GA2001 Econometrics
Solution to Problem Set 3

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September 26, 2025

Problem 1.

1. Since we have proved $X = 0$a.s. $\Rightarrow \mathbb{E}X = 0$ in our recitation. I only need to prove that

$$\mathbb{E}X = 0 \Rightarrow X = 0 \text{ a.s.}$$

(Roadmap: I will use Markov inequality to show that $\mathbb{E}X = 0 \Rightarrow X = 0$ a.s..)

Proof For any $\epsilon > 0$, by Markov inequality, we have

$$\mathbb{P}(|X| > 0) \leq \frac{\mathbb{E}|X|}{\epsilon} \text{ for any } \epsilon > 0$$

Since $\mathbb{E}X = 0$, we have $\mathbb{E}|X| = 0$. Notice that $\mathbb{P}(X > 0) = \bigcup_{n=1}^{\infty} \mathbb{P}(|X| > \frac{1}{n})$, we have

$$\mathbb{P}(X > 0) \leq \sum_{n=1}^{\infty} \mathbb{P}(|X| > \frac{1}{n}) \leq 0$$

Therefore, we conclude that $\mathbb{P}(X > 0) = 0$ and thus $X = 0$ a.s. ■

2. Consider a random variable that maps "Head" and "Tail" to $Y = 1$ and $Y = -1$ respectively. We have $\mathbb{E}Y = 0$, but $Y \neq 0$ everywhere. Therefore, the claim "$X = 0$ a.s. $\Leftrightarrow \mathbb{E}X = 0$" fails if X includes both positive and negative values.

Problem 2.

1. Since $f(\cdot, \theta)$ is $\mu-$integrable, we have that $f(\omega, \theta_n)$ and $f(\omega, \theta_0)$ are measurable. Since fg are measurable if f and g are measurable, we can conclude that $f_n'(\omega) = \frac{f(\omega,\theta_n)-f(\omega,\theta_0)}{\theta_n-\theta_0}$ is measurable. ■

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2. Since f'_n is measurable for each n, and we know that **measurability is preserved by limiting operations on sequences of functions**. Therefore, $\frac{\partial f(\cdot,\theta_0)}{\partial\theta} = \lim_{n\to\infty} f'_n(\omega)$ is measurable. ■

Since g is $\mu-$integrable, by Dominated Convergence Theorem, we have $\frac{\partial f(\cdot,\theta_0)}{\partial\theta}$ is $\mu-$integrable. ■

3. (Roadmap: I will use Intermediate Value Theorem to prove this part.)

Proof Since $f(\omega,\cdot)$ is differentiable for each ω, by Intermediate Value Theorem, there exists $\theta^*_n \in [\theta_0,\theta_n]$ such that

$$\frac{f(\omega,\theta_n)-f(\omega,\theta_0)}{\theta_n-\theta_0} = \frac{\partial f(\omega,\theta^*_n)}{\partial\theta}$$

By Assumption A.3, we complete the proof that $|f'_n| < g$ for every n. ■

4. From part 3, we showed that

$$\frac{h(\theta_n)-h(\theta_0)}{\theta_n-\theta_0} = \int_\Omega f'_n(\omega)d\mu(\omega)$$

It follows that

$$\frac{dh(\theta_0)}{d\theta} = \lim_{n\to\infty}\int_\Omega f'_n(\omega)d\mu(\omega)$$

By Dominated Convergence Theorem (here I use the result from part 3 that $|f'_n| < g$ for every n), we have

$$\frac{dh(\theta_0)}{d\theta} = \int_\Omega \lim_{n\to\infty} f'_n(\omega)d\mu(\omega)$$

As a result, we conclude that

$$\frac{dh(\theta_0)}{d\theta} = \int_\Omega \frac{\partial f(\omega,\theta_0)}{\partial\theta}d\mu(\omega)$$

. ■

Problem 3.

1.

$$\begin{aligned}\int_{[0,\infty)} \mathbb{P}\{X>t\}d\lambda(t) &= \int_{\mathbb{R}} \mathbb{P}\{X>t\}\mathbf{1}_{[0,\infty)}(t)d\lambda(t)\\ &= \int_{\mathbb{R}}\left[\int_\Omega \mathbf{1}_{(\omega,t):X(\omega)\geq t\geq 0}(\omega,t)d\mathbb{P}(\omega)\right]d\lambda(t)\end{aligned}$$

where the second equality uses the definition of $\mathbb{P}(X>t)=\int_\Omega \mathbf{1}_{X(\omega)>t}d\mathbb{P}(\omega)$.

2. Per Fubini's theorem, the integrals of $\int_{\mathbb{R}}\left[\int_\Omega \mathbf{1}_{(\omega,t):X(\omega)\geq t\geq 0}(\omega,t)d\mathbb{P}(\omega)\right]d\lambda(t)$ are inter-changable.

3. For fixed ω, we have

$$\int_{\mathbb{R}} \mathbf{1}_{(\omega,t):X(\omega)\geq t\geq 0}(\omega,t)d\lambda(t) = \int_0^{X(\omega)} 1 d\lambda(t) = X(\omega)$$

Therefore, we can complete the proof by showing that

$$\int_{\mathbb{R}} \left[\int_{\Omega} \mathbf{1}_{(\omega,t):X(\omega)\geq t\geq 0}(\omega,t)d\mathbb{P}(\omega)\right] d\lambda(t) = \int_{\Omega} X(\omega)d\mathbb{P}(\omega) = \mathbb{E}X$$

4. Note that for a function (in this case, a random variable), we can decompose it into its positive and negative parts.

$$f = f^+ - f^- \quad \text{where } f^+ = \max\{f,0\} \text{ and } f^- = \max\{-f,0\}$$

Therefore, we have

$$\mathbb{E}(Y) = \mathbb{E}(Y^+) - \mathbb{E}(Y^-) = \int_0^{\infty} \mathbb{P}\{Y^+ > t\}d\lambda(t) - \int_0^{\infty} \mathbb{P}\{Y^- > t\}d\lambda(t)$$

■

Problem 4.

1. Define a sequence of function $f_i : \mathbb{R} \to [0,\infty)$ as follow:

$$f_i(x) = \begin{cases} f(x), & x \in [0,i] \\ 0, & \text{otherwise} \end{cases} \tag{0.1}$$

Clearly, $f_i(x) \leq f_{i+1}(x)$ for all $x \in \mathbb{R}$.

For each $f_i : \mathbb{N} \to [0,\infty)$, we can represent its integral with respect to counting measure using simple functions:

$$\int_{\Omega} f_i(\omega)d\mu(\omega) = \left(\sum_{k=1}^{i} f_i(k)\mu_c(k)\right) + \left(\sum_{k=i+1}^{\infty} f_i(k)\mu_c(k)\right) = \sum_{k=1}^{i} f_i(k) \tag{0.2}$$

Note: The counting measure $\mu_c(k) = 1$ for all $k \in \mathbb{N}$.

Then, applying Monotone Convergence Theorem, we have

$$\int_{\Omega} f(\omega)d\mu(\omega) = \lim_{i\to\infty} \int_{\Omega} f_i(\omega)d\mu(\omega) = \lim_{i\to\infty} \sum_{k=1}^{i} f_i(k) = \sum_{k=1}^{\infty} f(k) \tag{0.3}$$

■

2. Following the result of part 1, we have the following relation in $\mathbb{N}^2$:

$$\int_{\mathbb{N}^2} f d\mu(\omega) = \sum_{(m,n)\in\mathbb{N}^2} f(m,n)$$

which is equivalent to both

$$\sum_{m\in\mathbb{N}} \sum_{n\in\mathbb{N}} a_{mn} \quad \text{and} \quad \sum_{n\in\mathbb{N}} \sum_{m\in\mathbb{N}} a_{mn}$$

Replacing $f(m,n)$ with a_{mn}, then we finished the proof. ■