Notes on General Equilibrium Theory

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Chapter 1

Math Preliminaries

1.1 Differential Topology

Definition 1.1.1 (Transversality)**.** Let $f : \mathbb{R}^M \to \mathbb{R}^N$, with $M > N$. We say f is **transversal** at 0 (denoted as $f \pitchfork 0$), if

$$\forall x, \text{ such that } f(x) = 0$$

, we have the Jacobian matrix

$$Df = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_m} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_m} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_m} \end{bmatrix}$$

is full rank, namely, $\text{rank}(Df) = N$.

Example Consider $M = 2$, and $N = 1$. Define

$$f(x, y) = x^2 + y^2 - 1$$

The zero set is the unit circle. $x^2 + y^2 = 1$.

Since $n = 1$, the Jacobian is a single row vector (the gradient):

$$Df(x, y) = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{bmatrix} = \begin{bmatrix} 2x & 2y \end{bmatrix}$$

The rank of a 1×2 matrix is 1 (full rank) unless all entries are zero.

$$Df = [0, 0] \text{ iff } x = 0 \text{ and } y = 0.$$

However, the origin is **not** in the zero set. Therefore, for every point actually on the circle ($f = 0$), the gradient is non-zero. Hence, we have

$$f \pitchfork 0$$

Example 2: Non-transveral

$$g(x, y) = y^2 - x^3$$

The set of points where $y^2 = x^3$. This describes a semicubical parabola which has a sharp "cusp" (singular point) at the origin.

The Jacobian is given by

$$Dg(x, y) = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{bmatrix} = \begin{bmatrix} -3x^2 & 2y \end{bmatrix}$$

It follows that $Df(0, 0)$ at the point $(0, 0)$: Therefore, its rank is **0** and thus g is not transversal at 0.

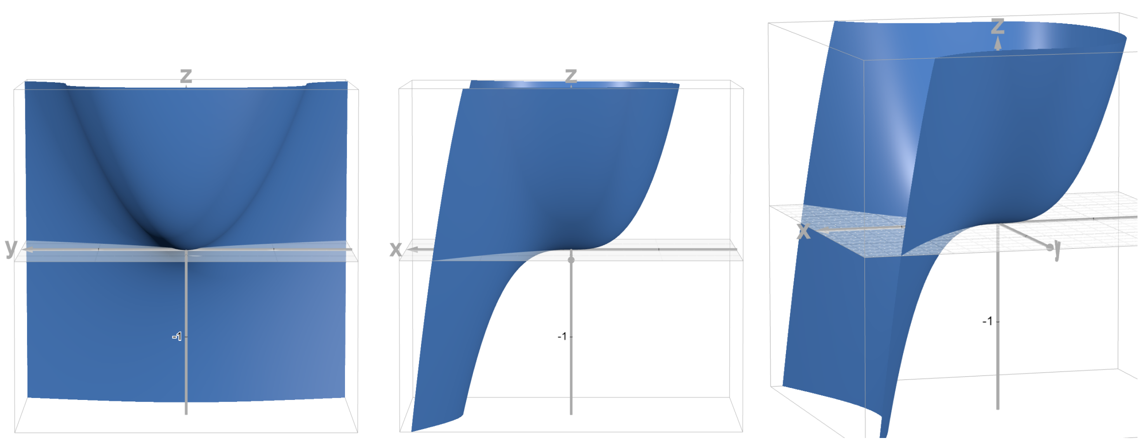

Figure 1.1.1: A function that is not transversal at the origin

NOTES: This function depicts $g(x, y) = y^2 - x^3$, which is not transversal at the origin due to the cusp there.

Proposition 1 (Transversality Theorem)**.** *Let $f : \mathbb{R}^M \to \mathbb{R}^N$ be C^∞, with $M > N$. And decompose $x = [x1, x2]$, where $x1 \in \mathbb{R}^{M-N}$, and $x2 \in \mathbb{R}^N$. For almost every $x1$, if*

$f(x1, x2) = 0,$ *we have*

$$rank(D_{x2}f) = \begin{bmatrix} \frac{\partial f_1}{\partial x_{2,1}} & \frac{\partial f_1}{\partial x_{2,2}} & \cdots & \frac{\partial f_1}{\partial x_{2,N}} \\ \frac{\partial f_2}{\partial x_{2,1}} & \frac{\partial f_2}{\partial x_{2,2}} & \cdots & \frac{\partial f_2}{\partial x_{2,N}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_N}{\partial x_{2,1}} & \frac{\partial f_N}{\partial x_{2,2}} & \cdots & \frac{\partial f_N}{\partial x_{2,N}} \end{bmatrix} = N.$$

Remarks: We will use transverality theorem to prove a bunch of important results in GE theory, such as the genericity of regular economies.

Chapter 2

Abstract Exchange Economies

2.1 Welfare Analysis

Proposition 2 (First Welfare Theorem)**.** *In an abstract exchange economy characterized by $w \in \mathbb{R}^{S+1}_{++}$, any competitive equilibrium allocation $x \in \mathbb{R}^{S+1}_{+}$ is Pareto efficient.*

Proof: We prove by contradiction. Suppose there exists y that Pareto dominates x, then $y^i \succeq_P x^i \forall i$, and $y^j \succ_P x^j$ for some j. Then by the Walras Law and the Law of Revealed Prferences, we have

$$p_0(y_0^i - w_0^j) + \sum_s p_s(y_s^i - w_s^j) \geq p_0(x_0^i - w_0^i) + \sum_s p_s(x_s^i - w_s^i) = 0, \quad \text{and}$$

$$p_0(y_0^j - w_0^j) + \sum_s p_s(y_s^j - w_s^j) > p_0(x_0^j - w_0^j) + \sum_s p_s(x_s^j - w_s^j) = 0$$

Summing over i, we have

$$p_0 \sum_i (y_0^i - w_0^i) + \sum_s p_s \sum_i (y_s^i - w_s^i) > 0$$

Since p_0, p_s are strictly positive, there exists an s such that

$$\sum_i (y_s^i - w_s^i) > 0,$$

which implies that y is not feasible. ■

The converse of the First welfare theorem is the Second welfare theorem below. **Interpretation:** In words, the Second Welfare Theorem states that any Pareto efficient allocation can be decentralized as a competitive equilibrium via transfer (AKA redistribution of initial endowment).

Proposition 3 (Second Welfare Theorem)**.** *For any economy characterized by $w \in \mathbb{R}^{LI}_{++}$, any Pareto efficient allocation $x \in \mathbb{R}^{LI}_{+}$ can be decentralized as a competitive equilibrium via price, $p \in \mathbb{R}^{L}_{++}$, under redistributioned endowments with $\sum_i w_i + t_i = w$ (i.e., $\sum_i t_i = 0$).*

Proof Let $x_i = w_i + t_i$. First, define the strict upper contour set for agent i,

$$B^i(x_i) := \{y_i \in \mathbb{R}^L_+ : U(y_i) > U(x_i)\}$$

Then, define the aggregate strict upper contour set as

$$B(x) := \left\{ \sum_i y_i \in \mathbb{R}^L_+ : y_i \in B^i(x_i) \forall i \right\}$$

Since $B^i(x_i)$ is convex, the aggregate upper contour set is also convex. In addition, $(\sum_i x_i) \notin B(x)$ due to the strictness of the upper contour set. Then, by the Separting Hyperplane theorem, there exists a nonzero price vector p, such that

$$\inf_{a \in B(x)} pa \geq p \sum_i x_i$$

We will use this equation to establish that (1) prices are strictly positive,[1] and (2) under such price, no individual will trade away from x_i.

Part 1. Suppose there exists some l such that $p_l < 0$, by the fact that utility is **strictly** monotonic, consumers can increase their consumption of l arbitrarily large such that $pa < p \sum_i x_i$. Then, we have a contradiction.

Now suppose there exists some l such that $p_l = 0$, then by the continuity of $B(x)$, there exists an allocation y such that $py < p \sum_i x_i$, where y is defined by

$$y_k = x_k - \epsilon, \text{ and } y_l = x_l + \delta$$

with sufficiently small *epsilon* and sufficiently large δ. Again, we obtain a contradiction.

Part 2. Finally, we prove that, any bundle in $B^i(x_i)$ is unaffordable under $p \in \mathbb{R}^L_{++}$. Since $px_i > 0$, there exists a strictly cheaper bundel such that

$$p\underline{x}_i < px_i$$

[1]Intuitively, valued goods should not have non-positive prices.

It follows that $\alpha y_i + (1-\alpha)\underline{x}_i$ ¡ p x$_i$ for some $y \in B^i(x_i)$ and $\alpha \in (0,1)$. But by the convexity of $B^i(x_i)$, we have $\alpha y_i + (1-\alpha)\underline{x}_i := \tilde{x}_i \in B^i(x_i)$. Then, summing over individuals, we have

$$p\tilde{x}_i + \sum_{j \neq i} p x_j < p \sum_j x_j,$$

which contradicts to the separating hyperplane condition. ■

2.2 Consumption Externalities and Pigouvian Taxes

In the presence of consumption externalities, an individual's utility depends on both her and others' consumptions. As a result, the centralized allocation (i.e., Negishi's solution) deviates from the decentralized one. To induce the decentralized allocation to be Pareto Efficient, one can introduce the so-called Pigouvian taxes, so that the taxes make individuals behave *as if* they care about others. The following derives the tax formula.

For simplicity, we assume that the goods 1 does not have consumption exteranlity. First, the Negishi problem is give by[2]

$$\begin{aligned} \max_{x \in \mathbb{R}^{LI}} \quad & \sum_i \alpha^i U^i(x_1^i, \{x_l^i\}, \{x_l^{-i}\}) \\ \text{subject to:} \quad & \sum_i x_l^i \leq w_l, \quad \forall l \end{aligned}$$

Note that there are L constraints, so we introduce L Lagrangean Multipliers in the equation below

$$\mathcal{L} = \sum_i \alpha^i U^i(x_1^i, \{x_l^i\}, \{x_l^{-i}\}) + \sum_l \lambda_l \left(w_l - \sum_i x_l^i \right)$$

The first-order conditions are given by

$$\begin{aligned} [x_1^i] \qquad & \alpha^i \frac{\partial U^i}{\partial x_1^i} = \lambda_1 \\ [x_l^i] \qquad & \sum_j \alpha^j \frac{\partial U^j}{\partial x_l^i} = \lambda_l \quad \text{for } l \neq 1 \end{aligned}$$

[2]We use superscripts to denote individuals, and subscripts to index goods.

The decentralized consumer's problem is given by

$$\begin{aligned} \max_{x^i \in \mathbb{R}^I} \quad & U^i(x_1^i, \{x_l^i\}, \{x_l^{-i}\}) \\ \text{subject to:} \quad & \sum_l p_l(1+\tau_l^i)x^i \leq \sum_l p_l w_l^i, \quad \forall l \end{aligned}$$

The Lagrangean is given by

$$\mathcal{L}^i = U^i(x_1^i, \{x_l^i\}, \{x_l^{-i}\}) + \mu^i \sum_l \left[p_l w_l^i - p_l(1+\tau_l^i)x^i\right]$$

The first-order conditions are given by

$$\begin{aligned} [x_1^i] \qquad & \frac{\partial U^i}{\partial x_1^i} = \mu^i p_1 = \mu^i \qquad \text{normalization using} \quad p_1 = 1 \\ [x_l^i] \qquad & \frac{\partial U^i}{\partial x_l^i} = \mu^i p_l(1+\tau_l^i) \quad \text{for } l \neq 1 \end{aligned}$$

Combining the these two conditions, we have

$$\frac{\partial U^i}{\partial x_l^i} = \frac{\partial U^i}{\partial x_1^i} p_l(1+\tau_l^i)$$

Note that we can decompose the second FOC of the Negishi problem as

$$\frac{\partial U^i}{\partial x_l^i} + \sum_{j \neq i} \frac{\alpha^j}{\alpha^i} \frac{\partial U^j}{\partial x_l^i} = \frac{\lambda_l}{\alpha^i} \quad \text{for } l \neq 1$$

Therefore, we can equate the equation above with the decentralized condition and compute the tax formula:

$$\underbrace{\frac{\lambda_l}{\alpha_i} - \sum_{j \neq i} \frac{\alpha^j}{\alpha^i} \frac{\partial U^j}{\partial x_l^i}}_{\text{Negishi's}} = \underbrace{\frac{\partial U^i}{\partial x_1^i} p_l(1+\tau_l^i)}_{\text{Decentralized consumer's}}$$

It follows that

$$\tau_l^i = \frac{\lambda_l}{p_l \alpha_i \frac{\partial U^i}{\partial x_1^i}} - 1 - \sum_{j \neq i} \frac{\alpha^j}{p_l \alpha^i} \frac{\frac{\partial U^j}{\partial x_l^i}}{\frac{\partial U^i}{\partial x_1^i}}$$

Now assuming $\alpha^i = \frac{1}{\mu^i} = \frac{1}{\frac{\partial U^i}{\partial x_1^i}}$, and $p_l = \lambda_l$, the we have

$$\tau_l^i = -\frac{1}{p_l} \sum_{j \neq i} \frac{\frac{\partial U^j}{\partial x_l^i}}{\frac{\partial U^j}{\partial x_1^j}}.$$

Chapter 3

Financial Market Economies

3.1 Arrow-Debreu Equilibrium

Definition 3.1.1. An Arrow-Debreu equilibrium is defined by an allocation $x_0, \{x_s\}$, such that

1. given the goods prices ϕ_0, ϕ_s, the allocation solves the consumer's problem below:

$$\begin{aligned} \max_{x_o, \{x_s\}} \quad & u_0(x_0) + \sum_s \text{Prob}_s u_s(x_s) \\ \text{s.t.} \quad & \phi_0(x_0 - w_0^i) + \sum_s \phi_s(x_s - w_s^i) = 0 \end{aligned}$$

2. Goods market clears:

$$\sum_i x^i \leq \sum_i w^i$$

3.2 No-Arbitrage

There are multiple definitions of the No Arbitrage condition. Essentially, they all require zero profit on the spot market at time 0, and the future financial market no matter what the future state will be. Formally, the first definition is given by

Definition I (No Arbitrage): Let $A \in \mathbb{R}^{S\times J}$ be a future return matrix. Let $z \in \mathbb{R}^J$ be a portfolio. $q^T \in \mathbb{R}^J$ be a price vector of the portfolio.[a] For $W = \begin{bmatrix} -q \\ A \end{bmatrix}$, there is no $z \in \mathbb{R}^J$ such that $Wz > 0$.

- **Remark:** Intuitively, $Wz \in \mathbb{R}^{S+1}$ denotes the profit of buying/selling assets on the spot market and all the future financial markets.

[a]In this note, only q is a row vector, while other vectors are columns.

Example: Consider $W = \begin{bmatrix} -3 & -4 \\ 5 & 0 \\ 2 & 2 \\ 3 & 1 \end{bmatrix}$. Then take $z = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ (namely, buying 1 unit of asset 1 and selling 1 unit of asset 2), we have $Wz = \begin{bmatrix} 1 \\ 5 \\ 0 \\ 2 \end{bmatrix}$, which means the agent can make profit 1 at time 0 (on the spot market), and 5, 0, 2 at time 1 if $s = 1, 2, 3$, respectively.

The No Arbitrage condition above is (mathematically) equivalent to the following definition

Definition II (No Arbitrage):

$$\text{span}(W) \cap \mathbb{R}_+^{S+1} = \{\mathbf{0}\},$$

where $\text{span}(W) = \{Wz, \forall z \in \mathbb{R}^J\}$ denotes the profit of all posible portfolio.

With these two definitions in mind, we now introduce the No-Arbitrage theorem

Defintion (No-Arbitrage Theorem)

$$\text{span}(W) \cap \mathbb{R}_+^{S+1} = \{\mathbf{0}\} \Leftrightarrow \exists \hat{\pi} \in \mathbb{R}_{++}^{S+1} \text{ such that } \hat{\pi}\tau = 0, \forall \tau \in \text{span}(W)$$

Remarks:

- We require $\hat{\pi} \in \mathbb{R}_{++}^{S+1}$ (strictly positive) due to the need of normalization (see below).

Intuitively, $\hat{\pi}$ serves as a machinery that summarizes the agent's profit at time 0 and time

1, taking probabilities and discount factors into considerations:

$$\hat{\pi}_0[Wz]_0 + \sum_s \hat{\pi}_s[Wz]_s = 0, \quad \forall z \in \mathbb{R}^J$$

Since $[Wz]_0 = -\sum_j q_j z_j$, after the normalization $\pi_s = \frac{\hat{\pi}_s}{\hat{\pi}_0}$, we have

$$-\sum_j q_j z_j + \sum_s \pi_s \sum_j a_{sj} z_j = 0,$$

where π_s denotes the price of one unit of consumption good delivered in state s at time 1. In matrix form, we have $(-q + \pi^T A)z = 0 \Rightarrow q = \pi^T A$.

Defining $\pi_s = m_s \mathrm{Prob}_s$, we have the **Fundamental Equation of Asset Pricing**:

$$q = \mathbb{E}\left(mA\right). \tag{3.2.1}$$

Defintion (No-Arbitrage Theorem)

$$\text{span}(W) \cap \mathbb{R}_{+}^{S+1} = \{\mathbf{0}\} \Leftrightarrow \exists \hat{\pi} \in \mathbb{R}_{++}^{S+1} \text{ such that } \hat{\pi}\tau = 0, \forall \tau \in \text{span}(W)$$

Proof of the No-Arbitrage Theorem

($\Rightarrow$) We prove this direction in two steps:

1. "Strict positivity". First, we define a simplex in the space of $\mathbb{R}^{S+1}$:

$$\Delta = \{\tau \in \mathbb{R}^{S+1} : \sum_{s=0}^{S} \tau_s = 1\}$$

By the definition of No-Arbitrage condition (i.e., the LHS of the theorem), we have

$$\text{span}(W) \cap \Delta = \emptyset$$

Now, invoking the **Strong Separating Hyperplane Theorem** (this is the essence of the proof), then there exists $\hat{\pi}$, such that

$$\sup_{\tau \in \text{span}(W)} \hat{\pi}\tau < \inf_{\tau \in \Delta} \hat{\pi}\tau \tag{3.2.2}$$

To prove $\hat{\pi} \in \mathbb{R}_{++}^{S+1}$, suppose there exists a $\hat{\pi}$ whose s^{th} component, $\pi_s \leq 0$, then since the base vector ($e_s \in \Delta$), we have

$$\inf_{\tau \in \Delta} \hat{\pi}\tau \leq e_s\tau \leq 0$$

However, $0 \in \text{span}(W)$ (in words, no buying and selling in the asset market), which implies that

$$\sup_{\tau \in \text{span}(W)} \hat{\pi}\tau \geq \hat{\pi}0 = 0$$

Hence, we have the $0 \leq LHS < RHS \leq 0$ in equation (**??**), a contradiction.

2. "Orthorgonality".. Suppose there exists a $\tau \in \text{span}(W)$ such that $\hat{\pi}\tau \neq 0$. Then, we can find an α such that $\hat{\pi}\alpha\tau$ is arbitrarily large. (In words, agent can make infinite profit if there is an opportunity for arbitrage). However, the RHS of the theorem is bounded above (because Δ is a compact set). Then we have a contradiction. Hence,

$$\hat{\pi}\tau = 0 \forall \tau \in \text{span}(W)$$

($\Leftarrow$) Now suppose there exists an arbitrage opportunity $\tau^* \in \text{span}(W)$ such that $\tau^* \in \mathbb{R}^{S+1}\backslash\{0\}$. Since $\hat{\pi} \in \mathbb{R}^{S+1}_{++}$, we have $\hat{\pi}\tau^* > 0$. However, this contradicts to the assumption that $\hat{\pi}\text{span}(W) = 0$ as $\tau^* \in \text{span}(W)$. ■

Proposition. (The degree of freedom of stochastic discounts) Given $A \in \mathbb{R}^{S\times J}$ of rank J and $q \in \mathbb{R}^J$, the set of viable stochastic discounts is an $\mathbb{R}^{S-J}$ strictly positive subspace of $\mathbb{R}^J$, namely,

$$R(q; A) = \{\pi \in \mathbb{R}^S_{++} : q = \pi^T A\} = \mathbb{R}^{S-J}_{++}$$

In words, once you find a viable π, there are only $S - J$ elements in that π you can vary.

3.3 Arrow Theorem

3.3.1 Budget constraints equivalence

Proposition 4. *The following two budget constraints are equivalent under No-Arbitrage:*

1. $p_0(x^i_0 - w^i_0) + qz = 0$ *and* $p_s(x^i_s - w^i_s) = a_s z$
2. $p_0(x^i_0 - w^i_0) + \sum_s \pi_s p_s(x^i_s - w^i_s) = 0$ *and,*

$$\begin{bmatrix} \vdots \\ p_s(x^i_s - w^i_s) \\ \vdots \end{bmatrix} \in span(A)$$

Remarks: When the asset market is complete $\text{span}(A) = \mathbb{R}^J = \mathbb{R}^S$.

Proof Suppose q satisfies the No-Arbitrage condition, we have the following asset pricing equation

$$q = \pi A, \quad \text{with } q \in \mathbb{R}^J, \pi \in \mathbb{R}^S, A \in \mathbb{R}^{S\times J}$$

Now the first equation of the first set of constraints becomes

$$p_0(x^i_0 - w^i_0) + qz = p_0(x^i_0 - w^i_0) + \pi Az = 0,$$

which is equivalent to

$$p_0(x^i_0 - w^i_0) + \sum_s \pi_s a_s z = 0$$

Using the the second equation of the first set of constraints to replace $a_s z$, we have

$$p_0(x_0^i - w_0^i) + \sum_s \pi_s p_s(x_s^i - w_s^i) = 0$$

To complete the proof, notice that the second equation of the second set of constraints is simply the matrix for of

$$p_s(x_s^i - w_s^i) = a_s z$$

$\Rightarrow$

$$\begin{bmatrix} \vdots \\ p_s(x_s^i - w_s^i) \\ \vdots \end{bmatrix} = \begin{bmatrix} \vdots \\ a_s \\ \vdots \end{bmatrix} z$$

which belongs to the span of A. ■

3.3.2 Arrow's Theorem

Theorem 1 ((Arrow's Theorem)). *Under the No-Arbitrage condition, any Arrow-Debreu equilibrium* $(x_0^*, \{x_s^*\}, \{p_0^*, p_s^*\})$ *can be supported as a Financial Market Equilibrium (FME)* $(x_0^*, \{x_s^*\}, z^*, \{p_0^*, q^*\})$*, where* z^* *is the portfolio that satisfies the budget constraints in the FME, and* q^* *satisfies the No-Arbitrage condition.*

Proof Since goods market clearing implies financial market clearing, the market clearing conditions in the FME is the same as the one in Arrow-Debreu equilibrium. Now, we only need to discuss the equivalence of the budget constraints between these two economies.

Since the market in complete, the spaning condition in the FME is not binding. Then the budget constraints in the FME reduce to

$$p_0(x_0^i - w_0^i) + \sum_s \pi_s p_s(x_s^i - w_s^i) = 0$$

Now using $p_0 = \phi_0$ and $p_s = \pi_s \phi_s$, we can immediately see the equivalence between the AD equilibrium and the FME. ■

Chapter 4

General Equilibirum with Incomplete Markets (GEI)

In this chapter, we introduce several examples of the so-called "Pecuniary Externalities", where the prices have dual roles in the economy:

- They clear the markets;
- They also affect the restrictions on agents' choice set. Concretely, the consumption allocation is restricted by:

$$\begin{bmatrix} \vdots \\ x_s^i \\ \vdots \end{bmatrix} \in B^i(p_1, \ldots, p_S).$$

An important proposition that relates to the second role of prices in the GEI is the following:

Proposition 5. *Constrained Pareto Efficiency* $\Rightarrow B^i \perp p$.

Equivalently, If B^i depends on prices p, the equilibrium allocation is generally Constrained Inefficient.[a]

[a] An exception is homothetic preferences.

The converse is not true for the proposition above if $L > 1$. But by the Diamond's theorem, we have if $L = 1$ (single good economy), then

Proposition 6. $B^i \perp p \Leftrightarrow$ *Constrained Pareto Efficiency* $\quad$ *when* $\; L = 1$.

Consider an economy with agents $i \in \mathcal{I}$, two dates $t = 0, 1$, and states $s \in \mathcal{S}$ at date 1 with probabilities π_s. Let $z \in R^K$ be the asset portfolio vector with date-0 price vector q. The asset payoff in state s is $a_s \in R^K$.

Consumer i's Problem:

$$\begin{aligned} \max_{\{x^i, z^i\}} \quad & u^i(x_0^i) + \sum_{s \in \mathcal{S}} \pi_s u^i(x_s^i) \\ \text{s.t.} \quad & p_0(x_0^i - w_0^i) + q \cdot z^i = 0 \\ & p_s(x_s^i - w_s^i) = a_s \cdot z^i, \quad \forall s \in \mathcal{S} \end{aligned}$$

4.1 Sources of Inefficiency

4.1.1 Bid–Ask Spread (Transaction Costs)

Inefficiency Analysis:

- **Real Costs:** If spreads represent resource costs, the allocation is Second-Best Efficient.
- **Market Power/Rents:** If spreads represent rents, the outcome is inefficient relative to the frictionless benchmark (impedes risk sharing).

Model Modification: We distinguish between long (z^L) and short (z^S) positions. Let q^a be the ask price and q^b be the bid price ($q^a \geq q^b$).

- **Positions:** $z = z^L - z^S$ with $z^L, z^S \geq 0$.
- **Date 0 Budget:**
$$p_0(x_0 - w_0) + q^a \cdot z^L - q^b \cdot z^S = 0$$
- **State s Budget:**
$$p_s(x_s - w_s) = a_s \cdot (z^L - z^S)$$

4.1.2 Exogenous Borrowing Limits

Inefficiency Analysis:

- Relative to First-Best: **Yes**, it rules out mutually beneficial intertemporal trades.
- Relative to Constrained Set: Typically Constrained Pareto Efficient (CPE) if the planner faces the same enforceability constraint.

Model Modification: Impose a cap $B^i \geq 0$ on the value of short positions (or borrowing).

- **Constraint:**
$$q \cdot z \geq -B^i \quad \text{(Net value limit)}$$

- *Alternatively, with Bid-Ask spreads:*
$$q^b \cdot z^S \leq B^i$$

4.1.3 Default (Limited Enforcement)

Inefficiency Analysis:

- **Constrained Inefficiency:** Generally yes. Default introduces deadweight losses (ψ) and pecuniary externalities (via price-dependent recovery rates).

Model Modification: Agents choose collateralized short positions. Let r_s be the actual repayment on short positions in state s.

- **Positions:** $z = z^L - z^S$, with $z^L, z^S \geq 0$.
- **Repayment Constraint (Limited Liability):**
$$0 \leq r_s \leq a_s \cdot z^S$$

- **Date 0 Budget:**
$$p_0(x_0 - w_0) + q \cdot (z^L - z^S) = 0$$

- **State s Budget:**
$$p_s(x_s - w_s) = a_s \cdot z^L - r_s - \underbrace{\psi_s(a_s \cdot z^S - r_s)}_{\text{Optional deadweight cost}}$$

- **Pricing:** Lenders are rational, so q reflects the anticipated repayment schedule $\{r_s\}$.

4.1.4 Collateral (Secured Borrowing)

Inefficiency Analysis:

- **Pecuniary Externalities:** Yes. The value of collateral depends on spot prices (P_s^k). When prices fall, constraints tighten, causing fire-sales which lower prices further. Competitive equilibria are generally Constrained Inefficient.

Model Modification: Introduce a durable, collateralizable asset $k \geq 0$ with date-0 price p_0^k and state-s price P_s^k. Only a fraction $\mu_s \in (0, 1]$ is pledgeable.

- **Date 0 Budget:**
$$p_0(x_0 - w_0) + p_0^k k + q \cdot (z^L - z^S) = 0$$

- **Collateral Constraint (per state s):**
$$r_s \leq \mu_s P_s^k k$$

- **State s Budget:**
$$p_s(x_s - w_s) = a_s \cdot z^L - r_s + P_s^k k$$
Note: The term $P_s^k k$ represents the resale value of the collateral asset held by the borrower, net of the repayment r_s.

4.1.5 Information Asymmetry

Inefficiency Analysis:

- **General Inefficiency:** Markets fail to reach the informationally constrained frontier due to adverse selection (pooling) or moral hazard (contract externalities).

Model Modification (Two Approaches):

1. **Non-verifiable States (Radner):** Contracts cannot condition on s, but only on a public signal $y = y(s)$. Payoffs a_y and prices p_y are defined over signals, effectively pooling states with the same y.

2. **Incentive Compatibility (Mechanism Design):** Keep state-contingent goods but impose Truth-Telling constraints. If state s is private info, for any s, t with same observables:
$$u(x_s) - T_s \geq u(x_t) - T_t$$
where T_s is the net financial transfer in state s.

Chapter 5

Assignment and Search

Proposition 7. *The dual assignment and the primal assignment problems yield the same solution.*

Proof The Primal problem is given by

$$\begin{aligned} \max_{\pi^{ih}} \quad & \sum_i \sum_h \pi^{ih} U^{ih} \\ \text{s.t.} \quad & \sum_h \pi^{ih} = w^i, \quad \forall i \\ & \sum_i \pi^{ih} = f^h, \quad \forall h \end{aligned}$$

The Dual problem is given by

$$\begin{aligned} \min_{w^i, f^h} \quad & \sum_i u^i w^i + \sum_h v^h f^h \\ \text{s.t.} \quad & u^i + v^h \geq U^{ih}, \quad \forall i, h \end{aligned}$$

The Lagragean for the Primal problem is

$$\mathcal{L}^P = \sum_i \sum_h \pi^{ih} U^{ih} + \sum_h v^h \left(f^h - \sum_i \pi^{ih} \right) + \sum_i u^i \left(w^i - \sum_h \pi^{ih} \right)$$

The Lagragean for the Dual problem is

$$\mathcal{L}^D = \sum_i u^i w^i + \sum_h v^h f^h + \sum_i \sum_h \pi^{ih} \left(U^{ih} - u^i - v^h \right)$$

Obivously, when choosing proper Lagragean Multipliers, namely, using $\{v^h\}, \{u^i\}$ as the

LM for the Primal problem and $\{\pi^{ih}\}$ for the Dual problem, these two Lagragean coincide with each other and thus yield the same solutions. ■

Proposition 8. *If U^{ih} is supermodular, then the optimal matching is assortative.*

Proof Mathematically, it is a direct implication of the Hardy-Littlewood-Polya inequality, which says

$$\langle x, y \rangle \leq \langle \hat{x}, \hat{y} \rangle,$$

where $\hat{x}, \hat{y}$ are the decreasing rearrangements of x, y respectively.

Now, we prove the proposition by contradiction with inductive steps. Assume that the optimal mathching is not assortative; concretely, assume that worker i matches with firm $\sigma_*(i) \neq i$. Then, using the supermodularity of U, we have

$$\sum_{i \neq 1 \neq \sigma_*{}^{-1}(1)} U^{i\sigma_*(i)} + U^{11} + U^{\sigma_*{}^{-1}(1)\sigma_*(1)} > \sum_i U^{i\sigma_*(i)}.$$

In the k^{th} step, we have

$$\sum_{i \neq k \neq \sigma_*{}^{-1}(k)} U^{i\sigma_*(i)} + U^{kk} + U^{\sigma_*{}^{-1}(k)\sigma_*(k)} > \sum_i U^{i\sigma_*(i)}.$$

Iterate this procedure for all i, we reach a contradiction. ■