Mathematics for Computer Science

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Contents

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Notations

- $\forall$: For all, for every possible choice of n
- $\mathbb{N}$: Set of Natural Numbers i.e. $\{0, 1, 2, 3, 4 \ldots\}$
- $\mathbb{R}$: Set of all real numbers
- $\mathbb{Z}$: Set of all integers
- $\mathbb{P}$: Prime number is a number that is only divisible by 1 and itself.
- $\exists$: There exists, there is at least one possible choice of n.
- $\nexists$: There does not exist, there is no possible choice of n.
- $\sin(\theta)$: Sine function
- $\cos(\theta)$: Cosine function
- ∇f : Gradient of function f
- $\int_a^b f(x)\,dx$: Definite integral of f from a to b

1 Introduction to Proofs

A *Proof* is a method for ascertaining (find (something) out for certain; make sure of) the **truth**. Examples of ways in which we ascertain proof:

1. Experimentation and observation (bedrock of physics)
2. Counterexamples
3. Sampling (repeating a process a large number of times)
4. Judge/Jury system
5. Religion
6. Word of boss (authority)
7. Inner conviction

1.1 Mathematical Proof

A *Mathematical Proof* is a verification of a **Proposition** by a chain of **logical deductions** from a set of **axioms**.

1.2 Proposition

A Proposition is a statement that is either true or false. Examples:

- $2 + 3 = 5$
- $\forall n \in \mathbb{N}, n^2 + n + 41$ is a prime number.

1.2.1 Predicate

$$\forall n \in \mathbb{N}, \underbrace{n^2 + n + 41}_{\text{Predicate}}$$

A Predicate is a Proposition whose truth is determined by the values of its variables.

In order to determine if this Proposition is true, we need to make sure that this Predicate is true for every Natural Number $\mathbb{N}$. Lets verify the Predicate by plugging in some values of Natural Numbers. The predicate if false, even though it worked for the first 40 numbers.

n	n^2+n+41	Prime
0	41	✓
1	43	✓
2	47	✓
3	53	✓
$\vdots$	$\vdots$	$\vdots$
20	461	✓
$\vdots$	$\vdots$	$\vdots$
39	1601	✓
40	$1681 = 41^2$	$\times$
41	.	$\times$

1.3 Euler's Theorem

Example 1.1. $a^4 + b^4 + c^4 = d^4$ *has no positive integer solutions.*

This proposition was unsolved for centuries. It was conjectured to be true by Euler in 1769. It was disproved recently using very big numbers. Hence the correct Proposition would be :

$$\exists \quad a, b, c, d \in \mathbb{N}^+, \quad \underbrace{a^4 + b^4 + c^4 = d^4}_{\text{Predicate}}$$

We care about such these proof problems because they involve *Factoring large integers*. It is a way to break crypto systems like RSA which are used for everything we do electronically today.

Example 1.2. *The regions in any map can be coloured in 4 colours such that adjacent regions have different colours.*

2 Induction