Chapter 14

14.1 $h(y) = \int_0^\infty x e^{-x(1+y)} dy = \frac{1}{(1+y)^2}$

$\phi(x|y) = x e^{-x(1+y)} (1+y)^2$

$E(x|y) = (1+y)^2 \int_0^\infty x^2 e^{-x(1+y)} dx \qquad z = x(1+y)$

$= \int_0^\infty z^2 e^{-z} \frac{dz}{1+y} = \frac{\Gamma(3)}{1+y} = \frac{2}{1+y}$

14.2 $g(x) = \frac{2}{5}\int_0^1 (2x+3y)dy = \frac{2}{5}\left(2x + \frac{3}{2}\right)$

$w(y|x) = \frac{\frac{2}{5}(2x+3y)}{\frac{2}{5}\left(2x+\frac{3}{2}\right)} = \frac{2x+3y}{2x+\frac{3}{2}}$

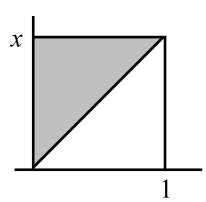
$\mu_{Y|x} = \frac{1}{2x+\frac{3}{2}} \int_0^1 y(2x+3y)dy = \frac{x+1}{2x+\frac{3}{2}} = \frac{2(x+1)}{4x+3}$

$h(y) = \frac{2}{5}\int_0^1 (2x+3y)dx = \frac{2}{5}(1+3y)$

$\phi(x|y) = \frac{\frac{2}{5}(2x+3y)}{\frac{2}{5}(1+3y)} = \frac{2x+3y}{1+3y}$

$\mu_{x|Y} = \frac{1}{1+3y}\int_0^1 x(2x+3y)dx = \frac{\frac{2}{3}+\frac{3}{2}y}{1+3y} = \frac{4+9y}{6(1+3y)}$

14.3



$g(x) = \int_x^1 6x\ dy = 6x(1-x),\ w(y|x) = \frac{6x}{6x(1-x)} = \frac{1}{1-x}$

$E(Y|x) = \frac{1}{1-x}\int_x^1 y\ dy = \frac{1-x^2}{2(1-x)} = \frac{1+x}{2}$

$h(y) = \int_0^y 6x\ dx = 3y^2 \qquad \phi(x|y) = \frac{2x}{y^2}$

$E(x|y) = \frac{2}{y^2}\int_0^y x^2\ dx = \frac{2}{y^2}\cdot\frac{y^3}{3} = \frac{2y}{3}$

14.4 $f(x,y) = \dfrac{2x}{(1+x+xy)^2}$

$$g(x) = \int_0^\infty \frac{2x}{(1+x+xy)^2}dy \qquad u = 1+x+xy \qquad du = x\ dy$$

$$= \int_{1+x}^\infty \frac{2\ du}{u^2} = \frac{1}{u^2}\bigg|_{1x}^\infty = \frac{1}{(1+x)^2}$$

$$w(y|x) = \frac{2x(1+x)^2}{(1+x+xy)^3}$$

$$E(Y|x) = 2x(1+x)^2 \int_0^\infty \frac{y\ dy}{(1+x+xy)^2} \qquad u = 1+x+xy \quad du = x\ dy$$

$$= 2x(1+x)^2 \int_{1+x}^\infty \frac{u-(1+x)}{x} \cdot \frac{du}{xu^3} \qquad v = \frac{u-(1+x)}{x}$$

$$= \frac{2(1+x)^2}{x}\left[-\frac{1}{u} + \frac{(1+x)}{2u^2}\right]_{1+x}^\infty = \frac{1+x}{x}$$

$$E(Y^2|x) = 2x(1+x)^2 \int_0^\infty \frac{y^2\ dy}{(1+x+xy)^3} \to \infty$$

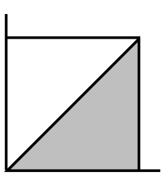
14.5 $\mu_{x|1} = 0 \cdot \dfrac{10}{21} + 1 \cdot \dfrac{10}{21} + 2 \cdot \dfrac{1}{21} = \dfrac{12}{21} = \dfrac{4}{7}$

$$\mu_{Y|0} = 0 \cdot \frac{5}{28} + 1 \cdot \frac{15}{28} + 2 \cdot \frac{15}{56} + 3 \cdot \frac{1}{56} = \frac{63}{56} = \frac{9}{8}$$

14.6 $m(x,y) = \dfrac{xy}{36}$, $g(x) = \dfrac{x}{6}$, so $w(y|x) = \dfrac{y}{6}$

$$E(Y|x) = \sum_{y=1}^{3} \frac{y^2}{6} = \frac{1}{6}(1+4+9) = \frac{14}{6} = \frac{7}{3}$$

14.7



$$f(x,y) = 2 \qquad g(x) = 2\int_0^x dx = 2x$$

$$h(y) = 2\int_y^1 dx = 2(1-y)$$

(a) $w(y|x) = \dfrac{2}{2x} = \dfrac{1}{x}$, $\mu_{Y|x} = \dfrac{1}{x}\displaystyle\int_0^x y\ dy = \dfrac{1}{x} \cdot \dfrac{x^2}{2} = \dfrac{x}{2}$

$$\mu_{x|y} = \frac{1}{1-y}\int_y^1 x\ dx = \frac{1}{1-y} \cdot \frac{1}{2}(1-y^2) = \frac{1+y}{2}$$

(b) $$E(x^mY^n) = 2\int_0^1\int_0^x x^m y^n dy\ dx = 2\int_0^1 x^m\left[\frac{y^{n+1}}{n+1}\right]_0^x dx = \frac{2}{n+1}\int_0^1 x^{m+n+1}dx$$

$$= \frac{2}{(n+1)(m+n+2)}$$

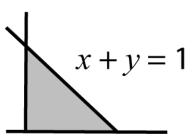
$$E(x) = \frac{2}{3},\ E(Y) = \frac{1}{3},\ E(x^2) = \frac{1}{2},\ E(Y^2) = \frac{1}{6},\ E(xY) = \frac{1}{4}$$

$$\sigma_1^2 = \frac{1}{18},\ \sigma_2^2 = \frac{1}{18},\ \sigma_{12} = \frac{1}{36},\ \rho = \frac{1/38}{\sqrt{\frac{1}{18}\cdot\frac{1}{18}}} = \frac{1}{2}$$

$$\mu_{Y|x} = \frac{1}{3} + \frac{1}{2}\left(x - \frac{2}{3}\right) = \frac{x}{2}$$

$$\mu_{x|y} = \frac{2}{3} + \frac{1}{2}\left(y - \frac{1}{3}\right) = \frac{1+y}{2}$$

14.8



$$g(x) = 24x\int_0^{1-x} y\ dy = 12x(1-x)^2$$

$$\phi(y|x) = \frac{24xy}{12x(1-x)^2} = \frac{2y}{(1-x)^2}$$

$$\mu_{Y|x} = \frac{2}{(1-x)^2}\int_0^{1-x} y^2\ dx = \frac{2}{(1-x)^2}\cdot\frac{(1-x)^3}{3} = \frac{2}{3}(1-x)$$

$$E(x^mY^n) = \int_0^1\int_0^{1-x} 24x^{m+1}y^{n+1}dy\ dx = \frac{24}{n+2}\int_0^1 x^{m+1}(1-x)^{n+2}dx$$

$$= \frac{24}{n+2}\cdot\frac{(m+1)!(n+2)!}{(m+n+4)!}\quad \text{by definition of Beta function}$$

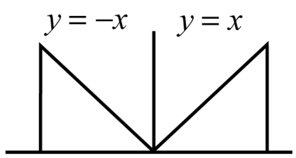
$$= \frac{24(m+1)!(n+1)!}{(m+n+4)!}$$

$$E(x) = \frac{2}{5},\ E(Y) = \frac{2}{5},\ E(x^2) = \frac{1}{5},\ E(Y^2) = \frac{1}{5},\ E(xY) = \frac{2}{15}$$

$$\sigma_1^2 = \frac{1}{25},\ \sigma_2^2 = \frac{1}{25},\ \sigma_{12} = -\frac{2}{75},\ \rho = -\frac{2}{3}$$

$$\mu_{Y|x} = \frac{2}{5} - \frac{2}{3}\left(x - \frac{2}{5}\right) = \frac{2}{3}(1-x)$$

14.9



$E(x)=0,\ E(xY)=0 \rightarrow$ uncorrelated

$$E(x^m y^n)=\int_0^1\int_0^x x^m y^n dy\ dx+\int_{-1}^0\int_0^{-x} x^m y^n dy\ dx$$

$$=\int_0^1 \frac{x^{m+n+1}}{n+1}dx+(-1)^{n+1}\int_{-1}^0 \frac{x^{m+n+1}}{n+1}dx=\frac{1-(-1)^{m+1}}{(n+1)(m+n+2)}$$

$E(x)=0,\ E(Y)=\frac{1}{3},\ E(xY)=0$

$\therefore \sigma_{12}=0 \rightarrow$ uncorrelated

$$h(y)=\int_{-y}^{y} dx=2y,\quad 0<y<1$$

$$g(x)=\begin{cases}\int_{-x}^{1} dy=1+x \text{ for } -1<x<0\\ \int_{x}^{1} dy=1-x \text{ for } 0<x<1\end{cases}$$

$$\phi(y|x)=\begin{cases}\frac{1}{1+x} \text{ for } -1<x\le 0 \text{ and } -x<y<1\\ \frac{1}{1-x} \text{ for } 0<x<1 \text{ and } x<y<1\end{cases}$$

14.10 $\operatorname{var}(Y|x)=E(Y^2|x)-[E(Y|x)]^2$

multiply by $g(x)$ and integrate over x

$$\int \operatorname{var}(Y|x)\ g(x)\ dx=\int\{g(x)\{E(Y^2|x)-[E(Y|x)]^2\}dx$$

$$\operatorname{var}(Y|x)=E(Y^2)-\int g(x)[E(Y|x)]^2dx$$

$$=E(Y^2)-[E(Y)]^2-\left\{\int g(x)[E(Y|x)]^2dx-E(Y)^2\right\}$$

$$=\operatorname{var}(Y)-\operatorname{var}E(Y|x)$$

$$=\sigma_2^2-\operatorname{var}\left[\mu_2+\rho\frac{\sigma_2}{\sigma_1}(x-\mu_1)\right]$$

$$=\sigma^2-\rho^2\frac{\sigma_2^2}{\sigma_1^2}\sigma_1^2=\sigma_2^2(1-\rho^2)$$

14.11 $\text{var}\left(\frac{x}{\sigma_2}+\frac{Y}{\sigma_2}\right)=\frac{\sigma_1^2}{\sigma_1^2}+\frac{2\sigma_{12}}{\sigma_1\sigma_2}+\frac{\sigma_2^2}{\sigma_2}=2(1+\rho)$

$\text{var}\left(\frac{x}{\sigma_1}-\frac{Y}{\sigma_2}\right)=\frac{\sigma_1^2}{\sigma_2^2}-\frac{2\sigma_{12}}{\sigma_1\sigma_2}+\frac{\sigma_2^2}{\sigma_2^2}=2(1-\rho)$

$1+\rho\geq 0 \qquad \rho\geq -1$ and $1-\rho\geq 0 \qquad \rho\leq 1$

$-1\leq\rho\leq 1$

14.12 $\int x_3 g(x_3|x_1,x_2)dx_3=\alpha+\beta_1(x_1-\mu_1)+\beta_2(x_2-\mu_2)$

multiply by $h(x_1,x_2)$ and integrate over x_1,x_2 and x_3

$\mu_2=\alpha+0+0=\alpha$

multiply by $(x_1-\mu_1)h(x_1,x_2)$ and integrate

$\sigma_{13}=\beta_1\sigma_1^2+\beta_2\sigma_{12}$

multiply by $(x_2-\mu_2)h(x_1,x_2)$ and integrate

$\sigma_{23}=\beta_1\sigma_{12}+\beta_2\sigma_2^2$

solve for β_1 and β_2

$\beta_1=\frac{\sigma_{23}\sigma_2^2-\sigma_{12}\sigma_{13}}{\sigma_1^2\sigma_2^2-\sigma_{12}^2}$ and $\beta_2=\frac{\sigma_{23}\sigma_1^2-\sigma_{12}\sigma_{13}}{\sigma_1^2\sigma_2^2-\sigma_{12}^2}$

14.13 $q=\sum_{i=1}^{n}[y_1=\hat{\beta}x_i]^2$

$\frac{dq}{d\beta}=\sum_{i=1}^{n}(-2)x_i[y_i-\hat{\beta}x_i]=0 \qquad \hat{\beta}=\frac{\sum_{i=1}^{n}x_iy_i}{\sum_{i=1}^{n}x_i^2}$

14.14 $\sum y=\hat{\alpha}n+\hat{\beta}\sum x$

$\sum xy-\hat{\alpha}\sum x+\hat{\beta}\sum x^2$

$$\hat{\alpha}=\frac{\begin{vmatrix}\sum y & \sum x\\ \sum xy & \sum x^2\end{vmatrix}}{\begin{vmatrix}n & \sum x\\ \sum x & \sum x^2\end{vmatrix}}=\frac{\left(\sum x^2\right)\left(\sum y\right)-\left(\sum x\right)\left(\sum xy\right)}{n\left(\sum x^2\right)-\left(\sum x\right)^2}$$

14.15 In previous exercise also $\hat{\beta} = \dfrac{\begin{vmatrix} n & \sum y \\ \sum x & \sum xy \end{vmatrix}}{n\left(\sum x^2\right) - \left(\sum x\right)^2} = \dfrac{n\left(\sum xy\right) - \left(\sum x\right)\left(\sum y\right)}{n\left(\sum x^2\right) - \left(\sum x\right)^2}$

letting $\sum x = 0$ yields $\hat{\alpha} = \dfrac{\left(\sum x^2\right)\left(\sum y\right)}{n\left(\sum x^2\right)} = \dfrac{\sum y}{n}$

$$\hat{\beta} = \frac{n\left(\sum xy\right)}{n\left(\sum x^2\right)} = \frac{\sum xy}{\sum x^2}$$

14.16 $q = \sum_{i=1}^{n} e_i^2 = 2\sum(y - \alpha - \beta x - \gamma x^2)$ differentiating partially with respect to α, β and γ and setting the resulting derivatives to zero to obtain the maximum likelihood estimates, we obtain

$$\frac{\partial q}{\partial \alpha} = 2\sum_{i=1}^{n}\left(y_i - \alpha - \beta x_i - \gamma x_i^2\right)(-1) = 0,$$

$$\frac{\partial q}{\partial \beta} = 2\sum_{i=1}^{n}\left(y_i - \alpha - \beta x_i - \gamma x_i^2\right)(-x_i) = 0, \text{ and}$$

$$\frac{\partial q}{\partial \gamma} = 2\sum_{i=1}^{n}\left(y_i - \alpha - \beta x_i - \gamma x_i^2\right)(-x_i^2) = 0.$$

Omitting the subscripts and limits of summation, we can write these equations in the usual normal-equation form:

$$\sum y = \alpha \cdot n + \beta \sum x + \gamma \sum x^2$$
$$\sum xy = \alpha \sum x + \beta \sum x^2 + \gamma \sum x^3$$
$$\sum x^2 y = \alpha \sum x^2 + \beta \sum x^3 + \gamma \sum x^4$$

14.17 $\sum[y - (\hat{\alpha} - \hat{\beta}x)]^2 = \sum(y_i - \overline{y} + \hat{\beta}\overline{x} - \hat{\beta}x_i]^2$

$$= \sum[(y_i - \overline{y}) - \hat{\beta}(x_i - \overline{x})]^2$$
$$= S_{yy} - 2\hat{\beta}S_{xy} + \hat{\beta}^2 S_{xx}$$
$$= S_{yy} - 2\hat{\beta}S_{xy} + \hat{\beta}\left(\frac{S_{xy}}{S_{xx}}\right)S_{xx}$$
$$= S_{yy} - \hat{\beta}S_{xy}$$

14.18 by Theorem 14.3 $E\left(\frac{n\hat{\sigma}^2}{\sigma^2}\right) = n-2$

(a) $E(\hat{\sigma}^2) = \frac{n-2}{n}\sigma^2 \neq \sigma^2$ QED

(b) $E\left(\frac{n\hat{\sigma}^2}{n-2}\right) = \frac{n}{n-2}\cdot\frac{n-2}{n}\cdot\sigma^2 = \sigma^2$

14.19 **(a)** $s_e = \hat{\sigma}\sqrt{\frac{n}{n-2}}$ $t = \frac{\hat{\beta}-\beta}{s_e/\sqrt{S_{xx}}}$

(b) $\hat{\beta} \pm t_{\alpha/2,n-2}\cdot\frac{s_e}{\sqrt{S_{xx}}}$

14.20 $\hat{\alpha} = \overline{y} - \hat{\beta}\overline{x}$ with $\hat{\beta} = \sum\left(\frac{x_i-\overline{x}}{S_{xx}}\right)y$ from text

(a) $\hat{\alpha} = \frac{\sum y_i}{n} - \sum\overline{x}\left(\frac{x_i-\overline{x}}{S_{xx}}\right)y_i$

$$\sum\left[\frac{1}{n} - \frac{(x_i-\overline{x})}{S_{xx}}y_i\ \overline{x}\right] = \sum_{i=1}^{n}\frac{S_{xx}+n\overline{x}^2-n\overline{x}x_i}{nS_{xx}}y_i$$

(b) Use corollary to Theorem 4.14 and Exercise 7.58
Since $\hat{A}$ is linear combination of y's $\rightarrow \hat{\alpha}$ has normal distribution.

$$E(\hat{\alpha}) = \sum\left[\frac{SS_{xx}+n\overline{x}^2-n\overline{x}x_i}{nS_{xx}}\right]E(Y_i)$$

$$= \sum\left[\frac{SS_{xx}+n\overline{x}^2-n\overline{x}x_i}{nS_{xx}}\right](\alpha+\beta x_i)$$

$$= \frac{\alpha}{nS_{xx}}\sum[S_{xx}-n\overline{x}(x_i-\overline{x})] + \beta\sum\left[\frac{(S_{xx}+n\overline{x}^2)x_i}{nS_{xx}} - \frac{n\overline{x}x_i^2}{nS_{xx}}\right]$$

$$= \frac{\alpha}{nS_{xx}}\sum S_{xx} + \beta\sum\left[\frac{(S_{xx}+n\overline{x}^2)n\overline{x}}{nS_{xx}} - \frac{\overline{x}}{S_{xx}}\sum x_i^2\right]$$

$$= \alpha + \frac{\beta\overline{x}}{S_{xx}}\left[S_{xx}+n\overline{x}^2-\sum x_i^2\right] = \alpha$$

$$\operatorname{var}(\hat{\alpha}) = \sum\left[\frac{S_{xx}+n\overline{x}^2-n\overline{x}x_i}{nS_{xx}}\right]^2\sigma^2$$

$$= \sum\left[\frac{S_{xx}+n\overline{x}(x_i-\overline{x})}{nS_{xx}}\right]^2\sigma^2 = \frac{1}{n} + \frac{n^2\overline{x}S_{xx}}{n^2S_{xx}^2}\cdot\sigma^2$$

$$= \frac{(S_{xx}+n\overline{x}^2)\sigma^2}{nS_{xx}}$$

14.21 $a_i = \dfrac{S_{xx} - n\overline{x}(x_i - \overline{x})}{nS_{xx}}$ $\qquad b_i = \dfrac{x_i - \overline{x}}{S_{xx}}$

$$\text{cov}(\hat{A}, \hat{B}) = \sum a_i b_i \sigma^2 = \frac{\sigma^2}{nS_{xx}^2} \sum [S_{xx} - n\overline{x}(x_i - \overline{x})](x_i - \overline{x})$$

$$= \frac{\sigma^2}{nS_{xx}^2}[-n\overline{x}S_{xx}] = -\frac{\overline{x}}{S_{xx}}\sigma^2$$

14.22 $z = \dfrac{\hat{\alpha} - \alpha}{\sqrt{\dfrac{(S_{xx} + n\overline{x}^2)}{nS_{xx}}} \cdot \sigma} = \dfrac{(\hat{\alpha} - \alpha)\sqrt{nS_{xx}}}{\sigma\sqrt{S_{xx} + n\overline{x}^2}}$ has standard normal distribution and is independent of Z.

Also $\dfrac{n\hat{\sigma}^2}{\sigma^2}$ has χ^2 distribution with $n-2$ degrees of freedom.

$$t = \frac{(\hat{\alpha} - \alpha)\sqrt{nS_{xx}}}{\sigma\sqrt{S_{xx} + n\overline{x}^2}} + \sqrt{\frac{n\hat{\sigma}^2 / \sigma^2}{n-2}} = \frac{(\hat{\alpha} - \alpha)\sqrt{(n-2)S_{xx}}}{\hat{\sigma}^2\sqrt{S_{xx} + n\overline{x}^2}}$$

has t distribution with $n-2$ degrees of freedom

14.23 $\hat{Y}_0 = \hat{A} + \hat{B}x_0$ is sum of independent normal random variables and according to Ex. 7.58 has normal distribution

$$E(\hat{A}) + x_0 E(\hat{B}) = \alpha + x_0\beta = E(\hat{Y}_0 \big| x_0)$$

$$\text{var}(\hat{Y}_0 \big| x_o) = \text{var}(\hat{A}) + x_0^2 \text{ var}(\hat{B}) + 2x_0 \text{ cov}(\hat{A}, \hat{B})$$

$$= \frac{(S_{xx} + n\overline{x}_2)\sigma^2}{nS_{xx}} + x_0^2 \cdot \frac{\sigma^2}{S_{xx}} + 2x_0\left(-\frac{\overline{x}}{S_{xx}}\sigma^2\right)$$

$$= \sigma^2\left[\frac{1}{n} + \frac{\overline{x}^2}{S_{xx}} + \frac{x_0^2}{S_{xx}} - \frac{2x_0\overline{x}}{S_{xx}}\right] = \sigma^2\left[\frac{1}{n} + \frac{(\overline{x} - x_0)^2}{S_{xx}}\right]$$

Using Theorem 14.3,

$$t = \frac{\hat{y}_0 - (\alpha + x_0\beta)}{\sigma\sqrt{\dfrac{1}{n} + \dfrac{(\overline{x} - x_0)^2}{S_{xx}}}} \div \sqrt{\frac{n\hat{\sigma}^2 / \sigma^2}{n-2}} = \frac{[\hat{y} + (\alpha + x_0\beta)]\sqrt{n-2}}{\hat{\sigma}\sqrt{1 + \dfrac{n(x - \overline{x}_0)^2}{S_{xx}}}}$$

has t distribution with $n-2$ degrees of freedom.

14.24 confidence limits are

$$\hat{y}_0 \pm t_{\alpha/2, n-2} \cdot \frac{\hat{\sigma}}{\sqrt{n-2}}\sqrt{1 + \frac{n(\overline{x} - x_0)^2}{S_{xx}}}$$

by substituting expression for t from Exercise 14.31 into $-t_{\alpha/2,n-2} < t < t_{\alpha/2,n-2}$ and solving by simple algebra.

14.25 $E[Y_0-(\hat{A}+\hat{B}x_0)]=(\alpha+\beta x_0)-(\alpha+\beta x_0)=0$

$$\operatorname{var}[Y_0-(A+Bx_0)]=\sigma^2+\operatorname{var}(\hat{A})+x_0^2\operatorname{var}(\hat{B})-2x_0\operatorname{cov}(\hat{A},\hat{B})$$

$$=\sigma^2+\frac{(S_{xx}+n\overline{x}^2)\sigma^2}{nS_{xx}}+\frac{\sigma^2}{S_{xx}}x_0^2-\frac{2x_0\overline{x}}{S_{xx}}\sigma^2$$

$$=\sigma^2\left[1+\frac{1}{n}+\frac{\overline{x}^2+x_0^2-2x_0\overline{x}}{S_{xx}}\right]=\sigma^2\left[1+\frac{1}{n}+\frac{(\overline{x}-x_0)^2}{S_{xx}}\right]$$

$$t=\frac{[y_0-(\hat{\alpha}+\hat{\beta}x_0)]}{\sigma\sqrt{1+\frac{1}{n}+\frac{(\overline{x}-x_0)^2}{S_{xx}}}}+\sqrt{\frac{n\hat{\sigma}^2/\sigma^2}{n-2}}=\frac{[\hat{y}-(\alpha+\beta x_0)]\sqrt{n-2}}{\hat{\sigma}\sqrt{1+n+\frac{n(\overline{x}-x_0)^2}{S_{xx}}}}$$

14.26 Simple algebra leads to the following limits of prediction:

$$\hat{y}_0\pm t_{\alpha/2,n-2}\cdot\frac{\hat{\sigma}}{\sqrt{n-2}}\sqrt{1+n+\frac{n(\overline{x}_0-x)^2}{S_{xx}}}$$

14.28 $t=\dfrac{\hat{\beta}-\beta}{\hat{\sigma}}\sqrt{\dfrac{(n-2)S_{xx}}{n}}=\left(1-\dfrac{\beta}{\hat{\beta}}\right)\dfrac{\hat{\beta}}{\sigma}\sqrt{\dfrac{(n-2)S_{xx}}{n}}$

$$=\left(1-\frac{\beta}{\hat{\beta}}\right)\frac{S_{xy}}{S_{xx}}\frac{\sigma^2}{\sqrt{1-r^2}}\sqrt{\frac{(n-2)S_{xx}}{n}}$$

$$=\left(1-\frac{\beta}{\hat{\beta}}\right)\frac{r}{\sqrt{1-r^2}}\sqrt{n-2}\qquad\text{QED}$$

14.29 $1-\dfrac{\beta}{\hat{\beta}}=\pm t_{a/2,n-2}\dfrac{\sqrt{1-r^2}}{r\sqrt{n-2}}$

$$\frac{\beta}{\hat{\beta}}=1\pm t_{a/2,n-2}\frac{\sqrt{1-r^2}}{r\sqrt{n-2}}$$

$$\beta=\hat{\beta}\left[1\pm t_{\alpha/2,n-2}\frac{\sqrt{1-r^2}}{r\sqrt{n-2}}\right]\qquad\text{QED}$$

14.30 $t=\dfrac{r\sqrt{n-2}}{\sqrt{1-r^2}}$ $\qquad u=r^2$

$$t^2=\frac{r^2(n-2)}{1-r^2}\qquad du=2r\ dr$$

$$2t\frac{dt}{dr^2}=\frac{n-2}{(1-r^2)^2}\qquad r^2=\frac{t^2}{n-2+t^2}$$

$$\frac{dt}{dr^2}=\frac{(n-2)}{(1-r^2)^2}\cdot\frac{\sqrt{1-r^2}}{2r\sqrt{n-2}}$$

$$g(r^2)=\frac{\sqrt{n-2}}{2r(1-r^2)\sqrt{1-r^2}}\cdot k\left(1+\frac{t^2}{n-2}\right)^{-(n-1)/2}$$
$$=\frac{\sqrt{(n-2)}k}{2r(1-r^2)\sqrt{1-r^2}}\left[1+\frac{r^2}{1-r^2}\right]^{-(n-1)/2}$$
$$=\frac{K}{r(1-r^2)\sqrt{1-r^2}}(1-r^2)^{(n-1)/2}$$
$$=K(r^2)^{-1/2}(1-r^2)^{(n-4)/2} \qquad \text{beta distribution}$$

$$\alpha-1=-\frac{1}{2} \qquad \beta-1=\frac{n-4}{2}$$

$$\mu=\frac{\alpha}{\alpha+\beta}=\frac{\frac{1}{2}}{\frac{1}{2}+\frac{n-2}{n}}=\frac{1}{n-1}$$

14.31 $$-z_{\alpha/2}\le\frac{\sqrt{n-3}}{2}\ln\frac{(1+r)(1-\rho)}{(1-r)(1+\rho)}\le z_{\alpha/2}$$
$$-\frac{2z_{\alpha/2}}{\sqrt{n-3}}\le\ln\frac{(1+r)(1-\rho)}{(1-r)(1+\rho)}\le\frac{2z_{\alpha/2}}{\sqrt{n-3}}$$
$$e^{-(2z_{\alpha/2})/\sqrt{n-3}}\le\frac{(1+r)(1-\rho)}{(1-r)(1+\rho)}\le e^{(2z_{\alpha/2})/\sqrt{n-3}}$$
$$\frac{(1-r)}{(1+r)}e^{-(2z_{\alpha/2})/\sqrt{n-3}}\le\frac{1-\rho}{1+\rho}\le\frac{(1-r)}{(1+r)}e^{(2z_{\alpha/2})/\sqrt{n-3}}$$
$$1+\rho\cdot\frac{(1-r)}{1+r}e^{-(2z_{\alpha/2})/\sqrt{n-3}}\le\frac{1-\rho}{1+\rho}\le\frac{(1-r)}{(1+r)}e^{(2z_{\alpha/2})/\sqrt{n-3}}$$
$$\rho\left[1+\frac{1-r}{1+r}e^{-(2z_{\alpha/2})/\sqrt{n-3}}\right]\le1-\frac{(1-r)}{(1+r)}e^{-(2z_{\alpha/2})/\sqrt{n-3}}$$
$$\rho\le\frac{1+r-(1-r)e^{-(2z_{\alpha/2})/\sqrt{n-3}}}{1+r+(1-r)e^{-(2z_{\alpha/2})/\sqrt{n-3}}} \quad \text{and}$$
$$\rho\left[1+\frac{1-r}{1+r}e^{(2z_{\alpha/2})/\sqrt{n-3}}\right]\ge1-\frac{(1-r)}{(1+r)}e^{(2z_{\alpha/2})/\sqrt{n-3}}$$
$$p\ge\frac{1+r-(1-r)e^{(2z_{\alpha/2})/\sqrt{n-3}}}{1+r+(1-r)e^{(2z_{\alpha/2})/\sqrt{n-3}}}$$
$$\frac{1+r-(1-r)e^{(2z_{\alpha/2})/\sqrt{n-3}}}{1+r+(1-r)e^{(2z_{\alpha/2})/\sqrt{n-3}}}\le\rho\le\frac{1+r-(1-r)e^{-(2z_{\alpha/2})/\sqrt{n-3}}}{1+r+(1-r)e^{-(2z_{\alpha/2})/\sqrt{n-3}}}$$

14.32 Substitute $$S_{xx} = \sum_{i=1}^{r} x_i^2 f_i - \frac{1}{n}\left[\sum_{i=1}^{r} x_i f_i\right]^2$$

$$S_{yy} = \sum_{j=1}^{r} y_j^2 f_j - \frac{1}{n}\left[\sum_{j=1}^{r} y_j f_j\right]^2$$

and

$$S_{xy} = \sum_{i=1}^{r}\sum_{j=1}^{r} x_i y_j f_{ij} - \frac{1}{n}\left[\sum_{i=1}^{r} x_i f_i\right]\left[\sum_{j=1}^{r} y_j f_j\right]$$

into $r = \dfrac{S_{xy}}{\sqrt{S_{xx}S_{yy}}}$

14.33 $q = (Y - Xb)'(Y - Xb)$

$= \{Y' - (Xb)'\}\{Y - Xb\}$

$= Y'Y - Y'Xb - (Xb)'Y + (Xb)'Xb$

since $Y'Xb$ is $|X|$, a number, not a matrix, $Y'Xb = (Xb)'Y$

$q = Y'Y - 2Y'Xb + b'X'Xb$

vector of partial derivatives is

$-2(Y'X)' + 2X'Xb = -2X'Y = 2X'Xb$

put equal to zero yields

$-2X'Y + 2X'Xb = 0$

$b = (X'X)^{-1}X'Y$ QED

14.34 $L(b,\sigma^2) = \dfrac{1}{(2\pi\sigma^2)^{n/2}} e^{-(1/2\sigma^2)(Y-xb)'(Y-Xb)}$

To maximize L minimize $(Y - Xb)'(Y - Xb)$ as in Ex 14.33

(a) $\therefore$ maximum likelihood estimates = least square estimates

(b) as in simple regression

$$\ln L = -\frac{n}{2}\ln 2\pi - \frac{n}{2}\ln\sigma^2 - \frac{1}{2\sigma^2}(Y - Xb)'(Y - Xb)$$

$$\frac{\partial \ln L}{\partial \sigma^2} = -\frac{n}{2}\cdot\frac{1}{\sigma^2} + \frac{1}{2\sigma^4}(Y - Xb)'(Y - Xb) = 0$$

together with $\dfrac{\partial \ln L}{\partial b} = 0$ we get

$\hat{\sigma}^2 = \dfrac{1}{n}(Y - XB)'(Y - XB)$ QED

14.35 $(Y - XB)'(Y - XB) = [(Y - X(X'X)^{-1}X'Y]'[Y - X(X'X)^{-1}X'Y]$

$= Y'[I - X(X'X)^{-1}X'][I - X(X'X)^{-1}X']Y$

$= Y'[I - X(X'X)^{-1}X']Y$

$= Y'Y - Y'X(X'X)^{-1}X'Y$

$= Y'Y - B'X'Y$ QED

14.36 $\hat{B} = (X'X)^{-1} X'Y$

(a) $E(\hat{B}) = (X'X)^{-1} X'E(Y)$

$= (X'X)^{-1} X'XB = B$

$E(\hat{B}_i) = \hat{B}_i$ for $i = 0,1,2,\ldots k$

(b) $\text{var}(\hat{B}) = (X'X)^{-1} X' \text{var}(Y)[(X'X)^{-1} X']'$

$= (X'X)^{-1} X'\sigma^2[(X'X)^{-1} X']'$

$= \sigma^2 (X'X)^{-1}$

$\text{var}(\hat{B}_i) = c_{i1}\sigma^2$ for $0,1,2,\ldots k$

(c) $\text{cov}(\hat{B}) = (X'X)^{-1} X \,\text{cov}(Y)[(X'X)^{-1} X']'$

$= (X'X)^{-1}\sigma^2 I[(X'X)^{-1} X']'$

$= \sigma^2 (x'x)^{-1}$

$\text{cov}(\hat{B}_i, \hat{B}_j) = c_{ij}\sigma^2$ for $i \neq j = 0,1,\ldots k$

14.38 $\hat{\beta}_i - t_{\alpha/2,n-k-1}\hat{\sigma}^2\sqrt{\dfrac{n|c_{ii}|}{n-k-1}} \le \beta_i \le \hat{\beta}_i + t_{\alpha/2,n-k-1}\hat{\sigma}\sqrt{\dfrac{n|c_{ii}|}{n-k-1}}$

14.39 (a) $B'X_0 = (\hat{\alpha}\hat{\beta})(X_0') = \hat{\alpha} + \hat{\beta}x_{01} = \hat{y}_0$

$$(X'X)^{-1}\begin{pmatrix} \dfrac{S_{xx} + n\bar{x}^2}{nS_{xx}} & -\dfrac{\bar{x}}{S_{xx}} \\ -\dfrac{\bar{x}}{S_{xx}} & \dfrac{1}{S_{xx}} \end{pmatrix}$$

$$X_0'(X'X)^{-1} = \frac{S_{xx} + n\bar{x}^2 - nx_0\bar{x}}{nS_{xx}}, \frac{-\bar{x} + x_0}{S_{xx}}$$

$$X_0'(X'X)^{-1} X_0 = \frac{S_{xx} + n\bar{x}^2 - nx_0\bar{x} - nx_0\bar{x} + nx_0^2}{nS_{xx}}$$

$$n[X_0'(X'X)^{-1} X_0] = 1 + \frac{n(x_0 - \bar{x})^2}{S_{xx}}$$

$$t = \frac{(\hat{y}_0 - \mu_{Y|x_0})\sqrt{n-2}}{\hat{\sigma}\sqrt{1 + \dfrac{n(x_0 - \bar{x})^2}{S_{xx}}}}$$

(b) confidence limits are $B'X_0 \pm t_{\alpha/2,n-k-1}\hat{\sigma}\sqrt{\dfrac{n[X_0'(X'X)^{-1} X_0}{n-k-1}}$

14.40 **(a)** From 14.39

$$B'X_0 = \hat{\alpha} + \hat{\beta}X_0$$

$$X_0'(X'X)^{-1}X_0 = \frac{S_{xx} + n(x_0 - \bar{x})^2}{nS_{xx}}$$

$$n[1 + X_0'(X'X)^{-1}X_0] = \frac{nS_{xx} + S_{xx} + n(x_0 - \bar{x})^2}{S_{xx}} = n + 1 + \frac{n(x_0 - \bar{x})^2}{S_{xx}}$$

$$t = \frac{[(y_0 - (\hat{\alpha} + \hat{\beta}x_0)]\sqrt{n-2}}{\hat{\sigma}\left[1 + n\frac{n(x_0 - \bar{x})^2}{S_{xx}}\right]}$$

(b) confidence limits are $B'X_0 \pm t_{\alpha/2, n-k-1}\hat{\sigma}\sqrt{\dfrac{n[1 + X_0'(X'X)^{-1}X_0]}{n-k-1}}$

14.41 **(a)** $n = 5$, $\sum x = 7.69$, $\sum x^2 = 14.0225$, $\sum y = 447.9$, $\sum xy = 697.608$ Thus,
$S_{xx} = 14.0225 - (7.69)^2 / 5 = 2.1953 = 2.1953$ and
$S_{xy} = 697.608 - (7.69)(447.9)/5 = 8.7378.$
Finally, $\hat{\beta} = \dfrac{8.7378}{2.1953} = 3.98$ and $\hat{\alpha} = \dfrac{447.9}{5} - 3.98\dfrac{7.69}{5} = 83.46$.

(b) If x= 1.3, y is estimated as $\hat{y} = 83.46 + (3.98)(1.3) = 88.63.$

14.42 **(a)** $n = 7$, $\sum x = 70$, $\sum x^2 = 812$, $\sum y = 68$, $\sum y^2 = 952$, $\sum xy = 862$

$$S_{xx} = 812 - \frac{1}{7}(70)^2 = 812 - 700 = 112$$

$$S_{xy} = 862 - \frac{1}{7}(70)68 = 862 - 680 = 182$$

$$S_{yy} = 962 - \frac{1}{7}(68)^2 = 962 - 650.5714 = 301.4286$$

$$\hat{\beta} = \frac{182}{112} = 1.625 \qquad \hat{\alpha} = \frac{68}{7} - (1.625)10$$

$$= 9.7143 - 16.25 = -6.5357$$

(a) $\hat{y} = -6.5357 + 1.625x$

(b) $\hat{y} = -6.5357 + 1.625(7) = 4.8393$

14.43 $n = 12$, $\sum x = 854$, $\sum x^2 = 64.222$, $\sum y = 876$, $\sum y^2 = 65{,}850$, $\sum xy = 64{,}346$

$$S_{xx} = 64{,}222 - \frac{1}{12}(854)^2 = 64{,}222 - 60{,}776.333 = 3445.67$$

$$S_{xy} = 64{,}346 - \frac{1}{12}(854)(876) = 64{,}346 - 62{,}342 = 2004$$

$$\hat{\beta} = \frac{2004}{3445.67} = 0.5816 \qquad \hat{\alpha} = 73 - (0.5816)(71.1667) = 31.609$$

(a) $\hat{y} = 31.609 + 0.5816x$

(b) $\hat{y} = 31.609 + 0.5816(84) = 80.45$

14.44 $n = 12,\ \sum x = 507,\ \sum x^2 = 22,265,\ \sum y = 144,\ \sum y^2 = 1802,\ \sum xy = 6314$

$$S_{xx} = 22,265 - \frac{1}{12}(507)^2 = 844.25$$

$$S_{xy} = 6314 - \frac{1}{12}(507)(144) = 230$$

$$\hat{\beta} = \frac{230}{844.25} = 0.2724,\quad \hat{\alpha} = \frac{144}{12} - (0.2724)\frac{507}{12} = 0.4911$$

(a) $\hat{y} = 0.4911 + 0.2724x$

(b) $\hat{y} = 0.4911 + (0.2724)(38) = 10.8423$

14.45 $n = 6,\ \sum x = 42,\ \sum x^2 = 364,\ \sum y = 7.8,\ \sum y^2 = 10.68,\ \sum xy = 48.6$

$$S_{xx} = 364 - \frac{1}{6}(42)^2 = 70,\quad S_{xy} = 48.6 - \frac{1}{6}(42)(7.8) = -6$$

$$\hat{\beta} = \frac{-6}{70} = -0.0857 \text{ and } \hat{\alpha} = \frac{7.8}{6} - (-0.0857)\frac{42}{6} = 1.8999$$

(a) $\hat{y} = 1.8999 - 0.0857x$

(b) $\hat{y} = 1.8999 - 0.0857(5) = 1.4714$

14.46

x	y	$x'y$
−3	1	−3
−2	3	−6
−1	6	−6
0	8	
1	14	14
2	16	32
3	20	60
	68	91

$$\hat{\alpha} = \frac{68}{7} = 9.7143$$

$$\hat{\beta} = \frac{91}{28} = 3.25$$

(a) $\hat{y} = 9.7143 + 3.25x$ (coded)

(b) $\hat{y} = 9.7143 + 3.25(-1.5) = 9.7413 - 4.875$
$= 4.8393$

14.47

x	y	xy
−5	1.8	−9.0
−3	1.5	−4.5
−1	1.4	−1.4
1	1.1	1.1
3	1.1	3.3
5	0.9	4.5
	7.8	−6.0

$$\sum x^2 = 70,\quad \hat{\alpha} = \frac{7.8}{6} = 1.3$$

$$\hat{\beta} = \frac{-6}{70} = -0.0857$$

(a) $\hat{y} = 1.3 - 0.0857x$ (coded)

(b) $\hat{y} = 1.3 - (0.0857)(-2) = 1.4714$

14.48

x	y	xy
−2	1.4	−2.8
−1	2.1	−2.1
0	2.6	
1	3.5	3.5
2	3.7	7.4
	13.3	6.0

$$\sum x^2 = 10,\ \hat{\alpha} = \frac{13.3}{5} = 2.66$$

$$\hat{\beta} = \frac{6}{10} = 0.6$$

$$\hat{y} = 2.66 + 0.6x \ \text{(coded)}$$

Sixth year $\hat{y} = 2.66 + 0.6(3) = 4.46$ million dollars

14.49

x	y	$y' = \log x$	xy'
1	2.0	0.3010	
2	2.4	0.3802	
4	5.1	0.7077	
5	7.3	0.8634	
6	9.4	0.9732	
8	18.3	1.2625	
26		4.4880	24.1484

$$\sum x^2 = 146$$

$$4.4880 = 6\log\hat{a} + 26\log\hat{\beta}$$

$$24.1484 = 25(\log\hat{\alpha}) + 146\log\hat{\beta}$$

$$\log\hat{\alpha} = \frac{\begin{vmatrix} 4.4880 & 26 \\ 24.1484 & 146 \end{vmatrix}}{\begin{vmatrix} 6 & 26 \\ 26 & 146 \end{vmatrix}} = \frac{27.3896}{200}$$

$$= 0.13695 \qquad \hat{\alpha} = 1.371$$

$$\log\hat{\beta} = \frac{\begin{vmatrix} 6 & 4.4880 \\ 26 & 24.1484 \end{vmatrix}}{200} = \frac{28.2024}{200} = 0.1410$$

$$\hat{\beta} = 1.383 \qquad \hat{y} = 1.371(1.383)^x$$

14.50 $x' = \log x$ $\qquad y' = \log y$

x	y	x'	y'
50	108	1.6990	2.0334
100	53	2.0000	1.7243
250	24	2.3679	1.3802
500	9	2.6990	0.9542
1,000	5	3.0000	0.6990

$$n = 5 \qquad \sum x' = 11.7659$$

$$\sum (x')^2 = 28.77815 \qquad \sum y' = 6.7911$$

$$\sum x'y' = 14.8439$$

$$S_{x'x'} = 28.77815 - \frac{1}{5}(11.7659)^2 = 28.77815 - 27.68728 = 1.0909$$

$$S_{x'y'} = 14.8439 - \frac{1}{5}(11.7659)(6.7911) = 14.8439 - 15.9807 = -1.1368$$

$$\hat{\beta} - \frac{-1.1368}{1.0909} = -1.0421 \qquad \log\hat{\alpha} = \frac{6.7911}{5} + (1.0421)\frac{11.7659}{5}$$

$$= 1.3582 + 2.4522 = 3.8104 \qquad \hat{\alpha} = 6{,}460$$

(a) $\hat{y} = 6,450x^{-1.0421}$

(b) $\log \hat{y} = 3.8104 - 1.0421(2.4771) = 3.8104 - 2.5814 = 1.2290$

$\hat{y} = 17.3 \quad (\$17.30)$

Since the calculations in Exercises 14.51 through 14.61 are fairly extensive, answers may differ substantially due to rounding.

14.51 $n = 7,\ \hat{\beta} = 1.625,\ S_{xx} = 112,\ S_{xy} = 182,\ S_{yy} = 301.4286$

1. $H_0: \beta = 1.25,\ H_1: \beta > 1.25,\ \alpha = 0.01$
2. Reject null hypothesis if $t \ge t_{0.01,5} = 3.365$
3. $\hat{\sigma} = \sqrt{\frac{1}{7}[301.4286 - (1.625)182]} = 0.9007$

   $t = \frac{(1.625 - 1.25)}{0.9007}\sqrt{\frac{5(112)}{7}} = (0.4163)(8.9443) = 3.7235$
4. Since $3.7235 > 3.365$, null hypothesis must be rejected.

14.52 $n = 12,\ \hat{\beta} = 0.2724,\ S_{xx} = 844.25,\ S_{xy} = 230$

$S_{yy} = 1802 - \frac{1}{12}(144)^2 = 1802 - 1728 = 74$ from Ex 14.18

$\hat{\sigma} = \sqrt{\frac{1}{12}[74 - (0.2724)230]} = 0.9725$

$t = \frac{0.2724 - 0.350}{0.9725}\sqrt{\frac{10(844.25)}{12}} = -\frac{0.0776}{0.9725}(26.5244) = -2.12$

1. $H_0: \beta = 0.350,\ H_1: \beta < 0.350,\ \alpha = 0.05$
2. Reject null hypothesis if $t \le -t_{0.05,10} = -1.812$
3. $t = -2.12$
4. Since $t = -2.12 < -1.812$, null hypothesis must be rejected.

14.53 $n = 8,\ \sum x = 1447.5,\ \sum x^2 = 264,290.5,\ \sum y = 1864.5,\ \sum y^2 = 439,901.6,$

$SS_{XX} = 264,290.5 - \frac{1}{8}(1447.5)^2 = 2383.469$

$SS_{XY} = 340,915.9 - \frac{1}{8}(1447.5)(1804.5) = 3557.911$

$S_{YY} = 439,901.6 - \frac{1}{8}(1864.5)^2 = 5356.599$

(a) $\hat{\beta} = \frac{3557.911}{2393.469} = 14,927$

$\hat{\alpha} = \frac{1864.5}{8} - (1.4927)\frac{1447.5}{8} = -37.023$

$\hat{y} = -37.023 + 1.4927x$

(b) 1. $H_0: \beta = 1.30,\ \ H_1: \beta > 1.30,\ \ \alpha = 0.05$

2. Reject null hypothesis if $t \ge t_{0.05,6} = 1.943$

3. $\hat{\sigma} = \sqrt{\frac{1}{8}[535.599 - (1.4927)(3557.911)]} = 2.3866$

$$t = \frac{1.4927 - 1.30}{2.3866}\sqrt{\frac{6}{8}(2383.469)} = 3.413$$

4. Since $t = 3.414 > 1.943$, null hypothesis must be rejected.

14.54 $n = 12,\ \ S_{xx} = 3445.67,\ \ S_{xy} = 2004$

$\hat{\beta} = 0.5816$ from Ex. 14.43

$$S_{yy} = 65{,}850 - \frac{1}{12}(876)^2 = 1902$$

$$\hat{\sigma} = \sqrt{\frac{1}{12}[1902 - (0.5816)(2004)]} = 7.8341$$

confidence limits are $0.5816 \pm (3.169)(7.8341)\sqrt{\frac{12}{10(3445.67)}}$

$0.5816 \pm (3.169)(7.8341)(0.01866)$

0.5816 ± 0.4632

$0.1184 < \beta < 1.0448$

14.55 $n = 6,\ \ \hat{\beta} = -0.0857,\ \ S_{xx} = 70,\ \ S_{xy} = -6$

$$S_{yy} = 10.68 - \frac{1}{6}(7.8)^2 = 0.54$$

$$\hat{\sigma} = \sqrt{\frac{1}{6}(0.54 - (-0.0857)(-6)} = 0.06557$$

confidence limits are $-0.0857 \pm (3.747)(0.06557)\sqrt{\frac{6}{4(70)}}$

-0.0857 ± 0.0360

$-0.1217 < \beta < -0.0497$

14.56 $n = 10,\ \ S_{xx} = 376,\ \ S_{xy} = 1305,\ \ \hat{\alpha} = 21.69,\ \ \hat{\beta} = 3.471$

$$S_{yy} = 36{,}562 - \frac{1}{10}(564)^2 = 4752.4$$

1. $H_0: \alpha = 21.50,\ \ H_1: \alpha \ne 21.50,\ \ \alpha = 0.01$

2. Reject null hypothesis if $t \le -3.355$ or $t \ge 3.355$ $\quad (t_{0.05,8})$

3. $\hat{\sigma} = \sqrt{\frac{1}{10}[4752.4 - (3.471)(1305)]} = 4.7196$

$$t = \frac{(21.69 - 21.50)\sqrt{8(376)}}{4.7196\sqrt{376 + 10(37.6)^2}} = 0.0183$$

4. Since $t = 0.0183$ falls between -3.355 and 3.355, null hypothesis cannot be rejected.

14.57 $n = 6, \ \sum x = 9, \ \sum x^2 = 16.94, \ \sum y = 20.9, \ \sum y^2 = 80.47, \ \sum xy = 36.45$

$S_{xx} = 16.94 - \frac{1}{6}(9)^2 = 3.44$

$S_{xy} = 36.45 - \frac{1}{6}(9)(20.9) = 5.1$

$S_{yy} = 80.47 - \frac{1}{6}(20.9)^2 = 7.6683$

(a) $\hat{\beta} = \frac{5.1}{3.44} = 1.4826$ and $\hat{\alpha} = \frac{20.9}{6} - (1.4826)(1.5) - 1.2594$

$\hat{y} = 1.2594 - 1.4826x$

(b) 1. $H_0 : \alpha = 0.08, \ H_1 : \alpha > 0.08, \ \alpha = 0.01$

2. Reject null hypothesis if $t \geq -t_{0.01,4} = 3.747$

3. $\hat{\sigma} = \sqrt{\frac{1}{6}[7.6683 - (1.4826)(5.1)]} = 0.1336$

$$t = \frac{(1.2594 - 0.8)\sqrt{4(3.44)}}{(0.1336)\sqrt{3.44 + 6(1.5)^2}} = 3.10$$

4. Since $t = 3.10$ is less than 3.747, null hypothesis cannot be rejected.

14.58 $n = 7, \ \hat{\alpha} = -6.5357, \ S_{xx} = 112, \ \bar{x} = \frac{70}{7} = 10, \ \hat{\sigma} = 0.9007, \ t_{0.025,5} = 2.571$

$$-6.5357 \pm \frac{(2.571)(0.9007)\sqrt{112 + 7(10)^2}}{\sqrt{5(112)}}$$

$$-6.5367 \pm \frac{(2.3157)(28.4956)}{23.6643}$$

-6.5357 ± 2.7885

$-9.3242 < \alpha < -3.7472$

14.59 $n = 12, \ \hat{\alpha} = 31.609, \ \hat{\beta} = 0.5816, \ \mathrm{S}_{xx} = 3445.67, \ \hat{\sigma} = 7.8341, \ \bar{x} = \frac{854}{12} = 71.1667,$

$t_{0.005,10} = 3.169$

$$31.609 \pm \frac{(3.169)(7.8341)\sqrt{3445.67 + 12(71.1667)^2}}{\sqrt{10(3445.67)}}$$

$$31.609 \pm \frac{(24.8263)(253.4207)}{185.6252}$$

31.609 ± 33.8936

$-2.2846 < \alpha < 65.5026$

14.60 **(a)** $70.284 \pm (2.306)(4.720)\dfrac{\sqrt{1+\dfrac{10(14-10)^3}{376}}}{\sqrt{8}}$

$70.284 \pm (3.8482)\sqrt{1+0.4255}$

$70.284 \pm (3.8482)(1.1939)$

70.284 ± 4.5945

$65.6895 < \mu_{Y|14} < 74.8785$

(b) $70.284 \pm (3.8482)\sqrt{11.4255}$

70.284 ± 13.0075

Limits of prediction are 57.2765 and 83.2915

14.61 $n = 7,\ \ S_{xx} = 112,\ \ \bar{x} = 10,\ \ x_0 = 9,\ \ t_{0.005,5} = 4.032,\ \ \hat{\sigma} = 0.9007$ **,**

$\hat{y}_0 = -6.5357 + 1.625(9) = 8.0893$

(a) $8.0893 \pm \dfrac{(4.032)(0.9007)\sqrt{1+\dfrac{7(9-10)^2}{112}}}{\sqrt{5}}$

$8.0893 \pm (1.6421)\sqrt{1.0625}$

8.0893 ± 1.6741

$6.452 < \mu_{Y|9} < 9.7634$

(b) $8.0893 \pm (1.6241)\sqrt{8.0625}$

8.0893 ± 4.6116

Limits of prediction are 3.4777 and 12.7009

14.62 $\hat{y}_0 = -6.537 + 1.625 \cdot 20 = 25.963$

(a) The confidence limits are $25.963 \pm \dfrac{4.032 \cdot 0.9007\sqrt{1+\dfrac{7(20-10)^2}{112}}}{\sqrt{5}}$ or 25.963 ± 4.373

(b) The limits of prediction are $25.963 \pm \dfrac{4.032 \cdot 0.9007\sqrt{1+7+\dfrac{7(20-10)^2}{112}}}{\sqrt{5}}$ or 25.963 ± 13.709 .

14.63 **(a)** Using MINITAB

MTB> Regress C2 on 1 C1

The regression equation is

C2 = 2.20 + 13.3 C1

(b) We calculate: $\sum x = 45.8 \quad \sum x^2 = 260.46 \quad \sum xy = 3{,}558.42$

$\sum y = 630.0 \quad \sum y^2 = 48{,}735.06$

Therefore, $S_{xx} = 260.46 - (45.8)^2 / 10 = 50.70$

$S_{yy} = 48,735.06 - (630.0)^2 / 10 = 9,045.06$

$S_{xy} = 3,558.42 - (45.8)(630.0) / 10 = 673.02$

The 99% confidence limits for β are

$$\hat{\beta} = t_{\alpha/2,n-2}\hat{\sigma}\sqrt{\frac{n}{(n-2)S_{xx}}} : \text{ numerically, } 13.27 \pm (3.355)(3.38)\sqrt{\frac{10}{(8)(50.70)}}$$

where $t_{0.005,8} = 3.355$ Table IV) and

$$\hat{\sigma} = \sqrt{\frac{1}{10}[9,045.06 - (13.27)(673.02)]} = 3.38$$

Thus, 99% confidence limits for β are 13.27 ± 1.78, or (11.5, 15.1).

14.64 Using MINITAB

MTB> Regress C2 1 C1

The regression equation is

C2 = 1.09 + 0.0131 C1

(b) We calculate: $\sum x = 340 \quad \sum x^2 = 15,500 \quad \sum xy = 573.10$

$\sum y = 13.16 \quad \sum y^2 = 21.9072$

Therefore, $S_{xx} = 15,500 - (340)^2 / 8 = 1,050$

$S_{yy} = 21.9072 - (13.16)^2 / 8 = 0.259$

$S_{xy} = 573.10 - (340)(13.16) / 8 = 13.80$

To test $H_0 : \beta = 0.01;\ H_1 : \beta > 0.01$ we calculate

$$t = \frac{\hat{\beta} - \beta_0}{\hat{\sigma}}\sqrt{\frac{(n-2)S_{xx}}{n}} = \frac{0.013 - 0.010}{0.100}\sqrt{\frac{(6)(1,050)}{8}} = 0.84$$

where $\hat{\sigma} = \sqrt{\frac{1}{8}[0.259 - (0.013)(13.80)]} = 0.100$

Since $t_{0.05,6} = 3.707$, we cannot reject the null hypothesis at the 0.05 level of significance.

14.65 $n = 20, \ \sum x = 688, \ \sum x^2 = 24{,}282, \ \sum y = 703, \ \sum y^2 = 25{,}555, \ \sum xy = 24{,}582$

$$S_{xx} = 24{,}282 - \frac{1}{20}(688)^2 = 24{,}282 - 23{,}677.2 = 614.8$$

$$S_{yy} = 25{,}555 - \frac{1}{20}(703)^2 = 25{,}555 - 24{,}710.45 = 844.55$$

$$S_{xy} = 24{,}582 - \frac{1}{20}(688)(703) = 24{,}582 - 24{,}183.2 = 398.8$$

$$r = \frac{398.8}{\sqrt{(614)(844.55)}} = \frac{398.8}{720.5757} = 0.553$$

$$z = \frac{\sqrt{17}}{2}\ln\frac{1.553}{0.447} = (2.06)(\ln 3.474) = 2.06(1.24530) = 2.565$$

1. $H_0: \rho = 0; \ \ H_1: \rho \neq 0, \ \ \alpha = 0.05$
2. Reject null hypothesis is $z \leq -1.96$ or $z \geq 1.96$
3. $z = \dfrac{\sqrt{17}}{2}\ln\dfrac{1.553}{0.447} = 2.565$
4. Reject null hypothesis; value of r is significant.

14.66 $$\frac{1.553 - 0.447e^{2(1.96)/\sqrt{17}}}{1.553 + 0.447e^{0.951}} \leq \rho \leq \frac{1.553 - 0.447e^{-0.951}}{1.553 + 0.447e^{-0.951}}$$

$$\frac{1.553 - 0.447(2.59)}{1.553 + 0.447(2.59)} \leq \rho \leq \frac{1.553 - 0.447(0.386)}{1.553 + 0.447(0.386)}$$

$$\frac{0.395}{2.711} \leq \rho \leq \frac{1.380}{1.726} \qquad\qquad 0.15 \leq \rho \leq 0.80$$

14.67 $n = 33, \ \sum x = 2550, \ \sum x^2 = 238{,}960, \ \sum y = 861, \ \sum y^2 = 25{,}313, \ \sum xy = 74{,}476$

$$S_{xx} = 238{,}960 - 197.045.45 = 41{,}914.55$$

$$S_{yy} = 25{,}313 - 22{,}464.27 = 2{,}848.73$$

$$S_{xy} = 74{,}476 - 66{,}531.82 = 7{,}944.18$$

$$r = \frac{7944.18}{10927.18} = 0.727$$

1. $H_0: \rho = 0; \ \ H_1: \rho \neq 0, \ \ \alpha = 0.01$
2. Reject null hypothesis is $z \leq -2.575$ or $z \geq 2.575$
3. $z = \dfrac{\sqrt{30}}{2}\ln\dfrac{1.727}{0.273} = (2.739)\ln 6.326 = (2.739)(1.845) = 5.05$
4. Reject null hypothesis; value of r is significant.

14.68 $\dfrac{1.727-(0.273)e^{0.94}}{1.727+(0.273)e^{0.94}} \le \rho \le \dfrac{1.727-(0.273)e^{-0.94}}{1.727+(0.273)e^{-0.94}}$

$\dfrac{1.727-0.699}{1.727+0.699} \le \rho \le \dfrac{1.727-0.107}{1.727-0.107}$

$\dfrac{1.028}{2.426} \le \rho \le \dfrac{1.620}{1.834}$ $\qquad 0.42 \le \rho \le 0.88$

14.69 $\left(1-\dfrac{\beta}{3.471}\right)\dfrac{0.976\sqrt{8}}{\sqrt{1-0.976^2}} = \pm 2.306$

$\left(1-\dfrac{\beta}{3.471}\right)\dfrac{2.7605}{0.2178} = \pm 2.306$

$1-\dfrac{\beta}{3.471} = \pm 0.182$ $\qquad \dfrac{\beta}{3.471} = 1 \pm 0.182$

$2.84 \le \beta \le 4.10$

14.70

x	y
12	27
26	36
0	9
24	25
39	53
1	16
20	32
–4	3
14	24
35	63

$n = 10,\ \sum x = 167,\ \sum x^2 = 4755,\ \sum y = 288,\ \sum y^2 = 11{,}374,$
$\sum xy = 7112$

$S_{xx} = 4755 - \dfrac{1}{10}(167)^2 = 4755 - 2788.9 = 1966.1$

$S_{yy} = 11374 - \dfrac{1}{10}(288)^2 = 11374 - 8294.4 = 3079.6$

$S_{xy} = 7112 - \dfrac{1}{10}(167)(288) = 7112 - 4809.6 = 2302.4$

$r = \dfrac{2302.4}{\sqrt{(1966.1)(3079.6)}} = \dfrac{2302.4}{2460.65} = 0.936$

14.71

	23	28	33	38	43	
23	1					1
28		3	1			4
33		2	5	2		9
38			1	4	1	6
43			1	3		4
48					1	1
	1	5	8	9	2	25

$n = 25$

$\sum xf = 855 \qquad \sum x^2 f = 29{,}855$

$SS_{xx} = 29{,}855 - 29.241 = 614$

$\sum yf = 880 \qquad \sum y^2 f = 31{,}830$

$S_{yy} = 31{,}830 - 30{,}976 = 854$

$\sum xyf = 30{,}655$

$S_{xy} = 30{,}655 - \dfrac{1}{25}(855)(880)$

$= 30{,}655 - 30096 = 559$

$r = \dfrac{559}{\sqrt{(614)(854)}} = \dfrac{559}{724.1} = 0.772$

1. $H_0: \rho = 0;\ H_1: \rho \neq 0,\ \ \alpha = 0.05$
2. Reject null hypothesis is $z \le -1.96$ or $z \ge 1.96$
3. $z = \dfrac{\sqrt{22}}{2} \ln \dfrac{1.772}{0.228} = 4.81 > 1.96$
4. Reject null hypothesis; the value of r is significant.

14.72

	−2	−1	0	1	2	
−2	1					1
−1		3	1			4
0		2	5	2		9
1			1	4	1	6
2			1	3		4
3					1	1
	1	5	8	9	2	

$n = 25,\ \sum x = 6,\ \sum x^2 = 26$

$\sum y = 11,\ \sum y^2 = 39$

$$S_{xx} = 26 - \frac{1}{25}(6)^2 = 26 - 1.44 = 24.56$$

$$S_{yy} = 39 - \frac{1}{25}(11)^2 = 39 - 4.84 = 34.16$$

$$\sum fxy = 4 + 3 + 4 + 6 + 2 + 6 = 25$$

$$S_{xy} = 25 - \frac{1}{25}(6)(11) = 25 - 2.64 = 22.36$$

$$r = \frac{22.36}{\sqrt{(24.56)(34.16)}} = \frac{22.36}{28.9650} = 0.772$$

14.73

		x			
		−1	0	1	
y	−1	63	42	15	120
	0	58	61	31	150
	1	14	47	39	90
		135	150	75	400

$\sum xf = -60,\ \sum x^2 f = 210$

$$S_{xx} = 210 - \frac{1}{360}(-60)^2 = 210 - 10 = 200$$

$\sum yf = -30,\ \sum y^2 f = 210$

$$S_{yy} = 210 - \frac{1}{360}(-30)^2 = 210 - 2.5 = 207.5$$

$$\sum xyf = 63 - 15 - 14 + 29 = 63$$

$$S_{xy} = 63 - \frac{1}{360}(-60)(-30) = 63 - 5 = 58$$

$$r = \frac{58}{\sqrt{200(207.5)}} = \frac{58}{203.7} = 0.285$$

$$z = \frac{\sqrt{357}}{2} \ln \frac{1.285}{0.715} = 9.447 \ln 1.80 = 9.45(0.58779) = 5.55$$

$z = 5.55 > 2.575$ is significant

14.74

		x			
		-1	0	1	
y	-1	67	64	25	156
	0	42	76	56	174
	1	119	23	37	70
		119	163	118	400

$\sum xf = -1, \ \sum x^2 f = 237$

$S_{xx} = 237 - \frac{1}{400}(-1)^2 = 237 - 0 = 237$

$\sum yf = -86, \ \sum y^2 f = 226$

$$S_{yy} = 226 - \frac{1}{400}(-86)^2 = 226 - 18.49 = 207.51$$

$$\sum xyf = 67 - 25 - 10 + 37 = 69$$

$$S_{yy} = 69 - \frac{1}{400}(-1)(-86) = 69 - 0.215 = 68.8$$

$$r = \frac{68.8}{\sqrt{237(207.51)}} = \frac{68.8}{221.8} = 0.31$$

$$z = \frac{\sqrt{397}}{2}\ln\frac{1.31}{0.69} = (9.96)\ln 1.90 = (9.96)(0.64185) = 6.39$$

$z = 6.39 > 1.96 \rightarrow$ significant

14.75 (a) Using the data of Exercise 14.63 and MINITAB:

MTB> Correlate C1 C2
Correlation of C1 and C2 = 0.994

(b) $$z = \frac{\sqrt{n-3}}{2}\cdot\ln\frac{1+r}{1-r} = \frac{\sqrt{10-3}}{2}\cdot\ln\frac{1.994}{0.006} = 7.68$$

Since $z > z_{0.02,5} = 1.96$, we reject the null hypothesis of no correlation.

14.76 (a) Using the data of Exercise 14.64 and MINITAB:

MTB> Correlate C1 C2
Correlation of C1 and C2 = 0.837

(b) $$z = \frac{\sqrt{n-3}}{2}\cdot\ln\frac{1+r}{1-r} = \frac{\sqrt{8-3}}{2}\cdot\ln\frac{1.837}{0.163} = 2.71$$

Since $z > z_{0.005} = 2.575$, we reject the null hypothesis of no correlation.

14.77 (a) $\hat{\beta}_0 = 14.56, \ \hat{\beta}_1 = 30.109, \ \hat{\beta}_2 = 12.16$

$\hat{y} = 14.56 + 30.109x_1 + 12.16x_2$

(b) $\hat{y} = \$101.41$

14.78 (a) $\hat{\beta}_0 = -0.627, \ \hat{\beta}_1 = 0.0972, \ \hat{\beta}_2 = 0.662$

(b) $\hat{y} = 29.05$

14.79 **(a)** $\hat{\beta}_0 = -124.57,\ \hat{\beta}_1 = 1.659,\ \hat{\beta}_2 = 1.439$

(b) $\hat{y} = 63.24$

14.80 $\hat{\beta}_0 = 197.68,\ \hat{\beta}_1 = 37.19,\ \hat{\beta}_2 = -0.120$

$\hat{y} = 197.68 + 37.19x_1 - 0.120x_2;\ \hat{y} = 70.89$

14.81 $\hat{\beta}_0 = 69.73,\ \hat{\beta}_1 = 2.975,\ \hat{\beta}_2 = -11.97$

$\hat{y} = 69.73 + 2.975z_1 - 11.97z_2$ where the $z_1's$ and $z_2's$ are the coded values;

$\hat{y} = 71.2$ (difference due to rounding) $z_1 = 0.5,\ z_2 = 0$

14.82 $\hat{\beta}_0 = -2.33,\ \hat{\beta}_1 = 0.90,\ \hat{\beta}_2 = 1.27,\ \hat{\beta}_3 = 0.90$

$\hat{y} = -2.33 + 0.90x_1 + 1.27x_2 + 0.90x_3$

14.83 $\hat{\beta}_0 = 10.5,\ \hat{\beta}_1 = -2.0,\ \hat{\beta}_2 = 0.2$

$y = 10.5 - 2.0x + 0.2x^2$

$y = 5.95$

14.84 $\hat{\beta}_0 = 384.39,\ \hat{\beta}_1 = -36.00,\ \hat{\beta}_2 = 0.896$

$\hat{y} = 384.39 - 36.00x + 0.896x^2$

14.85 $t = 2.94$; the null hypothesis $\beta_2 = 0$ cannot be rejected. It is worthwhile to fit a parabola.

14.86 $2723 < \hat{\beta}_2 < 10,957$

14.87 $t = 0.16$; null hypothesis cannot be rejected

14.88 $13.7 < \beta_1 < 46.5$

14.89 $t = -4.18$ reject the null hypothesis

14.90 $0.244 < \beta_2 < 1.08$

14.91 $288,650 < \mu_{Y|3,2} < 296,220$

14.92 $292,785 \pm 19,048$, $(273,737 - 311,833)$

14.93 $74.5 < \mu_{Y|2.4,1.2} < 128.3$ (in \$1000)

14.94 101.4 ± 57.4, 44.0 and 158.8 (in \$1000)

14.97 (a) Using MINITAB, we enter the values of y in C1 and $x_1, \ldots x_3$ in C2,...C4.

MTB> Regress C1 on C2 C3 C4

The regression equation is

C1 = $-2.33 + 0.900$ C2 + 1.27 C3 + 0.900 C4

14.98 (a) Using MINITAB, we enter the values of y in C1 and $x_1, \ldots x_3$ in C2,...C4.

The regression equation is

C1 = 2,906 + 5.46 C2 + 20.1 C3 − 120 C4

(b) $\hat{y} = 2{,}906 + 5.46(90.0) + 20.1(65) - 120(20) = 2{,}304$

14.99 (a) Using statistical software to fit the plane, we obtain $\hat{y} = 170 - 1.39x_1 + 6.07x_2$.

(b) $R^2 = 0.367$; the regression equation explains only 36.7% of the variability of y.

(c) A computer-generated plot of the residuals against $\hat{y}$ shows an apparently random pattern.

(d) The correlation of x_1 and x_2 is -0.142, suggesting little or no multicollinearity, (This correlations is not significant at the 0.05 level of significance.

14.100(a) Using statistical software to fit the surface, we obtain

$$\hat{y} = 2{,}097 + 6.34x_1 + 12.9x_2 - 61.5x_3.$$

(b) A computer generated normal-scores plot suggests little departure from normality.

(c) A computer-generated plot of the residuals against $\hat{y}$ shows an apparently random pattern.

(d) The correlations among the independent variables are $r_{x_1x_2} = 0.133,\ r_{x_1x_3} = 0.344,\ r_{x_2x_3} = 0.192$. Since none of them is significant at the 0.05 level of significance, we conclude that there is little or no multicollinearity among the independent variables.

14.101(b) Using statistical software, we find $\hat{y} = 86.9 - 0.904x_1 + 0.508x_2 + 2.06x_2^2$.

(c) The correlations among the independent variables are $r_{x_1x_2} = -0.142,\ r_{x_1x_2^2} = -0.218,\ r_{x_2x_2^2} = 0.421$. Although the correlation between x_2 and x_2^2 is 0.421, a bit high, none of these correlations is significant at the 0.05 level.

(e) The standardized regression equation is

$$\hat{y} = 47.5 - 24.84x_1' + 15.0x_2' + 70.2(x_2')^2$$

(f) A computer generated plot of the residuals seems to be random. It is noted that the residuals are much smaller than those of Exercise 14.99.

14.102(b) Using statistical software, we find

$$\hat{y} = 11{,}024 - 98.2x_1 - 170x_2 + 2.70x_3 + 185x_1x_2.$$

(c) The correlation matrix is:

	x_1	x_2	x_3
x_2	0.133		
x_3	0.344	0.192	
x_1x_2	0.729	0.769	0.325

Standardization is strongly recommended as two of these correlations are high.

(e) The standardized regression equation is

$$\hat{y} = 2,218 - 261x_1' - 192x_2' + 4.2x_3' + 446x_1'x_2'.$$

The multiple correlation coefficient is 0.970, compared to 0.346 for Exercise.

(f) The new correlation matrix is:

	x_1'	x_2'	x_3'
x_2'	0.133		
x_3'	0.344	0.192	
$x_1'x_2'$	−0.515	−0.218	−0.452

Note the reduction in absolute value of the correlation coefficients involving $x_1'x_2'$.