Ex 3.20 Solve $a_n = 3a_{n-1} - 3a_{n-2} + a_{n-3} \quad \text{for } n > 2, \quad \text{with } a_0 = a_1 = 0, a_2 = 1$

$$\begin{aligned}
a_n &= 3a_{n-1} - 3a_{n-2} + a_{n-3} + \delta_{n2} \\
A(z) &= (3z - 3z^2 + z^3)A(z) + z^2 \\
A(z) &= \frac{z^2}{(1-z)^3} = \sum_{N=2}^{\infty} \frac{N(N-1)}{2} z^N
\end{aligned}$$

for $a_1 = 1$

$$\begin{aligned}
a_n &= 3a_{n-1} - 3_{n-2} + a_{n-3} + \delta_{n1} - 2\delta_{n2} \\
A(z) &= (3z - 3z^2 + z^3)A(z) + z - 2z^2 \\
A(z) &= \frac{z - 2z^2}{(1-z)^3}
\end{aligned}$$

simplify:

$$\begin{aligned}
\frac{z - 2z^2}{(1-z)^3} &= \frac{z - 2z^2}{2} \sum_{N=2}^{\infty} N(N-1) z^{N-2} \\
&= \sum_{N=2}^{\infty} \frac{N(N-1)}{2} z^{N-1} - \sum_{N=2}^{\infty} N(N-1) z^N \\
&= z + \sum_{N=2}^{\infty} \frac{(N+1)N}{2} z^N - \sum_{N=2}^{\infty} N(N-1) z^N \\
&= z + \sum_{N=2}^{\infty} \frac{N - N^2}{2} z^N
\end{aligned}$$

Ex 3.28 Find an expression for $[z^n] \frac{1}{\sqrt{1-z}} \log \frac{1}{1-z}$
since

$$\frac{\partial}{\partial \alpha} (1-z)^{-\alpha} = (1-z)^{-alpha} \log \frac{1}{1-z}$$

, we have:

$$\begin{aligned}
[z^n] \frac{1}{\sqrt{1-z}} \log \frac{1}{1-z} &= \frac{\partial}{\partial \alpha} \binom{-\alpha}{n} \bigg|_{\alpha = \frac{1}{2}} \\
&= \frac{\partial}{\partial \alpha} (-1)^n \frac{\alpha \cdot (\alpha+1) \cdots (\alpha + (n-1))}{n!} \bigg|_{\alpha = \frac{1}{2}} \\
&= (-1)^n \cdot \frac{\alpha \cdot (\alpha+1) \cdots (\alpha + (n-1))}{n!} [\frac{1}{\alpha} + \frac{1}{\alpha+1} + \cdots + \frac{1}{\alpha + (n-1)}] \bigg|_{\alpha = \frac{1}{2}} \\
&= (-1)^n \frac{1}{2^{2n}} \binom{2n}{n} 2[1 + \frac{1}{3} + \cdots + \frac{1}{2n-1}] \\
&= (-1)^n \frac{1}{2^{2n-1}} \binom{2n}{n} 2[H_{2n} - 2H_n]
\end{aligned}$$

Quiz: 1. $a_n = 9a_{n-1} - 20a_{n-2} \quad \text{for } n > 1, \quad \text{with } a_0 = 0, a_1 = 1, lim_{n\to\infty} \frac{a_n}{a_{n-1}}$

$$\begin{aligned}
a_n &= 9a_{n-1} - 20a_{n-2} + \delta_{n1} \\
A(z) &= (9z - 20z^2)A(z) + z \\
A(z) &= \frac{z}{1 - 9z + 20z^2} \\
A(z) &= \frac{\frac{z}{20}}{(z - \frac{1}{4})(z - \frac{1}{5})} \\
A(z) &= \frac{\frac{1}{4}}{z - \frac{1}{4}} - \frac{\frac{1}{5}}{z - \frac{1}{5}} \\
[z^N]A(z) = a_n &= (\frac{1}{4})^{n+1} - (\frac{1}{5})^{n+1} \\
\lim_{n\to\infty} \frac{a_n}{a_{n-1}} &= \frac{(\frac{1}{4})^{n+1} - (\frac{1}{5})^{n+1}}{(\frac{1}{4})^n - (\frac{1}{5})^n} = 5
\end{aligned}$$

2. The value of $\sum_{0\le k\le n} \binom{2k}{k}\binom{2n-2k}{n-k}$
we know that

$$\frac{1}{\sqrt{1-4z}} = \sum_{n\ge 0} \binom{2n}{n} z^n$$

, The sum is given by

$$[z^n](\frac{1}{1-4z})^2 = [z^n]\frac{1}{1-4z} = 4^n$$