Chapter 8

8.1 $a_1 = -\frac{1}{n},\ \ldots,\ a_r = 1 - \frac{1}{n}\ \ldots\ a_n = -\frac{1}{n}$

$b_1 = \frac{1}{n},\ \ldots,\ \mathrm{b}_r = \frac{1}{n}\ \ldots\ \mathrm{b}_n = \frac{1}{n}$

$$\mathrm{cov} = \left(-\frac{1}{n^2} + \ldots + \frac{1}{n} - \frac{1}{n^2} + \ldots - \frac{1}{n^2}\right)\sigma^2$$

$$= \left[\frac{1}{n} + n\left(-\frac{1}{n^2}\right)\right]\sigma^2 = \left(\frac{1}{n} - \frac{1}{n}\right)\sigma^2 = 0$$

8.2 $Y = \bar{x}_1 - \bar{x}_2$

(a) $$E(Y) = E(\bar{x}_1 - \bar{x}_2) = \frac{1}{n_1}\sum E(x_{1i}) - \frac{1}{n_2}\sum E(x_{2i})$$

$$= \frac{n_1}{n_1}\mu_1 - \frac{n_2}{n_2}\mu_2 = \mu_1 - \mu_2$$

(b) $$\mathrm{var}(Y) = \sum \frac{1}{n_1^2}\mathrm{var}(x_{1i}) + \sum \frac{1}{n_2^2}\mathrm{var}(x_{2i}) = \frac{1}{n_1^2}\cdot n_1\sigma_1^2 + \frac{1}{n_2^2}\cdot n_2\sigma_2^2 = \frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}$$

8.3 $$M_Y(t) = \prod_{i=1}^{n_1} M_{X_{1i}}\left(\frac{t}{n_1}\right) \cdot \prod_{j=1}^{n_2} M_{X_{2j}}\left(\frac{-t}{n_2}\right)$$

$$= \prod_{i=1}^{n_1} e^{\mu_1}(t/n_1) + (1/2)\sigma_1^2)(t/n_1)^2 \cdot \prod_{j=1}^{n_2} e^{\mu_2(-t/n_1)+(1/2)\sigma_2^2(-t/n_2)^2}$$

$$= e^{\mu_2 t(1/2)(\sigma_1^2/n_1)t^2} \cdot e^{\mu_2 t+(1/2)(\sigma_2^2/n_2)t^2}$$

$$= e^{(\mu_1-\mu_2)t+(1/2)[(\sigma_1^2/n_1)+(\sigma_2^2/n_2)]t^2}$$

$$\mu = \mu_1 - \mu_2$$

$$\sigma^2 = \frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}$$

8.4 $M_x = [1 + \theta(e^t - 1)]$

$M_{\bar{x}} = [1 + \theta(e^{t/n} - 1)]^n$

$$M' = n[1 + \theta(e^{t/n} - 1)]^{n-1}\frac{\theta}{n}e^{t/n} = \theta[1 + \theta(e^{t/n} - 1)]^{n-1}e^{t/n}$$

$M'(0) = \theta$

$$M'' = \theta[1 + \theta(e^{t/n} - 1)]^{n-1} \cdot \frac{1}{n}e^{t/n} + \theta e^{t/n}(n-1)[1 + \theta(e^{t/n} - 1)]^{n-2} \cdot \frac{\theta}{n}e^{t/n}$$

$$M''(0) = \frac{\theta}{n} + \frac{\theta^2(n-1)}{n}$$

$$\sigma^2 = \frac{\theta}{n} + \frac{\theta^2(n-1)}{n} - \theta^2 = \frac{\theta(n-1)}{n}$$

8.5 $E(Y) = \mu_1 - \mu_2 = \theta_1 - \theta_2$

$$\text{var}(Y) = \frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2} = \frac{\theta_1(1-\theta_1)}{n_1} + \frac{\theta_2(1-\theta_2)}{n_2}$$

Follows directly by substitution.

8.6 $M_{\bar{x}} = [1 + \theta(e^t - 1)]^n \qquad \mu = \theta \qquad \sigma = \sqrt{\dfrac{\theta(1-\theta)}{n}}$

$$M_{(\bar{x}-\mu)/\sigma} = e^{-\mu/\sigma} \cdot M_{\bar{x}}\left(\frac{t}{\sigma}\right) = e^{-\sqrt{[\theta/n(1-\theta)]}t} \cdot \left[1 + \theta\left(e^{t/\sqrt{n\theta(1-\theta)}} - 1\right)\right]^n$$

Use series expansion to show that as $n \to \infty$

$M_{(\bar{x}-\mu)/\sigma} \to e^{(1/2)t^2}$

8.7 (1) independent

(2) information bounded with $k = \dfrac{1}{2}$

(3) $$E(x_i) = \frac{1}{2}\left[1 - \left(\frac{1}{2}\right)^i\right] + \frac{1}{2}\left[\left(\frac{1}{2}\right)^i - 1\right] = 0$$

$$E(x_i)^2 = \frac{1}{2}\left[1 - \left(\frac{1}{2}\right)^i\right]^2 + \frac{1}{2}\left[\left(\frac{1}{2}\right)^i - 1\right]^2 = \left[1 - \left(\frac{1}{2}\right)^i\right]^2$$

$$= 1 - \left(\frac{1}{2}\right)^{i-1} + \left(\frac{1}{4}\right)^i$$

$$E(Y_n) = n - \frac{1 - \left(\frac{1}{2}\right)^n}{1 - \frac{1}{2}} + \frac{1}{4}\,\frac{1 - \left(\frac{1}{4}\right)^n}{1 - \frac{1}{4}}$$

$$\lim_{n\to\infty} E(Y_n) = \lim_{n\to\infty}\left(n - 2 + \frac{1}{3}\right) \to \infty \qquad \text{QED}$$

8.8 (1) independent
(2) uniformly bounded $k = 2$

(3) $$E(x_i) = \frac{1}{2 - \frac{1}{i}} \int_0^{2-(1/i)} x\,dx = \frac{1}{2 - \frac{1}{i}} \frac{\left(2 - \frac{1}{i}\right)^2}{2} = 1 - \frac{1}{2i}$$

$$E\left(x_i^2\right) = \frac{1}{2 - \frac{1}{i}} \int_0^{2-(1/i)} x^2\,dx$$

$$E\left(x_i^2\right) = \frac{1}{2 - \frac{1}{i}} \cdot \frac{\left(2 - \frac{1}{i}\right)^3}{3} = \frac{\left(2 - \frac{1}{i}\right)^2}{3} \qquad \sigma^2 = \frac{\left(2 - \frac{1}{i}\right)^2}{3} - \frac{\left(2 - \frac{1}{i}\right)^2}{4} = \frac{\left(2 - \frac{1}{i}\right)^2}{12}$$

$$\sigma_{Y_n}^2 = n - \left(\frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}\right) + \frac{1}{4}\left(1 + \frac{1}{4} + \frac{1}{9} \ldots + \frac{1}{n^2}\right)$$
$$> n - \left(\frac{1}{2} + \frac{1}{2} + \ldots \frac{1}{2}\right) + \frac{1}{4}\left(1 + \frac{1}{4} + \ldots \frac{1}{n^2}\right)$$
$$> \frac{n}{2} + \frac{1}{4}\left(1 + \frac{1}{4} + \ldots \frac{1}{n^2}\right) \to \infty$$

8.9 $C_i = E\left(|x_i|^2\right) = \left[1 - \left(\frac{1}{2}\right)^i\right]^3$

$$\sigma_i^2 = \left[1 - \left(\frac{1}{2}\right)^i\right]^2$$

$$\text{var}(Y_n) = \sum_{i=1}^{n}\left[1 - \left(\frac{1}{2}\right)^i\right]^2$$

Let $Q = [\text{var}(Y_n)]^{-3/2} \cdot \sum_{i=1}^{n} C_i$

$$= \frac{\sum_{i=1}^{n}\left[1 - \left(\frac{1}{2}\right)^i\right]^3}{\sum_{i=1}^{n}\left[1 - \left(\frac{1}{2}\right)^i\right]^{2\ 3/2}}$$

$$= \frac{n + \ldots}{\{n + \ldots\}^{3/2}} = \frac{n + \ldots}{n\sqrt{n} + \ldots}$$

$$\lim_{n\to\infty} Q = 0$$

8.10 $E(x_i) = 0$ $\qquad$ $\sigma^2 = \dfrac{\left(2 - \frac{1}{i}\right)^2}{4}$

$$\text{var}(Y_n) = n - \left(\frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}\right) + \frac{1}{4}\left(1 + \frac{1}{4} \ldots + \frac{1}{n^2}\right)$$

$$C_i = \int_0^{2-(1/i)} \frac{1}{2 - \frac{1}{i}} x^2\ dx = \frac{1}{4}\left(2 - \frac{1}{i}\right)^2 = 2 - \frac{1}{i} + \frac{3}{2} \cdot \frac{1}{i^2} - \frac{1}{4} \cdot \frac{1}{i^2}$$

$$\sum_{i=1}^{n} C_i = 2n - \left(1 + \frac{1}{2} + \frac{1}{3} \ldots \frac{1}{n}\right) + \frac{3}{2}\left(1 + \frac{1}{4} + \ldots \frac{1}{n^2}\right) - \frac{1}{4}\left(1 + \frac{1}{8} + \frac{1}{27} + \ldots \frac{1}{n^2}\right)$$

$$\frac{\sum C_i}{\left[(\text{var}(Y_n)^{3/2}\right]} = \frac{n - \left(\frac{1}{2} + \frac{1}{3} \ldots \frac{1}{4}\right) + \frac{1}{4}\left(1 + \frac{1}{4} \ldots \frac{1}{n^2}\right)}{\left\{2n - \left(1 + \frac{1}{2} \ldots\right) + \frac{3}{2}\left(1 + \frac{1}{4} \ldots\right) - \frac{1}{4}\left(1 + \frac{1}{8} \ldots\right)\right\}^{3/2}}$$

$$= \frac{n + \ldots}{k\sqrt{n}n + \ldots} \to 0 \qquad \text{when } n \to \infty$$

8.11 When we sample with replacement from a finite population we satisfy all the conditions for random sampling from an infinite population. The random variables $x_1,\ x_2,\ \ldots,\ x_n$ are independent and identically distributed.

8.12 Hypergeometric distribution applies to sampling without replacement from a finite population

$$\mu = \frac{k}{N}$$

Consider population of k 1's and $N - k$ 0's.

$$\mu = \frac{k}{N} \text{ and } \sigma^2 = \frac{k}{N} - \frac{k^2}{N^2} = \frac{k(N-k)}{N^2}$$

by theorem 8.6 $E(\bar{x}) = \dfrac{k}{N}$ and $\text{var}(\bar{x}) = \dfrac{k(N-k)}{nN^2} \cdot \dfrac{N-n}{N-1}$

and for $Y = n\bar{x}$ $\qquad E(Y) = \dfrac{nk}{N}$ and $\text{var}(Y) = \dfrac{k(N-k)}{N^2} \cdot \dfrac{N-n}{N-1}$

Then let $\theta = \dfrac{k}{N}$ $\qquad E(Y) = \theta$ and $\text{var}(Y) = n\theta(1-\theta)\dfrac{N-n}{N-1}$

Y is a random variable having the hypergeometric distribution.

8.15 **(a)** $\mu = \dfrac{1+2+3\ldots+N}{N} = \dfrac{N(N+1)}{2N} = \dfrac{N+1}{2} \qquad \mu_{\bar{x}} = \dfrac{N+1}{2}$

(b) $\sigma^2 = \dfrac{1^2+2^2+\ldots N^2}{N} - \dfrac{(N+1)^2}{4} = \dfrac{(N+1)(2N+1)}{6} - \dfrac{(N+1)^2}{4} = \dfrac{N^2-1}{12}$

$$\text{var}(\bar{x}) = \frac{N^2-1}{12n} \cdot \frac{N-n}{N-1} = \frac{(N+1)(N-n)}{12n}$$

(c) $\mu_Y = \dfrac{n(N+1)}{2}$ and $\text{var}(Y) = \dfrac{n^2(N+1)(N-n)}{12n} = \dfrac{n(N+1)(N-n)}{12}$

8.16 $\sum c = 130 \qquad \mu = 13 \qquad \sum (C-13)^2 = 256$

$$\sigma^2 = \frac{256}{10} = 25.6$$

8.17 $\sigma^2 = \sum_{i=1}^{N} (c_i = \mu)^2 \cdot \frac{1}{N}$

$$= \frac{1}{N}\left(\sum_{i=1}^{N} c_i^2 - 2\mu\sum_{i=1}^{N} c_i + N\mu^2\right)$$

$$= \frac{1}{N}\left(\sum_{i=1}^{N} c_i^2 - 2N\mu^2 + N\mu^2\right)$$

$$= \frac{\sum_{i=1}^{N} c_i^2}{N} - \mu^2$$

In Exercise 8.14 we have

$$\mu = (15 + 13 + \ldots + 9) \cdot \frac{1}{10} = 13.0;\ \ \sigma^2 = \frac{15^2 + 13^2 + \ldots + 9^2}{10} - (13.0)^2 = 25.8$$

8.18 $S^2 = \frac{\sum_{i=1}^{n} (X_i - \overline{X})^2}{n-1}$

$$= \frac{1}{n-1}\left(\sum_{i=1}^{n} X_i^2 - 2\overline{X}\sum_{i=1}^{n} X_i + n\overline{X})^2\right)$$

$$= \frac{1}{n-1}\left(\sum_{i=1}^{n} X_i^2 - 2n\overline{X}^2 + n\overline{X}^2\right)$$

$$= \frac{\sum_{i=1}^{n} X_i^2}{n-1} - \frac{n\overline{X}^2}{n-1}$$

From the given data we calculate

$$\sum_{i=1}^{8} X_i = 108; \qquad \sum_{i=1}^{8} X_i^2 = 1{,}486$$

$$S^2 = \frac{1{,}486}{7} - \frac{8 \cdot \left(\frac{108}{8}\right)^2}{7} = 4$$

8.19 Multiplying both sides of the last equation in Exercise 8.18 by n we have

$$nS^2 = \frac{n\sum_{i=1}^{n} X_i^2}{n-1} - \frac{(n\overline{X})^2}{n-1}$$

$$\therefore S^2 = \frac{n\sum_{i=1}^{n} X_i^2 - \left(\sum_{i=1}^{n} X_i\right)^2}{n(n-1)}$$

Substituting the data of Exercise 8.18 we obtain

$$S^2 = \frac{8(1,486)-(108)^2}{8(7)} = 4$$

8.20 $M_{x_i}(t) = (1-2t)^{-(1/2)v_i} \qquad Y = \sum x_i$

$$M_Y(t) = \prod_{i=1}^{n}(1-2t)^{-(1/2)v_i} = (1-2t)^{-(1/2)\sum v_i}$$

chi square with $\sum v_i$ degrees of freedom

8.21 $M_{x_i}(t)\cdot M_{x_2}(t) = M_{x_i+x_2}(t)$

$$(1-2t)^{-(1/2)v_2}\cdot M_{x_2}(t) = (1-2t)^{-(1/2)(v_1+v_2)}$$

$$M_{x_2}(t) = (1-2t)^{(1/2)v_2} \qquad \text{QED}$$

chi square with v_2 degrees of freedom

8.22 $\displaystyle\sum_{i=1}^{n}(x_i-\mu)^2 = \sum_{i=1}^{n}\left[(x_i-\overline{x})+(\overline{x}-\mu)\right]^2$

$$= \sum_{i=1}^{n}(x_i-\overline{x})^2 + 2\sum_{i=1}^{n}(x_i-\overline{x})(\overline{x}-\mu) + \sum_{i=1}^{n}(\overline{x}-\mu)^2$$

$$= \sum_{i=1}^{n}(x_i-\overline{x})^2 + 2(\overline{x}-\mu)\sum_{i=1}^{n}(x_i-\overline{x}) + n(\overline{x}-\mu)^2$$

$$= \sum_{i=1}^{n}(x_i-\overline{x})^2 + n(\overline{x}-\mu)^2 \qquad \text{QED}$$

8.23 $E\left[\dfrac{(n-1)S^2}{\sigma^2}\right] = n-1 \qquad E(S^2) = \dfrac{\sigma^2(n-1)}{n-1} = \sigma^2$

$$\text{var}\left[\frac{(n-1)S^2}{\sigma^2}\right] = 2(n-1) \qquad \text{var}(S^2) = \frac{\sigma^4}{(n-1)^2}\cdot 2(n-1) = \frac{2\sigma^4}{n-1}$$

8.24 Follows *directly* from central limit theorem

x_i has chi square distribution with 1 degree of freedom

$\mu = 1$ and $\sigma = \sqrt{2}$

8.25 From 8.24 with $z = \dfrac{Y_n - n}{\sqrt{2n}} \to N(0,1)$

Here Y_n is a Chi-Square random variable with *n* degrees of freedom.

8.26 $\mu = 50$ and $\sigma = \sqrt{2}\cdot 50 = 10 \qquad z = \dfrac{68-50}{50} = 1.8$

Probability is $0.5000 - 0.4641 = 0.0359$

8.27 $\sqrt{2x} - \sqrt{2v} < k$

$$\sqrt{2x} < k + \sqrt{2v}$$
$$2x < k^2 + 2k\sqrt{2v} + 2v$$
$$2x - 2v < k^2 + 2k\sqrt{2v}$$
$$\frac{x-v}{\sqrt{2v}} < \frac{k^2}{2\sqrt{2v}} + k$$

8.28 From 8.27 $P\left[\frac{x-v}{\sqrt{2v}} < k + \frac{k^2}{2\sqrt{2v}}\right] \to P\left[\frac{x-v}{\sqrt{2v}} < k\right] = P\left[\sqrt{2x} - \sqrt{2v} < k\right]$

Since $\frac{x-v}{\sqrt{2v}} \to N(0,\ 1)$ for $n \to \infty$, also $P\left[\sqrt{2x} - \sqrt{2v} < k\right] \to N(0,\ 1)$

Also, $z = \sqrt{2 \cdot 68} - \sqrt{2 \cdot 50} = 11.66 - 10 = 1.66$

$0.5000 - 0.4515 = 0.0485$

8.29 From 8.26 probability is 0.0359; % error = $\frac{0.0359 - 0.04596}{0.04596} \cdot 100 = -21.9\%$

From 8.27 probability is 0.0485; % error = $\frac{0.0485 - 0.04596}{0.04596} \cdot 100 = 5.53\%$

8.35 $f(x) = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{\pi n}\,\Gamma\left(\frac{n}{2}\right)}\left(1 + \frac{t^2}{n}\right)^{-(n+1)/2}$

$$\to \frac{\sqrt{2\pi\left(\frac{n-1}{2}\right)}\left(\frac{n-1}{2e}\right)^{(n-1)/2}}{\sqrt{\pi n}\sqrt{2\pi\left(\frac{n-2}{2}\right)}\left(\frac{n-2}{2e}\right)^{(n-2)/n}}\left(1 + \frac{t^2}{n}\right)^{-(n+1)/2} \qquad \mu = \frac{t^2}{n}$$

$$= \frac{k(n-1)^{n/2}}{\sqrt{n}(n-2)^{(n-1)/2}}(1+u)^{-[(t^2/2u)-(1/2)]}$$

$$f(x) = \frac{k(n-1)^{n/2}}{\sqrt{n}(n-2)^{(n-1)/2}}\left[(1+u)^{1/u}\right]^{-t^2/2}(1+u)^{-1/2}$$

$$= k\sqrt{\frac{(n-1)^n}{n(n-2)^{n-1}}}\left[(1+u)^{1/u}\right]^{-t^2/2}(1+u)^{-1/2}$$

$$\downarrow \qquad \downarrow \qquad \downarrow$$

$$1 \qquad e^{-t^2/1} \qquad 1$$

$= ke^{-t^2/2}$ QED

8.36 The Cauchy distribution

8.37 $F = \dfrac{u / v_1}{v / v_2}$ $\qquad w = v \qquad u = Fw\dfrac{v_1}{v_2} \qquad v = w$

$$\frac{\partial u}{\partial F} = w\frac{v_1}{v_2}, \quad \frac{\partial u}{\partial w} = F\frac{v_1}{v_2}, \quad \frac{\partial v}{\partial F} = 0, \quad \frac{\partial v}{\partial w} = 1$$

$$J = \begin{vmatrix} w\dfrac{v_1}{v_2} & F\dfrac{v_1}{v_2} \\ 0 & 1 \end{vmatrix} = w\frac{v_1}{v_2}$$

$$f(u,\ v) = k\ u^{(v_1-2)/2} v^{(v_2-2)/2} e^{-(1/2)(u+v)}$$

$$g(F,\ w) = k\left(Fw\frac{v_1}{v_2}\right)^{(v_1-2)/2} w^{(v_2-2)/2} e^{-(1/2)w[F(v_1/v_2)+1]} \cdot w\frac{v_1}{v_2}$$

$$= k'\ F^{(v_1-2)/2} w^{(v_1+v_2-2)/2} e^{-(1/2)w[F(v_1/v_2)+1]}$$

$$h(F) = k''\ F^{(v_1-2)/2} \int_0^\infty w^{[(v_1+v_2)/2]-1} e^{-(1/2)[F(v_1/v_2)+1]} dw$$

Gamma distribution with $\alpha = \dfrac{v_1 + v_2}{2}$

$$\beta = \frac{2}{\left(F\dfrac{v_1}{v_2} + 1\right)}$$

$$= CF^{(v_1-2)/2}\left(F\frac{v_1}{v_2} + 1\right)^{-(1/2)(v_1+v_2)} \qquad \text{QED}$$

8.38 Make use of the fact that $F = \dfrac{u / v_1}{v / v_2}$ where *u* and *v* are independent chi square random variables, so that $E(F) = \dfrac{v_2}{v_1} E(u) E\left(\dfrac{1}{v}\right) = \dfrac{v_2}{v_1} \cdot v_1 \cdot \dfrac{1}{v_2 - 2} = \dfrac{v_2}{v_2 - 2}$ QED

8.39 $$\left(1 + \frac{v_1}{v_2}F\right)^{-(1/2)(v_1+v_2)} = \left(1 + \frac{v_1}{v_2}F\right)^{[(v_2/v_1F)(-v_1F/2)-(1/2)v_1]}$$

$$\to e^{-v_1F/2} \therefore g(F) \to k\ F^{[(v_1/2)-1]} e^{-v_1F/2}$$

$$f(v_1F) = kF^{[(v_1/2)-1]}\ e^{-(1/2)F} \to x^2(v_1)$$

8.40 T defined as $T = \dfrac{Z}{\sqrt{Y/v}}$ in Theorem 8.12 where $Z + Y$ are independent.

$T^2 = \dfrac{Z^2}{Y/v}$ where $Z^2 = \chi^2(1)$ by Theorem 8.7 $\quad Y = \chi^2(v) \quad$ QED

8.41 $F = \dfrac{u/v_1}{v/v_2}$ in Theorem 8.14 $\left.\begin{array}{l} U \text{ is } \chi^2(v_1) \\ V \text{ is } \chi^2(v_1) \end{array}\right\}$ independent

$\dfrac{1}{F} = \dfrac{v(v_1)}{u(v)}$ is ratio of 2 chi square random variables with v_2 and v_1 degrees of freedom

So $\dfrac{1}{F}$ has F distribution with v_2 and v_1 degrees of freedom.

8.42 $x \to F(v_1, v_2)$

$y \to F(v_1, v_2)$ by Exercise 8.41

$P(x \geq F_{\alpha,v_1,v_2}) = \alpha$

$P(\frac{1}{Y} \geq F_{\alpha,v_1,v_2}) = \alpha$

$P\left(Y \leq \dfrac{1}{F_{\alpha,v_1,v_2}}\right) = \alpha$

$P\left(Y \leq F_{1-\alpha,v_2,v_1}\right) = \alpha \qquad \therefore F_{1-\alpha,v_2,v_1} = \dfrac{1}{F_{\alpha,v_1,v_2}}$

8.43 $f(y) = \dfrac{\Gamma\left(\frac{v_1}{2} + \frac{v_2}{2}\right)}{\Gamma\left(\frac{v_1}{2}\right)\Gamma\left(\frac{v_2}{2}\right)} y^{(v_1/2)-1}(1-y)^{(v_2/2)-1}$

$x = \dfrac{v_2 y}{v_1(1-y)} \to y = \dfrac{v_1 x}{v_2 + v_1 x} \to \dfrac{dy}{dx} = \dfrac{v_2 v_1}{(v_2 + x v_1)^2}$

$$g(x) = \frac{\Gamma\left(\frac{v_1}{2} + \frac{v_2}{2}\right)}{\Gamma\left(\frac{v_1}{2}\right)\Gamma\left(\frac{v_2}{2}\right)} \cdot \left(\frac{v_1 x}{v_2 + v_1 x}\right)^{(v_1/2)} - 1\left(\frac{v_2}{v_2 + v_1 x}\right)^{(v_1/2)-1} \cdot \left(\frac{v_2 v_1}{x v_2 + x v_1}\right)^2$$

$$= \frac{\Gamma\left(\frac{v_1 + v_2}{2}\right)}{\Gamma\left(\frac{v_1}{2}\right)\Gamma\left(\frac{v_2}{2}\right)} \left(v_1^{v_1/2} v_2^{v_2/2} x^{(v_1/2)-1}\right) \cdot \frac{1}{(v_2 + v_1 x)^{(1/2)(v_1+v_2)}}$$

$$g(x) = \frac{\Gamma\left(\frac{v_1 + v_2}{2}\right)}{\Gamma\left(\frac{v_1}{2}\right)\Gamma\left(\frac{v_2}{2}\right)} \left(\frac{v_1}{v_2}\right)^{v_1/2} x^{(v_1/2)-1} \left(1 + \frac{v_1}{v_2} x\right)^{-(1/2)(v_1+v_2)} \qquad \text{QED}$$

8.44 Substituting into formula of Theorem 8.14 yields

$$g(F)=\frac{\Gamma(4)}{\Gamma(2)\Gamma(2)}\cdot F(1+F)^{-4}\cdot\frac{6F}{(1+F)^4}$$

Since $\frac{1}{F}$ has same distribution as F by Ex. 8.41

$$\text{probability} = 2\int_2^\infty \frac{6F}{(1+F)^4}dF \quad \text{let } u=1+F \qquad du=dF$$

$$=2\int_3^\infty \frac{6(u-1)}{u^4}du=\frac{14}{27}$$

8.45 $g_1(y_1)=n\frac{1}{8}e^{-y_1/\theta}\left[\int_{y_1}^\infty \frac{1}{\theta}e^{-x/\theta}dx\right]^{n-1} = \frac{n}{\theta}e^{-y_1/\theta}\left[e^{-y_1/\theta}\right]^{n-1}$

$=\frac{n}{\theta}e^{-y_1 n/\theta}$ for $y_1>0$ and $g_1(y_1)=0$ elsewhere

$g_1(y_n)=n\frac{1}{\theta}e^{-y_n/\theta}\left[\int_0^{y_1}\frac{1}{\theta}e^{-x/\theta}dx\right]^{n-1} = \frac{n}{\theta}e^{-y_n/\theta}\left[1-e^{y_n/\theta}\right]^{n-1}$

for $y_n>0$ and $g_n(y_n)=0$ elsewhere

$$h(\overline{x})=\frac{(2m+1)!}{m!\,m!}\left[\int_0^{\overline{x}}\frac{1}{\theta}e^{-x/\theta}dx\right]^m\frac{1}{\theta}e^{-\overline{x}/\theta}\left[\int_{\overline{x}}^\infty\frac{1}{\theta}e^{-x/\theta}dx\right]^m$$

$$=\frac{(2m+1)!}{m!\,m!}\left[1-e^{-\overline{x}/\theta}\right]^m\frac{1}{\theta}e^{-\overline{x}/\theta}\left[e^{-\overline{x}/\theta}\right]^m$$

$=\frac{(2m+1)!}{m!\,m!\,\theta}e^{-\overline{x}(m+1)/\theta}\left[1-e^{-x/\overline{\theta}}\right]^m$ for $\overline{x}>0$ and $h(\overline{x})=0$ elsewhere

8.46 $g_1(y_1)=n\cdot 1\cdot\left[\int_{y_1}^1 dx\right]^{n-1} = n(1-y_1)^{n-1}$ for $0<y_1<1$ $g_1(y_1)=0$ elsewhere

$g_n(y_n)=n\cdot 1\cdot\left[\int_0^{y_n}dx\right]^{n-1} = ny_n^{n-1}$ for $0<y_n<1$ $g_n(y_n)=0$ elsewhere

8.47 $h(\overline{x})=\frac{(2m+1)!}{m!\,m!}\left[\int_0^{\overline{x}}dx\right]^m\cdot 1\cdot\left[\int_{\overline{x}}^1 dx\right]^m = \frac{(2m+1)!}{m!\,m!}\overline{x}(1-\overline{x})^m$

for $0<\overline{x}<1$ $h(\overline{x})=0$ elsewhere

8.48 $E(y_1) = n\int_0^1 y_1(1-y_1)^{n-1}dy_1$ let $u = 1-y_1$

$$= n\int_0^1 (1-u)u^{n-1}du = n\left[\frac{1}{n}-\frac{1}{n+1}\right] = 1-\frac{n}{n+1} = \frac{1}{n+1}$$

$E(y_1^2) = n\int_0^1 y_1^2(1-y_2)^{n-1}dy_1$ $u = 1-y$

$$= n\int_0^1 (1-u)^2 u^{n-1}du = n\left[\frac{1}{n}-\frac{2}{n+1}+\frac{1}{n+2}\right] = \left[\frac{2}{(n+1)(n+2)}\right]$$

$$\text{var}(y_2) = \frac{2}{(n+1)(n+2)} - \left(\frac{1}{n+1}\right)^2 = \frac{n}{(n+1)(n+2)}$$

8.49 $g_1(y_1) = n\ 12y_1^2(1-y_1)\left[12\int_{y_1}^1 x^2(1-x)\ dx\right]^{n-1}$

$= 12\ ny_1^2(1-y_1)\left[1-4y_1^3+3y_1^4\right]^{n-1}$ for $0 < y_1 < 1$ $g_1(y_1) = 0$ elsewhere

$g_n(y_n) = n\ 12y_n^2(1-y_n)\left[12\int_0^{y_n} x^2(1-x)\ dx\right]^{n-1}$

$= 12\ ny_n^2(1-y_n)y_n^{3(n-1)}(4-3y_n)^{n-1}$

$= 12\ ny_n^{3n-1}(1-y_n)(4-3y_n)^{n-1}$ for $0 < y_n < 1$ $g_n(y_n) = 0$ elsewhere

8.50 $h(\overline{x}) = \dfrac{(2m+1)!}{m!\ m!}\left[12\int_0^{\overline{x}} x^2(1-x)\ dx\right]^m 12\overline{x}^2(1-\overline{x})\left[12\int_{\overline{x}}^1 x^2(1-x)\ dx\right]^m$

$= \dfrac{12(2m+1)!}{m!\ m!}\overline{x}^{3m+2}(1-\overline{x})[4-3\overline{x}]^m[1-4\overline{x}^3+3\overline{x}^4]^m$

$h(\overline{x}) = 0$ elsewhere

8.51 **(a)**

	y_2	$g_1(y_1)$
1 and 2		
1 and 3	1	4/10
1 and 4	2	3/10
1 and 5	3	2/10
2 and 3	4	1/10
2 and 4		
2 and 5		
3 and 4		
3 and 5		
4 and 5		

(b)

			y_1	$g_1(y_1)$
11	31	51		
12	32	52	1	9/25
13	33	53	2	7/25
14	34	54	3	5/25
15	35	55	4	3/25
21	41		5	1/25
22	42			
23	43			
24	44			
25	45			

8.52 **(a)** $$g(y_1,\ y_n) = n(n-1)\frac{1}{\theta^2}e^{-y/\theta}e^{-y_n/\theta}\left[\int_{y_1}^{y_n}\frac{1}{\theta}e^{-x/\theta}\ dx\right]^{n-2}$$

$$= \frac{n(n-1)}{\theta^2}e^{-(1/\theta)(y_1+y_n)}\left[e^{-y_1/\theta}-e^{-y_n/\theta}\right]^{n-2} \quad \text{for } 0 < y_1 < y_n < \infty$$

$g(y_1,\ y_n) = 0$ elsewhere

(b) $$g(y_1,\ y_n) = n(n-1)\left[\int_{y_1}^{y_n} dx\right]^{n-2}$$

$$= n(n-1)(y_n - y_1)^{n-2} \quad \text{for } 0 < y_2 < y_n < 1$$

$g(y_1,\ y_n) = 0$ elsewhere

8.53 From 8.48 $E(y_1) = n\int_0^1 y_1(1-y_1)^{n-1}dy_1 = \frac{1}{n+1}$

and $E(Y_n) = n\int_0^1 y_n^n\ dy_n = \frac{n}{n+1}$

$$E(Y_1,\ Y_n) = n(n-1)\int_0^1\int_0^{y_n} y_1 y_n(y_n - y_1)^{n-2}dy_1\ dy_n = \frac{1}{n+2}$$

$$\text{cov}(Y_1,\ Y_2) = \frac{1}{n+2} - \frac{1}{n+1}\frac{n}{n+1} = \frac{(n+1)^2 - n(n+2)}{(n+2)(n+1)^2} = \frac{1}{(n+1)^2(n+2)}$$

8.54 $$h(y_1,\ R) = n(n-1)f(y_1)f(y_1+R)\left[\int_{y_1}^{y_1+R} f(x)\ dx\right]^{n-2}$$

Let $y_n = y_1 + R$

and transform holding y_1 fixed. $\frac{dR}{dy_n} = 1$

8.55 $$h(y_1,\ R) = n(n-1)\frac{1}{\theta^2}e^{-y_1/\theta}e^{-(y_1+R)/\theta}\left[\int_{y_1}^{y_1+R}\frac{1}{\theta}\mathrm{e}^{-x/\theta}\ dx\right]^{n-2}$$

$$= \frac{n(n-1)}{\theta^2}e^{-y_1(n-1)/\theta}e^{-(y_1+R)/\theta}\left[1-e^{-R/\theta}\right]^{n-2}$$

$$= \frac{n(n-1)}{\theta^2}e^{-y_1 n/\theta}e^{-R/\theta}\left[1-e^{-R/\theta}\right]^{n-2}$$

$$= \underbrace{\frac{n}{\theta}e^{-y_2 n/\theta}}_{g(y_1)}\cdot\underbrace{\frac{n-1}{\theta}e^{-R/\theta}\left[1-e^{-R/\theta}\right]^{n-2}}_{f(R)} \qquad \text{independent}$$

$$f(R) = \frac{n-1}{\theta}e^{-R/\theta}\left[1-e^{-R/\theta}\right]^{n-2} \quad \text{for } R > 0$$

$g(R) = 0$ elsewhere

8.56 $h(y_1,\ R) = n(n-1)\left[\int\limits_{y_1}^{y_1+R} dx\right]^{n-2} = n(n-1)R^{n-2} \qquad 0 < R < 1 - y_1 < 1$

and 0 elsewhere

$$g(R) = n(n-1)R^{n-2}\int\limits_{0}^{1-R} dy = n(n-1)R^{n-2}(1-R) \qquad 0 < R < 1;\ \ = 0 \text{ elsewhere}$$

8.57 $E(R) = n(n-1)\int\limits_0^1 R^{n-1}(1-R)\ dR = n(n-1)\cdot\dfrac{1}{n(n+1)} = \dfrac{n-1}{n+1}$

$$E(R^2) = n(n-1)\int\limits_0^1 R^{n}(1-R)\ dR = n(n-1)\cdot\frac{1}{(n+1)(n+2)} = \frac{n(n-1)}{(n+1)(n+2)}$$

$$\sigma^2 = \frac{n(n-1)}{(n+1)(n+2)} - \frac{(n-1)^2}{(n+1)^2} = \frac{n(n-1)(n+1)-(n+2)(n-1)^2}{(n+1)^2(n+2)} = \frac{2(n-}{(n+1)^2(n+2)}$$

8.58 **(a)** $p = \int\limits_{y_1}^{y_n} f(x)\ dx \qquad \dfrac{dp}{dy_n} = f(y_n)$

$$h(y_1,\ p) = n(n-1)f(y_1)f(y_n)p^{n-2}\frac{1}{f(y_1)} = n(n-1)f(y_2)p^{n-2}$$

(b) $w = \int\limits_{-\infty}^{y_1} f(x)\ dx \qquad \dfrac{dw}{dy_1} = f(y_1)$

$$\phi(w,\ p) = n(n-1)f(y_2)p^{n-2}\frac{1}{f(y_2)} = n(n-1)p^{n-2}$$

$$w > 0,\ p > 0,\ w + p < 1$$

$$\phi(w,\ p) = 0 \text{ elsewhere}$$

(c) $g(p) = \int\limits_0^{1-p} n(n-1)p^{n-2}\ dw = n(n-1)p^{n-2}(1-p) \qquad 0 < p < 1 \qquad g(p) = 0 \text{ elsewhere}$

8.59 Density of P is same density as R obtained in Exercise 8.56, so the formula for the mean and the variance are the same as those obtained in Exercise 8.57. When n is large $E(p) \to 1$ and $\text{var}(p) \to 0$.

8.60 **(a)** $\binom{12}{3} = \frac{12 \cdot 11 \cdot 10}{6} = 220$

(b) $\binom{20}{3} = \frac{20 \cdot 19 \cdot 18}{6} = 1140$

(c) $\binom{50}{3} = \frac{50 \cdot 49 \cdot 48}{6} = 19,600$

8.61 **(a)** $\frac{1}{\binom{12}{4}} = \frac{1}{495}$ **(b)** $\frac{1}{\binom{22}{5}} = \frac{120}{12 \cdot 21 \cdot 20} = \frac{1}{77}$

8.62 $\frac{\binom{49}{2}}{\binom{50}{3}} = \frac{49!}{2!\ 47!} \cdot \frac{47!\ 3!}{50!} = \frac{3}{50} = 0.06$

8.63 **(a)** It is divided by 2 $\sqrt{120/30} = 2$
(b) It is divided by 1.5 $\sqrt{180/80} = 1.5$
(c) It is multiplied by 3 $\sqrt{450/50} = 3$
(d) It is multiplied by 2.5 $\sqrt{250/40} = 2.5$

8.64 **(a)** $\frac{200-5}{200-1} = 0.9799$; **(b)** $\frac{300-50}{300-1} = 0.8361$; **(c)** $\frac{800-200}{800-1} = 0.7509$

8.65 **(a)** $n = 100,\ \mu = 75,\ \sigma = 16,\ \therefore \sigma_{\bar{x}} = \frac{16}{\sqrt{100}} = 1.6$

$P\left(\left|\bar{X} - 75\right| < 5 \cdot 1.6\right) \geq 1 - \frac{1}{5^2} = \frac{24}{25} = 0.96$

(b $Z_1 = \frac{67-75}{\frac{16}{\sqrt{100}}} = -5$; $Z_2 = \frac{83-75}{\frac{16}{\sqrt{100}}} = 5$

From Table III, $P(67 < \bar{x} < 83) = 2 \cdot 0.4999997 = 0.9999994$

8.66 $\sigma_{\bar{x}} = \frac{6.3}{9} = 0.7$ $\frac{129.4-128}{0.7} = 2$

(a) Probability is at most $\frac{1}{4}$

(b) $1 - 2(0.4772) = 1 - 0.9544 = 0.0455$

8.67 $\sigma_{\bar{x}} = 0.7\sqrt{\frac{400-81}{400-1}} = 0.7(0.8941) = 0.626$ $z = \frac{1.4}{0.626} = 2.24$

$1 - 2(0.4875) = 0.025$

8.70 $\sigma_{\bar{x}} = \frac{6.8}{8} = 0.85$

(a) $z = \frac{52.9 - 51.4}{0.85} = 1.765$

$0.5 - 0.4612 = 0.0388$

(b) $\frac{52.3 - 51.4}{0.85} = 1.06$

$\frac{50.5 - 51.4}{0.85} = -1.05$

$2(0.3554) = 0.7108$

(c) $\frac{50.6 - 51.4}{0.85} = -0.94$

$0.5 - 0.3264 = 0.1736$

8.71 $\sigma_{\bar{x}} = \frac{25}{\sqrt{100}} = 2.5$ $z = \frac{3}{2.5} = 1.2$

$1 - 2(0.3849) = 1 - 0.7698 = 0.2302$

8.72 $\sigma_{\bar{x}_1 - \bar{x}_2} = \sqrt{\frac{20^2}{400} + \frac{30^2}{400}} = \sqrt{1 + 2.25} = 1.803$ $k = 10$

$k\sigma = 18.03$ The value of $\bar{x}_1 - \bar{x}_2$ will fall between -18.03 and 18.03.

8.73 $z = 2.57$ $k = 2.57(1.803) = 4.63$

8.74 $\mu_{\bar{x}_1 - \bar{x}_2} = 78 - 75 = 3$ $\sigma_{\bar{x}_1 - \bar{x}_2} = \sqrt{\frac{150}{30} + \frac{200}{50}} = 3$

$z = \frac{4.8 - 3}{3} = 0.6$ $0.5 - 0.2257 = 0.2743$

8.75 $E(\hat{\theta}) = 0.70$ $\text{var}(\hat{\theta}) = \frac{0.70(0.30)}{84} = 0.0025$ $\sigma = 0.05$

(a) $k = \frac{0.06}{0.05} = 1.2$

Probability is at least $1 - \frac{1}{1.2^2} = 0.3056$

(b) $2(0.3849) = 0.7698$

8.76 $1-\dfrac{1}{k^2}=1-0.9375=0.0625,\ k=4$

$$\sigma=\sqrt{\frac{(0.4)(0.6)}{500}+\frac{(0.25)(0.75)}{400}}=\sqrt{0.00048+0.00047}=0.0308$$

It will fail between $0.40-0.25\pm k\sigma=0.15\pm 4(0.0308)=0.15\pm 0.1232$
0.0268 and 0.2732

8.77 $n=5 \qquad \sigma^2=25 \qquad y=\dfrac{4s^2}{25}\rightarrow\chi^2(4)$

$f(y)=\dfrac{1}{4}y\,e^{-y/2} \qquad s^2=20 \qquad y=\dfrac{80}{25}=\dfrac{16}{5}=3.2$

$s^2=30 \qquad y=\dfrac{120}{25}=\dfrac{24}{5}=4.8$

$$\text{probability}=\frac{1}{4}\int_{3.2}^{4.8}y\,e^{-y/2}\,dy=\left[-e^{-y/2}\left(\frac{1}{2}y+1\right)\right]\Bigg|_{3.2}^{4.8}$$

$$=-3.4e^{-2.4}+2.6e^{-1.6}=-3.4(0.091)+2.6(0.202)=0.216$$

8.78 $n=16 \qquad \sigma^2=25 \qquad y=\dfrac{15s^2}{25}=0.6s^2$

has chi-square distribution with 15 degrees of freedom
probability $\left[y\ge 0.6(54.668)\right]=P(y\ge 32.801)=0.005$
probability $\left[y\le 0.6(12.102)\right]=P(y\le 7.2612)=0.05$
total probability = 0.055

8.79 $\sigma^2=4 \qquad n=9 \qquad y=\dfrac{8s^2}{4}=2s^2$

probability $\left[y\ge 2(7.7535)\right]=P(y\ge 15.507)$ 8 degrees of freedom
$=0.5$ (Table V)

8.80 $t=\dfrac{47-42}{7/\sqrt{25}}=3.57$ Since 3.57 exceeds $t_{0.005,\ 24}=2.797$ for $v=24$,
result is highly unlikely and conjecture is probably false.

8.81 $t=\dfrac{27.8-28.5}{1.8/\sqrt{12}}=-\dfrac{0.7}{1.8/3.464}=-1.347$

Since this value is fairly small (close to $-t_{0.10,\ 11}$) the data tend to support the claim.

8.82 $F = \dfrac{s_1^2/12}{s_2^2/18} = 1.5\dfrac{s_1^2}{s_2^2}$

$$P\left(\frac{s_1^2}{s_2^2} > 1.16\right) = P\left[1.5\frac{s_1^2}{s_2^2} > (1.16)(1.5)\right] = P(F > 1.74)$$

for 60, 30 degrees of freedom

From Table V $F_{0.05,\ 60,\ 30} = 1.74$ So probability is 0.05.

8.83 $F = \dfrac{s_1^2}{s_2^2}$

$P\left(\dfrac{s_1^2}{s_2^2} < 4.03\right) = 1 - P(F > 4.03)$ with 9 and 14 degrees of freedom

From Table VI $F_{0.01,\ 9,\ 14} = 4.03$

So probability = $1 - 0.01 = 0.99$

8.84 Giving the MINITAB commands

MTB> CDF 1.363;

SUBC> T 11.

We obtain 0.8999, which verifies that $t_{1,\ 11} = 1 - 0.8999 = 0.1001$ The remaining four values also can be verified to within an error of at most 0.0001.

8.85 Following the procedure of Exercise 8.84, but using 21 in place of 11, we verify all five table entries to four decimal places.

8.86 Using the MINITAB commands

MTB> CDF 4.21;

SUBC> F 7 6.

We obtain 0.9501, verifying the entry in Table V to within an error of 0.0001. The remaining entries are similarly verified to within an error of at most 0.2

8.87 Following the procedure of Exercise 8.86, but using

SUBC> F 12 17

We obtain 0.9900. The remaining entries are similarly verified

8.88 From 8.46 $g_1(y_1) = n(1 - y_1)^{n-1}$ $\quad y < y_1 < 1$

probability = $n\displaystyle\int_{0.2}^{1} (1 - y_1)^{n-1}dy_1 = (1 - y_1)^n \Big|_{1}^{0.2}$

$= (0.8)^4 = 0.4096$

8.89 $g(y_n) = 36y_n^2(1-y_n)(4-y_n^3-3y_n)^2 \quad$ for $n = 3$

$$= 36[16y^8 - 40y^9 + 33y^{10} - 9y^{11}]$$

$$\text{probability} = \int_0^{0.9} g(y_n)dy_n = 0.851$$

8.90 $g(R) = 20R^2(1-R) \qquad$ for $0 < R < 1$

$$\text{probability} = 20\int_{0.75}^{1} (R^3 - R^4)\ dR = (5R^4 - 4R^5)\Big|_{0.75}^{1} = 0.3672$$

8.91 $g(p) = n(n-1)p^2(1-p) \qquad 0 < p < 1$

$$= 90p^3(1-p)$$

$$\text{probability} = 90\int_{0.8}^{1} p^8(1-p)\ dp = (10p^9 - 9p^{10})\Big|_{0.8}^{1}$$

$$= 1 - 1.3422 + 0.9664 = 0.6242$$

8.92 $g(p) = n(n-1)p^{n-2}(1-p)$

$$\alpha = n(n-1)\int_0^p p^{n-2}(1-p)\ dp = np^{n-1} - (n-1)p^n = p^{n-1}\left[n-(n-1)p\right]$$

$$p^{n-1} = \frac{\alpha}{n-(n-1)p}$$

for $\alpha = 0.05$ and $p = 0.90 \qquad (0.90)^{n-1} = \dfrac{0.05}{n-(n-1)0.9} = \dfrac{1}{2n+18}$

$$n = \frac{1}{2} + \frac{1}{4}\cdot\frac{1.90}{0.10}\cdot 9.488 = 0.5 + 45.068 = 45.568 \text{ rounded up to } n = 46$$

8.93 The top cans have less pressure on them and may be less prone to damage.

8.94 **(a)** The sample, without the "bad" parts, will make the lathe seem better than it is.
(b) The sample is representative of product produced by the lather after inspection.

8.95 The sample is more likely to include longer sections than shorter ones; they take more time to pass the inspection station.

8.96 A systematic sample (e.g. every so many millimeters) may produce results always near the top or bottom of a wave, over- or understating the oxide thickness. To avoid this kind of problem, it is best to choose the locations to sample at random.