2.6.2 Summary

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Summary

A linear transformation is a vector function that has the following two properties:

- Transforming a scaled vector is the same as scaling the transformed vector:

$$L(\alpha x) = \alpha L(x)$$

- Transforming the sum of two vectors is the same as summing the two transformed vectors:

$$L(x + y) = L(x) + L(y)$$

$L : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation if and only if (iff) for all $u, v \in \mathbb{R}^n$ and $\alpha, \beta \in \mathbb{R}$

$$L(\alpha u + \beta v) = \alpha L(u) + \beta L(v).$$

If $L : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, then

$$L(\beta_0 x_0 + \beta_1 x_1 + \cdots + \beta_{k-1} x_{k-1}) = \beta_0 L(x_0) + \beta_1 L(x_1) + \cdots + \beta_{k-1} L(x_{k-1}).$$

A vector function $L : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation if and only if it can be represented by an $m \times n$ **matrix**, which is a very special two dimensional array of numbers (elements).

The **set of all real valued** $m \times n$ **matrices** is denoted by $\mathbb{R}^{m \times n}$.

Let A is the matrix that represents $L : \mathbb{R}^n \to \mathbb{R}^m$, $x \in \mathbb{R}^n$, and let

$$
\begin{array}{rcll}
A & = & \left(\begin{array}{c|c|c|c} a_0 & a_1 & \cdots & a_{n-1} \end{array} \right) & (a_j \text{ equals the } j\text{th column of } A) \\
 & = & \left(\begin{array}{cccc} \alpha_{0,0} & \alpha_{0,1} & \cdots & \alpha_{0,n-1} \\ \alpha_{1,0} & \alpha_{1,1} & \cdots & \alpha_{1,n-1} \\ \vdots & \vdots & & \vdots \\ \alpha_{m-1,0} & \alpha_{m-1,1} & \cdots & \alpha_{m-1,n-1} \end{array} \right) & (\alpha_{i,j} \text{ equals the } (i,j) \text{ element of } A). \\
x & = & \left(\begin{array}{c} \chi_0 \\ \chi_1 \\ \vdots \\ \chi_{n-1} \end{array} \right) &
\end{array}
$$

Then

- $A \in \mathbb{R}^{m \times n}$.
- $a_j = L(e_j) = Ae_j$ (the jth column of A is the vector that results from transforming the unit basis vector e_j).
- $L(x) = L(\sum_{j=0}^{n-1} \chi_j e_j) = \sum_{j=0}^{n-1} L(\chi_j e_j) = \sum_{j=0}^{n-1} \chi_j L(e_j) = \sum_{j=0}^{n-1} \chi_j a_j$
.
-

$$
\begin{array}{rcl}
Ax & = & L(x) \\
& = & \left(\begin{array}{c|c|c|c} a_0 & a_1 & \cdots & a_{n-1} \end{array} \right) \begin{pmatrix} \chi_0 \\ \chi_1 \\ \vdots \\ \chi_{n-1} \end{pmatrix} \\
& = & \chi_0 a_0 + \chi_1 a_1 + \cdots + \chi_{n-1} a_{n-1} \\
& = & \chi_0 \begin{pmatrix} \alpha_{0,0} \\ \alpha_{1,0} \\ \vdots \\ \alpha_{m-1,0} \end{pmatrix} + \chi_1 \begin{pmatrix} \alpha_{0,1} \\ \alpha_{1,1} \\ \vdots \\ \alpha_{m-1,1} \end{pmatrix} + \cdots + \chi_{n-1} \begin{pmatrix} \alpha_{0,n-1} \\ \alpha_{1,n-1} \\ \vdots \\ \alpha_{m-1,n-1} \end{pmatrix} \\
& = & \begin{pmatrix} \chi_0 \alpha_{0,0} + \chi_1 \alpha_{0,1} + \cdots + \chi_{n-1} \alpha_{0,n-1} \\ \chi_0 \alpha_{1,0} + \chi_1 \alpha_{1,1} + \cdots + \chi_{n-1} \alpha_{1,n-1} \\ \vdots \\ \chi_0 \alpha_{m-1,0} + \chi_1 \alpha_{m-1,1} + \cdots + \chi_{n-1} \alpha_{m-1,n-1} \end{pmatrix} \\
& = & \begin{pmatrix} \alpha_{0,0} & \alpha_{0,1} & \cdots & \alpha_{0,n-1} \\ \alpha_{1,0} & \alpha_{1,1} & \cdots & \alpha_{1,n-1} \\ \vdots & \vdots & & \vdots \\ \alpha_{m-1,0} & \alpha_{m-1,1} & \cdots & \alpha_{m-1,n-1} \end{pmatrix} \begin{pmatrix} \chi_0 \\ \chi_1 \\ \vdots \\ \chi_{n-1} \end{pmatrix}.
\end{array}
$$

How to check if a vector function is a linear transformation:

- Check if $f(0) = 0$. If it isn't, it is **not** a linear transformation.
- If $f(0) = 0$ then *either*:
  - Prove it is or isn't a linear transformation from the definition:
    - Find an example where $f(\alpha x) \neq \alpha f(x)$ or $f(x + y) \neq f(x) + f(y)$. In this case the function is *not* a linear transformation; or
    - Prove that $f(\alpha x) = \alpha f(x)$ and $f(x + y) = f(x) + f(y)$ for all α, x, y.

  or

- Compute the *possible* matrix A that represents it and see if $f(x) = Ax$. If it is equal, it is a linear transformation. If it is not, it is not a linear transformation.

Mathematical induction is a powerful proof technique about natural numbers. (There are more general forms of mathematical induction that we will not need in our course.)

The following results about summations will be used in future weeks:

- $\sum_{i=0}^{n-1} i = n(n-1)/2 \approx n^2/2$.
- $\sum_{i=1}^{n} i = n(n+1)/2 \approx n^2/2$.
- $\sum_{i=0}^{n-1} i^2 = (n-1)n(2n-1)/6 \approx \frac{1}{3}n^3$.