XVA Projects M2MO 2024-25: Gap Risk

What follows is a simplified version of Albanese, Armenti, and Crépey (2020, Appendix), with GitHub - yarmenti/CCVA[1]: codes of the paper *XVA Metrics for CCP Optimization*). It illustrates the transfer of counterparty credit risk into liquidity funding risk induced by extensive collateralization.

We denote by Φ and ϕ the standard normal cumulative distribution and density functions. All equations are written with respect to the survival measure of the reference bank (which XVA metrics are calculated) associated with an underlying "fininsurance" probability measure.

1 Market and Credit Model

As common asset driving all our clearing member portfolios, we consider a stylized swap with strike rate $\bar{S}$ and maturity T on an underlying (FX or interest) rate process S. At discrete time points T_l such that $0 < T_1 < T_2 < ... < T_d = T$, the swap pays an amount $h_l(S_{T_{l-1}} - \bar{S})$, where $h_l = T_l - T_{l-1}$. The underlying rate process S is supposed to follow standard Black-Scholes dynamics with risk-neutral drift κ and volatility σ, so that the process $\widehat{S}_t = e^{-\kappa t} S_t$ is a Black martingale with volatility σ. For $t \in [T_0 = 0, T_d = T]$, we denote by l the index such that $T_{l_t - 1} \leq t < T_{l_t}$. The mark-to-market of a long (i.e. floating receiving) position in this swap, with dividend process

$$D_t = \mathrm{Nom} \sum_{l=1}^{d} \mathbf{1}_{T_l \leq t} h_l (S_{T_{l-1}} - \bar{S}),$$

is given by

$$\begin{aligned}
\mathrm{MtM}_t &= \mathbb{E}_t \int_t^T \beta_t^{-1} \beta_s dD_s \\
&= \mathrm{Nom} \times \mathbb{E}_t \big[\beta_t^{-1} \beta_{T_{l_t}} h_{l_t} (S_{T_{l_t - 1}} - \bar{S}) + \sum_{l=l_t+1}^{d} \beta_t^{-1} \beta_{T_l} h_l (S_{T_{l-1}} - \bar{S}) \big] \qquad (1) \\
&= \mathrm{Nom} \times \Big(\beta_t^{-1} \beta_{T_{l_t}} h_{l_t} (S_{T_{l_t - 1}} - \bar{S}) + \beta_t^{-1} \sum_{l=l_t+1}^{d} \beta_{T_l} h_l \big(e^{\kappa T_{l-1}} \widehat{S}_t - \bar{S} \big) \Big),
\end{aligned}$$

by the martingale property of the Black process $\widehat{S}$. The following numerical parameters are used:

[1] CCVA is an acronym for Central Clearing Valuation Adjustment

$$r = 2\%,\ S_0 = 100,\ \kappa = 12\%,\ \sigma = 20\%,\ h_l = 3 \text{ months},\ T = 5 \text{ years}.$$

The nominal (Nom) of the swap is set so that each leg has a time-0 mark-to-market of one (i.e. 10^4 basis points). Figure 1 shows the resulting mark-

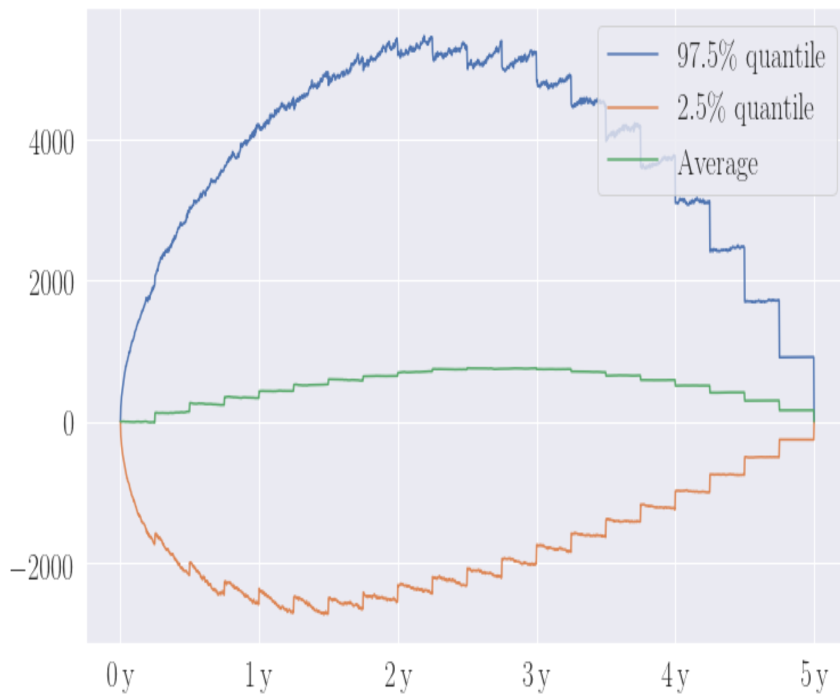

Figure 1: Mean and 2.5% and 97.5% quantiles, in basis points as a function of time, of the process MtM in (1), calculated by Monte Carlo simulation of 5000 Euler paths of the process S. □

to-market process MtM in (1).

In addition, we assume independent exponential default times with a common and constant intensity $\gamma = \gamma_1 = 100bps$ for the bank and its client, with zero recoveries. We assume for simplicity that the margins of each clearing member are updated in continuous time, in particular $\mathrm{VM} = \mathrm{MtM}$, while the bank and its client are non-default, and are stopped before their first-to-default time, until the liquidation of their portfolio occurs after a period of length δ = one week. We denote by $t^\delta = t + \delta$ and by $\Delta_t = \int_{[t-\delta,t]} \beta_t^{-1}\beta_t dD_s$ the cumulative contractual cash flows of the swap accrued at the risk-free rate, accumulated over a past period of length δ.

By (1),

$$\beta_{t^\delta}(\mathrm{MtM}_{t^\delta} + \Delta_{t^\delta}) - \beta_t \mathrm{MtM}_t = \mathrm{Nom} \times f(t) \times (\widehat{S}_{t^\delta} - \widehat{S}_t), \tag{2}$$

with $f(t) = \sum_{l=l_{t^\delta}+1}^{d} \beta_{T_l} h_l e^{\kappa T_{l-1}}$.

Remark 1.1 At least, (2) holds whenever there is no coupon date between t and t^δ. the term $\beta_{T_{l_t}} h_{l_t}(S_{T_{l_t-1}} - \bar{S})$ in (1) induces a small and centered difference

$$\mathrm{Nom} \times h_{l_{t^\delta}} \beta_{T_{l_{t^\delta}}} \left(S_{T_{l_t}} - e^{\kappa T_{l_t}} \widehat{S}_t \right) \tag{3}$$

between the left hand side and the right hand side in (2). As $\delta \approx$ a few days, a coupon between t and t^δ is the exception rather than the rule. Moreover the resulting error (3) is not only rare but also small and centered. As all XVA numbers are time and space averages over future scenarios, we can and do neglect this feature in our numerics. □

2 CVA w/o RIM

Perfect variation margining (erodes the FVA but) leaves room for gap risk CVA.

Lemma 2.1 *Denoting the client's default time by τ_1, we have, for $t \geq 0$,*

$$\mathbb{E}_t \Big[\big(\beta_{t^\delta}(\mathrm{MtM}_{t^\delta} + \Delta_{t^\delta}) - \beta_t \mathrm{MtM}_t \big)^+ \Big] = \mathrm{Nom}\, A(t)\, \widehat{S}_t, \tag{4}$$

where

$$A(t) = f(t) \left[\Phi\left(\frac{\sigma\sqrt{\delta}}{2} \right) - \Phi\left(-\frac{\sigma\sqrt{\delta}}{2} \right) \right]. \tag{5}$$

Proof. In view of (2),

$$\big(\beta_{t^\delta}(\mathrm{MtM}_{t^\delta} + \Delta_{t^\delta}) - \beta_t \mathrm{MtM}_t \big)^+ = \mathrm{Nom} \times f(t) \left(\widehat{S}_{t^\delta} - \widehat{S}_t \right)^+ .$$

Hence the result follows from the Black formula. □

Proposition 2.1 *The* CVA *of the bank is given, at time 0, by*

$$\mathrm{CVA}_0 = \mathrm{Nom}\, S_0 \int_0^T A(t) e^{-\int_0^t \gamma_1(s) ds} \gamma_1(t) dt. \tag{6}$$

Proof. We have

$$
\begin{aligned}
\mathrm{CVA}_0 &= \mathbb{E}\left[\left(\beta_{\tau_1^\delta}(\mathrm{MtM}_{\tau_1^\delta} + \Delta_{\tau_1^\delta}) - \beta_{\tau_1}\mathrm{MtM}_{\tau_1}\right)^+\right] \\
&= \mathbb{E}\left[\mathbb{E}_{\tau_1}\left(\left(\beta_{\tau_1^\delta}(\mathrm{MtM}_{\tau_1^\delta} + \Delta_{\tau_1^\delta}) - \beta_{\tau_1}\mathrm{MtM}_{\tau_1}\right)^+\right)\right] \\
&= \mathbb{E}\int_0^T \mathbb{E}_t\left[\left(\beta_{t^\delta}(\mathrm{MtM}_{t^\delta} + \Delta_{t^\delta}) - \beta_t\mathrm{MtM}_t\right)^+\right] e^{-\int_0^t \gamma_1(s)ds}\gamma_1(t)dt,
\end{aligned}
\tag{7}
$$

by application of Crépey et al. (2014, Lemma 13.7.5 page 331-332). We conclude the proof by an application of Lemma 2.1 and of the martingale property of $\widehat{S}$. □

3 MVA

To erode gap risk and the ensuing CVA, one can introduce initial margins, which on the other hand induce a funding cost for posting these margins. We thus assume (focusing on the viewpoint of the reference bank)

$$
\beta_t\mathrm{PIM}_t = \left(\mathbb{V}\mathrm{a}\mathbb{R}_t^a\left(- \beta_{t^\delta}(\mathrm{MtM}_{t^\delta} + \Delta_{t^\delta}) + \beta_t\mathrm{MtM}_t\right)\right)^+, \tag{8}
$$

for some PIM quantile level a.

Lemma 3.1 *For $t \le T$, we have*

$$
\mathbb{V}\mathrm{a}\mathbb{R}_t^a\left(- \beta_{t^\delta}(\mathrm{MtM}_{t^\delta} + \Delta_{t^\delta}) + \beta_t\mathrm{MtM}_t\right) = \mathrm{Nom} \times B(t) \times \widehat{S}_t, \tag{9}
$$

where

$$
B(t) = f(t) \times \left(1 - e^{\sigma\sqrt{\delta}\Phi^{-1}(1-a) - \frac{\sigma^2}{2}\delta}\right). \tag{10}
$$

Proof. This follows from (8) and (2) in view of the Black model used for $\widehat{S}$. □

Proposition 3.1 *The (unsecured borrowing)* MVA *of the bank is given, at time 0, by*

$$
\mathrm{MVA}_0 = \mathrm{Nom}\, S_0 \int_0^T B(t)\gamma(t)e^{-\int_0^t \gamma_1(s)ds}dt. \tag{11}
$$

Proof. We have

$$\begin{aligned}\mathrm{MVA}_0 &= \mathbb{E}\int_0^T \beta_t\gamma(t)\mathrm{PIM}_t\mathbb{1}_{\{t<\tau_1\}}dt \\ &= \mathbb{E}\int_0^T \beta_t\gamma(t)\mathrm{PIM}_t e^{-\int_0^t \gamma_1(s)ds}dt,\end{aligned} \tag{12}$$

by application of Crépey et al. (2014, Lemma 13.7.5 page 331-332). We conclude the proof by an application of Lemma 3.1 and of the martingale property of $\widehat{S}$. □

4 CVA w/ RIM

We assume likewise

$$\beta_t\mathrm{RIM}_t = \Big(\mathbb{V}\mathrm{a}\mathbb{R}_t^a\Big(\beta_{t^\delta}(\mathrm{MtM}_{t^\delta}+\Delta_{t^\delta}) - \beta_t\mathrm{MtM}_t\Big)\Big)^+. \tag{13}$$

With the same notation as in Lemma 3.1, we have much like (9):

$$\mathbb{V}\mathrm{a}\mathbb{R}_t^a\Big(\beta_{t^\delta}(\mathrm{MtM}_{t^\delta}+\Delta_{t^\delta}) - \beta_t\mathrm{MtM}_t\Big) = \mathrm{Nom}\times\bar{B}(t)\times\widehat{S}_t, \tag{14}$$

where $\bar{B}(t) = f(t)\times\big(e^{\sigma\sqrt{\delta}\Phi^{-1}(a)-\frac{\sigma^2}{2}\delta}-1\big)$.

Lemma 4.1 *We have, for $s\le T$:*

$$\mathbb{E}_t\left[(\beta_{t^\delta}(\mathrm{MtM}_{t^\delta}+\Delta_{t^\delta}) - \beta_s(\mathrm{MtM}_t+\mathrm{RIM}_t))^+\right] = \mathrm{Nom}\times C(t)\times\widehat{S}_t, \tag{15}$$

where

$$C(t) = (1-a)\times f(t)\times e^{-\frac{\sigma^2\delta}{2}}\big(e^{\sigma\sqrt{\delta}\frac{\phi(\Phi^{-1}(a))}{1-a}} - e^{\sigma\sqrt{\delta}\Phi^{-1}(a)}\big). \tag{16}$$

Proof. In view of (13)-(14) and (2), the conditional version of the identity

$$\mathbb{E}[X\mathbb{1}_{X\ge\mathbb{V}\mathrm{a}\mathbb{R}^a(X)}] = (1-a)\mathbb{E}\mathbb{S}^a(X)$$

yields

$$\begin{aligned}&\mathbb{E}_t\left[(\beta_{t^\delta}(\mathrm{MtM}_{t^\delta}+\Delta_{t^\delta}) - \beta_t(\mathrm{MtM}_t+\mathrm{RIM}_t))^+\right] \\ &= (1-a)\times(\mathbb{E}\mathbb{S}_t^a - \mathbb{V}\mathrm{a}\mathbb{R}_t^a)\left(\beta_{t^\delta}(\mathrm{MtM}_{t^\delta}+\Delta_{t^\delta}) - \beta_t\mathrm{MtM}_t\right) \\ &= \mathrm{Nom}\times(1-a)\times f(t)\left[\mathbb{E}\mathbb{S}_t^a(\widehat{S}_{t^\delta}-\widehat{S}_t) - \mathbb{V}\mathrm{a}\mathbb{R}_t^a\left(\widehat{S}_{t^\delta}-\widehat{S}_t\right)\right].\end{aligned} \tag{17}$$

The result follows in view of the Black model used for $\widehat{S}$. □

Proposition 4.1 *The* CVA^{rim} *of the bank w/ RIM is given, at time 0, by*

$$\mathrm{CVA}_0^{rim} = \mathrm{Nom}\, S_0 \int_0^T C(t) e^{-\int_0^t \gamma_1(s)ds} \gamma_1(t) dt. \tag{18}$$

Proof. We have

$$\begin{aligned}
\mathrm{CVA}_0^{rim} &= \mathbb{E}\left[\left(\beta_{\tau_1^\delta}(\mathrm{MtM}_{\tau_1^\delta} + \Delta_{\tau_1^\delta}) - \beta_{\tau_1}\mathrm{MtM}_{\tau_1} - \beta_{\tau_1}\mathrm{RIM}_{\tau_1}\right)^+\right] \\
&= \mathbb{E}\left[\mathbb{E}_{\tau_1}\left(\left(\beta_{\tau_1^\delta}(\mathrm{MtM}_{\tau_1^\delta} + \Delta_{\tau_1^\delta}) - \beta_{\tau_1}\mathrm{MtM}_{\tau_1} - \beta_{\tau_1}\mathrm{RIM}_{\tau_1}\right)^+\right)\right] \\
&= \mathbb{E}\int_0^T \mathbb{E}_t\left[\left(\beta_{t^\delta}(\mathrm{MtM}_{t^\delta} + \Delta_{t^\delta}) - \beta_t\mathrm{MtM}_t - \beta_t\mathrm{RIM}_t\right)^+\right] e^{-\int_0^t \gamma_1(s)ds}\gamma_1(t)dt,
\end{aligned} \tag{19}$$

by application of Crépey et al. (2014, Lemma 13.7.5 page 331-332). We conclude the proof by an application of Lemma 2.1 and of the martingale property of $\widehat{S}$. □

5 Project Topics

Under jupyter python notebook / colab with GPU / torch as in the tutorial:

5.1 *Warm up for All*: CVA w/o RIM

1 Linear regress vs. train a (one layer) linear neural network to learn the left-hand side in (4) at grid times t.

2 Validate by (all at each grid time t)

   a. the formula (4),

   b. twin Monte Carlo for conditional expectations,

   c. nested Monte Carlo.

3 Deduce CVA_0 by numerical integration and Monte Carlo based on your predictors plugged in the last line in (7). Validate the result by the explicit formula (6).

4 Replacing "Linear regress vs. train a (one layer) linear neural network to learn" by "Polynomial vs. (nonlinear) NN train", redo 1.–3. above for an at-the-money call instead of the above swap, i.e. $\mathrm{MtM}_t = call^{bs}(t, S_t; T, K)$ with $K = S_0$ (and $\Delta = 0$ as there are

there no intermediary cash flows involved). You then lose the closed formulas, but the twin and the nested Monte Carlo still allow validating your predictors.

5 Note that if it was only for computing CVA_0, learning the conditional expectations in (7) is in fact unnecessary (as the tower rule allows replacing $\mathbb{E}_t$ by $\mathbb{E}$ in the last line of (7)). Use this observation to validate the CVA_0 of point 4 by a companion Monte Carlo procedure.

Hereafter, instead, learning is necessary.

5.2 MVA for PIM

1 Detail the proofs of Lemma 3.1 and Proposition 3.1.

2 Train a (one layer) linear neural network[2] to pinball-learn the left-hand side in (9) at grid times t.

3 Validate by (all at each grid time t)

a. the formula (9),

b. twin Monte Carlo for quantile regression,

c. nested Monte Carlo.

4 Deduce MVA_0 by numerical integration and Monte Carlo based on your learners plugged in the last line in (12). Validate your MVA by the deterministic formula (11).

5 Replacing "Train a (one layer) linear neural network to pinball-learn" by "Polynomial vs. NN pinball-train", redo 1.–3. above for an at-the-money call instead of the above swap, i.e. $\mathrm{MtM}_t = call^{bs}(t, S_t; T, K)$ with $K = S_0$ (and $\Delta = 0$ as there are there no intermediary cash flows). You then lose the closed formulas, but the twin and the nested Monte Carlo still allow validating your predictors.

6 Redo the above for different levels of a simulteneously by randomizing a the way detailed Sections 5 and 7 of Barrera, Crépey, Gobet, Nguyen, and Saadeddine (2022) (see in particular their figure 2).

[2]in fact in this linear swap case one could also proceed by linear programming, cf. https://en.wikipedia.org/wiki/Quantile_regression, "Computation of estimates for regression parameters".

5.3 CVA w/ RIM

1 Detail the proofs of Lemma 4.1 and Proposition 4.1.

2 Train a (one layer) linear neural network[3] to pinball-learn (followed by linear regression for the ES) the expression of the second line in (17) at grid times t.

3 Validate by (all at each grid time t)

a. the formula (15),

b. twin Monte Carlo for quantile regression,

c. nested Monte Carlo.

4 Deduce CVA_0^{rim} by numerical integration and Monte Carlo based on your learners plugged in the last line in (19). Validate the result by the deterministic formula (18).

5 Replacing “Train a (one layer) linear neural network to pinball-learn” by “Polynomial vs. NN pinball-train”, redo 1.–3. above for an at-the-money call instead of the above swap, i.e. $\mathrm{MtM}_t = call^{bs}(t, S_t; T, K)$ with $K = S_0$ (and $\Delta = 0$ as there are there no intermediary cash flows). You then lose the closed formulas, but the twin and the nested Monte Carlo still allow validating your predictors.

6 Redo the above for different levels of a simulteneously by randomizing a the way detailed Sections 5 and 7 of Barrera, Crépey, Gobet, Nguyen, and Saadeddine (2022).

References

Albanese, C., Y. Armenti, and S. Crépey (2020). XVA metrics for CCP optimization. *Statistics & Risk Modeling 37*(1-2), 25–53.

Barrera, D., S. Crépey, E. Gobet, H.-D. Nguyen, and B. Saadeddine (2022). Statistical Learning of value-at-risk and expected shortfall, arXiv:2209.06476v2 and https://perso.lpsm.paris/∼crepey/.

[3] in fact in this linear swap case one could also proceed by linear programming, cf. https://en.wikipedia.org/wiki/Quantile_regression, “Computation of estimates for regression parameters”.

Crépey, S., T. R. Bielecki, and D. Brigo (2014). *Counterparty Risk and Funding: A Tale of Two Puzzles*. Chapman & Hall/CRC Financial Mathematics Series.