GA2001 Econometrics
Solution to Problem Set 2

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Problem 1.

1. Take $S \in \{A \subseteq \Omega' : f^{-1}(A) \in \mathcal{F}\}$, we have $S^c = \Omega' \backslash S$, and $f^{-1}(\Omega' \backslash S) = \Omega \backslash f^{-1}(S) \in \mathcal{F}$. (Closed under complementation.) Then, we can also show that $f^{-1}(\bigcup_{i \in \mathbb{N}} S_i) = \bigcup_{i \in \mathbb{N}} f^{-1}(S_i) \in \mathcal{F}$. (Closed under countable union.) This concludes that $\mathcal{P}$ is a σ-algebra.

2. Since $\mathcal{F}$ is a sigma-algebra, $\sigma(\mathcal{F}) \subseteq \mathcal{F}$. It follows that, for any $A \in \mathcal{F}$, we have $A \in \sigma(\mathcal{F})$, therefore, $\mathcal{F} \backslash \mathcal{A}$-measurable implies $\mathcal{F} \backslash \sigma(\mathcal{A})$-measurable.

Problem 2.

1. By the definition of measurable functions, we have for any $A \in \mathcal{G}$, $f^{-1}(A) \in \mathcal{F}$, and for any $B \in \mathcal{H}$, $g^{-1}(B) \in \mathcal{G}$. Now, consider a function $f^{-1}(g^{-1}(.))$, we have for any $B \in \mathcal{H}$, $f^{-1}(g^{-1}(B)) \in \mathcal{F}$ because $g^{-1}(B) \in \mathcal{G}$. $\blacksquare$

2. Let $h : \mathcal{B}(\mathbb{R}^2) \to \mathcal{B}(\mathbb{R})$. Since for each $B \in \mathcal{B}(\mathbb{R})$, we can find $h^{-1}(B) \in \mathcal{B}(\mathbb{R}^2)$, h is a $\mathcal{B}(\mathbb{R}^2) \backslash \mathcal{B}(\mathbb{R})$-measurable function. By **1.**, we know that the composition of measurable functions is measurable. Therefore, $\omega \circ h$ is $\mathcal{F} \backslash \mathcal{B}(\mathbb{R})$-measrable. In conclusion, $f + g$ and fg are Borel-measurable. $\blacksquare$

3. Since X is a random variable, we know that X is a $\Omega \backslash \mathbb{R}$-measurable function. Also, in the clase, we prove that

$$\mathcal{F}_X \subseteq \mathcal{F}_Y \text{ iff } \exists g : \mathbb{R}^{\dim Y} \to \mathbb{R}^{\dim X}$$

Notice that $\mathbb{R}_+ \subseteq \mathbb{R}$, there exists a function $g_+ : \mathbb{R} \to \mathbb{R}_+$. Since both X and g_+ are measurable, by Problem 2. (1), we have $(X \circ g_+)$ is $\Omega \backslash \mathbb{R}_+$-measurable. Thus, $X^+(\omega)$ is a random variable. Similarily, we have $(X \circ g_-)$ is $\Omega \backslash \mathbb{R}_-$-measurable, where $g_- : \mathbb{R} \to \mathbb{R}_-$. $\blacksquare$

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$$
\begin{tikzcd}
 & & \mathbb{R}_+ \\
\Omega \arrow[r, "X"] & \mathbb{R} \arrow[ur, "g_+"] \arrow[dr, "g_-"'] & \\
 & & \mathbb{R}_-
\end{tikzcd}
$$

Problem 3.

1. (a) "Non-negativity": $P_X(B) \geq 0$ for any $B \in \mathcal{B}(\mathbb{R})$ as the codomain is $[0, 1]$.

(b) "Countable-additivity": Let $\{B_i\} \in \mathcal{B}(\mathbb{R})$ be a collection of disjoint sets. $P_X(\sqcup B_i) = P(X^{-1}(\sqcup B_i)) = P(\sqcup X^{-1}(B_i)) = \sum_i P(X^{-1}(B_i)) = \sum_i P_X(B_i)$.

(c) Finally, we have $P_X(\mathbb{R}) = P(X^{-1}(\mathbb{R})) = P(\Omega) = 1$. The last equation, $P(\Omega) = 1$, holds because P is a probability space.

2. Let $s, t \in \mathbb{R}$ with $s > t$. We have

$$
\begin{aligned}
F_X(s) - F_X(t) &= P_X(-\infty, s] - P_X(-\infty, t] \\
&= P(X^{-1}(-\infty, s]) - P(X^{-1}(-\infty, t]) \\
&= P(X^{-1}(t, s]) > 0
\end{aligned}
$$

Take $t \in \mathbb{R}$, consider a sequence of intervals such that $\{A_n : (-\infty, t + \frac{1}{n})\}_{n=1}^{\infty}$. Note that A_n is decreasing, i.e., $A_1 \supset A_2 \supset A_3 \ldots$. Therefore, we have $\lim_{x \to t^+} F_X(x) = F_X(\bigcap_{n=1}^{\infty} A_n) = F_X((-\infty, t]) = F_X(t)$. **Thus, F_X is right-continous**.

Take $m, t \in \mathbb{R}$ and $m > t$, we have $F_X((\infty, m]) = F_X((\infty, t]) + F_X((t, m])$. As $t \to -\infty$, $F_X((\infty, m]) = F_X((t, m])$, therefore,

$$F_X((-\infty, t]) = F_X((\infty, m]) - F_X((t, m]) = 0.$$

As $t \to \infty$, $F_X(t) = P_X(\mathbb{R}) = 1$.

Problem 4.

1. Note that $\mathbb{E}(Y) = Y$ if Y is constant. If follows that $\mathbb{E}(XY) = Y\mathbb{E}(X) = \mathbb{E}(Y)\mathbb{E}(X)$. Note that the first equality is due to definition of expectation. Similarily, $\mathbb{E}(X = YY) = Y\mathbb{E}(Y) = \mathbb{E}(Y)\mathbb{E}(Y)$. Therefore, if Y is constant, then X and Y are independent. Y and Y are independent. ■

2. I apply Fubini's theorem to prove that if X and Y are independent then $\mathbb{E}(f(X)g(Y)) = \mathbb{E}(f(X))\mathbb{E}(g(Y))$.

Proof

$$\begin{aligned}
\mathbb{E}(f(X)g(Y)) &= \int_{\mathbb{R}^2} f(x)g(y)(\mu \times v)(dx \times dy) \\
&= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(x)g(y)\mu dx \right) v dy \\
&= \int_{\mathbb{R}} g(y) \left(\int_{\mathbb{R}} f(x)\mu dx \right) v dy \\
&= \int_{\mathbb{R}} g(y)\mathbb{E}(f(X)) v dy \\
&= \mathbb{E}(f(X)) \int_{\mathbb{R}} g(y) v dy \\
&= \mathbb{E}(f(X))\mathbb{E}(g(Y))
\end{aligned}$$

■