Statistical and Mathematical Methods

Statistical and Mathematical Methods for Data Science
DS5003

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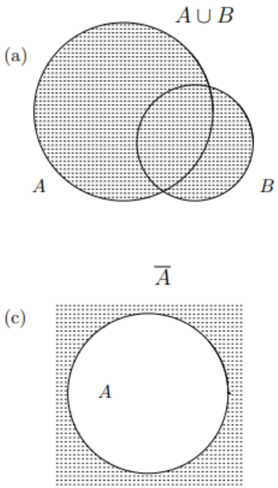
Probability Review

SET Operations



EVENTS

- A collection of all elementary results, or outcomes of an experiment, is called a sample space
- Any set of outcomes is an event. Thus, events are subsets of the sample space.
- A sample space of N possible outcomes yields 2^N possible events.
- A tossed die can produce one of 6 possible outcomes:
  - 1 dot through 6 dots.
  - Each outcome is an event.
  - There are other events:
    - observing an even number of dots,
    - an odd number of dots,
    - a number of dots less than 3, etc.

Types of events

Events A and B are **disjoint** if their intersection is empty,

$$A \cap B = \varnothing.$$

Events $A_1, A_2, A_3, \ldots$ are **mutually exclusive** or **pairwise disjoint** if any two of these events are disjoint, i.e.,

$$A_i \cap A_j = \varnothing \quad \text{for any} \quad i \neq j\,.$$

Events $A, B, C, \ldots$ are **exhaustive** if their union equals the whole sample space, i.e.,

$$A \cup B \cup C \cup \ldots = \Omega.$$

Example of Events

- Mutually exclusive events will never occur at the same time.
- Occurrence of any one of them eliminates the possibility for all the others to occur.
- Exhaustive events cover the entire Ω, so that "there is nothing left."
- Among any collection of exhaustive events, at least one occurs for sure.
- When a card is pooled from a deck at random, the four suits are at the same time disjoint and exhaustive.
- Any event A and its complement $\overline{A}$ represent a classical example of disjoint and exhaustive events.

Occurrence of Events

- $\overline{E_1 \cup E_2 \cup \ldots E_n} = \overline{E_1} \cap \overline{E_2} \cap \ldots \cap \overline{E_n}$
- $\overline{E_1 \cap E_2 \cap \ldots E_n} = \overline{E_1} \cup \overline{E_2} \cup \ldots \cup \overline{E_n}$
- Graduating with a GPA of 4.0 is an intersection of getting an A in each course.
- Its complement, graduating with a GPA below 4.0, is a union of receiving a grade below A at least in one course.
- A complement to "nothing" is "something,"
- "not everything" means "at least one is missing"

Introduction to Probability

Assume a sample space Ω and a sigma-algebra of events $\mathfrak{M}$ on it. **Probability**

$$\boldsymbol{P} \; : \; \mathfrak{M} \to [0, 1]$$

is a function of events with the domain $\mathfrak{M}$ and the range $[0, 1]$ that satisfies the following two conditions,

(Unit measure) The sample space has unit probability, $\boldsymbol{P}(\Omega) = 1$.

(Sigma-additivity) For any finite or countable collection of *mutually exclusive* events $E_1, E_2, \ldots \in \mathfrak{M}$,

$$\boldsymbol{P}\left\{E_1 \cup E_2 \cup \ldots\right\} = \boldsymbol{P}(E_1) + \boldsymbol{P}(E_2) + \ldots .$$

Different Cases

A sample space Ω consists of all possible outcomes, therefore, it occurs for sure. On the contrary, an empty event $\varnothing$ never occurs. So,

$$\boldsymbol{P}\{\Omega\} = 1 \text{ and } \boldsymbol{P}\{\varnothing\} = 0.$$

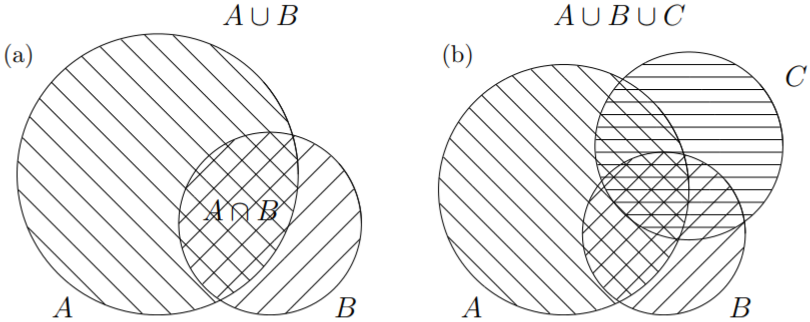
Consider an event that consists of some finite or countable collection of mutually exclusive outcomes,

$$E = \{\omega_1, \omega_2, \omega_3, ...\}.$$

Summing probabilities of these outcomes, we obtain the probability of the entire event,

$$\boxed{\boldsymbol{P}\{E\} = \sum_{\omega_k \in E} \boldsymbol{P}\{\omega_k\} = \boldsymbol{P}\{\omega_1\} + \boldsymbol{P}\{\omega_2\} + \boldsymbol{P}\{\omega_3\} \dots}$$

Union of Two and Three Events



(a) Union of two events. (b) Union of three events.

Union of Two and Three Events

Probability of a union

$$\boldsymbol{P}\{A \cup B\} = \boldsymbol{P}\{A\} + \boldsymbol{P}\{B\} - \boldsymbol{P}\{A \cap B\}$$

For mutually exclusive events,

$$\boldsymbol{P}\{A \cup B\} = \boldsymbol{P}\{A\} + \boldsymbol{P}\{B\}$$

$$\begin{aligned}\boldsymbol{P}\{A \cup B \cup C\} &= \boldsymbol{P}\{A\} + \boldsymbol{P}\{B\} + \boldsymbol{P}\{C\} - \boldsymbol{P}\{A \cap B\} - \boldsymbol{P}\{A \cap C\} \\ &\quad - \boldsymbol{P}\{B \cap C\} + \boldsymbol{P}\{A \cap B \cap C\}.\end{aligned}$$

Examples

- If a job sent to a printer appears first in line with probability 60%, and second in line with probability 30%, then with probability 90% it appears either first or second in line.
- During some construction, a network blackout occurs on Monday with probability 0.7 and on Tuesday with probability 0.5. Then, does it appear on Monday or Tuesday with probability 0.7+0.5=1.2?
- suppose there is a probability 0.35 of experiencing network blackouts on both Monday and Tuesday. Then the probability of having a blackout on Monday or Tuesday equals

Complement Rule

- A and $\overline{A}$ are exhaustive, hence $A \cup \overline{A} = \Omega$
- $P\{A\} + P\{\overline{A}\} = P\{A\} \cup P\{\overline{A}\} = P\{\Omega\} = 1$.
- $P\{\overline{A}\} = 1 - P\{A\}$
- If a system appears protected against a new computer virus with probability 0.7, then it is exposed to it with probability
- Suppose a computer code has no errors with probability 0.45. Then, it has at least one error with probability

Independent events

Events $E_1, \ldots, E_n$ are **independent** if they occur independently of each other, i.e., occurrence of one event does not affect the probabilities of others.

Independent events

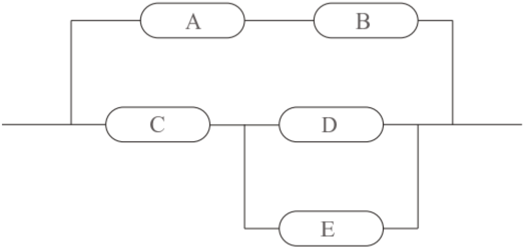
$$\boxed{\boldsymbol{P}\{E_1 \cap \ldots \cap E_n\} = \boldsymbol{P}\{E_1\} \cdot \ldots \cdot \boldsymbol{P}\{E_n\}}$$

(Reliability of backups). There is a 1% probability for a hard drive to crash. Therefore, it has two backups, each having a 2% probability to crash, and all three components are independent of each other. The stored information is lost only in an unfortunate situation when all three devices crash. What is the probability that the information is saved?

Example

Suppose that a shuttle's launch depends on three key devices that operate independently of each other and malfunction with probabilities 0.01, 0.02, and 0.02 respectively. If any of the key devices malfunctions, the launch will be postponed. Compute the probability for the shuttle to be launched on time, according to its schedule.

Example



(Techniques for solving reliability problems). Calculate reliability of the system in Figure if each component is operable with probability 0.92 independently of the other components.

Equally likely outcomes

Equally likely outcomes

$$\boxed{\boldsymbol{P}\{E\} = \frac{\text{number of favorable outcomes}}{\text{total number of outcomes}} = \frac{\mathcal{N}_F}{\mathcal{N}_T}}$$

- Tossing a die results in 6 equally likely possible outcomes, identified by the number of dots from 1 to 6.
- **P**{1}
- **P**{ odd number of dots}
- **P**{ less than 5 }
- A card is drawn from a bridge 52-card deck at random. Compute the probability that the selected card is a spade.

Examples

- You are meeting a young family with two children. What is the probability that those are two girls?
- A family has two children. You met one of them, Lev, and he is a boy. What is the probability that the other child is also a boy.
- There are 20 computers in a store. Among them, 15 are brand new and 5 are refurbished. Six computers are purchased for a student lab. From the first look, they are indistinguishable, so the six computers are selected at random. Compute the probability that among the chosen computers, two are refurbished.

Conditional Probability

Conditional probability of event A given event B is the probability that A occurs when B is *known to occur.*

$$\| \ \boldsymbol{P}\{A \mid B\} \quad = \quad \text{conditional probability of } A \text{ given } B \ \|$$

$$\boldsymbol{P}\{A \mid B\} = \frac{\text{number of outcomes in } A \cap B}{\text{number of outcomes in } B} = \frac{\boldsymbol{P}\{A \cap B\}}{\boldsymbol{P}\{B\}}$$

Conditional probability $\boxed{\boldsymbol{P}\{A \mid B\} = \frac{\boldsymbol{P}\{A \cap B\}}{\boldsymbol{P}\{B\}}}$

Intersection, general case $\boxed{\boldsymbol{P}\{A \cap B\} = \boldsymbol{P}\{B\}\,\boldsymbol{P}\{A \mid B\}}$

Events A and B are **independent** if occurrence of B does not affect the probability of A, i.e.,

$$\boldsymbol{P}\{A \mid B\} = \boldsymbol{P}\{A\}. \qquad \boldsymbol{P}\{A \cap B\} = \boldsymbol{P}\{A\}\,\boldsymbol{P}\{B\}$$

Example

- Ninety percent of flights depart on time. Eighty percent of flights arrive on time. Seventy-five percent of flights depart on time and arrive on time.
  - Eric is meeting Alyssa's flight, which departed on time. What is the probability that Alen will arrive on time?
  - Evan has met Alen, and she arrived on time. What is the probability that her flight departed on time?
  - Are the events, departing on time and arriving on time, independent?
- A box contains a total of 100 DVDs that were manufactured on two machines. Of them, 60 were manufactured on Machine I. Of the total DVDs, 15 are defective. Of the 60 DVDs that were manufactured on Machine I, 9 are defective. Let D be the event that a randomly selected DVD is defective, and let A be the event that a randomly selected DVD was manufactured on Machine I. Are events D and A independent?

Bayes Rule

Bayes Rule
$$\boxed{\boldsymbol{P}\{B \mid A\} = \frac{\boldsymbol{P}\{A \mid B\}\boldsymbol{P}\{B\}}{\boldsymbol{P}\{A\}}}$$

- (Situation on a midterm exam). On a midterm exam, students X, Y , and Z forgot to sign their papers. Professor knows that they can write a good exam with probabilities 0.8, 0.7, and 0.5, respectively. After the grading, he notices that two unsigned exams are good and one is bad. Given this information, and assuming that students worked independently of each other, what is the probability that the bad exam belongs to student Z?

Law of Total Probability and Bayes Rule

Bayes Rule

$$\boldsymbol{P}\{B \mid A\} = \frac{\boldsymbol{P}\{A \mid B\}\,\boldsymbol{P}\{B\}}{\boldsymbol{P}\{A\}}$$

Law of Total Probability

$$\boldsymbol{P}\{A\} = \sum_{j=1}^{k} \boldsymbol{P}\{A \mid B_j\}\,\boldsymbol{P}\{B_j\}$$

In case of two events $(k = 2)$,

$$\boldsymbol{P}\{A\} = \boldsymbol{P}\{A \mid B\}\,\boldsymbol{P}\{B\} + \boldsymbol{P}\{A \mid \overline{B}\}\,\boldsymbol{P}\{\overline{B}\}$$

Bayes Rule for two events

$$\boldsymbol{P}\{B \mid A\} = \frac{\boldsymbol{P}\{A \mid B\}\,\boldsymbol{P}\{B\}}{\boldsymbol{P}\{A \mid B\}\,\boldsymbol{P}\{B\} + \boldsymbol{P}\{A \mid \overline{B}\}\,\boldsymbol{P}\{\overline{B}\}}$$

Example

5 (Diagnostics of computer codes). A new computer program consists of two modules. The first module contains an error with probability 0.2. The second module is more complex; it has a probability of 0.4 to contain an error, independently of the first module. An error in the first module alone causes the program to crash with probability 0.5. For the second module, this probability is 0.8. If there are errors in both modules, the program crashes with probability 0.9. Suppose the program crashed. What is the probability of errors in both modules?