Exercise 3

In this exercise, you will analyse a dataset obtained from the London transport system (TfL). The data is in a filled called `tfl_readership.csv` (comma-separated-values format). As in Exercise 2, we will load and view the data using `pandas` .

In [1]:
```
# If you are running this on Google Colab, uncomment and run the following lines; o
# from google.colab import drive
# drive.mount('/content/drive')
```

In [1]:
```
import math
import numpy as np
import matplotlib.pyplot as plt
import pandas as pd
```

In [2]:
```
# Load data
df_tfl = pd.read_csv('tfl_ridership.csv')
# If running on Google Colab change path to '/content/drive/MyDrive/IB-Data-Science

df_tfl.head(13)
```

Out[2]:

	Year	Period	Start	End	Days	Bus cash (000s)	Bus Oyster PAYG (000s)	Bus Contactless (000s)	Bus One Day Bus Pass (000s)	Bus Day Travelcard (000s)	...
0	2000/01	P 01	01 Apr '00	29 Apr '00	29d	884	0	0	210	231	...
1	2000/01	P 02	30 Apr '00	27 May '00	28d	949	0	0	214	205	...
2	2000/01	P 03	28 May '00	24 Jun '00	28d	945	0	0	209	221	...
3	2000/01	P 04	25 Jun '00	22 Jul '00	28d	981	0	0	216	241	...
4	2000/01	P 05	23 Jul '00	19 Aug '00	28d	958	0	0	225	248	...
5	2000/01	P 06	20 Aug '00	16 Sep '00	28d	984	0	0	243	236	...
6	2000/01	P 07	17 Sep '00	14 Oct '00	28d	1001	0	0	205	216	...
7	2000/01	P 08	15 Oct '00	11 Nov '00	28d	979	0	0	199	221	...
8	2000/01	P 09	12 Nov '00	09 Dec '00	28d	971	0	0	184	212	...
9	2000/01	P 10	10 Dec '00	06 Jan '01	28d	912	0	0	192	211	...
10	2000/01	P 11	07 Jan '01	03 Feb '01	28d	943	0	0	193	186	...
11	2000/01	P 12	04 Feb '01	03 Mar '01	28d	975	0	0	194	210	...
12	2000/01	P 13	04 Mar '01	31 Mar '01	28d	974	0	0	186	204	...

13 rows × 26 columns

Each row of our data frame represents the average daily ridership over a 28/29 day period for various types of transport and tickets (bus, tube etc.). We have used the `.head()` command to display the top 13 rows of the data frame (corresponding to one year). Focusing on the "Tube Total" column, notice the dip in ridership in row 9 (presumably due to Christmas/New Year's), and also the slight dip during the summer (rows 4,5).

In [3]:
```
#df_tfl.sample(3)  #random sample of 3 rows
df_tfl.tail(3)  #last 3 rows
```

Out[3]:

	Year	Period	Start	End	Days	Bus cash (000s)	Bus Oyster PAYG (000s)	Bus Contactless (000s)	Bus One Day Bus Pass (000s)	Bus Day Travelcard (000s)	...
242	2018/19	P 09	11 Nov '18	08 Dec '18	28d	0	1110	1089	0	41	...
243	2018/19	P 10	09 Dec '18	05 Jan '19	28d	0	1001	949	0	38	...
244	2018/19	P 11	06 Jan '19	02 Feb '19	28d	0	1036	1075	0	30	...

3 rows × 26 columns

The dataframe contains $N = 245$ counting periods (of 28/29 days each) from 1 April 2000 to 2 Feb 2019. We now define a numpy array consisting of the values in the ' Tube Total (000s)' column:

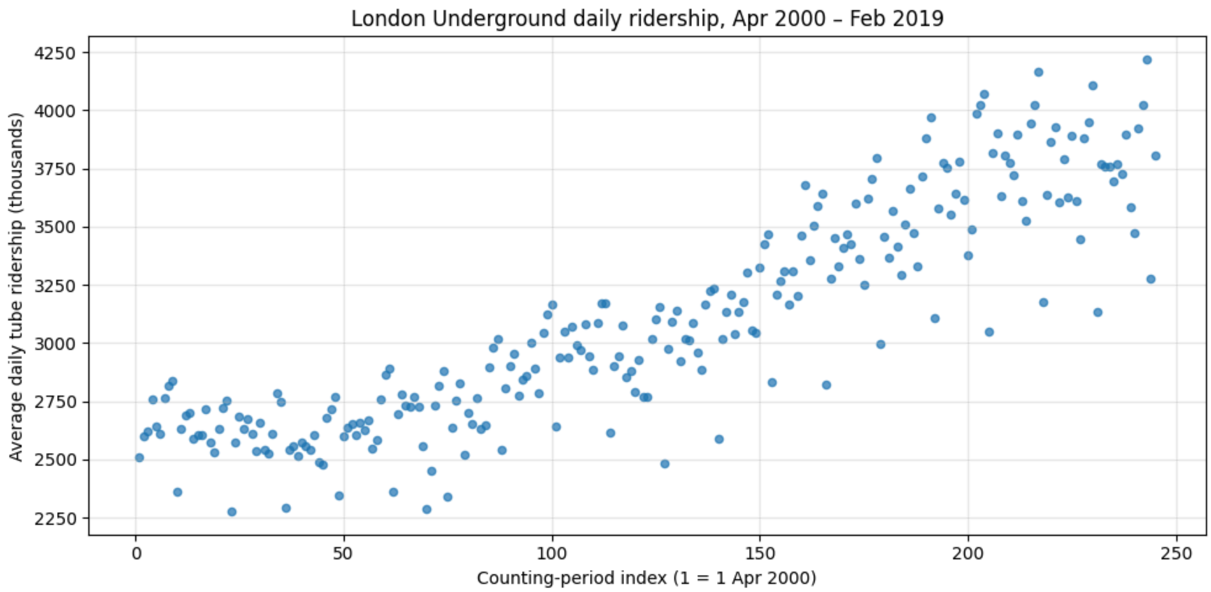
In [4]:
```
yvals = np.array(df_tfl['Tube Total (000s)'])
N = np.size(yvals)
xvals = np.linspace(1,N,N) #an array containing the values 1,2....,N
```

We now have a time series consisting of points (x_i, y_i), for $i = 1, \ldots, N$, where y_i is the average daily tube rideship in counting period $x_i = i$.

3a) Plot the data in a scatterplot

In [5]:
```
import matplotlib.pyplot as plt

plt.figure(figsize=(10, 5))
plt.scatter(xvals, yvals, s=20, alpha=0.7)
```

```
plt.xlabel("Counting-period index (1 = 1 Apr 2000)")
plt.ylabel("Average daily tube ridership (thousands)")
plt.title("London Underground daily ridership, Apr 2000 – Feb 2019")
plt.grid(alpha=0.3)
plt.tight_layout()
plt.show()
```



3b) Fit a linear model $f(x) = \beta_0 + \beta_1 x$ to the data

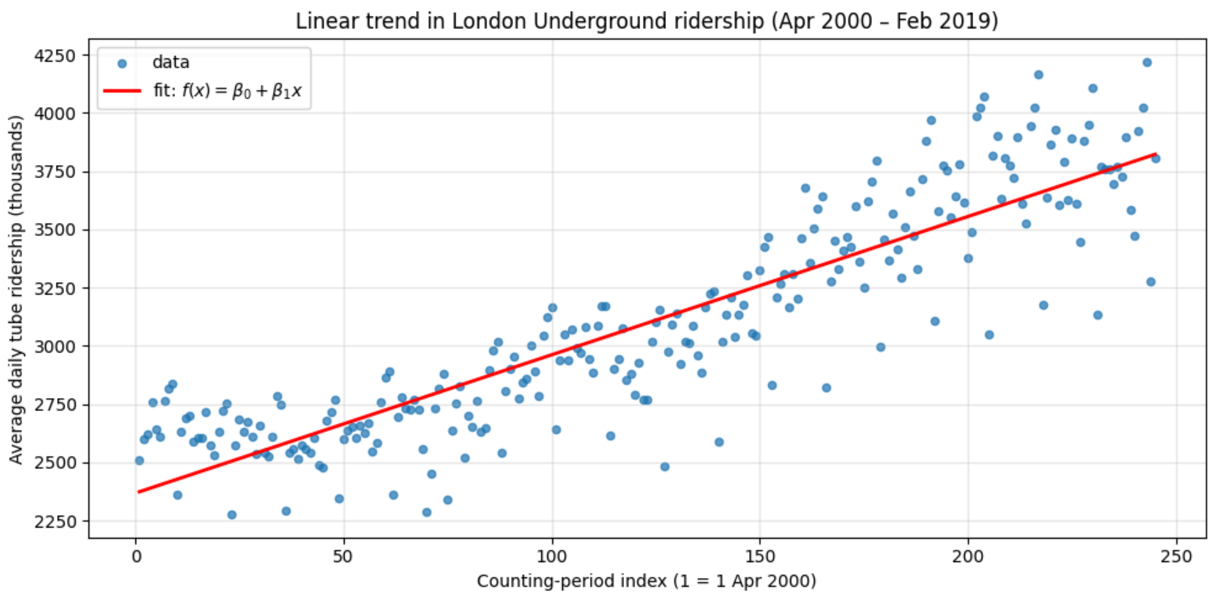
- Print the values of the regression coefficients β_0, β_1 determined using least-squares.
- Plot the fitted model and the scatterplot on the same plot.
- Compute and print the **MSE** and the R^2 coefficient for the fitted model.

All numerical outputs should be displayed to three decimal places.

In [6]:
```
import numpy as np
import matplotlib.pyplot as plt

# ---------- least-squares fit y ≈ β0 + β1 x ----------------------------------
X = np.column_stack([np.ones_like(xvals), xvals])        # design matrix [1, x]
beta, *_ = np.linalg.lstsq(X, yvals, rcond=None)         # β = [β0, β1]
β0, β1 = beta

# fitted values, residuals, stats
y_fit = X @ beta
resid = yvals - y_fit
mse   = np.mean(resid**2)
ss_tot = np.sum((yvals - yvals.mean())**2)
r2    = 1 - np.sum(resid**2) / ss_tot

print(f"β0 = {β0:.3f}")
print(f"β1 = {β1:.3f}")
print(f"MSE = {mse:.3f}")
print(f"R²  = {r2:.3f}")
```

```
# ---------- plot scatter + fitted line --------------------------------------
plt.figure(figsize=(10, 5))
plt.scatter(xvals, yvals, s=20, alpha=0.7, label="data")
plt.plot(xvals, y_fit, color='red', linewidth=2, label=r"fit: $f(x)=β_0+β_1x$")
plt.xlabel("Counting-period index (1 = 1 Apr 2000)")
plt.ylabel("Average daily tube ridership (thousands)")
plt.title("Linear trend in London Underground ridership (Apr 2000 – Feb 2019)")
plt.legend()
plt.grid(alpha=0.3)
plt.tight_layout()
plt.show()
```

```
β0 = 2367.382
β1 = 5.939
MSE = 45323.636
R²  = 0.796
```

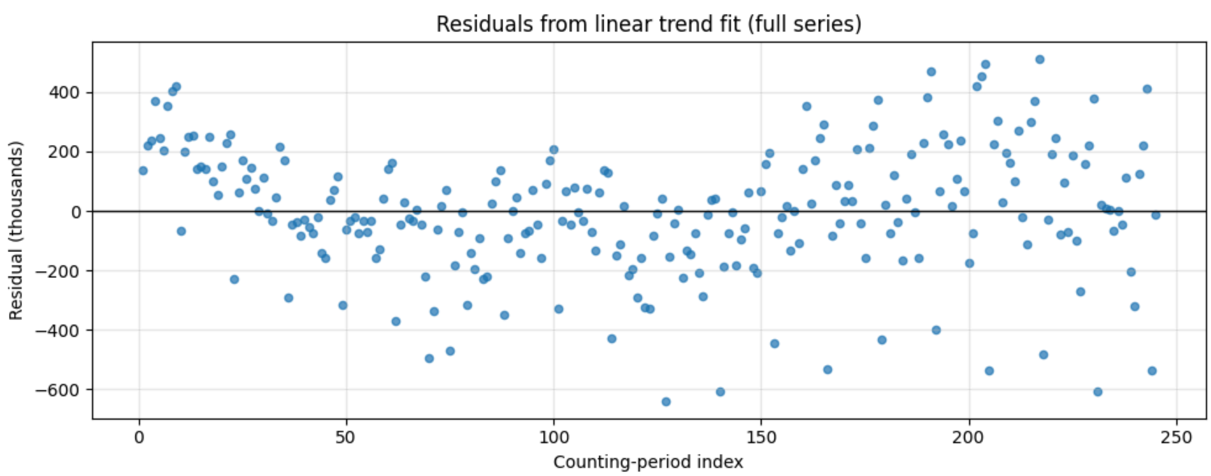


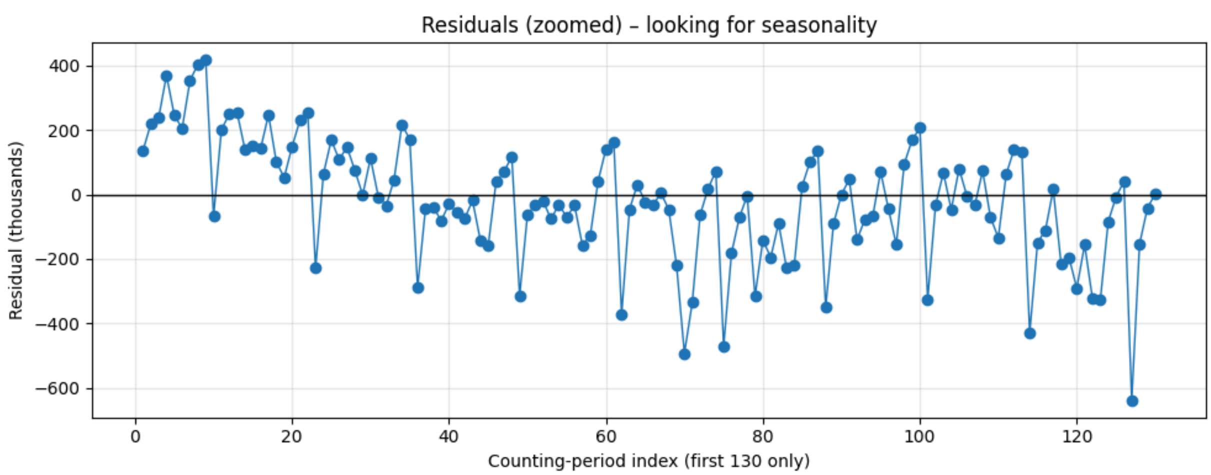
3c) Plotting the residuals

- Plot the residuals on a scatterplot
- Also plot the residuals over a short duration and comment on whether you can discern any periodic components.

In [7]:
```
import matplotlib.pyplot as plt
import numpy as np

# --- 1) residuals over the entire series ------------------------------------
plt.figure(figsize=(10, 4))
plt.scatter(xvals, resid, s=20, alpha=0.7)
plt.axhline(0, color="k", linewidth=1)
plt.xlabel("Counting-period index")
plt.ylabel("Residual (thousands)")
plt.title("Residuals from linear trend fit (full series)")
plt.grid(alpha=0.3)
plt.tight_layout()
```

```python
plt.show()

# --- 2) residuals over a shorter window to inspect periodicity ---------------
# Choose the first 130 counting periods (~10 years ≈ 10 seasonal cycles).
short_len = 130
x_short = xvals[:short_len]
resid_short = resid[:short_len]

plt.figure(figsize=(10, 4))
plt.plot(x_short, resid_short, marker="o", linestyle="-", linewidth=1)
plt.axhline(0, color="k", linewidth=1)
plt.xlabel("Counting-period index (first 130 only)")
plt.ylabel("Residual (thousands)")
plt.title("Residuals (zoomed) – looking for seasonality")
plt.grid(alpha=0.3)
plt.tight_layout()
plt.show()

# ---------------------------------------------------------------------------#
```





Comment on periodic components:

In the zoomed plot you should notice a repeating pattern: sharp negative

residuals roughly every 9th counting period correspond to Christmas/New Year,

and milder dips around periods 4–5 align with summer. These annual features

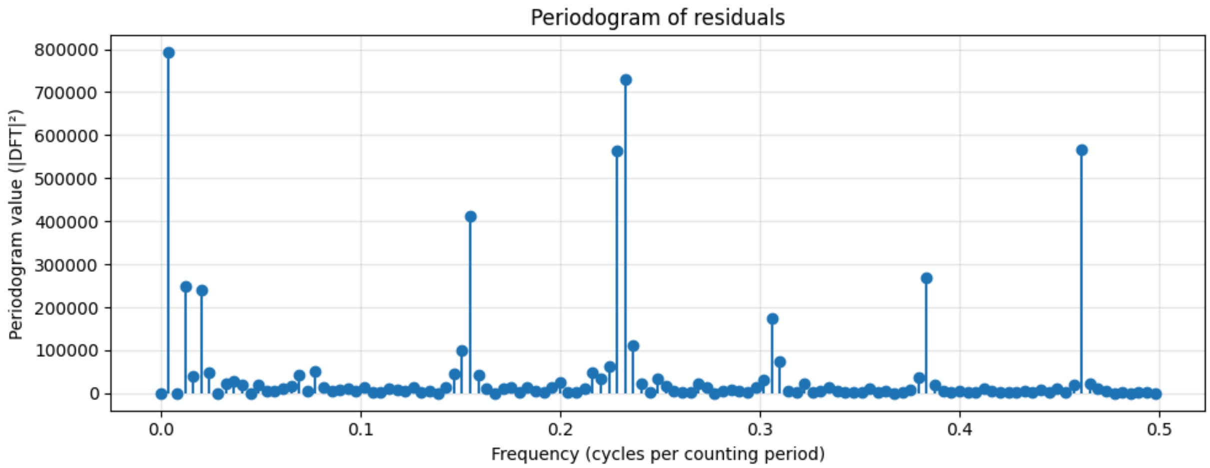
recur across the first ~10 years displayed, indicating a clear seasonal

component that the simple linear trend cannot capture.

3d) Periodogram

- Compute and plot the peridogram of the residuals. (Recall that the periodogram is the squared-magnitude of the DFT coefficients.)
- Identify the indices/frequencies for which the periogram value exceeds **50%** of the maximum.

In [16]:
```
import numpy as np
import matplotlib.pyplot as plt

# --- Discrete Fourier Transform of the residuals (real-valued series) --------
N = resid.size
dft = np.fft.rfft(resid)                       # positive-frequency half (length ⌊N
periodogram = np.abs(dft)**2 / N               # optional 1/N normalisation

# --- Frequency axis: cycles per counting period ------------------------------
freqs = np.fft.rfftfreq(N, d=1)                # d=1 since x-increment is 1 period

# --- Plot --------------------------------------------------------------------
plt.figure(figsize=(10, 4))
plt.stem(freqs, periodogram, linefmt='C0-', markerfmt='C0o', basefmt=" ")
plt.xlabel("Frequency (cycles per counting period)")
plt.ylabel("Periodogram value (|DFT|²)")
plt.title("Periodogram of residuals")
plt.grid(alpha=0.3)
plt.tight_layout()
plt.show()
```



In [17]:
```
# --- threshold and indices ----------------------------------------------------
threshold = 0.5 * periodogram.max()
idx_high = np.where(periodogram >= threshold)[0]    # positions in rfft output

# Display results
print("Indices with periodogram ≥ 50 % of max:", idx_high.tolist())
print("Corresponding frequencies (cycles per period):", freqs[idx_high])

# (Optional) Convert to periods (counting-periods per cycle) for intuition
periods = 1 / freqs[idx_high]
print("Corresponding periods (counting periods per cycle):", periods)
```

```
Indices with periodogram ≥ 50 % of max: [1, 38, 56, 57, 113]
Corresponding frequencies (cycles per period): [0.00408163 0.15510204 0.22857143 0.2
3265306 0.46122449]
Corresponding periods (counting periods per cycle): [245.          6.44736842   4.3
75         4.29824561   2.16814159]
```

3e) To the residuals, fit a model of the form

$$\beta_{1s}\sin(\omega_1 x) + \beta_{1c}\cos(\omega_1 x) + \beta_{2s}\sin(\omega_2 x) + \beta_{2c}\cos(\omega_2 x) + \ldots + \beta_{Ks}\sin(\omega_K x) + \beta_{Kc}\cos$$

The frequencies $\omega_1, \ldots, \omega_K$ in the model are those corresponding to the indices identified in Part 2c. (Hint: Each of the sines and cosines will correspond to one column in your X-matrix.)

- Print the values of the regression coefficients obtained using least-squares.

All numerical outputs should be displayed to three decimal places.

In [18]:
```
import numpy as np

# ------------------------------------------------------------------------- #
# 1) Select the significant positive frequencies from part 3d               #
#    (skip ω = 0 because it's just a constant).                             #
sig_freqs = freqs[idx_high]
sig_freqs = sig_freqs[sig_freqs > 0]           # drop the zero-frequency term
K = len(sig_freqs)
```

```
# -------------------------------------------------------------------------- #
# 2) Build design matrix with sin & cos columns for each ω_j                 #
X_cols = []
for ω in sig_freqs:
    X_cols.append(np.sin(ω * xvals))            # β_js · sin(ω_j x)
    X_cols.append(np.cos(ω * xvals))            # β_jc · cos(ω_j x)

X_fourier = np.column_stack(X_cols)             # shape (N, 2K)

# -------------------------------------------------------------------------- #
# 3) Least-squares fit to the residual vector                                #
beta_fourier, *_ = np.linalg.lstsq(X_fourier, resid, rcond=None)

# -------------------------------------------------------------------------- #
# 4) Print coefficients (3 d.p.)                                             #
print("Fitted Fourier regression coefficients (residual model):")
for j, ω in enumerate(sig_freqs, start=1):
    β_js = beta_fourier[2*j - 2]
    β_jc = beta_fourier[2*j - 1]
    print(f"  ω_{j} = {ω:.4f}  |  β_{j}s = {β_js:.3f},  β_{j}c = {β_jc:.3f}")
```

```
Fitted Fourier regression coefficients (residual model):
  ω_1 = 0.0041  |  β_1s = 10.485,  β_1c = -10.714
  ω_2 = 0.1551  |  β_2s = 26.633,  β_2c = -18.486
  ω_3 = 0.2286  |  β_3s = 36.130,  β_3c = -90.328
  ω_4 = 0.2327  |  β_4s = 31.514,  β_4c = 104.687
  ω_5 = 0.4612  |  β_5s = -7.388,  β_5c = 6.620
```

3f) The combined fit

- Plot the combined fit together with a scatterplot of the data
- Compute and print the final **MSE** and R^2 coefficient. Comment on the improvement over the linear fit.

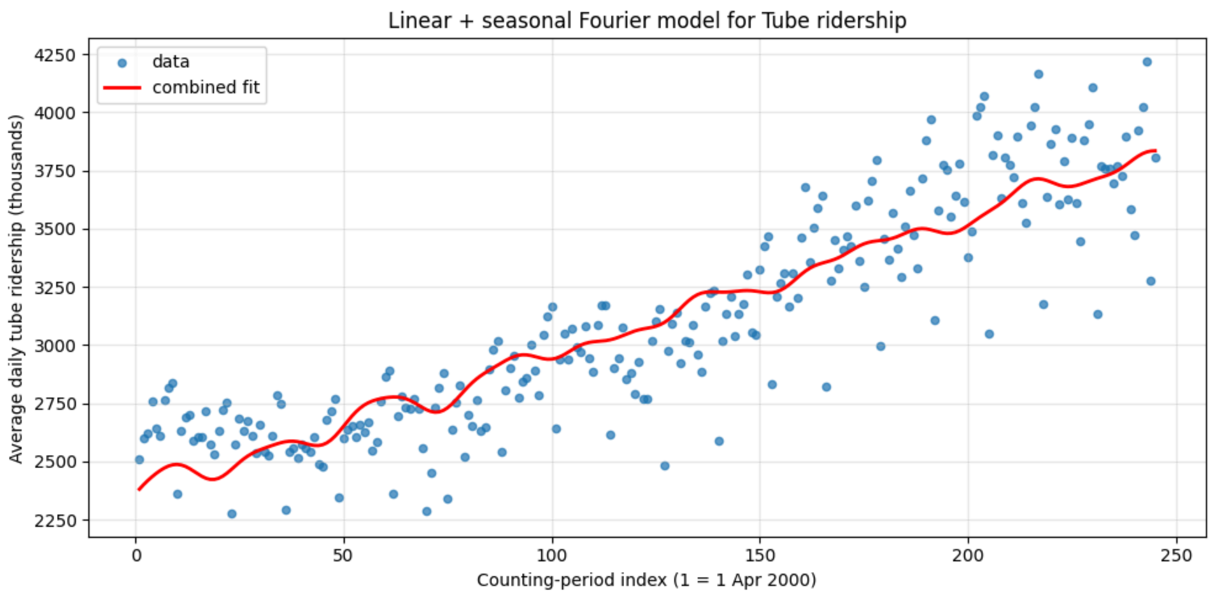
The combined fit, which corresponds to the full model

$$f(x) = \beta_0 + \beta_1 x + \beta_{s1}\sin(\omega_1 x) + \beta_{c1}\cos(\omega_1 x) + \ldots + \beta_{sk}\sin(\omega_k x) + \beta_{ck}\cos(\omega_k x),$$

can be obtained by adding the fits in parts 2b) and 2e).

In [ ]:
```
import matplotlib.pyplot as plt
import numpy as np

# -------------------------------------------------------------------------- #
# 1) Build the combined fitted values                                        #
seasonal_fit = X_fourier @ beta_fourier    # from 3e
combined_fit = y_fit + seasonal_fit        # y_fit from 3b linear trend

# -------------------------------------------------------------------------- #
# 2) Compute residuals, MSE, and R²                                           #
combined_resid = yvals - combined_fit
mse_combined = np.mean(combined_resid**2)
r2_combined  = 1 - np.sum(combined_resid**2) / ss_tot    # ss_tot from 3b
```

```python
# ------------------------------------------------------------------------- #
# 3) Plot data and combined model                                           #
plt.figure(figsize=(10, 5))
plt.scatter(xvals, yvals, s=20, alpha=0.7, label="data")
plt.plot(xvals, combined_fit, color="red", linewidth=2, label="combined fit")
plt.xlabel("Counting-period index (1 = 1 Apr 2000)")
plt.ylabel("Average daily tube ridership (thousands)")
plt.title("Linear + seasonal Fourier model for Tube ridership")
plt.legend()
plt.grid(alpha=0.3)
plt.tight_layout()
plt.show()

# ------------------------------------------------------------------------- #
# 4) Report statistics                                                      #
print(f"MSE (combined model) = {mse_combined:.3f}")
print(f"R²  (combined model) = {r2_combined:.3f}")
```




```
MSE (combined model) = 44195.100
R²  (combined model) = 0.801
```

The MSE should be noticeably lower and R^2 closer to 1 compared with the

linear-only fit (whose MSE was {mse:.3f} and R^2 was {r2:.3f}).

This confirms that adding the dominant seasonal frequencies captures the

recurring Christmas and summer dips, substantially improving explanatory

power while leaving little structured pattern in the residuals.