Chapter 7

7.1 $G(y) = P(Y \le y) = P(\ln X \le y) = P(X \le e^y)$

$$= \int_0^{e^y} \frac{1}{8} e^{-x/\theta} dx = -e^{-x/\theta} \Big|_0^{e^y} = 1 - e^{-(1/\theta)e^y}$$

$$g(y) = \frac{1}{8} e^y e^{-(1/\theta)e^y} \text{ for } -\infty < y < \infty$$

7.2 $G(y) = P(Y \le y) = P(X^2 \le y) = P(X \le \sqrt{y})$

$$= \int_0^{\sqrt{y}} 2xe^{-x^2} dx \qquad\qquad u = x^2 \qquad du = 2x\ \ dx$$

$$= \int_0^{y} e^{-u} du = -e^{-u} \Big|_0^{y} = 1 - e^{-y}$$

(a) $G(y) = \begin{cases} 1 - e^{-y} & y > 0 \\ 0 & \text{elsewhere} \end{cases}$

(b) $g(y) = \dfrac{dG(y)}{dy} = e^{-y}$ for $y > 0$ and 0 elsewhere

7.3 $G(y) = P(Y \le y) = P(\sqrt{X} \le y) = P(X \le y^2)$

$$= \int_0^{y^2} dx = y^2 \text{ for } 0 < y < 1$$

$g(y) = 2y$ for $0 < y < 1$ and 0 elsewhere

7.4 $G(z) = P(Z \le z) = P(X^2 + Y^2 + z^2)$

$$= \int_0^{z} \int_0^{\sqrt{z^2 - y^2}} 4xye^{-(x^2+y^2)} dx\ dy \qquad\qquad \text{let } u = x^2 \text{ and } v = y^2$$

$= 1 - (1 + z^2)e^{-z^2}$ for $z > 0$ and $G(z) = 0$ elsewhere

$g(z) = -(1 + z^2)e^{-z^2}(-2z) - 2z\ e^{-z^2}$

$= 2z^3 e^{-z^2}$ for $z > 0$ and elsewhere

7.5 $G(y) = P(Y \le y) = P(X_1 + X_2 \le y)$

$$= \int_0^y \int_0^{y-x_2} \frac{1}{\theta_1} e^{-x_1/\theta_1} \frac{1}{\theta_2} e^{-x_2/\theta_2}\, dx_2\, dx_2$$

$$= \int_0^y \left[\frac{1}{\theta_2} e^{-x_2/\theta_2} - \frac{1}{\theta_2} e^{-x_2/\theta_2} e^{-(y-x_2)/\theta_1} \right] dx_2$$

(a) $\theta_1 \ne \theta_2$

$$g(y) = \frac{1}{\theta_1 - \theta_2}\left[e^{-y/\theta_1} - e^{y/\theta_2} \right] \qquad y > 0$$

(b) $\theta_1 = \theta_2 = \theta$

$$G(y) = \int_0^y \left[\frac{1}{\theta_2} e^{-x_2/\theta_2} - \frac{1}{\theta_2} e^{-y/\theta_2} \right] dx_2$$

$$= 1 - e^{-y/\theta} - y \frac{1}{\theta} e^{-y/\theta}$$

$$g(y) = \frac{1}{\theta^2} y e^{-y/\theta} \qquad y > 0$$

7.6 **(a)** $F(y) = 0,$ **(b)** $F(y) = \frac{1}{2} y^2,$ **(c)** $F(y) = 1 - \frac{1}{2}(2-y)^2,$ **(d)** $F(y) = 1$

$f(y) = 0, f(y) = y, f(y) = 2 - y, f(y) = 0$

7.7 $G(Z) = P(Z \le z) = P\left(\dfrac{X_1}{X_1 + X_2} \le z \right)$

$$x = xz + yz$$
$$yz = x(1-z)$$
$$y = \frac{x(1-z)}{z}$$

$$= \int_0^\infty \int_0^{x(1-z)/z} e^{-x} e^{-y} dy\, dx$$

$$= \int_0^\infty e^{-x} \int_0^{x(1-z)/z} e^{-y} dy\, dx$$

$$= \int_0^\infty \int_{x(1-z)/z}^\infty e^{-x} e^{-y} dy\, dx = \int_0^\infty e^{-x} \int_{x(1-z)/z}^\infty e^{-y} dy\, dx$$

$$= \int_0^\infty e^{-x} \left[e^{-x(1-z)/z} \right] dx \int_0^\infty e^{-x/z}\, dx = z$$

$g(z) = 1$ QED

7.8 $P(Z \le z) = P\left(\frac{X+Y}{2} \le z\right)$

$$= \int_0^{2z}\int_0^{2z-x} e^{-x}e^{-y}dy\ dx = \int_0^{2z} e^{-x}\left[-e^{-y}\right]\Big|_0^{2z-x} dx$$

$$= \int_0^{2z} e^{-x}[1-e^{x-2z}]dx = \int_0^{2z}(e^{-x}-e^{-2z})dx$$

$$= [-e^{-x} - xe^{-2z}]\Big|_0^{2z} = -e^{-2z} - 2z\ e^{-2z} + 1$$

$$g(z) = 2e^{-2z} - 2e^{-2z} + 4ze^{-2z} = 4ze^{-2z}$$

7.9 $h(0) = \frac{\binom{3}{0}\binom{3}{2}}{\binom{6}{2}} = \frac{3}{15} = \frac{1}{5},\quad h(1) = \frac{\binom{3}{1}\binom{3}{1}}{15} = \frac{9}{15} = \frac{3}{5}$

$$h(2) = \frac{\binom{3}{2}\binom{3}{0}}{\binom{6}{2}} = \frac{3}{15} = \frac{1}{5}$$

$$x - (2-x)$$
$$2x - 2$$

X	0	1	2
Y	-2	0	2

Y	-2	0	2
$h(y)$	$\frac{1}{5}$	$\frac{3}{5}$	$\frac{1}{5}$

7.10

X	0	1	2
Z	1	0	1

Z	0	1
$h(z)$	$\frac{3}{5}$	$\frac{2}{5}$

7.11 $f(0) = 1\cdot\frac{8}{27} = \frac{8}{27},\quad f(1) = 3\cdot\frac{1}{3}\cdot\frac{4}{9} = \frac{12}{27},\quad f(2) = 3\cdot\frac{1}{9}\cdot\frac{2}{3} = \frac{6}{27},\quad f(3) = 1\cdot\frac{1}{27} = \frac{1}{27}$

(a)

$x = 0$	1	2	3
$y = 0$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{3}{4}$

y	0	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{3}{4}$
$g(y)$	$\frac{8}{27}$	$\frac{12}{27}$	$\frac{6}{27}$	$\frac{1}{27}$

(b)

$x=0$	1	2	3
$y=1$	0	1	16

y	0	1	16
$g(y)$	$\frac{12}{27}$	$\frac{14}{27}$	$\frac{1}{27}$

7.12 $f(x)=\theta(1-\theta)^{x-1},\qquad x=1,\ 2,\ 3,\ \ldots\qquad x-1=\dfrac{-1-y}{5}$

$y=4-5x\qquad x=\dfrac{4-y}{5}\qquad x-1=\dfrac{-(1+y)}{5}$

$g(y)=\theta(1-\theta)^{-(1+y)/5}$ for $y=-1,\ -6,\ -11,\ -16,\ \ldots$

7.13

X	2	3	4	5	6	7	8	9	10	11	12
$f(x)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

$g(0)\quad =\dfrac{2}{36}+\dfrac{5}{36}+\dfrac{4}{36}+\dfrac{1}{36}=\dfrac{12}{36}=\dfrac{1}{3}$

$g(1)\quad =\dfrac{3}{36}+\dfrac{6}{36}+\dfrac{3}{36}\qquad =\dfrac{12}{36}=\dfrac{1}{3}$

$g(2)\quad =\dfrac{1}{36}+\dfrac{4}{36}+\dfrac{5}{36}+\dfrac{2}{36}=\dfrac{12}{36}=\dfrac{1}{3}$

7.14 $g(z)=\dfrac{dx}{dz}\cdot f(x)\qquad x-\mu=\sigma z\qquad x-\sigma z+\mu\qquad \dfrac{dx}{dz}=\sigma$

$g(z)=\sigma\cdot\dfrac{1}{\sqrt{2\pi}\sigma}e^{-(1/2)[(x-\mu)/\sigma]^2}=\dfrac{1}{\sqrt{2\pi}}e^{-(1/2)z^2}$ QED

7.15 $f(x)=2xe^{-x^2}\qquad y=x^2\qquad 1=2x\dfrac{dx}{dy}$

$$g(y)=\frac{1}{2x}\cdot 2xe^{-x^2}=\begin{cases}e^{-y} & \text{for } y>0\\ 0 & \text{elsewhere}\end{cases}$$

7.16 $y=\dfrac{2x}{1+2x},\ y(1+2x)=2x\qquad 1+2x=\dfrac{1}{1-y}$

$y=2x(1-y)\qquad 2x=\dfrac{y}{1-y}\qquad x=\dfrac{y}{2(1-y)^2}$

$g(y)=\dfrac{dx}{dy}f(x)\qquad 2\dfrac{dx}{dy}=\dfrac{(1-y)+y}{(1-y)^2}=\dfrac{1}{(1-y)^2}\qquad \dfrac{dx}{dy}=\dfrac{1}{2(1-y)^2}$

$g(y)=\dfrac{k\,y^3(1-y)^2}{8(1-y)^3}\cdot\dfrac{1}{2(1-y)^2}=\dfrac{k}{16}y^3(1-y)$

Beta distribution with $\alpha=4$ and $\beta=2$

$\dfrac{k}{16}=\dfrac{\Gamma(6)}{\Gamma(2)\Gamma(4)}=\dfrac{5!}{1!\ 3!}=20,\ k=320$

7.17 $f(x) = \dfrac{x}{2} \qquad 0 < x < 2$

$y = x^3 \qquad 1 = 3x^2 \dfrac{dx}{dy}$

$g(y) = \dfrac{1}{3x^2} \cdot \dfrac{x}{2} = \dfrac{1}{6y^{1/3}}$

$$g(y) = \begin{cases} \dfrac{1}{6} y^{-2/3} & \text{for } 0 < y < 8 \\ 0 & \text{elsewhere} \end{cases}$$

7.18 $f(x) = 1 \qquad 0 < x < 1 \qquad y = -2\ln x \qquad 1 = \dfrac{-2}{x\dfrac{dx}{dy}}$

$g(y) = e^{-(1/2)y} \qquad 0 < y < \infty \qquad \dfrac{dx}{dy} = -\dfrac{x}{2}$

$\alpha = 1$ and $\beta = 2 \qquad -\dfrac{1}{2} y = \ln x \qquad x = e^{-(1/2)y}$

7.19 $f(x) = 1 \qquad 0 < x < 1$

$y = x^{-1/\alpha},\ x = y^{-\alpha},\ \dfrac{dx}{dy} = -\alpha y^{-(1+a)}$

$g(y) = 1 \cdot \alpha y^{-(1+\alpha)} = \dfrac{\alpha}{y^{1+\alpha}}$ for $x > 1$

7.20 **(a)** $Y = |x| \qquad g(y) = f(y) + f(-y)$

$$= \begin{cases} 3y^2 & 0 < y < 1 \\ 0 & \text{elsewhere} \end{cases}$$

(b) $z = y^2 \qquad 1 = 2y \cdot \dfrac{dy}{dz}$

$$h(z) = \frac{1}{\sqrt{z}} \cdot 3z = \begin{cases} \dfrac{3}{2\sqrt{z}} & \text{for } 0 < z < 1 \\ 0 & \text{elsewhere} \end{cases}$$

7.21 $f(x) = \dfrac{1}{4} \qquad \alpha = 1 \qquad \beta = 3$

(a) $y = |x|$

$$g(y) = \begin{cases} \dfrac{1}{2} & \text{for } 0 < y < 1 \\ \dfrac{1}{4} & \text{for } 1 < y < 3 \end{cases}$$

(b) $z = y^4 \qquad 1 = 4y^3 \dfrac{dy}{dz}$

$$g(z) = \begin{cases} \dfrac{1}{4\,z^{3/4}} \cdot \dfrac{1}{2} = {}_{8}\, z^{-3/4} & 0 < z \le 1 \\[2ex] \dfrac{1}{4\,z^{3/4}} \cdot \dfrac{1}{4} = \dfrac{1}{16} z^{-3/4} & 1 < z < 81 \end{cases}$$

7.22

		x_1: 1	2	3
x_2	1	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$
	2	$\frac{2}{36}$	$\frac{4}{36}$	$\frac{6}{36}$
	3	$\frac{3}{36}$	$\frac{6}{36}$	$\frac{9}{36}$

(a)

x_1x_2	1	2	3	4	6	9
$g(x_1x_2)$	$\frac{1}{36}$	$\frac{4}{36}$	$\frac{6}{36}$	$\frac{4}{36}$	$\frac{6}{36}$	$\frac{9}{36}$

(b)

x_1 / x_2	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{2}{3}$	1	$\frac{3}{2}$	2	3
$h(x_1 / x_2)$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{6}{36}$	$\frac{14}{36}$	$\frac{6}{36}$	$\frac{2}{36}$	$\frac{3}{36}$

7.23 (a)

		y_1: 1	2	3	4	5	6
y_2	−2				$\frac{3}{36}$		
	−1			$\frac{2}{36}$		$\frac{6}{36}$	
	0		$\frac{1}{36}$		$\frac{4}{36}$		$\frac{9}{36}$
	1			$\frac{2}{36}$		$\frac{6}{36}$	
	2				$\frac{3}{36}$		

(b)

y_1	2	3	4	5	6
$g(y_1)$	$\frac{1}{36}$	$\frac{4}{36}$	$\frac{10}{36}$	$\frac{12}{36}$	$\frac{9}{36}$

7.24 $f(x,y) = \dfrac{(x-y)^2}{7}$ $\quad x = 1, 2 \quad\quad y = 1, 2, 3$

$$\begin{array}{cc|ccc|} & & & y & \\ & & 1 & 2 & 3 \\ \hline x & 1 & 0 & \frac{1}{7} & \frac{4}{7} \\ & 2 & \frac{1}{7} & 0 & \frac{1}{7} \\ \hline \end{array}$$

(a)

$u = x + y$
$v = -x - y$

$$\begin{array}{cc|cccc|} & & & u & & \\ & & 2 & 3 & 4 & 5 \\ \hline & -2 & & & \frac{4}{7} & \\ v & -1 & & \frac{1}{7} & & \frac{1}{7} \\ & 0 & 0 & & 0 & \\ & 1 & & \frac{1}{7} & & \\ \hline \end{array}$$

(b)

u	2	3	4	5
$g(u)$	0	$\frac{2}{7}$	$\frac{4}{7}$	$\frac{1}{7}$

7.25

x_1	x_2	x_3		y_1	y_2	y_3	
2	0	0	1/16	2	2	0	$g(0,\ 0,\ 2) = \dfrac{25}{144}$
0	2	0	1/9	2	−2	0	$g(1,\ -1,\ 1) = \dfrac{5}{18}$
0	0	2	25/144	0	0	2	$g(1,\ 1,\ 1) = \dfrac{5}{24}$
1	1	0	1/6	2	0	0	$g(2,\ -2,\ 0) = \dfrac{1}{9}$
1	0	1	5/24	1	1	1	$g(2,\ 0,\ 0) = \dfrac{1}{6}$
0	1	1	5/18	0	−1	1	$g(2,\ 2,\ 0) = \dfrac{1}{16}$

7.26

		$X=0$	$X=1$	$X=2$
Y	0	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{12}$
	1	$\frac{2}{9}$	$\frac{1}{6}$	
	2	$\frac{1}{36}$		

(a)

u	0	1	2
	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{12}$
		$\frac{2}{9}$	$\frac{1}{6}$
			$\frac{1}{36}$
$f(u)$	$\frac{1}{6}$	$\frac{5}{9}$	$\frac{5}{18}$

(b)

v	0	1
$g(v)$	$\frac{5}{6}$	$\frac{1}{6}$

w	-2	-1	0	1	2
	$\frac{1}{36}$	$\frac{2}{9}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{12}$
		$\frac{1}{6}$			
$h(w)$	$\frac{1}{36}$	$\frac{2}{9}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{12}$

7.27 $f(x_1, x_2) = \binom{n_1}{x_1}\binom{n_2}{x_2}\theta^{x_1+x_2}(1-\theta)^{n_1+n_2-(x_1+x_2)}$

$x_1 + x_2 = y \qquad g(y) = \sum_{x_1=0}^{y}\binom{n_1}{x_1}\binom{n_2}{y-x_1}\theta^{y}(1-\theta)^{n_1+n_2-y}$

$$= \binom{n_1+n_2}{y}\theta^{y}(1-\theta)^{n_1+n_2-1-y}$$

7.28 $f(x_1, x_2) = \theta(1-\theta)^{x_1-1}\theta(1-\theta)^{x_2-1} \qquad x_1 + x_2 = y$

$g(y) = k\theta^2(1-\theta)^{y-2} \qquad b^*(y;\ 2,\ \theta) = (y-1)\cdot\theta^2(1-\theta)^{y-2}$

k is number of ways in which $x_1 + x_2 = y$ (with y fixed)

which is $y-1 \qquad g(y) = (y-1)\theta^2(1-\theta)^{y-2} = \binom{y-1}{1}\theta^2(1-\theta)^{y-2}$

7.29 $\frac{1}{2\pi}e^{-(1/2)(x^2+y^2)}$ $\qquad z = x + y$

$\frac{1}{2\pi}e^{-(1/2)[x^2+(z-x)^2]}$

$\frac{1}{2\pi}e^{-(1/2)[(x-z)^2/(1/2)]}\cdot e^{-(1/2)(z^2/2)}$

$\frac{\sqrt{2}}{\sqrt{2\pi}}e^{-(1/2)[(x-z/2)/(1/\sqrt{2})]^2}\cdot\frac{\sqrt{2}}{\sqrt{2\pi}}e^{-(1/2)(z/\sqrt{2})^2}$

$\frac{1}{\sqrt{2\pi}}e^{-(1/2)(z/\sqrt{2})^2}$

normal $\mu = 0$ $\qquad \sigma^2 = 2$

7.30 $f(x,\ y) = 12xy(1-y)$ $\qquad z = xy^2$ $\qquad 1 = \frac{dx}{dz}y^2$

$g(z,\ y) = 12\cdot\frac{z}{y^2}(1-y)\cdot\frac{1}{y^2}$

$= 12(y^{-3}-y^2)$ $\qquad$ bounded by $z = 0,\ u = 1,\ \ z = u^2$

$h(z) = 12z\int_{\sqrt{z}}^{1}(y^{-3}-y^{-2})dy = 12z\left[\frac{y^{-2}}{-2}-\frac{y^{-1}}{-1}\right]\Bigg|_{\sqrt{z}}^{1}$

$= 12z\left[-\frac{1}{2}+1+\frac{1}{2z}-\frac{1}{\sqrt{z}}\right]$

$= 6z+6-12\sqrt{z}$ $\qquad 0 < z < 1$

0 $\qquad$ elsewhere

7.31 $z = xy^2$ $\qquad x = \frac{z}{u^2}$ $\qquad \frac{\partial x}{\partial u} = \frac{-2z}{u^2}$ $\qquad \frac{\partial y}{\partial u} = 1$

$u = y$ $\qquad y = u$ $\qquad \frac{\partial x}{\partial z} = \frac{1}{u^2}$ $\qquad \frac{\partial y}{\partial z} = 0$

$J = \begin{vmatrix} \frac{-2z}{u^2} & \frac{1}{u^2} \\ 1 & 0 \end{vmatrix} = \frac{1}{u^2}$

$g(z,\ u) = 12\frac{z}{u^2}u(1-u)\cdot\frac{1}{u^2} = 12z(u^{-3}-u^{-2})$

from here same as in 7.30

7.32 $f(x_1, x_2) = \dfrac{1}{\pi^2(1+x_1^2)(1+x_2^2)}$ $\qquad y = x_1 + x_2$

$g(x_1, y_2) = \dfrac{1}{\pi^2(1+x_1^2)[1+(y_1-x_2)^2]}$

Use partial fractions to perform necessary integration

Result is $g(y) = \dfrac{1}{\pi}\dfrac{2}{4+y_1^2}$

$-\infty < y_1 < \infty$ Cauchy distribution

7.34 $g(u, y) = \begin{array}{ll} \frac{1}{2} & \text{over region bounded by } y = 0,\ u = y, \text{ and } 2y - u = 0 \\ 0 & \text{elsewhere} \end{array}$

$-2 < u < 0 \quad h(u) = \displaystyle\int_0^{(1/2)(u+2)} \frac{1}{2}dy = \frac{1}{4}(u+2)$

$0 < u < 2 \quad h(u) = \displaystyle\int_u^{(1/2)(u+2)} \frac{1}{2}dy = \frac{1}{4}(2-u)$

elsewhere it is 0

7.35 $u = y - x,\ v = x \qquad \dfrac{\partial u}{\partial x} = -1 \qquad \dfrac{\partial u}{\partial y} = 1 \qquad \dfrac{\partial v}{\partial x} = 1 \qquad \dfrac{\partial v}{\partial y} = 0 \qquad \begin{vmatrix} -1 & 1 \\ 1 & 0 \end{vmatrix} = -1$

$$f(u,v) = \begin{cases} \frac{1}{2} & \text{over the region bounded by } v = 0,\ u = -v, \text{ and } 2v + u = 2 \\ 0 & \text{elsewhere} \end{cases}$$

$$g(u) = \int_0^{(1/2)(2-u)} \frac{1}{2}dv = \frac{1}{4}(2-u) \quad \text{for } 0 < u < 2$$

$$g(u) = \int_{-u}^{(1/2)(2-u)} \frac{1}{2}dv = \frac{1}{2}\left[\frac{1}{2}(2-u) + u\right]$$

$$= \frac{1}{4}(2+u) \quad \text{for } -2 < u < 0$$

7.36 $f(x_1, x_2) = 4x_1x_2 \qquad y_1 = x_1^2 \qquad y_2 = x_1x_2$

$x_1 = \sqrt{y} \qquad \dfrac{\partial x_1}{\partial y_1} = \dfrac{1}{2\sqrt{y_1}} \qquad \dfrac{\partial x_1}{\partial y_2} = 0$

$x_2 = y_2 / \sqrt{y_1} \qquad \dfrac{\partial x_2}{\partial y_1} = -\dfrac{1}{2}y_2 y_1^{-3/2} \qquad \dfrac{\partial x_2}{\partial y_2} = \dfrac{1}{\sqrt{y_1}}$

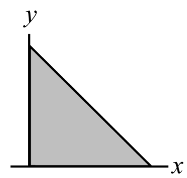
$$g(y_1, y_2) = 4\sqrt{y_1}\,\frac{y_2}{\sqrt{y_1}}\cdot\frac{1}{2y_1} = \frac{2y_2}{y_1} \qquad J = \begin{vmatrix} \frac{1}{2\sqrt{y_1}} & 0 \\ -\frac{1}{2}y_2 y_1^{-3/2} & \frac{1}{\sqrt{y_1}} \end{vmatrix} = \frac{1}{2y_1}$$

over region bounded by $y = 1,\ y_2 = 0,$ and $y_1 = y_2^2$

7.37 $f(x,\ y) = 24xy$

$z = x + y \quad w = x \rightarrow x = w$

and $y = z - w$



$$\frac{\partial x}{\partial w} = 1 \qquad \frac{\partial x}{\partial z} = 0$$

$$\frac{\partial y}{\partial \omega} = -1 \qquad \frac{\partial y}{\partial z} = 1 \qquad J = \begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix} = 1$$

$$g(w, z) = \begin{cases} 24w(z - w) & \text{over region bounded by } w = 0,\ z = 1, \text{ and } z = w \\ 0 & \text{elsewhere} \end{cases}$$

7.38 **(a)** $u = \dfrac{x}{x + y}$ and $v = x + y$

$$x = uv \qquad \frac{\partial x}{\partial u} = v \qquad \frac{\partial x}{\partial v} = u$$

$$y = v(1 - u) \qquad \frac{\partial y}{\partial u} = -v \qquad \frac{\partial y}{\partial v} = 1 - u$$

$$J = \begin{vmatrix} v & u \\ -v & (1 - u) \end{vmatrix} = v(1 - u) + uv = v$$

$$f(x,\ y) = \frac{1}{[\beta^{\alpha}\Gamma(\alpha)]^2}\, x^{\alpha-1} y^{\alpha-1} e^{-(1/\beta)(x+y)}$$

$$g(u,\ v) = \frac{1}{\beta^{2\alpha}\Gamma(\alpha)]^2}\,[u(1 - u)]^{\alpha-1} v^{2\alpha-1} e^{-(1/\beta)v}$$

for $0 < u < 1,\ 0 < v < \infty$

(b)
$$h(u) = \frac{1}{\beta^{2\alpha}\Gamma(\alpha)]^2}\,[u(1 - u)]^{\alpha-1}\int_0^{\infty} v^{2\alpha-1} e^{-(1/\beta)v}\,dv$$

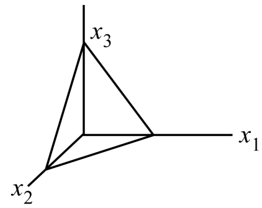
$$= \frac{1}{\beta^{2\alpha}\Gamma(\alpha)]^2}\cdot\beta^{2\alpha}\Gamma(2\alpha)\cdot[u(1 - u)]^{\alpha-1}$$

$$= \frac{\Gamma(2\alpha)}{\Gamma(\alpha)\Gamma(\alpha)}\,u^{\alpha-1}(1 - u)^{\alpha-1} \qquad \text{for } 0 < u < 1$$

U has beta distribution with $\beta = \alpha$

7.39 $y = x_1 + x_2 + x_3$

$g(x_1,\ x_2,\ y) = e^{-y} \qquad x_1 > 0,\ x_2 > 0,\ y > 0$



$$h(y) = \int_0^y \int_0^{y-x_2} e^{-y}\,dx_1 dx_2 = \begin{cases} \frac{1}{2} y^2 e^{-y} & \text{for } y > 0 \quad x_1 + x_2 \le y \\ 0 & \text{elsewhere} \end{cases}$$

7.40 $g(y,\ x_3) = h(y)$ as given in Example 7.13

(a) $$g(y,\ u) = h(y)\cdot 1 = \begin{cases} y & \mathrm{I} + \mathrm{II} \\ 2-y & \mathrm{III} + \mathrm{IV} \\ 0 & \text{elsewhere} \end{cases}$$

(b) $$h(u) = \int_0^u g(y,\ u)\ dy = \int_0^u y\ dy = \frac{u^2}{2} \qquad \text{for } 0 < u < 1$$

$$h(u) = \int_{u-1}^1 y\ dy + \int_1^u (2-y)\ dy = \frac{1}{2}u^2 - \frac{3}{2}(u-1)^2 \qquad 1 < u < 2$$

$$h(u) = \int_{u-1}^2 (2-y)\ dy = \frac{1}{2}u^2 - \frac{3}{2}(u-1)^2 + \frac{3}{2}(u-2)^2 \qquad 2 < u < 3$$

$h(u) = 0$ elsewhere; $h(1) = \frac{1}{2},\ h(2) = \frac{1}{2}$ will make it continuous

7.41 $M_Y = [1+\theta(e^t-1)^{n_1}[1+\theta(e^t-1)]^{n_2}$

$= [1+\theta(e^t-1)]^{n_1+n_2}$

Y is random variable having binomial distribution with the parameter θ and $n_1 + n_2$.

7.42 $$M_Y = \left[\frac{\theta e^t}{1-e^t(1-\theta)}\right]^k = \frac{\theta^k e^{kt}}{[1-e^t(1-\theta)]^k}$$

7.43 $M_X = (1-\beta t)^{-\alpha}$

$M_Y = (1-\beta)^{-\alpha n}$

Y is a random variable having gamma distribution with the parameter α and β.

7.44 $M_X = e^{\mu t + (1/2)t^2\sigma^2}$

$M_Y = \prod e^{\mu_i t + (1/2)t^2\sigma_i^2} = e^{t\left(\sum \mu_i\right) + (1/2)t^2\left(\sum \sigma_i^2\right)}$

Y is a random variable having normal distribution with $\mu = \sum \mu_i$ and $\sigma^2 = \sum \sigma_i^2$

7.45 Let $Z_i = a_i X_i$

$M_{Z_i} = M_{x_i}(a_i t)$

since $Y = \sum Z_i$

$M_Y = \prod M_{x_i}(a_i t)$ QED

7.46 $M_{x_i} = e^{\mu_i t + (1/2)t^2 \alpha_1^2}$ $Y = \sum a_i X_i$

$M_Y \prod e^{\mu_i a_i t + (1/2)t^2 a_i^2 \sigma_i^2}$

This is normal distribution with $\mu = \sum a_i \mu_i$ and variance $\sigma^2 = \sum a_i^2 \sigma_i^2$

7.47 $G(v) = P(V \le v) = P(SP \le v)$

$$= \int_{0.2}^{0.4} 5p \int_{0}^{v/p} e^{-sp} ds\ dp = \int_{0.2}^{0.4} 5p \left[-\frac{1}{p} e^{-sp} \right] \Bigg|_{0}^{v/p} dp$$

$$= \int_{0.2}^{0.4} 5[1 - e^{-v}] dp = 1 - e^{-v}$$

$g(v) = e^{-v}$ for $v > 0$ and 0 elsewhere

7.48 $x + y = 2u$

$$G(u) = \int_0^{2u} \int_0^{2u-x} \left[-\frac{1}{30} e^{-x/30} \right] \left[-\frac{1}{30} e^{-y/30} \right] dy\ dx$$

$$= 1 - e^{-u/15} - \frac{u}{15} e^{-u/15} \qquad y > 0$$

$g(u) = \dfrac{u}{255} e^{-u/15}$ for $y > 0$ and 0 elsewhere

7.49 $z = x - y$

for $0 < z < 5$ $\qquad G(z) = \displaystyle\int_{10}^{20} \int_{x-z}^{x} \frac{1}{25} \left(\frac{20 - x}{x} \right) dy\ dx$

$$= \frac{1}{25} z(20 \ \ln\ 2 - 10)$$

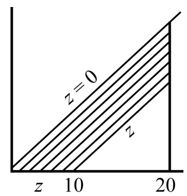
$g(z) = \dfrac{1}{25}(20 \ \ln\ 2 - 10)$ and 0 elsewhere

for $5 < z < 10$

$G(z) = 1 - \displaystyle\int_{2z}^{20} \int_{x/2}^{x-z} \frac{1}{25} \left(\frac{20 - x}{x} \right) dy\ dx$ leads to

$g(z) = \dfrac{1}{25} \left(2z - 20 - 20 \ \ln \dfrac{z}{10} \right)$ for $5 < z < 10$

7.50 $f(x,\ y) = \dfrac{1}{200}$ $\qquad 0 < y < x \qquad z = x - y$



$$G(z) = 1 - \frac{(20-z)^2}{z} \cdot \frac{1}{200} \qquad 0 < z < 20$$

$$g(z) = -\frac{2(20-z)(-1)}{2} \cdot \frac{1}{200} = \frac{20-z}{200}$$

for $0 < z < 20$
0 elsewhere

7.51 for $0 < y < 1$

$$G(y) = \int_0^y \int_0^{y-x_1} \frac{3}{11}(5x_1 + x_2)dx_2\ dx_1 = \frac{3}{11}y^3$$

$$g(y) = \frac{9}{11}y^2$$

for $1 < y < 2$

$$G(y) = 1 - \int_0^{2-y} \int_{y-x_2}^{2(1-x_2)} \frac{3}{11}(5x_1 + x_2)dx_1\ dx_2$$

$$= 1 - \frac{1}{11}(1+7y)(2-y)^2$$

$$g(y) = \frac{3(2-y)(7y-4)}{11}$$

7.52 $f(v) = kv^2 e^{-\beta v^2} \qquad v > 0$

$$E = \frac{1}{2}mv^2 \qquad 1 = \frac{1}{2}m \cdot 2v\frac{dv}{dE} = mv\frac{dv}{dE} \qquad v = \sqrt{\frac{2}{m}E}$$

$$g(E) = \frac{k}{m}v\ e^{-\beta 2E/m} = KE^{1/2}e^{-cE} \quad \text{which is a gamma distribution}$$

7.53 $f(x,\ y) = \dfrac{1}{\pi} \qquad 0 < x^2 + y^2 < 1 \qquad r^2 = x^2 + y^2$

$$g(r, y) = \frac{4}{\pi}\frac{dx}{dr} \qquad 2r = \frac{dx}{dr} \qquad \frac{dx}{dr} = \frac{r}{x}$$

$$= \frac{4}{\pi} \cdot \frac{r}{x} = \frac{1}{\pi} \cdot \frac{r}{\sqrt{r^2 - y^2}}$$

$$h(r) = \frac{4}{\pi}\int_0^r \frac{r\ dy}{\sqrt{r^2 - y^2}} = \frac{4}{\pi}\int_0^r \frac{dy}{\sqrt{r^2 - y^2}} = \frac{4r}{\pi} \cdot \sin^{-1}\frac{y}{r}\Big|_0^r$$

$$= \frac{4r}{\pi} \cdot (\sin^{-1}1 - \sin^{-1}0) = \frac{4r}{\pi}\left[\frac{\pi}{2} - 0\right]$$

$$= 2r \text{ for } 0 < r < 1$$

7.54 $f(x, y) = \frac{2}{5}(2x + 3y)$ $\quad 0 < x < 1,\ 0 < y < 1$ $\quad z = \frac{x+y}{z}$

$g(z, y) = \frac{2}{5}[4z + y] \cdot 2$ $\quad 2z = x + y$

$z = \frac{dx}{dz}$

$$= \begin{cases} \frac{4}{5}(4z + y) \text{ over } y = 0,\ y = 1,\ 2z = y, \text{ and } 2z = y + 1 \\ 0 \text{ elsewhere} \end{cases}$$

$h(z) = \frac{4}{5}\int_0^{2z}(4z + y)dy = 8z^2$ $\quad$ for $0 < z < \frac{1}{2}$

$h(z) = \frac{4}{5}\int_{2z-1}^{1}(4z + y)dy = 8z(1 - z)$ $\quad$ for $\frac{1}{2} < z < 1$

$h(z) = 0$ elsewhere

Also, let $h\left(\frac{1}{2}\right) = 2$

7.55 $f(p, s) = 5pe^{-ps}$ $\quad 0.2 < p < 0.4$ and $s > 0$

$v = sp$ $\quad s = \frac{v}{w}$ $\quad \frac{\partial s}{\partial v} = \frac{1}{w},\ \frac{\partial s}{\partial w} = -\frac{v}{w^2},\ \frac{\partial p}{\partial v} = 0,\ \frac{\partial p}{\partial w} = 1$

$w = p$ $\quad p = \mathrm{w}$ $\quad J = \begin{vmatrix} \frac{1}{w} & -\frac{v}{w^2} \\ 0 & 1 \end{vmatrix} = \frac{1}{w}$

$g(v, w) = 5we^{-v} \cdot \frac{1}{w} = 5e^{-v}$ $\quad$ for $0.2 < w < 0.4$ and $v > 0$

$h(v) = 5e^{-v}\int_{0.2}^{0.4} dw = e^{-v}$ $\quad$ for $v > 0$

7.56 Using MINITAB, we generate 10 "pseudo-random" numbers in C1 having the standard normal distribution with the following commands:

MTB> Random 10 C1
SUBC> Normal 0.0 1.0.

7.57 First the computer generates 10 "pseudo-random" numbers on the interval (0, 1). For example, for numbers to two decimal places, the interval (0, 1) is regarded as the union of the subintervals (-0.0050, 0.0049), (0.0050, 0.0149), ..., (0.9950, 1.049), corresponding to the numbers 0.00, ..., 0.01, ..., 1.00, respectively. Since there are 101 such intervals (numbers) each one is chosen with probability 1/101.. Then, the required numbers are generated with the inverse of the probability integral transformation.

7.58 Total number of calls per hour is random variable having Poisson distribution with parameter $\lambda = 2.1 + 10.9 = 13$. From Table II

(a) 0.1021

(b) 0.0002 + 0.0008 + 0.0027 + 0.0070 + 0.0152 = 0.0259

7.59 Total number of inquiries is a random variable having Poisson distribution with $\lambda = 3.6 + 5.8 + 4.6 = 14$. From Table II

(a) 0.0001 + 0.0004 + 0.0013 + ... + 0.0473 = 0.1093

(b) 0.0989 + 0.0866 + ... + 0.0286 = 0.3817

(c) 0.0554 + 0.0409 + ... + 0.0001 = 0.1728

7.60 Six inquiries with $\lambda_2 = 5.8$ $\quad p(6;\ 5.8) = 0.1601$ Table ii

Eight inquiries with $\lambda = 8.2$ $\quad p(8;\ 8.2) = 0.1392$

$$(0.1601)(0.1392) = 0.0222$$

7.61 **(a)** $p(2;\ 3.3) = 0.2008$

(b) $p(5;\ 6.6) = 0.1420$

(c) p(at least 12; 9.9) = 0.0928 + 0.0707 + ... + 0.0001 = 0.2919

7.62 **(a)** $p(4;\ 3.2) = 0.1781$

(b) p(at least 2; 4.8) = $1 - (0.0082 + 0.0395) = 0.9523$

(c) p(at most 3; 6.4) = $0.0017 + 0.0106 + 0.0340 + 0.0726 = 0.1189$

7.63 **(a)** Gamma with $\alpha = 2$ and $\beta = 5$

$$\frac{1}{5^2 \cdot 1} t \int_0^8 x\ e^{-x/5} dx = 0.475$$

(b) Gamma with $\alpha = 3$ and $\beta = 5$

$$\frac{1}{5^3 \cdot 2!} \int_{12}^{\infty} x^2\ e^{-x/5} dx = 0.570$$

7.64 **(a)** $\dfrac{1}{9} \displaystyle\int_{20}^{\infty} e^{-x/9} dx = e^{-20/9} = e^{-2.22} = 0.1086$

(b) Gamma with $\alpha = 2$ and $\beta = 9$

$$\frac{1}{81 \cdot 1} \int_{20}^{\infty} x\ e^{-x/9} dx = 0.3492$$

(c) Gamma with $\alpha = 3$ and $\beta = 9$

$$\frac{1}{9^3 \cdot 2} \int_{20}^{\infty} x^2\ e^{-x/9} dx = 0.6168$$

7.65 $f(x) = \binom{3}{x} \left(\frac{1}{6}\right)^x \cdot \left(\frac{5}{6}\right)^{3-x}$, $x = 0,\ 1,\ 2,\ 3$. For $x^2 > 2$, $x > 1$. The probability that $x > 1$ is given by $3 \cdot \left(\frac{1}{6}\right)^2 \cdot \left(\frac{5}{6}\right) + \left(\frac{1}{6}\right)^3 = \frac{16}{216} = \left(\frac{2}{27}\right)$

7.66 $P(x>1)=\int_{1}^{\infty} 0.5\cdot e^{-0.5x} ex = e^{-0.5}$.

7.67 **(a)** $\frac{1}{k}=\int_0^6 \left(1-\frac{d}{5}\right) dd = 2.5, \;\therefore k=\frac{2}{5}$.

(b) $A=\pi\frac{d^2}{4} \therefore d=\frac{2\sqrt{A}}{\sqrt{\pi}}$. Thus, $dA=\frac{\pi}{2}d\cdot dd$; $dd=\frac{dA}{d}\frac{2}{\pi}=\frac{1}{\sqrt{\pi}}A^{-1/2}dA$.

Substituting for d in $\int\left(1-\frac{d}{5}\right)dd$, we obtain

$\int\left(1-\frac{2\sqrt{A}}{5\pi}\right)\cdot\frac{1}{\sqrt{\pi}}\frac{1}{\sqrt{A}}dA=\int\left(\frac{1}{\sqrt{\pi A}}-\frac{2}{5\pi^{3/2}}\right)dA$ so that the integrand is

$g(A)=\pi^{-1/2}A^{-1/2}-\frac{2}{5}\pi^{-3/2}$ for $0<A<25\pi/4$, and $g(A)=0$ elsewhere.

7.69 $f(x)=\frac{1}{\sqrt{2\pi}\sigma}e^{-(x-\mu)^2/2\sigma^2}$. Substituting $y=\ln x$, with $x=e^y$ and $dx=e^y dy$, we obtain $g(y)=\frac{1}{\sqrt{2\pi}\sigma}\cdot y^{-1}e^{(\ln y-\mu)^2/2\sigma^2}$ for $y>0$, and $g(y)=0$ elsewhere.

7.70 Since $G=\log\frac{I_o}{I_i}$, and G is normally distributed with the mean 1.8 and the standard deviation 0.05, we calculate $z=\frac{6-1.8}{0.05}=84$ and conclude that the probability of the gain exceeding 6 is negligible.