PS3 Solutions

Thursday, November 11, 2021 4:47 PM

Q1 a) $f(x,y,z) = x^2 + y^2 - z^2 - 2c_x x - 2c_y y + (c_x^2 + c_y^2 + r^2)$

$$= \begin{bmatrix} x & y & z & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & -c_x \\ 0 & 1 & 0 & -c_y \\ 0 & 0 & -1 & 0 \\ -c_x & -c_y & 0 & c_x^2 + c_y^2 + r^2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

b) $\nabla f(x,y,z) = \left(\frac{\partial f}{\partial x}(x,y,z), \frac{\partial f}{\partial y}(x,y,z), \frac{\partial f}{\partial z}(x,y,z)\right)$

$$= (2(x-c_x), 2(y-c_y), -2z)$$

Normalize:

$$\|\nabla f\| = 2\sqrt{(x-c_x)^2 + (y-c_y)^2 + z^2}$$

$$\frac{\nabla f}{\|\nabla f\|} = \frac{(x-c_x, y-c_y, -z)}{\sqrt{(x-c_x)^2 + (y-c_y)^2 + z^2}}$$

c) Consider fixed values of $z \geq 0$:

$$(x-c_x)^2 + (y-c_y)^2 = z^2 - r^2$$

Traces a circle of radius $\sqrt{z^2 - r^2}$ with center (c_x, c_y, z)

No solutions for $z < r$, so $z_{min} = r$, $z_{max} = +\infty$

kinda bleh notation, I know

$$x(\theta, z) = c_x + \sqrt{z^2 - r^2}\cos(\theta)$$
$$y(\theta, z) = c_y + \sqrt{z^2 - r^2}\sin(\theta)$$
$$z(\theta, z) = z$$

d) $$\text{normal} = \frac{\partial \vec{n}}{\partial \theta} \times \frac{\partial \vec{n}}{\partial z} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -\sqrt{z^2-r^2}\sin(\theta) & \sqrt{z^2-r^2}\cos(\theta) & 0 \\ \frac{z}{\sqrt{z^2-r^2}}\cos(\theta) & \frac{z}{\sqrt{z^2-r^2}}\sin(\theta) & 1 \end{vmatrix}$$

$$= \left(\sqrt{z^2-r^2}\cos(\theta), \sqrt{z^2-r^2}\sin(\theta), -z\right)$$

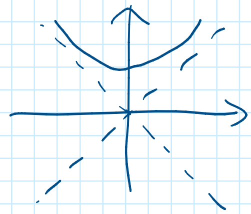
$$\|\text{normal}\|^2 = (z^2-r^2)\cos^2\theta + (z^2-r^2)\sin^2\theta + z^2$$
$$= 2z^2 - r^2$$

$$\text{unit normal} = \frac{1}{2z^2-r^2}\left(\sqrt{z^2-r^2}\cos(\theta), \sqrt{z^2-r^2}\sin(\theta), -z\right)$$

e) Recall that

e) Recall that

$$x = \sinh\lambda, \quad y = \cosh\lambda \quad \text{for } \lambda\in\mathbb{R} \quad \text{traces out top half of a hyperbola}$$



& that $\cosh^2\lambda - \sinh^2\lambda = 1$

This is the intersection with the two-sheeted hyperboloid along a plane. Thus, we may use a parameter θ to denote the half-plane in question.

$$\begin{aligned} x(\theta,\lambda) &= c_x + r\sinh(\lambda)\cos(\theta) \\ y(\theta,\lambda) &= c_y + r\sinh(\lambda)\sin(\theta) \qquad \text{for } \lambda \geq 0 \\ z(\theta,\lambda) &= r\cosh(\lambda) \end{aligned}$$

$$\text{normal} = \frac{\partial\vec{p}}{\partial\theta}\times\frac{\partial\vec{p}}{\partial\lambda} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -r\sinh(\lambda)\sin(\theta) & r\sinh(\lambda)\cos(\theta) & 0 \\ r\cosh(\lambda)\cos(\theta) & r\cosh(\lambda)\sin(\theta) & r\sinh(\lambda) \end{vmatrix}$$

$$= \left(r^2\sinh^2(\lambda)\cos(\theta),\ r^2\sinh^2(\lambda)\sin(\theta),\ -r^2\cosh(\lambda)\sinh(\lambda)\right)$$

$$\|\text{normal}\|^2 = r^4\left(\sinh^4(\lambda) + \cosh^2(\lambda)\sinh^2(\lambda)\right) = r^4\sinh^2(\lambda)\cosh(2\lambda)$$

$$\text{unit normal} = \left(\frac{\sinh(\lambda)\cos\theta}{\sqrt{\cosh(2\lambda)}},\ \frac{\sinh(\lambda)\sin\theta}{\sqrt{\cosh(2\lambda)}},\ -\frac{\cosh(\lambda)}{\sqrt{\cosh(2\lambda)}}\right)$$

<u>Q2</u> a) $p(u) = p_0 + (p_1 - p_0)u$

$$= (2,0,0) + u(-2,0,6) = \begin{bmatrix} 2-2u \\ 0 \\ 6u \end{bmatrix}$$

b) $p(u)$ parametrizes the center point, while v parametrizes the elliptic cross-section

$$P(u,v) = p(u) + \begin{bmatrix} r_x\cos(2\pi v) \\ r_y\sin(2\pi v) \\ 0 \end{bmatrix} = \begin{bmatrix} 2-2u + r_x\cos(2\pi v) \\ r_y\sin(2\pi v) \\ 6u \end{bmatrix}$$

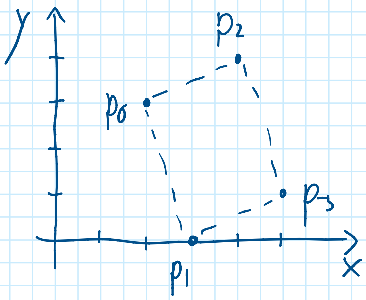
c) $$\text{normal} = \frac{\partial p}{\partial v}\times\frac{\partial p}{\partial u} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -2\pi r_x\sin(2\pi v) & 2\pi r_y\cos(2\pi v) & 0 \end{vmatrix}$$

c) normal $= \frac{\partial}{\partial v} \times \frac{\partial}{\partial u} = \begin{vmatrix} i & j & k \\ -2\pi r_x \sin(2\pi v) & 2\pi r_y \cos(2\pi v) & 0 \\ -2 & 0 & 6 \end{vmatrix}$

$$= 2\pi\left(6r_y\cos(2\pi v),\, 6r_x\sin(2\pi v),\, 4r_y\cos(2\pi v)\right)$$

Q3

a)



b) $p'(0) = 3(p_1 - p_0)$ (from lecture)

$= 3(1,-3) = (3,-9)$

c) C^0 continuity? Yes, as (p_3 of second curve) = (p_0 of first curve)

Are first derivatives equal? For second curve:

$$p'(1) = 3(p_3 - p_2) = 3(2,-6) = (6,-18) \neq (3,-9)$$

NO

d) If we change p_2 of second curve to $(1,6)$ we have:

$$3(p_3 - p_2) = 3(1,-3) = (3,-9)$$

Q4 a) Assuming $P(u)=au^3+bu^2+cu+d$, the constraints give us:

$$P(0)=d=p_k$$

$$P(1)=a+b+c+d=p_{k+1}$$

$$P'(u)=3au^2+2bu+c \quad \begin{cases} P'(0)=c=-\tfrac{1}{2}(1+b)p_{k-1}+bp_k+\tfrac{1}{2}(1-b)p_{k+1} \\ P'(1)=3a+2b+c=-\tfrac{1}{2}(1+b)p_k+bp_{k+1}+\tfrac{1}{2}(1-b)p_{k+2} \end{cases}$$

This implies:

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} P(0) \\ P(1) \\ P'(0) \\ P'(1) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\tfrac{1}{2}(1+b) & b & \tfrac{1}{2}(1-b) & 0 \\ 0 & -\tfrac{1}{2}(1+b) & b & \tfrac{1}{2}(1-b) \end{bmatrix} \times \begin{bmatrix} p_{k-1} \\ p_k \\ p_{k+1} \\ p_{k+2} \end{bmatrix}$$

(ran out of space a bit)

inverse (from text/notes)

denoted A

$$\Rightarrow \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix} \begin{bmatrix} p_{k-1} \\ p_k \\ p_{k+1} \\ p_{k+2} \end{bmatrix}$$

denoted B

$$P(u) = \begin{bmatrix} u^3 & u^2 & u & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} u^3 & u^2 & u & 1 \end{bmatrix} \underbrace{\begin{bmatrix} & & \\ & B & \\ & & \end{bmatrix} \begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix}}_{M_c} \begin{bmatrix} p_{k-1} \\ p_k \\ p_{k+1} \\ p_{k+2} \end{bmatrix}$$

$$\Rightarrow \boxed{M_c = BA}$$

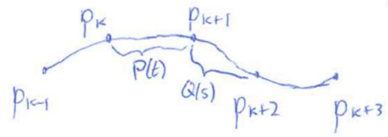
b)
$$M_c = \begin{bmatrix} -\tfrac{1}{2}(b+1) & \tfrac{1}{2}(b+3) & \tfrac{1}{2}(b-3) & \tfrac{1}{2}(1-b) \\ b+1 & -\tfrac{1}{2}(3b+5) & 2 & \tfrac{1}{2}(b-1) \\ -\tfrac{1}{2}(b+1) & b & \tfrac{1}{2}(1-b) & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

note swap in term order

$$\boxed{\begin{aligned} P(u) &= \left[-\tfrac{1}{2}(b+1)u^3+(b+1)u^2-\tfrac{1}{2}(b+1)u\right]p_{k-1} \\ &+ \left[\tfrac{1}{2}(b+3)u^3-\tfrac{1}{2}(3b+5)u^2+bu+1\right]p_k \\ &+ \left[\tfrac{1}{2}(b-3)u^3+2u^2+\tfrac{1}{2}(1-b)u\right]p_{k+1} \\ &+ \left[\tfrac{1}{2}(1-b)u^3+\tfrac{1}{2}(b-1)u^2\right]p_{k+2} \end{aligned}}$$

blending functions

c) A sthematic (not an actual spline)



For two adjacent segments $P(t)$ & $Q(S)$:

C_0 continuity? Yes, $P(1) = Q(0) = p_{k+1}$

C_1 continuity? Yes:

YES

$$P'(1) = -\tfrac{1}{2}(1+b)p_k + b\,p_{k+1} + \tfrac{1}{2}(1-b)p_{k+2} = Q'(0)$$

(note the indices shift for Q)

d) Changing b changes the linear combination of control points that define $P'(0)$ & $P'(1)$, so the direction changes, as well as the magnitude.

See Fig. 19 in text for examples.