Chapter 6

6.1 $$\int_{\alpha}^{\alpha+p(\beta-\alpha)} \frac{1}{\beta-a}\,dx = \frac{1}{\beta-\alpha}[\alpha+p(\beta-\alpha)-\alpha] = p$$

6.2 $$\mu = \frac{1}{\beta-\alpha}\int_{\alpha}^{\beta}\frac{1}{\beta-a}\,x\,dx = \frac{1}{\beta-\alpha}\left(\frac{\beta^2}{2}-\frac{\alpha^2}{2}\right) = \frac{1}{2(\beta-\alpha)}\cdot(\beta-\alpha)(\beta+\alpha) = \frac{\alpha+\beta}{2}$$

$$\mu_2' = \frac{1}{\beta-\alpha}\int_{\alpha}^{\beta} x^2\,dx = \frac{1}{3(\beta-\alpha)}(\beta^3-\alpha^3) = \frac{1}{3}(\beta^2+\alpha\beta+\alpha^2)$$

$$\sigma^2 = \frac{1}{3}(\beta^2+\alpha\beta+\alpha^2) - \frac{(\alpha+\beta)^2}{4} = \frac{1}{12}[4\beta^2+4\alpha\beta+4\alpha^2-3\alpha^2-6\alpha\beta-3\beta^2)$$
$$= \frac{1}{12}(\beta^2-2\alpha\beta+\alpha^2) = \frac{1}{12}(\beta-\alpha)^2$$

6.3 $$F(x) = \frac{1}{\beta-\alpha}\int_{\alpha}^{x} dx = \frac{x-\alpha}{\beta-\alpha} \qquad f(x) = \begin{cases} 0 & x \le \alpha \\ \dfrac{x-\alpha}{\beta-\alpha} & \alpha < x < \beta \\ 1 & \beta \le x \end{cases}$$

6.4 $$\mu_r = \frac{1}{\beta-\alpha}\int_{\alpha}^{\beta}\left[x-\frac{\alpha-\beta}{2}\right]^r dx = \frac{1}{(\beta-\alpha)2^r}\int_{\alpha}^{\beta}[2x-(\alpha+\beta)]^r\,dx$$
$$= \frac{1}{(\beta-\alpha)2^r}\left[\frac{[2x-(\alpha+\beta)]^{r+1}}{2(r+1)}\right]\Bigg|_{\alpha}^{\beta}$$
$$= \frac{1}{(\beta-\alpha)2^r}\cdot\frac{(\beta-\alpha)^{r+1}-(-1)^{r+1}(\beta-\alpha)^{r+1}}{2(r+1)}$$

(a) $= 0$ when r is odd

(b) $= \dfrac{1}{(\beta-\alpha)2^{r+3}(r+1)}2(\beta-\alpha)^{r+1} = \dfrac{1}{r+1}\left(\dfrac{\beta-\alpha}{2}\right)^r$ when r is even

6.5 $$\mu_1 = 0,\ \mu_2 = \frac{1}{3}\frac{(\beta-\alpha)^2}{4} = \frac{(\beta-\alpha)^2}{12},\ \mu_3 = 0,\ \mu_4 = \frac{1}{5}\left(\frac{\beta-\alpha)}{2}\right)^4 = \frac{1}{80}(\beta-\alpha)^4$$

$$\alpha_3 = 0 \text{ and } \alpha_4 = \frac{\frac{1}{80}(\beta-\alpha)^4}{\frac{(\beta-\alpha)^4}{144}} = \frac{9}{5}$$

6.6 Intergals do not exist.

6.7 $\Gamma(\alpha)=\int_0^\infty x^{\alpha-1}e^{-x}dx$ $\quad u=x^{\alpha-1}$

$dv=e^{-x}\ dx$

$=-x^{\alpha-1}e^{-x}\Big|_0^\infty+(\alpha-1)\int_0^\infty x^{\alpha-2}e^{-x}\ dx$ $\quad du=(\alpha-1)x^{\alpha-2}\ dx$

$v=-e^{-x}$

$=(\alpha-1)\Gamma(\alpha-1)$ QED

6.8 $y=\frac{1}{2}z^2$ $\quad \Gamma(\alpha)=\int_0^\infty y^{\alpha-1}e^{-y}dy=\int_0^\infty\left(\frac{z^2}{2}\right)^{\alpha-1}e^{-(1/2)z^2}z\ dz$

$dy=z\ dz$

$$=2^{1-\alpha}\int_0^\infty z^{2\alpha-1}er^{-(1/2)z^2}\ dz$$

6.9 $x=r\cos\theta \quad y=r\sin\theta \quad dx\ dy=r\ dr\ d\theta$

$$\left[\Gamma\left(\frac{1}{2}\right)\right]^2=2\int_0^{\pi/2}\int_0^\infty re^{-(1/2)r^2}\ dr\ d\theta=\pi\int_0^\infty re^{-(1/2)r^2}\ dr$$

$u=-\frac{1}{2}r^2$

$$=\pi\int_0^\infty -e^u\ du=-\pi[e^u]\Big|_0^\infty=\pi \qquad \text{QED}$$

$du=-r\ dr$

6.10 **(a)** $\alpha=2,\ \beta=3,\ x>4,\ p=\int_4^\infty\frac{1}{9\cdot 1}x\ e^{-x/3}\ dx=\frac{1}{9}\int_4^\infty x\ e^{-x/3}\ dx$

$$=\frac{1}{9}\left[\frac{e^{-x/3}}{1/9}\left(-\frac{1}{3}x-1\right)\right]=e^{-4/3}\left(\frac{7}{3}\right)=\frac{7}{3}e^{-4/3}=\frac{7}{3}(0.2645)=0.6171$$

(b) $\alpha=3,\ \beta=4,\ p=\int_4^\infty\frac{1}{64\cdot 2}x^2e^{-x/4}\ dx=\frac{1}{128}\int_4^\infty x^2e^{-x/4}\ dx=0.7818$

6.11 $\frac{\partial}{\partial x}=x^{\alpha-1}\left(-\frac{1}{\beta}e^{-x/\beta}\right)+e^{-x/\beta}(\alpha-1)x^{\alpha-2}$

$$=x^{\alpha-2}e^{-x/\beta}\left(-\frac{x}{\beta}+\alpha-1\right)=0 \qquad x=\beta(\alpha-1)$$

$0<\alpha<1$ function $\to\infty$ when $x\to 0$

$\alpha=1$ function has absolute maximum at $x=0$.

6.13 $M = (1-\beta t)^{-\alpha} = 1 - \alpha(-\beta t) + \alpha(\alpha+1)\dfrac{(-\beta t)^2}{2} - \alpha(a+1)(\alpha+2)\dfrac{(-\beta t)^2}{3!}$

$= 1 + \alpha\beta t + \alpha(\alpha+1)\dfrac{\beta^2 t^2}{2!} + \dfrac{\alpha(\alpha+1)(\alpha+2)\beta^2 t^2}{3!} + \alpha(\alpha+1)(\alpha+2)(a+3)\dfrac{\beta^4 t^4}{4!} + \ldots$

$\mu_1' = \alpha\beta,\ \mu_2' = \alpha(\alpha+1)\beta^2,\ \mu_3' = \alpha(\alpha+1)(\alpha+2)\beta^3$

$\mu_4' = \alpha(\alpha+1)(\alpha+2)(\alpha+3)\beta^4$

6.14 $\mu_3 = (\alpha+1)(\alpha+2)\beta^3 - 3\alpha(\alpha+1)\beta^2\alpha\beta + 2\alpha^3\beta^3$

$= \alpha\beta^3[(\alpha+1)(\alpha+2) - 3\alpha(\alpha+1) + 2\alpha^2] = \alpha\beta^3[2] = 2\alpha\beta^3$

$\alpha_3 = \dfrac{2\alpha\beta^2}{(\alpha\beta^2)^{3/2}} = \dfrac{2}{\sqrt{\alpha}}$

$\mu_4 = \alpha(\alpha+1)(\alpha+2)(\alpha+3)\beta^4 - 4\alpha(\alpha+1)(\alpha+2)\beta^3 \cdot \alpha\beta + 6\alpha(\alpha+1)\beta^2 \cdot \alpha^2\beta^2 - 3\alpha^4\beta^4$

$= 2\beta^4[(\alpha+1)(\alpha+2)(\alpha+3) - 4\alpha(\alpha+1)(\alpha+2) + 6\alpha^2(\alpha+1) - 3\alpha^3] = \alpha\beta^4$

$\alpha^4 = \dfrac{\alpha\beta^4(3\alpha+6)}{\alpha^2\beta^4} = 3 + \dfrac{6}{\alpha}$

6.15 $f(x) = \dfrac{1}{\theta}e^{-x/\theta} \qquad p = \displaystyle\int_0^{-\theta\ln(1-p)} \dfrac{1}{\theta}e^{-x/\theta}d\theta = [-e^{-x/\theta}]\Big|_0^{-\theta\ln(1-p)}$

$= 1 - e^{\ln(1-p)} = 1 - (1-p) = p$

6.16 $\dfrac{p(x \geq t+T)}{P(x \geq T)} = \dfrac{e^{-(t+T)/\theta}}{e^{-T/\theta}} = e^{-t/\theta} = p(x \geq t)$

6.17 $M_x = (1-\theta t)^{-1} \qquad M_{x-\theta} = e^{-\theta t}(1-\theta t)^{-1} = \dfrac{e^{-\theta t}}{1-\theta t}$

6.18 $\left(1 - \theta t + \dfrac{\theta^2 t^2}{2!} - \dfrac{\theta^3 t^3}{3!} + \dfrac{\theta^4 t^4}{4!}\ldots\right)(1 + \theta t + \theta^2 t^2 + \theta^3 t^3 + \theta^4 t^4 \ldots)$

$1 + \left(1 + \dfrac{1}{2} - 1\right)\theta^2 t^2 + \left(-\dfrac{1}{6} + \dfrac{1}{2} - 1 + 1\right)\theta^3 t^3 + \left(\dfrac{1}{24} - \dfrac{1}{6} + \dfrac{1}{2} - 1 + 1\right)\theta^4 t^4 \ldots$

$1 + \dfrac{\theta^2 t^2}{2!} + 2 \cdot \dfrac{\theta^3 t^3}{3!} + \dfrac{9\theta^4 t^4}{4!} + \ldots \qquad \alpha_3 = \dfrac{2\theta^3}{\theta^3} = 2 \qquad \alpha_4 = \dfrac{9\theta^4}{\theta^4} = 9$

6.19 $\alpha = \frac{\nu}{2},\ \beta = 2$ See 6.11

From 6.11 $x = \beta(\alpha - 1) = 2\left(\frac{\nu}{2} - 1\right) = \nu - 2$

$0 < \nu < 2$ function $\to \infty$ when $x \to 0$
$\nu = 2$ function has absolute maximum at $x = 0$

6.20 $\mu = 2\alpha \int_0^\infty x^2\ e^{-\alpha x^2}\ dx$ $u = \alpha x^2$ $du = 2\alpha x\ dx$

$$= \frac{1}{\sqrt{\alpha}} \int_0^\infty u^{1/2} e^{-u}\ du = \frac{1}{\sqrt{a}} \Gamma\left(\frac{3}{2}\right) = \frac{1}{\sqrt{\alpha}} \cdot \frac{1}{2} \cdot \sqrt{\pi} = \frac{1}{2}\sqrt{\frac{\pi}{\alpha}}$$

$$\mu_2' = 2\alpha \int_0^\infty x^3\ e^{-\alpha x^2\ dx} = \frac{1}{\alpha} \qquad \sigma^2 = \frac{1}{\alpha} - \frac{1}{4} \cdot \frac{\pi}{\alpha} - \frac{1}{\alpha}\left(1 - \frac{\pi}{4}\right)$$

6.21 $\mu_r' = \alpha \int_1^\infty x^{r-\alpha-1}\ dx$ exists only if $r - \alpha - 1 < 1$
$r < \alpha - 2$

6.22 $\mu_1' = \alpha \int_1^\infty x^{-\alpha}\ dx = \alpha \left. \frac{x^{1-\alpha}}{1-\alpha} \right|_1^\infty = \frac{\alpha}{\alpha - 1}$

6.23 **(a)** $k \int_0^\infty x^{\beta-1} e^{-\alpha x^\beta}\ dx = 1$ let $u = \alpha x^\beta$ $du = \alpha\beta x^{\beta-1}\ dx$

$$= k \int_0^\infty \frac{1}{\alpha\beta} e^{-u}\ du = \frac{k}{\alpha\beta} = 1 \qquad k = \alpha\beta$$

(b) $\mu = \alpha\beta \int_0^\infty x^\beta\ e^{-\alpha x^\beta}\ dx$

$$= \alpha^{-1/\beta} \int u^{1/\beta}\ e^{-u}\ du = \alpha^{-1/\beta} \Gamma\left(1 + \frac{1}{\beta}\right)$$

6.24 **(a)** $f(x) = \frac{1}{\theta} e^{-x/\theta}$ $F(t) = \int_0^{t/\theta} e^{-u}\ du = 1 - e^{-t/\theta} = 1 - e^{-t/\theta}$

$$\frac{f(t)}{1 - F(t)} = \frac{\frac{1}{\theta} e^{-t/\theta}}{e^{-t/\theta}} = \frac{1}{\theta}$$

(b) $F(t) = \alpha\beta \int_0^t x^{\beta-1}\ e^{-\alpha x^\beta}\ dx = 1 - e^{-\alpha t^\beta}$

$$\frac{f(t)}{1-F(t)} = \frac{\alpha\beta t^{\beta-1}\ e^{-\alpha t^\beta}}{e^{-\alpha t^\beta}} = \alpha\beta t^{\beta-1}$$

6.25 **(a)** $\dfrac{\Gamma(6)}{\Gamma(2)\Gamma(4)} \int_0^1 x(1-x)^3\ dx = 20\left[\dfrac{x^2}{2} - x^3 + \dfrac{3x^4}{4} - \dfrac{x^5}{5}\right]\Bigg|_0^1$

$$= 20\left(\frac{1}{2} - 1 + \frac{3}{4} - \frac{1}{5}\right) = 20 \cdot \frac{1}{20} = 1$$

(b) $\dfrac{\Gamma(6)}{\Gamma(3)\Gamma(3)} \int_0^1 x^2(1-x)^2\ dx = 30\left[\dfrac{1}{3} - \dfrac{1}{2} + \dfrac{1}{5}\right] = 30 \cdot \dfrac{1}{30} = 1$

6.26 $f(x) = kx^{\alpha-1}(1-x)^{\beta-1}$

$$\frac{df}{dx} = k\ x^{\alpha-1}(\beta-1)(1-x)^{\beta-2}(-1) + k(1-x)^{\beta-1}(\alpha-1)x^{\alpha-2}$$

$$= k\ x^{\alpha-2}(1-x)^{\beta-2}[-x(\beta-1) + (\alpha-1)(1-x)]$$

$x(2-\alpha-\beta) = 1-\alpha$ and $x = \dfrac{\alpha-1}{\alpha+\beta-2}$

6.28 $\mu_2' = \dfrac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 x^{\alpha+1}(1-x)^{\beta-1}\ dx$

$$= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \cdot \frac{\Gamma(\alpha+2)\Gamma(\beta)}{\Gamma(\alpha+\beta+2)} = \frac{(\alpha+1)\alpha}{(\alpha+\beta+1)(\alpha+\beta)}$$

6.29 $\mu = \dfrac{}{\alpha+\beta}$ $\qquad \sigma^2 = \dfrac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$

$\alpha+\beta = \dfrac{\alpha}{\mu} \qquad \sigma^2 = \mu(1-\mu)\dfrac{1}{\alpha+\beta+1}$

$\alpha+\beta+1 = \dfrac{\mu(1-\mu)}{\sigma^2},\ \ \dfrac{\alpha}{\mu} = \dfrac{\mu(1-\mu)}{\sigma^2} - 1,\ \ \alpha = \mu\left[\dfrac{\mu(1-\mu)}{\sigma^2} - 1\right]$

$\beta = \dfrac{\alpha}{\mu} - \alpha = \alpha\left(\dfrac{1}{\mu} - 1\right) = \dfrac{\alpha(1-\mu)}{\mu}$

$$= (1-\mu)\left[\frac{\mu(1-\mu)}{\sigma^2} - 1\right]$$

6.30 **(a)**

$$\frac{1}{f(x)}\frac{df(x)}{dx}=\frac{d-x}{bx}=\frac{d}{bx}-\frac{1}{5}$$

$$\ln f(x)-\frac{d}{b}\ln x=-\frac{1}{b}x+c$$

$$\ln\frac{f(x)}{x^{b/d}}=-\frac{1}{b}x,\ f(x)=kx^{b/d}e^{-(1/b)x}$$

(b)

$$\frac{1}{f(x)}\frac{df(x)}{dx}=-\frac{1}{b}\ln f(x)=-\frac{1}{b}x+c \qquad f(x)=ke^{-(1/b)x}$$

(c)

$$\frac{1}{f(x)}\frac{df(x)}{dx}=-\frac{d-x}{cx(1-x)}=\frac{-d/c}{x(1-x)}+\frac{1/c}{(1-x)}$$

$$\frac{-d/c}{x(1-x)}=\frac{A}{x}+\frac{B}{1-x}=\frac{A(1-x\ +Bx}{x(1-x)} \qquad A=-d/c=B$$

$$\frac{1}{f(x)}\frac{df(x)}{dx}=\frac{-d/c}{x}-\frac{d/c}{1-x}+\frac{1/c}{1-x}=\frac{-d/c}{x}-\frac{(d-1)/c}{1-x}$$

$$\ln f(x)=-\frac{d}{c}\ln x+\frac{(d-1)}{c}\ln(1-x)$$

$$f(x)=k\ x^{-d/c}(1-x)^{(d-1)/c}$$

6.31

$$f(x)=\frac{1}{\sqrt{2\pi}\sigma}e^{-(1/2)[(x-\mu)/\sigma]^2} \qquad \ln f(x)=-\ln\sqrt{2\pi}\sigma-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2$$

(a)

$$\ln f(x)=k-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2$$

$$\frac{1}{f(x)}\frac{df(x)}{dx}=-\frac{1}{\sigma}\left(\frac{x-\mu}{\sigma}\right) \qquad -\frac{1}{\sigma}\left(\frac{x-\mu}{\sigma}\right)=0 \qquad x=\mu$$

(b)

$$\frac{df(x)}{dx}=-\left(\frac{x-\mu}{\sigma^2}\right)f(x)$$

$$\frac{d^2f(x)}{dx}=-\frac{1}{\sigma^2}f(x)-\left(\frac{x-\mu}{\sigma^2}\right)\cdot\left[-\left(\frac{x-\mu}{\sigma^2}\right)f(x)\right]$$

$$=-\frac{f(x)}{\sigma^2}\left[1-\left(\frac{x-\mu}{\sigma}\right)^2\right]=0$$

$$\left(\frac{x-\mu}{\sigma}\right)^2=1 \qquad \frac{x-\mu}{\sigma}=\pm1 \qquad x=\mu\pm\sigma$$

6.32

$$\frac{1}{f(x)}\frac{df(x)}{dx}=\frac{d-x}{a} \qquad \ln f(x)=-\frac{(d-x)^2}{2a}+c$$

$$f(x)=ke^{-1/2a}(x-d)^2 \qquad \text{QED}$$

6.33 $M''' = [(\mu+\sigma^2 t)^2 + \sigma^2](\mu+\sigma^2 t)M + M[2(\mu+\sigma^2 t)\sigma^2]$

$= M(\mu+\sigma^2 t)[(\mu+\sigma^2 t)^2 + 3\sigma^2]$

$M'''(0) = \mu(\mu^2 + 3\sigma^2) = \mu^3 + 3\mu\sigma^2]$

$M'''' = M(\mu+\sigma^2 t)[2\sigma^2(\mu+\sigma^2 t)] + M[(\mu+\sigma^2 t)^2 + 3\sigma^2]\sigma^2$

$\quad + (\mu+\sigma^2 t)[(\mu+\sigma^2 t)^2 + 3\sigma^2](\mu+\sigma^2 t)M$

$M''''(0) = 3\sigma^4 + 6\sigma^2\mu^2 + \mu^4$

$\mu_3 = \mu^3 + 3\mu\sigma^2 - 3(\mu^2+\sigma^2)\mu + 2\mu^3 = 0$

$\mu_4 = 3\sigma^4 + 6\sigma^2\mu^2 + \mu^4 - 4(\mu^3 + 3\mu\sigma^2)\mu + 6\mu^2(\mu^2+\sigma^2) - 3\mu^4 = 3\sigma^4$

6.35 $\alpha_3 = 0$ and $\alpha_4 = \dfrac{3\sigma^4}{\sigma^4} = 3$

6.36 $M_x(t) = e^{\mu t + (1/2)\sigma^2 t^2}$

$M_{(x-\mu)/\sigma} = e^{-(\mu/\sigma)t} \cdot e^{\mu(t/\sigma) + (1/2)\sigma^2(t/\sigma)^2} = e^{(1/2)t^2}$

6.37 $E(x) = \mu,\ E(x^2) = \sigma^2 + \mu^2,\ E(x^3) = \mu^3 + 3\mu\sigma^2$

$\operatorname{cov}(x, x^2) = (\mu^3 + 3\mu\sigma^2) - \mu(\sigma^2 + \mu^2) = 2\mu\sigma^2$

for standard normal distribution $\mu = 0 \to \operatorname{cov}(x, x^2) = 0$

6.38 $M = e^{(1/2)t^2} = 1 + \dfrac{\left(\frac{1}{2}t^2\right)}{1!} + \dfrac{\left(\frac{1}{2}t^2\right)^2}{2!} + \ldots \underbrace{\dfrac{\left(\frac{1}{2}t^2\right)^{r/2}}{(r/2)!}}_{\downarrow}$

$$\frac{t^r}{2^{r/2}(r/2)!} = \frac{r!}{2^{r/2}(r/2)!} \cdot \frac{t^r}{r!}$$

(a) $\mu_r = 0$ since coefficient of t with r odd is zero.

(b) $\mu_r = \dfrac{r!}{(r/2)!} \dfrac{1}{2^{r/2}}$ read off for r even.

6.39 $M_{x-\mu} = e^{-\mu t} M_x(t) \qquad K_x(t) = -\mu t + \ln M_x(t)$

$$M_x(t) = 1 + \mu_1' t + \mu_2' \frac{t^2}{2!} + \mu_3' \frac{t^3}{3!} + \mu_4' \frac{t^4}{4!}$$

$$\ln M_x(t) = \ln\left[1 + \left(\mu_1' t + \mu_2' \frac{t^2}{2!} + \mu_3' \frac{t^3}{3!} + \mu_4' \frac{t^4}{4!} + \ldots\right)\right]$$

$$\ln(1+z) = z - \frac{1}{2}z^2 + \frac{1}{3}z^3 - \frac{1}{4}z^4 + \ldots$$

$$K_x(t) = 1 - \mu t + \left[\mu_1' t + \mu_2' \frac{t^2}{2!} + \mu_3' \frac{t^3}{3!} + \mu_4' \frac{t^4}{4!} + \ldots\right]$$

$$-\frac{1}{2}\left\{\mu_1' t + \mu_2' \frac{t^2}{2!} + \mu_3' \frac{t^3}{3!} + \ldots\right\}^2$$

$$+\frac{1}{3}\left\{\mu_1' t + \mu_2' \frac{t^2}{2!} + \mu_3' \frac{t^3}{3!} + \ldots\right\}^3$$

$$-\frac{1}{4}\left\{\mu_1' t + \mu_2' \frac{t^2}{2!} + \mu_3' \frac{t^3}{3!} + \ldots\right\}^4$$

$$= \frac{t^2}{2!}\left[\mu_2' - (\mu_1')\right]^2 + \frac{t^2}{3!}\left[\mu_2'\mu_1' + 2(\mu_1')^2\right] + \frac{t^2}{4!}\left[\mu_4' - 3(\mu_2')^2 - 4\mu_1'\mu_3' + 12(\mu_1')^2 \mu_2' - 6(\mu_1')^4\right] + \ldots$$

(a) $K_2 = \mu_2$, **(b)** $K_3 = \mu_3$, **(c)** $K_4 = \mu_4 - 3\mu_2^2$

6.40 $M_{x-\mu} = e^{-\mu t} M_x(t) = e^{-\mu t + \mu t + (1/2)t^2\sigma^2}$

$$\ln M_{x-\mu}(t) = \frac{1}{2}t^2\sigma^2$$

$$K_x(t) = \frac{1}{2}t^2\sigma^2$$

$K_1 = 0,\ K_2 = \sigma^2;\ K_r = 0 \text{ for } r > 2$

6.41 $M_x(t) = e^{\lambda(e^t - 1)} \qquad \mu = \lambda,\ \sigma = \sqrt{\lambda}$

$$M_{(x-\mu)/\sigma}(t) = e^{-(\mu/\sigma)t} M_x\left(\frac{t}{\sigma}\right) = e^{-\sqrt{\lambda}t} e^{\lambda(e^{t/\sigma} - 1)}$$

$$\ln M_{(x-\mu)/\sigma}(t) = -\sqrt{\lambda}t + \lambda(e^{t/\sigma} - 1)$$

$$= -\sqrt{\lambda}t + \lambda(e^{t/\sqrt{x}} - 1)$$

$$= -\sqrt{\lambda}t + \lambda\left[\frac{t}{\sqrt{\lambda}} + \frac{t^2}{2\lambda} + \frac{t^3}{3\lambda\sqrt{\lambda}} + \ldots\right]$$

$$= -\sqrt{\lambda}t + \sqrt{\lambda}t + \frac{t^2}{2} + \frac{t^3}{3\sqrt{\lambda}} + \ldots$$

$\lambda \to \infty \qquad = \frac{1}{2}t^2$

6.42 $M_x(t) = (1-\beta t)^{-\alpha}$ $\mu = \alpha\beta,\ \sigma = \beta\sqrt{\alpha}$

$$M_{(x-\mu)/\sigma} = e^{-\sqrt{\alpha}t}\left(1-\frac{t}{\sqrt{\alpha}}\right)^{-\alpha}$$

$$\ln M_{(x-\mu)/\alpha} = -\sqrt{\alpha}t - \alpha\ln\left(1-\frac{t}{\sqrt{\alpha}}\right) \qquad \ln(1+z) = +z + \frac{z^2}{2} + \frac{z^3}{3} + \ldots$$

$$= -\sqrt{\alpha}t + \alpha\left[\frac{t}{\sqrt{a}} - \frac{t^2}{2\alpha} + \frac{t^3}{3\alpha\sqrt{\alpha}}\ldots\right] = +\frac{t^2}{2} \text{ when } \alpha \to \infty$$

6.43 **(a)** Constant terms of $g(x)$ and $h(y)$ are $\dfrac{1}{\sigma_1\sqrt{2\pi}}$ and $\dfrac{1}{\sigma_2\sqrt{2\pi}}$

Constant term of $f(x,y) = \dfrac{1}{2\pi\sigma_2\sigma_2\sqrt{1-p^2}}$

If independent then $\dfrac{1}{2\pi\sigma_1\sigma_2\sqrt{1-p^2}} = \dfrac{1}{\sigma_1\sqrt{2\pi}}\cdot\dfrac{1}{\sigma_2\sqrt{2\pi}}\sqrt{1-p^2} = 1,\ p = 0$

(b) Substitute $p = 0$ into $f(x, y)$ and it becomes product of $g(x)$ and $h(y)$.

6.44 Substitute $y = a + bx$ into $f(x, y)$

6.45 **(a)** $\mu_1 = -2,\ \mu_2 = 1;$ Let k^2 be suitable constant.

$\dfrac{k^2}{\sigma_1^1} = 1,\ \dfrac{k^2}{\sigma_2^2} = 4,\ \dfrac{2pk^2}{\sigma_1\sigma_2} = 2.8,$ so that $\sigma_1 = k,\ \sigma_2 = \dfrac{k}{2}$ and $\dfrac{2pk^2}{k^2/2} = 2.8,$

$$4p = 2.8,\ p = 0.7$$

$$-\frac{1}{2(1-p^2)} = \frac{-1}{2(0.51)} = \frac{-1}{1.02}$$

$$-\frac{1}{102}\left[\left(\frac{x+2}{10}\right)^2 - 2.8\left(\frac{x+2}{10}\right)\left(\frac{y-1}{10}\right) + \left(\frac{y-1}{5}\right)^2\right]$$

so that $\sigma_1 = 10$ and $\sigma_2 = 5$

6.46 Equating coefficients of $x^2,\ xy,$ and y^2 with those of bivariate normal density

$27 = (1-\rho^2)\sigma_1^2$ multiply first and third and divide by square of second

$$-27 = \frac{(1-\rho^2)\sigma_1\sigma_2}{\rho}$$

$27 = 4(1-\rho^2)\sigma_2^2$ $\qquad \dfrac{27\cdot 27}{(-27)^2} = \dfrac{4(1-\rho^2)^2\sigma_1^2\sigma_2^2}{(1-\rho^2)^2\sigma_1^2\sigma_2^2}\cdot\rho^2$

$\rho^2 = \dfrac{1}{4} \qquad \rho = \pm\dfrac{1}{2}$

from second equation must be $\rho = -\dfrac{1}{2}$

$$\sigma_1^2 = \frac{27}{0.75} = 36,\ \sigma_1 = 6$$

$$\sigma_2^2 = \frac{27}{4(0.75)} = 9,\ \sigma_2 = 3$$

6.47 $\mu_1 = 2,\ \mu_2 = 5,\ \sigma_1 = 3,\ \sigma_2 = 6,\ p = \dfrac{2}{3}$

$$\mu_{Y|1} = 5 + \frac{2}{3}\cdot\frac{6}{3}(1-2) = 5 - \frac{4}{3} = \frac{11}{3}$$

$$\sigma_{Y|1}^2 = 36\left(1 - \frac{4}{9}\right) = \frac{36\cdot 5}{9} = 20 \qquad \sigma_{Y|1} = \sqrt{20} = 4.47$$

6.48 $U = X + Y,\ V = X - Y$

$$E(U) = \mu_1 + \mu_2,\ E(V) = \mu_1 = \mu_2$$

$$\sigma_U^2 = \sigma_1^2 + \sigma_2^2 + 2\rho\sigma_1\sigma_2$$

$$\sigma_V^2 = \sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2$$

$$E(UV) = E[(X+Y)(X-Y)] = E(X^2 - Y^2) = \sigma_1^2 + \mu_1^2 - \sigma_2^2 - \mu_2^2$$

$$\operatorname{cov}(UV) = \sigma_1^2 + \mu_1^2 - \sigma_2^2 - \mu_2^2 - (\mu_1 + \mu_2)(\mu_1 - \mu_2) = \sigma_1^2 - \sigma_2^2$$

$$\rho = \frac{\sigma_1^2 - \sigma_2^2}{\sqrt{\left(\sigma_1^2 + \sigma_2^2 + 2\rho\sigma_1\sigma_2\right)\left(\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2\right)}}$$

$$\rho = \frac{\sigma_1^2 - \sigma_2^2}{\sqrt{\left(\sigma_1^2 - \sigma_2^2\right)^2 - 4\rho^2\sigma_1^2\sigma_2^2}}$$

6.49 **(a)** $M(t_1, t_2) = e^{t_1\mu_1 + t_2\mu_2 + (1/2)[\sigma_1^2 t_1^2 + 2\rho\sigma_1\sigma_2 t_1 t_2 + \sigma_2^2 t_2^2]} = e^Q$

$$\frac{\partial}{\partial t_1} = (\mu_1 + \sigma_1^2 t_1 + \rho\sigma_1\sigma_2 t_2)e^Q = \mu_1 \text{ at } t_1 = t_2 = 0$$

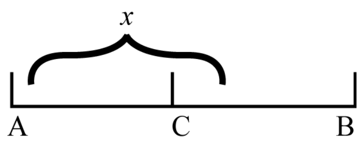
(b) $\dfrac{\partial^2}{\partial t_1^2} = (\mu_1 + \sigma_1^2 t_1 + \rho\sigma_1\sigma_2 t_2)^2 e^Q + \sigma_1^2 e^Q = (\mu_1^2 + \sigma_1^2) = \sigma_1^2 + \mu_1^2$ at $t_1 = t_2 = 0$

(c) $\dfrac{\partial^2}{\partial t_1 \partial t_2} = (\mu_1 + \sigma_1^2 t_1 + \rho\sigma_1\sigma_2 t_2)e^Q(\mu_2 + \sigma_2^2 t_2 + \rho\sigma_1\sigma_2 t_2) + \rho\sigma_1\sigma_2 \cdot e^Q$

$$= \mu_1\mu_2 + \rho\sigma_1\sigma_2 \text{ at } t_1 = t_2 = 0$$

6.50 **(a)** $\dfrac{0.003 - (0.002)}{0.03} = \dfrac{0.005}{0.030} = \dfrac{1}{6}$; **(b)** $\dfrac{2(0.1)}{0.03} = \dfrac{2}{3}$

6.51



$$x + (a - x) > \frac{a}{2} \qquad a > \frac{a}{2}$$

$$x + \frac{a}{2} > a - x \qquad x > \frac{a}{4}$$

$$(a - x) + \frac{a}{2} > x \qquad x < \frac{3}{4}a$$

Probability is $\frac{1}{2}$

$\alpha = -0.015$ and $\beta = 0.015$, $\beta - \alpha = 0.03$

6.52 $\alpha = 3,\ \beta = 2$

$$\rho = \frac{1}{8 \cdot 2}\int_{12}^{\infty} x^2 e^{-x/2}\ dx = \frac{1}{16}\left[\frac{x^2 3^{-(1/2)x}}{-1/2} - \frac{2}{-1/2}\cdot\frac{e^{(-1/2)x}}{1/4}\left(-\frac{1}{2}x - 1\right)\right]\Bigg|_{12}^{\infty}$$

$$= \frac{1}{16}\left[-2x^2 e^{-(1/2)x} + 16e^{-(1/2)x}\left(\frac{1}{2}x + 1\right)\right]\Bigg|_{12}^{\infty}$$

$$= \frac{1}{16}\left[288e^{-6} + 16e^{-6}0.7\right] = 25e^{-6} = 25(0.002479) = 0.062$$

6.53 $\mu = \alpha\beta = 80 \cdot 2\sqrt{n} = 160\sqrt{n}$

$$E = 160\sqrt{n} - 8n \qquad \frac{dE}{dn} = \frac{160}{2\sqrt{n}} - 8 = 0 \qquad n = 100$$

6.54 **(a)** $$\int_0^{24} \frac{1}{120} e^{-(1/120)x} dx = -e^{-x/120}\Big|_0^{24} = 1 - e^{-0.2} = 1 - 0.8187 = 0.1813$$

(b) $$\int_{180}^{\infty} \frac{1}{120} e^{-1/120} dx = -e^{-x/120}\Big|_{180}^{\infty} = e^{-1.5} = 0.2231$$

6.55 **(a)** $$\int_{20}^{\infty} \frac{1}{40} e^{-(1/40)x} dx = -e^{-x/40}\Big|_{20}^{\infty} = e^{-1/2} = 0.6065$$

(b) $$\int_0^{30} \frac{1}{40} e^{-(1/40)x} dx = -e^{-x/40}\Big|_0^{30} = 1 - e^{-3/4} = 1 - 0.4724 = 0.5276$$

6.56 $\lambda = 0.4$ per hour $\displaystyle\int_2^{\infty} 0.4e^{-0.4t} dt = -e^{-0.4t}\Big|_2^{\infty} = e^{-0.8} = 0.4493$

6.57 $\lambda = 1.2$ per hour $\displaystyle\int_1^{\infty} 1.2e^{-1.2t} dt = -e^{-1.2t}\Big|_1^{\infty} = e^{-1.2} = 0.1827$

6.58 $\alpha = 2,\ \beta = 9$

$$90\int_0^{0.1} x(1-x)^8\ dx \qquad y = 1-x \qquad dy = -dx$$

$$= 90\int_{0.9}^{1} y^8(1-y)\ dy = 90\left[\frac{1}{9} - \frac{1}{10} - \frac{(0.9)^9}{9} + \frac{(0.9)^{10}}{10}\right] = 0.2463$$

6.59 $\lambda = 0.5 \int_3^{\infty} e^{-0.5t} dt = -e^{-0.5t}\Big|_3^{B} = e^{-1.5} = 0.2231$

6.60 $\alpha = 1,\ \beta = 4$

(a) $\mu = \frac{1}{1+4} = \frac{1}{5}$

(b) $\frac{\Gamma(5)}{\Gamma(1)\Gamma(4)} \int_{0.25}^{1} (1-x)^3 dx = 4\int_0^{0.75} y^2 dy \qquad y = 1-x \quad dy = -dx$

$$= 4 \cdot \frac{y^4}{4}\Big|_0^{0.75} = (0.75)^4 = \left(\frac{3}{4}\right)^4 = \frac{81}{256} = 0.3164$$

6.61 $\alpha = 0.025,\ \beta = 0.5$

(a) $\mu = (0.025)^{-2}\Gamma(3) = \frac{2}{(0.025)^2} = 3200$ hours

(b) $\alpha\beta \int_{4000}^{\infty} x^{\beta-1} e^{-\alpha x^{\beta}} dx \qquad y = \alpha x^{\beta} \qquad y = 0.025 \cdot \sqrt{4000} = 1.58 \quad dy = \alpha\beta x^{\beta-1} dx$

$$= \int_{1.58}^{\infty} e^{-y} dy = e^{-1.58} = 0.2060$$

6.62 **(a)** $0.5 + 0.4082 = 0.9082$
(b) $0.5 + 0.2852 = 0.7852$
(c) $0.3888 - 0.2088 = 0.1800$
(d) $0.4713 + 0.1700 = 0.6413$

6.63 **(a)** $0.5 - 0.3729 = 0.1271$
(b) $0.5 + 0.1406 = 0.6406$
(c) $0.1772 - 0.359 = 0.1413$
(d) $0.2190 + 0.3686 = 0.5876$

6.64 **(a)** $z_1 = 1.48$
(b) $z_2 = -0.74$
(c) $z_3 = 0.55$
(d) $z_4 = 2.17 \qquad 0.4850$

6.65 **(a)** $z = 1.92$
(b) $z = 2.22$
(c) $z = 1.12$ 0.3686
(d) $z = \pm 1.44$ 0.4251

6.66 **(a)** $2(0.3413) = 0.6826$
(b) $2(0.4772) = 0.9544$
(c) $2(0.4987) = 0.9974$
(d) $2(0.49997) = 0.99994$

6.67 **(a)** $z_{0.05} = 1.645$ 0.4500
(b) $z_{0.025} = 1.96$ 0.475
(c) $z_{0.01} = 2.33$ 0.49
(d) $z_{0.005} = 2.575$ 0.495

6.68 **(a)** Using MINITAB and entering $= -2.159$ and 0.5670 into C1, then giving the commands

```
MTB> CDF C1;
SUBC> Normal 1.786 1.0416
```

we get

K	P(X LESS THAN OR = K)
−2.1590	0.3601
0.5670	0.9881

Thus the required probability is $0.9881 - 0.3601 = 0.6280$

(b) $z_1 = \dfrac{-2.159 + 1.786}{1.0416} = -0.958$ $z_2 = \dfrac{0.5670 + 1.786}{1.0416} = 2.25$

The corresponding cumulative probabilities are obtained from Table II (with interpolation) to be 0.3602 and 0.9881. Thus the required probability is $0.9881 - 0.3602 = 0.6279$

6.69 **(a)** Using MINITAB and entering 8.626 into C1,

```
MTB> CDF C1;
SUBC> Normal 5.853 1.361
   K    P(X LESS THAN OR = K)
8.626   .9792
```

Thus, the required probability is $1 - 0.9792 = 0.0208$.

(b) $z = \dfrac{8.625 - 5.853}{1.361} = 2.0367; \quad \therefore\ p = 0.5 - 0.47915 = 0.02085$

6.70 **(a)** $z = \dfrac{44.5 - 37.6}{4.6} = 1.5$ $0.5 - 0.4332 = 0.0668$

(b) $z = \dfrac{35 - 37.6}{4.6} = -.565$ $0.5 - 0.214 = 0.2860$

(c) $z_1 = \dfrac{30 - 37.6}{4.6} = -1.65$ $0.4505 + 0.1985 = 0.6490$

$z_2 = \dfrac{40 - 37.6}{4.6} = 0.52$

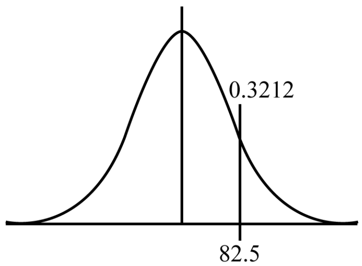
6.71 **(a)** $z = \dfrac{16-15.40}{0.48} = 1.25$ $\qquad 0.5-0.3944 = 0.1056$

(b) $z = \dfrac{14.2-15.4}{0.48} = -2.5$ $\qquad 0.5-0.4938 = 0.0062$

(c) $z_1 = \dfrac{15-15.4}{0.48} = -0.83$ $\qquad 2(0.2967) = 0.5934$

$z_2 = 0.83$

6.72



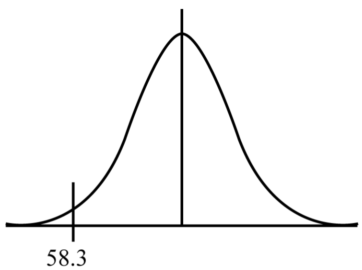
$\dfrac{82.5-\mu}{10} = 0.92$

$82.5-\mu = 9.2$

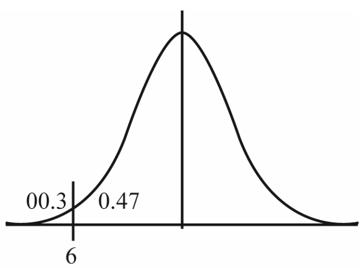
$\mu - 73.3$

$z = \dfrac{58.3-73.3}{10} = -1.5$

$0.5+0.4332 = 0.9332$



6.73



$z = -1.88$ $\qquad \dfrac{6-\mu}{0.05} = -1.88$

$6-\mu = 0.094$

$\mu = 6.094$ ounces

6.74 **(a)** $n\theta = 3.2,\ n(1-\theta) = 15.68$, No

(b) $n\theta = 6.5,\ n(1-\theta) = 58.5$, Yes

(c) $n\theta = 117.6,\ n(1-\theta) = 2.4$, No

6.75 **(a)** $n\theta = 7.5,\ n(1-\theta) = 142.5$, Yes

(b) $\mu = 7.5,\ \sigma^2 = 150(0.05)(0.95) = 7.125,\ \sigma = 2.6693$

$z_1 = \dfrac{0.5-7.5}{2.6693} = -2.6224,$ $\qquad z_2 = \dfrac{1.5-7.5}{2.6693} = -2.2478$

Probability = $0.4956-0.4877 = 0.0079$

(c) $\dfrac{0.0079-0.0036}{0.0036}\cdot 100 = 119\%$

6.76 $n = 14,\ x = 7,\ \theta = \frac{1}{2},\ z_1 = \frac{6.5-7}{1.871} = -0.27,\ z_2 = \frac{7.5-7}{1.871} = 0.27$

$\rho = 2(0.1064) = 0.2128$ Table yields 0.2095

6.77 $\lambda = 7.5,\ p(1;\ 7.5) = \frac{7.5^1 e^{-7.5}}{1!} = 7.5(0.00055) = 0.0041$

6.78 $n = 120,\ \theta = -0.23$

$\mu = 27.6,\ \sigma = \sqrt{21.25} = 4.61$

$z = \frac{32.5-27.6}{4.61} = 1.06$

$0.5 - 0.3554 = 0.1446$

6.79 $n = 225,\ \theta = 0.2,\ \mu = 45,\ \sigma = 6$

$z = \frac{40.5-45}{6} = -0.75$

$0.5 - 0.2734 = 0.2266$

6.80 **(a)** $\mu = 50,\ \sigma = 5,\ z = \frac{51.5-50}{5} = 0.3$

49 to 51 $2(0.1179) = 0.2358 = 0.24$

(b) $\mu = 500,\ \sigma = 15.81,\ z = \frac{510.5-500}{15.81} = 0.664$

490 to 510 $2(0.2454) = 0.49$

(c) $\mu = 5000,\ \sigma = 50,\ z = \frac{5100.5-5000}{50} = 2.01$

4900 to 5100 $2(0.4778) = 0.96$