GA2001 Econometrics
Solution to Problem Set 1

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Problem 1.

(a) The Probability Space of an Urns-Balls Experiment

The probability space for an n-urns-k-balls experiement with $n > k$ is given by

- $\Omega = \{(x_1, x_2, \ldots, x_n) : \sum_{i=1}^{n} x_i = k \text{ and } x_i \in \mathbb{Z}\}$ with $|\Omega| = \binom{k+n-1}{n-1}$.
- $\mathcal{F} = 2^{\Omega}$, the power set of Ω.
- For any event $A \in \mathcal{F}$, $P(A) = \frac{|A|}{\binom{k+n-1}{n-1}}$, where $|A|$ is the cardinality of set A.

(b) The Law of Total Probability

Since $(B_i)_{i\in\mathbb{N}}$ is a partition of Ω, $(A \bigcap B_i)$ are disjoint across i. Therefore, $\sum_{i=1}^{\infty} P(A \bigcap B_i) = P(\cup_i (A \bigcap B_i)) = P(A)$. ■

(c)

Denote $P(B_i)$ the probability that exactly i urns are empty. Then the probability of all empty urns are located to the left of non-empty urns is given by

$$P(A) = \frac{1}{\binom{k+n-1}{n-1}} \left(\sum_{i=1}^{k} \frac{1}{P(B_{n-i})} \right)$$

Problem 2.

Let $\mathcal{I} := \{(a, b); a, b \in \mathbb{R}\}$ and $\mathcal{F} = \sigma(\mathcal{I})$.

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(a)

Note that $(c,d] = \bigcap_n^{\infty}(c, d+\frac{1}{n})$. Since $(c, d+\frac{1}{n}) \in \mathcal{F}$, and the countable intersection is closed in σ-algebra, $\{(c,d]\} \subseteq \mathcal{F}$ We need to show that $\mathcal{F}$ is strictly larger than $\{(c,d] : c < d; a, b \in \mathbb{R}\}$. In particular, $[a,b) \in \mathcal{F}$ (because $[a,b) = \cup_{n=1}^{\infty}(a-\frac{1}{n}, b)$), while $[a,b)$ cannot be generated by $(c,d]$ with either countable union or intersection. Therefore, we have $\{(c,d] : c < d; a, b \in \mathbb{R}\} \subset \mathcal{F}$.

Similarly, $(a,\infty) = \cup_n^{\infty}(a,n)$, therefore, $\{(a,\infty)\} \subseteq \mathcal{F}$. However, $\{b\} \in \mathcal{F}$ while it cannot be generated by (a,∞) with either countable union or intersection. Thus, $\{(a,\infty) : a \in \mathbb{R}\} \subset \mathcal{F}$. ■

(b)

To show that both $\mathcal{A} = \{[a,b] : a < b; a, b \in \mathbb{R}\}$ and $\mathcal{B} = \{(-\infty, b] : b \in \mathbb{R}\}$ generate $\mathcal{F}$, we need to show that $\sigma(\mathcal{A}) = \mathcal{F}$ and $\sigma(\mathcal{B}) = \mathcal{F}$.

Part 1. First, we show that $\mathcal{I} \subseteq \sigma(\mathcal{A})$. Pick any $(a,b) \in \sigma(\mathcal{I})$,

$$(a,b) = \cup^{\infty}[a+\frac{1}{n}, b-\frac{1}{n}] \in \sigma(\mathcal{A})$$

Therefore, $\sigma(\mathcal{I}) \subseteq \sigma(\mathcal{A})$.

Pick any $[a,b] \in \sigma(\mathcal{A})$, one can show that $[a,b] = \{a\} \cup (a,b) \cup \{b\} \in \sigma(\mathcal{I})$.[1] Therefore, $[a,b] \in \sigma(\mathcal{I})$. Thus, $\sigma(\mathcal{A}) \subseteq \sigma(\mathcal{I})$. It concludes that, $\sigma(\mathcal{I}) = \sigma(\mathcal{A})$.

Part 2. Pick $(-\infty, b] \in \sigma(\mathcal{B})$, it follows that

$$(-\infty, b] = \mathbb{R} \backslash (b, \infty) \in \mathcal{F}$$
$$\Rightarrow \sigma(\mathcal{B}) \subseteq \mathcal{F}$$

Pick $(a,b) \in \mathcal{F}$, it follows that

$$(a,b) = \mathbb{R} \backslash (-\infty, a] \bigcap \left(\cup_{n=1}^{\infty} (-\infty, b-\frac{1}{n}] \right) \in \sigma(\mathcal{B})$$
$$\Rightarrow \mathcal{F} \subseteq \sigma(\mathcal{B})$$

Therefore, $\mathcal{F} = \sigma(\mathcal{B})$.

(c)

False. Because this set doesn't include singleton.

[1] $\{a\} \in \sigma(\mathcal{I})$ because $\{a\} = \bigcap_{n=1}^{\infty}(a, a+\frac{1}{n})$.

Problem 3.

The set $\mathcal{L} := \{(x, y) \in \mathbb{R}^2 : x = y; 0 \le x \le 1\}$ can be obtained by countable intersection of open sets in $\mathbb{R}^2$. Concretely,

$$\mathcal{L} = \bigcap_{n\in\mathbb{N}} \{(x,y) \in \mathbb{R}^2 : y > x + \frac{1}{n} \text{ and } x \in (-\frac{1}{n}, 1 + \frac{1}{n})\} \bigcap$$
$$\bigcap_{n\in\mathbb{N}} \{(x,y) \in \mathbb{R}^2 : y < x - \frac{1}{n} \text{ and } x \in (-\frac{1}{n}, 1 + \frac{1}{n})\}$$

Problem 4.

Consider the following sequence:

$$A_1 = \{1, 2, \ldots, \}$$
$$A_2 = \{2, 3, \ldots, \}$$
$$\ldots$$

It follows that $\bigcap_{i\in\mathbb{N}} A_i = \emptyset$ (because suppose $n \in \mathbb{N}$ is in A_n, there exists A_m, s.t. $n \notin A_m$ for $i \ge m$.) Now we use length measure. Therefore, we have $\mu(\bigcap_{i\in\mathbb{N}} A_i) = 0$ while $\lim_{i\to\infty} \mu(A_i) = \infty$. (In fact, $\mu(A_i) = \infty, \forall i$.)

Problem 5.

Recall that in $(\mathbb{R}^k, \mathbb{B}^k)$, the Lebesgue measure λ is

$$\lambda(B) \equiv \inf_{B \subseteq \bigcup_{j=1}^{\infty} \{\times_{i=1}^{k} (a_{ij}, b_{ij})\}} \sum_{j=1}^{\infty} \pi_{j=1}^{k} (b_{ji} - a_{ij})$$

(a.) Since $a - a = 0$, we have $\lambda(\{a\}) = 0$

(b.) Since every countable set A can be enumerated using singletons, $\mu(A) = 0$ for every countable set $A \in (B)(\mathbb{R})$.

(C.) Similar to **(b.)**, $\lambda((a, b)) = \lambda([a, b]) - \lambda(\{a\}) - \lambda(\{b\}) = b - a - 0 - 0$. Following this logic, $\lambda([a, b)) = \lambda((a, b]) = b - a$.

Problem 6.

Note that $A\Delta B = (A\backslash B) \cup (B\backslash A)$. Since $(A\backslash B) \cap (B\backslash A) = \emptyset$, $P\Big((A\backslash B) \cup (B\backslash A)\Big) = P(A\backslash B) + P(B\backslash A)$.

$$\begin{aligned}
|P(A) - P(B)| &= |P\Big((A\backslash B) \cup (A \cap B)\Big) - P\Big((B\backslash A) \cup (A \cap B)\Big)| \\
&= |P(A\backslash B) + P(A \cap B) - P(B\backslash A) - P(A \cap B)| \\
&= |P(A\backslash B) - P(B\backslash A)| \\
&\leq |P(A\backslash B) + P(B\backslash A)| = |P(A\Delta B)|
\end{aligned}$$

■