7.2.1 When Gaussian Elimination Works

7.2.1 When Gaussian Elimination Works

Video

eventually it can become 0,

then this algorithm fails.

But if a 0 there is never encountered, then we

know that this algorithm completes, and therefore computes an LU factorization.

Once we're done with that, and we are confronted with the right-hand side,

then we first execute a triangular solve for the unit lower triangular matrix.

And notice that that only involves multiplication and subtraction.

So there is no opportunity for that to fail.

And finally when we're done with that, we

take the solution that comes

0:00 / 8:00 1.0x

out of solving with the lower triangular

matrix, and we feed that

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Homework 7.2.1.1

1/1 point (graded)

Let $L \in \mathbb{R}^{1 \times 1}$ be a unit lower triangular matrix.

$Lx = b$, where x is the unknown and b is given, has a unique solution.

Always

Answer: Always

Explanation

Answer: Always

Since L is 1×1, it is a scalar:

$$(1)(\chi_0) = (\beta_0).$$

From basic algebra we know that then $\chi_0 = \beta_0$ is the unique solution.

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Answers are displayed within the problem

Homework 7.2.1.2

2/2 points (graded)

Give the solution of $\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} \chi_0 \\ \chi_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

1

Answer: 1

0

Answer: 0

Answer: The above translates to the system of linear equations

$$\begin{array}{rcrcl} \chi_0 & & & = & 1 \\ 2\chi_0 & + & \chi_1 & = & 2 \end{array}$$

which has the solution

$$\begin{pmatrix} \chi_0 \\ \chi_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 - (2)(1) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

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Answers are displayed within the problem

Homework 7.2.1.3

3/3 points (graded)

Give the solution of $\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 2 & 1 \end{pmatrix} \begin{pmatrix} \chi_0 \\ \chi_1 \\ \chi_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

1 **Answer:** 1

0 **Answer:** 0

4 **Answer:** 4

Answer: A clever way of solving the above is to slice and dice:

$$\underbrace{\left(\begin{array}{cc|c} 1 & 0 & 0 \\ 2 & 1 & 0 \\ \hline -1 & 2 & 1 \end{array}\right)\left(\begin{array}{c} \chi_0 \\ \chi_1 \\ \hline \chi_2 \end{array}\right)}_{\left(\begin{array}{c} \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} \chi_0 \\ \chi_1 \end{pmatrix} \\ \hline \begin{pmatrix} -1 & 2 \end{pmatrix}\begin{pmatrix} \chi_0 \\ \chi_1 \end{pmatrix} + \begin{pmatrix} \chi_2 \end{pmatrix} \end{array}\right)} = \underbrace{\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}}_{\left(\begin{array}{c} \begin{pmatrix} 1 \\ 2 \end{pmatrix} \\ \hline 3 \end{array}\right)}$$

Hence, from the last exercise, we conclude that

$$\begin{pmatrix} \chi_0 \\ \chi_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

We can then compute χ_2 by substituting in:

$$\begin{pmatrix} -1 & 2 \end{pmatrix}\begin{pmatrix} \chi_0 \\ \chi_1 \end{pmatrix} + \begin{pmatrix} \chi_2 \end{pmatrix} = 3$$

So that

$$\chi_2 = 3 - \begin{pmatrix} -1 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 3 - (-1) = 4.$$

Thus, the solution is the vector

$$\begin{pmatrix} \chi_0 \\ \chi_1 \\ \chi_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix}.$$

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Answers are displayed within the problem

Homework 7.2.1.4

1/1 point (graded)

Let $L \in \mathbb{R}^{2\times 2}$ be a unit lower triangular matrix.

$Lx = b$, where x is the unknown and b is given, has a unique solution.

Always ▾ ✔ **Answer:** Always

Explanation

Answer: Always

Since L is 2×2, the linear system has the form

$$\left(\begin{array}{cc} 1 & 0 \\ \lambda_{1,0} & 1 \end{array}\right)\left(\begin{array}{c} \chi_0 \\ \chi_1 \end{array}\right) = \left(\begin{array}{c} \beta_0 \\ \beta_1 \end{array}\right).$$

But that translates to the system of linear equations

$$\begin{array}{ccc} \chi_0 & & = & \beta_0 \\ \lambda_{1,0}\chi_0 & +\chi_1 & = & \beta_1 \end{array}$$

which has the unique solution

$$\left(\begin{array}{c} \chi_0 \\ \chi_1 \end{array}\right) = \left(\begin{array}{c} \beta_0 \\ \beta_1 - \lambda_{1,0}\chi_0 \end{array}\right).$$

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Answers are displayed within the problem

Homework 7.2.1.5

1/1 point (graded)

Let $L \in \mathbb{R}^{3\times 3}$ be a unit lower triangular matrix.

$Lx = b$, where x is the unknown and b is given, has a unique solution.

Always ▾ ✔ **Answer:** Always

Explanation

Answer: Always

Notice

$$\underbrace{\left(\begin{array}{cc|c} 1 & 0 & 0 \\ \lambda_{1,0} & 1 & 0 \\ \hline \lambda_{2,0} & \lambda_{2,1} & 1 \end{array}\right)\left(\begin{array}{c} \chi_0 \\ \chi_1 \\ \hline \chi_2 \end{array}\right)}_{\left(\begin{array}{c} \left(\begin{array}{cc} 1 & 0 \\ \lambda_{1,0} & 1 \end{array}\right)\left(\begin{array}{c} \chi_0 \\ \chi_1 \end{array}\right) \\ \hline \left(\begin{array}{cc} \lambda_{2,0} & \lambda_{2,1} \end{array}\right)\left(\begin{array}{c} \chi_0 \\ \chi_1 \end{array}\right) + \left(\begin{array}{c} \chi_2 \end{array}\right) \end{array}\right)} = \underbrace{\left(\begin{array}{c} \beta_0 \\ \beta_1 \\ \beta_2 \end{array}\right)}_{\left(\begin{array}{c} \left(\begin{array}{c} \beta_0 \\ \beta_1 \end{array}\right) \\ \hline \beta_2 \end{array}\right)}$$

Hence, from the last exercise, we conclude that the unique solutions for χ_0 and χ_1 are

$$\left(\begin{array}{c} \chi_0 \\ \chi_1 \end{array}\right) = \left(\begin{array}{c} \beta_0 \\ \beta_1 - \lambda_{1,0}\beta_0 \end{array}\right).$$

We can then compute χ_2 by substituting in:

$$\left(\begin{array}{cc} \lambda_{2,0} & \lambda_{2,1} \end{array}\right)\left(\begin{array}{c} \chi_0 \\ \chi_1 \end{array}\right) + \left(\begin{array}{c} \chi_2 \end{array}\right) = \beta_2$$

So that

$$\chi_2 = \beta_2 - \left(\begin{array}{cc} \lambda_{2,0} & \lambda_{2,1} \end{array}\right)\left(\begin{array}{c} \beta_0 \\ \beta_1 - \lambda_{1,0}\beta_0 \end{array}\right)$$

Since there is no ambiguity about what χ_2 must equal, the solution is unique:

$$\left(\begin{array}{c} \chi_0 \\ \chi_1 \\ \chi_2 \end{array}\right) = \left(\begin{array}{c} \beta_0 \\ \beta_1 - \lambda_{1,0}\beta_0 \\ \beta_2 - \left(\begin{array}{cc} \lambda_{2,0} & \lambda_{2,1} \end{array}\right)\left(\begin{array}{c} \beta_0 \\ \beta_1 - \lambda_{1,0}\beta_0 \end{array}\right) \end{array}\right).$$

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Answers are displayed within the problem

Homework 7.2.1.6

1/1 point (graded)

Let $L \in \mathbb{R}^{n \times n}$ be a unit lower triangular matrix.

$Lx = b$, where x is the unknown and b is given, has a unique solution.

Always ▾ ✔ **Answer:** Always

Explanation

Always

The last exercises were meant to make you notice that this can be proved with a proof by induction on the size, n, of L.

Base case: $n = 1$. In this case $L = (1)$, $x = (\chi_1)$ and $b = (\beta_1)$. The result follows from the fact that $(1)(\chi_1) = (\beta_1)$ has the unique solution $\chi_1 = \beta_1$.

Inductive step: Inductive Hypothesis (I.H.): Assume that $Lx = b$ has a unique solution for all $L \in \mathbb{R}^{n \times n}$ and right-hand side vectors b.

We now want to show that then $Lx = b$ has a unique solution for all $L \in \mathbb{R}^{(n+1)\times(n+1)}$ and right-hand side vectors b.

Partition

$$L \rightarrow \left(\begin{array}{c|c} L_{00} & 0 \\ \hline l_{10}^T & \lambda_{11} \end{array}\right), \quad x \rightarrow \left(\begin{array}{c} x_0 \\ \hline \chi_1 \end{array}\right) \quad \text{and} \quad b \rightarrow \left(\begin{array}{c} b_0 \\ \hline \beta_1 \end{array}\right),$$

where, importantly, $L_{00} \in \mathbb{R}^{n \times n}$. Then $Lx = b$ becomes

$$\underbrace{\left(\begin{array}{c|c} L_{00} & 0 \\ \hline l_{10}^T & \lambda_{11} \end{array}\right) \left(\begin{array}{c} x_0 \\ \hline \chi_1 \end{array}\right)}_{\left(\begin{array}{c} L_{00}x_0 \\ \hline l_{10}^T x_0 + \lambda_{11}\chi_1 \end{array}\right)} = \left(\begin{array}{c} b_0 \\ \hline \beta_1 \end{array}\right)$$

or

$$\begin{array}{c} L_{00}x_0 = b_0 \\ \hline l_{10}^T x_0 + \lambda_{11}\chi_1 = \beta_1 \end{array}.$$

By the Inductive Hypothesis, we know that $L_{00}x_0 = b_0$ has a unique solution. But once x_0 is set, $\lambda_{11}\chi_1 = \beta_1 - l_{10}^T x_0$ uniquely determines χ_1.

By the **Principle of Mathematical Induction**, the result holds.

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Answers are displayed within the problem

Homework 7.2.1.7

1/1 point (graded)

The proof for the last exercise suggests an alternative algorithm (Variant 2) for solving $Lx = b$ when L is unit lower triangular. Use the below partial algorithm to state this alternative algorithm.

Algorithm: $[b] := \text{LTRSV\_UNB\_VAR2}(L, b)$

Partition $L \rightarrow \left(\begin{array}{c|c} L_{TL} & 0 \\ \hline L_{BL} & L_{BR} \end{array}\right)$, $b \rightarrow \left(\begin{array}{c} b_T \\ \hline b_B \end{array}\right)$

where L_{TL} is 0×0, b_T has 0 rows

while $m(L_{TL}) < m(L)$ **do**

Repartition

$$\left(\begin{array}{c|c} L_{TL} & 0 \\ \hline L_{BL} & L_{BR} \end{array}\right) \rightarrow \left(\begin{array}{c|c|c} L_{00} & 0 & 0 \\ \hline l_{10}^T & \lambda_{11} & 0 \\ \hline L_{20} & l_{21} & L_{22} \end{array}\right), \left(\begin{array}{c} b_T \\ \hline b_B \end{array}\right) \rightarrow \left(\begin{array}{c} b_0 \\ \hline \beta_1 \\ \hline b_2 \end{array}\right)$$

where λ_{11} is 1×1, β_1 has 1 row

Continue with

$$\left(\begin{array}{c|c} L_{TL} & 0 \\ \hline L_{BL} & L_{BR} \end{array}\right) \leftarrow \left(\begin{array}{c|c|c} L_{00} & 0 & 0 \\ \hline l_{10}^T & \lambda_{11} & 0 \\ \hline L_{20} & l_{21} & L_{22} \end{array}\right), \left(\begin{array}{c} b_T \\ \hline b_B \end{array}\right) \leftarrow \left(\begin{array}{c} b_0 \\ \hline \beta_1 \\ \hline b_2 \end{array}\right)$$

endwhile

Next, implement it, yielding

- `[ b_out ] = Ltrsv_unb_var2( L, b )`

You can check that they compute the right answers with the script in

- test_Ltrsv_unb_var2.m

Unfortunately, PictureFLAME does not work for this problem.

done/skip

Algorithm:

Algorithm: $[b] := \text{LTRSV\_UNB\_VAR2}(L, b)$

Partition $L \rightarrow \left(\begin{array}{c|c} L_{TL} & 0 \\ \hline L_{BL} & L_{BR} \end{array}\right)$, $b \rightarrow \left(\begin{array}{c} b_T \\ \hline b_B \end{array}\right)$

where L_{TL} is 0×0, b_T has 0 rows

while $m(L_{TL}) < m(L)$ **do**

Repartition

$$\left(\begin{array}{c|c} L_{TL} & 0 \\ \hline L_{BL} & L_{BR} \end{array}\right) \rightarrow \left(\begin{array}{c|c|c} L_{00} & 0 & 0 \\ \hline l_{10}^T & \lambda_{11} & 0 \\ \hline L_{20} & l_{21} & L_{22} \end{array}\right), \left(\begin{array}{c} b_T \\ \hline b_B \end{array}\right) \rightarrow \left(\begin{array}{c} b_0 \\ \hline \beta_1 \\ \hline b_2 \end{array}\right)$$

where λ_{11} is 1×1 , β_1 has 1 row

$$\beta_1 := \beta_1 - l_{10}^T b_0$$

Continue with

$$\left(\begin{array}{c|c} L_{TL} & 0 \\ \hline L_{BL} & L_{BR} \end{array}\right) \leftarrow \left(\begin{array}{c|c|c} L_{00} & 0 & 0 \\ \hline l_{10}^T & \lambda_{11} & 0 \\ \hline L_{20} & l_{21} & L_{22} \end{array}\right), \left(\begin{array}{c} b_T \\ \hline b_B \end{array}\right) \leftarrow \left(\begin{array}{c} b_0 \\ \hline \beta_1 \\ \hline b_2 \end{array}\right)$$

endwhile

Implementation: `Ltrsv_unb_var2.m`

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Answers are displayed within the problem

Homework 7.2.1.8

1/1 point (graded)

Let $L \in \mathbb{R}^{n \times n}$ be a unit lower triangular matrix.

$Lx = 0$, where 0 is the zero vector of size n, has the unique solution $x = 0$.

Always ▾ Answer: Always

Explanation

Always/Sometimes/Never **Answer:** Always

Obviously $x = 0$ is a solution. But a previous exercise showed that when L is a unit lower triangular matrix, $Lx = b$ has a unique solution for *all* b. Hence, it has a unique solution for $b = 0$.

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Answers are displayed within the problem

Homework 7.2.1.9

1/1 point (graded)

Let $U \in \mathbb{R}^{1 \times 1}$ be an upper triangular matrix with no zeroes on its diagonal.

$Ux = b$, where x is the unknown and b is given, has a unique solution.

Always ▾ ✔ **Answer:** Always

Explanation

Answer: Always

Since U is 1×1, it is a nonzero scalar

$$(\upsilon_{0,0})(\chi_0) = (\beta_0).$$

From basic algebra we know that then $\chi_0 = \beta_0 / \upsilon_{0,0}$ is the unique solution.

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Answers are displayed within the problem

Homework 7.2.1.10

2/2 points (graded)

Give the solution of $\begin{pmatrix} -1 & 1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} \chi_0 \\ \chi_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

0 ✔ **Answer:** 0

1 ✔ **Answer:** 1

Answer: The above translates to the system of linear equations

$$\begin{array}{rrcl} -1\chi_0 & +\chi_1 & = & 1 \\ & 2\chi_1 & = & 2 \end{array}$$

which has the solution

$$\begin{pmatrix} \chi_0 \\ \chi_1 \end{pmatrix} = \begin{pmatrix} (1-\chi_1)/(-1) \\ 2/2 \end{pmatrix} = \begin{pmatrix} (1-(1))/(-1) \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

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Answers are displayed within the problem

Homework 7.2.1.11

3/3 points (graded)

Give the solution of $\begin{pmatrix} -2 & 1 & -2 \\ 0 & -1 & 1 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} \chi_0 \\ \chi_1 \\ \chi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$

-1 ✔ **Answer:** -1

0 ✔ **Answer:** 0

1 ✔ **Answer:** 1

Answer: A clever way of solving the above is to slice and dice:

$$\underbrace{\left(\begin{array}{c|cc} -2 & 1 & -2 \\ \hline 0 & -1 & 1 \\ 0 & 0 & 2 \end{array}\right)\left(\begin{array}{c} \chi_0 \\ \hline \chi_1 \\ \chi_2 \end{array}\right)}_{\left(\begin{array}{c} -2\chi_0 + \begin{pmatrix} 1 & -2 \end{pmatrix}\begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} \\ \hline \begin{pmatrix} -1 & 1 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} \end{array}\right)} = \underbrace{\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}}_{\left(\begin{array}{c} 0 \\ \hline \begin{pmatrix} 1 \\ 2 \end{pmatrix} \end{array}\right)}$$

Hence, from the last exercise, we conclude that

$$\begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

We can then compute χ_0 by substituting in:

$$-2\chi_0 + \begin{pmatrix} 1 & -2 \end{pmatrix}\begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} = 0$$

So that

$$-2\chi_2 = 0 - \begin{pmatrix} 1 & -2 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0 - (-2) = 2.$$

Thus, the solution is the vector

$$\begin{pmatrix} \chi_0 \\ \chi_1 \\ \chi_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}.$$

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Answers are displayed within the problem

Homework 7.2.1.12

1/1 point (graded)

Let $U \in \mathbb{R}^{2\times 2}$ be an upper triangular matrix with no zeroes on its diagonal.

$Ux = b$, where x is the unknown and b is given, has a unique solution.

Always ▾ ✔ **Answer:** Always

Explanation

Answer: Always

Since U is 2×2, the linear system has the form

$$\begin{pmatrix} \upsilon_{0,0} & \upsilon_{0,1} \\ 0 & \upsilon_{1,1} \end{pmatrix} \begin{pmatrix} \chi_0 \\ \chi_1 \end{pmatrix} = \begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix}.$$

But that translates rto the system of linear equations

$$\begin{array}{rcl} \upsilon_{0,0}\chi_0 \quad +\upsilon_{0,1}\chi_1 & = & \beta_0 \\ \upsilon_{1,1}\chi_1 & = & \beta_1 \end{array}$$

which has the unique solution

$$\begin{pmatrix} \chi_0 \\ \chi_1 \end{pmatrix} = \begin{pmatrix} (\beta_0 - \upsilon_{0,1}\chi_1)/\upsilon_{0,0} \\ \beta_1/\upsilon_{1,1} \end{pmatrix}.$$

Submit

Answers are displayed within the problem

Homework 7.2.1.13

1/1 point (graded)

Let $U \in \mathbb{R}^{3\times 3}$ be an upper triangular matrix with no zeroes on its diagonal.

$Ux = b$, where x is the unknown and b is given, has a unique solution.

Always ▾ ✔ **Answer:** Always

Explanation

Answer: Always
Notice

$$\underbrace{\left(\begin{array}{c|cc} \upsilon_{0,0} & \upsilon_{0,1} & \upsilon_{0,2} \\ \hline 0 & \upsilon_{1,1} & \upsilon_{1,2} \\ 0 & 0 & \upsilon_{2,2} \end{array}\right)\left(\begin{array}{c} \chi_0 \\ \hline \chi_1 \\ \chi_2 \end{array}\right)}_{\left(\begin{array}{c} \upsilon_{0,0}\chi_0 + \left(\begin{array}{cc} \upsilon_{0,1} & \upsilon_{0,2} \end{array}\right)\left(\begin{array}{c} \chi_1 \\ \chi_2 \end{array}\right) \\ \hline \left(\begin{array}{cc} \upsilon_{1,1} & \upsilon_{1,2} \\ 0 & \upsilon_{2,2} \end{array}\right)\left(\begin{array}{c} \chi_1 \\ \chi_2 \end{array}\right) \end{array}\right)} = \underbrace{\left(\begin{array}{c} \beta_0 \\ \hline \beta_1 \\ \beta_2 \end{array}\right)}_{\left(\begin{array}{c} \beta_0 \\ \hline \left(\begin{array}{c} \beta_1 \\ \beta_2 \end{array}\right) \end{array}\right)}$$

Hence, from the last exercise, we conclude that the unique solutions for χ_0 and χ_1 are

$$\left(\begin{array}{c} \chi_1 \\ \chi_2 \end{array}\right) = \left(\begin{array}{c} (\beta_1 - \upsilon_{1,2}\chi_2)/\upsilon_{1,1} \\ \beta_2/\upsilon_{2,2} \end{array}\right).$$

We can then compute χ_0 by substituting in:

$$\upsilon_{0,0}\chi_0 + \left(\begin{array}{cc} \upsilon_{0,1} & \upsilon_{0,2} \end{array}\right)\left(\begin{array}{c} \chi_1 \\ \chi_2 \end{array}\right) = \beta_0$$

So that

$$\chi_0 = \left(\beta_0 - \left(\begin{array}{cc} \upsilon_{0,1} & \upsilon_{0,2} \end{array}\right)\left(\begin{array}{c} \chi_1 \\ \chi_2 \end{array}\right)\right)/\upsilon_{0,0}$$

Since there is no ambiguity about what χ_2 must equal, the solution is unique:

$$\left(\begin{array}{c} \chi_0 \\ \chi_1 \\ \chi_2 \end{array}\right) = \left(\begin{array}{c} \left(\beta_0 - \left(\begin{array}{cc} \upsilon_{0,1} & \upsilon_{0,2} \end{array}\right)\left(\begin{array}{c} \chi_1 \\ \chi_2 \end{array}\right)\right)/\upsilon_{0,0} \\ (\beta_1 - \upsilon_{1,2}\chi_2)/\upsilon_{1,1} \\ \beta_2/\upsilon_{2,2} \end{array}\right).$$

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Answers are displayed within the problem

Homework 7.2.1.14

1/1 point (graded)

Let $U \in \mathbb{R}^{n \times n}$ be an upper triangular matrix with no zeroes on its diagonal.

$Ux = b$, where x is the unknown and b is given, has a unique solution.

Always ▾ ✔ **Answer:** Always

Explanation

Answer: Always

Again, the last exercises were meant to make you notice that this can be proved with a proof by induction on the size, n, of U.

Base case: $n = 1$. In this case $U = (\upsilon_{11})$, $x = (\chi_1)$ and $b = (\beta_1)$. The result follows from the fact that $(\upsilon_{11})(\chi_1) = (\beta_1)$ has the unique solution $\chi_1 = \beta_1/\upsilon_{11}$.

Inductive step: Inductive Hypothesis (I.H.): Assume that $Ux = b$ has a unique solution for all upper triangular $U \in \mathbb{R}^{n \times n}$ that do not have zeroes on their diagonal, and right-hand side vectors b.

We now want to show that then $Ux = b$ has a unique solution for all upper triangular $U \in \mathbb{R}^{(n+1)\times(n+1)}$ that do not have zeroes on their diagonal, and right-hand side vectors b.

Partition

$$U \rightarrow \left(\begin{array}{c|c} \upsilon_{11} & u_{12}^T \\ \hline 0 & U_{22} \end{array}\right), \quad x \rightarrow \left(\begin{array}{c} \chi_1 \\ \hline x_2 \end{array}\right) \quad \text{and} \quad b \rightarrow \left(\begin{array}{c} \beta_1 \\ \hline b_2 \end{array}\right),$$

where, importantly, $U_{22} \in \mathbb{R}^{n \times n}$. Then $Ux = b$ becomes

$$\underbrace{\left(\begin{array}{c|c} \upsilon_{11} & u_{12}^T \\ \hline 0 & U_{22} \end{array}\right)\left(\begin{array}{c} \chi_1 \\ \hline x_2 \end{array}\right)}_{\left(\begin{array}{c} \upsilon_{11}\chi_1 + u_{12}^T x_2 \\ \hline U_{22}x_2 \end{array}\right)} = \left(\begin{array}{c} \beta_1 \\ \hline b_2 \end{array}\right)$$

or

$$\begin{array}{c} \upsilon_{11}\chi_1 + u_{12}^T x_2 = \beta_1 \\ \hline U_{22}x_2 = b_2 \end{array}.$$

By the Inductive Hypothesis, we know that $U_{22}x_2 = b_2$ has a unique solution. But once x_2 is set, $\chi_1 = (\beta_1 - u_{12}^T x_2)/\upsilon_{11}$ uniquely determines χ_1.

By the **Principle of Mathematical Induction**, the result holds.

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Answers are displayed within the problem

Homework 7.2.1.15

1/1 point (graded)

Let $U \in \mathbb{R}^{n \times n}$ be an upper triangular matrix with no zeroes on its diagonal.

$Ux = 0$, where 0 is the zero vector of size n, has the unique solution $x = 0$.

Always ▾ ✔ **Answer:** Always

Explanation

Answer: Always
Obviously $x = 0$ is *a* solution. But a previous exercise showed that when U is an upper triangular matrix with no zeroes on its diagonal, $Ux = b$ has a unique solution for *all* b. Hence, it has a unique solution for $b = 0$.

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Homework 7.2.1.16

1/1 point (graded)

Let $A \in \mathbb{R}^{n \times n}$.

If Gaussian elimination completes and the resulting triangular system has no zero coefficients on the diagonal (U has no zeroes on its diagonal). then there is unique solution x to $Ax = b$ for all $b \in \mathbb{R}$.

Always ▾ ✔ **Answer:** Always

Explanation

Answer: Always

We already argued that under the stated conditions, the LU factorization algorithm computes $A \rightarrow LU$ as do the algorithms for solving $Lz = b$ and $Ux = z$. Hence, **a** solution is computed. Now we address the question of whether this is a unique solution.

To show uniqueness, we assume there there are two solutions, r and s, and then show that $r = s$. Assume that $Ar = b$ and $As = b$. Then

$$A \underbrace{(r - s)}_{w} = Ar - As = b - b = 0.$$

Let's let $w = r - s$ so that we know that $Aw = 0$. Now, IF we can show that the assumptions imply that $w = 0$, then we know that $r = s$.

Now, $Aw = 0$ means $(LU)w = 0$ which them means that $L(Uw) = 0$. But, since L is unit lower triangular, by one of the previous exercises we know that $Lz = 0$ has the unique solution $z = 0$. Hence $Uw = 0$. But, since U is an upper triangular matrix with no zeroes on its diagonal, we know from a previous exercise that $w = 0$. Hence $r = s$ and the solution to $Ax = b$ is unique.

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Answers are displayed within the problem