Note 1.23
We find first the generating function for the most compact codes that satisfy the requirement, i.e all the sequence of binary strings that do not have consecutive 1s. Denote this class as W, then we have:

$$W = \{\epsilon\} + \{1\} \times (\{\epsilon\} + 0 \times W) + \{0\} \times W$$

Let 0 and 1 have weight 1, and denote the OGF as T(z).

$$0^*(100^*)^* \mapsto \frac{1}{1-z} \cdot \frac{1}{1 - \frac{z^2}{1-z}} =: f(z)$$

First note that:

$$\begin{aligned} f(z) &:= \frac{1}{1-z-z^2} \\ &= \sum_{k=0}^{\infty} F_k z^k \end{aligned}$$

where F_k the k-th *Fibonacci number* is. Then:

$$\begin{aligned} T(z) &= 1 + z(1 + zT(z)) + zT(z) \\ \Rightarrow T(z) &= \frac{1+z}{1-z-z^2} \\ &= \sum_{n=0}^{\infty} F_n \cdot z^n + \sum_{n=0}^{\infty} F_n z^{n+1} \\ &= F_0 z + \sum_{n=0}^{\infty} (F_{n+1} + F_n) \cdot z^{n+1} \\ &= z + \sum_{n=1}^{\infty} F_{n+1} z^n \\ &= \sum_{n=0}^{\infty} F_{n+1} z^n \end{aligned}$$

Since this the best compact code we can achieve, given any other code scheme C, with $2^n \leq \sum_{j=0}^{L} C_j \leq \sum_{j=0}^{L} F_{j+1}$ since

$$F_1 + F_2 + \cdots + F_L + F_{L+1} = F_{L+3} - 2$$

We have:

$$F_{L+3} \geq 2^n + 2$$

for large L,

$$F_{L+3} \approx \frac{1}{\sqrt{5}} \varphi^{L+3}$$

so:

$$
\begin{aligned}
\frac{1}{\sqrt{5}}\varphi^{L+3} &\geq 2^n + 2 \\
\Rightarrow \log_2 \varphi \cdot (L+3) &\geq \log_2 \sqrt{5} + \log_2(2^n+2) \\
\Rightarrow L &\geq \lambda n + O(1), \quad \lambda = \frac{1}{\log_2 \varphi}
\end{aligned}
$$

Note 1.43
Differentiate

$$
\begin{aligned}
\frac{H^{'}(z)}{H(z)} &= (\log(z))' + \sum -H_m \frac{1}{1-z^m}' \\
\Rightarrow \frac{zH'(z)}{H(z)} &= 1 + \sum H_m \frac{mz^m}{1-z^m} \\
\Rightarrow \sum_{n=1}^{\infty} nH_n z^n &= \sum_{n=1}^{\infty} nH_n z^n + \sum_{m=1}^{\infty} H_m \frac{mz^m}{1-z^m} \cdot \sum_{l=1}^{\infty} H_l z^l \\
&= \sum_{n=1}^{\infty} nH_n z^n + \sum_{m=1}^{\infty} H_m (mz^m \sum_{k=0}^{\infty} z^{mk}) \cdot \sum_{l=1}^{\infty} H_l z^l \\
&= \sum_{n=1}^{\infty} nH_n z^n + (\sum_{m=1}^{\infty} H_m \sum_{k=0}^{\infty} mz^{m(k+1)}) \cdot \sum_{l=1}^{\infty} H_l z^l \\
&= \sum_{n=1}^{\infty} nH_n z^n + \sum_{n=1}^{\infty} \sum_{m(k+1)=j} H_m \cdot m \cdot H_{n-j} \cdot z^{m(k+1)+z^{n-j}} \\
&= \sum_{n=1}^{\infty} nH_n z^n + \sum_{n=1}^{\infty} (\sum_{m(k+1)=j} H_m \cdot m) \cdot H_{n-j} \cdot z^j \\
&= \sum_{n=1}^{\infty} nH_n z^n + (\sum_{n=1}^{\infty} (\sum_{m|j} H_m \cdot m) \cdot H_{n-j}) \cdot z^j \\
\Rightarrow (n-1) \cdot H_n &= \sum_{j=1}^{n-1} (\sum_{m|j} (H_m \cdot m)) \cdot H_{n-j}
\end{aligned}
$$

Program 1.1: see **coins.java** The optimum is reached with the four tuple $(1, 2, 3, 4)$

Program 1.2: see **cayleyPolya.py**

use python code to test large data, the ratio of H_n/H_{n-1} increases monotonically to the *beta* value stated in the book. ($\doteq 2.95576$)