PhD Econometrics I (First Half) matias.guizzo.altube@nyu.edu
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Recitation 1 - Suggested Solutions

Exercise 1. An economy that lasts T periods is described by $(Y_t)_{t=1}^T$, where each $Y_t \in \mathbb{R}^K$ is a (random) state variable.
(a) Viewing the economy $(Y_t)_{t=1}^T$ as an outcome of an experiment, formulate an appropriate sample space and σ-algebra $(\Omega, \mathcal{F})$.
(b) An economist, despite being trained at NYU, does not observe Y_t, but instead only observes $h(Y_t)$ for some $\mathcal{F}/\mathcal{B}(\mathbb{R})$-measurable function $h : \mathbb{R}^K \to \mathbb{R}$. Formulate an appropriate sample space and σ-algebra from the economist's perspective.

Suggested solution. (a) $(\Omega, \mathcal{F}) = ((\mathbb{R}^K)^T, \mathcal{B}((\mathbb{R}^K)^T))$.
(b) $(\mathbb{R}^T, \mathcal{B}(\mathbb{R}^T))$. Alternatively, define $\eta : \mathbb{R}^{KT} \to \mathbb{R}^T$ as $\eta(Y_1, \ldots, Y_T) = (h(Y_1), \ldots, h(Y_T))$. Then, we can use the measurable space $(\mathbb{R}^T, \mathcal{H})$, where $\mathcal{H} := \{H \in \mathbb{R}^T : \eta^{-1}(H) \in \mathcal{B}(\mathbb{R}^{KT})\}$.

Exercise 2. Let $\{\mathcal{F}_\gamma : \gamma \in G\}$ be a collection of σ-algebras on Ω, of which G is possibly uncountable. Show that $\mathcal{G} := \bigcap_{\gamma \in G} \mathcal{F}_\gamma$ is a σ-algebra. Using the definition of generated σ-algebra, explain why $\sigma(\mathcal{H})$ for any $\mathcal{H} \in 2^\Omega$ is non-empty and prove that $\sigma(\mathcal{H})$ is an σ-algebra.

Suggested solution. First, we want to show that $\mathcal{G}$ is non-empty. $\Omega \in \mathcal{F}_\gamma \forall \gamma$ by definition. Therefore, $\Omega \in \bigcap_{\gamma \in G} \mathcal{F}_\gamma$.
Second, we want $\mathcal{G}$ to be closed under complementation. Suppose $A \in \bigcap_{\gamma \in G} \mathcal{F}_\gamma$, then $A \in \mathcal{F}_\gamma \forall \gamma$. Since each $\mathcal{F}_\gamma$ is a σ-algebra, $A^c \in \mathcal{F}_\gamma \forall \gamma$, from which it follows $A^c \in \bigcap_{\gamma \in G} \mathcal{F}_\gamma$.
Finally, suppose $A_1, A_2, \cdots \in \bigcap_{\gamma \in G} \mathcal{F}_\gamma$, then $A_1, A_2, \cdots \in \mathcal{F}_\gamma \forall \gamma$. Since each $\mathcal{F}_\gamma$ is a σ-algebra, $\bigcup_{i=1}^\infty A_i \in \mathcal{F}_\gamma \forall \gamma \implies \bigcup_{i=1}^\infty A_i \in \bigcap_{\gamma \in G} \mathcal{F}_\gamma$, making $\mathcal{G}$ closed under countable unions, and a σ-algebra.
Let $\Sigma(\mathcal{H}) := \{\mathcal{F} : \mathcal{H} \subseteq \mathcal{F} \text{ and } \mathcal{F} \text{ is a } \sigma\text{-algebra}\}$. Then, $\sigma(\mathcal{H}) = \bigcap \Sigma(\mathcal{H})$. The intersection is non-empty because $\Sigma(\mathcal{H})$ is non-empty, in particular $2^\Omega \in \Sigma(\mathcal{H})$. Thus $\sigma(\mathcal{H})$ exists. From the previous part $\sigma(\mathcal{H})$ is a σ-algebra.

Exercise 3. Let $(\Omega, \mathcal{F}, P)$ be a probability space.
(a) Suppose $A, B \in \mathcal{F}$. Show that if $A \subseteq B$ then $P(A) \leqslant P(B)$ and $P(B \backslash A) = P(B) - P(A)$.
(b) Suppose $A_1, A_2, \dots \in \mathcal{F}$. Show that P is *countably subadditivite:*

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) \leqslant \sum_{i=1}^{\infty} P(A_i).$$

Explain why this implies P is *finitely subadditivite*: $P(\bigcup_{i=1}^{n} A_i) \leqslant \sum_{i=1}^{n} P(A_i)$.

Suggested solution. (a) $A \subseteq B \implies B = A \sqcup (B \backslash A)$. By countable subadditivity,

$$P(B) = P(A) + P(B \backslash A).$$

Since $P(B \backslash A) \geqslant 0$ it follows that $P(B) \geqslant P(A)$. Since P is finite, we can re-arrange the previous display to get $P(B \backslash A) = P(B) - P(A)$.
(b) Construct the sets $B_1 = A_1, B_i = A_i \backslash (A_1 \cup \cdots \cup A_{i-1})$ for $i \geqslant 2$, and note that $B_i \subseteq A_i$ and $\bigcup_{i=1}^{\infty} A_i = \bigsqcup_{i=1}^{\infty} B_i$. So

$$\begin{aligned} P\left(\bigcup_{i=1}^{\infty} A_i\right) &= P\left(\bigsqcup_{i=1}^{\infty} B_i\right) \\ &= \sum_{i=1}^{\infty} P(B_i) && \text{countable additivity} \\ &\leqslant \sum_{i=1}^{\infty} P(A_i) && \text{Exercise 3a.} \end{aligned}$$

Finite subadditivity follows from applying countable subadditivity to the sequence $(A_1, \dots, A_n, \emptyset, \emptyset, \dots)$.

Exercise 4. Let $(\Omega, \mathcal{F})$ be a measurable space and suppose $\mu : \Omega \to [0, \infty]$ a *finitely additive* map, i.e.,

$$A_1, \ldots, A_n \in \mathcal{F} \text{ disjoint } \implies \mu\left(\bigsqcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \mu(A_i).$$

Show that μ is *countably additive* if and only if the following condition holds: For any (A_i) non-decreasing sequence of sets in $\mathcal{F}$ (i.e., $A_1 \subseteq A_2 \subseteq \cdots$) it holds that $\mu(\bigcup_{i=1}^{\infty} A_i) = \lim_{i\to\infty} \mu(A_i)$.

Suggested solution. ($\implies$) Let (A_i) be such that $A_1 \subseteq A_2 \subseteq \cdots$.

Case 1: If there is some j such that $\mu(A_j) = +\infty$, since μ is $\subseteq$-increasing (analogue of Exercise 3a) and $A_j \subseteq A_{j+k}$, then $\mu(A_{j+k}) = +\infty$ for every $k \geqslant 1$ as well. Also $A_j \subseteq \bigcup_{i=1}^{\infty} A_i \implies \mu(\bigcup_{i=1}^{\infty} A_i) = +\infty$. Therefore $\mu(\bigcup_{i=1}^{\infty} A_i) = +\infty = \lim_{n\to\infty} \mu(A_n)$.

Case 2: Assume $\mu(A_i) < +\infty$ for all i. Put $A_0 := \emptyset$. The sets $R_i = A_i \backslash A_{i-1}$ partition $\bigcup_{i=1}^{\infty} A_i$, so

$$\begin{aligned}
\mu\left(\bigcup_{i=1}^{\infty} A_i\right) &= \mu\left(\bigsqcup_{i=1}^{\infty} R_i\right) \\
&= \sum_{i=1}^{\infty} \mu(R_i) && \text{countable additivity (hypothesis)} \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \mu(A_i \backslash A_{i-1}) \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} [\mu(A_i) - \mu(A_{i-1})] && \text{Exercise 3a} \\
&= \lim_{n\to\infty} \mu(A_n)
\end{aligned}$$

($\impliedby$) Let $A_1, A_2, \cdots \in \mathcal{F}$ be disjoint. Consider the sets $B_i = A_1 \sqcup \cdots \sqcup A_i$, which form a non-decreasing sequence and clearly $\bigsqcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} B_i$. So

$$\begin{aligned}
\mu\left(\bigsqcup_{i=1}^{\infty} A_i\right) &= \mu\left(\bigcup_{i=1}^{\infty} B_i\right) \\
&= \lim_{n\to\infty} \mu(B_n) && \text{by hypothesis} \\
&= \lim_{n\to\infty} \mu(A_1 \sqcup \cdots \sqcup A_n) \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} P(A_i) && \text{finite additivity}
\end{aligned}$$

Exercise 5. Let $\mathcal{G}$ be an algebra on Ω and $\mu : \mathcal{G} \to [0, \infty]$ be a countably additive map. Define the outer measure $\mu^* : 2^\Omega \to [0, \infty]$ with respect to μ as

$$\mu^*(A) = \inf\left\{ \sum_{i=1}^{\infty} \mu(A_i) : A_1, A_2, \cdots \in \mathcal{G} \text{ ; } A \subseteq \bigcup_{i=1}^{\infty} A_i \right\}.$$

Show that $\mu^*(A) = \mu(A)$ for every $A \in \mathcal{G}$. (*Hint:* prove/use the fact that μ is countably subadditive.)

Suggested solution. Let $A \in \mathcal{G}$. Denote $M(A) := \{\sum_{i=1}^{\infty} \mu(A_i) : A_1, A_2, \cdots \in \mathcal{G} \text{ ; } A \subseteq \bigcup_{i=1}^{\infty} A_i\}$.
Take the sequence $A, \emptyset, \emptyset, \cdots$. Then, it is clear that $\mu(A) \in M(A)$, as $A \subseteq \bigcup_{i=1}^{\infty} A_i = A$.
Therefore, $\mu(A) \geqslant \inf M(A) = \mu^*(A)$.
To show the other inequality, it suffices to show $\mu(A) \leqslant m$ for every $m \in M(A)$, i.e., for any arbitrary (A_i) in $\mathcal{G}$ such that $A \subseteq \bigcup_{i=1}^{\infty} A_i$, we want to show $\mu(A) \leqslant \sum_{i=1}^{\infty} \mu(A_j)$.
Observe that since $A \subseteq \bigcup_{i=1}^{\infty} A_i$, A can be expressed as $A = \bigcup_{i=1}^{\infty} (A \cap A_i)$. Thus

$$\begin{aligned}
\mu(A) &= \mu\left(\bigcup_{i=1}^{\infty} (A \cap A_i) \right) & \\
&\leqslant \sum_{i=1}^{\infty} \mu(A \cap A_i) & \text{countable subadditivity} \\
&\leqslant \sum_{i=1}^{\infty} \mu(A_i) & A \cap A_i \subseteq A_i.
\end{aligned}$$

Thus, we conclude that $\mu(A) \leqslant \mu^*(A)$, therefore $\mu(A) = \mu^*(A)$.