CS 520: HW 5

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Problem 0: Homework Checklist

I did this homework by myself.

Problem 1: Steepest Descent

The gradient of the function is

$$\nabla f(\mathbf{x}) = Q\mathbf{x} - b.$$

Since $\mathbf{x}^*$ is the minimal point of the function, we have

$$\nabla f(\mathbf{x})\Big|_{\mathbf{x}=\mathbf{x}^*} = Q\mathbf{x}^* - b = 0.$$

Suppose $\mathbf{x}_0 - \mathbf{x}^* = k\mathbf{p}$ for some $k \neq 0$, where $\mathbf{p}$ is an eigenvector of Q. Start at $\mathbf{x}_0$, consider steepest descent

$$\begin{aligned}\min_\alpha f(\mathbf{x}_0 - \alpha(Q\mathbf{x}_0 - b)) &= \min_\alpha \left[\frac{1}{2}(\mathbf{x}_0 - \alpha(Q\mathbf{x}_0 - b))^T Q(\mathbf{x}_0 - \alpha(Q\mathbf{x}_0 - b)) - (\mathbf{x}_0 - \alpha(Q\mathbf{x}_0 - b))^T b\right] \\ &= \min_\alpha \frac{1}{2}\alpha^2(Q\mathbf{x}_0 - b)^T Q(Q\mathbf{x}_0 - b) - \alpha(Q\mathbf{x}_0 - b)^T(Q\mathbf{x}_0 - b) + c(\mathbf{x}_0).\end{aligned}$$

It is a quadratic function of α and thus is minimized at

$$\alpha = \frac{(Q\mathbf{x}_0 - b)^T(Q\mathbf{x}_0 - b)}{(Q\mathbf{x}_0 - b)^T Q(Q\mathbf{x}_0 - b)}.$$

And we get the point after a single iteration

$$\begin{aligned}\mathbf{x}_1 &= \mathbf{x}_0 - \alpha(Q\mathbf{x}_0 - b) \\ &= \mathbf{x}^* + k\mathbf{p} - \frac{(Q\mathbf{x}^* + kQ\mathbf{p} - b)^T(Q\mathbf{x}^* + kQ\mathbf{p} - b)}{(Q\mathbf{x}^* + kQ\mathbf{p} - b)^T Q(Q\mathbf{x}^* + kQ\mathbf{p} - b)}(Q\mathbf{x}^* + kQ\mathbf{p} - b) \\ &= \mathbf{x}^* + k\mathbf{p} - k \cdot \frac{k^2(Q\mathbf{p})^T(Q\mathbf{p})Q}{k^2(Q\mathbf{p})^T Q(Q\mathbf{p})}\mathbf{p} \\ &= \mathbf{x}^* + k\mathbf{p} - k \cdot \frac{\lambda^3\mathbf{p}^T\mathbf{p}}{\lambda^3\mathbf{p}^T\mathbf{p}}\mathbf{p} = \mathbf{x}^*,\end{aligned}$$

the last line is from the property of eigenvector: $Q\mathbf{p} = \lambda\mathbf{p}$ for an eigenvalue λ.

Hence, the algorithm converges in a single iteration.

Problem 2: LPs in Standard Form

Let $\mathbf{e} = (1, \ldots, 1)$. As we have shown in the last homework, the problem can be written as a constrained smooth problem:

$$\min_{\mathbf{t},\mathbf{x}} e^T\mathbf{t} \quad \text{s.t. } t_i \geq a_i^T\mathbf{x} - b_i,\ t_i \geq -0.5.$$

Now, we introduce slack variable and then we can write it as a standard LP:

$$\min_{\mathbf{t},\mathbf{s},\mathbf{x}} e^T\mathbf{t} \quad \text{s.t. } \mathbf{t} = A\mathbf{x} - b + \mathbf{s},\ \mathbf{t} \geq 0,\ \mathbf{s} \geq 0,$$

where $A = (a_1, \ldots, a_n)^T$, $b = (b_1 - 0.5, \ldots, b_n - 0.5)^T$ and $\geq$ is elementwise.

Problem 3: Duality

The KKT conditions of the first problem are:

$$\begin{aligned}
&c - A^T\lambda - s = 0\\
&b - A\mathbf{x} \leq 0\\
&\mathbf{x}^Ts = 0\\
&\lambda^T(A\mathbf{x} - b) = 0\\
&\mathbf{x} \geq 0\\
&s \geq 0\\
&\lambda \geq 0.
\end{aligned}$$

The KKT conditions of the second problem are

$$\begin{aligned}
&-b + Ax - s = 0\\
&A^T\lambda \leq c\\
&\lambda \geq 0\\
&\mathbf{x}^T(c - A^T\lambda) = 0\\
&\mathbf{x} \geq 0\\
&s \geq 0
\end{aligned}$$

They are equivalent once we let $s = c - A^T\lambda$.

Problem 4: Geometry of LPs

It is not a basic feasible point because

$$B = \begin{pmatrix} 1 & 4 & 7\\ 2 & 5 & 8\\ 3 & 6 & 9 \end{pmatrix}$$

is not invertible ($\det B = 0$).

Problem 5: Using the Geometry

Use negation. Otherwise $x_i^+, x_i^- > 0$. By the definition of BFP, the constraint matrix, which is of the form $(a_1, \ldots, a_n, \mathbf{0}, \mathbf{0})$, must satisfies $B = (a_{n_1}, \ldots, a_{n_m}, \mathbf{0}, \mathbf{0})$ is nonsingular, where a_{n_j}'s are a set of columns of A_1 corresponding to the non-zero entries of $\mathbf{x}$. This yields a contradiction because the columns of B are not linearly independent.