\ 0 & 1 & -3\\ 0 & 0 & 0\end{bmatrix} \simeq \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & -3\\ 0 & 0 & 0\end{bmatrix}$

So, $\boxed{x_1=0,\quad x_2=3x_3}$ and x_3 is a free variable. $\boxed{\underline{x}=x_3\begin{bmatrix}0\\3\\1\end{bmatrix}}$

For any value of $x_3\neq 0$ we have a non-trivial Sol.

b) $\begin{bmatrix}1 & -1 & 3 & 0\\ 1 & 1 & -h & 2\\ 1 & -1 & h & 0\end{bmatrix} \simeq \begin{bmatrix}1 & -1 & 3 & 0\\ 0 & 2 & -3-h & 2\\ 0 & 0 & -3+h & 0\end{bmatrix}$. • For all values of h last column is not a pivot column hence system is consistent for all h.

•• For $\boxed{h\neq 3}$ we have $\begin{bmatrix}1 & -1 & 3 & 0\\ 0 & 2 & -3-h & 2\\ 0 & 0 & 1 & 0\end{bmatrix}$

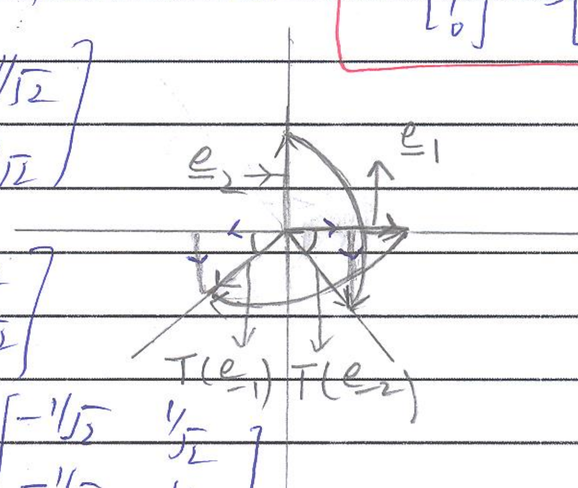
$\simeq \begin{bmatrix}1 & -1 & 0 & 0\\ 0 & 2 & 0 & 2\\ 0 & 0 & 1 & 0\end{bmatrix} \simeq \begin{bmatrix}1 & 0 & 0 & 1\\ 0 & 1 & 0 & 1\\ 0 & 0 & 1 & 0\end{bmatrix}$ So for $\boxed{h\neq 3}$ unique Sol. is $\boxed{x_1=x_2=1,\ x_3=0}$ $\boxed{\underline{x}=\begin{bmatrix}1\\1\\0\end{bmatrix}}$.

••• For $\boxed{h=3}$ last two columns are not pivot columns, which means system is consistent and has ∞'ly many Sols:

$\begin{bmatrix}1 & -1 & 3 & 0\\ 0 & 2 & -6 & 2\\ 0 & 0 & 0 & 0\end{bmatrix} \simeq \begin{bmatrix}1 & -1 & 3 & 0\\ 0 & 1 & -3 & 1\\ 0 & 0 & 0 & 0\end{bmatrix} \simeq \begin{bmatrix}1 & 0 & 0 & 1\\ 0 & 1 & -3 & 1\\ 0 & 0 & 0 & 0\end{bmatrix}$

So $\boxed{x_1=1,\quad x_2=1+3x_3}$ x_3 is a free variable. $\boxed{\underline{x}=\begin{bmatrix}1\\1\\0\end{bmatrix}+x_3\begin{bmatrix}0\\3\\1\end{bmatrix}}$

Q2 - a) $T(\underline{e}_1)=\begin{bmatrix}-\cos\pi/4\\ -\sin\pi/4\end{bmatrix}=\begin{bmatrix}-1/\sqrt{2}\\ -1/\sqrt{2}\end{bmatrix}$



$T(\underline{e}_2)=\begin{bmatrix}\cos\pi/4\\ -\sin\pi/4\end{bmatrix}=\begin{bmatrix}1/\sqrt{2}\\ -1/\sqrt{2}\end{bmatrix}$

So, $A=\begin{bmatrix}T(\underline{e}_1) & T(\underline{e}_2)\end{bmatrix}=\begin{bmatrix}-1/\sqrt{2} & 1/\sqrt{2}\\ -1/\sqrt{2} & -1/\sqrt{2}\end{bmatrix}$.

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b) Let α & β be the weights, then $\alpha\begin{bmatrix}1\\-1\\2\end{bmatrix}+\beta\begin{bmatrix}3\\1\\2\end{bmatrix}=\begin{bmatrix}1\\2\\k\end{bmatrix}$

$\Rightarrow \alpha+3\beta=1;\ -\alpha+\beta=2 \Rightarrow \beta=3/4$ & $\alpha=1-3\beta=1-\frac{9}{4}=-5/4$

So, $K=2(\alpha+\beta)=2(-5/4+3/4)=-5/2+3/2=-1$

So, $\boxed{K=-1}$, $\boxed{\alpha=-5/4}$, $\boxed{\beta=3/4}$

c) $T(\underline{x})=A\underline{x}=\begin{bmatrix}1&6&7\\2&4&6\\4&2&6\\6&1&7\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}x_1+6x_2+7x_3\\2x_1+4x_2+6x_3\\4x_1+2x_2+6x_3\\6x_1+x_2+7x_3\end{bmatrix}$

Q3 a) A matrix is called an elementary matrix if it can be reduced to an identity matrix by using one elementary row operation.

i) Yes, inv is $\begin{bmatrix}1&0&0&0\\0&1&0&3\\0&0&1&0\\0&0&0&1\end{bmatrix}$ ii) Yes, inv is $\begin{bmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\1&0&0&1\end{bmatrix}$

iii) Not an elementary matrix iv) Yes, inv is $\begin{bmatrix}-1/10&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}$

b) $\left[\begin{array}{ccc|ccc}1&2&3&1&0&0\\1&2&1&0&1&0\\2&1&3&0&0&1\end{array}\right]$

$\simeq\left[\begin{array}{ccc|ccc}1&2&3&1&0&0\\0&0&-2&-1&1&0\\0&-3&-3&-2&0&1\end{array}\right]$ $R_2=R_2-R_1$, $R_3=R_3-2R_1$

$\simeq\left[\begin{array}{ccc|ccc}1&2&3&1&0&0\\0&1&1&2/3&0&-1/3\\0&0&1&1/2&-1/2&0\end{array}\right]$ R_{23}, $R_2=-\frac{1}{3}R_2$, $R_3=-\frac{1}{2}R_3$

$\simeq\left[\begin{array}{ccc|ccc}1&2&0&-1/2&3/2&0\\0&1&0&1/6&1/2&-1/3\\0&0&1&1/2&-1/2&0\end{array}\right]$ $R_1=R_1-3R_3$, $R_2=R_2-R_3$

$\simeq\left[\begin{array}{ccc|ccc}1&0&0&-5/6&1/2&2/3\\0&1&0&1/6&1/2&-1/3\\0&0&1&1/2&-1/2&0\end{array}\right]$ $R_1=R_1-2R_2$

So $A^{-1}=\begin{bmatrix}-5/6&1/2&2/3\\1/6&1/2&-1/3\\1/2&-1/2&0\end{bmatrix}$ $\underline{x}=A^{-1}\begin{bmatrix}14\\8\\13\end{bmatrix}=\begin{bmatrix}1\\2\\3\end{bmatrix}$

c) From A to A^{-1}, eight row operations are performed. Five of them are of the form $R_\alpha=R_\alpha-\beta R_\gamma$ which has no effect on $|A|$. Now the three operations shown in Red are establishing the chain between $|A|$ & $|I|$

$|A|=(-1)\frac{1}{(-\frac{1}{3})}\frac{1}{(-\frac{1}{2})}|I|$

$=-6|I|$

$=-6.$

$< \wedge 4$

Q4) a) Letting $G(y)$ denote the value of the distribution function of Y at y, then we can write:

$$G(y) = P(Y \le y)$$
$$= P(X^3 \le y)$$
$$= P(X \le y^{1/3}) = \int_0^{y^{1/3}} \frac{x}{4.5}\,dx$$
$$= \frac{x^2}{9}\Big|_0^{y^{1/3}} = \frac{1}{9}\,y^{2/3}$$

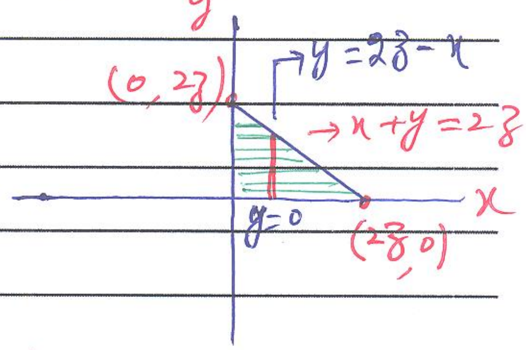
and hence $g(y) = G'(y) = \dfrac{2}{27}\,y^{-1/3} \qquad 0 < y \le \cancel{9}\,27$

$$\text{So, } g(y) = \begin{cases} \dfrac{2}{27}\,y^{-1/3} & 0 < y \le \cancel{9}\,27 \\ 0 & \text{otherwise.} \end{cases}$$

b) Let $F(z)$ be the distribution function of Z at z, then we can write:

$$F(z) = P(Z \le z) = P\left(\frac{X+Y}{2} \le z\right) = P(X+Y \le 2z)$$

$$\text{So } F(z) = \int_{x=0}^{2z}\int_{y=0}^{2z-x} e^{-(x+y)}\,dy\,dx$$



$$= \int_0^{2z} e^{-(x+y)}\Big|_{2z-x}^{0}\,dx$$

$$= \int_0^{2z}\left(e^{-x} - e^{-(\cancel{x}+2z-\cancel{x})}\right)dx$$

$$= \int_0^{2z}\left(e^{-x} - e^{-2z}\right)dx = e^{-x}\Big|_{2z}^{0} + xe^{-2z}\Big|_{2z}^{0}$$

$$= 1 - e^{-2z} - 2z e^{-2z} = 1 - (1+2z)e^{-2z}$$

$$\text{So } f(z) = F'(z) = 2(1+2z)e^{-2z} - 2e^{-2z} = 4z e^{-2z}.$$

$$\text{i.e. } f(z) = \begin{cases} 4z e^{-2z} & z > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Confirmation $\int_0^\infty f(z)\,dx = \int_0^\infty 4z e^{-2z}\,dz = \int(-2z)\,d(e^{-2z})$

$$= -2z e^{-2z}\Big|_0^\infty - \int_0^\infty (-2)e^{-2z}\,dz = 0 - 0 - \left[e^{-2z}\right]_0^\infty = 0-0-0+1 = 1.$$

Q5 - a i) $\int_0^1\int_0^1 x\,dx\,dy + c\int_0^1\int_0^1 y^2\,dy\,dx = \int_0^1 \frac{1}{2}\,dy + c\int_0^1 \frac{1}{3}\,dx = \frac{1}{2} + c\frac{1}{3} = 1$

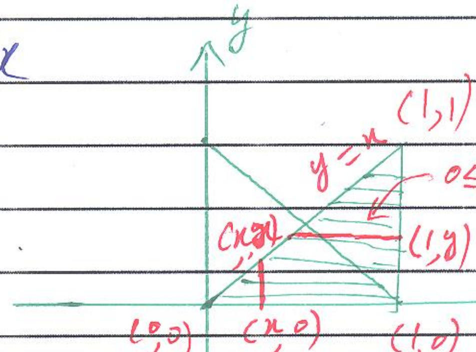
$\Rightarrow c/3 = 1/2 \Rightarrow \boxed{c = 3/2}$.

ii) $P(0 \le X \le \frac{1}{2}, 0 \le Y \le \frac{1}{2}) = \int_0^{1/2}\int_0^{1/2} x\,dx\,dy + c\int_0^{1/2}\int_0^{1/2} y^2\,dy\,dx$

$= \int_0^{1/2} \frac{1}{8}\,dy + c\int_0^{1/2} \frac{1}{3}\times\frac{1}{8}\,dx = \frac{1}{16} + \frac{c}{48} = \frac{1}{16} + \frac{3}{2}\times\frac{1}{48} = \frac{2}{32} + \frac{1}{32} = \boxed{\frac{3}{32}}$.

b) i) $1 = \iint f(x,y)\,dx\,dy = c\int_{x=0}^{1} x^2 \int_{y=0}^{x} y\,dy\,dx$

$= c\int_0^1 x^2 \frac{x^2}{2}\,dx = \frac{c}{10} x^5 \Big|_0^1 = \frac{c}{10}$.



$\Rightarrow \boxed{c = 10}$

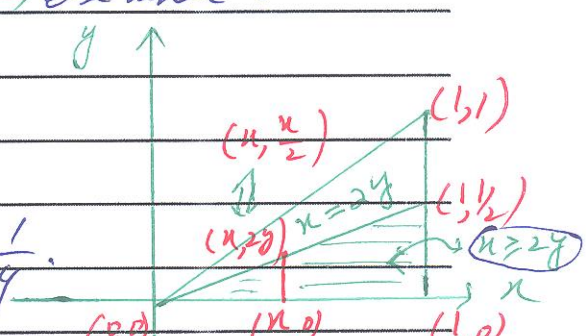
ii) $g(x) = cx^2 \int_{y=0}^{x} y\,dy = c\frac{x^4}{2} = \begin{cases} 5x^4 & 0 \le x \le 1 \\ 0 & \text{elsewhere} \end{cases}$

$h(y) = cy\int_y^1 x^2\,dx = cy\frac{x^3}{3}\Big|_y^1 = \frac{cy}{3}(1-y^3) = \frac{10}{3}y(1-y^3)$

So $h(y) = \begin{cases} \frac{10}{3}y(1-y^3) & 0 \le y \le 1 \\ 0 & \text{elsewhere.} \end{cases}$

iii) $P(X \ge 2Y) = c\int_{x=0}^{1} x^2 \int_{y=0}^{x/2} y\,dy\,dx$

$= c\int_0^1 x^2 \frac{1}{2}\frac{x^2}{4}\,dx = \frac{c}{8}\frac{1}{5} = \frac{10}{40} = \frac{1}{4}$.

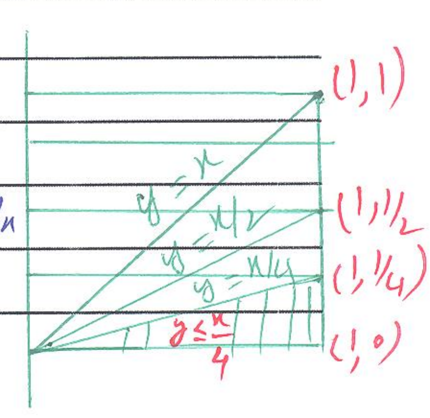


iv) $P(X \ge 4Y \mid X \ge 2Y) = \frac{P(X \ge 4Y, X \ge 2Y)}{P(X \ge 2Y)}$

$= P(X \ge 4Y) / P(X \ge 2Y)$

$P(X \ge 4Y) = c\int_{x=0}^{1} x^2 \int_{y=0}^{x/4} y\,dy\,dx = c\int_0^1 x^2 \frac{1}{2}\frac{x^2}{16}\,dx$

$= \frac{c}{32}\frac{1}{5} = \frac{10}{32}\times\frac{1}{5} = \frac{1}{16}$



$\therefore$ Req. prob $= \frac{1}{16}\times\frac{4}{1} = \frac{1}{4}$.

Q5-c i) $g_X(x) = \int_{y=0}^{2}\left(\frac{x^2}{4}+\frac{y^2}{4}+\frac{xy}{6}\right)dy = \frac{x^2 y}{4}+\frac{y^3}{12}+\frac{xy^2}{12}\Big|_{y=0}^{2}$

$$= \begin{cases} \frac{x^2}{2}+\frac{2}{3}+\frac{x}{3} & 0\le x\le 1 \\ 0 & \text{otherwise} \end{cases}$$

$h(y) = \int_{x=0}^{1}\left(\frac{x^2}{4}+\frac{y^2}{4}+\frac{xy}{6}\right)dx = \frac{x^3}{12}+\frac{y^2}{4}x+\frac{x^2 y}{12}\Big|_{x=0}^{1}$

$$= \begin{cases} y^2/4 + y/12 + 1/12 & 0\le y\le 2 \\ 0 & \text{otherwise.} \end{cases}$$

ii) $f(x|y) = f(x,y)/h(y)$

$$= \begin{cases} \left(\frac{x^2}{4}+\frac{y^2}{4}+\frac{xy}{6}\right)\Big/\left(\frac{y^2}{4}+\frac{y}{12}+1/12\right) & 0\le y\le 2 \\ 0 & \text{otherwise.} \end{cases}$$

$$\begin{cases} \frac{3x^2+3y^2+2xy}{3y^2+y+1} & 0\le x\le 1,\ 0\le y\le 2 \\ 0 & \text{otherwise.} \end{cases}$$

iii) $P(X<\frac{1}{2} \mid Y=y) = \frac{1}{\left(\frac{y^2}{2}+\frac{y}{12}+\frac{1}{12}\right)} \int_0^{1/2}\left(\frac{x^2}{4}+\frac{y^2}{4}+\frac{xy}{6}\right)dx$

$$= \frac{1}{\left(\frac{y^2}{2}+\frac{y}{12}+\frac{1}{12}\right)} \left[\frac{x^3}{12}+\frac{y^2}{4}x+\frac{y x^2}{12}\right]_{x=0}^{1/2}$$

$$= \frac{1}{\left(\frac{y^2}{4}+\frac{y}{12}+\frac{1}{12}\right)}\left(\frac{1}{96}+\frac{y^2}{8}+\frac{y}{48}\right)$$

$$= \frac{\frac{3}{2}y^2+\frac{y}{4}+1/8}{3y^2+y+1}$$

Note that, as we expect, $P(X<\frac{1}{2} \mid Y=y)$ depends on $y \in [0,2]$.