6.2.1 Reducing a System of Linear Equations to an Upper Triangular System

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Dr. Robert van de Geijn: So we're going to look

at how to reduce a system of linear equations

to an upper triangular system, and this is probably a method

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Homework 6.2.1.1

3/3 points (graded)

Practice reducing a system of linear equations to an upper triangular system of linear equations by visiting the "Practice with Gaussian Elimination" webpage we created for you. For now, only work with the top part of that webpage.

Problem 1 in that webpage starts with the system of linear equations

$$\begin{array}{rcrcrcrcr} 1 & \chi_0 & + & 1 & \chi_1 & + & 2 & \chi_2 & = & -1 \\ 3 & \chi_0 & + & 1 & \chi_1 & + & 7 & \chi_2 & = & -7 \\ 1 & \chi_0 & + & 7 & \chi_1 & + & 1 & \chi_2 & = & 7 \end{array}$$

and yields the upper triangular system

$$\begin{array}{rcrcrcrcr} 1 & \chi_0 & + & 1 & \chi_1 & + & 2 & \chi_2 & = & -1 \\ & & & \alpha_{1,1} & \chi_1 & + & 1 & \chi_2 & = & \beta_1 \\ & & & & & & \alpha_{2,2} & \chi_2 & = & -4 \end{array}$$

Enter the values for $\alpha_{1,1}, \alpha_{2,2}, \text{ and } \beta_1$ below:

$\alpha_{1,1} =$

-2

✔ **Answer:** -2

$\alpha_{2,2} =$

2

Answer: 2

$\beta_1 =$

-4

Answer: -4

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Answers are displayed within the problem

Homework 6.2.1.2

3/3 points (graded)

$$\begin{array}{rcrcrcr} -2\chi_0 & + & \chi_1 & + & 2\chi_2 & = & 0 \\ 4\chi_0 & - & \chi_1 & - & 5\chi_2 & = & 4 \\ 2\chi_0 & - & 3\chi_1 & - & \chi_2 & = & -6 \end{array}$$

$$\begin{pmatrix} \chi_0 \\ \chi_1 \\ \chi_2 \end{pmatrix} =$$

-1 **Answer:** -1

2 **Answer:** 2

-2 **Answer:** -2

Answer:

$$\begin{array}{rcrcrcr} -2\chi_0 & + & \chi_1 & + & 2\chi_2 & = & 0 \\ 4\chi_0 & - & \chi_1 & - & 5\chi_2 & = & 4 \\ 2\chi_0 & - & 3\chi_1 & - & \chi_2 & = & -6 \end{array} \longrightarrow \begin{array}{rcrcrcr} -2\chi_0 & + & \chi_1 & + & 2\chi_2 & = & 0 \\ & & \chi_1 & - & \chi_2 & = & 4 \\ & & -2\chi_1 & + & \chi_2 & = & -6 \end{array} \longrightarrow$$

$$\begin{array}{rcrcrcr} -2\chi_0 & + & \chi_1 & + & 2\chi_2 & = & 0 \\ & & \chi_1 & - & \chi_2 & = & 4 \\ & & & & -\chi_2 & = & 2 \end{array} \longrightarrow \left\{ \begin{array}{ccc} -\chi_2 = 2 & \Rightarrow & \chi_2 = -2 \\ \chi_1 - (-2) = 4 & \Rightarrow & \chi_1 = 2 \\ -2\chi_0 + (2) + 2(-2) = 0 & \Rightarrow & \chi_0 = -1 \end{array} \right.$$

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Homework 6.2.1.3

3/3 points (graded)

Compute the coefficients γ_0, γ_1, and γ_2 so that

$$\sum_{i=0}^{n-1} i = \gamma_0 + \gamma_1 n + \gamma_2 n^2$$

Hint: let $p_2(n) = \gamma_0 + \gamma_1 n + \gamma_2 n^2$. Evaluate $p_2(0), p_2(1)$, and $p_2(2)$ by plugging n into the expression. Then also evaluate $\sum_{i=0}^{n-1} i$. This then gives you three equations in three unknowns (the coefficients). Then you solve!

In other words: if $n = 0$ then

$$\gamma_0 + \gamma_1 \times 0 + \gamma_2 \times 0^2 = \sum_{i=0}^{0-1} i$$

or

$$\gamma_0 + 0 \times \gamma_1 + 0 \times \gamma_2 = 0$$

since $\sum_{i=0}^{-1} i = 0$ (because the sum over an "empty range" is defined to equal zero). That is your first equation. Similarly, create the second equation and third equation by setting $n = 1$ and $n = 2$, respectively. Then solve your system of linear equations with three equations in three unknowns.

$\gamma_0 =$

0

✔ **Answer:** 0

$\gamma_1 =$

-.5

✔ **Answer:** -.5

$\gamma_2 =$

.5

✔ **Answer:** .5

Answer: Earlier in this course, as an example when discussing proof by induction and then again later when discussing the cost of a matrix-vector multiplication with a triangular matrix and the solution of a triangular system of equations, we encountered

$$\sum_{i=0}^{n-1} i.$$

Now, you may remember that the equalled some quadratic (second degree) polynomial in n, but not what the coefficients of that polynomial were:

$$\sum_{i=0}^{n-1} i = \gamma_0 + \gamma_1 n + \gamma_2 n^2,$$

for some constant scalars γ_0, γ_1, and γ_2. What if you wanted to determine what these coefficients are? Well, you now know how to solve linear systems, and we now see that determining the coefficients is a matter of solving a linear system.

Starting with

$$p_2(n) = \sum_{i=0}^{n-1} i = \gamma_0 + \gamma_1 n + \gamma_2 n^2,$$

we compute the value of $p_2(n)$ for $n = 0, 1, 2$:

$$\begin{array}{lclclclclcl}
p_2(0) = \sum_{i=0}^{(0)-1} i & = & \gamma_0(0)^0 & + & \gamma_1(0) & + & \gamma_2(0)^2 & = & 0 & = & 0 \\
p_2(1) = \sum_{i=0}^{(1)-1} i & = & \gamma_0(1)^0 & + & \gamma_1(1) & + & \gamma_2(1)^2 & = & 0 & = & 0 \\
\sum_{i=0}^{(2)-1} i & = & \gamma_0(2)^0 & + & \gamma_1(2) & + & \gamma_2(2)^2 & = & 0+1 & = & 1
\end{array}$$

or, in matrix notation,

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{pmatrix} \begin{pmatrix} \gamma_0 \\ \gamma_1 \\ \gamma_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

One can then solve this system to find that

$$\begin{pmatrix} \gamma_0 \\ \gamma_1 \\ \gamma_2 \end{pmatrix} = \begin{pmatrix} 0 \\ -\frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$$

so that

$$\sum_{i=0}^{n-1} i = \frac{1}{2} n^2 - \frac{1}{2} n$$

which equals the $n(n-1)/2$ that we encountered before.

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Answers are displayed within the problem

Homework 6.2.1.4

4/4 points (graded)

Compute the coefficients $\gamma_0, \gamma_1, \gamma_2$ and γ_3 so that

$$6 \sum_{i=0}^{n-1} i^2 = \gamma_0 + \gamma_1 n + \gamma_2 n^2 + \gamma_3 n^3$$

(Note: the "6" is there to make it so the solution only involves integers.)

$\gamma_0 =$

0

Answer: 0

$\gamma_1 =$

1

Answer: 1

$\gamma_2 =$

-3

Answer: -3

$\gamma_3 =$

2

Answer: 2

Answer: (Note: $\sum_{i=0}^{-1}$ anything $= 0$.)

$$\begin{array}{lclclclclcll}
\sum_{i=0}^{(0)-1} i^2 & = & \gamma_0 & + & \gamma_1(0) & + & \gamma_2(0)^2 & + & \gamma_3(0)^3 & = & 0 & = & 0 \\
\sum_{i=0}^{(1)-1} i^2 & = & \gamma_0 & + & \gamma_1(1) & + & \gamma_2(1)^2 & + & \gamma_3(1)^3 & = & 0^2 & = & 0 \\
\sum_{i=0}^{(2)-1} i^2 & = & \gamma_0 & + & \gamma_1(2) & + & \gamma_2(2)^2 & + & \gamma_3(2)^3 & = & 0^2+1^2 & = & 1 \\
\sum_{i=0}^{(3)-1} i^2 & = & \gamma_0 & + & \gamma_1(3) & + & \gamma_2(3)^2 & + & \gamma_3(3)^3 & = & 0^2+1^2+2^2 & = & 5
\end{array}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \end{pmatrix} \begin{pmatrix} \gamma_0 \\ \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 5 \end{pmatrix}$$

Notice that $\gamma_0 = 0$. So

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & 4 & 8 \\ 3 & 9 & 27 \end{pmatrix} \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 5 \end{pmatrix}$$

Solution:

$$\begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{pmatrix} = \frac{1}{6} \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix}$$

so that

$$\sum_{i=0}^{n-1} i^2 = \frac{1}{6}n - \frac{1}{2}n^2 + \frac{1}{3}n^3.$$

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Answers are displayed within the problem