Chapter 10

10.1 $E\left[\sum a_i x_i\right] = \sum a_i E(x_i) = \sum a_i \mu = \mu \sum a_i$

$$\therefore \sum_{i=1}^{n} a_i = 1$$

10.2 $E[k_1\hat{\theta}_1 + k_2\hat{\theta}_2] = k_1\theta + k_2\theta = \theta,\ \ k_1 + k_2 = 1$

10.3 $h(\tilde{x}) = \dfrac{(2m+1)!}{m!\ m!}\left[\displaystyle\int_{-\infty}^{\tilde{x}} f(x)\ dx\right]^m f(\tilde{x})\left[\displaystyle\int_{\tilde{x}}^{\infty} f(x)\ dx\right]^m$

$$h(\tilde{x}) = \frac{(2m+1)!}{m!\ m!}\left[\int_{\theta-(1/2)}^{\tilde{x}} dx\right]^m \cdot 1 \cdot \left[\int_{\tilde{x}}^{\theta+(1/2)} dx\right]^m$$

$$= \frac{(2m+1)!}{m!m!}\left(\tilde{x} - \theta + \frac{1}{2}\right)^m \left(\theta + {}_{2} - \tilde{x}\right)^m \qquad\qquad m = 1$$

$$h(\tilde{x}) = 6(\left(\tilde{x} - \theta + \frac{1}{2}\right)\left(\theta + \frac{1}{2} - \tilde{x}\right)$$

$$E(\tilde{x}) = 6\int_{\theta-(1/2)}^{\theta+(1/2)} \tilde{x}\left(\tilde{x} - \theta + \frac{1}{2}\right)\left(\theta + \frac{1}{2} - \tilde{x}\right)d\tilde{x}$$

let $u = \tilde{x} - \theta + \dfrac{1}{2}$

$$= 6\int_0^1 \left(u + \theta - \frac{1}{2}\right)u(1-u)du = \theta$$

10.4 $h(\bar{x}) = \dfrac{6}{8} e^{-2\bar{x}/\theta}\left[1 - e^{-\bar{x}/\theta}\right]$

$$E[\bar{x}] = \frac{6}{\theta}\int_0^\infty \tilde{x}\ e^{-2\tilde{x}/\theta}\left[1 - e^{-\tilde{x}/\theta}\right]d\tilde{x}$$

$$= \frac{6}{\theta}\int_0^\infty \tilde{x}\ e^{-2\tilde{x}/\theta}d\tilde{x} - \frac{6}{\theta}\int_0^\infty \tilde{x}\ e^{-3\tilde{x}/\theta}d\tilde{x}$$

$$= \frac{5}{6}\theta \qquad \therefore \text{ biased}$$

Use gamma integrals.

10.5 $$E\left[\frac{1}{n}\sum(x_i-\mu)^2\right]=\frac{1}{n}\left[\sum_{i=1}^{n}E\left[(x_i-\mu\right]^2\right]$$
$$=\frac{1}{n}\sum_{i=1}^{n}\sigma^2=\frac{1}{n}\cdot n\sigma^2=\sigma^2$$

10.6 $E(\overline{x})=\mu \qquad \text{var}(\overline{x})=\dfrac{\sigma^2}{n}$

$$E(\overline{x}^2)=\frac{\sigma^2}{n}+\mu^2 \ \rightarrow\ \mu^2 \ \text{ as } n\rightarrow\infty$$

10.7 $$E\left(\frac{x+1}{n+2}\right)=\frac{1}{n+2}E(x+1)=\frac{1}{n+2}(n\theta+1)=\frac{n}{n+2}\theta+\frac{1}{n+2}$$
$E\rightarrow\theta$ when $n\rightarrow\infty$, so is asymptotically unbiased

10.8 $$g_1(y_1)=n\ e^{-(y_1-\delta)}\left[\int_{y_1}^{\infty}e^{-(x-\delta)}\ dx\right]^{n-1}$$
$$=n\ e^{-(y_1-\delta)}\cdot e^{-(n-1)(y_2-\delta)}$$
$$=n\ e^{-n(y_1-\delta)}$$
$$E(y_1)=n\int_{\delta}^{\infty}y_1\ e^{-n(y_1-\delta)}\ dy_1 \qquad \text{let } u=y_1-\delta$$
$$=n\int_{0}^{\infty}(u+\delta)e^{-nu}du=\frac{1}{n}+\delta$$

The unbiased estimate is $Y_1-\dfrac{1}{n}$ $\qquad E(Y_1)\rightarrow\delta$ as $n\rightarrow\infty$

10.9 $$g_1(y_1)=n\cdot\frac{1}{\beta}\left[\int_{y_1}^{\beta}\frac{1}{\beta}\ dx\right]^{n-1}=\frac{n}{\beta^n}(\beta-y_1)^{n-1}$$
$$E(Y_1)=\frac{n}{\beta^n}\int_{0}^{\beta}y_1(\beta-y_1)^{n-1}dy_1 \qquad u=\frac{y_1}{\beta} \qquad du=\frac{dy_1}{\beta}$$
$$=\frac{b}{\beta^n}\int_{0}^{1}\beta u(\beta-\beta u)^{n-1}\beta\ du=n\beta\int_{0}^{1}u(1-u)^{n-1}du=\frac{\beta}{n+1}$$
Unbiased estimate is $(n+1)Y_1$

10.10 $$E\left[\sum_{i=1}^{n}\frac{x_i^2}{n}\right]=\frac{1}{n}\sum_{i=1}^{n}E\left(x_i^2\right)$$
$$=\frac{1}{n}\sum_{i=1}^{n}(\sigma^2+\mu^2)=\frac{1}{n}\sum_{i=1}^{n}\sigma^2=\sigma^2$$

10.11 $E\left[n\cdot\frac{x}{n}\cdot\left(1-\frac{x}{n}\right)\right]=E(x)-\frac{1}{n}E(x^2)$

$$=n\theta-\frac{1}{n}\left[n\theta(1-\theta)+n^2\theta^2\right]$$
$$=(n-1)\theta(1-\theta)\neq n\theta(1-\theta) \qquad \text{biased}$$

10.12 (a) $n-1$ values before y_n in $\binom{y_n-1}{n-1}$ ways.

$$f(y_n)=\frac{\binom{y_n-1}{n-1}}{\binom{k}{n}} \qquad \text{for } y_n=n,\ldots,k$$

(b) $E(Y_n)=\sum_{y_n=n}^{k} y_n\cdot\frac{\binom{y_n-1}{n-1}}{\binom{k}{n}}=\frac{n}{\binom{k}{n}}\sum_{y_n=n}^{k}\binom{y_n}{n}=\frac{n}{\binom{k}{n}}\binom{k+1}{n+1}$

$$=\frac{n(k+1)}{n+1} \qquad \text{see Exercise 1.15 or Theorem 1.11, respectively}$$

$$E\left[\frac{n+1}{n}\cdot Y_n-1\right]=\frac{n+1}{n}\cdot\frac{n(k+1)}{n+1}-1=k \qquad \text{QED}$$

10.13 $E(\hat{\theta}^2)=\text{var}(\hat{\theta})+E(\hat{\theta})^2=\text{var}(\hat{\theta})+\theta^2$
$E(\tilde{\theta}^2)>\theta^2$ since $\text{var}(\tilde{\theta})>0$

10.14 $f(x;\theta)=\theta^x(1-\theta)^{1-x} \qquad E(x)=\theta \qquad E(x^2)=\theta$

$\ln f(x;\theta)=x\ln\theta+(1-x)\ln(1-\theta)$

$$\frac{\partial\ln f(x;\theta)}{\partial\theta}=\frac{x}{\theta}-\frac{1-x}{1-\theta}=\frac{x-\theta}{\theta(1-\theta)}$$

$$E\left[\left(\frac{\partial\ln f(x;\theta)^2}{\partial\theta}\right)\right]=\frac{1}{\theta^2(1-\theta)^2}E(x-\theta)^2=\frac{1}{\theta(1-\theta)}$$

$\frac{1}{n\cdot E}=\frac{\theta(1-\theta)}{n}=\text{var}\left(\frac{x}{u}\right)$ when x is binomial random variable.

$\therefore \frac{x}{n}$ is minimum variance estimator

$$E\left(\frac{x}{n}\right)=\frac{n\theta}{n}=\theta$$

$\therefore$ unbiased

10.15 $f(x;\lambda)=\dfrac{\lambda^x e^{-\lambda}}{x!}$ $\quad\mu=\lambda\quad\sigma^2=\lambda\quad\text{var}(\overline{x})=\dfrac{\lambda}{n}$

$E(\overline{x})=\lambda\rightarrow$ unbiased

$\ln f=x\ln\lambda-\lambda-\ln x!$

$$\frac{\partial\ln f}{\partial\lambda}=\frac{x}{\lambda}-1\qquad E\left[\left(\frac{\partial\ln f}{\partial\lambda}\right)^2\right]=\frac{E(x^2)}{\lambda^2}-\frac{2}{\lambda}E(x)+1$$

$$=\frac{\lambda+\lambda^2}{\lambda^2}-\frac{2}{\lambda}\lambda+1=\frac{1}{\lambda}$$

$$\frac{1}{nE}=\frac{\lambda}{n}=\text{var}(\overline{x})$$

$\therefore\ \overline{x}$ is minimum variance unbiased estimator

10.16 $\text{var}(\hat{\theta}_1)=3\,\text{var}(\hat{\theta}_2)$

$E(a_1\hat{\theta}_1+a_2\hat{\theta}_2)=a_1\theta+a_2\theta=\theta\rightarrow a_1+a_2=1$

$\text{var}=a_i^2\,\text{var}(\hat{\theta}_1)+a_2^2\,\text{var}(\hat{\theta}_2)$

$\text{var}=3a_i^2\,\text{var}(\hat{\theta}_2)+a_2^2\,\text{var}(\hat{\theta}_2)=(3a_1^2+a_2^2)\,\text{var}(\hat{\theta}_2)$

$=[3a_1^2+(1-a_1)^2]\,\text{var}(\hat{\theta}_2)$

$$\frac{\partial}{\partial a_1}=6a_2+2(1-a_1)(-1)$$

$$=8a_1-2=0\qquad a_1=\frac{1}{4}\quad a_2=\frac{3}{4}$$

10.17 $f(x;\theta)=\dfrac{1}{\theta}e^{-x/\theta}\quad E(x)=\theta\quad E(x^2)=2\theta^2\quad\sigma^2=\theta^2$

$E(\overline{x})=\theta\rightarrow$ unbiased $\quad\text{var}(\overline{x})=\dfrac{\theta^2}{n}$

$$\ln f=-\ln\theta-\frac{x}{\theta}$$

$$\frac{\partial\ln f}{\partial\theta}=-\frac{1}{\theta}+\frac{x}{\theta^2}=\frac{x-\theta}{\theta^2}$$

$$E\left[\left(\frac{\partial\ln f}{\partial\theta}\right)^2\right]=\frac{1}{\theta^4}E(x-\theta)^2-\frac{1}{\theta^2}$$

$\dfrac{1}{nE}=\dfrac{\theta^2}{n}=\text{var}(\overline{x})\quad\therefore\ \overline{x}$ is minimum variance unbiased estimator

10.18 $E(Y_n)=\dfrac{n}{n+1}\beta,\ E(Y_n^2)=\dfrac{n\beta^2}{n+2},\ \text{var}(Y_n)=\dfrac{n\beta^2}{(n+2)(n+1)^2}$

let $B=\dfrac{n+1}{n}\cdot Y_n$

$E(B)=\dfrac{n+1}{n}\cdot\dfrac{n}{n+1}\cdot\beta=\beta\rightarrow$ unbiased

$$\operatorname{var}(B) = \frac{(n+1)^2}{n^2} \cdot \frac{n\beta^2}{(n+2)(n+1)^2} = \frac{\beta^2}{n(n+2)}$$

$$\frac{1}{nE\left(\frac{\partial \ln f(X)}{\partial \beta}\right)} = \frac{1}{n\frac{1}{\beta^2}} = \frac{\beta^2}{n} > \frac{\beta^2}{n(n+2)} = \operatorname{var}(B)$$

so the Cramèr-Rao inequality is not satisfied.

10.19 (a) $$\frac{\partial \ln f(x)}{\partial \theta} = \frac{1}{f(x)} \frac{\partial f(x)}{\partial \theta} \qquad\qquad \frac{\partial f(x)}{\partial \theta} = \frac{\partial \ln f(x)}{\partial \theta} \cdot f(x)$$

$$\therefore \int \frac{\partial \ln f(x)}{\partial \theta} \cdot f(x)dx = 0$$

(b) $$\frac{\partial^2 \ln f(x)}{\partial \theta^2} \cdot f(x) + \frac{\partial \ln f(x)}{\partial \theta} \cdot \frac{\partial \ln f(x)}{\partial \theta} \cdot f(x)$$

$$\int \frac{\partial^2 \ln f(x)}{\partial \theta^2} \cdot f(x)dx = -\int \left[\frac{\partial \ln f(x)}{\partial \theta}\right]^2 f(x)dx$$

$$E\left[\left(\frac{\partial \ln f(x)}{\partial \theta}\right)^2\right] = -E\left[\left(\frac{\partial \ln f(x)}{\partial \theta}\right)\right]^2$$

10.20 $$\frac{\partial \ln f(x)}{\partial \mu} = \frac{1}{\sigma}\left(\frac{x-\mu}{\sigma}\right) \quad \text{from Example 10.5}$$

$$\frac{\partial^2 \ln f(x)}{\partial \mu^2} = -\frac{1}{\sigma^2}$$

$$\frac{1}{nE\left[\left(\frac{\partial \ln f(x)}{\partial \mu}\right)^2\right]} = \frac{1}{n\left(-\frac{-1}{\sigma^2}\right)} = \frac{\sigma^2}{n}$$

10.21 (a) $$E[w\bar{x}_1 + (1-w)\bar{x}_2] = w\mu + (1-w)\mu = \mu$$

(b) $$\operatorname{var}[w\bar{x}_1 + (1-w)\bar{x}_2] = w^2\frac{\sigma_1^2}{n} + (1-w)^2\frac{\sigma_2^2}{n}$$

$$\frac{d}{dw} = 2w\frac{\sigma_1^2}{n} + 2(1-w)(-1)\frac{\sigma_2^2}{n} = 0$$

$$w\left(\sigma_1^2 + \sigma_2^2\right) = \sigma_2^2 \qquad\qquad w = \frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2}$$

10.22 $$\operatorname{var} 1 = w^2\frac{\sigma_1^2}{n} + (1-w)^2\frac{\sigma_2^2}{n}$$

$$w = \frac{1}{2} \qquad\qquad \operatorname{var} = \frac{\sigma_1^2}{4n} + \frac{\sigma_2^2}{4n} = \frac{1}{4n}(\sigma_1^2 + \sigma_2^2)$$

$$\text{var}\,2 = \left(\frac{\sigma_2^2}{\sigma_1^2+\sigma_2^2}\right)^2 \frac{\sigma_1^2}{n} + \left(\frac{\sigma_1^2}{\sigma_1^2+\sigma_2^2}\right)^2 \frac{\sigma_2^2}{n}$$

$$= \frac{\sigma_1^2\sigma_2^2}{n}\left[\frac{\sigma_2^2}{\left(\sigma_1^2+\sigma_2^2\right)} + \frac{\sigma_1^2}{\left(\sigma_1^2+\sigma_2^2\right)^2}\right] = \frac{\sigma_1^2\sigma_2^2}{n(\sigma_1^2+\sigma_2^2)}$$

$$\text{efficiency} = \frac{\dfrac{\sigma_1^2\cdot\sigma_2^2}{n(\sigma_1^2+\sigma_2^2)^2}}{\dfrac{1}{4n}(\sigma_1^2+\sigma_2^2)} = \frac{\sigma_1^2\sigma_2^2}{n(\sigma_1^2+\sigma_2^2)}\cdot\frac{4n}{\sigma_1^2+\sigma_2^2}$$

$$= \frac{4\sigma_1^2\sigma_2^2}{(\sigma_1^2+\sigma_2^2)^2}$$

10.23 $\text{var} = w^2\dfrac{\sigma^2}{n_1} + (1-w)^2\dfrac{\sigma^2}{n_2}$

$$\frac{d}{dw} = \frac{2w\sigma^2}{n_1} - \frac{2(1-w)\sigma^2}{n_2} = 0$$

$$\frac{w}{n_1} = \frac{1-w}{n_2} \qquad w = \frac{n_1}{n_1+n_2}$$

10.24 For $w = \dfrac{1}{2}$ $\qquad \text{var} = \dfrac{1}{4}\cdot\dfrac{\sigma^2}{n_1} + \dfrac{1}{4}\dfrac{\sigma^2}{n_2} = \dfrac{\sigma^2}{4}\left(\dfrac{1}{n_1}+\dfrac{1}{n_2}\right)$

For $w = \dfrac{n_1}{n_1+n_2}$ $\qquad \text{var} = \left(\dfrac{n_1}{n_1+n_2}\right)^2\dfrac{\sigma^2}{n_1} + \left(\dfrac{n_2}{n_1+n_2}\right)^2\dfrac{\sigma^2}{n_2}$

$$= \frac{\sigma^2}{(n_1+n_2)^2}(n_1+n_2) = \frac{\sigma^2}{n_1+n_2}$$

$$\text{Efficiency} = \frac{\dfrac{\sigma^2}{n_1+n_2}}{\dfrac{\sigma^2}{4}\left(\dfrac{1}{n_1}+\dfrac{1}{n_2}\right)} = \frac{4n_1n_2}{(n_1+n_2)^2}$$

10.25 $\text{var}\left(\dfrac{x_1+2x_2+x_3}{4}\right) = \dfrac{1}{16}\sigma^2 + \dfrac{1}{4}\sigma^2 + \dfrac{1}{16}\sigma^2 = \dfrac{3}{8}\sigma^2 \qquad \text{var}(\overline{x}) = \dfrac{\sigma^2}{3}$

$$\text{Efficiency} = \frac{\dfrac{\sigma^2}{3}}{\dfrac{3}{8}\sigma^2} = \frac{8}{9}$$

10.26 $\mu = \theta$ and $\sigma^2 = \theta^2$ $\quad$ $\text{var}(\overline{x}) = \dfrac{\theta^2}{2}$

From Ex. 8.4 $\quad g_1(y_1) = \dfrac{2}{\theta} e^{-2y_1/\theta}$ $\quad$ for $y_1 > 0$

$\text{var}(Y_1) = \left(\dfrac{\theta}{2}\right)$ $\quad E(2Y_1) = \theta$ unbiased

$\text{var}(Y_1) = \left(\dfrac{\theta}{2}\right)^2 = \dfrac{\theta^2}{4}$ $\quad \text{var}(2Y_1) = 4 \cdot \dfrac{\theta^2}{4} = \theta^2$

Efficiency $= \dfrac{\theta^2 / 2}{\theta^2} = \dfrac{1}{2}$

10.27 $g_n(y_n) = \dfrac{n}{\beta^n} y_n^{n-1}$

$$E(Y_n) = \frac{n}{\beta^n} \int y_n^n dy_n \qquad 0 < y_n < \beta$$

$$= \frac{n}{\beta^n} \cdot \frac{\beta^{n+1}}{n+1} = \frac{\beta n}{n+1}$$

$$E(Y_n)^2 = \frac{n}{\beta^n} \int_0^\beta y_n^{n+1} dy_n = \frac{n}{\beta^n} \cdot \frac{\beta^{n+2}}{n+2} = \frac{n\beta^2}{n+2}$$

$$\text{var}(Y_n) = \frac{n\beta^2}{n+2} - \frac{n^2\beta^2}{(n+1)^2} = \frac{\beta^2\left[n(n+1)^2 - n^2(n+2)\right]}{(n+2)(n+1)^2}$$

$$= \frac{n\beta^2}{(n+2)(n+1)^2}$$

$Z = Y_n \cdot \dfrac{n+1}{n}$ $\quad E(Z) = \dfrac{n+1}{n} \cdot \dfrac{n\beta}{n+1} = \beta$ unbiased

$$\text{var}(Z) = \left(\frac{n+1}{n}\right)^2 \cdot \frac{n\beta^2}{(n+2)(n+1)^2} = \frac{\beta^2}{n(n+2)} \quad \text{QED}$$

10.28 $Y = \overline{x} - 1$ $\quad \text{var}(Y) = \text{var}(\overline{x}) = \dfrac{\theta^2}{n} = \dfrac{1}{n}$

$Z = Y_1 - \dfrac{1}{n}$ $\quad g_1(y_1) = ne^{-n(y_1 - \delta)}$

$$E(Y_1) = \frac{1}{n} + \delta$$

$$E(Y_1^2) = n\int_\delta^\infty y_1^2 e^{-n(y_1-\delta)} dy_1 \qquad u = y_1 - \delta$$

$$= n\int_0^\infty (u+\delta)^2 e^{-nu} du = \frac{2}{n^2} + \frac{2\delta}{n} + \delta^2$$

$$\mathrm{var}(Y_1) = \frac{2}{n^2} + \frac{2\delta}{n} + \delta^2 - \left(\frac{1}{n} + \delta\right)^2 = \frac{1}{n^2}$$

$$\text{efficiency} = \frac{\mathrm{var}(Z)}{\mathrm{var}(Y)} = \frac{\left(\frac{1}{n}\right)^2}{\frac{1}{n}} = \frac{1}{n}$$

10.29 Continue from Exercise 10.12

$$E\left[Y_n(Y_n+1)\right] = \frac{1}{\binom{k}{n}} \sum_{y_n=n}^{k} y_n(y_n+1)\binom{y_n-1}{n-1} = \frac{n(n+1)}{\binom{k}{n}} \sum_{y_n=n}^{k} \binom{y_n+1}{n+1}$$

$$= \frac{n(n+1)}{\binom{k}{n}} \cdot \binom{k+2}{n+2} \qquad \text{Exerxise 1.15 or } \sum_{i=n}^{k}\binom{i}{n} = \binom{k+1}{n+1}$$

$$= \frac{n(k+1)(k+2)}{n+2}$$

$$\mathrm{var}(Y_n) = \frac{n(k+1)(k+2)}{n+2} - E(Y_n^2) - E(Y_n)^2$$

$$= \frac{n(k+1)(k+2)}{n+2} - \frac{n(k+1)}{n+1} - \frac{n^2(k+1)^2}{(n+1)^2}$$

$$\mathrm{var}(\left(\frac{n+1}{n}\cdot Y_n - 1\right) = \frac{(n+1)^2}{n^2}\mathrm{var}(Y_n)$$

$$= \frac{(k+1)\left[(k+2)(n+1)^2 - (n+1)(n+2) - (k+1)n(n+2)\right]}{n(n+2)}$$

$$= \frac{(k+1)(k-n)}{n(n+2)}$$

$$E(x) = \frac{k+1}{2},\ E(x^2) = \frac{(k+1)(2k+1)}{6},\ \sigma^2 = \frac{k^2-1}{12}$$

for population

$$E(2\overline{x}-1) = 2E(\overline{x}) - 1 = 2\frac{(k+1)}{2} - 1 = k \qquad \text{unbiased}$$

$$\mathrm{var}(\overline{x}) = \frac{(k^2-1)}{12n}\cdot\frac{k-n}{k-1} = \frac{(k+1)(k-n)}{12n}$$

$$\mathrm{var}(2\overline{x}-1) = \frac{(k+1)(k-n)}{3n},\ \text{efficiency } = \frac{\frac{(k+1)(k-n)}{n(n+2)}}{\frac{(k+1)(k-n)}{3n}} = \frac{3}{n+2}$$

(a) efficiency = $\frac{3}{4}$; **(b)** efficiency = $\frac{3}{5}$

10.30 (a) $E(x)=\int_0^1 x\ dx=\frac{1}{2},\ E(x^2)=\int_0^1 x^2\ dx=\frac{1}{3},\ \text{var}(x)=\frac{1}{3}-\frac{1}{4}=\frac{1}{12}$

$\text{var}(\overline{x})=\frac{1/12}{3}=\frac{1}{36}$

(b) $g_1(y_1)=3(1-y_1)^2 \qquad 0<y_1<1$

$g_3(y_3)=3y_3^2 \qquad 0<y_3<1$

$f(y_1,y_3)=6(y_3-y_1) \qquad 0<y_1<y_3<1$

$E(Y_1)=3\int_0^1 y_1(1-y_1)^2 dy_1=\frac{1}{4}$

$E(Y_1^2)=3\int_0^1 y_1^2(1-y_1)^2 dy_1=\frac{1}{10}$

$E(Y_3)=3\int_0^1 y_3^2\ dy_3=\frac{3}{4},\ E(Y_3^2)=3\int_0^1 y_3^4\ dy_3=\frac{3}{5}$

$E(Y_1Y_3)=b\int_0^1\int_0^{y_3} y_1y_3(y_3-y_1)\ dy_1\ dy_3=\frac{1}{5}$

$\text{var}(Y_1)=\frac{1}{10}-\frac{1}{16}=\frac{3}{80},\quad \text{var}(Y_3)=\frac{3}{5}-\frac{9}{16}=\frac{3}{80}$

$\text{cov}(Y_1,Y_3)=\frac{1}{5}-\frac{3}{16}=\frac{1}{80}$

(c) $E\left(\frac{Y_1+Y_3}{2}\right)=\frac{1}{2}\left(\frac{1}{4}+\frac{3}{4}\right)=\frac{1}{2}\rightarrow$ unbiased

$\text{var}\left(\frac{Y_1+Y_3}{2}\right)=\frac{1}{4}\cdot\frac{3}{80}+\frac{1}{4}\cdot\frac{3}{80}+\frac{1}{2}\cdot\frac{1}{80}=\frac{1}{40}$

Since $\frac{1}{40}$ is less than $\frac{1}{36}$ midrange here is more efficient than the mean.

10.31 $E(\hat{\theta})=\theta+b(\theta)$

$E\left[(\hat{\theta}-\theta)^2\right]=E(\hat{\theta}^2)-2\theta E(\hat{\theta})+\theta^2=E(\hat{\theta})^2-2\theta\left[\theta+b(\theta)\right]+\theta^2$

$=E(\hat{\theta}^2)-\theta^2-2\theta b(\theta)$

$\text{var}(\hat{\theta})=E(\hat{\theta}^2)-\left[\theta+b(\theta)\right]^2=E(\hat{\theta}^2)-\theta^2-2\theta b(\theta)-\left[b(\theta)\right]^2$

$=E\left[(\hat{\theta}-\theta^2)\right]-\left[b(\theta)\right]^2$

$\therefore E\left[(\hat{\theta}-\theta)^2\right]=\text{var}(\hat{\theta})-\left[b(\theta)\right]^2$

10.32 $\text{var}(\hat{\theta}_1) = \dfrac{\theta(1-\theta)}{n} = \dfrac{1}{4n}$

$E(\hat{\theta}_2) = \dfrac{n\theta+1}{n+2}$ for $\theta = \dfrac{1}{2}$ $E(\hat{\theta}_2) = \dfrac{1}{2} \rightarrow$ unbiased

$\text{variance}(\hat{\theta}_2) = \dfrac{n\theta(1-\theta)}{(n+2)^2} = \dfrac{n}{4(n+2)^2}$ = mean square error

$E\left[(\hat{\theta}_2 - \theta)^2\right] - \left(\dfrac{1}{3} - \dfrac{1}{2}\right)^2 = \dfrac{1}{36}$

(a) $\dfrac{n}{4(n+2)^2} < \dfrac{1}{4n}$ $\qquad n^2 < (n+2)^2$

for all values of n

(b) $\dfrac{1}{36} < \dfrac{1}{4n}$, $4n < 36$, $n < 9$

10.33 $g_1(y_1) = n\left[\displaystyle\int_{y_1}^{\alpha+1} f(x)\,dx\right]^{n-1}$ $\qquad f(x) = 1 \quad \alpha < x < \alpha+1$

$\qquad 0 \quad$ elsewhere

$= n(\alpha+1-y_1)^{n-1}$ $\qquad$ for $\alpha < y_1 < \alpha+1$

0 $\quad$ elsewhere

$P\left(|Y_1 - \alpha| < c\right) = \displaystyle\int_{\alpha}^{\alpha+c} n(\alpha+1-y_1)^{n-1}dy_1$

$= 1^n - (1-c)^n \rightarrow 1$ when $n \rightarrow \infty$ with c fixed. QED

10.34 $E(\alpha+1-Y_1) = n\displaystyle\int_{\alpha}^{\alpha+1} (\alpha+1-y_n)^n dy = \dfrac{n}{n+1}$

$E(\alpha+1-Y_1)^2 = n\displaystyle\int_{\alpha}^{\alpha+1} (\alpha+1-y)^{n+1} dy = \dfrac{n}{n+2}$

$E(Y_1) = \alpha+1-\dfrac{n}{n+2} = \alpha + \dfrac{1}{n+1}$

$E(\left(Y_1 - \dfrac{1}{n+1}\right) = \alpha \rightarrow$ unbiased

$\text{var}(\alpha+1-Y_1) = \dfrac{n}{(n+2)} - \left(\dfrac{n}{n+1}\right)^2 = \dfrac{n}{(n+2)(n+1)^2}$

$\text{var}\left(Y_1 - \dfrac{1}{n+1}\right) = \dfrac{n}{(n+2)(n+1)^2} \rightarrow 0$ as $n \rightarrow \infty$ $\qquad$ QED

10.35 $g_n(y_n) = \frac{n}{\beta^n} y_n^{n-1} \qquad 0 < y_n < \beta$

$$P\left(|Y_n - \beta| < c\right) = \frac{n}{\beta^n}\int_{\beta-c}^{\beta} y_n^{n-1} dy_n = \frac{1}{\beta^n}\left[\beta^n - \left(\beta - c\right)^n\right]$$

$$= 1 - \left(\frac{\beta - c}{\beta}\right)^n \to 1$$

when $n \to \infty$ with c fixed.

10.36 $\bar{x}$ is consistent estimate of the mean of any population with a finite variance. Since θ is the mean and $\sigma^2 = \theta^2$ if follows that $\bar{x}$ is consistent estimate of θ.

10.37 For any single observation and for $c = \theta$, $P(|X - \theta| < \theta) = 1 - e^{-2\theta/\theta} = e^{-2}$ does not converge to 0, so X_n is not consistent for θ.

10.38 Shown is (a) of 10.21 that it is unbiased. From 10.22 variance is $\frac{\sigma_1^2\sigma_2^2}{n\left(\sigma_1^2 + \sigma_2^2\right)} \to 0$

So it is consistent by Theorem 10.3.

10.39 $\text{Var}\left(\frac{X+1}{n+2}\right) = \frac{1}{(n+2)^2} Var(X) = \frac{n\theta(1-\theta)}{(n+2)^2} \qquad \to \quad \theta \text{ as } n \to \infty$

asymptotically unbiased

$$\text{Var}\left(\frac{X+1}{n+2}\right) = \frac{1}{(n+2)^2}\text{var}(x) = \frac{n\theta(1-\theta)}{n^2}$$

$$= \frac{\theta(1-\theta)}{n} \to 0 \text{ as } n \to \infty \qquad \text{QED}$$

10.40 $E(Y_n) = \frac{n}{n+1}\beta \to \beta$ as $n \to \infty$ $\therefore$ asymptotically unbiased

From Example 10.6 (see Exercise 10.27)

$$\text{var}(Y_n) = \frac{n}{n+1} \cdot \frac{\beta^2}{n(n+2)} = \frac{\beta^2}{(n+1)(n+2)} \to 0 \text{ as } n \to \infty$$

consistent by Theorem 10.3

10.41 **(a)** $P\left(|x - \mu| < c\right)\frac{n-1}{n} P\left(|x-\mu| < c\right) + \frac{1}{n} P\left(|n^2 - \mu| < c\right) \quad 1 + 0 = 1$

since $\bar{x}$ is known to be consistent and $\frac{n-1}{n} \to 1$

(b) Let estimate be x

$E(x) = \mu \cdot \frac{n+1}{n} + n^2 \cdot \frac{1}{n} = \mu \cdot \frac{n+1}{n} + n \neq \mu$

not unbiased and *not* asymptotically unbiased.

10.42 $f(x_1, x_2, \ldots, x_n) = \frac{1}{\theta^n} e^{-\left[\frac{1}{\theta}\sum_{i=1}^{n} x_i\right]} = \underbrace{\frac{1}{\theta^n} e^{-(1/\theta)n\bar{x}}}_{g(\hat{\theta},\theta)}$

Since the joint density depends only on θ and $\bar{x}$, $\bar{x}$ is a sufficient estimator of θ.

10.43 $f(x_1, x_2) = \binom{n_1}{x_1}\binom{n_2}{x_2}\theta^{x_1+x_2}(1-\theta)^{(n_1+n_2)-(x_1+x_2)}$

$$\hat{\theta} = \frac{x_1 + x_2}{n_1 + n_2}$$

$$= \underbrace{\binom{n_1}{x_1}\binom{n_2}{x_2}}_{h(x_1,x_2)}\underbrace{\theta^{(n_1+n_2)\hat{\theta}}(1-\theta)^{(n_1+n_2)(1-\hat{\theta})}}_{(\hat{\theta},\theta)}$$

$\therefore$ by theorem, estimator is sufficient.

10.44 Try $x_1 = 0$ and $x_2 = 1$

$$f(0,1) = \binom{2}{0}\binom{2}{1}\theta(1-\theta)^3 = 2\theta(1-\theta)^2$$

$Y = \frac{x_1 + 2x_2}{n_1 + 2n_2} = \frac{2}{6} = \frac{1}{3}$ only possibilities $\begin{matrix} x_1 = 0 & x_2 = 1 \\ x_1 = 2 & x_2 = 0 \end{matrix}$

$\therefore$ by theorem, estimator is sufficient.

$$f(2,0) = \binom{2}{2}\binom{2}{0}\theta^2(1-\theta)^2 = \theta^2(1-\theta)^2$$

$$f\left(0,1 \middle| Y = \frac{1}{3}\right) = \frac{2\theta(1-\theta)^3}{2\theta(1-\theta)^2 + \theta^2(1-\theta)^2} = \frac{2(1-\theta)}{2(1+\theta)+\theta}$$

$$= \frac{2-2\theta}{2-\theta}$$ not independent of θ

$\therefore$ Y not sufficient

10.45 $f(x_1, \ldots x_n) = \frac{1}{\beta^n}$ $\quad g(y_n) = \frac{n}{\beta^n} y_n^{n-1}$

$$f(x_1, \ldots x_n | Y_n) = \frac{\frac{1}{\beta^n}}{\frac{n}{\beta^n} y_n^{n-1}} = \frac{1}{n\, y_n^{n-1}}$$

independent of β

$\therefore$ sufficient

10.46 $f(x_1, x_2) = \frac{\lambda^{x_1+x_2} e^{-2\lambda}}{x_1!\, x_2!}$ $\quad \bar{x} = \frac{x_1 + x_2}{2}$

$$\underbrace{\lambda^{2\bar{x}} e^{-\lambda}}_{g(\bar{x},\lambda)} \cdot \underbrace{\frac{1}{x_1!\, x_2!}}_{(x_1,x_2)}$$

satisfies Theorem 10.3

$\therefore$ sufficient

10.47 Try $x_1 = 0,\ x_2 = 1,\ x_3 = 0,\ Y = 2$

The only possibility is $x_1 = 1,\ x_2 = 0,\ x_3 = 1$

$f(0,1,0) = \theta(1-\theta)^2$

$f(1,0,1) = \theta^2(1-\theta)$

$$f(0,1,0 \mid Y = 2) = \frac{\theta(1-\theta)^2}{\theta(1-\theta)^2 + \theta^2(1-\theta)} = 1-\theta$$

not independent of $\theta \rightarrow$ not sufficient

10.48 $f(x) = \theta(1-\theta)^{x-1}$

$f(x_1, \ldots x_n) = \theta^n (1-\theta)^{\sum x - n} = \theta^n (1-\theta)^{n\bar{x}-n}$

Depends only on θ and $\bar{x} \rightarrow$ sufficient

10.49 $f(x_1 \ldots x_n) = \dfrac{1}{(2\pi)^{n/2}\sigma^n} e^{-(1/2)\left[\sum (x_i-\mu)^2\right]/\sigma^2} = \dfrac{1}{(2\pi)^{n/2}\sigma^n} e^{-(n/2\sigma^2)\hat{\sigma}^2}$

Depends only on σ^2 and $\hat{\sigma}^2 \rightarrow$ sufficient.

10.50 $\hat{\mu} = m_1',\ \mu^2 + \sigma^2 = m_2'$

$\hat{\sigma}^2 = m_2' - (m_1')^2$

10.51 $m_1' = \mu = \theta \qquad \hat{\theta} = m_1'$

10.52 $\mu = \dfrac{p}{2},\ \hat{\beta} = 2m_1'$

10.53 $\mu = \lambda \qquad \hat{\lambda} = m_1'$

10.54 $\beta = 1 \qquad \mu = \dfrac{\alpha}{\alpha+1} \qquad \dfrac{\alpha}{\alpha+1} = m_1' \qquad \alpha = \alpha m_1' + m_1'$

$\alpha(1-m_1') = m_1',\ \hat{\alpha} = \dfrac{m_1'}{1-m_1'}$

10.55 $\mu = \dfrac{2}{\theta^2}\displaystyle\int_0^\theta x(\theta - x)dx = \dfrac{\theta}{3},\ \hat{\theta} = 3m_1'$

10.56 $\mu = \dfrac{1}{\theta}\displaystyle\int_\delta^\infty x e^{-(1/\theta)(x-\delta)}dx = \dfrac{1}{\theta}\int_0^\infty (u+\delta)e^{-(1/\theta)u}du = \theta + \delta$

$u = x - \delta$

$\mu_2' = \dfrac{1}{\theta}\displaystyle\int_\delta^\infty x^2 e^{-(1/\theta)(x-\delta)}dx = \dfrac{1}{\theta}\int_0^\infty (u+\delta)^2 e^{-(1/\theta)u}du = 2\theta^2 + 2\delta\theta + \delta^2$

$m_1' = \delta + \theta,\ m_2' = 2\theta^2 + 2\delta\theta + \delta^2 = \theta^2 + (\theta+\delta)^2 = \theta^2 + (m_1')^2$

$\hat{\theta} = \sqrt{m_2' - (m_1')^2}$ and $\hat{\delta} - m_1' = \sqrt{m_2' - (m_1')^2}$

10.57 $\frac{\alpha+\beta}{2} = m_1'$ $\qquad \frac{1}{12}(\beta-\alpha)^2 + \frac{1}{4}(\alpha+\beta)^2 = m_2'$

$m_2' = \frac{1}{12}(\beta-\alpha)^2 + (m_1')^2 \quad (\beta-\alpha)^2 = 12[m_2' - (m_1')^2]$

$$\beta-\alpha = 2\sqrt{3[m_2'-(m_1')^2]}$$

$$\beta+\alpha = 2m_1$$

add $\quad \hat{\beta} = m_1' + \sqrt{3[m_2'-(m_1')^2]}$

subtract $\quad \hat{\alpha} = m_1' - \sqrt{3[m_2'-(m_1')^2}$

10.58 $\mu = 38$ $\qquad m_1' = \frac{n_0 \cdot 0 + n_1 \cdot 1 + n_2 \cdot 2 + n_3 \cdot 3}{N} = 3\theta$

$\hat{\theta} = \frac{n_1 + 2n_2 + 3n_2}{3N}$

10.59 $L(\lambda) = \frac{\lambda^{\sum x} e^{-n\lambda}}{\prod x_i!}$ $\qquad \ln L(\lambda) = \left(\sum x\right) - (\ln\lambda) - n\lambda - \ln\prod x_i!$

$$\frac{d\ln L(\lambda)}{d\lambda} = \frac{\sum x}{\lambda} - n = 0$$

$$\hat{\lambda} = \frac{\sum x}{n} = \bar{x}$$

10.60 $b(x;\alpha) = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha)\Gamma(1)} x^{\alpha-1} = \alpha x^{\alpha-1}$

$L(\alpha) = \alpha^n (\prod x_i)^{\alpha-1}$ $\qquad \ln L(\alpha) = n\ln(n) + (\alpha-1)\sum \ln x_i$

$$\frac{d\ln L(\alpha)}{d\alpha} = \frac{n}{\alpha} + \sum \ln x_i$$

$$\alpha = \frac{-n}{\sum_{i=1}^{n} \ln x_i}$$

10.61 $f(x) = \frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta}$ $\qquad \alpha = 2$

$$= \frac{1}{\beta^2} x\, e^{-x/\beta}$$

$L(\beta) = \frac{1}{\beta^{2n}}\left(\prod x_i\right) e^{-(1/\beta)\sum x}$ $\qquad \ln L(\beta) = -2n\ln\beta + \ln\prod x_i - \frac{1}{\beta}\sum x$

$$\frac{d\ln L(\beta)}{d\beta} = \frac{-2n}{\beta} + \frac{1}{\beta^2}\sum x = 0$$

$$\beta = \frac{\sum x}{2n} = \frac{\bar{x}}{2}$$

10.62 $f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(1/2)[(x-\mu)/\sigma]^2}$ $\quad L(\sigma) = \frac{1}{(2\pi)^n \sigma^n} e^{-(1/2\sigma^2)\sum(x-\mu)^2}$

$$\ln L(\sigma) = -\frac{n}{2}\ln 2\pi - n\ln\sigma - \frac{1}{2\sigma^2}\sum(x-\mu)^2$$

$$\frac{d\ln L(\sigma)}{d\sigma} = -\frac{n}{\sigma} + \frac{1}{\sigma_2^2}\sum(x-\mu)^2 = 0$$

$$\hat{\sigma}^2 = \frac{\sum(x-\mu)^2}{n} \text{ and } \hat{\sigma} = \sqrt{\frac{\sum(x-\mu)^2}{n}}$$

10.63 (a) $\mu = \frac{1}{8} = m_1'$ $\quad \hat{\theta} = \frac{1}{m_1'} = \frac{1}{\bar{x}}$

(b) $g(x) = \theta(1-\theta)^{x-1}$ $\quad L(\theta) = \theta^n(1-\theta)^{\sum x - n}$

$$\ln L(\theta) = n\ln\theta + \left(\sum x - n\right)\ln(1-\theta)$$

$$\frac{d\ln L(\theta)}{d\theta} = \frac{n}{\theta} + \left(\sum x - n\right)\left(\frac{-1}{1-\theta}\right) = 0 \qquad \hat{\theta} = \frac{n}{\sum x} = \frac{1}{\bar{x}}$$

10.64 $f(x) = 2\alpha x e^{-\alpha x^2}$ $\quad L(\alpha) = 2^n \alpha^n \left(\prod x_i\right) e^{-\alpha\left(\sum x^2\right)}$

$$\ln L(\alpha) = n\ln 2 + n\ln\alpha + \ln\prod x_i - \alpha\left(\sum x^2\right)$$

$$\frac{\ln L(\alpha)}{d\alpha} = \frac{n}{\alpha} - \sum x^2 = 0 \qquad \hat{\alpha} = \frac{n}{\sum x^2}$$

10.65 $f(x) = \frac{\alpha}{x^{\alpha+1}}$ $\quad L(a) = \frac{\alpha^n}{\left(\prod x_i\right)^{\alpha+1}}$

$$\ln L(\alpha) = n\ln\alpha - (\alpha+1)\ln(\prod x_i)$$

$$\frac{dL(\alpha)}{d\alpha} = \frac{n}{\alpha} - \ln\prod x_i = \frac{n}{\alpha} - \sum \ln x_i = 0$$

$$\bar{\alpha} = \frac{n}{\sum \ln x_i}$$

10.66 $f(x) = \frac{1}{8} e^{-(x-\delta)/\theta}$

$$L(\theta,\delta) = \frac{1}{\theta^n} e^{-(1/\theta)\sum(x-\delta)}$$

Maximized with respect to δ let $\hat{\delta}$ be y_1 (smallest sample value)

$\hat{\delta} = y_1$

$$\ln L(\theta,\delta) = -n\ln\theta - \frac{1}{\theta}\sum(x-\delta)$$

$$\frac{d\ln L(\theta,6)}{\partial\theta} = -\frac{n}{\theta} + \frac{1}{\sigma^2}\cdot\sum(x-\delta) \qquad \hat{\theta} = \frac{\sum x}{n} - \hat{\delta} \qquad \hat{\theta} = \bar{x} - y_1$$

10.67 $f(x) = \frac{1}{\beta - \alpha}$ $\qquad L(\alpha, \beta) = \frac{1}{(\beta - \alpha)^n}$

To maximize $\hat{\alpha} = y_1$, and $\hat{\beta} = y_n$

10.68 $L = [(1-\theta)^3]^{n_0}[3\theta(1-\theta)^2]^{n_1}[3\theta^2(1-\theta)]^{n_2}[\theta^3]^{n_3}$

$$= 3^{n_1+n_2}\theta^{n_1+2n_2+3n_3}(1-\theta)^{3n_0+2n_1+n_2}$$

$$\ln L = (n_1+n_2)\ln 3 + (n_1+2n_2+3n_3)\ln\theta + (3n_0+2n_1+n_2)\ln(1-\theta)$$

$$\frac{dL}{d\theta} = \frac{n_1+2n_2+3n_3}{\theta} - \frac{3n_0+2n_1+n_2}{1-\theta}$$

$$(n_1+2n_2+3n_3)(1-\theta) = (3n_0+2n_1+n_2)\theta$$

$$\theta(3n_0+3n_1+3n_2+3n_3) = n_1+2n_2+3n_3$$

$$\hat{\theta} = \frac{n_1+2n_2+3n_3}{3N}$$

10.69 $f(x) = \frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta}$

(a) $L(\beta) = \frac{1}{\beta^{n\alpha}[\Gamma(\alpha)]^n}\left(\prod x_i\right)^{\alpha-1} e^{-(1/\beta)\sum x_i}$

$$\ln L(\beta) = -n\alpha\ln\beta - n\ln\Gamma(\alpha) + (\alpha-1)\ln\prod x_i - \frac{1}{\beta}\sum x_i$$

$$\frac{d\ln L(\beta)}{d\beta} = \frac{-n\alpha}{\beta} + \frac{1}{\beta^2}\sum x_i \qquad \hat{\beta} = \frac{\sum x_i}{n\alpha} = \frac{\overline{x}}{\alpha}$$

(b) $\tau = \left(\frac{2\overline{x}}{\alpha} - 1\right)^2$

10.70 $L(\alpha, \beta) = \left(\sqrt{2\pi}\right)^{-2n} e^{-(1/2)\sum[v-(\alpha+\beta)]^2 - (1/2)\sum[w-(\alpha-\beta)]^2}$

$$\ln L(\alpha,\beta) = k - \frac{1}{2}\sum[v-(\alpha+\beta)]^2 - \frac{1}{2}\sum[w-(\alpha-\beta)]^2$$

$$\frac{\partial \ln L}{\partial\alpha} = \sum(v-(\alpha+\beta)] + \sum(w-(\alpha-\beta)] = 0$$

$$\sum v + \sum w - 2n\alpha = 0 \qquad \hat{\alpha} = \frac{\sum v + \sum w}{2n} = \frac{\overline{v}+\overline{w}}{2}$$

$$\frac{\partial \ln L}{\partial\beta} = \sum(v-(\alpha+\beta)] - \sum(w-(\alpha-\beta)] = 0$$

$$\sum v + \sum w - 2n\beta = 0 \qquad \hat{\beta} = \frac{\sum v - \sum w}{2v} = \frac{\overline{v}-\overline{w}}{2}$$

10.71 $V \; n_1 \, \mu_1 \, \sigma$

$W \; n_2 \, \mu_2 \, \sigma$

$$L = \frac{1}{\left(\sqrt{2\pi}\right)^{n_1+n_2} \sigma^{n_1+n_2}} e^{-(1/2\sigma^2)\sum (v-\mu_1)^2 - (1/2\sigma^2)\sum (w-\mu_2)^2}$$

$$\ln L = k - (n_1 + n_2)\ln\sigma - \frac{1}{2\sigma^2}\sum (v-\mu_1)^2 - \frac{1}{2\sigma^2}\sum (w-\mu_2)^2$$

$$\frac{\partial \ln L}{\partial \mu_1} = +\frac{1}{2\sigma^2} \cdot 2\sum (v-\mu_1) = 0 \qquad \hat{\mu}_1 = \bar{v}$$

$$\frac{\partial \ln L}{\partial \mu_2} = +\frac{1}{2\sigma^2} \cdot 2\sum (w-\mu_2) = 0 \qquad \mu_2' = \bar{w}$$

$$\frac{\partial L}{\partial \sigma} = -\frac{n_1+n_2}{\sigma} + \frac{1}{\sigma^3}\left[\sum (v-\mu_1)^2 + \sum (w-\mu_2)^2\right]$$

$$\hat{\sigma}^2 = \frac{\sum (v-\bar{v})^2 + \sum (w-\hat{w})^2}{n_1+n_2}$$

10.72 Any value $\hat{\theta}$ will do so long as

$$\hat{\theta} - \frac{1}{2} \le y_1 \quad \text{and} \quad y_n < \hat{\theta} + \frac{1}{2}$$

$$\hat{\theta} \le y_2 + \frac{1}{2} \quad \text{and} \quad \hat{\theta} \ge y_n - \frac{1}{2}$$

$$y_n - \frac{1}{2} \le \hat{\theta} \le y_1 + \frac{1}{2}$$

10.73 (a) It is if $Y_n - \frac{1}{2} \le \frac{1}{2}(Y_1 + Y_n) \le Y_1 + \frac{1}{2}$

make use of $\boxed{Y_1 \le Y_n \le Y_1 + 1}$

$$\frac{1}{2}(Y_1 + Y_n) \le \frac{1}{2}(Y_1 + Y_1 + 1) = Y_1 + \frac{1}{2}$$

$$\frac{1}{2}(Y_1 + Y_n) \ge \frac{1}{2}(Y_n + Y_n - 1) = Y_n - \frac{1}{2}$$ both conditions are satisfied

(b) Suppose $Y_2 = Y_1 + 1$ let $n = 2$

$$\frac{1}{3}(Y_1 + 2Y_2) = \frac{1}{3}(3Y_1 + 2) = Y_1 + \frac{2}{3} \not\le Y_2 + \frac{1}{2}$$

not max likelihood estimate

10.74 $E(\theta|x) = \dfrac{x+\alpha}{\alpha+\beta+n}$ where $\alpha = \theta_0\left[\dfrac{\theta_0(1-\sigma_0^2)}{\sigma_0^2} - 1\right]$

$\beta = (1-\theta_0)\left[\dfrac{\theta_0(1-\theta_0)}{\sigma_0^2} - 1\right]$ $\qquad\qquad \alpha+\beta = \dfrac{\theta_0(1-\theta_0)}{\sigma_0^2} - 1$

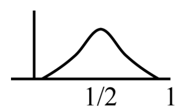
$$E(\theta|x) = \frac{x}{n}\cdot\frac{n}{\alpha+\beta+n} + \frac{\alpha}{\alpha+\beta+n}$$

$$= \frac{x}{n}\cdot\frac{n}{\dfrac{\theta_0(1-\theta_0)}{\sigma_o^2} - 1 + n} + \frac{\theta_0\left[\dfrac{\theta_0(1-\theta_0)}{\sigma_0^2} - 1\right]}{\dfrac{\theta_0(1-\theta_0)}{\sigma_0^2} - 1 + n}$$

$= \dfrac{x}{n}\cdot w + \theta_0(1-w)$ where $w = \dfrac{n}{n + \dfrac{\theta_0(1-\theta_0)}{\sigma_0^2} - 1}$

10.75 $\mu = \dfrac{\alpha}{\alpha+\beta} = \dfrac{40}{40+40} = \dfrac{1}{2}$ $\qquad \sigma^2 = \dfrac{40\cdot 40}{80^2\cdot 81} = \dfrac{1}{324}$

$\sigma = \dfrac{1}{18}$



Distribution is symmetrical about $x = \dfrac{1}{2}$

The function as well as its derivatives are 0 at $x = 0$ and 1, and with $k = 3$ in Chebyshev's Theorem

$\dfrac{8}{9}$ of area under curve falls between $\dfrac{1}{2} \pm \dfrac{1}{6} = \dfrac{1}{3}$ and $\dfrac{2}{3}$

10.76 $\mu_1 = \bar{x}\cdot\dfrac{n\sigma_0^2}{n\sigma_0^2+\sigma^2} + \mu_0\cdot\dfrac{\sigma^2}{n\sigma_0^2+\sigma^2} = \bar{x}w + \mu_0(1-w)$

$w = \dfrac{n\sigma_0^2}{n\sigma_0^2+\sigma^2} = \dfrac{n}{n + \dfrac{\sigma^2}{\sigma_0^2}}$ $\qquad$ QED

10.77 $f(x|\lambda) = \dfrac{\lambda^x e^{-\lambda}}{x!}$

(a) $f(x,\lambda) = \dfrac{\lambda^x e^{-\lambda}}{x!}\cdot\dfrac{1}{\beta^\alpha\Gamma(\alpha)}\lambda^{\alpha-1}e^{-\lambda/\beta}$

$g(x) = \dfrac{x^{\alpha-1}e^{-\beta}}{x!\beta^\alpha\Gamma(a)}\displaystyle\int_0^\infty \lambda^x e^{-\lambda}d\lambda$ $\qquad$ gamma distribution with $\alpha = x+1$ and $\beta = 1$

$$= \frac{x^{\alpha-1}e^{-\beta}}{x!\beta^\alpha\Gamma(\alpha)}\cdot\Gamma(x+1) = \frac{x^{\alpha-1}e^{-\beta}x!}{x!\beta^\alpha\Gamma(\alpha)} = \frac{x^{\alpha-1}e^{-\beta}}{\beta^\alpha\Gamma(\alpha)}$$

$$f(x,\lambda) = \frac{\lambda^{x+\alpha-1} e^{-\lambda[1+(1/\beta)]}}{x!\beta^{\alpha}\Gamma(\alpha)}$$

$$g(x) = \frac{1}{x!\beta^{\alpha}\Gamma(\alpha)} \int_0^{\infty} \lambda^{x+\alpha-1} e^{-\lambda(\beta+1)/\beta}$$

gamma distribution with $x+\alpha$ and $\dfrac{\beta}{\beta+1}$

$$g(x) = \frac{\left(\dfrac{\beta}{\beta+1}\right)^{x+\alpha} \Gamma(x+\alpha)}{x!\beta^{\alpha}\Gamma(\alpha)}$$

$$\phi(\lambda|x) = \frac{\lambda^{x+\alpha-1} e^{-\lambda(\beta+1)/\beta}}{x!\beta^{\alpha}\Gamma(\alpha)} \cdot \frac{x!\beta^{\alpha}\Gamma(\alpha)}{\left(\dfrac{\beta}{\beta+1}\right)^{x+\alpha} \Gamma(x+\alpha)}$$

$$= \frac{1}{\left(\dfrac{\beta}{\beta+1}\right)^{x+\alpha} \Gamma(x+\alpha)} . \lambda^{x+\alpha-1} e^{-\lambda(\beta+1)/\beta}$$

gamma distribution with parameters

$x+\alpha$ and $\dfrac{\beta}{\beta+1}$

(b) $E(\Lambda|x) = \dfrac{(x+\alpha)\beta}{\beta+1}$ from Theorem 6.3

10.78 $\dfrac{25}{75}(27.6) + \dfrac{50}{75}(38.1) = 34.6$

10.79 $\dfrac{9}{13}(26.0) + \dfrac{4}{13}(32.5) = 28$

10.80 $\dfrac{4}{3} \cdot 210 - 1 = 279$

10.81 $\hat{\alpha} = \dfrac{n\bar{x}^2}{\sum (x-\bar{x})^2}$ $\qquad \hat{\beta} = \dfrac{\sum (x-\bar{x})^2}{n\bar{x}}$

or $\hat{\alpha} = \dfrac{(m_1')^2}{m_2' - (m_1')^2}$ $\qquad \hat{\beta} = \dfrac{m_2' - (m_1')^2}{m_1'}$

$\sum x = 86.4$ and $\sum x^2 = 756.52$

$m_1' = \dfrac{86.4}{12} = 7.2$ and $m_2' = \dfrac{756.52}{12} = 63.0433$

$$\hat{\alpha} = \frac{(7.2)^2}{63.0433 - (7.2)^2} = \frac{51.84}{63.0433 - 51.84} = 4.627$$

$$\hat{\beta} = \frac{63.0433 - (7.2)^2}{7.2} = 1.556$$

10.82 $\hat{\theta} = m_1'$ $\quad \sum x = 201{,}000 \quad \hat{\theta} = \dfrac{201{,}000}{5} = 40{,}200 \text{ miles}$

10.83 The likelihoods are $\dfrac{\binom{3}{1}\binom{N-3}{3}}{\binom{N}{4}}$

N	*Likelihood*	N	*Likelihood*
9	$\dfrac{\binom{3}{1}\binom{6}{3}}{\binom{9}{4}} = \dfrac{3\cdot 20}{126} = 0.4762$	12	$\dfrac{\binom{3}{1}\binom{9}{3}}{\binom{12}{4}} = \dfrac{3\cdot 84}{495} = 0.5091$
10	$\dfrac{\binom{3}{1}\binom{7}{3}}{\binom{10}{4}} = \dfrac{3\cdot 35}{210} = 0.5000$	13	$\dfrac{\binom{3}{1}\binom{10}{3}}{\binom{13}{4}} = \dfrac{3\cdot 120}{715} = 0.5035$
11	$\dfrac{\binom{3}{1}\binom{8}{3}}{\binom{11}{4}} = \dfrac{3\cdot 56}{330} = 0.5091$	14	$\dfrac{\binom{3}{1}\binom{11}{3}}{\binom{14}{4}} = \dfrac{3\cdot 165}{1001} = 0.4945$

Likelihood greatest for $N = 11$ or $N = 12$

10.84 $\hat{\theta} = 3m_1' \quad \sum x = 0.39 \quad m_1' = \dfrac{0.39}{6} = 0.065 \quad \hat{\theta} = 3\cdot\dfrac{0.39}{6} = 0.195$

10.85 $\sum x = 5524, \quad \sum x^2 = 2{,}570{,}176 \quad n = 12$

$m_1' = 460.3333 \quad m_2' = 214{,}181.3333$

$\hat{\theta} = \sqrt{214{,}181.3333 - 211{,}906.7471} = 47.69$

$\hat{\delta} = 460.3333 - 47.69 = 412.64$

10.86 $\hat{\delta} = y_1 = 403 \quad \hat{\theta} = 460.33 - 403 = 57.33$

10.87 $n = 8 \quad \sum x = 63.1 \quad \sum x^2 = 541.55 \quad m_1' = \dfrac{63.1}{8} = 7.8875$

$m_2' = \dfrac{541.55}{8} = 67.69375$

$$\hat{\alpha} = 7.8875 - \sqrt{3(67.69375 - 62.2126)}$$
$$= 7.8875 - 4.0550 = 3.83$$
$$\hat{\beta} = 7.8875 + 4.0550 = 11.9427 = 11.95$$

10.88 $\hat{\alpha} = 4.1$ and $\hat{\beta} = 11.5$

$\hat{\alpha} = y_1 \qquad \hat{\beta} = y_n$

10.89 $\hat{\alpha} = \dfrac{n}{\sum \ln x_i} = \dfrac{n(0.4343)}{\sum \log_{10} x}$

$\hat{\alpha} = \dfrac{15(0.4343)}{66.24567} = 0.098$

$\log_{10} x = 4.37840$

$n = 15$

4.37840
4.33244
4.42160
4.39445
4.52634
4.38917
4.46538
4.55871
4.35025
4.33244
4.45179
4.42813
4.49693
4.35603
4.36361
66.24567

10.90 $n = 3 \qquad N = 20 \qquad n_0 = 11 \qquad n_1 = 7 \qquad n_2 = 2 \qquad n_3 = 0$

$\hat{\theta} = \dfrac{7 + 2 \cdot 2 + 3 \cdot 0}{3 \cdot 20} = \dfrac{11}{60}$

10.91 1, 3, 5, 1, 2, 1, 3, 7, 2, 4, 4, 8, 1, 3, 6, 5, 2, 1, 6, 2

$\sum x = 67 \qquad \hat{\theta} = \dfrac{20}{67} = 0.30$

10.92 $\sum v = 107.4 \qquad \sum v^2 = 116{,}108 \qquad n_1 = 10$

$\sum w = 674 \qquad \sum w^2 = 76{,}246 \qquad n_2 = 6$

$\hat{\mu}_1 = \dfrac{1074}{10} = 107.4 \qquad \hat{\mu}_2 = \dfrac{674}{6} = 112.3$

$\hat{\sigma}^2 = \dfrac{116{,}108 - 115{,}347.6 + 76{,}246 - 75{,}712.7}{16} = \dfrac{1{,}293.7}{16} = 80.86$

10.93 $n = 100 \qquad \theta_0 = 0.20 \qquad \sigma_0 = 0.04 \qquad x = 38$

$E(\theta|38) = \dfrac{38}{100} w + 0.20(1 - w)$

$w = \dfrac{100}{99 + \dfrac{(0.2)(0.8)}{(0.04)^2}} = \dfrac{100}{99 + 100} = 0.5025$

$E(\theta|38) = 0.38(0.5025) + 0.20(0.4975) = 0.29$

10.94 $\theta_0 = 0.74$ $\sigma_0 = 0.03$ $n = 30$ $x = 18$

(a) $\hat{\theta} = 0.74$

(b) $\hat{\theta}_n = \frac{x}{n} = \frac{18}{30} = 0.60$

(c) $w = \frac{30}{29 + \frac{(0.74)(0.26)}{(0.03)^2}} = \frac{30}{29 + 213.8} = \frac{30}{242.8} = 0.1236$

$\hat{\theta} = (0.1236)(0.60) + (0.8764)(0.74) = 0.72$

10.95 $\mu_1 = 715$ $\sigma_1 = 9.5$ $z = \frac{712 - 715}{9.5} = -0.32$

$z = \frac{725 - 715}{9.5} = 1.05$

$p = 0.1255 + 0.3531 = 0.4786$

10.96 $\mu_0 = 65.2$ $\sigma_0 = 1.5$ $z = \frac{63 - 65.2}{1.5} = -1.47$

$z = \frac{68 - 65.2}{1.5} = 1.87$

(a) $p = 0.4292 + 0.4693 = 0.8985$

(b) $w + \frac{40}{40 + \frac{7.4^2}{1.5^2}} = \frac{40}{64.34} = 0.62$ $\mu_1 = (0.62)72.9 + (0.38)65.2$
$= 69.97$

$\frac{1}{\sigma_1^2} = \frac{40}{7.4^2} + \frac{1}{1.5^2} = 0.730 + 0.444 = 1.174$ $\sigma_1^2 = 0.92$

$z = \frac{63 - 70}{0.92} = -7.6$

$z = \frac{68 - 70}{0.92} = -2.18$

$p = 0.5000 - 0.4854 = 0.0146$

10.97 (a) $\hat{\mu} = \alpha\beta = 50 \cdot 2 = 100$

(b) $\hat{\mu} = \bar{x} = 112$

(c) $\hat{\mu} = \mu_1 = \frac{2(50 + 112)}{3} = 108$

10.98 $n = \frac{z^2\sigma^2}{E^2} = \left(\frac{2.575 \cdot 4.2}{0.5}\right)^2 = 467.9.$ Rounding up to the next integer, $n = 468$.

10.99 $z = \dfrac{E}{\sigma / \sqrt{n}} = \dfrac{6 \cdot 15}{1} = 9.0$; yes.

10.100 The sample is more likely to include longer sections than shorter ones; They take more time to pass the inspection station.

10.101 Heads of households may tend to have somewhat different political opinions than other members of the household who are likely to be younger and/or of a different sex.