GA2001 Econometrics
Solution to Problem Set 4

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Problem 1.

Proof Since $\phi \neq \psi \Leftrightarrow (\phi > \psi) \cup (\psi > \phi)$. For $\phi > \psi$, denote $X_1 = \phi - \psi$, since X_1 is non-negative, we can invoke the claim proved in exercise 3 to get that

$$X_1 = 0 \text{ a.s.}$$

Similarly, for $\psi > \phi$, denote $X_2 = \psi - \phi$, we can also get that

$$X_2 = 0 \text{ a.s.}$$

Therefore, we conclude that proof that

$$\int_A \phi d\mathbb{P} = \int_A \psi d\mathbb{P} \Rightarrow \phi = \psi \text{ a.s.}$$

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Problem 2.

1. By Chebyshev inequality, we have

$$\mathbb{P}(|S_n - S| > \epsilon) \leq \frac{\mathbb{E}|S_n - S|^r}{\epsilon^r}$$

Since $\frac{\mathbb{E}|S_n - S|^r}{\epsilon^r} \to 0$, we have $\mathbb{P}(|S_n - S|) \to 0$. This completes the proof that $S_n \xrightarrow{P} S$.

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2. Given $\bar{X}_n = \frac{1}{n}\sum_{i=1}^n X_i$ and $\mathbb{E}(X) = \mu$, we have

$$\begin{aligned}
\mathbb{E}|\frac{1}{n}\sum_{i=1}^{n}(X_i-\mu)|^r &= \frac{1}{n^r}\mathbb{E}|\sum_{i=1}^{n}(X_i-\mu)|^r \\
&\leq \frac{1}{n^r}\mathbb{E}\sum_{i=1}^{n}|(X_i-\mu)|^r \\
&\leq \mathbb{E}\Big[\max_i |X_i-\mu|\Big] < \infty
\end{aligned}$$

It follows that $\frac{1}{n}\sum_{i=1}^n X_i \xrightarrow{P} \mu$. Similarly, we have

$$\begin{aligned}
\mathbb{E}|\frac{1}{n^{1/2}}\sum_{i=1}^{n^{1/2}}(X_i-\mu)|^r &= \frac{1}{n^{r/2}}\mathbb{E}|\sum_{i=1}^{n^{1/2}}(X_i-\mu)|^r \\
&\leq \frac{1}{n^{r/2}}\mathbb{E}\sum_{i=1}^{n^{1/2}}|(X_i-\mu)|^r \\
&\leq \mathbb{E}\Big[\max_i |X_i-\mu|\Big] < \infty
\end{aligned}$$

It follows that $\frac{1}{n^{1/2}}\sum_{i=1}^{n^{1/2}} X_i \xrightarrow{P} \mu$. Also, $\frac{1}{\ln(n)}\sum_{i=1}^{\ln(n)} X_i \xrightarrow{P} \mu$.

Remark: We prefer $\bar{X}_n = \frac{1}{n}\sum_{i=1}^n X_i$ because $\sqrt(n)(\bar{X}_n - \mu) \xrightarrow{d} N(0,\sigma^2)$ per CLT.

3. Proof Because

$$\mathrm{Var}(Y_i) = \mathbb{E}Y_i^2 - (\mathbb{E}Y_i)^2 \leq \mathbb{E}Y_i^2 \leq M,$$

we have $\sigma_i^2 := \mathrm{Var}(Y_i) \leq M$ for every i. Since Y_i are independent,

$$\mathrm{Var}\left(\sum_{i=1}^{n}(Y_i)\right) = \sum_{i=1}^{n}\sigma_i^2.$$

Hence

$$\mathrm{Var}\Big(\frac{1}{n}\sum_{i=1}^{n}(Y_i-\mathbb{E}Y_i)\Big) = \frac{1}{n^2}\sum_{i=1}^{n}\sigma_i^2 \leq \frac{nM}{n^2} \to 0.$$

It follows that

$$\mathbb{E}\Big[\Big(\frac{1}{n}\sum_{i=1}^{n}(Y_i-\mathbb{E}Y_i)\Big)^2\Big],$$

by the result proved in part 1, $\frac{1}{n}\sum_{i=1}^n (Y_i - \mathbb{E}Y_i) \xrightarrow{\mathbb{P}} 0$. ■

Problem 3.

1.

For $X_n = \text{Bern}(\frac{1}{n})$ and $X = \text{Bern}(0)$. $X_n \xrightarrow{P} X$ because of strong law of large number.

2.

For $X_n = \text{Unif}(\frac{1}{n}, \frac{2}{n}, \dots, \frac{n-1}{n}, 1)$ and $X = \text{U}[0,1]$. $X_n \xrightarrow{d} X$ because of weak law of large number.

Problem 4.

1.

Proof Given the functional form of $g(X)$, the objective function is given by

$$\min_{a,b} \mathbb{E}\left[(Y - (a + bX))^2\right]$$

The F.O.C with respect to a and b are

$$0 = -2\mathbb{E}\left[(Y - a - bX)\right]$$

$$0 = -2\mathbb{E}\left[X(Y - a - bX)\right]$$

Then we have $\hat{a} = \mathbb{E}[Y] - \hat{b}\mathbb{E}[X]$, and

$$\begin{aligned} 0 &= \mathbb{E}\left[X(Y - \mathbb{E}[Y] + \hat{b}\mathbb{E}[X] - \hat{b}X)\right] \\ &= \mathbb{E}\left[X(Y - \mathbb{E}[Y] + \hat{b}(\mathbb{E}[X] - X))\right] \end{aligned}$$

$$\Rightarrow$$

$$\hat{b} = \frac{\mathbb{E}\left[X(Y - \mathbb{E}[Y])\right]}{\mathbb{E}\left[X(X - \mathbb{E}[X])\right]} = \frac{\text{Cov}(X,Y)}{\text{Var}(X)}$$

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2.

Suppose function $f : \{0,1\} \to \mathbb{R}$ is given by

$$f(x) = \begin{cases} \alpha & \text{if } x = 0 \\ \beta & \text{if } x = 1 \end{cases}$$

which can be expressed as

$$\tilde{f}(x) = \alpha + (\beta - \alpha)x$$

3.

Given $\mathbb{P}\{X = 1\} = \mathbb{P}\{X = 0\} = \frac{1}{2}$, $\mathbb{E}[Y] = 0$, and $\mathbb{E}[XY] = 1$ we have

$$\mathbb{E}[X] = \frac{1}{2}, \quad \mathrm{Var}(X) = \frac{1}{4}, \quad \mathrm{Cov}(X, Y) = 1$$

Therefore, $\beta^* = 4$ and $\alpha^* = -2$.

$$\mathbb{E}[Y|X] = \alpha^* + \beta^* X = -2 + 4X$$

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