To prove the f(n) = O(g(n)), we can prove "There exists positive constants c and n_0 such that 0 ≤ f(n) ≤ cg(n) for all $n \geq n_0$". If f(n) is 0, then f(n) = O(g(n)) if $g(n) \geq 0$. If f(n) is not 0, previous statement can be rewrite to there exists positive constants c and n_0 such that $\frac{cg(n)}{f(n)} \geq 1 \geq 0$ for all $n \geq n_0$.

(a) By applying l L'Hˆopital's rule, we have

$$\lim_{n\to\infty}\frac{f(n)}{g(n)} = \lim_{n\to\infty}\frac{\log_2 n}{n^{\frac{1}{10}}} = \lim_{n\to\infty}\frac{g'(n)}{f'(n)} = \lim_{n\to\infty}\frac{\frac{1}{nln2}}{\frac{1}{10}*n^{-\frac{9}{10}}} = \lim_{n\to\infty}\frac{10n^{\frac{9}{10}}}{nln2} = \lim_{n\to\infty}\frac{10}{n^{\frac{1}{10}}ln2} \to 0$$

So, we get $\lim_{n\to\infty}\frac{f(n)}{g(n)} \to 0$, which means $\lim_{n\to\infty}\frac{g(n)}{f(n)} \to \infty$. So, there exist a n_0, let c = 1, $\frac{cf(n)}{g(n)} = \frac{f(n)}{g(n)} \geq 1$ for all $n \geq n_0$. Thus, f(n)=O(g(n)).

(b)

$$\frac{g(n)}{f(n)} = \frac{2^{n\log n^2}}{n^n} = \frac{2^{2n\log n}}{2^{\log n^n}} = \frac{2^{2n\log n}}{2^{\mathrm{n}\log n}} = 2^{n\log n}.$$

Let c = 1 and $n_0 = 4$, $\frac{cg(n_0)}{f(n_0)} = 2^8 \geq 1$. Because $2^{n\log n}$ is monotonically increasing, there exist c = 1 and $n_0 = 4$ such that $\frac{cg(n)}{f(n)} \geq 1 \geq 0$ for all $n \geq n_0$. Therefore, f(n) = O(g(n)).

(c)

$$\frac{g(n)}{f(n)} = \frac{n^{1+\sin(\pi n)}}{n} = n^{\sin(\pi n)}$$

. since n should always be integer, $\sin(\pi n) = 0$, $n^{\sin(\pi n)} = 1$. Thus, there let c = 1 and $n_0 = 1$, $\frac{cg(n)}{f(n)} = 1$ and $\frac{cf(n)}{g(n)} = 1$ for all $n \geq n_0$. Therefore, both f(n) = O(g(n)) and g(n) = O(f(n)).