Quantum teleportation

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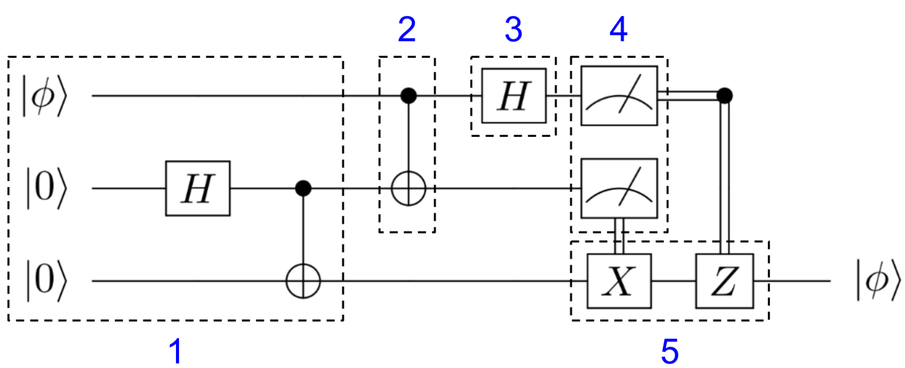
The quantum teleportation protocol was first described in a paper from C. H. Bennett, G. Brassard, C. Crépeau, R. Jozsa, A. Peres, and W. K. Wootters published in 1993. It allows the transfer of an unknown quantum state of a qubit between two parties.

1 The protocol

In order to do so, the two parties, Alice and Bob, must first share an entangled pair of qubits in a maximally entangled state. Alice can then transfer to Bob the unknown quantum state of an auxiliary qubit by measuring her share of the entangled pair and the auxiliary qubit she wants to transfer the state. Depending on the outcome of these measurements, she then tells Bob, through classical communication, which gates he should apply to his qubit in order to convert the state of his share of the entangled pair of qubit into the unknown state Alice wished to transfer.

Let's take a look at the details of the quantum teleportation protocol as depicted in the circuit bellow.



1. Let us Assume that Alice (A) and Bob (B) each have a qubit from a shared entangled pair whose state vector is:

$$|\Phi+\rangle = \frac{|0\rangle_A |0\rangle_B + |1\rangle_A |1\rangle_B}{\sqrt{2}}$$

   Alice wants to transfer to Bob the unknown quantum state $|\phi\rangle$ of another qubit:

$$|\phi\rangle = \alpha |0\rangle + \beta |1\rangle$$

The state vector $|\psi_1\rangle$ of the qubit triplet is then:

$$\begin{aligned}
|\psi_1\rangle &= |\phi\rangle \otimes |\Phi+\rangle \\
&= \left(\alpha |0\rangle_A + \beta |1\rangle_A\right) \otimes \left(\frac{|0\rangle_A |0\rangle_B + |1\rangle_A |1\rangle_B}{\sqrt{2}}\right) \\
&= \frac{\alpha}{\sqrt{2}} |0\rangle_A |0\rangle_A |0\rangle_B + \frac{\alpha}{\sqrt{2}} |0\rangle_A |1\rangle_A |1\rangle_B + \frac{\beta}{\sqrt{2}} |1\rangle_A |0\rangle_A |0\rangle_B + \frac{\beta}{\sqrt{2}} |1\rangle_A |1\rangle_A |1\rangle_B
\end{aligned}$$

2. Next, Alice applies a CNOT gate to her two qubits using the qubit originally in the unknown state $|\phi\rangle$ as the control and the qubit from the entangled pair as the target:

$$|\psi_2\rangle = \frac{\alpha}{\sqrt{2}} |0\rangle_A |0\rangle_A |0\rangle_B + \frac{\alpha}{\sqrt{2}} |0\rangle_A |1\rangle_A |1\rangle_B + \frac{\beta}{\sqrt{2}} |1\rangle_A |1\rangle_A |0\rangle_B + \frac{\beta}{\sqrt{2}} |1\rangle_A |0\rangle_A |1\rangle_B$$

3. Then, Alice applies a Hadamard gate to the qubit originally in the unknown state $|\phi\rangle$:

$$\begin{aligned}
|\psi_3\rangle &= \frac{\alpha}{\sqrt{2}}\left(\frac{|0\rangle_A + |1\rangle_A}{\sqrt{2}}\right)\left(|0\rangle_A |0\rangle_B + |1\rangle_A |1\rangle_B\right) + \frac{\beta}{\sqrt{2}}\left(\frac{|0\rangle_A - |1\rangle_A}{\sqrt{2}}\right)\left(|1\rangle_A |0\rangle_B + |0\rangle_A |1\rangle_B\right) \\
&= \frac{\alpha}{2}\left(|0\rangle_A |0\rangle_A |0\rangle_B + |0\rangle_A |1\rangle_A |1\rangle_B + |1\rangle_A |0\rangle_A |0\rangle_B + |1\rangle_A |1\rangle_A |1\rangle_B\right) \\
&\quad + \frac{\beta}{2}\left(|0\rangle_A |1\rangle_A |0\rangle_B + |0\rangle_A |0\rangle_A |1\rangle_B - |1\rangle_A |1\rangle_A |0\rangle_B - |1\rangle_A |0\rangle_A |1\rangle_B\right) \\
&= \frac{1}{2} |0\rangle_A |0\rangle_A \left(\alpha |0\rangle_B + \beta |1\rangle_B\right) + \frac{1}{2} |0\rangle_A |1\rangle_A \left(\alpha |1\rangle_B + \beta |0\rangle_B\right) \\
&\quad + \frac{1}{2} |1\rangle_A |0\rangle_A \left(\alpha |0\rangle_B - \beta |1\rangle_B\right) + \frac{1}{2} |1\rangle_A |1\rangle_A \left(\alpha |1\rangle_B - \beta |0\rangle_B\right)
\end{aligned}$$

4. Alice measures her two qubits in the computational basis and communicate the measurement outcome a_0a_1 to Bob using a classical channel. Bob's qubit collapses to one of the four following states:

a_0a_1	State of Bob's qubit
00	$\alpha \|0\rangle + \beta \|1\rangle$
01	$\alpha \|1\rangle + \beta \|0\rangle$
10	$\alpha \|0\rangle - \beta \|1\rangle$
11	$\alpha \|1\rangle - \beta \|0\rangle$

5. Depending on the values of the classical bits which Bob receives from Alice, he will apply one of four possible unitary transformations to his qubit:

- If $a_0a_1 = 00$:

$$\alpha |0\rangle + \beta |1\rangle \xrightarrow{I} \alpha |0\rangle + \beta |1\rangle = |\phi\rangle$$

- If $a_0a_1 = 01$:

$$\alpha |1\rangle + \beta |0\rangle \xrightarrow{X} \alpha |0\rangle + \beta |1\rangle = |\phi\rangle$$

- If $a_0a_1 = 10$:

$$\alpha |0\rangle - \beta |1\rangle \xrightarrow{Z} \alpha |0\rangle + \beta |1\rangle = |\phi\rangle$$

- If $a_0 a_1 = 11$:

$$\alpha |1\rangle - \beta |0\rangle \xrightarrow{X} \alpha |0\rangle - \beta |1\rangle \xrightarrow{Z} \alpha |0\rangle + \beta |1\rangle = |\phi\rangle$$

Therefore, Bob is always able to reproduce the state $|\phi\rangle$ on his qubit.

2 Implementation

As an example use of the teleportation protocol, we would like you to write a Qiskit program to set qubit `q[0]` (belonging to Alice) of a quantum circuit `teleport` in state $|\phi\rangle = \frac{|0\rangle + i|1\rangle}{\sqrt{2}}$, and then teleport this state to qubit `q[2]` (belonging to Bob) by first entangling `q[2]` with `q[1]` (belonging to Alice) as detailed in the protocol.

To help you a bit in this task, we have prepared the following code to fill:

```python
from qiskit import QuantumRegister, ClassicalRegister, QuantumCircuit
from qiskit.providers.basic_provider import BasicSimulator
from qiskit.visualization import plot_histogram

# Creating registers
q = QuantumRegister(name='q', size=3)
c = ClassicalRegister(name='c', size=3)

# Creates the quantum circuit
teleport = QuantumCircuit(q, c)

# Step 0 - Instantiate qubit 0 in the state to be teleported:
...
...
teleport.barrier()  # just a visual aid

# Step 1 - Make the shared entangled state in between qubit 1 and qubit 2
...
...
teleport.barrier()  # just a visual aid

# Step 2 & 3: Alice applies a series of gates on qubit 0 and qubit 1
...
...

# Step 4 - Alice measures qubit 0 and qubit 1
...
...

# Step 5 - Bob applies a unitary transformation on his qubit
# based on the outcome of Alice's measurements
```

```
teleport.x(...).c_if(..., ...)
teleport.z(...).c_if(..., ...)
teleport.barrier()  # just a visual aid

# Visualizing your teleportation circuit
teleport.draw()
```

How can Bob quickly check if his qubit is actually in state $|\phi\rangle = \dfrac{|0\rangle + i\,|1\rangle}{\sqrt{2}}$?

Implement your solution...

```
# Step 6 - Bob checks if he got the right state
...
...
...
```

...and then run a simulation:

```
backend = BasicSimulator()
result = backend.run(teleport, shots=8192).result()
counts = result.get_counts()
print(f"The counts for circuit teleport are: {counts}")
plot_histogram(counts)
```

Is your implementation successful? More generally speaking, for an arbitrary state $|\phi\rangle$, how can Bob verify the validity of the transfer?