# MEASURE AND INTEGRATION

## Math 631

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# Contents

# 1 Measure Theory and Integration

## 1.1 Elementary Sets and Measure

The motivation to start this study is to lay a foundation to assign a measure to arbitrary sets $E \in \mathbb{R}^d$. The set can be nice set such as intervals, boxes, and polygons, but it can also be any generic or strange set that we wish to study. We want a measure that is well behaved under limits (e.g. Lebesgue measure is well behaved under pointwise limits).

**Definition 1.1.1.** *Finite Intervals:* $(a, b), (a, b], [a, b), [a, b]$ *for* $a, b \in \mathbb{R}$. *The length is* $b - a$ *and it can be zero.*

**Definition 1.1.2.** *Box:* $B \in \mathbb{R}^d$ *is a Cartesian product of intervals* $B = I_1 \times ... \times I_d$

**Definition 1.1.3.** *Elementary Set is any subset of* $\mathbb{R}^d$ *that is a finite union of boxes.*

We denote the set of elementary sets as $\mathcal{E}(\mathbb{R}^d)$. It does not include unbounded sets.

**Proposition 1.1.1.** *(Boolean closure) The set* $\mathcal{E}(\mathbb{R}^d)$ *is closed taking unions* $(E \cup F)$*, intersects* $(E \cap F)$*, differences* $(E \setminus F)$ *and symmetric differences* $(E \Delta F = (E \setminus F) \cup (F \setminus E))$*, as well as translations:* $E \in \mathcal{E}(\mathbb{R}^d), x \in \mathbb{R}^d \Rightarrow E + x = \{y + x : y \in E\} \in \mathcal{E}(\mathbb{R}^d)$.

*Proof.* Let $E, F \in \mathcal{E}(\mathbb{R}^d)$, then each of them is a finite union of boxes. $E = \bigcup_{i=1}^n B_i$, $F = \bigcup_{i=1}^m B_i'$.
Denote the collection of $\{B_i\}_{i=1}^n$ and $\{B_i'\}_{i=1}^m$ as a new set of boxes:
$\{C_i\}_{i=1}^{n+m} = \{B_1, ..., B_n, B_1', ..., B_m'\}$ Then,

$$\begin{aligned} E \cup F &= (\bigcup_{i=1}^n B_i) \cup (\bigcup_{i=1}^m B_i') \\ &= \bigcup_{i=1}^{n+m} C_i \end{aligned}$$

Thus, $E \cup F$ is a finite union of boxes and is an elementary set.

$$\begin{aligned} E \cap F &= (\bigcup_{i=1}^n B_i) \cap (\bigcup_{i=1}^m B_i') \\ &= (B_1 \cap (\bigcup_{i=1}^m B_i')) \cup ... \cup (B_n \cap (\bigcup_{i=1}^m B_i') \\ &= ((B_1 \cap B_1') \cup ... \cup (B_1 \cap B_m')) \cup ... \cup ((B_n \cap B_1') \cup ... \cup (B_n \cap B_m')) \\ &= (B_1 \cap B_1') \cup ... \cup (B_1 \cap B_m') \cup ... \cup (B_n \cap B_1') \cup ... \cup (B_n \cap B_m') \end{aligned}$$

Now we prove that the intersection of two boxes is a box. Suppose $B_i = I_1^i \times ... \times I_d^i$, $B_j' = I_1^{'j} \times ... \times I_d^{'j}$, then $B_i \cap B_j' = (I_1^i \cap I_1^{'j}) \times ... \times (I_d^i \cap I_d^{'j})$. Since the intersection of two intervals is an interval (degenerate or non-degenerate), by definition, $B_i \cap B_j'$ is a box.
Then, by (2), $E \cap F$ is a finite union of $n \times m$ boxes, and it is an elementary set.

Now we look at difference.

$$\begin{aligned}
E \setminus F &= E \cap F^C \\
&= (\bigcup_{i=1}^{n} B_i) \cap (\bigcup_{i=1}^{m} B_i')^C \\
&= (\bigcup_{i=1}^{n} B_i) \cap (\bigcap_{i=1}^{m} B_i^{'C}) \\
&= (B_1 \cap (\bigcap_{i=1}^{m} B_i^{'C})) \cup ... \cup (B_n \cap (\bigcap_{i=1}^{m} B_i^{'C})) \\
&= ((B_1 \cap B_1^{'C}) \cap ... \cap (B_1 \cap B_m^{'C})) \cup ... \cup ((B_n \cap B_1^{'C}) \cap ... \cap (B_n \cap B_m^{'C})) \\
&= ((B_1 \setminus B_1^{'}) \cap ... \cap (B_1 \setminus B_m^{'})) \cup ... \cup ((B_n \setminus B_1^{'}) \cap ... \cap (B_n \setminus B_m^{'}))
\end{aligned}$$

Since $B_i$ and $B_j^{'}$ are boxes, $B_i \setminus B_j^{'}$ are elementary. Then, by the first two parts that we just proved, each $((B_i \setminus B_1^{'}) \cap ... \cap (B_i \setminus B_m^{'}))$ is elementary, and their union is elementary. Thus, $E \setminus F$ is an elementary set.
Next, since $E$ and $F$ are elementary, $E \setminus F$ is elementary and $F \setminus E$ is also elementary. By the part we just proved, their union $(E \setminus F) \cup (F \setminus E)$ is also elementary. Thus, $E \Delta F$ is an elementary set.
Next,

$$\begin{aligned}
E + x &= \bigcup_{i=1}^{n} B_i + x \\
&= \bigcup_{i=1}^{n} (I_1^i + x) \times ... \times (I_d^i + x) \\
&= \bigcup_{i=1}^{n} I_1^{'i} \times ... \times I_d^{'i} \\
&= \bigcup_{i=1}^{b} B_{'i}
\end{aligned}$$

where we have translation of each interval by $x$ in the second and third equality. Thus, $E + x$ is an elementary set.
Thus, we finished the proof that $\mathcal{E}(\mathbb{R}^d)$ is closed under union, intersection, difference, symmetric difference, and translation. □

Note that this does not form an algebra. Now it is time to give the elementary set a measure.

**Lemma 1.1.1.** *Let $E \in \mathbb{R}^d$ be an elementary set.*
*(1). $E$ can be expressed (partitioned) as the finite union of disjoint boxes, that is, $E = \bigcup_{i=1}^{n} B_i$ for $B_i$ pairwise disjoint.*
*(2). For any two such partitions $E = \bigcup_{i=1}^{n} B_i = \bigcup_{i=1}^{m} B_i'$, we have $\sum_{i=1}^{n} |B_i| = \sum_{i=1}^{m} |B_i'|$. We denote this value by $m(E) = m^d(E)$, the elementary measure of $E$. It is independent of partitions.*

*Proof.* Begin with (i).
For the case $d = 1$, $E = \bigcup_{i=1}^n I_i$. Place the $2n$ endpoints of these intervals in increasing order (discarding repititions) and relable them to be $c_1 \le c_2 \le ... \le c_{2n}$.
Let $J_1, ...J_{4n-1}$ be disjoint intervals formed by these endpoints where

$$J_i = \{c_i\} \ for\ 1 \le i \le 2n$$

$$J_i = (c_{i-2n}, c_{i-2n+1}) \ for\ 2n+1 \le i \le 4n-1$$

Basically we have $2n$ endpoints and $2n-1$ open intervals between endpoints by doing this. Then, each $I_i$ can be expressed as some subcollection of $J_1, ..., J_{4n-1}$. Then, $E = \bigcup_{k: J_k \cap E \neq \emptyset} J_k$.
For general cases $d \ge 2$ Now we have $E = \bigcup_{i=1}^n B_i$ where $B_i = I_i^1 \times ... \times I_i^d$. For each $m$ such that $1 \le m \le d$, we apply the $d = 1$ case to get a family of disjoint intervals $\{J_k^m\}_{k=1}^{n_m}$ such that $\bigcup_{i=1}^n I_i^m = \bigcup_{i=1}^{n_m} J_k^m$.
Then, we can get $n_1 \times ... \times n_d$ pairwise disjoint boxes and each of them is represented as $\tilde{B}_{k_1...k_d} = J_{k_1}^1 \times ... \times J_{k_d}^d$ for $(k_1, ..., k_d) \in \{1, ..., n_1\} \times ... \times \{1, ..., n_d\}$.
Then, each $B_i$ is a union of a subcollection of $\{\tilde{B}_{k_1...k_d}\}$, and thus $E$ is expressed as a finite union of pairwise disjoint boxes.

Now let's prove (ii).
First we notice that for any interval $I$, we have that

$$|I| = \lim_{N \to +\infty} \frac{1}{N} \#(I \cap \frac{1}{N}\mathbb{Z})$$

where $\frac{1}{N}\mathbb{Z} = \{\frac{k}{N} : k \in \mathbb{Z}\}$.
I don't know how to prove this. One way to look at it is to take a sample of rational points in $(a, b)$ with the distance between each adjacent pair to be $\frac{1}{N}$. Another way is through an example where $a = 2$, $b = 4$. Then, $(a, b) \cap \frac{1}{N}\mathbb{Z} = \{z : z \in (aN, bN), z \in \mathbb{Z})\}$. When $n = 1$, $\# = 1$; $n = 2$, $\# = 3$, ... Then the cardinality is equal to $(b-a)N - 1 = 2N - 1$ with each $N$. Then $\lim_{N \to +\infty} \frac{1}{N}((b-a)N - 1) = b - a$.
By taking Cartesian product,

$$|B| = \lim_{N \to +\infty} \frac{1}{N^d} \#(B \cap \frac{1}{N}\mathbb{Z}^d)$$

For $E = \cup_{i=1}^n B_i$ with pairwise disjoint $B_i$,

$$\begin{aligned} \frac{1}{N^d} \#(E \cap \frac{1}{N}\mathbb{Z}^d) &= \sum_{i=1}^n \frac{1}{N^d} \#(B_i \cap \frac{1}{N}\mathbb{Z}^d) \\ &\to \sum_{i=1}^n |B_i| \ as\ N \to +\infty \end{aligned}$$

The LHS is independent of partitions, thus $m(E) = \lim_{N \to +\infty} \frac{1}{N^d} \#(E \cap \frac{1}{N}\mathbb{Z}^d)$. □

Now we have the property of the elementary measure:

(i). $m(E) \geq 0$

(ii). Finite additivity: $m(E \cup F) = m(E) + m(F)$ for disjoint $E, F \in \mathcal{E}(\mathbb{R}^d)$. By induction, $m(E_1 \cup ... \cup E_n) = \sum_{i=1}^n m(E_i)$ for disjoint $\{E_i\}$.

(iii). $m(\emptyset) = 0$.

(iv). $m(B) = |B|$ $\forall$ box $B$.

(v). Monotonicity: $E \subset F$, then $m(E) \leq m(F)$.

(vi). Sub-additivity: $m(E \cup F) \leq m(E) + m(F)$ for any $E, F \in \mathcal{E}(\mathbb{R}^d)$. By induction: $m(E_1 \cup ... \cup E_n) \leq \sum_{i=1}^n m(E_i)$.

(vii). $E \in \mathcal{E}(\mathbb{R}^d)$, $x \in \mathbb{R}$, then $m(E + x) = m(E)$.

*Proof.* (v). $F = E \cup (F \setminus E)$, so $m(F) \overset{\text{(ii)}}{=} m(E) + m(F \setminus E) \overset{\text{(i)}}{\geq} m(E)$.
(vi). Finite sub-additivity: $E \cup F = E \cup (F \setminus E)$, therefore $m(E \cup F) \overset{\text{(ii)}}{=} m(E) + m(F \setminus E) \overset{\text{(v)}}{\leq} m(E) + m(F)$.
(vii). $E$ can be partitioned into finite disjoint boxes $E = \bigcup_{i=1}^n B_i$, where each $B_i = I_i^1 \times ... \times I_i^d$. Then, $E' = E + x = \bigcup_{i=1}^n B_i + x = \bigcup_{i=1}^n I_i^1 \times ... \times I_i^d + x = \bigcup_{i=1}^n (I_i^1 + x) \times ... \times (I_i^1 + x) = \bigcup_{i=1}^n B_i'$. $|B_i| = |B_i'|$. Then, because of disjoint, $m(E) = \sum_{i=1}^n |B_i| = \sum_{i=1}^n |B_i'| = m(E')$. $\square$

## 1.2 Jordan Measure

More advanced sets such as triangle, disk, or rotated boxes can be measured by approaching from without and within by elementary sets.

**Definition 1.2.1.** *(Jordan Measure) Let $E \in \mathbb{R}^d$ be a bounded set.*
*Its Jordan inner measure is $m_J(E) = \sup_{A \subset E,\ A\ elementary} m(A)$.*
*Its Jordan outer measure is $m^J(E) = \inf_{E \subset B,\ B\ elementary} m(B)$.*
*If $m_J(E) = m^J(E)$, then $E$ is Jordan measurable.*
*Let $\mathcal{J}(\mathbb{R}^d)$ be class of Jordan measurable sets. For $E \in \mathcal{J}(\mathbb{R}^d)$, define Jordan measure as $m(E) := m_J(E) = m^J(E)$.*

Note that (i). unbounded sets are not Jordan measurable. (ii). The Jordan measure of $E \in \mathcal{E}(\mathbb{R}^d)$ is equal to the elementary measure of $E$.
Here gives the characterization of Jordan Measure.

**Proposition 1.2.1.** *Let $E \in \mathbb{R}^d$ be bounded. Then the following are equivalent: (TFAE):*
*(1). $E \in \mathcal{J}(\mathbb{R}^d)$.*
*(2). $\forall \epsilon > 0$, $\exists A, B \in \mathcal{E}(\mathbb{R}^d)$ with $A \subset E \subset B$, such that $m(B \setminus A) < \epsilon$.*
*(3). $\forall \epsilon > 0$, $\exists A \in \mathcal{E}(\mathbb{R}^d)$ such that $m^J(A \Delta E) < \epsilon$.*

*Proof.* $\square$

**Proposition 1.2.2.** *Let $E$, $F \in \mathcal{J}(\mathbb{R}^d)$, then,*
*(1). $E \cup F$, $E \cap F$, $E \setminus F$, $E \Delta F \in \mathcal{J}(\mathbb{R}^d)$.*
*(2). $m(E) \geq 0$.*
*(3). Finite Additivity: If $E \cap F = \emptyset$, then $m(E \cup F) = m(E) + m(F)$.*
*(4). Monotonicity: If $E \in F$, then $m(E) \leq m(F)$.*

*(5). Finite subadditivity:* $m(E \cup F) \leq m(E) + m(F)$.
*(6). Translation Invariabce:* $m(E + x) = m(E)$.

*Proof.* $E \cap F \in \mathcal{J}(\mathbb{R}^d)$:
Using (3) from Proposition 2, let $A, B, C, D \in \mathcal{E}(\mathbb{R}^d)$ with $A \in E \in B$, $C \in E \in D$, such that $m(B \setminus A) < \epsilon$, $m(D \setminus C) < \epsilon$. Then we claim that $m((B \cap D)\ (A \cap C)) \leq 2\epsilon$.

$$\begin{aligned}(B \cap D) \setminus (A \cap C) &= (B \cap D) \cap (A \cap C)^C \\ &= B \cap D \cap (A^C \cup C^C) \\ &= (B \cap D \cap A^C) \cup (B \cap D \cap C^C) \\ &\subset (B \cap A^C) \cup (D \cap C^C) \\ &= (B \setminus A) \cup (D \setminus C)\end{aligned}$$

Then, $m((B \cap D) \setminus (A \cap C)) \leq m((B \setminus A) \cup (D \setminus C)) \leq 2\epsilon$.
Since $A \cap C \subset E \cap F \subset B \cap D$, we have $E \cap F$ is also Jordan measurable. □

There are sets that are not Jordan measurable. For example, $E = [0, 1] \cap \mathcal{Q}$ is not Jordan measurable. This is because $E$ contains no open interval, any elementary set $A \in E$ can only be a finite union of singletons, therefore, $m_J(E) = 0$. However, we can prove that $m_J(E) = 1$.
Similarly, there exists open and bounded sets that are not Jordan measurable, and there exists compact sets that are not Jordan measurable. So, we need Lebegue measure.

## 1.3 Lebesgue Measure

**Definition 1.3.1.** *The lebesgue outer measure (exterior measure) of* $E \in \mathbb{R}^d$ *is*

$$m^*(E) := \inf\{\sum_{i=1}^{\infty} |B_i| : B_1, B_2, ...boxes, E \subset \bigcup_{i=1}^{\infty} B_i\}$$

Note that $m^*(E) \leq m^J(E)$ since it is the infimum over a bigger set.
For $E = [0, 1] \cap \mathcal{Q} = \{q_1, q_2, ...\}$, by taking boxes $B_i = \{q_i\}$, we get the Lebesgue measure $m^*(E) \leq \sum_{i=1}^{\infty} |B_i| = \sum_{i=1}^{\infty} 0 = 0$.

Similarly, $m^*(E) = 0$ for any countable $E$, just take $\{x_i\}$ as boxes. Or, take boxes $B_i = (q_i - \frac{\epsilon}{2^i}, q_i + \frac{\epsilon}{2^i})$, then $m^*(E) \leq m(\bigcup_{i=1}^{\infty} B_i) \leq \sum_{i=1}^{\infty} |B_i| = 2\epsilon$. Since $\epsilon$ is arbitrary, we have $m^*(E) = 0$.

**Definition 1.3.2.** *Lebesgue Measurability: A set* $E \in \mathbb{R}^d$ *is Lebesgue measurable if* $\forall \epsilon > 0$, $\exists$ *an open set* $U \in \mathbb{R}^d$, $E \subseteq U$, *such that* $m^*(U \setminus E) < \epsilon$.

We denote the class of all Lebesgue measurable sets by $\mathcal{L}(\mathbb{R}^d)$. for $E \in \mathcal{L}(\mathbb{R}^d)$, its Lebesgue measure is $m(E) := m^*(E)$. Now we gives the properties of the Lebesgue outer measure (the outer measure axioms).

### 1.3.1 Properties of Lebesgue Outer Measure

**Proposition 1.3.1.** *(The outer measure axioms)*

1. $m^*(\emptyset) = 0$.

2. *Monotonicity: if $E \subset F \subseteq \mathbb{R}^d$, then $m^*(E) \leq m^*(F)$.*

3. *$\sigma$-subadditivity: If $E_1, E_2, ... \subset \mathbb{R}^d$ is a countable sequence of sets, then, $m^*(\bigcup_{n=1}^{\infty} E_i) \leq \sum_{i=1}^{\infty} m^*(E_i)$.*

*Proof.* (3).
We know that,

$$m^*(E_i) = \inf\{\sum_{j=1}^{\infty} |B_j^i| : B_1^i, B_2^i, ...boxes, E_i \subset \bigcup_{j=1}^{\infty} B_j^i\}$$

Then, for each $i$, there exists $B_1^i, B_2^i, ...$ to be boxes, such that $E_i \subset \bigcup_{j=1}^{\infty} B_j^i$ and

$$\sum_{j=1}^{\infty} |B_j^i| \leq m^*(E_i) + \frac{\epsilon}{2^i}$$

Since $E_i \subset \bigcup_{j=1}^{\infty} B_j^i$, we have $\bigcup_{i=1}^{\infty} E_i \subset \bigcup_{i=1}^{\infty} \bigcup_{j=1}^{\infty} B_j^i$, then we have

$$\begin{aligned}
m^*(\bigcup_{i=1}^{\infty} E_i) &\leq m^*(\bigcup_{i=1}^{\infty} \bigcup_{j=1}^{\infty} B_j^i) \text{ by monotonicity} \\
&\leq \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |B_j^i| \text{ by definition of Lebesgue measure: infimum} \\
&= \sum_{i=1}^{\infty} (\sum_{j=1}^{\infty} |B_j^i|) \text{ by Tonelli's Theorem for series} \\
&= \sum_{i=1}^{\infty} (m^*(E_i) + \frac{\epsilon}{2^i}) \\
&= \sum_{i=1}^{\infty} m^*(E_i) + \epsilon
\end{aligned}$$

Since $\epsilon$ is arbitrary, we then have $m^*(\bigcup_{n=1}^{\infty} E_i) \leq \sum_{i=1}^{\infty} m^*(E_i)$. $\square$

Also, (3) and (1) together give the finite subadditivity property by just let $\emptyset = E_{k+1} = E_{k+2} = ...$, that is, $m^*(\bigcup_{i=1}^{k} E_i) \leq \sum_{i=1}^{k} m^*(E_i)$.

**Lemma 1.3.1.** *Finite additivity for separated sets: Let $E, F \subset \mathbb{R}^d$ be such that $dist(E, F) := \inf\{|x - y| : x \in E, y \in F\} > 0$, then $m^*(E \cup F) = m^*(E) + m^*(F)$.*

*Proof.* First, we prove $\leq$. This is natual from $\sigma$-additivity: $m^*(E \cup F) \leq m^*(E) + m^*(F)$.
Next, we prove $\geq$. Without loss of generality, assume that $m^*(E \cup F) < +\infty$.
Let $\epsilon > 0$ be arbitrary, then, by definition there exists a countable collection of boxes $B_1, B_2, ...$ such that,

$$E \cup F \subset \bigcup_{i=1}^{\infty} B_i$$

$$\sum_{i=1}^{\infty} |B_i| \leq m^*(E) + \epsilon$$

Fix $\delta \in (0, dist(E, F))$. By subdividing these boxes into finer boxes $B_i'$, we may assume that $diam(B_i') < \delta$. Then, some of these boxes have intersection with $E$ while others have

intersection with $F$.
Let $I = \{i : B'_i \cap E \neq \emptyset\}$, $J = \{j : B'_j \cap F \neq \emptyset\}$. Then $B'_i \cap B'_j = \emptyset$ cause otherwise we would have a box with diameter bigger than $\delta$.
Then, $m^*(E) \leq \sum_{i \in I} |B'_i|$, $m^*(F) \leq \sum_{j \in J} |B'_j|$.

$$\begin{aligned} m^*(E) + m^*(F) &\leq \sum_{i \in I \cup J} |B'_i| \\ &\leq \sum_{i=1}^{\infty} |B_i| \\ &\leq m^*(E \cup F) + \epsilon \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, we have $m^*(E) + m^*(F) \leq m^*(E \cup F)$.
Now that we have $\leq$ and $\geq$, we have $=$. $\square$

**Lemma 1.3.2.** *Outer measurability for elementary sets: Let $E \in \mathcal{E}(\mathbb{R}^d)$, then $m^*(E) = m(E)$, the elementary measure.*

*Proof.* First we prove $\leq$. We already know that $m^*(E) \leq m^J(E) = m(E)$, thus $\leq$ holds.
Next we prove $\geq$.
Consider first the case where $E$ is closed, then, $E$ is compact. Then we can use the Heine-Borel Theorem which states that any covering of a compact set by a collection of open sets contains a finite subcovering.
Take a covering of $E$ by boxes: $E \subset \bigcup_{i=1}^{\infty} B_i$ such that $\sum_{i=1}^{\infty} |B_i| \leq m^*(E) + \epsilon$. For each box $B_i$, find an open box $B'_i$ such that $B_i \subset B'_i$ and $|B'_i| \leq |B_i| + \frac{\epsilon}{2^i}$.
Then, $\sum_{i=1}^{\infty} |B'_i| \leq \sum_{i=1}^{\infty} |B_i| + \epsilon \leq m^*(E) + 2\epsilon$.
Using Heine-Borel Theory, there is a finite $N$ such that $E \subset \bigcup_{i=1}^{N} B'_i$. Then,

$$\begin{aligned} m(E) &\leq \sum_{i=1}^{N} |B_i| \\ &\leq \sum_{i=1}^{\infty} |B_i| \\ &\leq m^*(E) + 2\epsilon \end{aligned}$$

Since $\epsilon > 0$ arbitrary, we have $\geq$.
Now consider the case where $E$ is not closed. Then, write $E$ as a finite union of disjoint boxes $E = \bigcup_{i=1}^{k} Q_i$, which need not be closed.
Let $\epsilon > 0$ be arbitrary, and for each $j \in \{1, ..., k\}$, find a closed sub-box $Q'_j \subset Q_j$ such that $|Q'_j| \geq |Q_j| - \frac{\epsilon}{k}$. Then, by the previous discussion and finite additivity of elementary

measure, we have

$$\begin{aligned}m^*(\bigcup_{j=1}^k Q'_j) &= m(\bigcup_{j=1}^k Q'_j)\\ &= \sum_{j=1}^k m(Q'_j)\\ &\geq \sum_{j=1}^k m(Q_j) - \epsilon\\ &= m(E) - \epsilon\end{aligned}$$

Also, $\bigcup_{j=1}^k Q'_j \subset E$, so by monotonicity, we have

$$\begin{aligned}m^*(E) &\geq m^*(\bigcup_{j=1}^k Q'_j)\\ &\geq m(E) - \epsilon\end{aligned}$$

Then $m(E) \leq m^*(E) + \epsilon$. Since $\epsilon > 0$ arbitrary, $\geq$ holds. $\square$

**Lemma 1.3.3.** *(Outer measure of countable unions of almost disjoint boxes): Let* $E = \bigcup_{i=1}^\infty B_i$ *be a countable union of almost disjoint boxes, then* $m^*(E) = \sum_{i=1}^\infty |B_i|$. *Almost disjoint means that* $B_i^o \cap B_j^o = \emptyset\ \forall i \neq j$ *(topological interior doesn't intersect).*

*Proof.* From countable sub-additivity and **Lemma 1.3.2**,

$$m^*(E) \leq \sum_{i=1}^\infty m^*(B_i) = \sum_{i=1}^\infty |B_i|$$

Therefore, it suffices to show that

$$m^*(E) \geq \sum_{i=1}^\infty |B_i|$$

Notice that for each $N \in \mathbb{N}$,

$$E \supset \bigcup_{i=1}^N B_i$$

Then,

$$\begin{aligned}m^*(E) &\geq m^*(\bigcup_{i=1}^N B_i)\\ &= m(\bigcup_{i=1}^N B_i)\\ &= \sum_{i=1}^N |B_i|\end{aligned}$$

Let $N \to \infty$, we have $m^*(E) \geq \sum_{i=1}^\infty |B_i|$ Therefore we conclude the proof. $\square$

From this lemma we have a corollary.

**Corollary 1.3.1.** *If $E = \bigcup_{i=1}^{\infty} B_i = \bigcup_{i=1}^{\infty} B_i'$, $(B_i)_{i\in\mathbb{N}}$ and $(B_i')_{i\in\mathbb{N}}$ are almost disjoint boxes, then $\sum_{i=1}^{\infty} |B_i| = \sum_{i=1}^{\infty} |B_i'|$.*

**Lemma 1.3.4.** *An open set $U \subseteq \mathbb{R}^d$ is the countable union of almost disjoint boxes. (in fact, the countable union of almost disjoint closed boxes).*

*Proof.* For $n \in \mathbb{Z}$, let $\mathcal{Q}_n$ be the collection of all closed cubes of the form

$$[\frac{k_1}{2^n}, \frac{k_1+1}{2^n}] \times ... \times [\frac{k_d}{2^n}, \frac{k_d+1}{2^n}] \text{ for some integers } k_1, ..., k_d$$

Define $\mathcal{Q}_{\geq 0} := \bigcup_{n=1}^{\infty} \mathcal{Q}_n$ to be the union of all dyadic cubes of side length $\leq 1$. Notice that $\mathcal{Q}_{\geq 0}$ has a tree structure, that is, for each $Q \in \mathcal{Q}_n$, $\exists! Q' \in \mathcal{Q}_{n-1}$ such that $Q \subset Q'$.
Given these, we have the dyadic nesting property: $\forall Q_1, Q_2 \in \mathcal{Q}_{\geq 0}$ with $Q_1^o \cap Q_2^o \neq \emptyset$, either $Q_1 \subseteq Q_2$ or $Q_2 \subseteq Q_1$.
Since $U$ is open, $\forall x \in U$, $\exists$ open ball $B(x,r) \subset U$. Therefore, $\exists$ closed $Q \in \mathcal{Q}_{\geq 0}$ such that $x \in Q \subseteq E$. Then, let $Q_U = \{Q \subset \mathcal{Q}_{\geq 0} : Q \subseteq U\}$. Then,

$$U = \bigcup_{Q \in \mathcal{Q}_U} Q \text{ with } \mathcal{Q}_U \text{ being countable}$$

To get almost disjoint subcollection, take $\mathcal{Q}_U^* \subseteq \mathcal{Q}_U$ to be a subcollection of maximal elements with respect to set inclusion, which means that they are not contained in any other cube in $\mathcal{Q}_U$.

$$\mathcal{Q}_U^* := \{Q \in \mathcal{Q}_{\geq 0} : Q \subseteq U, Q' \nsubseteq U \text{ for any } Q' \in \mathcal{Q}_{\geq 0} \text{ and } Q' \supset Q\}$$

First we see that if $Q \subseteq U$ then $Q \subseteq \mathcal{Q}_U$, then $\mathcal{Q}_U \subseteq \mathcal{Q}_U^*$. Together with the definition of $\mathcal{Q}_U^*$, we see that $\mathcal{Q}_U^* = \mathcal{Q}_U$. Second, by dyadic nesting property, every cube in $\mathcal{Q}$ is contained in exactly one maximal cube in $\mathcal{Q}^*$, and that any two such maximal cubes in $\mathcal{Q}^*$ are almost disjoint. Thus, $U = \bigcup_{Q \in \mathcal{Q}^8} Q$ are almost disjoint, and also countable. $\square$

**Lemma 1.3.5.** *(Outer regularity): For any $E \subseteq \mathbb{R}^d$,*

$$m^*(E) = \inf_{E \subset U, U \text{ } open} m^*(U)$$

*Proof.* ($\leq$): it is easy to see from monotonicity that $\forall U \supset E$, $m^*(E) \leq m*(U)$, thus

$$m^*(E) \leq \inf_{E \subset U, U \text{ } open} m^*(U)$$

Therefore it suffices to prove that

$$m^*(E) \geq \inf_{E \subset U, U \text{ } open} m^*(U)$$

By definition of the outer Lebesgue measure,

$$m^*(E) = \inf\{\sum_{i=1}^{\infty} |B_i| : E \subset \bigcup_{i=1}^{\infty} B_i, B_1, B_2, ...boxes\}$$

Then, $\forall \epsilon > 0$, $\exists B'_1, B'_2...$ such that

$$\sum_{i=1}^{\infty} |B'_i| \leq m^*(E) + \epsilon$$

Enlarge each box $B'_i$ to be an open box $B'_i \subset B''_i$ such that

$$|B''_i| \leq |B'_i| + \frac{\epsilon}{2^i}$$

Thus, $E \subset \bigcup_{i=1}^{\infty} B''_i$ where $\bigcup_{i=1}^{\infty} B''_i$ is open.

$$\begin{aligned}\sum_{i=1}^{\infty} |B''_i| &\leq \sum_{i=1}^{\infty} |B'_i| + \epsilon \\ &= m^*(E) + 2\epsilon\end{aligned}$$

Since $\bigcup_{i=1}^{\infty} B''_i$ is open, by countable sub-additivity and the definition of infimum,

$$\inf_{E \subset U, U \ open} m^*(U) \leq m^*(\bigcup_{i=1}^{\infty} B''_i) \leq \sum_{i=1}^{\infty} |B''_i| \leq m^*(E) + 2\epsilon$$

Since $\epsilon > 0$ is arbitrary, we have

$$m^*(E) \geq \inf_{E \subset U, U \ open} m^*(U)$$

□

### 1.3.2 Lebesgue Measurability

There are plenty of Lebesgue measurable sets, as we can see from the following proposition.

**Proposition 1.3.2.** *(Existence of Lebesgue measurable sets). Let $E \subseteq \mathbb{R}^d$, then $E \subset \mathcal{L}(\mathbb{R}^d)$ if*

1. *$E$ is open.*
2. *$E$ is closed.*
3. *$E$ is a null set, i.e. $m^*(E) = 0$.*
4. *$E = \emptyset$.*
5. *if $E \in \mathcal{L}(\mathbb{R}^d)$, then $\mathbb{R}^d \setminus E \in \mathcal{L}(\mathbb{R}^d)$.*
6. *$E = \bigcup_{i=1}^{\infty} E_i \in \mathcal{L}(\mathbb{R}^d)$ where $E_i \in \mathcal{L}(\mathbb{R}^d)$.*
7. *$E = \bigcap_{i=1}^{\infty} E_i \in \mathcal{L}(\mathbb{R}^d)$ where $E_i \in \mathcal{L}(\mathbb{R}^d)$.*

*Proof.* (1) is immediate from definition. By **Lemma 1.3.4**, write $E$ as $E = \bigcup_{i=1}^{\infty} B_i$ where $B_i$ are disjoint boxes. Expand each $B_i$ to be an open box $B'_i \supset B_i$ such that $\forall \epsilon > 0$,

$$|B'_i| \leq |B_i| + \frac{\epsilon}{2^i}$$

Then, by $\sigma$-additivity and Lemma 1.3.3,

$$m^*(\bigcup_{i=1}^{\infty} B_i') \leq \sum_{i=1}^{\infty} |B_i'| \leq m(E) + \epsilon$$

Therefore

$$m^*(\bigcup_{i=1}^{\infty} B_i' \setminus E) \leq \epsilon$$

Thus, we found an open set $\bigcup_{i=1}^{\infty} B_i' \supset E$, such that $m^*(\bigcup_{i=1}^{\infty} B_i' \setminus E) \leq \epsilon$.
(3) and (4) are immediate.
Since $E_i \in \mathcal{L}(\mathbb{R}^d)$, $\exists$ open set $E_i'$ such that $E_i \subset E_i'$, $m^*(E_i') \leq m^*(E_i) + \frac{\epsilon}{2^i}$. Then, $\bigcup_{i=1}^{\infty} E_i \subset \bigcup_{i=1}^{\infty} E_i'$ where $\bigcup_{i=1}^{\infty} E_i'$ is open. Since

$$\begin{aligned}
\bigcup_{i=1}^{\infty} E_i' \setminus \bigcup_{i=1}^{\infty} E_i &= \bigcup_{i=1}^{\infty} E_i' \cap (\bigcap_{i=1}^{\infty} E_i^C) \\
&= \bigcup_{i=1}^{\infty} (E_i' \cap (\bigcap_{i=1}^{\infty} E_i^C)) \\
&= \bigcup_{i=1}^{\infty} (E_i' \cap (\bigcap_{i=1}^{\infty} E_i^C)) \\
&\subset \bigcup_{i=1}^{\infty} (E_i' \cap E_i^C) \\
&= \bigcup_{i=1}^{\infty} (E_i' \setminus E_i)
\end{aligned}$$

Then, by monotonicity and $\sigma$-additivity,

$$\begin{aligned}
m^*(\bigcup_{i=1}^{\infty} E_i' \setminus \bigcup_{i=1}^{\infty} E_i) &\leq m^*(\bigcup_{i=1}^{\infty} (E_i' \setminus E_i)) \\
&\leq \sum_{i=1}^{\infty} m^*(E_i' \setminus E_i) \\
&= \sum_{i=1}^{\infty} m^*(E_i') - m^*(E_i) \\
&\leq \epsilon \qquad (1)
\end{aligned}$$

Therefore, $\bigcup_{i=1}^{\infty} E_i \in \mathcal{L}(\mathbb{R}^d)$.
(2). First, we can express each closed set as $E = \bigcup_{i=1}^{\infty}$ for $E_n$ closed and bounded (for example, $E_n = \overline{B(0,n)} \cap E$ for $n = 1, 2, ...$). Then by (6), it suffices to verify the claim when $E$ is closed and bounded, hense compact.
By Lemma 1.3.5, $\exists U \supset E$ open, such that $m^*(U) \leq m^*(E) + \epsilon$. Therefore, it suffices to show that $m^*(U \setminus E) \leq \epsilon$.
If w have finite additivity for $m^*$, then we have $m^*(U \setminus E) + m^*(E) = m^*(U) \leq m^*(E) + \epsilon$ and then $m^*(U \setminus E) \leq \epsilon$. But we don't have it, so we should instead fo the following.
Since $U \setminus E$ is also open, by Lemma 1.3.4, $U \setminus E = \bigcup_{i=1}^{\infty} Q_i$ where $Q_i$ are almost disjoint closed boxes. Then by Lemma 1.3.3, $m^*(U \setminus E) = \sum_{i=1}^{\infty} |Q_i|$.

We truncate the sum: for any finite $N \in \mathbb{N}$, $\bigcup_{i=1}^N Q_i$ is closed and disjoint from $E$. From **Exercise 1.2.4** (Let $E, F \subset \mathbb{R}^d$ be disjoint closed sets, with at least one of $E, F$ being compact. Then $dist(E, F) > 0$), since $E$ is compact and $\bigcup_{i=1}^N Q_i$ is closed, we have $dist(E, \bigcup_{i=1}^N Q_i) > 0$. Then by Lemma 1.3.1,

$$\begin{aligned} m^*(\bigcup_{i=1}^N Q_i) + m^*(E) &= m^*(E \cup \bigcup_{i=1}^N Q_i) \\ &\leq m^*(U) \\ &\leq m^*(E) + \epsilon \end{aligned}$$

$$\sum_{i=1}^N |Q_i| = m^*(\bigcup_{i=1}^N Q_i) \leq \epsilon$$

Let $N \to \infty$,

$$m^*(U \setminus E) = \sum_{i=1}^\infty |Q_i| = m^*(\bigcup_{i=1}^\infty Q_i) \leq \epsilon$$

Therefore $E \in \mathcal{L}(\mathbb{R}^d)$.
(5). Since $E \in \mathcal{L}(\mathbb{R}^d)$, for every $n \in \mathbb{N}$, $\exists U_n \supset E$ such that $m^*(U_n \setminus E) < \frac{1}{n}$.
Let $F_n := U_n^C$, then $(\mathbb{R}^d \setminus E) \supset F_n$ for all $n$. Since

$$(\mathbb{R}^d \setminus E) \setminus F_n = (\mathbb{R}^d \setminus E) \cap F_n^C = (\mathbb{R}^d \setminus E) \cap U_n = U_n \setminus E$$

we have

$$m^*((\mathbb{R}^d \setminus E) \setminus F_n) < \frac{1}{n}$$

Let $F := \bigcup_{i=1}^\infty F_n$, then $(\mathbb{R}^d \setminus E) \supset F$. From monotonicity, we have

$$m^*((\mathbb{R}^d \setminus E) \setminus F) \leq m^*((\mathbb{R}^d \setminus E) \setminus F_n) < \frac{1}{n} \; \forall n \in \mathbb{N}$$

Taking $n \to \infty$, we have $m^*((\mathbb{R}^d \setminus E) \setminus F) = 0$, thus $(\mathbb{R}^d \setminus E) \setminus F$ is a null set, and is Lebesgue measurable. Therefore, $\mathbb{R}^d \setminus E$ is the union of this null set and $F$. Since by definition $F = \bigcup_{i=1}^\infty U_n^C$ where $U_n^C$ is closed, $F$ is Lebesgue measurable. Therefore, by (6), $\mathbb{R}^d \setminus E \in \mathcal{L}(\mathbb{R}^d)$.
(7). Since $E_i \in \mathcal{L}(\mathbb{R}^d)$, we have $E_i^C \in \mathcal{L}(\mathbb{R}^d)$ and $\bigcup_{i=1}^\infty E_i^C \in \mathcal{L}(\mathbb{R}^d)$. Therefore $(\bigcap_{i=1}^\infty E_i)^C \in \mathcal{L}(\mathbb{R}^d)$ and $\bigcap_{i=1}^\infty E_i \in \mathcal{L}(\mathbb{R}^d)$. □

For $E \in \mathcal{L}(\mathbb{R}^d)$, its Lebesgue measure is defined to be $m(E) := m^*(E)$, and it has the following properties, which is significantly better than Lebesgue outer measure.

**Proposition 1.3.3.** *(The measure axioms)*

1. *$m(\emptyset) = 0$.*

2. *($\sigma$-additivity) For a countable sequence of disjoint sets $E_1, E_2 ... \in \mathcal{L}(\mathbb{R}^d)$,*

$$m(\bigcup_{i=1}^\infty E_i) = \sum_{i=1}^\infty m(E_n)$$

*Proof.* (1). is trivial.

(2). *Case 1.* $E_n$ is compact.

Then, by **Exercise 1.2.4**, $dist(E_i, E_j) > 0$, and

$$m(\bigcup_{i=1}^{N} E_i) = \sum_{i=1}^{N} m(E_i)$$

By monotonicity,

$$m(\bigcup_{i=1}^{\infty} E_i) \geq m(\bigcup_{i=1}^{N} E_i) = \sum_{i=1}^{N} m(E_i)$$

Let $N \to \infty$,

$$m(\bigcup_{i=1}^{\infty} E_i) \geq \sum_{i=1}^{\infty} m(E_i)$$

Also from $\sigma$-subadditivity,

$$m(\bigcup_{i=1}^{\infty} E_i) \leq \sum_{i=1}^{\infty} m(E_i)$$

Therefore we have

$$m(\bigcup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} m(E_i)$$

*Case 2.* $E_n$ is not compact but bounded.

For each $E_n$, it can be written as the union of a compact set $U_n$ and a set with outer measure $\frac{\epsilon}{2^n}$. Thus,

$$m(E_n) \leq m(U_n) + \frac{\epsilon}{2^n}$$

$$\sum_{n=1}^{\infty} m(E_n) \leq \sum_{n=1}^{\infty} m(U_n) + \epsilon$$

We just showed that for compact set,

$$\sum_{n=1}^{\infty} m(U_n) = m(\bigcup_{i=1}^{\infty} U_n)$$

and by monotonicity,

$$m(\bigcup_{i=1}^{\infty} U_n) \leq m(\bigcup_{i=1}^{\infty} E_n)$$

Thus,

$$\sum_{n=1}^{\infty} m(E_n) \leq m(\bigcup_{i=1}^{\infty} E_n) + \epsilon$$

Since $\epsilon > 0$ arbitrary, we have

$$\sum_{n=1}^{\infty} m(E_n) \leq m(\bigcup_{i=1}^{\infty} E_n)$$

Also from $\sigma$-subadditivity, we have

$$\sum_{n=1}^{\infty} m(E_n) \geq m(\bigcup_{i=1}^{\infty} E_n)$$

Thus

$$\sum_{n=1}^{\infty} m(E_n) = m(\bigcup_{i=1}^{\infty} E_n)$$

*Case 3.* $E_n$ is not compact and not closed.
Decompose $\mathbb{R}^d$ into annulis, for $m = 1, 2, ...,$

$$A_m := \{x \in \mathcal{R} : m - 1 \leq |x| \leq m\}$$

Then, each $E_n$ can be written as $E_n = \bigcup_{m=1}^{\infty} E_n \cap A_m$ for $E_n \cap A_m$ bounded, measurable, and disjoint.
Then, by previous argument,

$$m(E_n) = \sum_{m=1}^{\infty} m(E_n \cap A_m)$$

Also, for $E_n \cap A_m$ bounded, measurable, and disjoint,

$$\bigcup_{n=1}^{\infty} E_n = \bigcup_{n=1}^{\infty} \bigcup_{m=1}^{\infty} E_n \cap A_m$$

Then

$$m(\bigcup_{n=1}^{\infty} E_n) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} m(E_n \cap A_m) = \sum_{n=1}^{\infty} m(E_n)$$

□

**Theorem 1.3.1.** *(Monotone convergence theorem for measurable sets).*

*(i)* *(Upward monotone convergence) Let $E_1 \subset E_2 \subset \cdots \subset \mathbb{R}^n$ be a countable non-decreasing sequence of Lebesgue measurable sets. Then $m\left(\bigcup_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} m(E_n)$. (Hint: Express $\bigcup_{n=1}^{\infty} E_n$ as the countable union of the lacunae $E_n \setminus \bigcup_{n'=1}^{n-1} E_{n'}$.)*

*(ii)* *(Downward monotone convergence) Let $\mathbb{R}^d \supset E_1 \supset E_2 \supset \ldots$ be a countable non-increasing sequence of Lebesgue measurable sets. If at least one of the $m(E_n)$ is finite, Then $m\left(\bigcap_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} m(E_n)$.*

*(iii)* *Give a counterexample to show that the hypothesis that at least one of the $m(E_n)$ is finite in the downward monotone convergence theorem cannot be dropped.*

*Proof.* (1). Let $E_0 = \emptyset \subset E_1$. By expressing each finite union and countable union as the finite or countable union of the lacunae form,

$$\bigcup_{n=1}^{2} E_n = (E_2 \setminus E_1) \cup E_1$$

$$\bigcup_{n=1}^{3} E_n = (E_3 \setminus \bigcup_{n'=1}^{2} E_{n'}) \cup (E_2 \setminus E_1) \cup E_1$$

$$\bigcup_{n=1}^{4} E_n = (E_4 \setminus \bigcup_{n'=1}^{3} E_{n'}) \cup (E_3 \setminus \bigcup_{n'=1}^{2} E_{n'}) \cup (E_2 \setminus E_1) \cup E_1$$

......

$$\bigcup_{n=1}^{N} E_n = \bigcup_{k=1}^{N} (E_k \setminus \bigcup_{n'=1}^{k-1} E_{n'})$$

$$\bigcup_{n=1}^{\infty} E_n = \bigcup_{k=1}^{\infty} (E_k \setminus \bigcup_{n'=1}^{k-1} E_{n'})$$

Since each $E_k \in \mathcal{L}(\mathbb{R}^d)$, we have $\bigcup_{n'=1}^{k-1} E_{n'} \in \mathcal{L}(\mathbb{R}^d)$ and $E_k \setminus \bigcup_{n'=1}^{k-1} E_{n'} \in \mathcal{L}(\mathbb{R}^d)$, and any countable union of the latter is Lebesgue measurable as well. Also, $E_k \setminus \bigcup_{n'=1}^{k-1} E_{n'}$ and $E_j \setminus \bigcup_{n'=1}^{j-1} E_{n'}$ are disjoint for any $k \neq j$.
Hence, from countable additivity,

$$\begin{aligned}
m(\bigcup_{n=1}^{\infty} E_n) &= m(\bigcup_{k=1}^{\infty} (E_k \setminus \bigcup_{n'=1}^{k-1} E_{n'})) \\
&= \sum_{i=1}^{\infty} m(E_k \setminus \bigcup_{n'=1}^{k-1} E_{n'}) \\
&= \lim_{n\to\infty} \sum_{k=1}^{n} m(E_k \setminus \bigcup_{n'=1}^{k-1} E_{n'}) \\
&= \lim_{n\to\infty} \sum_{k=1}^{n} (m(E_k) - m(\bigcup_{n'=1}^{k-1} E_{n'})) \\
&= \lim_{n\to\infty} \sum_{k=1}^{n} (m(E_k) - m(E_{k-1})) \\
&= \lim_{n\to\infty} (m(E_n) - m(E_0)) \\
&= \lim_{n\to\infty} m(E_n)
\end{aligned}$$

(2).Since $E_1, E_2, ...$ are all Lebesgue measurable, $E_1$ $E_n$, $\bigcap_{n=2}^{\infty} E_n$ are Lebesgue measurable, then $\bigcup_{n=2}^{\infty} E_1 \setminus E_n$ and $E_1 \setminus \bigcap_{n=2}^{\infty} E_n$ are also Lebesgue measurable, and from basic set calculation we know that

$$\bigcup_{n=2}^{\infty} E_1 \setminus E_n = E_1 \setminus \bigcap_{n=2}^{\infty} E_n$$

Then, using conclusion from (1),

$$\begin{aligned}
m(E_1 \setminus \bigcap_{n=2}^{\infty} E_n) &= m(\bigcup_{n=2}^{\infty} E_1 \setminus E_n) \\
&= \lim_{n\to\infty} m(E_1 \setminus E_n) \\
&= \lim_{n\to\infty} m(E_1) - m(E_n) \\
&= m(E_1) - \lim_{n\to\infty} m(E_n)
\end{aligned}$$

Since $\bigcap_{n=2}^{\infty} E_n \subset E_1$, we have

$$m(E_1 \setminus \bigcap_{n=2}^{\infty} E_n) = m(E_1) - m(\bigcap_{n=2}^{\infty} E_n)$$

Now we prove $\bigcap_{n=2}^{\infty} E_n = \bigcap_{n=1}^{\infty} E_n$. $\forall x \in \bigcap_{n=2}^{\infty} E_n$, $x \in E_2$, since $E_2 \in E_1$, $x \in \bigcap_{n=1}^{\infty} E_n$. $\forall y \in \bigcap_{n=1}^{\infty} E_n$, $y \in E_2, E_3, ...$, so $y \in \bigcap_{n=2}^{\infty} E_n$. Then,

$$\lim_{n\to\infty} m(E_n) = m(\bigcap_{n=2}^{\infty} E_n) = m(\bigcap_{n=1}^{\infty} E_n)$$

(3). Consider the sequence $E_n := \mathbb{R}_+/[0,n]$. Clearly none of te $m(E_n)$ is finite. We have $m\left(\bigcap_{n=1}^{\infty} E_n\right) = m(\emptyset) = 0$. On the other hand,

$$\forall n \in \mathbb{N}, m(E_n) = \infty;$$

thus, the sequence of measures does not converge.

$\square$

**Theorem 1.3.2.** *(Dominated Convergence Theorem). We say that a sequence $E_n$ of sets in $\mathbb{R}^d$ converges pointwise to another set $E$ in $\mathbb{R}^d$ if the indicator functions $1_{E_n}$ converge pointwise to $1_E$.*

*(i) If the $E_n$ are all Lebesgue measurable, and converge pointwise to $E$, then $E$ is Lebesgue measurable also.*

*(Hint: use the identity $1_E(x) = \liminf_{n\to\infty} 1_{E_n}(x)$ or $1_E(x) = \limsup_{n\to\infty} 1_{E_n}(x)$ to write $E$ in terms of countable unions and intersections of the $E_n$.)*

*(ii) (Dominated convergence theorem) Suppose that the $E_n$ are all contained in another Lebesgue measurable set $F$ of finite measure. Then $m(E_n)$ converges to $m(E)$.*

*(Hint: use the upward and downward monotone convergence theorems, Theorem 1.3.1.)*

*(iii) Give a counterexample to show that the dominated convergence theorem fails if the $E_n$ are not contained in a set of finite measure, even if we assume that the $m(E_n)$ are all uniformly bounded.*

*Proof.* (i). If $x \in E$, we have $\mathbf{1}_E(x) = 1$,

$$\lim_{n\to\infty} \inf_{k\geq n} \mathbf{1}_{E_k}(x) = 1$$

This means that, $\forall \epsilon > 0$, $\exists N$, when $n \geq N$, $|\inf_{n\geq N} \mathbf{1}_{E_n}(x) - 1| < \epsilon$. Since inf is non-decreasing and is less than 1, we have $\inf_{n\geq N} \mathbf{1}_{E_n}(x) > 1-\epsilon$. Since $\epsilon > 0$ arbitrary, we have $\inf_{n\geq N} \mathbf{1}_{E_n}(x) = 1$. Then, $\mathbf{1}_{E_n}(x) = 1 \ \forall \ n \geq N$, which means that $x \in \bigcap_{n\geq N} E_n$. Since $\forall \epsilon > 0$ we can pick an $N$, $x \in \bigcap_{n\geq N} E_n \in \bigcup_{N\subset\mathbb{N}} \bigcap_{n\geq N} E_n$.
If $x \in \bigcup_{N\in\mathbb{N}} \bigcap_{n\geq N} E_n$, then $\exists N$, such that $x \in \bigcap_{n\geq N} E_n$. This means that $\mathbf{1}_{E_n}(x) = 1 \ \forall \ n \geq N$. Then, $\mathbf{1}_E(x) = \lim_{n\to\infty} \mathbf{1}_{E_n}(x) = 1$ because of pointwise convergence. Then $x \in E$.
Therefore, we have shown the following two sets are equivalent.

$$E = \bigcup_{N\in\mathbb{N}} \bigcap_{n\geq N} E_n$$

Also, if $x \in E$, we have $\mathbf{1}_E(x) = 1$,

$$\lim_{n\to\infty} \sup_{k\geq n} \mathbf{1}_{E_k}(x) = 1$$

This means that $\forall \epsilon > 0$, $\forall N$, $\exists k$, such that when $k \geq N$, $\mathbf{1}_{E_k}(x) \leq 1 - \epsilon$. Since $\epsilon > 0$ is arbitrary, $\mathbf{1}_{E_k}(x) = 1$ and $x \in E_k$. Thus, $\forall \epsilon > 0$, $x \in \bigcap_{N\in\mathbb{N}} E_k$ for some $k \geq N$. Thus, $x \in \bigcap_{N\in\mathbb{N}} \bigcup_{k\geq N} E_k$.
If $x \in \bigcap_{N\in\mathbb{N}} \bigcup_{k\geq N} E_k$, $x \in \bigcup_{k\geq N} E_k$ for all $N \in \mathbb{N}$. This means that $\forall N \in \mathbb{N}$, $\exists k_N \geq N$, such that $x \in E_{k_N}$ and $\mathbf{1}_{E_{k_N}}(x) = 1$. By pointwise convergence and the fact that $\mathbf{1}_{E_{k_N}}(x)$ is a subsequence of the convergent sequence $\mathbf{1}_{E_N}(x)$, we have $\mathbf{1}_E(x) = \lim_{N\to\infty} \mathbf{1}_{E_N}(x) = \lim_{N\to\infty} \mathbf{1}_{E_{k_N}}(x) = 1$. Then, $x \in E$. Therefre, we have shown the following two sets are equibalent,

$$E = \bigcap_{N\in\mathbb{N}} \bigcup_{k\geq N} E_k$$

Then we represented $E$ as either a countable union or a countable intersection of Lebesgue measurable sets, and $E$ is Lebesgue measurable.

(ii). Since

$$\bigcap_{n\geq 1} E_n \subset \bigcap_{n\geq 2} E_n \subset \bigcap_{n\geq 3} E_n \subset ...$$

ans they are all lebesgue measurable, we have

$$\begin{aligned} m(E) &= m(\bigcup_{N\in\mathbb{N}} \bigcap_{n\geq N} E_n) \\ &= \lim_{N\to\infty} m(\bigcap_{n\geq N} E_n)) \\ &\leq \lim_{N\to\infty} m(E_N) \end{aligned}$$

by monotonicity.
Similarly, since

$$\bigcup_{n\geq 1} E_n \supset \bigcup_{n\geq 2} E_n \supset \bigcup_{n\geq 3} E_n \supset ...$$

and $\bigcup_{n\geq k} E_n \subset F \ \forall k \in \mathbf{N}$, $F$ is a set with finite Lebesgue measure, we have $m(\bigcup_{n\geq k} E_n)$

is all finite for all $k$. Then,

$$\begin{aligned} m(E) &= m(\bigcap_{N\in\mathbb{N}} \bigcup_{k\geq N} E_k) \\ &= \lim_{N\to\infty} m(\bigcup_{n\geq N} E_n) \\ &\geq \lim_{N\to\infty} m(E_N) \end{aligned}$$

by monotonicity.
Now we have both $\leq$ and $\geq$, we conclude $m(E) = \lim_{N\to\infty} m(E_N)$.

(iii). Consider the sequence $E_n := \mathbb{R}_+/[0,n]$. Clearly non of the $E_n$ is contained in a set of finite measure. We have $m\left(\bigcap_{n=1}^{\infty} E_n\right) = m(\emptyset) = 0$. On the other hand,

$$\forall n \in \mathbb{N}, m(E_n) = \infty;$$

thus, the sequence of measures does not converge. □

**Theorem 1.3.3.** *(Inner regularity). Let $E \subset \mathbb{R}^d$ be Lebesgue measurable. Then*

$$m(E) = \sup_{K\subset E, K\ compact} m(K).$$

*Proof.* Since $K$ is compact, $K$ is Lebesgue measurable. Therefore, by monotonicity, for all $K \subset E$, $m(E) \geq m(K)$. Therefore,

$$m(E) \geq \sup_{K\subset E, K \text{ compact}} m(K)$$

Thus it suffices to prove that

$$m(E) \leq \sup_{K\subset E, K \text{ compact}} m(K)$$

Write $E = \bigcup_{i=1}^{\infty} B_i$ where $B_i$ are almost disjoint boxes. Then,

$$\begin{aligned} m(E) &= m(\bigcup_{i=1}^{\infty} B_i) \\ &= \sum_{i=1}^{\infty} |B_i|. \end{aligned}$$

Shrink each $B_i$ to $B_i'$ where $B_i' \subset B_i'$ and $\partial B_i' \cap \partial B_i = \emptyset$ and $|B_i| \leq |B_i'| + \frac{\epsilon}{2^i}$ for arbigrary $\epsilon > 0$. Then,

$$\sum_{i=1}^{\infty} |B_i| \leq \sum_{i=1}^{\infty} |B_i'| + \epsilon$$

Also, $B_i' \cap B_j' = \emptyset$, so $m(\bigcup_{i=1}^{\infty} B_i') = \sum_{i=1}^{\infty} |B_i'|$.
Let $\overline{B_i'} = B_i' \cup \partial B_i'$ and it is a closed set. By our previous construction, $\overline{B_i'} \cap \overline{B_j'} = \emptyset$. Also,

from monotonicity and **Exercise 1.1.18**,

$$m(\bigcup_{i=1}^{\infty} B_i') \leq m(\bigcup_{i=1}^{\infty} \overline{B_i'})$$

Since $\bigcup_{i=1}^{\infty} \overline{B_i'}$ is the countable union of closed and bounded sets, it is closed and bounded, thus compact. Therefore, by the definition of supremum and the fact that $\bigcup_{i=1}^{\infty} B_i' \subset \bigcup_{i=1}^{\infty} \overline{B_i'} \subset \bigcup_{i=1}^{\infty} B_i = E$, we have

$$\begin{aligned}
m(E) &= \sum_{i=1}^{\infty} |B_i| \\
&\leq \sum_{i=1}^{\infty} |B_i'| + \epsilon \\
&= m(\bigcup_{i=1}^{\infty} \overline{B_i'}) + \epsilon \\
&\leq \sup_{K \in E, K \text{ compact}} m(K) + \epsilon
\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, we have $m(E) \leq \sup_{K \in E, K \text{ compact}} m(K)$.
Now we have $\leq$ and $\geq$, we have $m(E) = \sup_{K \in E, K \text{ compact}} m(K)$. □

**Theorem 1.3.4.** *(Outer measure is not finitely additive). Then there exist disjoint bounded subsets $E, F$ of the real line such that*

$$m^*(E \cup F) \neq m^*(E) + m^*(F).$$

*(Hint: Show that the set constructed in the proof of the above proposition has positive outer measure.)*

*Proof.* Consider the set that we constructed in the proof of **Proposition 1.2.18**: $E := \{x_C : C \in \mathbb{R}/\mathbb{Q} \text{ and } x_C \in C \cap [0,1]\}$, and $\tilde{E} := \bigcap_{q \in \mathbb{Q} \cap [-1,1]} E + q$. We know by countable subadditivity that

$$\begin{aligned}
m^*(\tilde{E}) &\leq \sum_{q \in \mathbb{Q} \cap [-1,1]} m^*(E + q) \\
&= \sum_{q \in \mathbb{Q} \cap [-1,1]} m^*(E) \\
&= \begin{cases} 1 & \text{if } m^*(E) = 0, \\ +\infty & \text{if } m^*(E) > 0. \end{cases}
\end{aligned}$$

Also $m^*(\tilde{E}) \in [1, 3]$. This contradict with the case $m^*(E) = 0$, so it can only be that $m^*(E) > 0$ and it equals to some positive real number.
Let $n \in \mathbb{N}$ large enought so that $m^*(E) \geq \frac{1}{n}$. If $m^*$ is finitely additive, then for a subset

$F \subset \mathbb{Q} \cap [-1,1]$ with $\#F = 3n$, we have

$$\begin{aligned} m^*(\bigcup_{q\in F} E+q) &= \sum_{q\in F} m^*(E+q) \\ &= \sum_{q\in F} m^*(E) \\ &= 3n \times m^*(E) \\ &> 3 \end{aligned}$$

However, by monotonicity, $m^*(\bigcup_{q\in F} E+q) \le m^*(\bigcup_{q\in\mathbb{Q}\cap[-1,1]} E+q) \le 3$, contradiction! Thus, Lebesgue outer measure is not finitely additive. □

### 1.3.3 Non-Measurable Sets

Of course, there are non-measurable sets in $\mathbb{R}^d$.

**Proposition 1.3.4.** $\exists E \subset [0,1]$, $E \notin \mathcal{L}(\mathbb{R}^d)$.

*Proof.* We use the fact that $(\mathbb{Q}, +)$ is a subgroup of $(\mathbb{R}, +)$, and it partitions $\mathbb{R}$ into disjoint cosets $x+\mathbb{Q}$. This create a quotient group $\mathbb{R}/\mathbb{Q} := \{x+\mathbb{Q} : x \in \mathbb{R}\}$. Each coset$C = x+\mathbb{Q}$ of $\mathbb{R}/\mathbb{Q}$ is dense in $\mathbb{R}$, so it has non-empty intersection with $[0,1]$.
By axiom of choice, select $x_C \in C \cap [0,1]$ from each $C \in \mathbb{R}/\mathbb{Q}$. Let $E := \{x_C : C \in \mathbb{R}/\mathbb{Q}\}$ be the collection of all these coset representatives. By construction, $E \subset [0,1]$.
*Claim 1.* $[0,1] \subseteq \bigcup_{q\in\mathbb{Q}\cap[-1,1]} E+q$. Indeed, for any $y \in [0,1]$, $\exists C \in \mathbb{R}/\mathbb{Q}$ such that $y \in C$. Then, $y - x_C$ is rational. Since $y, x_C \in [0,1]$, we have $|x_C - y| \le 1$. Let $q = y - x_C$, since $x_C \in E$, we have $y \in E+q$.
*Claim 2.* For $q_1 \neq q_2 \in \mathbb{Q}$, $(E+q_1)\cap(E+q_2) = \emptyset$. Indeed, if $z \in (E+q_1)\cap(E+q_2)$, then $z = x_1+q_1 = x_2+q_2$ for $x_1, x_2 \in E$. Then, $x_1 = x_2 + (q_2 - q_1)$ where $q_2 - q_1$ is rational. Then, $x_1$ and $x_2$ are in the same coset $C$, then $x_1 = x_2 = x_C$, then $q_1 \neq q_2$, contradiction.
Suppose $E \in \mathcal{L}(\mathbb{R})$, then $E+q \in \mathcal{L}(\mathbb{R})\ \forall q \in \mathbb{Q}$, and $\tilde{E} := \bigcup_{q\in\mathbb{Q}\cap[-1,1]} E+q$. By monotonicity and *Claim 1*, $1 = m([0,1]) \le m(\tilde{E})$. Also since $\tilde{E} \subset [-1,2]$, we have $m(\tilde{E}) \in [1,3]$. By $\sigma$-additivity and *Claim 2* and transfomation invariante,

$$m(\tilde{E}) = \sum_{q\in\mathbb{Q}\cap[-1,1]} m(E+q) = \sum_{q\in\mathbb{Q}\cap[-1,1]} m(E)$$

If $m(E) = 0$ then $m(\tilde{E}) = 0$. If $m(E) > 0$ then $m(\tilde{E}) = +\infty$. Contradiction! □

## 1.4 Lebesgue Integral

### 1.4.1 Integration of Simple Functions

**Definition 1.4.1.** *(Simple function): A complex valued simple function $f : \mathbb{R}^d \to \mathbb{C}$ is a finite linear combination*

$$f = \sum_{k=1}^{n} c_k \mathbf{1}_{E_k}$$

*for $E_k \in \mathcal{L}(\mathbb{R}^d)$, $c_k \in \mathbb{C}$.*
*An unsigned simple function just tales $c_k \in [0, +\infty)$. For an indicator function,*

$$\int_{\mathbb{R}^d} \mathbf{1}_E(x)dx = m(E)$$

**Definition 1.4.2.** *(Integral of a simple function): for $f = \sum_{k=1}^n c_k \mathbf{1}_{E_k}$,*

$$Simp \int_{\mathbb{R}^d} f := \sum_{k=1}^{\infty} c_k m(E_k)$$

**Lemma 1.4.1.** *Let $k, k' \geq 0$ be natual numbers, $c_1, ..., c_k, c_1', ..., c_{k'}' \in [0, +\infty]$. Let $E_1, ...E_k, E_1', ..., E_{k'} \subset \mathbb{R}^q$ be in $\mathcal{L}(\mathbb{R}^d)$ such that*

$$\sum_{i=1}^{k} c_i \mathbf{1}_{E_i} = \sum_{i=1}^{k'} c_i' \mathbf{1}_{E_i'}) \ (*)$$

*holds identically on $\mathbb{R}^d$. Then,*

$$\sum_{i=1}^{k} c_i m(E_i) = \sum_{i=1}^{k'} c_i' m(E_i')$$

*Proof.* First, $\{E_1, ..., E_k, E_1', ..., E_{k'}'\}$ partitions $\mathbb{R}^d$ into $2^{k+k'}$ disjoint sets using finite Boolean algebra, each of which is an intersection of some of the $E_1, ..., E_k, E_1', ..., E_{k'}'$ and their compliments. Letting go empty sets, we are left with $m$ non-empty disjoint sets $A_1, ..., A_m$ for some $0 \leq m \leq 2^{k+k'}$. $A_i \in \mathcal{L}(\mathbb{R}^d)$ for $i \in \{1, ..., m\}$.
Then, $E_i = \bigcup_{j \in J_i} A_j$, $E_{i'}' = \bigcup_{j \in J_{i'}'} A_{j'}$ for all $i = 1, ..., k$ and $j' = 1, ..., k'$ and some subsets $J_i, J_{i'}'$. By finite additivity, $m(E_i) = \sum_{j \in J_i} m(A_i)$, $m(E_{i'}') = \sum_{j \in J_{i'}'} m(A_j)$. Thus we need to prove that

$$\sum_{i=1}^{k} c_i \sum_{j \in J_i} m(A_i) = \sum_{i'=1}^{k'} c_i' \sum_{j \in J_{i'}'} m(A_i)$$

Fix $1 \leq j \leq m$, evaluate $(*)$ at a point $x$ in the non-empty set $A_j$. Then, at such point,

$$\mathbf{1}_{E_i}(x) = \mathbf{1}_{J_i}(j)$$

$$\mathbf{1}_{E_i'}(x) = \mathbf{1}_{J_{i'}'}(j)$$

By $(*)$,

$$\sum_{i=1}^{k} c_i \mathbf{1}_{J_i}(j) = \sum_{i'=1}^{k'} c_{i'}' \mathbf{1}_{J_{i'}'}(j)$$

Multiply by $m(A_j)$,

$$\sum_{i=1}^{k} c_i \mathbf{1}_{J_i}(j) m(A_j) = \sum_{i'=1}^{k'} c_{i'}' \mathbf{1}_{J_{i'}'}(j) m(A_j)$$

Sum up $j = 1, ..., m$,

$$\sum_{i=1}^{k} c_i \sum_{j=1}^{m} \mathbf{1}_{J_i}(j) m(A_j) = \sum_{i'=1}^{k'} c_{i'}' \sum_{j=1}^{m} \mathbf{1}_{J_{i'}'}(j) m(A_j)$$

$$\sum_{i=1}^{k} c_i \sum_{j \in J_i} m(A_i) = \sum_{i'=1}^{k'} c_i' \sum_{j \in J'_{i'}} m(A_i)$$

□

**Definition 1.4.3.** *(Almost everywhere and support): A property $P(x)$ of $x \in \mathbb{R}^d$ holds (Lebesgue) almost everywhere (a.e.) if $\{x : P(x)$ does not hold$\}$ is a null set, that is, $m^*(\{x : P(x)$ does not hold$\}) = 0$.*
*The support of a function $f$ is $\{x \in \mathbb{R} : f(x) \neq 0\}$.*

### 1.4.2 Measurable Functions

By extending the class of unsigned simple functions to the larger class of unsigned Lebesgue measurable functions, we can complete the unsigned simple integral to the unsigned Lebesgue integral.

**Definition 1.4.4.** *(Unsigned measurable function) An unsinged function $f$ is Lebesgue measurable if it is the pointwise limit of unsigned simple functions, i.e., if $\exists f_1, f_2, f_3, ... : \mathbb{R}^d \to [0, +\infty]$ of unsigned simple functions such that $f_n(x) \to f(x)$ $\forall x \in \mathbb{R}^d$.*

This definition has some equivalent forms.

**Lemma 1.4.2.** *(Equivalent Notions of Measurability). Let $f : \mathbb{R}^d \to [0, +\infty]$, the followings are equivalent.*

1. *$f$ is Lebesgue measurable .*
2. *$f$ is the pointwise a.e. limit of unsigned simple functions $f_n$. Thus $\lim_{n\to\infty} f_n(x)$ exists and $f(x) = \lim_{n\to\infty} f_n(x)$ for all $x \in \mathbb{R}^d$.*
3. *For every interval $I \subset [0, +\infty)$, the set $f^{-1}(I) := \{x \in \mathbb{R}^d : f(x) \in I\}$ is Lebesgue measurable. $f^{-1}(I) \in \mathcal{L}(\mathbb{R}^d)$.*

*Proof.* (i) ⇒ (ii) is immediate from definition, so is (ii) *Rightarrow* (i).
(ii) ⇒ (iii): Assume that $\exists$ simple functions $f_n \to f$ pointwise a.e.. Then, for almost every $x \in \mathbb{R}^d$ and $N \in \mathbb{N}$,

$$f(x) = \lim_{n\to\infty} f_n(x) = \limsup_{n\to\infty} f_n(x) = \inf_{N>0} \sup_{n\geq N} f_n(x) := \tilde{f}(x)$$

Let $\lambda > 0$ be arbitrary, and denote $\{g > \lambda\} := \{x \in \mathbb{R}^d : g(x) > \lambda\}$ for $g : \mathbb{R}^d \to [0, +\infty]$, we have for $M, N \in \mathbb{N}$,

$$\begin{aligned} \{\tilde{f} > \lambda\} &= \bigcup_{M>0} \bigcap_{N>0} \{x \in \mathbb{R}^d : \sup_{n\geq N} f_n(x) > \lambda\} \\ &= \bigcup_{M>0} \bigcap_{N>0} \bigcup_{n\geq N} \{x \in \mathbb{R}^d : f_n(x) > \lambda\} \end{aligned} \tag{2}$$

Since $f_n$ is unsigned simple, $\{x \in \mathbb{R}^d : f_n(x) > \lambda\} \in \mathcal{L}(\mathbb{R}^d)$. By (6) and (7) of Proposition 1.3.2., $\{\tilde{f} > \lambda\} \in \mathcal{L}(\mathbb{R}^d)$. Also, $\{f > \lambda\}$ and $\{\tilde{f} > \lambda\}$ differs by a null set, so $\{f > \lambda\} \in$

$\mathcal{L}(\mathbb{R}^d)$. Thus we have proved that $f^{-1}(I) \in \mathcal{L}(\mathbb{R}^d)$ for $I = (\lambda, +\infty)$.
Note that $\{f \geq \lambda\} = \bigcap_{\lambda' \in \mathcal{Q}, \lambda' < \lambda} \{f \geq \lambda'\}$. Then by (7) of Proposition 1.3.2., $\{f \geq \lambda\} \in \mathcal{L}(\mathbb{R}^d)$.
Note that

$$f^{-1}([a,b]) = \{f \geq a\} \setminus \{f > b\}$$
$$f^{-1}([a,b)) = \{f \geq a\} \setminus \{f \geq b\}$$
$$f^{-1}((a,b]) = \{f > a\} \setminus \{f > b\}$$
$$f^{-1}((a,b)) = \{f > a\} \setminus \{f \geq b\}$$

By Proposition 1.3.2., they are all Lebesgue measurable.
(iii) $\Rightarrow$ (i): Let $f : \mathbb{R}^d \to [0, +\infty]$ with $f^{-1}(I) \in \mathcal{L}(\mathbb{R}^d)$ $\forall I \subset \mathbb{R}^d$ as an interbal.
For each $n \geq 1$ and $x \in \mathbb{R}^d$, set

$$f_n(x) = \max_{m \in \mathbb{Z}} \{m2^{-n} : m2^{-n} \leq \min(f_(x), n) \mathbf{1}_{\overline{B(0,n)}}(x)\}$$

Then, $f_1 \leq f_2 \leq \ldots$ pointwise, and $f(x) = \sup_{n \in \mathbb{N}} f_n$ $\forall x \in \mathbb{R}^d$. Each $f_n$ takes finitely many values and for any $c \in [0, +\infty)$, $f_n^{-1}(c) = f^{-1}(I_c) \cap \overline{B(0,n)}$ for some interval $I_c$ that is measurable. Thus $f_n$ is simple, and is bounded and has finite measure support, and the claim follows. □

**Theorem 1.4.1.** *(Functions that are Measurable).*

1. *Every continuous function $f : \mathbb{R}^d \to [0, +\infty]$ is measurable.*
2. *The supremum, infimum, limit superior, or limit inferior of unsigned measurable functions is unsigned measurable.*

*Proof.* (1). Fist, divide the $\mathbb{R}^d$ into dyadic cubes:

$$\begin{aligned} \mathbb{R} &= \bigcup_{k_1 \in \mathbb{Z}} \bigcup_{k_2 \in \mathbb{Z}} \ldots \bigcup_{k_d \in \mathbb{Z}} [\frac{k_1}{2^n}, \frac{k_1+1}{2^n}) \times [\frac{k_2}{2^n}, \frac{k_2+1}{2^n}) \times \ldots \times [\frac{k_d}{2^n}, \frac{k_d+1}{2^n}) \ \forall n \in \mathbb{N} \\ &= \bigcup_{k_1 \in \mathbb{Z}} \bigcup_{k_2 \in \mathbb{Z}} \ldots \bigcup_{k_d \in \mathbb{Z}} B_{k_1,k_2,\ldots,k_d} \ \forall n \in \mathbb{N} \end{aligned}$$

Then, define an unsigned simple function $f_n$ can be written as

$$f_n(x) = \sum_{k_1 \in \mathbb{Z}} \sum_{k_2 \in \mathbb{Z}} \ldots \sum_{k_d \in \mathbb{Z}} \mathbf{1}_{B_{k_1,k_2,\ldots,k_d}}(x) \inf_{B_{k_1,k_2,\ldots,k_d}} f(x)$$

For all $x \in \mathbb{R}^d$, $x$ has to be in one of the dyadic cubes. When $x \in B_{k_1,k_2,\ldots,k_d}$, $f_n(x) = \inf_{B_{k_1,k_2,\ldots,k_d}} f(x)$. Thus it suffices to show that $f_n \to f$ pointwise a.e. in this case.
It is easy to see that $f_n \geq 0$. By the definition of infimum,

$$\inf_{B_{k_1,k_2,\ldots,k_d}} f(x) \leq f(x) \ \forall x \in B_{k_1,k_2,\ldots,k_d}$$

$\forall \epsilon > 0$, $\exists x' \in B_{k_1,k_2,\ldots,k_d}$, such that $f(x') \leq \inf_{B_{k_1,k_2,\ldots,k_d}} f(x) + \epsilon$. Together with the last inequality we have

$$0 \leq f(x') - \inf_{B_{k_1,k_2,\ldots,k_d}} f(x) \leq \epsilon$$

$$|f(x') - \inf_{B_{k_1,k_2,...,k_d}} f(x)| \leq \epsilon$$

Notice that, two points inside the closure of $B_{k_1,k_2,...,k_d}$ has distance less than $\frac{\sqrt{d}}{2^n}$. Thus, by continuity, $\exists \delta > 0$, pick $n$ large enough that $n > log_2 \frac{\sqrt{d}}{\delta}$, so that $|x - x'| < \frac{\sqrt{d}}{2^n} < \delta$ implies $|f(x) - f(x')| \leq \epsilon$, $\forall x \in B_{k_1,k_2,...,k_d}$.
Therefore, $\forall x \in B_{k_1,k_2,...,k_d}$,

$$\begin{aligned}|f(x) - \inf_{B_{k_1,k_2,...,k_d}} f(x)| &\leq |f(x) - f(x')| + |f(x') - \inf_{B_{k_1,k_2,...,k_d}} f(x)| \\ &\leq 2\epsilon\end{aligned}$$

This means that, $\forall \epsilon > 0$, we found that when $n > log_2 \frac{\sqrt{d}}{\delta}$, $|f(x) - \inf_{B_{k_1,k_2,...,k_d}} f(x)| = |f(x) - f_n(x)| \leq 2\epsilon$. $f_n \to f$. Therefore, for all $x \in \mathbb{R}^d$ (equivalent to $X$ in arbitrary cube), $f_n \to f$ pointwise a.s. Then, $f$ is Lebesgue measurable.
(2). Let $f_n : \mathbb{R}^d \to [0, +\infty]$ be unsigned measurable functions and $\sup_{n \in \mathbb{N}} f_n$ be the supremum. Then, by Lemma 1.4.2. (3), it suffices to show that $\{x \in \mathbb{R}^d : \sup_{n \in \mathbb{N}} f_n(x) \geq \lambda\}$ is Lebesgue measurable for all $\lambda \in [0, +\infty]$.
First we show that

$$\{x \in \mathbb{R}^d : \sup_{n \in \mathbb{N}} f_n(x) \geq \lambda\} = \bigcup_{n \in \mathbb{N}} \{x \in \mathbb{R}^d : f_n(x) \geq \lambda\}$$

Let $y \in \bigcup_{n \in \mathbb{N}} \{x \in \mathbb{R}^d \ f_n(x) \geq \lambda\}$, then $\exists n' \in \mathbb{N}$ such that $f_{n'}(y) \geq \lambda$. By definition of supremum, $\sup_{n \in \mathbb{N}} f_n(x) \geq f_{n'}(y) \geq \lambda$, thus $y \in \{x \in \mathbb{R}^d : \sup_{n \in \mathbb{N}} f_n(x) \geq \lambda\}$.
Let $y \in \{x \in \mathbb{R}^d : \sup_{n \in \mathbb{N}} f_n(x) \geq \lambda\}$, then $\sup_{n \in \mathbb{N}} f_n(y) \geq \lambda$. $\forall \epsilon > 0$, $\exists N'$ such that $f_{n'}(y) \geq \sup_{N \in \mathbb{N}} f_N(y) - \epsilon$. Since $\epsilon > 0$ arbitrary, we have $f_{n'}(y) \geq \sup_{N \in \mathbb{N}} f_N(y)$. Thus $y \in \{x \in \mathbb{R}^d : \sup_{n \in \mathbb{N}} f_n(x) \geq \lambda\}$.
Thus we have proved that

$$\{x \in \mathbb{R}^d : \sup_{n \in \mathbb{N}} f_n(x) \geq \lambda\} = \bigcup_{n \in \mathbb{N}} \{x \in \mathbb{R}^d : f_n(x) \geq \lambda\}$$

Since each $f_n$ is Lebesgue integrable, $\{x \in \mathbb{R}^d : f_n(x) \geq \lambda\}$ is Lebesgue measurable, and the countable union is also Lebesgue measurable. Then, $\sup_{n \in \mathbb{N}} f_n(x)$ is Lebesgue integrable.

To show $\inf_{m \in \mathbb{N}}$ is Lebesgue integrable, it suffices to show that $\{x \in \mathbb{R}^d : \inf_{n \in \mathbb{N}} f_n(x) \leq \lambda\}$ is Lebesgue measurable for all $\lambda \in [0, +\infty]$.
First we show that

$$\{x \in \mathbb{R}^d : \inf_{n \in \mathbb{N}} f_n(x) \leq \lambda\} = \bigcap_{n \in \mathbb{N}} \{x \in \mathbb{R}^d : f_n(x) \leq \lambda\}$$

Let $x \in \bigcap_{n \in \mathbb{N}} \{x \in \mathbb{R}^d : f_n(x) \leq \lambda\}$, then $f_n(x) \leq \lambda \ \forall n \in \mathbb{N}$. Then, by definition of infimum, $\inf_{n \in \mathbb{N}} f_n(x) \leq f_n(x) \leq \lambda$, which means that $x \in \{x \in \mathbb{R}^d : \inf_{n \in \mathbb{N}} f_n(x) \leq \lambda\}$.
Let $x \in \{x \in \mathbb{R}^d : \inf_{n \in \mathbb{N}} f_n(x) \leq \lambda\}$, then it implies that $\forall \epsilon > 0$, $\forall n \in \mathbb{N}$, $\exists k \geq n$ such that $f_k(x) \leq \lambda + \epsilon$. Since $\epsilon > 0$ is arbitrary, $f_k \leq \lambda$. Then, $x \in \bigcap_{n \in \mathbb{N}} \{x \in \mathbb{R}^d : f_k(x) \leq \lambda\} \subseteq \bigcap_{n \in \mathbb{N}} \{x \in \mathbb{R}^d : f_n(x) \leq \lambda\}$.

Thus we have proved that

$$\{x \in \mathbb{R}^d : \inf_{n \in \mathbb{N}} f_n(x) \leq \lambda\} = \bigcap_{n \in \mathbb{N}} \{x \in \mathbb{R}^d : f_n(x) \leq \lambda\}$$

Since each $f_n$ is Lebesgue integrable, $\{x \in \mathbb{R}^d : f_n(x) \leq \lambda\}$ is Lebesgue measurable, and the countable intersection is also Lebesgue measurable. Then, $\inf_{n \in \mathbb{N}} f_n(x)$ is Lebesgue integrable.

To show that $\limsup_{n \to \infty} f_n$ is Lebesgue integrable, it suffices to show that $\limsup_{n \to \infty} \{x \in \mathbb{R}^d : f_n(x) \geq \lambda\}$ is Lebesgue measurable for all $\lambda \in [0, \infty]$.

$$\begin{aligned}\limsup_{n \to \infty} \{x \in \mathbb{R}^d : f_n(x) \geq \lambda\} &= \lim_{N \to \infty} \sup_{n \geq N} \{x \in \mathbb{R}^d : f_n(x) \geq \lambda\} \\ &= \bigcap_{N \in \mathbb{N}} \bigcup_{n \geq N} \{x \in \mathbb{R}^d : f_n(x) \geq \lambda\}\end{aligned}$$

As is proved before, $\{x \in \mathbb{R}^d : f_n(x) \geq \lambda\}$ is Lebesgue measurable, so that its countable intersection of countable union $\limsup_{n \to \infty} \{x \in \mathbb{R}^d : f_n(x) \geq \lambda\}$ is Lebesgue measurable, then $\limsup_{n \to \infty} f_n$ is Lebesgue integrable.

To show that $\liminf_{n \to \infty} f_n$ is Lebesgue integrable, it suffices to show that $\liminf_{n \to \infty} \{x \in \mathbb{R}^d : f_n(x) \leq \lambda\}$ is Lebesgue measurable for all $\lambda \in [0, \infty]$.

$$\begin{aligned}\liminf_{n \to \infty} \{x \in \mathbb{R}^d : f_n(x) \leq \lambda\} &= \lim_{N \to \infty} \inf_{n \geq N} \{x \in \mathbb{R}^d : f_n(x) \leq \lambda\} \\ &= \bigcup_{N \in \mathbb{N}} \bigcap_{n \geq N} \{x \in \mathbb{R}^d : f_n(x) \leq \lambda\}\end{aligned}$$

As is proved before, $\{x \in \mathbb{R}^d : f_n(x) \leq \lambda\}$ is Lebesgue measurable, so that its countable intersection of countable union $\liminf_{n \to \infty} \{x \in \mathbb{R}^d : f_n(x) \leq \lambda\}$ is Lebesgue measurable, then $\liminf_{n \to \infty} f_n$ is Lebesgue integrable. ☐

**Theorem 1.4.2.** *(Bounded unsigned measurable function is the uniform limit of bounded simple functions). Let $f : \mathbb{R}^d \to [0, +\infty]$. Then, $f$ is a bounded unsigned measurable function if and only if $f$ is the uniform limit of bounded simple functions.*

*Proof.* First we prove $\Leftarrow$.
Suppose $f_n$ is a sequence of bounded simple functions and $f_n \to f$ uniformly. Then, $\forall \epsilon > 0$, $\exists N$ such that when $n \geq N$, $|f_n(x) - f(x)| \leq \epsilon \; \forall x \in \mathbb{R}^d$. By triangle inequality we have $|f(x)| \leq \epsilon + |f_n(x)|$. Since $f_n(x)$ is bounded, it is clear that $f$ is also bounded. Also, uniform convergent induces pointwise convergence, and then $f$ is Lebesgue measurable by Lemma 1.4.2..
Then we prove $\Rightarrow$. By Lemma 1.4.2., since $f$ is unsigned Lebesgue measurable, $f$ is the supremum $f(x) = \sup_{n \in \mathbb{N}}$ of an increasing sequence $0 \leq f_1 \leq f_2 \leq ...$ of unsigned simple functions $f_n$, each of which are bounded with finite measure support.
By the definition of supremum, $\forall x$, $\forall \epsilon > 0$, $\exists n' \in \mathbb{N}$ such that $f_{n'}(x) \geq f(x) - \epsilon$. Also since $f_n$ is monotonic increasing, we have $|f_{n'}(x) - f(x)| \leq \epsilon$.
Since $f_{n'}$ is a unsigned simple function, it can take finite many values $c_1, c_2, ..., c_m$. Based

on these values, we divide the domain of $f_{n'}$ ($\mathbb{R}^d$) into $m$ parts,

$$\mathbb{R}^d = f_{n'}^{-1}(c_1) \cup f_{n'}^{-1}(c_2) \cup ... \cup f_{n'}^{-1}(c_m)$$

For each $f_{n'}^{-1}(c_i)$, by supremum, $\forall \epsilon > 0$, $\exists n_i$, such that, when $n \geq n_i$, $|f_n(x) - f(x)| \leq \epsilon \ \forall x \in f_{n'}^{-1}(c_i)$. By this we got a collection of $\{n_1, n_2, ..., n_m\}$.
Set $N := \max\{n_1, n_2, ..., n_m\}$, then we have $\forall \epsilon > 0$, $\exists N = \max\{n_1, n_2, ..., n_m\}$ such that when $n \geq N$, $|f_n(x) - f(x)| \leq \epsilon \ \forall x \in \mathbb{R}^d$. Thus we have $f_n \to f$ uniformly.
Also since $0 \leq f_n \leq \sup_{n \in \mathbb{N}} f_n = f$, $f$ is bounded, we have $f_n$ is bounded. □

### 1.4.3 Unsigned Lebesgue Integrals

Now let's integrate unsigned measurable functions.

**Definition 1.4.5.** *(Lower and upper unsigned Lebesgue integral). Let $f : \mathbb{R}^d \to [0, +\infty]$. Define the lower unsigned Lebesgue integral of $f$ as*

$$\underline{\int_{\mathbb{R}^d}} f := \sup_{0 \leq g \leq f, g \text{ simple}} \mathit{Simp} \int_{\mathbb{R}^d} g$$

*and the upper unsigned Lebesgue integral of $f$ as*

$$\overline{\int_{\mathbb{R}^d}} f := \inf_{f \leq h, h \text{ simple}} \mathit{Simp} \int_{\mathbb{R}^d} h$$

**Definition 1.4.6.** *(Unsigned lebesgue integral). Let $f : \mathbb{R}^d \to [0, \infty]$ be measurable. Define its unsigned Lebesgue integral as*

$$\int_{\mathbb{R}^d} f := \underline{\int_{\mathbb{R}^d}} f = \sup_{0 \leq g \leq f, g \text{ simple}} \mathit{Simp} \int_{\mathbb{R}^d} g$$

For $f : \mathbb{R}^d \to [0, \infty]$ measurable, bounded, and vanishing outside of a set of finite measure, the lower and upper Lebesgue integrals match. Also we have an important corollary. But first we have to prove a theorem.

**Theorem 1.4.3.** *Let $f : \mathbb{R}^d \to [0, +\infty]$ be measurable, bounded, and vanishing outside of a set of finite measure. Show that the lower and upper Lebesgue integrals of $f$ agree. (Hint: use Exercise 1.3.4.) There is a converse to this statement, but we will defer it to later notes. What happens if $f$ is allowed to be unbounded, or is not supported inside a set of finite measure?*

*Proof.* The very obvious way to prove this is by first proving $\leq$ then proving $\geq$.
For $\leq$: by definition,

$$\underline{\int_{\mathbb{R}^d}} f = \sup_{0 \leq g \leq f, g \text{ simple}} \text{Simp} \int_{\mathbb{R}^d} g$$

$$\overline{\int_{\mathbb{R}^d}} f = \inf_{h \geq f, g \text{ simple}} \text{Simp} \int_{\mathbb{R}^d} h$$

Since $g \leq h$ for every $g, h$ that satisfy the conditions, by definition we natually have $\underline{\int_{\mathbb{R}^d}} f \leq \overline{\int_{\mathbb{R}^d}} f$. Thus it suffices to show that $\underline{\int_{\mathbb{R}^d}} f \geq \overline{\int_{\mathbb{R}^d}} f$. However my original methods is too tedious

(I have to divide the finite measure support into finite sub-supports twice, take supremum and infimum of $f$ within each sub-support, etc.) So I'll just adopt another method.

Let $S$ be the finite measure support on which $f > 0$. By Theorem 1.4.2., we can find a sequence of unsigned simple functions $(g_n)_{n\in\mathbb{N}}$ such that (i) $0 \le g_1 \le g_2 \le ...$ (ii) has finite measure support $S$ (iii) $g_n \to f$ uniformly.
Pick a subsequence of this original sequence so that $\forall n \in \mathbb{N}$, $d_\infty(g_n, f) \le \frac{1}{n}$. Furthermore, construct another sequence of unsigned simple functions $(h_n)_{n\in\mathbb{N}}$ such that $h_n = g_n + \frac{2}{n}$.
Then, $h_n - f = g_n + \frac{2}{n} - f \ge \frac{1}{n} \ge 0$. $d_\infty(h_n, f) \le d_\infty(h_n, g_n) + d_\infty(g_n, f) \le \frac{3}{n}$.
Since now $\forall n \in \mathbb{N}$, $d_\infty(h_n, g_n) \le \frac{2}{n}$, we have $d_\infty(h_n, g_n) \to 0$ as $n \to \infty$. They converge uniformly to each other.
Pick a simple function $g' \le f$ that satisfies $\underline{\int_{\mathbb{R}^d}} f - \text{Simp} \int_{\mathbb{R}^d} g' \le \frac{1}{n}$, $d_\infty(g', f) \le \frac{1}{n}$.

$$\begin{aligned}
\underline{\int_{\mathbb{R}^d}} f - \text{Simp} \int_{\mathbb{R}^d} g_n &= \underline{\int_{\mathbb{R}^d}} f - \text{Simp} \int_{\mathbb{R}^d} g' + \text{Simp} \int_{\mathbb{R}^d} g' - \text{Simp} \int_{\mathbb{R}^d} g_n \\
&\le \frac{1}{n} + \text{Simp} \int_{\mathbb{R}^d} |g' - g_n| \\
&\le \frac{1}{n} + \text{Simp} \int_{\mathbb{R}^d} d_\infty(g', g_n)\mathbf{1}_S \\
&\le \frac{1}{n} + \text{Simp} \int_{\mathbb{R}^d} (d_\infty(g', f) + d_\infty(f, g_n))\mathbf{1}_S \\
&\le \frac{1}{n} + \frac{2}{n} m(S) \\
&\to 0 \text{ as } n \to \infty
\end{aligned}$$

Thus $\lim_{n\to\infty} \text{Simp} \int_{\mathbb{R}^d} g_n = \underline{\int_{\mathbb{R}^d}} f$.
Pick another simple function $h' \ge f$ such that $\text{Simp} \int_{\mathbb{R}^d} h' - \overline{\int_{\mathbb{R}^d}} f \le \frac{1}{n}$, $d_\infty(h', f) \le \frac{1}{n}$.

$$\begin{aligned}
\text{Simp} \int_{\mathbb{R}^d} h_n - \overline{\int_{\mathbb{R}^d}} f &= \text{Simp} \int_{\mathbb{R}^d} h_n - \text{Simp} \int_{\mathbb{R}^d} h' + \text{Simp} \int_{\mathbb{R}^d} h' - \overline{\int_{\mathbb{R}^d}} f \\
&\le \frac{1}{n} + \text{Simp} \int_{\mathbb{R}^d} |h' - h_n| \\
&\le \frac{1}{n} + \text{Simp} \int_{\mathbb{R}^d} d_\infty(h', h_n)\mathbf{1}_S \\
&\le \frac{1}{n} + \text{Simp} \int_{\mathbb{R}^d} (d_\infty(h', f) + d_\infty(f, h_n))\mathbf{1}_S \\
&\le \frac{1}{n} + \frac{2}{n} m(S) \\
&\to 0 \text{ as } n \to \infty
\end{aligned}$$

Thus $\lim_{n\to\infty} \text{Simp} \int_{\mathbb{R}^d} h_n = \overline{\int_{\mathbb{R}^d}} f$.
Since $\text{Simp} \int_{\mathbb{R}^d} (h_n - g_n) \le \text{Simp} \int_{\mathbb{R}^d} d_\infty(h_n, g_n)\mathbf{1}_S \to 0$ as $n \to \infty$, we have $\lim_{n\to\infty} \text{Simp} \int_{\mathbb{R}^d} h_n = \lim_{n\to\infty} \text{Simp} \int_{\mathbb{R}^d} g_n$. Thus

$$\underline{\int_{\mathbb{R}^d}} f = \lim_{n\to\infty} \text{Simp} \int_{\mathbb{R}^d} g_n = \lim_{n\to\infty} \text{Simp} \int_{\mathbb{R}^d} h_n = \overline{\int_{\mathbb{R}^d}} f$$

□

**Corollary 1.4.1.** *(Finite additivity of Lebesgue integral). Let $f, g : \mathbb{R}^d \to [0, +\infty]$ be measurable. Then,*

$$\int_{\mathbb{R}^d} (f + g) = \int_{\mathbb{R}^d} f + \int_{\mathbb{R}^d} g$$

We also have *Markov's Inequality*, which asserts that the Lebesgue integral of an unsigned measurable function controls how often that function can be large.

**Lemma 1.4.3.** *(Markov's Inequality). Let $f : \mathbb{R}^d \to [0, +\infty]$ be measurable. Then, for any $\lambda \in (0, +\infty)$,*

$$m(\{x \in \mathbb{R}^d : f(x) \geq \lambda\}) \leq \frac{1}{\lambda} \int_{\mathbb{R}^d} f(x)dx$$

*Proof.* First, notice that

$$\lambda \mathbf{1}_{\{x \in \mathbb{R}^d : f(x) \geq \lambda\}} \leq f(x)$$

Then, since $f$ is measurable,

$$\int_{\mathbb{R}^d} \lambda \mathbf{1}_{\{x \in \mathbb{R}^d : f(x) \geq \lambda\}} \leq \int_{\mathbb{R}^d} f(x)dx$$

By the definition of Lebesgue integral for simple function,

$$LHS = \lambda m(\{x \in \mathbb{R}^d : f(x) \geq \lambda\}) \leq \int_{\mathbb{R}^d} f(x)dx$$

Therefore we have

$$m(\{x \in \mathbb{R}^d : f(x) \geq \lambda\}) \leq \frac{1}{\lambda} \int_{\mathbb{R}^d} f(x)dx$$

□

### 1.4.4 Absolute Integrability

## 1.5 Citation

This is a citation[**?**].

## References