Chapter 4

4.1 **(a)** 0, 1, 4, 9

(b) $h(g_1) = f(0)$; $h(g_2) = f(-1) + f(1)$; $h(g_3) - f(-2) + f(2)$**;** $h(g_4) = f(3)$

(c) $0 \cdot f(0) + 1[f(-1) + f(1)] + 4[f(-2) + f(2)] + 9 \cdot f(3)$

4.2 Replace $\int$ by $\sum$

4.3 Replace $\sum$ by $\int$

4.4 Replace $\int$ by $\sum$

4.5 **(a)** $E(x) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x f(x, y)\, dy\, dx$; $E(x) = \int_{-\infty}^{\infty} x\, g(x)\, dx$

4.6 $$E(x) = (-1)\left(\frac{3}{7}\right) + 0\left(\frac{2}{7}\right) + 1\left(\frac{1}{7}\right) + 3\left(\frac{1}{7}\right) = \frac{1}{7}$$

4.7 $$E(Y) = \frac{1}{8}\int_2^4 (y^2 + y)\, dy = \frac{1}{8}\left[\frac{y^3}{3} + \frac{y^2}{2}\right]_2^4 = \frac{1}{8}\left(\frac{64}{3} + 8 - \frac{8}{3} - 2\right)$$
$$= \frac{1}{8}\left(\frac{56}{3} + 6\right) = \frac{1}{8} \cdot \frac{74}{3} = \frac{37}{12}$$

4.8 $$E(x) = \int_0^1 x^2\, dx + \int_1^2 (2x - x^2)\, dx = \frac{1}{3} + \left[x^2 - \frac{x^2}{3}\right]\Bigg|_1^2 = \frac{1}{3} + 4 - \frac{8}{3} - 1 + \frac{1}{3}$$
$$= 3 - \frac{6}{3} = 1$$

4.9 **(a)** $$E(x) = 0 \cdot \frac{1}{125} + 1 \cdot \frac{12}{125} + 2 \cdot \frac{48}{125} + 3 \cdot \frac{64}{125} = \frac{12 + 96 + 192}{125} = \frac{300}{125} = \frac{12}{5} = 2.4$$

$$E(x^2) = 0 \cdot \frac{1}{125} + 1 \cdot \frac{12}{125} + 4 \cdot \frac{48}{125} + 9 \cdot \frac{64}{125} = \frac{12 + 192 + 576}{125} = \frac{780}{125} = 6.24$$

(b) $$E[(3x+2)^2] = 9E(x^2) + 12E(x) + 4 = 56.16 + 28.8 + 4 = 88.96$$

4.10 **(a)** $E(x) = \int_1^3 \frac{1}{\ln 3}\,dx = \frac{2}{\ln 3},\ E(x^2) = \int_1^3 \frac{x}{\ln 3}\,dx = \frac{4}{\ln 3},\ E(x^3) = \int_1^3 \frac{x^2}{\ln 3}\,dx = \frac{26}{3(\ln 3)}$

(b) $\frac{26}{3(\ln 3)} + \frac{8}{\ln 3} - \frac{6}{\ln 3} + 1 = \frac{32}{3(\ln 3)} + 1$

4.11 **(a)** $E(x) = \int_0^1 \frac{x^2}{2}\,dx + \int_1^2 \frac{x}{2}\,dx + \int_2^3 \frac{3x - x^2}{2}\,dx = \frac{3}{2}$

$$E(x^2) = \int_0^1 \frac{x^3}{2}\,dx + \int_1^2 \frac{x^2}{2}\,dx + \int_2^3 \frac{3x^2 - x^3}{2}\,dx = \frac{8}{3}$$

$$E(x^2 - 5 + 3) = \frac{8}{3} - 5\cdot\frac{3}{2} + 3 = -\frac{11}{6}$$

4.12 $E(x) = 2,\quad E(Y) = \frac{19}{15}$, and $E(2x - Y) = \frac{60 - 19}{15} = \frac{41}{15} = 2\frac{11}{15}$

Marginal distributions

x	0	1	2	3
$g(x)$	1/10	1/5	3/10	2/5

y	0	1	2
h(y)	1/5	1/3	14/30

$$E(x) = \frac{1}{5} + \frac{6}{10} + \frac{6}{5} = \frac{20}{10} = 2$$

$$E(Y) = \frac{1}{3} + \frac{28}{30} = \frac{38}{30} = \frac{19}{15}$$

4.13 $E\left(\frac{x}{y}\right) = \int_0^1\int_0^y \frac{x}{y^2}\,dx\,dy = \int \frac{1}{2}\,dy = \frac{1}{2}$

4.14 $k = \frac{1}{54}$

for x $g(1) = \frac{1}{54}(1+2+2+4+3+6) = \frac{18}{54} = \frac{1}{3}$

$g(2) = \frac{2}{3}$

for y $h(1) = \frac{1}{54}(1+2+2+4) = \frac{1}{6};\ \ h(2) = \frac{1}{3},\ \ h(3) = \frac{1}{2}$

for z $\phi(1) = \frac{1}{3};\ \ \phi(2) = \frac{2}{3}$

$$E(x) = 1\cdot\frac{1}{3} + 2\cdot\frac{2}{3} = \frac{5}{3},\quad E(Y) = 1\cdot\frac{1}{6} + 2\cdot\frac{1}{3} + 3\cdot\frac{1}{2} = \frac{1+4+9}{6} = \frac{14}{6} = \frac{7}{3}$$

$$E(z) = \frac{5}{3},\quad E(u) = \frac{5}{3} + \frac{7}{3} + \frac{5}{3} = \frac{17}{3}$$

4.15 $\int_0^1\int_0^1\int_0^1 \frac{1}{3}(2x+3y+z)(x^2-yz)\,dx\,dy\,dz = \frac{1}{12}$

4.16 $E(2^X) = 2^x\left(\frac{1}{2}\right)^x = 1+1+1+1+\ldots = \infty$

So $E(2^x)$ does not exist.

4.17 $\mu_0 = \int (x-\mu)^0 f(x)\,dx = \int f(x)\,dx = 1$

$\mu_1 = \int (x-\mu)^1 f(x)\,dx = \int xf(x)\,dx - \mu\int f(x)\,dx - \mu - \mu = 0$

4.18 $\mu = (-2)\frac{1}{2} + (2)\frac{1}{2} = 0$, $u_2' = (-2)^2\frac{1}{2} + (2)^2\frac{1}{2} = 4$

$\sigma^2 = 4 - 0^2 = 4$

4.19 $\mu = \int_0^2 \frac{x^2}{2}dx = \frac{x^3}{6}\Big|_0^2 = \frac{4}{3}$, $\mu_2' = \int_0^2 \frac{x^3}{2}dx = \frac{x^4}{8}\Big|_0^2 = 2$

$\sigma^2 = 2 - \frac{16}{9} = \frac{2}{9}$

4.20 $\mu_r' = \frac{1}{\ln 3}\int_1^3 x^{r-1}dx = \frac{1}{\ln 3}\left[\frac{x^r}{r}\right]\Big|_1^3 = \frac{1}{r(\ln 3)}\cdot(3^r - 1) = \frac{3^r-1}{r(\ln 3)}$

$\mu = \frac{2}{\ln 3}$, $\mu_2' = \frac{8}{2(\ln 3)} = \frac{4}{\ln 3}$, $\sigma^2 = \frac{4}{\ln 3} - \frac{4}{(\ln 3)^2} = \frac{4(\ln 3 - 1)}{(\ln 3)^2}$

4.21 $E[ax+b] = aE(x)+b$

$E[(ax+b)^2] = E[(a^2x^2+2abx+b^2) = a^2E(x^2)+2abE(x)+b^2$

$\sigma^2 = a^2E(x^2)+2abE(x)+b^2-a^2[E(x)]^2-2abE(x)-b^2$

$= a^2\sigma^2$ QED

4.22 $\text{var}(2x+3) = 4\ \text{var}(x)$

$\mu = 1$ from Exercise 4.8

$\mu_2' = \int_0^1 x^3dx + \int_1^2 (2x^2-x^3)dx = \frac{1}{4} + \left[\frac{2x^3}{3} - \frac{x^4}{4}\right]\Big|_1^2$

$= \frac{1}{4} - \frac{16}{3} - 4 - \frac{2}{3} + \frac{1}{4} = \frac{7}{6}$, $\sigma^2 = \frac{7}{6} - 1 = \frac{1}{6}$

$\text{var}(2x+3) = 4\cdot\frac{1}{6} = \frac{2}{3}$

4.23 $E(z) = E\left(\frac{x-\mu}{\sigma}\right) = \frac{1}{\sigma}E(x-\mu) = \frac{1}{\sigma}(\mu-\mu) = 0$ exists

$$\text{var}(z) = E\left[\left(\frac{x-\mu}{\sigma}\right)^2\right] = \frac{1}{\sigma^2}E[(x-\mu)^2] = \frac{\sigma^2}{\sigma^2} = 1$$

4.24 $E(x) = \int_1^\infty 2x^{-2}dx = [-2x^{-1}] = \frac{2}{1} = 2$ exists

$\mu_2' = \int_1^\infty \frac{2}{x}dx = 2\ln x\Big|_1^\infty = \infty$ σ^2 does *not* exist

4.25 $\sum(x-\mu)^r f(x) = \sum x^r f(x) - \binom{r}{1}\mu\sum x^{r-1}f(x) + \binom{r}{2}\mu^2\sum x^{r-2}f(x)$

$$\ldots(-1)^{r-1}\mu^{r-1}\sum x\, f(x) + (-1)^r\mu^r\sum f(x)$$

$$= \sum x^r f(x) - \binom{r}{1}\mu\mu_{r-1}' + \binom{r}{2}\mu^2\mu_{r-2}^1 \ldots (-1)^{r-1}(r-1)\mu^r$$

$$\mu_3 = \mu_3' - 3\mu\mu_2' + 3\mu^2\cdot\mu - 1\mu^2 - \mu_3' - 2\mu\mu_2' + 2\mu^3$$

$$\mu_4 = \mu_4' - 4\mu\mu_3' + 6\mu^2\mu_2' - 4\mu^2\cdot\mu_2' + \mu^3 = \mu_4' - 4\mu\mu_2' + 6\mu^2\mu_2' - 3\mu^4$$

4.26 **(a)** $\mu = 1(0.05) + 2(0.15) + 3(0.30) + 4(0.30) + 5(0.15) + 6(0.05) = 3.50$

$$\mu_2' = 1^2(0.05) + 2^2(0.15) + 3^2(0.30) + 4^2(0.30) + 5^2(0.15) + 6^2(0.05) = 13.70$$

$$\mu_3' = 1^3(0.05) + 2^3(0.15) + 3^3(0.30) + 4^3(0.30) + 5^3(0.15) + 6^3(0.05) = 58.10$$

$\sigma^2 = 13.70 - 12.25 = 1.45$ $\mu_2 = 58.10 - 3(3.5)(13.7) + 2(3.5)^2 = 0$

$\alpha_3 = 0$

(b) $\mu = 3.5,\quad \mu_2' = 13.70$, $\mu_3' = 1(0.05) + 2^3(0.20) + \ldots + 6^2(0.05) = 57.8$

$\mu_3 = 57.8 - 3(3.5)(13.7) + 2(3.5)^3 = -0.3$

$$\alpha_3 = \frac{-0.3}{\left(\sqrt{1.45}\right)^3} = \frac{-0.3}{1.746} = -0.172$$

4.27 **(a)** $\mu = 0$ by symmetry, $\mu_2' = 0$ by symmetry

$\mu_2' = 9(0.06) + 4(0.09) + 1(0.10) + 0(0.50) + 1(0.10) + 4(0.09) + 9(0.06) = 2$

$\mu_4' = 81(0.06) + 16(0.09) + 1(0.10) + 0(0.50) + 1(0.10) + 16(0.09) + 81(0.06) = 12.8$

$\sigma^2 = 2$ and $\mu_4 = 12.8$; $\alpha_4 = \dfrac{12.8}{4} = 3.2$

(b) $\mu = 0$ and $\mu_3' = 0$ by symmetry

$\mu_2' = 9(0.04) + 4(0.11) + 1(0.20) + \ldots = 2$

$\mu_4' = 81(0.04) + 16(0.11) + 1(0.20) + \ldots = 10.4$

$\sigma^2 = 2$ and $\mu_4 = 10.4$; $\alpha_4 = \dfrac{10.4}{4} = 2.6$

4.29 $\mu = \int_0^a xf(x)\,dx + \int_a^\infty xf(x)\,dx \ge a\int_a^\infty f(x)\,dx = aP(x \ge a)$

$$\frac{\mu}{a} \ge P(x \ge a) \qquad \text{QED}$$

4.30 $P[(x-\mu)^2 \ge a] \le \dfrac{\sigma^2}{a}$ $\qquad a = k^2\sigma^2$

$P[(x-\mu)^2 \ge k^2\sigma^2] \le \dfrac{1}{k^2}$ or $P[|x = \mu| \ge k\sigma] \le \dfrac{1}{k^2}$

$P[|x-\mu| < k\sigma] \ge 1 - \dfrac{1}{k^2}$

4.31 **(a)** $1 - \dfrac{1}{k^2} = 0.95$, $\dfrac{1}{k^2} = 0.05 = \dfrac{1}{20}$, $k = \sqrt{20} = 4.47$

(b) $1 - \dfrac{1}{k^2} = 0.99$, $\dfrac{1}{k^2} = 0.01 = \dfrac{1}{100}$, $k = 10$

4.32 $P\left(|x-\mu| < k\sigma\right) \ge 1 - \dfrac{1}{k^2}$ $\qquad$ let $k\sigma = c$

$P\left(|x-\mu| < c\right) \ge 1 - \dfrac{\sigma^2}{c^2}$ $\qquad$ Probability is at least $1 - \dfrac{\sigma^2}{c^2}$

4.33 $M_x(t) = \int_0^t \left(e^{tx}\,dx\right) = \dfrac{e^{tx}}{t}\Big|_0^1 = \dfrac{e^t - 1}{t}$

$\dfrac{e^t - 1}{t} = 1 + \dfrac{t}{2!} + \dfrac{t^2}{3!} + \ldots \mu_1' = \dfrac{1}{2}$ and $\mu_2' = \dfrac{1}{3}$

$$\sigma^2 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$$

4.34 $M_x(t) = \sum 2\left(\frac{1}{3}\right)^x e^{tx} = \sum_1^{\infty} 2\left(\frac{e^t}{3}\right)^x = \frac{2\left(\frac{e^t}{3}\right)}{1-\left(\frac{e^t}{3}\right)} = \frac{2e^t}{3-e^t}$

$M'(t) = \frac{(3-e^t)2e^t - 2e^t(-e^t)}{(3-e^t)^2} = \frac{6e^t}{(3-e^t)^2}$

$M''(t) = \frac{(3-e^t)6e^t - 6e^t \cdot 2(3-e^t)(-e^t)}{(3-e^t)^4}$

$M'(0) = \frac{6}{4} = \frac{3}{2}$, $M''(0) = \frac{24 - 12 \cdot 2(-1)}{16} = 3$

$\mu_1' = \frac{3}{2}$ and $\mu_2' = 3$

$\sigma^2 = \mu_2' - (\mu_1')^2 = 3 - \frac{9}{4} = \frac{3}{4}$

4.35 $R_x(t) = \ln M_x(t)$, $R_x'(t) = \frac{1}{M_x(t)} \cdot M_x'(t)$, $R_x'(0) = \frac{M_x'(0)}{M_x(0)} = \frac{M}{1} = \mu$

$R''(t) = \frac{M_x(t) \cdot M_x''(t) - M_x'(t)M'(t)_x}{[M_x(t)]^2}$

$R''(0) = \frac{1 \cdot \mu_2' - \mu^2}{1^2} = \sigma^2$

$R_x(t) = 4(e^t - 1)$, $R_x'(t) = 4e^t$ and $R''(t) = 4e^t$

$\mu = 4$ and $\sigma^2 = 4$

4.36 $M_x(0) = 0 \neq 1$

4.37 $\frac{1}{2}\int_{-\infty}^{0} e^{tx}e^{x}\,dx + \frac{1}{2}\int_{0}^{\infty} e^{tx}e^{-x}\,dx$ $\qquad y = -x$

$cy = -dx$

$$\frac{1}{2}\int_{0}^{\infty} e^{-ty}e^{-y}\,dy + \frac{1}{2}\int_{0}^{\infty} e^{tx}e^{-x}\,dx = \frac{1}{2}\int_{0}^{\infty} e^{-(1+t)y}\,dy + \frac{1}{2}\int_{0}^{\infty} e^{-(1-t)x}\,dx$$

$$= \frac{\frac{1}{2}\left[e^{-(1+t)y}\right]\Big|_0^\infty}{-(1+t)} + \frac{\frac{1}{2}\left[e^{-(1-t)y}\right]\Big|_0^\infty}{-(1-t)} = \frac{1}{2}\left[\frac{1}{1+t} + \frac{1}{1-t}\right]$$

$$= \frac{1/2}{(1+t)(1-t)}[1-t+1+t] = \frac{1}{1-t^2}$$

4.38 $M_x(t) = 1 - t^2 + \frac{t^2}{2!} - \ldots$

(a) $\mu = 0$, $\sigma^2 = 2$

(b) $M_x'(t) = -(1-t^2)^{-2}(-2t) = \frac{2t}{(1-t^2)^2}$

$$M_x''(t) = \frac{(1-t^2)^2 2 - 2 + 2(1-t^2)(-2t)}{(1-t^2)^4} = \frac{2(1-t^2)^2 + 4t^2(1-t^2)}{(1-t^2)^4}$$

$M_x''(0) = 2$, $\sigma^2 = 2$

4.39 3. $M_{(x+a)/b}(t) = \int_{-\infty}^{\infty} e^{[(x+a)/b]t} f(x)\,dx = e^{at/b}\int_{-\infty}^{\infty} e^{xt/b} f(x)\,dx = e^{at/b}\cdot M_x\left(\frac{t}{b}\right)$ QED

1. Let $b = 1$;
2. Let $a = 0$ in above result.

4.40 $z = \frac{1}{4}(x-3)$, $a = -3$, $b = 4$

$M_z(t) = e^{-(3/4)t}\cdot e^{(3/4)t + (8/16)t^2} = e^{(1/2)t^2}$

$M_z(t) = 1 + \frac{1}{2}t^2 + \ldots$ $\qquad \mu = 0$ and $\sigma^2 = 1$

4.42 $(-3,-5)$, $(-1,1)$, $(1,1)$, $(3,5)$ probabilities are ¼

$E(X) = 0,\ E(Y) = 0,\ E(XY) = 15\cdot\frac{1}{4} + 1\cdot\frac{1}{4} + 1\cdot\frac{1}{4} + 15\cdot\frac{1}{4} = 8$

$\text{cov}(X,Y) = 8 - 0\cdot 0 = 8$

4.43 $E(X) = 0 \cdot \frac{56}{120} + 1 \cdot \frac{56}{120} + 2 \cdot \frac{8}{120} = \frac{72}{120} = 0.6$

$E(Y) = 0 \cdot \frac{35}{120} + 1 \cdot \frac{63}{120} + 2 \cdot \frac{21}{120} + 3 \cdot \frac{1}{120} = \frac{108}{120} = 0.9$

$E(XY) = 1 \cdot 1 \cdot \frac{1}{4} + 1 \cdot 2 \cdot \frac{1}{40} + 2 \cdot 1 \cdot \frac{1}{20} = \frac{16}{40} = 0.4$

$\text{cov}(X, Y) = 0.4 - (0.6)(0.9) = 0.4 - 0.54 = -0.14$

4.44
$$E(x_2) = \int_0^1 \int_0^1 \int_0^\infty x_2 (x_1 + x_2) e^{-x_3}\ dx_3\ dx_2\ dx_1 = \int_0^1 \int_0^1 (x_1^2 + x_1 x_2) dx_2\ dx_1$$

$$= \int_0^1 (x_1^2 x_2 + x_1 \frac{x_2^2}{2} \Big|_0^1 dx_1 = \int_0^1 (x_1^2 + \frac{1}{2} x_1) dx_1 = \frac{1}{3} + \frac{1}{4} = \frac{7}{12}$$

$$E(x_3) = \int_0^1 \int_0^1 \int_0^\infty x_3 e^{-x_3} (x_1 + x_2)\ dx_3\ dx_1\ dx_2 = \int_0^1 \int_0^1 (x_1 + x_2) dx_1\ dx_2$$

$$= \int_0^1 \left(\frac{1}{2} + x_2 \right) dx_2 = \frac{1}{2} + \frac{1}{2} = 1$$

$$E(x_2 x_3) = \int_0^\infty x_3 e^{-x_3} dx_3 \int_0^1 \int_0^1 (x_1^2 + x_1 x_2) dx_2\ dx_1$$

$$= \int_0^1 (x_1^2 + \frac{x_1}{2}) dx_2 = \frac{1}{3} + \frac{1}{4} = \frac{7}{12}$$

$$\text{cov}(x_1, x_3) = \frac{7}{12} - \frac{7}{12} \cdot 1 = 0$$

4.45
$$E(X) = \frac{1}{4} \int_0^1 \int_0^2 (2x^2 + xy)\ dy\ dx = \frac{1}{4} \int_0^1 (4x^2 + 2x)\ dx = \frac{1}{4} \left(\frac{4}{3} + 1 \right) = \frac{7}{12}$$

$$E(Y) = \frac{1}{4} \int_0^2 \int_0^1 (2xy + y^2)\ dx\ dy = \frac{1}{4} \int_0^2 (y + y^2)\ dy = \frac{1}{4} \left(2 + \frac{8}{3} \right) = \frac{14}{12}$$

$$E(XY) = \frac{1}{4} \int_0^1 \int_0^2 (2x^2 y + xy^2)\ dy\ dx = \frac{1}{4} \int_0^1 (4x^2 + \frac{8}{3} x)\ dx = \frac{1}{4} \left(\frac{4}{3} + \frac{4}{3} \right) = \frac{2}{3}$$

$$\text{cov}(X, Y) = \frac{2}{3} - \frac{7}{12} \cdot \frac{14}{12} = \frac{2}{3} - \frac{49}{72} = -\frac{1}{72}$$

4.46 **(a)** $f(-1,1)=\frac{1}{4},\ f(0,0)=\frac{1}{6},\ f(0,1)=0,\ f(1,0)=\frac{1}{12},\ f(1,1)=\frac{1}{2}$

$E(X)=-1\left(\frac{1}{4}\right)+0\left(\frac{1}{6}\right)+1\left(\frac{7}{12}\right)=\frac{1}{3}$

$E(Y)=0\left(\frac{1}{4}\right)+1\left(\frac{3}{4}\right)=\frac{3}{4}$

$E(XY)=-\frac{1}{4}+\frac{1}{2}=\frac{1}{4}$

$\text{cov}(X,Y)=\frac{1}{4}-\frac{1}{3}\cdot\frac{3}{4}=0$

(b) $f(0,0)=\frac{1}{6},\ g(0)h(0)=\frac{1}{6}\cdot\frac{1}{4}=\frac{1}{24},\ f(0,0)\neq g(0)h(0)$

4.47 **(a)** $E(U)=\int_{-1}^{0}(x+x^2)\,dx+\int_{0}^{1}(x-x^2)\,dx=-\frac{1}{2}+\frac{1}{3}+\frac{1}{2}-\frac{1}{3}=0$

$E(V)=\int_{-1}^{0}(x^2+x^3)\,dx+\int_{0}^{1}(x^2-x^3)\,dx=-\frac{1}{3}-\frac{1}{4}+\frac{1}{3}-\frac{1}{4}=\frac{2}{3}-\frac{1}{2}=\frac{1}{6}$

$E(UV)=\int_{-1}^{0}(x^3+x^4)\,dx+\int_{0}^{1}(x^3-x^4)\,dx=-\frac{1}{4}+\frac{1}{5}+\frac{1}{4}-\frac{1}{5}=0$

$\text{cov}(U,V)=0-0\cdot\frac{1}{6}=0$

not independent; in fact $V=U^2$.

4.48 **(a)** $\dfrac{\partial\int\ldots\int e^{\Sigma t_i x_i}f(x_1\ldots x_k)dx_1\ldots dx_k}{\partial t_i}$

$=\int\ldots\int x_i e^{\Sigma t_i x_i}f(x_1\ldots x_k)dx_1\ldots dx_k$

at $t_i's=0$

$=\int\ldots\int x_i f(x_1\ldots x_k)dx_1\ldots dx_k=\mu_i$

(b) same

(c) $M_{XY}(t_1,t_2)=\int_0^\infty\int_0^\infty e^{xt_1-x}e^{yt_2-y}\ dx\ dy=\int_0^\infty\int_0^\infty e^{x(t_1-1)}e^{y(t_2-1)}dx\ dy$

$=\frac{1}{t_1-1}\cdot\frac{1}{t_2-1}=\frac{1}{(1-t_1(1-t_2)}$

$\frac{\partial}{\partial t_1}=\frac{1}{(1-t_1)^2}\cdot\frac{1}{(1-t_2)}$

$\frac{\partial^2}{\partial t_1\,\partial t_2}=\frac{1}{(1-t_1)^2}\cdot\frac{1}{(1-t_2)^2}$

$E(X)=1$

$E(Y)=1$

$E(XY)=1$

$\text{cov}(X,Y)=0$

4.49 **(a)** $\mu_Y = 2(4) - 3(9) + 4(3) = -7$

$\sigma_Y^2 = 4(3) + 9(7) + 16(5) = 155$

(b) $\mu_Z = 1(4) + 2(9) - 1(3) = 19$

$\sigma_Z^2 = 1(3) + 4(7) + 1(5) = 36$

4.50 **(a)** $\mu_Y = -7, \quad \sigma_Y^2 = 155 - 12 - 48 + 48 = 143$

(b) $\mu_Z = 19, \quad \sigma_Z^2 = 36 + 4 + 6 + 8 = 54$

4.51 $E(x) = \frac{1}{3}\int_0^1\int_0^2 (x^2 + xy)\, dy\, dx = \frac{1}{3}\int_0^1 (2x^2 + 2x)\, dx = \frac{1}{3}\left(\frac{2}{3} + 1\right) = \frac{5}{9}$

$E(x^2) = \frac{1}{3}\int_0^1\int_0^2 (x^3 + x^2 y)\, dy\, dx = \frac{1}{3}\int_0^1 (2x^3 + 2x^2)\, dx = \frac{1}{3}\left(\frac{1}{2} + \frac{2}{3}\right) = \frac{7}{18}$

$\sigma_X^2 = \frac{7}{18} - \frac{25}{81} = \frac{63 - 50}{162} = \frac{13}{162}$

$E(Y) = \frac{1}{3}\int_0^2\int_0^1 (xy + y^2)\, dx\, dy = \frac{1}{3}\int_0^2 \left(\frac{1}{2}y + y^2\right) dy = \frac{1}{3}\left(1 + \frac{8}{3}\right) = \frac{11}{9}$

$E(Y^2) = \frac{1}{3}\int_0^2\int_0^1 (xy^2 + y^2)\, dx\, dy = \frac{1}{3}\int_0^2 \left(\frac{1}{2}y^2 + y^3\right) dy = \frac{1}{3}\left(\frac{4}{3} + 4\right) = \frac{16}{9}$

$\sigma_Y^2 = \frac{16}{9} - \frac{121}{81} = \frac{144 - 121}{81} = \frac{23}{81}$

$E(XY) = \frac{1}{3}\int_0^1\int_0^2 (x^2 y + xy^2)\, dy\, dx = \frac{1}{3}\int_0^1 \left(2x^2 + \frac{8}{3}x\right) dx = \frac{1}{3}\left(\frac{2}{3} + \frac{4}{3}\right) = \frac{2}{3}$

$\mathrm{cov}(X,Y) = \frac{2}{3} - \frac{5}{9}\cdot\frac{11}{9} = -\frac{1}{81}$

$\mathrm{var}(w) = 9\cdot\frac{13}{162} + 16\cdot\frac{23}{81} + 24\cdot\left(-\frac{1}{81}\right) = \frac{177 + 736 - 48}{162} = \frac{805}{162}$

4.53 $\mathrm{var}(X+Y) = \mathrm{var}(X) + \mathrm{var}(Y) + 2\,\mathrm{cov}(X,Y) \qquad a_1 = 1,\ a_2 = 1$

$\mathrm{var}(X-Y) = \mathrm{var}(X) + \mathrm{var}(Y) - 2\,\mathrm{cov}(X,Y) \qquad b_1 = 1,\ \mathrm{b}_2 = -1$

$\mathrm{cov}(X+Y, X-Y) = \mathrm{var}(X) - \mathrm{var}(Y) + 0\cdot\mathrm{cov}(X,Y) = \mathrm{var}(X) - \mathrm{var}(Y)$

4.54 $\mathrm{cov}(Y_1, Y_2) = (-2)5 + (-6)(4) + 12(7) + 7(3) + (-2)(-2)$

$= -10 - 24 + 84 + 21 + 4 = 75$

$a_1 = 1,\ a_2 = -2,\ a_3 = 3$

$b_1 = -2,\ b_2 = 3,\ b_3 = 4$

4.55 $\mathrm{cov}(Y,Z) = 2\cdot1\cdot3 - 3\cdot2\cdot7 - 4\cdot1\cdot5 = 6 - 42 - 20 = -56$

4.56 $F(-1\big|-1) = \dfrac{1}{5},\ f(1\big|-1) = \dfrac{4}{5}$

$$\mu_{x|-1} = (-1)\frac{1}{5} + 1\cdot\frac{4}{5} = \frac{3}{5}$$

$$\mu_2' = 1 - \frac{1}{5} + 1\cdot\frac{4}{5} = 1 \qquad\qquad \sigma^2_{x|-1} = 1 - \left(\frac{3}{5}\right)^2 = \frac{16}{25}$$

4.57 $f(z\big|1,2) = \phi(1\big|1,2) = \dfrac{1}{3},\ \phi(2\big|1,2) = \dfrac{2}{3}$

$$E(z^2\big|1,2) = 1^2\cdot\frac{1}{3} + 2^2\cdot\frac{2}{3} = \frac{1}{3} + \frac{8}{3} = 3$$

4.58 $f\left(y\big|\ \dfrac{1}{4}\right) = \dfrac{1}{6}(2y+1) \qquad 0 < y < 2)$

$$\mu_{Y|1/4} = \frac{1}{6}\int_0^2 (2y^2 + y)\ dy = \frac{1}{6}\left(\frac{16}{3} + 2\right) = \frac{1}{6}\cdot\frac{22}{3} = \frac{11}{9}$$

$$\mu_2' = \frac{1}{6}\int_0^2 (2y^3 + y^2)\ dy = \frac{1}{6}\left(8 + \frac{8}{3}\right) = \frac{1}{6}\cdot\frac{32}{3} = \frac{16}{9}$$

$$\sigma^2_{Y|1/4} = \frac{16}{9} - \frac{121}{81} = \frac{23}{81}$$

4.59 $f\left(x_2, x_3\big|\ \dfrac{1}{2}\right) = \left(x_2 + \dfrac{1}{2}\right)e^{-x_3} \qquad 0 < x_2 < 1 \text{ and } x_3 > \sigma$

$$E\left(x_2^2 x_3\big|\ \frac{1}{2}\right) = \int_0^1 \left(x_2^3 + \frac{x_2^2}{2}\right) dx_2 \int_0^\infty x_2 e^{-x_3}\ dx_3$$

$$= \left(\frac{1}{4} + \frac{1}{6}\right)\cdot 1 = \frac{5}{12}$$

4.60 **(a)** $f(x\big|a) \le x \le b) = \dfrac{f(x)}{F(b) - F(a)} \qquad a \le x < b)$

$$f(x\big|a \le x \le b) = \int_a^x \frac{f(x)}{F(b) - F(a)}\ dx = \frac{1}{F(b) - F(a)}\cdot F(x) - F(a)$$

$$= \frac{F(x) - F(a)}{F(b) - F(a)} \qquad a < x < b$$

(b) $f(x|a \le x \le b) = \dfrac{f(x)}{F(b)-F(a)}$

$$E[u(x)|a \le x \le b] = \frac{\int_a^b u(x)f(x)\,dx}{F(b)-F(a)} = \frac{\int_a^b u(x)f(x)\,dx}{\int_a^b f(x)\,dx}$$

4.61 **(a)** $E(0) = N(0) = 10^6 - 0.0001 = 100$

(b) $E(p|0) = N(0)P(p|D) = 100 \cdot 0.98 = 98$

(c) $E(p|\bar{D}) = N(\bar{D})P(p|\bar{D}) = 999{,}900 \cdot 0.03 = 29{,}997$

4.62 $3{,}000 \cdot \frac{3}{20} + 1{,}500 \cdot \frac{7}{20} + 0 \cdot \frac{7}{20} - 1{,}500 \cdot \frac{3}{20}$

$= \frac{1}{20}(9{,}000 + 10{,}500 - 4{,}500) = \frac{15{,}000}{20} = \$750$

4.63 $10 \cdot \frac{1}{3} = A \cdot \frac{2}{3},\ A = \$5.00$

4.64 **(a)** $1 \cdot \frac{5}{6} - 0.4 \cdot \frac{1}{6} = \frac{4.6}{6} = \$0.77$

(b) $2 \cdot \frac{4}{6} + 0.6 \cdot \frac{1}{6} - 0.8 \cdot \frac{1}{6} = \frac{7.8}{6} = \$1.30$

(c) $-1.2 \cdot \frac{1}{6} + 0.2 \cdot \frac{1}{6} + 1.6 \cdot \frac{1}{6} + 3 \cdot \frac{3}{6} = \$1.60$

(d) $-1.6 \cdot \frac{1}{6} - 0.2 \cdot \frac{1}{6} + 1.2 \cdot \frac{1}{6} + 2.6 \cdot \frac{1}{6} + 4 \cdot \frac{2}{6} = \$1.67$

(e) $-2 \cdot \frac{1}{6} - 0.6 \cdot \frac{1}{6} + 0.8 \cdot \frac{1}{6} + 2.2 \cdot \frac{1}{6} + 3.6 \cdot \frac{1}{6} + 5 \cdot \frac{1}{6} = \$1.50$

Expected profit is greatest if he bakes *four* cakes.

4.65 $E(x) = \int_{-1}^{5} \frac{x}{18}(x+1)\,dx = \frac{1}{18}\int_{-1}^{5}(x^2+x)\,dx = \frac{1}{18}\left[\frac{x^3}{3} + \frac{x^2}{2}\right]\Bigg|_{-1}^{5}$

$= \frac{1}{18}\left[\frac{125}{3} + \frac{25}{2} + \frac{1}{3} - \frac{1}{2}\right] = \frac{1}{18}\left[\frac{126}{3} + 12\right] = 3 = \$3{,}000$

4.66 $E(x) = \int_0^\infty \frac{x}{30} e^{-x/30} dx = 30$ or $30{,}0000$ kilometers

4.67 $E(x) = \int_0^\infty \frac{x^2}{9} e^{-x/3} dx = \frac{1}{9}\left[-3x^2 e^{-(1/3)x} - 18xe^{-(1/3)x} - 54e^{-(1/3)x}\right]\Big|_0^\infty$

$= \frac{1}{9} \cdot 54 = 6$ million liters

4.68 $E(ps) = \int_{0.2}^{0.2}\int_0^\infty 5p^2 s\ e^{-ps} ds\ dp = \int_{0.2}^{0.4} 5p^2 \cdot \frac{1}{p^2}\left[e^{-ps}(-ps-1)\right]\Big|_0^\infty dp$

$= 5\int_{0.2}^{0.4} dp = 1 = \$10{,}000$

4.69 p = probability Adam will win

$p \cdot b = (1-p)a,\ p(a+b) = a,\ p = \frac{a}{a+b}$

4.70 $\mu = 0 \cdot \frac{6}{11} + 1 \cdot \frac{9}{22} + 2 \cdot \frac{1}{22} = \frac{1}{2}$

$\mu_2' = 1 \cdot \frac{9}{22} + 4 \cdot \frac{1}{22} = \frac{13}{22}$ $\qquad \sigma^2 = \frac{13}{22} - \frac{1}{4} = \frac{26-11}{44} = \frac{15}{44}$

4.71 $\mu = \int_0^\infty \frac{1}{4} e^{-x/4} dx = \frac{1}{4} \cdot \frac{e^{-x/4}}{1/16}\left(-\frac{1}{4}x - 1\right)\Big|_0^\infty = 4$

$\mu_2' = \int_0^\infty \frac{1}{4} x^2 e^{-x/4} dx = \frac{1}{4}\left[-\frac{2}{\left(-\frac{1}{4}\right)^2}\right] = 32$

$\sigma^2 = 32 - 16 = 16$

4.72 $\mu = \frac{1}{288}\int_{-6}^{6} x(36-x^2)dx = \frac{1}{288}\left[18x^2 - \frac{x^4}{4}\right]_{-6}^{6} = 0$

$\mu_2' = \frac{1}{288}\int_{-6}^{6} x^2(36-x^2)dx = \frac{1}{288}\left(12x^3 - \frac{x^5}{5}\right)_{-6}^{6}$

$= \frac{1}{288}\left[12 \cdot 6^3 - \frac{1}{5}6^5 - 12(-6)^3 + \frac{1}{5}(-6)^5\right] = \frac{24 \cdot 6^3}{288} - \frac{2 \cdot 6^5}{288 \cdot 5} = 18 - 10.8 = 7.2$

$\sigma^2 = 7.2$

4.73 $g(0) = 0.4, .g(1) = 0.3,\ g(2) = 0.2,\ g(3) = 0.1$

$\mu = 0(0.4) + 1(0.3) + 2(0.2) + 3(0.1) = 1$ $\qquad \mu = 1$

$\mu_2' = 0^2(0.4) + 1^2(0.3) + 2^2(0.2) + 3^2(0.1) = 2$ $\quad \sigma^2 = 2 - 1^2 = 1$

4.74 **(a)** $P(x \geq 65) \leq \frac{41}{65} = 0.631$

(b) $P[(x-165) \leq 85)] \leq \frac{47}{85} = 0.553$

4.75 $\mu = 124,\ \sigma = 7.5,\ k(7.5) = 60,\ k = \frac{60}{7.5} = 8,\ p = 1 - \frac{1}{64} = \frac{63}{64},$ at least $\frac{63}{64}$

4.76 **(a)** $k = 6$ $\quad 0.260 \pm 6(0.005)$ $\quad$ between 0.230 and 0.290
0.030

(b) $k = 12$ $\quad 0.260 \pm 12(0.005)$ $\quad$ between 0.200 and 0.320
0.060

4.77 $\mu = 4,\ \sigma = 4$ at least $1 - \frac{1}{2.25} = \frac{1.25}{2.25} = \frac{5}{9}$

By Chebyshev's theorem probability $P(x < 10)$ is at least 59.

$$\int_0^{10} \frac{1}{4} e^{-(1/4)x} dx = -e^{-(1/4)x} \Big|_0^{10}$$

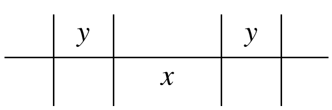
$$= 1 - e^{-2.5} = 1 - 0.0821 = 0.9179$$

4.78

	z	w	p	
(0, 0)	0	0	0.36	$E(0) = 0.60$
(1, 0)	1	1	0.24	$E(z) = 0(0.6) + 1(0.4) = 0.4$
(0, 1)	0	1	0.24	$E(w) = 0(0.36) + 1(0.48) + 2(0.16) = 0.8$
(1, 1)	1	2	0.16	$E(wz) = 0(0.36) + 1(0.24) + 0(0.24) + 2(0.16) = 0.56$

$\text{cov}(z, w) = 0.56 - 0.32 = 0.24$

4.79



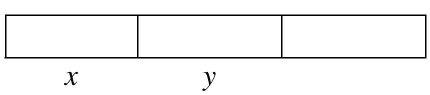
$\mu_x = 3$ $\quad \sigma_x = 0.02$

$\mu_y = 0.3$ $\quad \sigma_y = 0.005$ independent

$E(x + 2Y) = 3 + 2(0.3) = 3.6$

$\sigma^2_{x+2y} = (0.02)^2 + 4(0.005)^2 = 0.0005 \quad \sigma = \sqrt{0.0005} = 0.0224$

4.80



… for x $\quad \mu = 8 \quad \sigma = 0.1$

… for y $\quad \mu = 0.5 \quad \sigma = 0.03$

$z = \sum_{i=1}^{50} x_i + \sum_{j=1}^{49} y_j \quad E(z) = 50(8) + 49(0.5) = 424.5$ in.

$\text{var}(z) = 50(0.1)^2 + 49(0.03)^2 = 0.5441 \qquad \sigma_2 = 0.738$ in.

4.81 **(a)** X heads
Y getting 6
Z getting ace

$$E(X+Y+Z)=\frac{1}{2}+\frac{1}{6}+\frac{1}{13}=\frac{58}{78}=\frac{58}{78}=0.74$$
$$\text{var}(X+Y+Z)=\frac{1}{4}+\frac{5}{36}+\frac{12}{169}=0.46 \qquad \sigma=0.68$$

(b) $3x+2y+z \qquad E=3\cdot\frac{1}{2}+2\cdot\frac{1}{6}+\frac{1}{13}=\frac{117+26+6}{78}=\frac{149}{78}=1.91$

$$\sigma^2=3\cdot\frac{1}{4}+2\cdot\frac{5}{36}+\frac{12}{169}=1.099 \qquad \sigma=1.05$$

4.82 $\mu=5(0.5)+5(0.45)=4.75$
$\sigma^2=5(0.5)(5)+5(0.45)(0.55)=1.25+1.2375=2.4875$
$\sigma=1.58$

4.83 $\phi(0|0)=3/10,\ \phi(1|0)=6/10,\ \phi(2|0)=1/10$
$E(Y)=0(0.3)+1(0.6)+2(0.10)=0.8$

4.84 $\phi(y|12)=\frac{1}{6} \quad 6<y<12 \qquad \int_6^{12}\frac{y}{6}dy=\frac{1}{6}\left(\frac{y^2}{2}\right)\Bigg|_6^{12}=\frac{1}{6}(72-18)=\$9$

4.85 $$E=\frac{\int_1^\infty x\,f(x)\ dx}{\int_1^\infty f(x)\ dx}=\frac{N}{D}$$

$$N=\int_1^2\frac{x^2}{4}dx+\int_2^\infty\frac{4}{x^2}dx=\frac{x^3}{12}\Bigg|_1^2+\frac{-4}{x}\Bigg|_2^\infty=\frac{7}{12}+2=\frac{31}{12}$$

$$D=\int_1^2\frac{x}{4}dx+\int_2^\infty 4x^{-2}dx=\frac{x^2}{8}\Bigg|_1^2-\frac{2}{x^2}\Bigg|_2^\infty=\frac{1}{2}-\frac{1}{8}+\frac{1}{2}=\frac{7}{8}$$

$$E=\frac{31}{12}\cdot\frac{8}{7}=\frac{248}{84}=2.95\text{ min}$$