MEASURE AND INTEGRATION

Math 631

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Contents

1 Measure Theory and Integration

1.1 Elementary Sets and Measure

The motivation to start this study is to lay a foundation to assign a measure to arbitrary sets $E \in \mathbb{R}^d$. The set can be nice set such as intervals, boxes, and polygons, but it can also be any generic or strange set that we wish to study. We want a measure that is well behaved under limits (e.g. Lebesgue measure is well behaved under pointwise limits).

Definition 1.1.1. *(Finite Intervals). $(a, b), (a, b], [a, b), [a, b]$ for $a, b \in \mathbb{R}$. The length is $b - a$ and it can be zero.*

Definition 1.1.2. *(Box). $B \in \mathbb{R}^d$ is a Cartesian product of intervals $B = I_1 \times ... \times I_d$*

Definition 1.1.3. *Elementary Set is any subset of $\mathbb{R}^d$ that is a finite union of boxes.*

We denote the set of elementary sets as $\mathcal{E}(\mathbb{R}^d)$. It does not include unbounded sets.

Proposition 1.1.1. *(Boolean closure). The set $\mathcal{E}(\mathbb{R}^d)$ is closed taking unions ($E \cup F$), intersects ($E \cap F$), differences ($E \setminus F$) and symmetric differences ($E \Delta F = (E \setminus F) \cup (F \setminus E)$), as well as translations: $E \in \mathcal{E}(\mathbb{R}^d), x \in \mathbb{R}^d \Rightarrow E + x = \{y + x : y \in E\} \in \mathcal{E}(\mathbb{R}^d)$.*

Proof. Let $E, F \in \mathcal{E}(\mathbb{R}^d)$, then each of them is a finite union of boxes. $E = \bigcup_{i=1}^n B_i$, $F = \bigcup_{i=1}^m B_i'$.
Denote the collection of $\{B_i\}_{i=1}^n$ and $\{B_i'\}_{i=1}^m$ as a new set of boxes:
$\{C_i\}_{i=1}^{n+m} = \{B_1, ..., B_n, B_1', ..., B_m'\}$ Then,

$$\begin{aligned} E \cup F &= (\bigcup_{i=1}^n B_i) \cup (\bigcup_{i=1}^m B_i') \\ &= \bigcup_{i=1}^{n+m} C_i \end{aligned}$$

Thus, $E \cup F$ is a finite union of boxes and is an elementary set.

$$\begin{aligned} E \cap F &= (\bigcup_{i=1}^n B_i) \cap (\bigcup_{i=1}^m B_i') \\ &= (B_1 \cap (\bigcup_{i=1}^m B_i')) \cup ... \cup (B_n \cap (\bigcup_{i=1}^m B_i') \\ &= ((B_1 \cap B_1') \cup ... \cup (B_1 \cap B_m')) \cup ... \cup ((B_n \cap B_1') \cup ... \cup (B_n \cap B_m')) \\ &= (B_1 \cap B_1') \cup ... \cup (B_1 \cap B_m') \cup ... \cup (B_n \cap B_1') \cup ... \cup (B_n \cap B_m') \end{aligned}$$

Now we prove that the intersection of two boxes is a box. Suppose $B_i = I_1^i \times ... \times I_d^i$, $B_j' = I_1^{'j} \times ... \times I_d^{'j}$, then $B_i \cap B_j' = (I_1^i \cap I_1^{'j}) \times ... \times (I_d^i \cap I_d^{'j})$. Since the intersection of two intervals is an interval (degenerate or non-degenerate), by definition, $B_i \cap B_j'$ is a box.
Then, by (2), $E \cap F$ is a finite union of $n \times m$ boxes, and it is an elementary set.

Now we look at difference.

$$
\begin{aligned}
E \setminus F &= E \cap F^C \\
&= (\bigcup_{i=1}^{n} B_i) \cap (\bigcup_{i=1}^{m} B_i')^C \\
&= (\bigcup_{i=1}^{n} B_i) \cap (\bigcap_{i=1}^{m} B_i^{'C}) \\
&= (B_1 \cap (\bigcap_{i=1}^{m} B_i^{'C})) \cup ... \cup (B_n \cap (\bigcap_{i=1}^{m} B_i^{'C})) \\
&= ((B_1 \cap B_1^{'C}) \cap ... \cap (B_1 \cap B_m^{'C})) \cup ... \cup ((B_n \cap B_1^{'C}) \cap ... \cap (B_n \cap B_m^{'C})) \\
&= ((B_1 \setminus B_1^{'}) \cap ... \cap (B_1 \setminus B_m^{'})) \cup ... \cup ((B_n \setminus B_1^{'}) \cap ... \cap (B_n \setminus B_m^{'}))
\end{aligned}
$$

Since B_i and $B_j^{'}$ are boxes, $B_i \setminus B_j^{'}$ are elementary. Then, by the first two parts that we just proved, each $((B_i \setminus B_1^{'}) \cap ... \cap (B_i \setminus B_m^{'}))$ is elementary, and their union is elementary. Thus, $E \setminus F$ is an elementary set.
Next, since E and F are elementary, $E \setminus F$ is elementary and $F \setminus E$ is also elementary. By the part we just proved, their union $(E \setminus F) \cup (F \setminus E)$ is also elementary. Thus, $E \Delta F$ is an elementary set.
Next,

$$
\begin{aligned}
E + x &= \bigcup_{i=1}^{n} B_i + x \\
&= \bigcup_{i=1}^{n} (I_1^i + x) \times ... \times (I_d^i + x) \\
&= \bigcup_{i=1}^{n} I_1^{'i} \times ... \times I_d^{'i} \\
&= \bigcup_{i=1}^{b} B_{'i}
\end{aligned}
$$

where we have translation of each interval by x in the second and third equality. Thus, $E + x$ is an elementary set.
Thus, we finished the proof that $\mathcal{E}(\mathbb{R}^d)$ is closed under union, intersection, difference, symmetric difference, and translation. □

Note that this does not form an algebra. Now it is time to give the elementary set a measure.

Lemma 1.1.1. *Let $E \in \mathbb{R}^d$ be an elementary set.*
(1). E can be expressed (partitioned) as the finite union of disjoint boxes, that is, $E = \bigcup_{i=1}^{n} B_i$ for B_i pairwise disjoint.
(2). For any two such partitions $E = \bigcup_{i=1}^{n} B_i = \bigcup_{i=1}^{m} B_i'$, we have $\sum_{i=1}^{n} |B_i| = \sum_{i=1}^{m} |B_i'|$. We denote this value by $m(E) = m^d(E)$, the elementary measure of E. It is independent of partitions.

Proof. Begin with (i).
For the case $d = 1$, $E = \bigcup_{i=1}^n I_i$. Place the $2n$ endpoints of these intervals in increasing order (discarding repititions) and relable them to be $c_1 \leq c_2 \leq ... \leq c_{2n}$.
Let $J_1, ...J_{4n-1}$ be disjoint intervals formed by these endpoints where

$$J_i = \{c_i\} \ for \ 1 \leq i \leq 2n$$

$$J_i = (c_{i-2n}, c_{i-2n+1}) \ for \ 2n+1 \leq i \leq 4n-1$$

Basically we have $2n$ endpoints and $2n-1$ open intervals between endpoints by doing this. Then, each I_i can be expressed as some subcollection of $J_1, ..., J_{4n-1}$. Then, $E = \bigcup_{k:J_k \cap E \neq \emptyset} J_k$.
For general cases $d \geq 2$ Now we have $E = \bigcup_{i=1}^n B_i$ where $B_i = I_i^1 \times ... \times I_i^d$. For each m such that $1 \leq m \leq d$, we apply the $d = 1$ case to get a family of disjoint intervals $\{J_k^m\}_{k=1}^{n_m}$ such that $\bigcup_{i=1}^n I_i^m = \bigcup_{i=1}^{n_m} J_k^m$.
Then, we can get $n_1 \times ... \times n_d$ pairwise disjoint boxes and each of them is represented as $\tilde{B}_{k_1...k_d} = J_{k_1}^1 \times ... \times J_{k_d}^d$ for $(k_1, ..., k_d) \in \{1, ..., n_1\} \times ... \times \{1, ..., n_d\}$.
Then, each B_i is a union of a subcollection of $\{\tilde{B}_{k_1...k_d}\}$, and thus E is expressed as a finite union of pairwise disjoint boxes.

Now let's prove (ii).
First we notice that for any interval I, we have that

$$|I| = \lim_{N \to +\infty} \frac{1}{N} \#(I \cap \frac{1}{N}\mathbb{Z})$$

where $\frac{1}{N}\mathbb{Z} = \{\frac{k}{N} : k \in \mathbb{Z}\}$.
I don't know how to prove this. One way to look at it is to take a sample of rational points in (a, b) with the distance between each adjacent pair to be $\frac{1}{N}$. Another way is through an example where $a = 2$, $b = 4$. Then, $(a, b) \cap \frac{1}{N}\mathbb{Z} = \{z : z \in (aN, bN), z \in \mathbb{Z})\}$. When $n = 1$, $\# = 1$; $n = 2$, $\# = 3$, ... Then the cardinality is equal to $(b-a)N - 1 = 2N - 1$ with each N. Then $\lim_{N \to +\infty} \frac{1}{N}((b-a)N - 1) = b - a$.
By taking Cartesian product,

$$|B| = \lim_{N \to +\infty} \frac{1}{N^d} \#(B \cap \frac{1}{N}\mathbb{Z}^d)$$

For $E = \cup_{i=1}^n B_i$ with pairwise disjoint B_i,

$$\begin{aligned} \frac{1}{N^d} \#(E \cap \frac{1}{N}\mathbb{Z}^d) &= \sum_{i=1}^n \frac{1}{N^d} \#(B_i \cap \frac{1}{N}\mathbb{Z}^d) \\ &\to \sum_{i=1}^n |B_i| \ as \ N \to +\infty \end{aligned}$$

The LHS is independent of partitions, thus $m(E) = \lim_{N \to +\infty} \frac{1}{N^d} \#(E \cap \frac{1}{N}\mathbb{Z}^d)$. □

Theorem 1.1.1. *(Uniqueness of elementary measure). Let $d \geq 1$. Let $m' : \mathcal{E}(\mathbb{R}^d) \to \mathbb{R}^+$ be a map from the collection $\mathcal{E}(\mathbb{R}^d)$ of elementary subsets of $\mathbb{R}^d$ to the nonnegative reals* *that obeys the non-negativity, finite additivity, and translation invariance properties. Then there exists a constant $c \in \mathbb{R}^+$ such that $m'(E) = cm(E)$ for all elementary sets E. In particular, if we impose the additional normalisation $m'([0,1)^d) = 1$, then $m' \equiv m$. (Hint: Set $c := m'([0,1)^d)$, and then compute $m'([0,\frac{1}{n})^d)$ for any positive integer n.)*

Proof. In this proof we will use m to represent elementary measure.
First we observes that any $E \in \mathcal{E}(\mathbb{R}^d)$ can be expressed as a finite union of translated $[0,a)^d$ types sets together with with the boundary with zero measure. So we only need to work with type $[0,a)^d$ set.
Set $m'([0,1)^d) := c$. Then, it can be written as n^d finite disjoin unions of the translated $[0,\frac{1}{n})^d$. Therefore, by finite additivity, $m'([0,\frac{1}{n})^d) = \frac{m'([0,1)^d)}{n^d} = \frac{c}{n^d} = cm([0,\frac{1}{n})^d)$.
Without loss of generality, let the elementary set $E = \Pi_{i=1}^d [a_i, b_i) \in \mathbb{R}^d$. By translation invariance, $m'(E) = m'(E - (a_1, ..., a_d)) = m'(\Pi_{i=1}^d [0, b_i - a_i))$.
First, consider the case where $b_i - a_i$ is rational. Then it can be represented in the form $\frac{p_1}{n}, ..., \frac{p_d}{n}$ using some common numerator n. Then, by partition into disjoint sets and boundaries,

$$\prod_{i=1}^d [0, \frac{p_i}{n}) = \bigcup_{k_i \in \mathbb{Z}; 0 \le k_i \le p_i - 1} \prod_{i=1}^d [\frac{k_i}{n}, \frac{k_i+1}{n})$$
$$= \bigcup_{k \in \{(k_1,...,k_d): k_i \in \mathbb{Z}; 0 \le k_i \le p_i - 1\}} [0, \frac{1}{n})^d + k$$

Then, by finite additivity and zero measure on boundary,

$$\begin{aligned} m'(\prod_{i=1}^d [0, \frac{p_i}{n})) &= \sum_{k \in \{(k_1,...,k_d): k_i \in \mathbb{Z}; 0 \le k_i \le p_i - 1\}} m'([0, \frac{1}{n})^d + k) \\ &= \sum_{k \in \{(k_1,...,k_d): k_i \in \mathbb{Z}; 0 \le k_i \le p_i - 1\}} m'([0, \frac{1}{n})^d) \\ &= c \prod_{i=1}^d \frac{p_i}{n} \\ &= cm(\prod_{i=1}^d [0, \frac{p_i}{n})) \end{aligned} \tag{1}$$

Next, consider the case where $b_i - a_i$ is real. By the density of rationals, find two rational sequences $\{s_n^i\}_{n \in \mathbb{N}} < b_i - a_i < \{q_n^i\}_{n \in \mathbb{N}}$ such that $\lim_{n \to \infty} s_n^i = \lim_{i \to \infty} q_n^i = b_i - a_i$.
Then, we have $\prod_{i=1}^d [0, s_n^i) \subseteq \prod_{i=1}^d [0, b_i - a_i) \subseteq \prod_{i=1}^d [0, q_n^i)$.
Here we prove the monotonicity first. Let $A \subseteq B$, then, $m(B) = m((B \setminus A) \cup A) = m(B \setminus A) + m(A) \ge m(A)$. Then,

$$m'(\prod_{i=1}^d [0, s_n^i)) \le m'(\prod_{i=1}^d [0, b_i - a_i)) \le m'(\prod_{i=1}^d [0, q_n^i))$$

From the above calculation with the rationals we know that

$$cs_n^1 s_n^2 ... s_n^d \le m'(\prod_{i=1}^d [0, b_i - a_i)) \le cq_n^1 q_n^2 ... q_n^d$$

By the limiting and triangle rule, we have

$$\begin{aligned} m'(\prod_{i=1}^{d}[0,b_i-a_i)) &= c(b_1-a_1)(b_2-a_2)...(b_d-a_d) \\ &= cm(\prod_{i=1}^{d}[0,b_i-a_i)) \qquad (2) \end{aligned}$$

We already proved that for elementary set $m'(E) = m'([0,1)^d)m(E) = cm(E)$. Then, if $m'([0,1)^d) = 1$, we have $m' \equiv m$. □

Now we have the property of the elementary measure:

(i). $m(E) \geq 0$

(ii). Finite additivity: $m(E \cup F) = m(E) + m(F)$ for disjoint $E, F \in \mathcal{E}(\mathbb{R}^d)$. By induction, $m(E_1 \cup ... \cup E_n) = \sum_{i=1}^{n} m(E_i)$ for disjoint $\{E_i\}$.

(iii). $m(\emptyset) = 0$.

(iv). $m(B) = |B| \ \forall$ box B.

(v). Monotonicity: $E \subset F$, then $m(E) \leq m(F)$.

(vi). Sub-additivity: $m(E \cup F) \leq m(E) + m(F)$ for any $E, F \in \mathcal{E}(\mathbb{R}^d)$. By induction: $m(E_1 \cup ... \cup E_n) \leq \sum_{i=1}^{n} m(E_i)$.

(vii). $E \in \mathcal{E}(\mathbb{R}^d)$, $x \in \mathbb{R}$, then $m(E + x) = m(E)$.

Proof. (v). $F = E \cup (F \setminus E)$, so $m(F) \overset{\text{(ii)}}{=} m(E) + m(F \setminus E) \overset{\text{(i)}}{\geq} m(E)$.
(vi). Finite sub-additivity: $E \cup F = E \cup (F \setminus E)$, therefore $m(E \cup F) \overset{\text{(ii)}}{=} m(E) + m(F \setminus E) \overset{\text{(v)}}{\leq} m(E) + m(F)$.
(vii). E can be partitioned into finite disjoint boxes $E = \bigcup_{i=1}^{n} B_i$, where each $B_i = I_i^1 \times ... \times I_i^d$. Then, $E' = E + x = \bigcup_{i=1}^{n} B_i + x = \bigcup_{i=1}^{n} I_i^1 \times ... \times I_i^d + x = \bigcup_{i=1}^{n} (I_i^1 + x) \times ... \times (I_i^1 + x) = \bigcup_{i=1}^{n} B_i'$. $|B_i| = |B_i'|$. Then, because of disjoint, $m(E) = \sum_{i=1}^{n} |B_i| = \sum_{i=1}^{n} |B_i'| = m(E')$. □

1.2 Jordan Measure

More advanced sets such as triangle, disk, or rotated boxes can be measured by approaching from without and within by elementary sets.

Definition 1.2.1. *(Jordan Measure). Let $E \in \mathbb{R}^d$ be a bounded set.*
Its Jordan inner measure is $m_J(E) = \sup_{A \subset E,\ A\ elementary} m(A)$.
Its Jordan outer measure is $m^J(E) = \inf_{E \subset B,\ B\ elementary} m(B)$.
If $m_J(E) = m^J(E)$, then E is Jordan measurable.
Let $\mathcal{J}(\mathbb{R}^d)$ be class of Jordan measurable sets. For $E \in \mathcal{J}(\mathbb{R}^d)$, define Jordan measure as $m(E) := m_J(E) = m^J(E)$.

Note that (i). unbounded sets are not Jordan measurable. (ii). The Jordan measure of $E \in \mathcal{E}(\mathbb{R}^d)$ is equal to the elementary measure of E.
Here gives the characterization of Jordan Measure.

Proposition 1.2.1. *(Characterisation of Jordan measurability). Let $E \in \mathbb{R}^d$ be bounded. Then the following are equivalent: (TFAE):*
(1). $E \in \mathcal{J}(\mathbb{R}^d)$.
(2). $\forall \epsilon > 0$, $\exists A, B \in \mathcal{E}(\mathbb{R}^d)$ with $A \subset E \subset B$, such that $m(B \setminus A) < \epsilon$.
(3). $\forall \epsilon > 0$, $\exists A \in \mathcal{E}(\mathbb{R}^d)$ such that $m^J(A \Delta E) < \epsilon$.

Proof. (1) $\Rightarrow$ (2): Since E is Jordan measurable, $m_J(E) = m^J(E) = m(E)$. Then, $m(E) = \sup_{A \subset E,\ A\ elementary} m(A) = \inf_{E \subset B,\ B\ elementary} m(B)$. By definition, $\forall \epsilon > 0$, $\exists A' \subset E \subset B'$ such that $m(A') \geq m(E) - \frac{\epsilon}{2}$ and $m(B') \leq m(E) + \frac{\epsilon}{2}$. Since $A' \subset E \subset B'$, by finite additivity we have $m(B) = m(B' \cup A') = m((B' \setminus A') \cup A') = m(B' \setminus A') + m(A')$. Thus $m(B' \setminus A') = m(B') - m(A')$. By applying the two inequality we just got, $m(B' \setminus A') \leq \epsilon$.

(1) $\Rightarrow$ (3). Since E is Jordan measurable, $m(E) = m^J(E) = \inf_{E \subset A, A\ elementary} m(A)$. Then $\forall \epsilon > 0$, $\exists A'$ such that $E \subset A'$ and $m(A') \leq m(E) + \epsilon$. Since $E \subset A'$, we have $m(A' \setminus E) = m(A') - m(E) \leq \epsilon$. Also we have $m(E \setminus A') = m(\emptyset) = 0$.
We also have $m^J(A' \Delta E) = \inf_{A' \Delta E \subset B, B\ elementary} m(B)$. Take $B = A' \setminus E$ that we just found above. Clearly we have $A' \Delta E \subset A' \setminus E$. Then by the definition of infimum, $m^J(A' \Delta E) \leq m(A' \setminus E) \leq \epsilon$. Thus we found an elementary set A' such that $m^J(A \Delta E) \leq \epsilon$.

(2) $\Rightarrow$ (1): Let $\epsilon > 0$ be arbitrary. Then, by (2), $\exists A, B \subset \mathcal{E}(\mathbb{R}^d)$ and $A \subset E \subset B$, such that $m(B \setminus A) \leq \epsilon$. Since $m(B \setminus A) = m(B) - m(A)$, we have $m(B) \leq m(A) + \epsilon$. Since $\epsilon \geq 0$ is arbitrary, there exists A, B that satisfies all the above conditions and $m(B) \leq m(A)$.
Since A is elementary and $m(A)$ is elementary measure, we have $m(A)$ less than its least upper bound, that is,

$$m(A) \leq \sup_{A' \subset E, A'\ elementary} m(A')$$

Also, since B is elementary and $m(B)$ is elementary measure, we have $m(B)$ less than its least upper bound, that is,

$$m(B) \geq \inf_{E \subset B', B'\ elementary} m(B')$$

Combining $m(B) \leq m(A)$, we have

$$\inf_{E \subset B', B' \ elementary} m(B') \leq \sup_{A' \subset E, A' \ elementary} m(A')$$

Since for all elementary sets A' and B' such that $A' \subset E \subset B'$, by monotonicity of elementary measure, $m(A') \leq m(B')$. Then by the definition of supremum and infimum,

$$\sup_{A' \subset E, A' \ elementary} m(A') \leq \inf_{E \subset B', B' \ elementary} m(B')$$

Then we have

$$\sup_{A' \subset E, A' \ elementary} m(A') = \inf_{E \subset B', B' \ elementary} m(B')$$

And therefore E is Jordan measurable.

(3) $\Rightarrow$ (2): We know that $\forall \epsilon > 0$, $\exists A \in \mathcal{E}(\mathbb{R}^d)$ such that

$$m^J(A \Delta E) = \inf_{A \Delta E \subset C, C \ elementary} m(C) \leq \epsilon$$

Notice that, by subdividing and regrouping the almost disjoint boxes that consists of the elementary sets we just encountered, C can be written as $C = D \setminus F$ where $F, D \in \mathcal{E}(\mathbb{R}^d)$, $F \subset E \subset D$, $F \subset A \subset D$. To see that $A \Delta E \subset C$ still holds, note that $A \Delta E = (A \setminus E) \cup (E \setminus A) \subset (D \setminus E) \cup (E \setminus F) \subset (D \setminus F) \cup (D \setminus F) = D \setminus F = C$.
From the definition of infimum and the inequality above, $\exists B' \in \mathcal{E}(\mathbb{R}^d)$ such that $m(B') < m^J(A \Delta E) + \epsilon \leq 2\epsilon$. Also we have known that B' can be written as $B' = D' \setminus F'$ where $F', D' \in \mathcal{E}(\mathbb{R}^d)$, $F' \subset E \subset D'$, $F' \subset A \subset D'$. Therefore we have proved that (3) $\Rightarrow$ (2).

Since (1) $\Rightarrow$ (2), (2) $\Rightarrow$ (1), (1) $\Rightarrow$ (3), (3) $\Rightarrow$ (2), we have proved that these are equivalent. $\square$

Theorem 1.2.1. *(Regions under graphs are Jordan measurable). Let B be a closed box in $\mathbb{R}^d$, and let $f : B \to \mathbb{R}$ be a continuous function.*

1. *The graph $\{(x, f(x)) : x \in B\} \subset \mathbb{R}^{d+1}$ is Jordan measurable in $\mathbb{R}^{d+1}$ with Jordan measure zero. (Hint: on a compact metric space, continuous functions are uniformly continuous.)*
2. *The set $\{(x, t) : x \in B; 0 \leq t \leq f(x)\} \subset \mathbb{R}^{d+1}$ is Jordan measurable.*

Proof. We prove 1 first and then 2.

(1). Let $\epsilon > 0$ be arbitrary. Since f is continuous on the closed box B on a compact metric space, it is bounded and uniformly continuous. $\exists \delta > 0$, such that, $|x - c| \leq \delta \Rightarrow |f(x) - f(c)| \leq \epsilon$ for all $c, x \in B$.
Evenly subdivide B into n almost disjoint boxes $B = \bigcup_{i=1}^n B_i$, so that within each the euclidean distance between two points is less than δ, that is, $|x_i - c_i| \leq \delta$ for $x_i, c_i \in B_i$. Then we have $|f(x_i) - f(c_i)| \leq \epsilon$. Then, $|B_i| = \frac{|B|}{n}$. Then, within each box B_i,

$$\{(x, f(x)) | x \in B_i\} \subset B_i \times [\min_{x \in B_i} f(x), \max_{x \in B_i} f(x)]$$

$$|B_i \times [\min_{x\in B_i} f(x), \max_{x\in B_i} f(x)]| \leq \frac{|B|}{n}\epsilon$$

Then,

$$\begin{aligned}
\{(x, f(x))|x \in B\} &= \bigcup_{i=1}^{n} \{(x, f(x))|x \in B_i\} \\
&\subset \bigcup_{i=1}^{n} B_i \times [\min_{x\in B_i} f(x), \max_{x\in B_i} f(x)]
\end{aligned}$$

Since all of those $\mathbb{R}^{d+1}$ boxes are disjoint,

$$\begin{aligned}
m^J(\{(x, f(x))|x \in B\}) &= \inf m(\bigcup_{i=1}^{n} B_i \times [\min_{x\in B_i} f(x), \max_{x\in B_i} f(x)]) \\
&\leq m(\bigcup_{i=1}^{n} B_i \times [\min_{x\in B_i} f(x), \max_{x\in B_i} f(x)]) \\
&= \sum_{i=1}^{n} m(B_i \times [\min_{x\in B_i} f(x), \max_{x\in B_i} f(x)]) \\
&= \sum_{i=1}^{n} |B_i \times [\min_{x\in B_i} f(x), \max_{x\in B_i} f(x)]| \\
&= |B|\epsilon
\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, we have $m^J(\{(x, f(x))|x \in B\}) = 0$.
Also since

$$\begin{aligned}
m_J(\{(x, f(x))|x \in B\}) &= \sup_{A\in\{(x,f(x))|x\in B\},A \ elementary} m(A) \\
&= m(\emptyset) \\
&= 0
\end{aligned} \tag{3}$$

We have that the set is Jordan measurable with measure zero.

(2). In the proof of this, we will use the conclusion from **Exercise 1.1.5**.
Let $\epsilon > 0$ be arbitrary. Since f is continuous on the closed box B on a compact metric space, it is bounded and uniformly continuous. $\exists\delta > 0$, such that, $|x - c| \leq \delta \Rightarrow |f(x) - f(c)| \leq \epsilon$ for all $x, c \in B$.
Evenly subdivide B into n almost disjoint boxes $B = \bigcup_{i=1}^{n} B_i$, so that within each the euclidean distance between two points is less than δ, that is, $|x_i - c_i| \leq \delta$ for $x_i, c_i \in B_i$. Then we have $|f(x_i) - f(c_i)| \leq \epsilon$. Then, $|B_i| = \frac{|B|}{n}$.
Let A, B be two elementary sets such that

$$A = \bigcup_{i=1}^{n} B_i \times [0, \min_{x\in B_i} f(x)]$$

$$C = \bigcup_{i=1}^{n} B_i \times [0, \max_{x\in B_i} f(x)]$$

Clearly, sub-boxes of A are almost disjoint, sub-boxes of C are almost disjoint, and $A \subset \{(x,t)|x \in B, 0 \le t \le f(x)\} \subset C$. Then,

$$A \setminus C = \bigcup_{i=1}^{n} B_i \times [\min_{x \in B_i} f(x), \max_{x \in B_i} f(x)]$$

By finite additivity, we have

$$\begin{aligned} m(A \setminus C) &= m(\bigcup_{i=1}^{n} B_i \times [\min_{x \in B_i} f(x), \max_{x \in B_i} f(x)]) \\ &= \sum_{i=1}^{n} m(B_i \times [\min_{x \in B_i} f(x), \max_{x \in B_i} f(x)]) \\ &= \sum_{i=1}^{n} |B_i|| \max_{x \in B_i} f(x) - \min_{x \in B_i} f(x)| \\ &\le \sum_{i=1}^{n} \frac{|B|}{n}\epsilon \\ &= |B|\epsilon \end{aligned}$$

Since $\epsilon > 0$ arbitrary, $A \subset \{(x,t)|x \in B, 0 \le t \le f(x)\} \subset C$, we have $\{(x,t)|x \in B, 0 \le t \le f(x)\}$ to be Jordan measurable. □

Proposition 1.2.2. *Let E, $F \in \mathcal{J}(\mathbb{R}^d)$, then,*
(1). $E \cup F$, $E \cap F$, $E \setminus F$, $E \Delta F \in \mathcal{J}(\mathbb{R}^d)$.
(2). $m(E) \ge 0$.
(3). Finite Additivity: If $E \cap F = \emptyset$, then $m(E \cup F) = m(E) + m(F)$.
(4). Monotonicity: If $E \in F$, then $m(E) \le m(F)$.
(5). Finite subadditivity: $m(E \cup F) \le m(E) + m(F)$.
(6). Translation Invariabce: $m(E + x) = m(E)$.

Proof. $E \cap F \in \mathcal{J}(\mathbb{R}^d)$:
Using (3) from Proposition 2, let $A, B, C, D \in \mathcal{E}(\mathbb{R}^d)$ with $A \in E \in B$, $C \in E \in D$, such that $m(B \setminus A) < \epsilon$, $m(D \setminus C) < \epsilon$. Then we claim that $m((B \cap D)\ (A \cap C)) \le 2\epsilon$.

$$\begin{aligned} (B \cap D) \setminus (A \cap C) &= (B \cap D) \cap (A \cap C)^C \\ &= B \cap D \cap (A^C \cup C^C) \\ &= (B \cap D \cap A^C) \cup (B \cap D \cap C^C) \\ &\subset (B \cap A^C) \cup (D \cap C^C) \\ &= (B \setminus A) \cup (D \setminus C) \end{aligned}$$

Then, $m((B \cap D) \setminus (A \cap C)) \le m((B \setminus A) \cup (D \setminus C)) \le 2\epsilon$.
Since $A \cap C \subset E \cap F \subset B \cap D$, we have $E \cap F$ is also Jordan measurable. □

Theorem 1.2.2. *(Closure, interior, and topolitical boundary). Let $E \subset \mathbb{R}^d$ be a bounded set.*

1. *E and the closure $\overline{E}$ of E have the same Jordan outer measure.*

2. *E and the interior E° of E have the same Jordan inner measure.*

3. *E is Jordan measurable if and only if the topological boundary ∂E of E has Jordan outer measure zero.*

4. *The bullet-riddled square $[0,1]^2 \setminus \mathbb{Q}^2$, and set of bullets $[0,1]^2 \cap \mathbb{Q}^2$, both have Jordan inner measure zero and Jordan outer measure one. In particular, both sets are not Jordan measurable.*

Proof. (1). First, $E \subseteq \overline{E}$, which means that $m^J(E) \le m^J(\overline{E})$. Thus we only need to prove that $m^J(E) \ge m^J(\overline{E})$. Since

$$m^J(E) = \inf_{E \subset B, B \text{ elementary}} m(B)$$

For $\epsilon > 0$ there exists an elementary set B that covers E and write it as a finite union of almost disjoint boxes $B = \bigcup_{i=1}^n B_i$, such that

$$\sum_{i=1}^n m(B_i) \le m^J(E) + \epsilon$$

Since $\overline{E} \subseteq \overline{\bigcup_{i=1}^n B_i} \subseteq \bigcup_{i=1}^n \overline{B_i}$, we have $m^J(\overline{E}) \le \sum_{i=1}^n m(\overline{B_i}) = \sum_{i=1}^n m(B_i) \le m^J(E) + \epsilon$. Since $\epsilon > 0$ is arbitrary, we have $m^J(E) \le m^J(\overline{E})$.

(2). First, $E^o \subseteq E$, which means that $m_J(E^o) \le m_J(E)$. Thus we only need to prove that $m_J(E^o) \ge m_J(E)$. Since

$$m_J(E) = \sup_{B \subset E, B \text{ elementary}} m(B)$$

For $\epsilon > 0$ there exists an elementary set B that is covered by E and write it as a finite union of almost disjoint boxes $B = \bigcup_{i=1}^n B_i$, such that

$$\sum_{i=1}^n m(B_i) \ge m^J(E) - \epsilon$$

Since $\bigcup_{i=1}^n B_i^o \subseteq (\bigcup_{i=1}^n B_i)^o \subseteq E^o$, we have $m_J(E^o) \ge \sum_{i=1}^n m(B_i^o) = \sum_{i=1}^n m(B_i) \ge m_J(E) - \epsilon$. Since $\epsilon > 0$ is arbitrary, we have $m_J(E) \le m_J(E^o)$.

(3).

Notice that $\overline{E} \setminus E \subset \overline{E} \setminus E^o = \partial E$. Since $m^J(\partial E) = 0$, we have $m^J(\overline{E} \setminus E) = 0$. By definition

$$m^J(\overline{E} \setminus E) = \inf_{\overline{E} \setminus E \subset B, B \text{ elementary}} m(B)$$

we have $\forall \epsilon > 0$, $\exists B'$ elementary, $\overline{E} \setminus E \subset B'$ and $m^J(B') \le \epsilon$.
Notice that

$$E \subset (\overline{E} \setminus E) \cup E \subset B' \cup E$$

and that $B' \cup E$ is an elementary set. Then,

$$\begin{aligned} m^J((B' \cup E)\Delta E) &= m^J(((B' \cup E) \setminus E) \cup (E \setminus (B' \cup E))) \\ &= m^J((B' \cup E) \setminus E) \\ &= m^J(B') \\ &\leq \epsilon \end{aligned}$$

Therefore, we found an elementary set $B' \cup E$ such that its symmetric difference with E has Jordan outer measure less or equal than ϵ. Then, E is Jordan measurable.

(4). We use the property that the rational numbers are dense in $\mathbb{R}$, thus the rational pairs are also dense in $\mathbb{R}^2$.
For $A = [0,1]^2 \setminus \mathcal{Q}^2$, we have $m_J(A) = \sup_{B \subset A, B\ elementary} m(B)$. But since the rational pairs are dense, every non-empty elementary set $B \in [0,1]^2$ contains rational pairs and thus is not contained in A. Thus, $m_J(A) = m(\emptyset) = 0$. $m^J(A) = \inf_{A \subset C, C\ elementary} m(C)$. This value takes infimum when $C = [0,1]^2$, so $m^J(A) = 1$.
For $D = [0,1]^2 \cap \mathcal{Q}^2$, we have we have $m_J(D) = \sup_{E \subset A, E\ elementary} m(E)$. But since $\mathbb{R}^2$ without rational pairs are also dense, every non-empty elementary set $E \in [0,1]^2$ contains non-rational pairs and thus is not contained in D. Thus, $m_J(D) = m(\emptyset) = 0$. $m^J(D) = \inf_{D \subset F, F\ elementary} m(F)$. This value takes infimum when $F = [0,1]^2$, so $m^J(D) = 1$.

□

Theorem 1.2.3. *(Equivalence of Riemann integral and Darboux integral). Let $[a,b]$ be an interval, and $f : [a,b] \to \mathbb{R}$ be a bounded function. Then f is Riemann integrable if and only if it is Darboux integrable, in which case the Riemann integral and Darboux integrals are equal.*

Proof. Denote the set

$$E = \{(x,t) | x \in I, 0 \leq t \leq f(x)\}$$

First we prove that f is *Darboux Integrable* implies that the set E is Jordan measurable. From definition we know that

$$\overline{\int_a^b} f(x)dx = \inf_{E \subset B', B'\ elementary} m(B')$$

Where B' is a collection of almost disjoint boxes partitioned by almost disjoin intervals $I = \bigcup_{i=1}^n I_i$.

$$B' = \bigcup_{i=1}^n \{(x_i,t) | x_i \in I_i, 0 \leq t \leq h(x_i), h(x_i) = d_i \geq f(x_i)\ \forall x_i \in I_i\}$$

From this we know that $E \subset B'$. Also, $\forall \epsilon > 0$, $\exists B$ satisfying the conditions above such that

$$m(B) \leq \overline{\int_a^b} f(x)dx + \epsilon$$

From the definition we also know that

$$\underline{\int_a^b} f(x)dx = \sup_{A' \subset E, A' \ elementary} m(A')$$

Where A' is a collection of almost disjoint boxes partitioned by almost disjoin intervals $I = \bigcup_{i=1}^m I_i'$.

$$A' = \bigcup_{i=1}^{m} \{(x_i, t) | x_i \in I_i', 0 \leq t \leq g(x_i), g(x_i) = c_i \leq f(x_i) \ \forall x_i \in I_i'\}$$

From this we know that $A' \subset E$. Also, $\forall \epsilon > 0$, $\exists A$ satisfying the conditions above such that

$$m(A) \geq \underline{\int_a^b} f(x)dx - \epsilon$$

Clearly, $A \subset E \subset B$. Since f is Darboux integrable, we have

$$\begin{aligned} m(B \setminus A) &= m(B) - m(A) \\ &= (\overline{\int_a^b} f(x)dx + \epsilon) - (\underline{\int_a^b} f(x)dx - \epsilon) \\ &= 2\epsilon \end{aligned}$$

Thus, we have shown that *Darboux Integrable* means that (2) in **Exercise 1.1.5** has been satisfied. Thus, E is Jordan measurable. To deal with *Riemann Integrable*, we first collect the intervals that we just partitioned above $\{I_i\}_{i=1}^n$ and $\{I_i'\}_{i=1}^m$. Takes the endpoints of those intervals, order them and get a new (finer) subdivision of the interval $I = \bigcup_{i=1}^{n+m-3} I_i''$ that consists of $n + m - 3$ almost disjoint sub-intervals (this is clear to see). Then, take the set

$$F = \bigcup_{i=1}^{n+m-3} \{(x_i, t) | x_i \in I_i'', 0 \leq t \leq f(x_i^*) \ for \ some \ x_i^* \in I_i''\}$$

From its construction, it is clear to see that $A \subset F \subset B$. Then, $(E \setminus F) \subset (B \setminus A)$, $(F \setminus E) \subset (B \setminus A)$. Then, for $\epsilon > 0$, we have found an elementary set F such that $m^J(E \Delta F) \leq m(B \setminus A) \leq 2\epsilon$ by definition of Jordan outer measure and the monotonicity of elementary measure. Thus E is Jordan measurable

Notice that the Riemann sum of f on $I = [a, b]$ is just the elementary measure of F with partitions $\mathcal{P} : I = \bigcup_{i=1}^{n+m-3} I_i''$, and it equals its outer Jordan measure:

$$m^J(F) = m(F) = \sum_{i=1}^{n+m-3} f(x_i^*)|I_i''| = \mathcal{R}(f, \mathcal{P})$$

$$
\begin{aligned}
|m^J(F) - m^J(E)| &= |m^J(F) + m^J(E \setminus F) - (m^J(E \setminus F) + m^J(E))| \\
&= |m^J(F \cup E) - (m^J(E) + m^J(F \setminus E) - m^J(F \setminus E) + m^J(E \setminus F))| \\
&= |m^J(F \cup E) - (m^J(F \cup E) - m^J(F \setminus E) + m^J(E \setminus F))| \\
&= |m^J(F \setminus E) - m^J(E \setminus F)| \\
&\leq m^J(F \setminus E) + m^J(E \setminus F) \\
&= m^J(E \Delta F)
\end{aligned}
$$

Since we just proved that for $\epsilon > 0$ we can always find a set F such that $m^J(E\Delta F) \leq 2\epsilon$, and we established that $m^J(F) = \mathcal{R}(f, \mathcal{P})$, using the above inequality, we have $|\mathcal{R}(f, \mathcal{P}) - m^J(E)| \leq \epsilon$. Then, f is Riemann integrable when the set E is Jordan measurable.
Thus, we established the equivalence among Jordan measurability, Darboux integrability, and Riemann integrability. $\square$

There are sets that are not Jordan measurable. For example, $E = [0, 1] \cap \mathcal{Q}$ is not Jordan measurable. This is because E contains no open interval, any elementary set $A \in E$ can only be a finite union of singletons, therefore, $m_J(E) = 0$. However, we can prove that $m_J(E) = 1$.
Similarly, there exists open and bounded sets that are not Jordan measurable, and there exists compact sets that are not Jordan measurable. So, we need Lebegue measure.

1.3 Lebesgue Measure

Definition 1.3.1. *The lebesgue outer measure (exterior measure) of $E \in \mathbb{R}^d$ is*

$$m^*(E) := \inf\{\sum_{i=1}^{\infty} |B_i| : B_1, B_2, ...boxes, E \subset \bigcup_{i=1}^{\infty} B_i\}$$

Note that $m^*(E) \leq m^J(E)$ since it is the infimum over a bigger set.
For $E = [0,1] \cap \mathcal{Q} = \{q_1, q_2, ...\}$, by taking boxes $B_i = \{q_i\}$, we get the Lebesgue measure $m^*(E) \leq \sum_{i=1}^{\infty} |B_i| = \sum_{i=1}^{\infty} 0 = 0$.

Similarly, $m^*(E) = 0$ for any countable E, just take $\{x_i\}$ as boxes. Or, take boxes $B_i = (q_i - \frac{\epsilon}{2^i}, q_i + \frac{\epsilon}{2^i})$, then $m^*(E) \leq m(\bigcup_{i=1}^{\infty} B_i) \leq \sum_{i=1}^{\infty} |B_i| = 2\epsilon$. Since ϵ is arbitrary, we have $m^*(E) = 0$.

Definition 1.3.2. *(Lebesgue Measurability). A set $E \in \mathbb{R}^d$ is Lebesgue measurable if $\forall \epsilon > 0$, $\exists$ an open set $U \in \mathbb{R}^d$, $E \subseteq U$, such that $m^*(U \setminus E) < \epsilon$.*

We denote the class of all Lebesgue measurable sets by $\mathcal{L}(\mathbb{R}^d)$. for $E \in \mathcal{L}(\mathbb{R}^d)$, its Lebesgue measure is $m(E) := m^*(E)$. Now we gives the properties of the Lebesgue outer measure (the outer measure axioms).

1.3.1 Properties of Lebesgue Outer Measure

Proposition 1.3.1. *(The outer measure axioms).*

1. $m^*(\emptyset) = 0$.
2. *Monotonicity: if $E \subset F \subseteq \mathbb{R}^d$, then $m^*(E) \leq m^*(F)$.*
3. *σ-subadditivity: If $E_1, E_2, ... \subset \mathbb{R}^d$ is a countable sequence of sets, then, $m^*(\bigcup_{n=1}^{\infty} E_i) \leq \sum_{i=1}^{\infty} m^*(E_i)$.*

Proof. (3).
We know that,

$$m^*(E_i) = \inf\{\sum_{j=1}^{\infty} |B_j^i| : B_1^i, B_2^i, ...boxes, E_i \subset \bigcup_{j=1}^{\infty} B_j^i\}$$

Then, for each i, there exists $B_1^i, B_2^i, ...$ to be boxes, such that $E_i \subset \bigcup_{j=1}^{\infty} B_j^i$ and

$$\sum_{j=1}^{\infty} |B_j^i| \leq m^*(E_i) + \frac{\epsilon}{2^i}$$

Since $E_i \subset \bigcup_{j=1}^{\infty} B_j^i$, we have $\bigcup_{i=1}^{\infty} E_i \subset \bigcup_{i=1}^{\infty} \bigcup_{j=1}^{\infty} B_j^i$, then we have

$$\begin{aligned} m^*(\bigcup_{i=1}^{\infty} E_i) &\leq m^*(\bigcup_{i=1}^{\infty} \bigcup_{j=1}^{\infty} B_j^i) \text{ by monotonicity} \\ &\leq \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |B_j^i| \text{ by definition of Lebesgue measure: infimum} \\ &= \sum_{i=1}^{\infty} (\sum_{j=1}^{\infty} |B_j^i|) \text{ by Tonelli's Theorem for series} \\ &= \sum_{i=1}^{\infty} (m^*(E_i) + \frac{\epsilon}{2^i}) \\ &= \sum_{i=1}^{\infty} m^*(E_i) + \epsilon \end{aligned}$$

Since ϵ is arbitrary, we then have $m^*(\bigcup_{n=1}^{\infty} E_i) \leq \sum_{i=1}^{\infty} m^*(E_i)$.
Also, (3) and (1) together give the finite subadditivity property by just let $\emptyset = E_{k+1} = E_{k+2} = ...$, that is, $m^*(\bigcup_{i=1}^{k} E_i) \leq \sum_{i=1}^{k} m^*(E_i)$. □

Theorem 1.3.1. *(Distance of sets). Let $E, F \subset \mathbb{R}^d$ be disjoint closed sets, with at least one of E, F being compact. Then $dist(E, F) > 0$.*

Proof. Suppose, on the contrary, $d(E, F) = 0$. That is, $\forall \epsilon > 0$, $\exists x \in E$ and $y \in F$, such that $d(x, y) < \epsilon$.
Then we can construct sequences by axiom of countable choice, $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ such that $\forall n in \mathbb{N}$, $d(x_n, y_n) \leq \frac{1}{n}$.
Suppose E is compact, then by Bolzano-Weierstrass Theorem, $\{x_n\}_{n\in\mathbb{N}}$ has a convergent subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ that converges to x_0. Since E is closed, $x_0 \in E$. Then $\forall \epsilon > 0$, $\exists k \in \mathbb{N}$ such that $d(x_0, x_{n_k}) \leq \frac{1}{n_k} \leq \frac{\epsilon}{2}$. Also, by the previous construction we know that $d(x_{n_k}, y_{n_k}) \leq \frac{\epsilon}{2}$. Then,

$$d(x_0, y_{n_k}) \leq d(x_0, x_{n_k}) + d(x_{n_k}, y_{n_k}) < \epsilon$$

which means that $y_{n_k} \to x_0$. Also by closedness of F, $x_0 \in F$.
Contradiction to $E \cap F = \emptyset$. Therefore $d(E, F) > 0$.

□

Lemma 1.3.1. *(Finite additivity for separated sets). Let $E, F \subset \mathbb{R}^d$ be such that $dist(E, F) := \inf\{|x - y| : x \in E, y \in F\} > 0$, then $m^*(E \cup F) = m^*(E) + m^*(F)$.*

Proof. First, we prove $\leq$. This is natual from σ-additivity: $m^*(E \cup F) \leq m^*(E) + m^*(F)$.
Next, we prove $\geq$. Without loss of generality, assume that $m^*(E \cup F) < +\infty$.
Let $\epsilon > 0$ be arbitrary, then, by definition there exists a countable collection of boxes $B_1, B_2, ...$ such that,

$$E \cup F \subset \bigcup_{i=1}^{\infty} B_i$$

$$\sum_{i=1}^{\infty} |B_i| \leq m^*(E) + \epsilon$$

Fix $\delta \in (0, dist(E,F))$. By subdividing these boxes into finer boxes B'_i, we may assume that $diam(B'_i) < \delta$. Then, some of these boxes have intersection with E while others have intersection with F.
Let $I = \{i : B'_i \cap E \neq \emptyset\}$, $J = \{j : B'_j \cap F \neq \emptyset\}$. Then $B'_i \cap B'_j = \emptyset$ cause otherwise we would have a box with diameter bigger than δ.
Then, $m^*(E) \leq \sum_{i \in I} |B'_i|$, $m^*(F) \leq \sum_{j \in J} |B'_j|$.

$$\begin{aligned} m^*(E) + m^*(F) &\leq \sum_{i \in I \cup J} |B'_i| \\ &\leq \sum_{i=1}^{\infty} |B_i| \\ &\leq m^*(E \cup F) + \epsilon \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, we have $m^*(E) + m^*(F) \leq m^*(E \cup F)$.
Now that we have $\leq$ and $\geq$, we have $=$. □

Lemma 1.3.2. *(Outer measurability for elementary sets). Let $E \in \mathcal{E}(\mathbb{R}^d)$, then $m^*(E) = m(E)$, the elementary measure.*

Proof. First we prove $\leq$. We already know that $m^*(E) \leq m^J(E) = m(E)$, thus $\leq$ holds.
Next we prove $\geq$.
Consider first the case where E is closed, then, E is compact. Then we can use the Heine-Borel Theorem which states that any covering of a compact set by a collection of open sets contains a finite subcovering.
Take a covering of E by boxes: $E \subset \bigcup_{i=1}^{\infty} B_i$ such that $\sum_{i=1}^{\infty} |B_i| \leq m^*(E) + \epsilon$. For each box B_i, find an open box B'_i such that $B_i \subset B'_i$ and $|B'_i| \leq |B_i| + \frac{\epsilon}{2^i}$.
Then, $\sum_{i=1}^{\infty} |B'_i| \leq \sum_{i=1}^{\infty} |B_i| + \epsilon \leq m^*(E) + 2\epsilon$.
Using Heine-Borel Theory, there is a finite N such that $E \subset \bigcup_{i=1}^{N} B'_i$. Then,

$$\begin{aligned} m(E) &\leq \sum_{i=1}^{N} |B_i| \\ &\leq \sum_{i=1}^{\infty} |B_i| \\ &\leq m^*(E) + 2\epsilon \end{aligned}$$

Since $\epsilon > 0$ arbitrary, we have $\geq$.
Now consider the case where E is not closed. Then, write E as a finite union of disjoint boxes $E = \bigcup_{i=1}^{k} Q_i$, which need not be closed.
Let $\epsilon > 0$ be arbitrary, and for each $j \in \{1, ..., k\}$, find a closed sub-box $Q'_j \subset Q_j$ such that $|Q'_j| \geq |Q_j| - \frac{\epsilon}{k}$. Then, by the previous discussion and finite additivity of elementary

measure, we have

$$\begin{aligned}
m^*(\bigcup_{j=1}^{k} Q'_j) &= m(\bigcup_{j=1}^{k} Q'_j) \\
&= \sum_{j=1}^{k} m(Q'_j) \\
&\geq \sum_{j=1}^{k} m(Q_j) - \epsilon \\
&= m(E) - \epsilon
\end{aligned}$$

Also, $\bigcup_{j=1}^{k} Q'_j \subset E$, so by monotonicity, we have

$$\begin{aligned}
m^*(E) &\geq m^*(\bigcup_{j=1}^{k} Q'_j) \\
&\geq m(E) - \epsilon
\end{aligned}$$

Then $m(E) \leq m^*(E) + \epsilon$. Since $\epsilon > 0$ arbitrary, $\geq$ holds. □

Lemma 1.3.3. *(Outer measure of countable unions of almost disjoint boxes). Let* $E = \bigcup_{i=1}^{\infty} B_i$ *be a countable union of almost disjoint boxes, then* $m^*(E) = \sum_{i=1}^{\infty} |B_i|$. *Almost disjoint means that* $B_i^o \cap B_j^o = \emptyset \ \forall i \neq j$ *(topological interior doesn't intersect).*

Proof. From countable sub-additivity and Lemma 1.3.2.,

$$m^*(E) \leq \sum_{i=1}^{\infty} m^*(B_i) = \sum_{i=1}^{\infty} |B_i|$$

Therefore, it suffices to show that

$$m^*(E) \geq \sum_{i=1}^{\infty} |B_i|$$

Notice that for each $N \in \mathbb{N}$,

$$E \supset \bigcup_{i=1}^{N} B_i$$

Then,

$$\begin{aligned}
m^*(E) &\geq m^*(\bigcup_{i=1}^{N} B_i) \\
&= m(\bigcup_{i=1}^{N} B_i) \\
&= \sum_{i=1}^{N} |B_i|
\end{aligned}$$

Let $N \to \infty$, we have $m^*(E) \geq \sum_{i=1}^{\infty} |B_i|$ Therefore we conclude the proof. □

From this lemma we have a corollary.

Corollary 1.3.1. *If* $E = \bigcup_{i=1}^{\infty} B_i = \bigcup_{i=1}^{\infty} B_i'$, $(B_i)_{i\in\mathbb{N}}$ *and* $(B_i')_{i\in\mathbb{N}}$ *are almost disjoint boxes, then* $\sum_{i=1}^{\infty} |B_i| = \sum_{i=1}^{\infty} |B_i'|$.

Lemma 1.3.4. *An open set* $U \subseteq \mathbb{R}^d$ *is the countable union of almost disjoint boxes. (in fact, the countable union of almost disjoint closed boxes).*

Proof. For $n \in \mathbb{Z}$, let $\mathcal{Q}_n$ be the collection of all closed cubes of the form

$$[\frac{k_1}{2^n}, \frac{k_1+1}{2^n}] \times ... \times [\frac{k_d}{2^n}, \frac{k_d+1}{2^n}] \text{ for some integers } k_1, ..., k_d$$

Define $\mathcal{Q}_{\geq 0} := \bigcup_{n=1}^{\infty} \mathcal{Q}_n$ to be the union of all dyadic cubes of side length ≤ 1. Notice that $\mathcal{Q}_{\geq 0}$ has a tree structure, that is, for each $Q \in \mathcal{Q}_n$, $\exists! Q' \in \mathcal{Q}_{n-1}$ such that $Q \subset Q'$.
Given these, we have the dyadic nesting property: $\forall Q_1, Q_2 \in \mathcal{Q}_{\geq 0}$ with $Q_1^o \cap Q_2^o \neq \emptyset$, either $Q_1 \subseteq Q_2$ or $Q_2 \subseteq Q_1$.
Since U is open, $\forall x \in U$, $\exists$ open ball $B(x,r) \subset U$. Therefore, $\exists$ closed $Q \in \mathcal{Q}_{\geq 0}$ such that $x \in Q \subseteq E$. Then, let $Q_U = \{Q \subset \mathcal{Q}_{\geq 0} : Q \subseteq U\}$. Then,

$$U = \bigcup_{Q \in \mathcal{Q}_U} Q \text{ with } \mathcal{Q}_U \text{ being countable}$$

To get almost disjoint subcollection, take $\mathcal{Q}_U^* \subseteq \mathcal{Q}_U$ to be a subcollection of maximal elements with respect to set inclusion, which means that they are not contained in any other cube in $\mathcal{Q}_U$.

$$\mathcal{Q}_U^* := \{Q \in \mathcal{Q}_{\geq 0} : Q \subseteq U, Q' \nsubseteq U \text{ for any } Q' \in \mathcal{Q}_{\geq 0} \text{ and } Q' \supset Q\}$$

First we see that if $Q \subseteq U$ then $Q \subseteq \mathcal{Q}_U$, then $\mathcal{Q}_U \subseteq \mathcal{Q}_U^*$. Together with the definition of $\mathcal{Q}_U^*$, we see that $\mathcal{Q}_U^* = \mathcal{Q}_U$. Second, by dyadic nesting property, every cube in $\mathcal{Q}$ is contained in exactly one maximal cube in $\mathcal{Q}^*$, and that any two such maximal cubes in $\mathcal{Q}^*$ are almost disjoint. Thus, $U = \bigcup_{Q \in \mathcal{Q}^8} Q$ are almost disjoint, and also countable. ☐

Lemma 1.3.5. *(Outer regularity). For any* $E \subseteq \mathbb{R}^d$,

$$m^*(E) = \inf_{E \subset U, U \text{ open}} m^*(U)$$

Proof. ($\leq$): it is easy to see from monotonicity that $\forall U \supset E$, $m^*(E) \leq m*(U)$, thus

$$m^*(E) \leq \inf_{E \subset U, U \text{ open}} m^*(U)$$

Therefore it suffices to prove that

$$m^*(E) \geq \inf_{E \subset U, U \text{ open}} m^*(U)$$

By definition of the outer Lebesgue measure,

$$m^*(E) = \inf\{\sum_{i=1}^{\infty} |B_i| : E \subset \bigcup_{i=1}^{\infty} B_i, B_1, B_2, ...boxes\}$$

Then, $\forall \epsilon > 0$, $\exists B'_1, B'_2 ...$ such that

$$\sum_{i=1}^{\infty} |B'_i| \leq m^*(E) + \epsilon$$

Enlarge each box B'_i to be an open box $B'_i \subset B''_i$ such that

$$|B''_i| \leq |B'_i| + \frac{\epsilon}{2^i}$$

Thus, $E \subset \bigcup_{i=1}^{\infty} B''_i$ where $\bigcup_{i=1}^{\infty} B''_i$ is open.

$$\begin{aligned}\sum_{i=1}^{\infty} |B''_i| &\leq \sum_{i=1}^{\infty} |B'_i| + \epsilon \\ &= m^*(E) + 2\epsilon\end{aligned}$$

Since $\bigcup_{i=1}^{\infty} B''_i$ is open, by countable sub-additivity and the definition of infimum,

$$\inf_{E \subset U, U \ open} m^*(U) \leq m^*(\bigcup_{i=1}^{\infty} B''_i) \leq \sum_{i=1}^{\infty} |B''_i| \leq m^*(E) + 2\epsilon$$

Since $\epsilon > 0$ is arbitrary, we have

$$m^*(E) \geq \inf_{E \subset U, U \ open} m^*(U)$$

$\square$

1.3.2 Lebesgue Measurability

There are plenty of Lebesgue measurable sets, as we can see from the following proposition.

Proposition 1.3.2. *(Existence of Lebesgue measurable sets). Let $E \subseteq \mathbb{R}^d$, then $E \subset \mathcal{L}(\mathbb{R}^d)$ if*

1. *E is open.*
2. *E is closed.*
3. *E is a null set, i.e. $m^*(E) = 0$.*
4. *$E = \emptyset$.*
5. *if $E \in \mathcal{L}(\mathbb{R}^d)$, then $\mathbb{R}^d \setminus E \in \mathcal{L}(\mathbb{R}^d)$.*
6. *$E = \bigcup_{i=1}^{\infty} E_i \in \mathcal{L}(\mathbb{R}^d)$ where $E_i \in \mathcal{L}(\mathbb{R}^d)$.*
7. *$E = \bigcap_{i=1}^{\infty} E_i \in \mathcal{L}(\mathbb{R}^d)$ where $E_i \in \mathcal{L}(\mathbb{R}^d)$.*

Proof. (1) is immediate from definition. By Lemma 1.3.4., write E as $E = \bigcup_{i=1}^{\infty} B_i$ where B_i are disjoint boxes. Expand each B_i to be an open box $B'_i \supset B_i$ such that $\forall \epsilon > 0$,

$$|B'_i| \leq |B_i| + \frac{\epsilon}{2^i}$$

Then, by σ-additivity and Lemma 1.3.3,

$$m^*(\bigcup_{i=1}^{\infty} B_i') \leq \sum_{i=1}^{\infty} |B_i'| \leq m(E) + \epsilon$$

Therefore

$$m^*(\bigcup_{i=1}^{\infty} B_i' \setminus E) \leq \epsilon$$

Thus, we found an open set $\bigcup_{i=1}^{\infty} B_i' \supset E$, such that $m^*(\bigcup_{i=1}^{\infty} B_i' \setminus E) \leq \epsilon$.
(3) and (4) are immediate.
Since $E_i \in \mathcal{L}(\mathbb{R}^d)$, $\exists$ open set E_i' such that $E_i \subset E_i'$, $m^*(E_i') \leq m^*(E_i) + \frac{\epsilon}{2^i}$. Then, $\bigcup_{i=1}^{\infty} E_i \subset \bigcup_{i=1}^{\infty} E_i'$ where $\bigcup_{i=1}^{\infty} E_i'$ is open. Since

$$\begin{aligned}
\bigcup_{i=1}^{\infty} E_i' \setminus \bigcup_{i=1}^{\infty} E_i &= \bigcup_{i=1}^{\infty} E_i' \cap (\bigcap_{i=1}^{\infty} E_i^C) \\
&= \bigcup_{i=1}^{\infty} (E_i' \cap (\bigcap_{i=1}^{\infty} E_i^C)) \\
&= \bigcup_{i=1}^{\infty} (E_i' \cap (\bigcap_{i=1}^{\infty} E_i^C)) \\
&\subset \bigcup_{i=1}^{\infty} (E_i' \cap E_i^C) \\
&= \bigcup_{i=1}^{\infty} (E_i' \setminus E_i)
\end{aligned}$$

Then, by monotonicity and σ-additivity,

$$\begin{aligned}
m^*(\bigcup_{i=1}^{\infty} E_i' \setminus \bigcup_{i=1}^{\infty} E_i) &\leq m^*(\bigcup_{i=1}^{\infty} (E_i' \setminus E_i)) \\
&\leq \sum_{i=1}^{\infty} m^*(E_i' \setminus E_i) \\
&= \sum_{i=1}^{\infty} m^*(E_i') - m^*(E_i) \\
&\leq \epsilon
\end{aligned} \tag{4}$$

Therefore, $\bigcup_{i=1}^{\infty} E_i \in \mathcal{L}(\mathbb{R}^d)$.
(2). First, we can express each closed set as $E = \bigcup_{i=1}^{\infty}$ for E_n closed and bounded (for example, $E_n = \overline{B(0,n)} \cap E$ for $n = 1, 2, ...$). Then by (6), it suffices to verify the claim when E is closed and bounded, hense compact.
By Lemma 1.3.5., $\exists U \supset E$ open, such that $m^*(U) \leq m^*(E) + \epsilon$. Therefore, it suffices to show that $m^*(U \setminus E) \leq \epsilon$.
If w have finite additivity for m^*, then we have $m^*(U \setminus E) + m^*(E) = m^*(U) \leq m^*(E) + \epsilon$ and then $m^*(U \setminus E) \leq \epsilon$. But we don't have it, so we should instead fo the following.
Since $U \setminus E$ is also open, by Lemma 1.3.4, $U \setminus E = \bigcup_{i=1}^{\infty} Q_i$ where Q_i are almost disjoint closed boxes. Then by Lemma 1.3.3, $m^*(U \setminus E) = \sum_{i=1}^{\infty} |Q_i|$.

We truncate the sum: for any finite $N \in \mathbb{N}$, $\bigcup_{i=1}^N Q_i$ is closed and disjoint from E. From Theorem 1.3.1., since E is compact and $\bigcup_{i=1}^N Q_i$ is closed, we have $dist(E, \bigcup_{i=1}^N Q_i) > 0$. Then by Lemma 1.3.1,

$$\begin{aligned} m^*(\bigcup_{i=1}^N Q_i) + m^*(E) &= m^*(E \cup \bigcup_{i=1}^N Q_i) \\ &\leq m^*(U) \\ &\leq m^*(E) + \epsilon \end{aligned}$$

$$\sum_{i=1}^N |Q_i| = m^*(\bigcup_{i=1}^N Q_i) \leq \epsilon$$

Let $N \to \infty$,

$$m^*(U \setminus E) = \sum_{i=1}^\infty |Q_i| = m^*(\bigcup_{i=1}^\infty Q_i) \leq \epsilon$$

Therefore $E \in \mathcal{L}(\mathbb{R}^d)$.
(5). Since $E \in \mathcal{L}(\mathbb{R}^d)$, for every $n \in \mathbb{N}$, $\exists U_n \supset E$ such that $m^*(U_n \setminus E) < \frac{1}{n}$.
Let $F_n := U_n^C$, then $(\mathbb{R}^d \setminus E) \supset F_n$ for all n. Since

$$(\mathbb{R}^d \setminus E) \setminus F_n = (\mathbb{R}^d \setminus E) \cap F_n^C = (\mathbb{R}^d \setminus E) \cap U_n = U_n \setminus E$$

we have

$$m^*((\mathbb{R}^d \setminus E) \setminus F_n) < \frac{1}{n}$$

Let $F := \bigcup_{i=1}^\infty F_n$, then $(\mathbb{R}^d \setminus E) \supset F$. From monotonicity, we have

$$m^*((\mathbb{R}^d \setminus E) \setminus F) \leq m^*((\mathbb{R}^d \setminus E) \setminus F_n) < \frac{1}{n} \; \forall n \in \mathbb{N}$$

Taking $n \to \infty$, we have $m^*((\mathbb{R}^d \setminus E) \setminus F) = 0$, thus $(\mathbb{R}^d \setminus E) \setminus F$ is a null set, and is Lebesgue measurable. Therefore, $\mathbb{R}^d \setminus E$ is the union of this null set and F. Since by definition $F = \bigcup_{i=1}^\infty U_n^C$ where U_n^C is closed, F is Lebesgue measurable. Therefore, by (6), $\mathbb{R}^d \setminus E \in \mathcal{L}(\mathbb{R}^d)$.
(7). Since $E_i \in \mathcal{L}(\mathbb{R}^d)$, we have $E_i^C \in \mathcal{L}(\mathbb{R}^d)$ and $\bigcup_{i=1}^\infty E_i^C \in \mathcal{L}(\mathbb{R}^d)$. Therefore $(\bigcap_{i=1}^\infty E_i)^C \in \mathcal{L}(\mathbb{R}^d)$ and $\bigcap_{i=1}^\infty E_i \in \mathcal{L}(\mathbb{R}^d)$. ☐

For $E \in \mathcal{L}(\mathbb{R}^d)$, its Lebesgue measure is defined to be $m(E) := m^*(E)$, and it has the following properties, which is significantly better than Lebesgue outer measure.

Proposition 1.3.3. *(The measure axioms).*

1. $m(\emptyset) = 0$.

2. *(σ-additivity) For a countable sequence of disjoint sets $E_1, E_2 ... \in \mathcal{L}(\mathbb{R}^d)$,*

$$m(\bigcup_{i=1}^\infty E_i) = \sum_{i=1}^\infty m(E_n)$$

Proof. (1). is trivial.
(2). *Case 1.* E_n is compact.
Then, by Theorem 1.3.1., $dist(E_i, E_j) > 0$, and

$$m(\bigcup_{i=1}^{N} E_i) = \sum_{i=1}^{N} m(E_i)$$

By monotonicity,

$$m(\bigcup_{i=1}^{\infty} E_i) \geq m(\bigcup_{i=1}^{N} E_i) = \sum_{i=1}^{N} m(E_i)$$

Let $N \to \infty$,

$$m(\bigcup_{i=1}^{\infty} E_i) \geq \sum_{i=1}^{\infty} m(E_i)$$

Also from σ-subadditivity,

$$m(\bigcup_{i=1}^{\infty} E_i) \leq \sum_{i=1}^{\infty} m(E_i)$$

Therefore we have

$$m(\bigcup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} m(E_i)$$

Case 2. E_n is not compact but bounded.
For each E_n, it can be written as the union of a compact set U_n and a set with outer measure $\frac{\epsilon}{2^n}$. Thus,

$$m(E_n) \leq m(U_n) + \frac{\epsilon}{2^n}$$

$$\sum_{n=1}^{\infty} m(E_n) \leq \sum_{n=1}^{\infty} m(U_n) + \epsilon$$

We just showed that for compact set,

$$\sum_{n=1}^{\infty} m(U_n) = m(\bigcup_{i=1}^{\infty} U_n)$$

and by monotonicity,

$$m(\bigcup_{i=1}^{\infty} U_n) \leq m(\bigcup_{i=1}^{\infty} E_n)$$

Thus,

$$\sum_{n=1}^{\infty} m(E_n) \leq m(\bigcup_{i=1}^{\infty} E_n) + \epsilon$$

Since $\epsilon > 0$ arbitrary, we have

$$\sum_{n=1}^{\infty} m(E_n) \leq m(\bigcup_{i=1}^{\infty} E_n)$$

Also from σ-subadditivity, we have

$$\sum_{n=1}^{\infty} m(E_n) \geq m(\bigcup_{i=1}^{\infty} E_n)$$

Thus

$$\sum_{n=1}^{\infty} m(E_n) = m(\bigcup_{i=1}^{\infty} E_n)$$

Case 3. E_n is not compact and not closed.
Decompose $\mathbb{R}^d$ into annulis, for $m = 1, 2, ...,$

$$A_m := \{x \in \mathbb{R}^d : m - 1 \leq |x| \leq m\}$$

Then, each E_n can be written as $E_n = \bigcup_{m=1}^{\infty} E_n \cap A_m$ for $E_n \cap A_m$ bounded, measurable, and disjoint.
Then, by previous argument,

$$m(E_n) = \sum_{m=1}^{\infty} m(E_n \cap A_m)$$

Also, for $E_n \cap A_m$ bounded, measurable, and disjoint,

$$\bigcup_{n=1}^{\infty} E_n = \bigcup_{n=1}^{\infty} \bigcup_{m=1}^{\infty} E_n \cap A_m$$

Then

$$m(\bigcup_{n=1}^{\infty} E_n) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} m(E_n \cap A_m) = \sum_{n=1}^{\infty} m(E_n)$$

$\square$

Theorem 1.3.2. *(Monotone convergence theorem for measurable sets).*

(i) (Upward monotone convergence). Let $E_1 \subset E_2 \subset \cdots \subset \mathbb{R}^d$ be a countable non-decreasing sequence of Lebesgue measurable sets. Then $m\left(\bigcup_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} m(E_n)$. (Hint: Express $\bigcup_{n=1}^{\infty} E_n$ as the countable union of the lacunae $E_n \setminus \bigcup_{n'=1}^{n-1} E_{n'}$.)

(ii) (Downward monotone convergence) Let $\mathbb{R}^d \supset E_1 \supset E_2 \supset \ldots$ be a countable non-increasing sequence of Lebesgue measurable sets. If at least one of the $m(E_n)$ is finite, Then $m\left(\bigcap_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} m(E_n)$.

(iii) Give a counterexample to show that the hypothesis that at least one of the $m(E_n)$ is finite in the downward monotone convergence theorem cannot be dropped.

Proof. (1). Let $E_0 = \emptyset \subset E_1$. By expressing each finite union and countable union as the finite or countable union of the lacunae form,

$$\bigcup_{n=1}^{2} E_n = (E_2 \setminus E_1) \cup E_1$$

$$\bigcup_{n=1}^{3} E_n = (E_3 \setminus \bigcup_{n'=1}^{2} E_{n'}) \cup (E_2 \setminus E_1) \cup E_1$$

$$\bigcup_{n=1}^{4} E_n = (E_4 \setminus \bigcup_{n'=1}^{3} E_{n'}) \cup (E_3 \setminus \bigcup_{n'=1}^{2} E_{n'}) \cup (E_2 \setminus E_1) \cup E_1$$

......

$$\bigcup_{n=1}^{N} E_n = \bigcup_{k=1}^{N} (E_k \setminus \bigcup_{n'=1}^{k-1} E_{n'})$$

$$\bigcup_{n=1}^{\infty} E_n = \bigcup_{k=1}^{\infty} (E_k \setminus \bigcup_{n'=1}^{k-1} E_{n'})$$

Since each $E_k \in \mathcal{L}(\mathbb{R}^d)$, we have $\bigcup_{n'=1}^{k-1} E_{n'} \in \mathcal{L}(\mathbb{R}^d)$ and $E_k \setminus \bigcup_{n'=1}^{k-1} E_{n'} \in \mathcal{L}(\mathbb{R}^d)$, and any countable union of the latter is Lebesgue measurable as well. Also, $E_k \setminus \bigcup_{n'=1}^{k-1} E_{n'}$ and $E_j \setminus \bigcup_{n'=1}^{j-1} E_{n'}$ are disjoint for any $k \neq j$.
Hence, from countable additivity,

$$\begin{aligned}
m(\bigcup_{n=1}^{\infty} E_n) &= m(\bigcup_{k=1}^{\infty} (E_k \setminus \bigcup_{n'=1}^{k-1} E_{n'})) \\
&= \sum_{i=1}^{\infty} m(E_k \setminus \bigcup_{n'=1}^{k-1} E_{n'}) \\
&= \lim_{n\to\infty} \sum_{k=1}^{n} m(E_k \setminus \bigcup_{n'=1}^{k-1} E_{n'}) \\
&= \lim_{n\to\infty} \sum_{k=1}^{n} (m(E_k) - m(\bigcup_{n'=1}^{k-1} E_{n'})) \\
&= \lim_{n\to\infty} \sum_{k=1}^{n} (m(E_k) - m(E_{k-1})) \\
&= \lim_{n\to\infty} (m(E_n) - m(E_0)) \\
&= \lim_{n\to\infty} m(E_n)
\end{aligned}$$

(2).Since $E_1, E_2, ...$ are all Lebesgue measurable, E_1 E_n, $\bigcap_{n=2}^{\infty} E_n$ are Lebesgue measurable, then $\bigcup_{n=2}^{\infty} E_1 \setminus E_n$ and $E_1 \setminus \bigcap_{n=2}^{\infty} E_n$ are also Lebesgue measurable, and from basic set calculation we know that

$$\bigcup_{n=2}^{\infty} E_1 \setminus E_n = E_1 \setminus \bigcap_{n=2}^{\infty} E_n$$

Then, using conclusion from (1),

$$\begin{aligned} m(E_1 \setminus \bigcap_{n=2}^{\infty} E_n) &= m(\bigcup_{n=2}^{\infty} E_1 \setminus E_n) \\ &= \lim_{n\to\infty} m(E_1 \setminus E_n) \\ &= \lim_{n\to\infty} m(E_1) - m(E_n) \\ &= m(E_1) - \lim_{n\to\infty} m(E_n) \end{aligned}$$

Since $\bigcap_{n=2}^{\infty} E_n \subset E_1$, we have

$$m(E_1 \setminus \bigcap_{n=2}^{\infty} E_n) = m(E_1) - m(\bigcap_{n=2}^{\infty} E_n)$$

Now we prove $\bigcap_{n=2}^{\infty} E_n = \bigcap_{n=1}^{\infty} E_n$. $\forall x \in \bigcap_{n=2}^{\infty} E_n$, $x \in E_2$, since $E_2 \in E_1$, $x \in \bigcap_{n=1}^{\infty} E_n$. $\forall y \in \bigcap_{n=1}^{\infty} E_n$, $y \in E_2, E_3, ...$, so $y \in \bigcap_{n=2}^{\infty} E_n$. Then,

$$\lim_{n\to\infty} m(E_n) = m(\bigcap_{n=2}^{\infty} E_n) = m(\bigcap_{n=1}^{\infty} E_n)$$

(3). Consider the sequence $E_n := \mathbb{R}_+/[0,n]$. Clearly none of te $m(E_n)$ is finite. We have $m\left(\bigcap_{n=1}^{\infty} E_n\right) = m(\emptyset) = 0$. On the other hand,

$$\forall n \in \mathbb{N}, m(E_n) = \infty;$$

thus, the sequence of measures does not converge.

□

Theorem 1.3.3. *(Dominated Convergence Theorem). We say that a sequence E_n of sets in $\mathbb{R}^d$ converges pointwise to another set E in $\mathbb{R}^d$ if the indicator functions 1_{E_n} converge pointwise to 1_E.*

(i) If the E_n are all Lebesgue measurable, and converge pointwise to E, then E is Lebesgue measurable also.

(Hint: use the identity $1_E(x) = \liminf_{n\to\infty} 1_{E_n}(x)$ or $1_E(x) = \limsup_{n\to\infty} 1_{E_n}(x)$ to write E in terms of countable unions and intersections of the E_n.)

(ii) (Dominated convergence theorem) Suppose that the E_n are all contained in another Lebesgue measurable set F of finite measure. Then $m(E_n)$ converges to $m(E)$.

(Hint: use the upward and downward monotone convergence theorems, Theorem 1.3.1.)

(iii) Give a counterexample to show that the dominated convergence theorem fails if the E_n are not contained in a set of finite measure, even if we assume that the $m(E_n)$ are all uniformly bounded.

Proof. (i). If $x \in E$, we have $\mathbf{1}_E(x) = 1$,

$$\lim_{n\to\infty} \inf_{k\geq n} \mathbf{1}_{E_k}(x) = 1$$

This means that, $\forall\epsilon > 0$, $\exists N$, when $n \geq N$, $|\inf_{n\geq N} \mathbf{1}_{E_n}(x) - 1| < \epsilon$. Since inf is non-decreasing and is less than 1, we have $\inf_{n\geq N} \mathbf{1}_{E_n}(x) > 1-\epsilon$. Since $\epsilon > 0$ arbitrary, we have $\inf_{n\geq N} \mathbf{1}_{E_n}(x) = 1$. Then, $\mathbf{1}_{E_n}(x) = 1 \ \forall\ n \geq N$, which means that $x \in \bigcap_{n\geq N} E_n$. Since $\forall\epsilon > 0$ we can pick an N, $x \in \bigcap_{n\geq N} E_n \in \bigcup_{N\in\mathbb{N}} \bigcap_{n\geq N} E_n$.
If $x \in \bigcup_{N\in\mathbb{N}} \bigcap_{n\geq N} E_n$, then $\exists N$, such that $x \in \bigcap_{n\geq N} E_n$. This means that $\mathbf{1}_{E_n}(x) = 1 \ \forall\ n \geq N$. Then, $\mathbf{1}_E(x) = \lim_{n\to\infty} \mathbf{1}_{E_n}(x) = 1$ because of pointwise convergence. Then $x \in E$.
Therefore, we have shown the following two sets are equivalent.

$$E = \bigcup_{N\in\mathbb{N}} \bigcap_{n\geq N} E_n$$

Also, if $x \in E$, we have $\mathbf{1}_E(x) = 1$,

$$\lim_{n\to\infty} \sup_{k\geq n} \mathbf{1}_{E_k}(x) = 1$$

This means that $\forall\epsilon > 0$, $\forall N$, $\exists k$, such that when $k \geq N$, $\mathbf{1}_{E_k}(x) \geq 1 - \epsilon$. Since $\epsilon > 0$ is arbitrary, $\mathbf{1}_{E_k}(x) = 1$ and $x \in E_k$. Thus, $\forall\epsilon > 0$, $x \in \bigcap_{N\in\mathbb{N}} E_k$ for some $k \geq N$. Thus, $x \in \bigcap_{N\in\mathbb{N}} \bigcup_{k\geq N} E_k$.
If $x \in \bigcap_{N\in\mathbb{N}} \bigcup_{k\geq N} E_k$, $x \in \bigcup_{k\geq N} E_k$ for all $N \in \mathbb{N}$. This means that $\forall N \in \mathbb{N}$, $\exists k_N \geq N$, such that $x \in E_{k_N}$ and $\mathbf{1}_{E_{k_N}}(x) = 1$. By pointwise convergence and the fact that $\mathbf{1}_{E_{k_N}}(x)$ is a subsequence of the convergent sequence $\mathbf{1}_{E_N}(x)$, we have $\mathbf{1}_E(x) = \lim_{N\to\infty} \mathbf{1}_{E_N}(x) = \lim_{N\to\infty} \mathbf{1}_{E_{k_N}}(x) = 1$. Then, $x \in E$. Therefre, we have shown the following two sets are equibalent,

$$E = \bigcap_{N\in\mathbb{N}} \bigcup_{k\geq N} E_k$$

Then we represented E as either a countable union or a countable intersection of Lebesgue measurable sets, and E is Lebesgue measurable.

(ii). Since

$$\bigcap_{n\geq 1} E_n \subset \bigcap_{n\geq 2} E_n \subset \bigcap_{n\geq 3} E_n \subset ...$$

ans they are all lebesgue measurable, we have

$$\begin{aligned} m(E) &= m(\bigcup_{N\in\mathbb{N}} \bigcap_{n\geq N} E_n) \\ &= \lim_{N\to\infty} m(\bigcap_{n\geq N} E_n)) \\ &\leq \lim_{N\to\infty} m(E_N) \end{aligned}$$

by monotonicity.
Similarly, since

$$\bigcup_{n\geq 1} E_n \supset \bigcup_{n\geq 2} E_n \supset \bigcup_{n\geq 3} E_n \supset ...$$

and $\bigcup_{n\geq k} E_n \subset F \ \forall k \in \mathbb{N}$, F is a set with finite Lebesgue measure, we have $m(\bigcup_{n\geq k} E_n)$

is all finite for all k. Then,

$$
\begin{aligned}
m(E) &= m(\bigcap_{N\in\mathbb{N}}\bigcup_{k\geq N} E_k) \\
&= \lim_{N\to\infty} m(\bigcup_{n\geq N} E_n) \\
&\geq \lim_{N\to\infty} m(E_N)
\end{aligned}
$$

by monotonicity.
Now we have both $\leq$ and $\geq$, we conclude $m(E) = \lim_{N\to\infty} m(E_N)$.

(iii). Consider the sequence $E_n := \mathbb{R}_+/[0,n]$. Clearly non of the E_n is contained in a set of finite measure. We have $m\left(\bigcap_{n=1}^{\infty} E_n\right) = m(\emptyset) = 0$. On the other hand,

$$
\forall n\in\mathbb{N}, m(E_n) = \infty;
$$

thus, the sequence of measures does not converge. □

Theorem 1.3.4. *(Inner regularity). Let $E \subset \mathbb{R}^d$ be Lebesgue measurable. Then*

$$
m(E) = \sup_{K\subset E, K\ compact} m(K).
$$

Proof. Since K is compact, K is Lebesgue measurable. Therefore, by monotonicity, for all $K \subset E$, $m(E) \geq m(K)$. Therefore,

$$
m(E) \geq \sup_{K\subset E, K \text{ compact}} m(K)
$$

Thus it suffices to prove that

$$
m(E) \leq \sup_{K\subset E, K \text{ compact}} m(K)
$$

Write $E = \bigcup_{i=1}^{\infty} B_i$ where B_i are almost disjoint boxes. Then,

$$
\begin{aligned}
m(E) &= m(\bigcup_{i=1}^{\infty} B_i) \\
&= \sum_{i=1}^{\infty} |B_i|.
\end{aligned}
$$

Shrink each B_i to B_i' where $B_i' \subset B_i$ and $\partial B_i' \cap \partial B_i = \emptyset$ and $|B_i| \leq |B_i'| + \frac{\epsilon}{2^i}$ for arbigrary $\epsilon > 0$. Then,

$$
\sum_{i=1}^{\infty} |B_i| \leq \sum_{i=1}^{\infty} |B_i'| + \epsilon
$$

Also, $B_i' \cap B_j' = \emptyset$, so $m(\bigcup_{i=1}^{\infty} B_i') = \sum_{i=1}^{\infty} |B_i'|$.
Let $\overline{B_i'} = B_i' \cup \partial B_i'$ and it is a closed set. By our previous construction, $\overline{B_i'} \cap \overline{B_j'} = \emptyset$. Also,

from monotonicity and Theorem 1.2.2.,

$$m(\bigcup_{i=1}^{\infty} B_i') \leq m(\bigcup_{i=1}^{\infty} \overline{B_i'})$$

Since $\bigcup_{i=1}^{\infty} \overline{B_i'}$ is the countable union of closed and bounded sets, it is closed and bounded, thus compact. Therefore, by the definition of supremum and the fact that $\bigcup_{i=1}^{\infty} B_i' \subset \bigcup_{i=1}^{\infty} \overline{B_i'} \subset \bigcup_{i=1}^{\infty} B_i = E$, we have

$$\begin{aligned} m(E) &= \sum_{i=1}^{\infty} |B_i| \\ &\leq \sum_{i=1}^{\infty} |B_i'| + \epsilon \\ &= m(\bigcup_{i=1}^{\infty} \overline{B_i'}) + \epsilon \\ &\leq \sup_{K \in E, K \text{ compact}} m(K) + \epsilon \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, we have $m(E) \leq \sup_{K \in E, K \text{ compact}} m(K)$.
Now we have $\leq$ and $\geq$, we have $m(E) = \sup_{K \in E, K \text{ compact}} m(K)$. □

Theorem 1.3.5. *(Outer measure is not finitely additive). There exist disjoint bounded subsets E, F of the real line such that*

$$m^*(E \cup F) \neq m^*(E) + m^*(F).$$

(Hint: Show that the set constructed in the proof of the above proposition has positive outer measure.)

Proof. Consider the set that we will construct in the proof of Proposition 1.3.4.: $E := \{x_C : C \in \mathbb{R}/\mathbb{Q} \text{ and } x_C \in C \cap [0,1]\}$, and $\tilde{E} := \bigcap_{q \in \mathbb{Q} \cap [-1,1]} E + q$. We know by countable subadditivity that

$$\begin{aligned} m^*(\tilde{E}) &\leq \sum_{q \in \mathbb{Q} \cap [-1,1]} m^*(E+q) \\ &= \sum_{q \in \mathbb{Q} \cap [-1,1]} m^*(E) \\ &= \begin{cases} 1 & \text{if } m^*(E) = 0, \\ +\infty & \text{if } m^*(E) > 0. \end{cases} \end{aligned}$$

Also $m^*(\tilde{E}) \in [1,3]$. This contradict with the case $m^*(E) = 0$, so it can only be that $m^*(E) > 0$ and it equals to some positive real number.
Let $n \in \mathbb{N}$ large enought so that $m^*(E) \geq \frac{1}{n}$. If m^* is finitely additive, then for a subset

$F \subset \mathbb{Q} \cap [-1,1]$ with $\#F = 3n$, we have

$$\begin{aligned} m^*(\bigcup_{q\in F} E+q) &= \sum_{q\in F} m^*(E+q) \\ &= \sum_{q\in F} m^*(E) \\ &= 3n \times m^*(E) \\ &> 3 \end{aligned}$$

However, by monotonicity, $m^*(\bigcup_{q\in F} E+q) \le m^*(\bigcup_{q\in\mathbb{Q}\cap[-1,1]} E+q) \le 3$, contradiction! Thus, Lebesgue outer measure is not finitely additive. □

1.3.3 Non-Measurable Sets

Of course, there are non-measurable sets in $\mathbb{R}^d$.

Proposition 1.3.4. *$\exists E \subset [0,1]$, $E \notin \mathcal{L}(\mathbb{R}^d)$.*

Proof. We use the fact that $(\mathbb{Q},+)$ is a subgroup of $(\mathbb{R},+)$, and it partitions $\mathbb{R}$ into disjoint cosets $x+\mathbb{Q}$. This create a quotient group $\mathbb{R}/\mathbb{Q} := \{x+\mathbb{Q} : x \in \mathbb{R}\}$. Each coset$C = x+\mathbb{Q}$ of $\mathbb{R}/\mathbb{Q}$ is dense in $\mathbb{R}$, so it has non-empty intersection with $[0,1]$.
By axiom of choice, select $x_C \in C \cap [0,1]$ from each $C \in \mathbb{R}/\mathbb{Q}$. Let $E := \{x_C : C \in \mathbb{R}/\mathbb{Q}\}$ be the collection of all these coset representatives. By construction, $E \subset [0,1]$.
Claim 1. $[0,1] \subseteq \bigcup_{q\in\mathbb{Q}\cap[-1,1]} E+q$. Indeed, for any $y \in [0,1]$, $\exists C \in \mathbb{R}/\mathbb{Q}$ such that $y \in C$. Then, $y - x_C$ is rational. Since $y, x_C \in [0,1]$, we have $|x_C - y| \le 1$. Let $q = y - x_C$, since $x_C \in E$, we have $y \in E+q$.
Claim 2. For $q_1 \neq q_2 \in \mathbb{Q}$, $(E+q_1)\cap(E+q_2) = \emptyset$. Indeed, if $z \in (E+q_1)\cap(E+q_2)$, then $z = x_1+q_1 = x_2+q_2$ for $x_1, x_2 \in E$. Then, $x_1 = x_2 + (q_2 - q_1)$ where $q_2 - q_1$ is rational. Then, x_1 and x_2 are in the same coset C, then $x_1 = x_2 = x_C$, then $q_1 \neq q_2$, contradiction.
Suppose $E \in \mathcal{L}(\mathbb{R})$, then $E+q \in \mathcal{L}(\mathbb{R})\ \forall q \in \mathbb{Q}$, and $\tilde{E} := \bigcup_{q\in\mathbb{Q}\cap[-1,1]} E+q$. By monotonicity and *Claim 1*, $1 = m([0,1]) \le m(\tilde{E})$. Also since $\tilde{E} \subset [-1,2]$, we have $m(\tilde{E}) \in [1,3]$. By σ-additivity and *Claim 2* and transfomation invariante,

$$m(\tilde{E}) = \sum_{q\in\mathbb{Q}\cap[-1,1]} m(E+q) = \sum_{q\in\mathbb{Q}\cap[-1,1]} m(E)$$

If $m(E) = 0$ then $m(\tilde{E}) = 0$. If $m(E) > 0$ then $m(\tilde{E}) = +\infty$. Contradiction! □

1.4 Lebesgue Integral

1.4.1 Integration of Simple Functions

Definition 1.4.1. *(Simple function): A complex valued simple function $f : \mathbb{R}^d \to \mathbb{C}$ is a finite linear combination*

$$f = \sum_{k=1}^{n} c_k \mathbf{1}_{E_k}$$

for $E_k \in \mathcal{L}(\mathbb{R}^d)$, $c_k \in \mathbb{C}$.
An unsigned simple function just tales $c_k \in [0, +\infty)$. For an indicator function,

$$\int_{\mathbb{R}^d} \mathbf{1}_E(x)dx = m(E)$$

Definition 1.4.2. *(Integral of a simple function): for $f = \sum_{k=1}^{n} c_k \mathbf{1}_{E_k}$,*

$$Simp \int_{\mathbb{R}^d} f := \sum_{k=1}^{\infty} c_k m(E_k)$$

Lemma 1.4.1. *Let $k, k' \geq 0$ be natual numbers, $c_1, ..., c_k, c'_1, ..., c'_{k'} \in [0, +\infty]$. Let $E_1, ...E_k, E'_1, ..., E_{k'} \subset \mathbb{R}^q$ be in $\mathcal{L}(\mathbb{R}^d)$ such that*

$$\sum_{i=1}^{k} c_i \mathbf{1}_{E_i} = \sum_{i=1}^{k'} c'_i \mathbf{1}_{E'_i}) \ (*)$$

holds identically on $\mathbb{R}^d$. Then,

$$\sum_{i=1}^{k} c_i m(E_i) = \sum_{i=1}^{k'} c'_i m(E'_i)$$

Proof. First, $\{E_1, ..., E_k, E'_1, ..., E'_{k'}\}$ partitions $\mathbb{R}^d$ into $2^{k+k'}$ disjoint sets using finite Boolean algebra, each of which is an intersection of some of the $E_1, ..., E_k, E'_1, ..., E'_{k'}$ and their compliments. Letting go empty sets, we are left with m non-empty disjoint sets $A_1, ..., A_m$ for some $0 \leq m \leq 2^{k+k'}$. $A_i \in \mathcal{L}(\mathbb{R}^d)$ for $i \in \{1, ..., m\}$.
Then, $E_i = \bigcup_{j \in J_i} A_j$, $E'_{i'} = \bigcup_{j \in J'_{i'}} A_{j'}$ for all $i = 1, ..., k$ and $j' = 1, ..., k'$ and some subsets $J_i, J'_{i'}$. By finite additivity, $m(E_i) = \sum_{j \in J_i} m(A_i)$, $m(E'_{i'}) = \sum_{j \in J'_{i'}} m(A_j)$. Thus we need to prove that

$$\sum_{i=1}^{k} c_i \sum_{j \in J_i} m(A_i) = \sum_{i'=1}^{k'} c'_i \sum_{j \in J'_{i'}} m(A_i)$$

Fix $1 \leq j \leq m$, evaluate $(*)$ at a point x in the non-empty set A_j. Then, at such point,

$$\mathbf{1}_{E_i}(x) = \mathbf{1}_{J_i}(j)$$

$$\mathbf{1}_{E'_i}(x) = \mathbf{1}_{J'_{i'}}(j)$$

By $(*)$,

$$\sum_{i=1}^{k} c_i \mathbf{1}_{J_i}(j) = \sum_{i'=1}^{k'} c'_{i'} \mathbf{1}_{J'_{i'}}(j)$$

Multiply by $m(A_j)$,

$$\sum_{i=1}^{k} c_i \mathbf{1}_{J_i}(j) m(A_j) = \sum_{i'=1}^{k'} c'_{i'} \mathbf{1}_{J'_{i'}}(j) m(A_j)$$

Sum up $j = 1, ..., m$,

$$\sum_{i=1}^{k} c_i \sum_{j=1}^{m} \mathbf{1}_{J_i}(j) m(A_j) = \sum_{i'=1}^{k'} c'_{i'} \sum_{j=1}^{m} \mathbf{1}_{J'_{i'}}(j) m(A_j)$$

$$\sum_{i=1}^{k} c_i \sum_{j \in J_i} m(A_i) = \sum_{i'=1}^{k'} c'_i \sum_{j \in J'_{i'}} m(A_i)$$

□

Definition 1.4.3. *(Almost everywhere and support). A property $P(x)$ of $x \in \mathbb{R}^d$ holds (Lebesgue) almost everywhere (a.e.) if $\{x : P(x)$ does not hold$\}$ is a null set, that is, $m^*(\{x : P(x)$ does not hold$\}) = 0$.*
The support of a function f is $\{x \in \mathbb{R} : f(x) \neq 0\}$.

1.4.2 Measurable Functions

By extending the class of unsigned simple functions to the larger class of unsigned Lebesgue measurable functions, we can complete the unsigned simple integral to the unsigned Lebesgue integral.

Definition 1.4.4. *(Unsigned measurable function). An unsinged function f is Lebesgue measurable if it is the pointwise limit of unsigned simple functions, i.e., if $\exists f_1, f_2, f_3, ... : \mathbb{R}^d \to [0, +\infty]$ of unsigned simple functions such that $f_n(x) \to f(x)\ \forall x \in \mathbb{R}^d$.*

This definition has some equivalent forms.

Lemma 1.4.2. *(Equivalent Notions of Measurability). Let $f : \mathbb{R}^d \to [0, +\infty]$, the followings are equivalent.*

1. *f is Lebesgue measurable .*
2. *f is the pointwise a.e. limit of unsigned simple functions f_n. Thus $\lim_{n\to\infty} f_n(x)$ exists and $f(x) = \lim_{n\to\infty} f_n(x)$ for all $x \in \mathbb{R}^d$.*
3. *For every interval $I \subset [0, +\infty)$, the set $f^{-1}(I) := \{x \in \mathbb{R}^d : f(x) \in I\}$ is Lebesgue measurable. $f^{-1}(I) \in \mathcal{L}(\mathbb{R}^d)$.*

Proof. (i) $\Rightarrow$ (ii) is immediate from definition, so is (ii) *Rightarrow* (i).
(ii) $\Rightarrow$ (iii): Assume that $\exists$ simple functions $f_n \to f$ pointwise a.e.. Then, for almost every $x \in \mathbb{R}^d$ and $N \in \mathbb{N}$,

$$f(x) = \lim_{n\to\infty} f_n(x) = \limsup_{n\to\infty} f_n(x) = \inf_{N>0} \sup_{n\geq N} f_n(x) := \tilde{f}(x)$$

Let $\lambda > 0$ be arbitrary, and denote $\{g > \lambda\} := \{x \in \mathbb{R}^d : g(x) > \lambda\}$ for $g : \mathbb{R}^d \to [0, +\infty]$, we have for $M, N \in \mathbb{N}$,

$$\begin{aligned} \{\tilde{f} > \lambda\} &= \bigcup_{M>0} \bigcap_{N>0} \{x \in \mathbb{R}^d : \sup_{n \geq N} f_n(x) > \lambda + \frac{1}{M}\} \\ &= \bigcup_{M>0} \bigcap_{N>0} \bigcup_{n \geq N} \{x \in \mathbb{R}^d : f_n(x) > \lambda + \frac{1}{M}\} \end{aligned} \tag{5}$$

Since f_n is unsigned simple, $\{x \in \mathbb{R}^d : f_n(x) > \lambda\} \in \mathcal{L}(\mathbb{R}^d)$. By (6) and (7) of Proposition 1.3.2., $\{\tilde{f} > \lambda\} \in \mathcal{L}(\mathbb{R}^d)$. Also, $\{f > \lambda\}$ and $\{\tilde{f} > \lambda\}$ differs by a null set, so $\{f > \lambda\} \in \mathcal{L}(\mathbb{R}^d)$. Thus we have proved that $f^{-1}(I) \in \mathcal{L}(\mathbb{R}^d)$ for $I = (\lambda, +\infty)$.
Note that $\{f \geq \lambda\} = \bigcap_{\lambda' \in \mathcal{Q}, \lambda' < \lambda} \{f \geq \lambda'\}$. Then by (7) of Proposition 1.3.2., $\{f \geq \lambda\} \in \mathcal{L}(\mathbb{R}^d)$.
Note that

$$f^{-1}([a,b]) = \{f \geq a\} \setminus \{f > b\}$$

$$f^{-1}([a,b)) = \{f \geq a\} \setminus \{f \geq b\}$$

$$f^{-1}((a,b]) = \{f > a\} \setminus \{f > b\}$$

$$f^{-1}((a,b)) = \{f > a\} \setminus \{f \geq b\}$$

By Proposition 1.3.2., they are all Lebesgue measurable.
(iii) $\Rightarrow$ (i): Let $f : \mathbb{R}^d \to [0, +\infty]$ with $f^{-1}(I) \in \mathcal{L}(\mathbb{R}^d)$ $\forall I \subset \mathbb{R}^d$ as an interbal.
For each $n \geq 1$ and $x \in \mathbb{R}^d$, set

$$f_n(x) = \max_{m \in \mathbb{Z}} \{m2^{-n} : m2^{-n} \leq \min(f_(x), n) \mathbf{1}_{\overline{B(0,n)}}(x)\}$$

Then, $f_1 \leq f_2 \leq ...$ pointwise, and $f(x) = \sup_{n \in \mathbb{N}} f_n$ $\forall x \in \mathbb{R}^d$. Each f_n takes finitely many values and for any $c \in [0, +\infty)$, $f_n^{-1}(c) = f^{-1}(I_c) \cap \overline{B(0,n)}$ for some interval I_c that is measurable. Thus f_n is simple, and is bounded and has finite measure support, and the claim follows. □

Theorem 1.4.1. *(Functions that are Measurable).*

1. *Every continuous function $f : \mathbb{R}^d \to [0, +\infty]$ is measurable.*
2. *The supremum, infimum, limit superior, or limit inferior of unsigned measurable functions is unsigned measurable.*

Proof. (1). Fist, divide the $\mathbb{R}^d$ into dyadic cubes:

$$\begin{aligned} \mathbb{R} &= \bigcup_{k_1 \in \mathbb{Z}} \bigcup_{k_2 \in \mathbb{Z}} ... \bigcup_{k_d \in \mathbb{Z}} [\frac{k_1}{2^n}, \frac{k_1 + 1}{2^n}) \times [\frac{k_2}{2^n}, \frac{k_2 + 1}{2^n}) \times ... \times [\frac{k_d}{2^n}, \frac{k_d + 1}{2^n}) \ \forall n \in \mathbb{N} \\ &= \bigcup_{k_1 \in \mathbb{Z}} \bigcup_{k_2 \in \mathbb{Z}} ... \bigcup_{k_d \in \mathbb{Z}} B_{k_1, k_2, ..., k_d} \ \forall n \in \mathbb{N} \end{aligned}$$

Then, define an unsigned simple function f_n can be written as

$$f_n(x) = \sum_{k_1 \in \mathbb{Z}} \sum_{k_2 \in \mathbb{Z}} \dots \sum_{k_d \in \mathbb{Z}} \mathbf{1}_{B_{k_1,k_2,\dots,k_d}}(x) \inf_{B_{k_1,k_2,\dots,k_d}} f(x)$$

For all $x \in \mathbb{R}^d$, x has to be in one of the dyadic cubes. When $x \in B_{k_1,k_2,\dots,k_d}$, $f_n(x) = \inf_{B_{k_1,k_2,\dots,k_d}} f(x)$. Thus it suffices to show that $f_n \to f$ pointwise a.e. in this case.
It is easy to see that $f_n \geq 0$. By the definition of infimum,

$$\inf_{B_{k_1,k_2,\dots,k_d}} f(x) \leq f(x) \; \forall x \in B_{k_1,k_2,\dots,k_d}$$

$\forall \epsilon > 0$, $\exists x' \in B_{k_1,k_2,\dots,k_d}$, such that $f(x') \leq \inf_{B_{k_1,k_2,\dots,k_d}} f(x) + \epsilon$. Together with the last inequality we have

$$0 \leq f(x') - \inf_{B_{k_1,k_2,\dots,k_d}} f(x) \leq \epsilon$$

$$|f(x') - \inf_{B_{k_1,k_2,\dots,k_d}} f(x)| \leq \epsilon$$

Notice that, two points inside the closure of $B_{k_1,k_2,\dots,k_d}$ has distance less than $\frac{\sqrt{d}}{2^n}$. Thus, by continuity, $\exists \delta > 0$, pick n large enough that $n > log_2 \frac{\sqrt{d}}{\delta}$, so that $|x - x'| < \frac{\sqrt{d}}{2^n} < \delta$ implies $|f(x) - f(x')| \leq \epsilon$, $\forall x \in B_{k_1,k_2,\dots,k_d}$.
Therefore, $\forall x \in B_{k_1,k_2,\dots,k_d}$,

$$\begin{aligned} |f(x) - \inf_{B_{k_1,k_2,\dots,k_d}} f(x)| &\leq |f(x) - f(x')| + |f(x') - \inf_{B_{k_1,k_2,\dots,k_d}} f(x)| \\ &\leq 2\epsilon \end{aligned}$$

This means that, $\forall \epsilon > 0$, we found that when $n > log_2 \frac{\sqrt{d}}{\delta}$, $|f(x) - \inf_{B_{k_1,k_2,\dots,k_d}} f(x)| = |f(x) - f_n(x)| \leq 2\epsilon$. $f_n \to f$. Therefore, for all $x \in \mathbb{R}^d$ (equivalent to X in arbitrary cube), $f_n \to f$ pointwise a.s. Then, f is Lebesgue measurable.
(2). Since $\{f_n\}_{n \in \mathbb{N}}$ is unsigned measurable, $\{f_n \geq c\}$ and $\{f_n \leq c\}$ are measurable for all $c \geq 0$ by Lemma 1.4.2.. Thus, $\forall n \in \mathbb{N}$,

$$\begin{aligned} \{\sup_{k \geq n} f_k \leq c\} &= \{x \in \mathbb{R}^d : \sup_{k \geq n} f_k \leq c\} \\ &= \{x \in \mathbb{R}^d : f_k \leq c \; \forall k \geq n\} \\ &= \bigcap_{k \geq n} \{x \in \mathbb{R}^d : f_k \leq c\} \end{aligned}$$

$$\begin{aligned} \{\inf_{k \geq n} f_k \geq c\} &= \{x \in \mathbb{R}^d : \inf_{k \geq n} f_k \geq c\} \\ &= \{x \in \mathbb{R}^d : f_k \geq c \; \forall k \geq n\} \\ &= \bigcap_{k \geq n} \{x \in \mathbb{R}^d : f_k \geq c\} \end{aligned}$$

Since those are countable intersections of measurable sets, the functions

$$g_n := \sup_{k \geq n} f_k \;,\; h_n := \inf_{k \geq n} f_k$$

are unsigned measurable for all $n \in \mathbb{N}$. Take $n = 1$, then $\sup_{n\in\mathbb{N}} f_n$ and $\inf_{n\in\mathbb{N}} f_n$ are unsigned measurable.
Notice that g_n is monotonic decreasing in n and h_n is monotonic increasing in n. So

$$\lim_{n\to\infty} \sup_{k\geq n} f_k = \inf_{n\geq 1} \sup_{k\geq n} f_k = \inf_{n\geq 1} g_n$$

$$\lim_{n\to\infty} \inf_{k\geq n} f_k = \sup_{n\geq 1} \inf_{k\geq n} f_k = \sup_{n\geq 1} h_n$$

Since g_n and h_n are unsigned measurable, by the previous argument we have that the limsup and the liminf of f_n are unsigned measurable.

□

Theorem 1.4.2. *(Bounded unsigned measurable function is the uniform limit of bounded simple functions). Let $f : \mathbb{R}^d \to [0, +\infty]$. Then, f is a bounded unsigned measurable function if and only if f is the uniform limit of bounded simple functions.*

Proof. First we prove $\Leftarrow$.
Suppose f_n is a sequence of bounded simple functions and $f_n \to f$ uniformly. Then, $\forall \epsilon > 0$, $\exists N$ such that when $n \geq N$, $|f_n(x) - f(x)| \leq \epsilon \ \forall x \in \mathbb{R}^d$. By triangle inequality we have $|f(x)| \leq \epsilon + |f_n(x)|$. Since $f_n(x)$ is bounded, it is clear that f is also bounded. Also, uniform convergent induces pointwise convergence, and then f is Lebesgue measurable by Lemma 1.4.2..
Then we prove $\Rightarrow$. By Lemma 1.4.2., since f is unsigned Lebesgue measurable, f is the supremum $f(x) = \sup_{n\in\mathbb{N}}$ of an increasing sequence $0 \leq f_1 \leq f_2 \leq ...$ of unsigned simple functions f_n, each of which are bounded with finite measure support.
By the definition of supremum, $\forall x$, $\forall \epsilon > 0$, $\exists n' \in \mathbb{N}$ such that $f_{n'}(x) \geq f(x) - \epsilon$. Also since f_n is monotonic increasing, we have $|f_{n'}(x) - f(x)| \leq \epsilon$.
Since $f_{n'}$ is a unsigned simple function, it can take finite many values $c_1, c_2, ..., c_m$. Based on these values, we divide the domain of $f_{n'}$ ($\mathbb{R}^d$) into m parts,

$$\mathbb{R}^d = f_{n'}^{-1}(c_1) \cup f_{n'}^{-1}(c_2) \cup ... \cup f_{n'}^{-1}(c_m)$$

For each $f_{n'}^{-1}(c_i)$, by supremum, $\forall \epsilon > 0$, $\exists n_i$, such that, when $n \geq n_i$, $|f_n(x) - f(x)| \leq \epsilon \ \forall x \in f_{n'}^{-1}(c_i)$. By this we got a collection of $\{n_1, n_2, ..., n_m\}$.
Set $N := \max\{n_1, n_2, ..., n_m\}$, then we have $\forall \epsilon > 0$, $\exists N = \max\{n_1, n_2, ..., n_m\}$ such that when $n \geq N$, $|f_n(x) - f(x)| \leq \epsilon \ \forall x \in \mathbb{R}^d$. Thus we have $f_n \to f$ uniformly.
Also since $0 \leq f_n \leq \sup_{n\in\mathbb{N}} f_n = f$, f is bounded, we have f_n is bounded. □

1.4.3 Unsigned Lebesgue Integrals

Now let's integrate unsigned measurable functions.

Definition 1.4.5. *(Lower and upper unsigned Lebesgue integral). Let $f : \mathbb{R}^d \to [0, +\infty]$. Define the lower unsigned Lebesgue integral of f as*

$$\underline{\int_{\mathbb{R}^d}} f := \sup_{0\leq g\leq f, g \ simple} Simp \int_{\mathbb{R}^d} g$$

and the upper unsigned Lebesgue integral of f as

$$\overline{\int_{\mathbb{R}^d}} f := \inf_{f \leq h, h \ simple} Simp \int_{\mathbb{R}^d} h$$

Definition 1.4.6. *(Unsigned Lebesgue integral). Let $f : \mathbb{R}^d \to [0, \infty]$ be measurable. Define its unsigned Lebesgue integral as*

$$\int_{\mathbb{R}^d} f := \underline{\int_{\mathbb{R}^d}} f = \sup_{0 \leq g \leq f, g \ simple} Simp \int_{\mathbb{R}^d} g$$

For $f : \mathbb{R}^d \to [0, \infty]$ measurable, bounded, and vanishing outside of a set of finite measure, the lower and upper Lebesgue integrals match. Also we have an important corollary. But first we have to prove a theorem.

Theorem 1.4.3. *Let $f : \mathbb{R}^d \to [0, +\infty]$ be measurable, bounded, and vanishing outside of a set of finite measure. Show that the lower and upper Lebesgue integrals of f agree. (Hint: use Exercise 1.3.4.) There is a converse to this statement, but we will defer it to later notes. What happens if f is allowed to be unbounded, or is not supported inside a set of finite measure?*

Proof. The very obvious way to prove this is by first proving $\leq$ then proving $\geq$.
For $\leq$: by definition,

$$\underline{\int_{\mathbb{R}^d}} f = \sup_{0 \leq g \leq f, g \text{ simple}} \text{Simp} \int_{\mathbb{R}^d} g$$

$$\overline{\int_{\mathbb{R}^d}} f = \inf_{h \geq f, g \text{ simple}} \text{Simp} \int_{\mathbb{R}^d} h$$

Since $g \leq h$ for every g, h that satisfy the conditions, by definition we natually have $\underline{\int_{\mathbb{R}^d}} f \leq \overline{\int_{\mathbb{R}^d}} f$. Thus it suffices to show that $\underline{\int_{\mathbb{R}^d}} f \geq \overline{\int_{\mathbb{R}^d}} f$. However my original methods is too tedious (I have to divide the finite measure support into finite sub-supports twice, take supremum and infimum of f within each sub-support, etc.) So I'll just adopt another method.

Let S be the finite measure support on which $f > 0$. By Theorem 1.4.2., we can find a sequence of unsigned simple functions $(g_n)_{n \in \mathbb{N}}$ such that (i) $0 \leq g_1 \leq g_2 \leq ...$ (ii) has finite measure support S (iii) $g_n \to f$ uniformly.
Pick a subsequence of this original sequence so that $\forall n \in \mathbb{N}$, $d_\infty(g_n, f) \leq \frac{1}{n}$. Furthermore, construct another sequence of unsigned simple functions $(h_n)_{n \in \mathbb{N}}$ such that $h_n = g_n + \frac{2}{n}$. Then, $h_n - f = g_n + \frac{2}{n} - f \geq \frac{1}{n} \geq 0$. $d_\infty(h_n, f) \leq d_\infty(h_n, g_n) + d_\infty(g_n, f) \leq \frac{3}{n}$.
Since now $\forall n \in \mathbb{N}$, $d_\infty(h_n, g_n) \leq \frac{2}{n}$, we have $d_\infty(h_n, g_n) \to 0$ as $n \to \infty$. They converge uniformly to each other.

Pick a simple function $g' \leq f$ that satisfies $\underline{\int_{\mathbb{R}^d}} f - \text{Simp} \int_{\mathbb{R}^d} g' \leq \frac{1}{n}$, $d_\infty(g', f) \leq \frac{1}{n}$.

$$\begin{aligned}
\underline{\int_{\mathbb{R}^d}} f - \text{Simp} \int_{\mathbb{R}^d} g_n &= \underline{\int_{\mathbb{R}^d}} f - \text{Simp} \int_{\mathbb{R}^d} g' + \text{Simp} \int_{\mathbb{R}^d} g' - \text{Simp} \int_{\mathbb{R}^d} g_n \\
&\leq \frac{1}{n} + \text{Simp} \int_{\mathbb{R}^d} |g' - g_n| \\
&\leq \frac{1}{n} + \text{Simp} \int_{\mathbb{R}^d} d_\infty(g', g_n) \mathbf{1}_S \\
&\leq \frac{1}{n} + \text{Simp} \int_{\mathbb{R}^d} (d_\infty(g', f) + d_\infty(f, g_n)) \mathbf{1}_S \\
&\leq \frac{1}{n} + \frac{2}{n} m(S) \\
&\to 0 \text{ as } n \to \infty
\end{aligned}$$

Thus $\lim_{n\to\infty} \text{Simp} \int_{\mathbb{R}^d} g_n = \underline{\int_{\mathbb{R}^d}} f$.
Pick another simple function $h' \geq f$ such that $\text{Simp} \int_{\mathbb{R}^d} h' - \overline{\int_{\mathbb{R}^d}} f \leq \frac{1}{n}$, $d_\infty(h', f) \leq \frac{1}{n}$.

$$\begin{aligned}
\text{Simp} \int_{\mathbb{R}^d} h_n - \overline{\int_{\mathbb{R}^d}} f &= \text{Simp} \int_{\mathbb{R}^d} h_n - \text{Simp} \int_{\mathbb{R}^d} h' + \text{Simp} \int_{\mathbb{R}^d} h' - \overline{\int_{\mathbb{R}^d}} f \\
&\leq \frac{1}{n} + \text{Simp} \int_{\mathbb{R}^d} |h' - h_n| \\
&\leq \frac{1}{n} + \text{Simp} \int_{\mathbb{R}^d} d_\infty(h', h_n) \mathbf{1}_S \\
&\leq \frac{1}{n} + \text{Simp} \int_{\mathbb{R}^d} (d_\infty(h', f) + d_\infty(f, h_n)) \mathbf{1}_S \\
&\leq \frac{1}{n} + \frac{2}{n} m(S) \\
&\to 0 \text{ as } n \to \infty
\end{aligned}$$

Thus $\lim_{n\to\infty} \text{Simp} \int_{\mathbb{R}^d} h_n = \overline{\int_{\mathbb{R}^d}} f$.
Since $\text{Simp} \int_{\mathbb{R}^d} (h_n - g_n) \leq \text{Simp} \int_{\mathbb{R}^d} d_\infty(h_n, g_n) \mathbf{1}_S \to 0$ as $n \to \infty$, we have $\lim_{n\to\infty} \text{Simp} \int_{\mathbb{R}^d} h_n = \lim_{n\to\infty} \text{Simp} \int_{\mathbb{R}^d} g_n$. Thus

$$\underline{\int_{\mathbb{R}^d}} f = \lim_{n\to\infty} \text{Simp} \int_{\mathbb{R}^d} g_n = \lim_{n\to\infty} \text{Simp} \int_{\mathbb{R}^d} h_n = \overline{\int_{\mathbb{R}^d}} f$$

$\square$

Corollary 1.4.1. *(Finite additivity of Lebesgue integral). Let $f, g : \mathbb{R}^d \to [0, +\infty]$ be measurable. Then,*

$$\int_{\mathbb{R}^d} (f + g) = \int_{\mathbb{R}^d} f + \int_{\mathbb{R}^d} g$$

We also have *Markov's Inequality*, which asserts that the Lebesgue integral of an unsigned measurable function controls how often that function can be large.

Lemma 1.4.3. *(Markov's Inequality). Let $f : \mathbb{R}^d \to [0, +\infty]$ be measurable. Then, for any $\lambda \in (0, +\infty)$,*

$$m(\{x \in \mathbb{R}^d : f(x) \geq \lambda\}) \leq \frac{1}{\lambda} \int_{\mathbb{R}^d} f(x) dx$$

Proof. First, notice that

$$\lambda \mathbf{1}_{\{x\in\mathbb{R}^d : f(x)\geq\lambda\}} \leq f(x)$$

Then, since f is measurable,

$$\int_{\mathbb{R}^d} \lambda \mathbf{1}_{\{x\in\mathbb{R}^d : f(x)\geq\lambda\}} \leq \int_{\mathbb{R}^d} f(x)dx$$

By the definition of Lebesgue integral for simple function,

$$LHS = \lambda m(\{x \in \mathbb{R}^d : f(x) \geq \lambda\}) \leq \int_{\mathbb{R}^d} f(x)dx$$

Therefore we have

$$m(\{x \in \mathbb{R}^d : f(x) \geq \lambda\}) \leq \frac{1}{\lambda}\int_{\mathbb{R}^d} f(x)dx$$

□

1.4.4 Absolute Integrability

Now we define the absolutely convergent Lebesgue interval.

Definition 1.4.7. *(Absolute integrability). An almost everywhere defined measurable function $f : \mathbb{R}^d \to \mathbb{C}$ is absolutely integrable if the unsigned integral*

$$||f||_{L^1(\mathbb{R}^d)} := \int_{\mathbb{R}^d} |f(x)|dx < \infty$$

Denote $L^1(\mathbb{R}^d)$ to be the space of absoluely integrable functions.
If f is real valued and $||f||_{L^1(\mathbb{R}^d)} < \infty$, define its Lebesgue integral as

$$\int_{\mathbb{R}^d} f = \int_{\mathbb{R}^d} f_+ - \int_{\mathbb{R}^d} f_-$$

where $f_+ = \max\{f, 0\} \geq 0$ and $f_- = \max\{-f, 0\} \geq 0$.
If f is complex valued and $||f||_{L^1(\mathbb{R}^d)} < \infty$,

$$\begin{aligned} f &= Ref + iImf \\ &= (Ref)_+ - (Ref)_- + i[(Imf)_+ - (Imf)_-] \\ &= f_1 - f_2 + if_3 - if_4 \end{aligned}$$

where $f_1, f_2, f_3, f_4 : \mathbb{R}^d \to [0, +\infty]$.
When $||f||_{L^1(\mathbb{R}^d)} < \infty$, we can extend unsigned Lebesgue integral to such f by linearity:

$$\int_{\mathbb{R}^d} f = \int_{\mathbb{R}^d} f_1 - \int_{\mathbb{R}^d} f_2 + i\int_{\mathbb{R}^d} f_2 - i\int_{\mathbb{R}^d} f_4$$

Proposition 1.4.1. *(Integration is a linear operation from $L^1(\mathbb{R}^d)$ to $\mathbb{C}$).*

$$\int_{\mathbb{R}^d} (f+g) = \int_{\mathbb{R}^d} f + \int_{\mathbb{R}^d} g$$

$$\int_{\mathbb{R}^d} cf = c\int_{\mathbb{R}^d} f$$

for $c \in \mathbb{C}$.

From the pointwise triangle inequality

$$|f(x) + g(x)| \leq |f(x)| + |g(x)|$$

we have by monotonicity and linearity

$$\int_{\mathbb{R}^d} |f + g| \leq \int_{\mathbb{R}^d} (|f| + |g|) = \int_{\mathbb{R}^d} |f| + \int_{\mathbb{R}^d} |g|$$

that is,

$$||f + g||_{L^1(\mathbb{R}^d)} \leq ||f||_{L^1(\mathbb{R}^d)} + ||g||_{L^1(\mathbb{R}^d)}$$

Also, $\forall c \in \mathbb{C}$,

$$||cf||_{L^1(\mathbb{R}^d)} = |c|||f||_{L^1(\mathbb{R}^d)}$$

Therefore we say $L^1(\mathbb{R}^d \to \mathcal{C})$ is a complex vector space.

$||.||_{L^1(\mathbb{R}^d)}$ is a *seminorm*, since $||f||_{L^1(\mathbb{R}^d)} = 0$ does not lead to $f \equiv 0$ (i.e. $f(x) = 0\ \forall x \in \mathbb{R}^d$). Instead, lt leads to the following.

Proposition 1.4.2. $||f||_{L^1(\mathbb{R}^d)} = 0 \Rightarrow f = 0$ *a.e.*.

Proof. From Markov's Inequality, for arbitrary $\epsilon > 0$, we have pointwise

$$\lambda \mathbf{1}\{|f| \geq \lambda\} \leq |f|$$

Integrate both sides. Notice that LHS is a simple function.

$$\text{Simp}\int_{\mathbb{R}^d} \lambda \mathbf{1}\{|f| \geq \lambda\} \leq \sup_{0 \leq g \leq f|f|, g \text{ simple}} \text{Simp}\int_{\mathbb{R}^d} g = \underline{\int_{\mathbb{R}^d}} |f| = \int_{\mathbb{R}^d} f$$

$$m(\{|f| \geq \lambda\}) \leq \frac{1}{\lambda}\int_{\mathbb{R}^d} |f| = \frac{1}{\lambda}||f||_{L^1(\mathbb{R}^d)}$$

If $||f||_{L^1(\mathbb{R}^d)} = 0$, then $m(\{|f| \geq \lambda\}) = 0$, which means that $f = 0$ a.e.. ☐

To be precise, $L^1(\mathbb{R}^d)$ is the normed space of equivalent functions.

Definition 1.4.8. *(Equivalent functions). Let* $f, g \in \text{Ł}^1(\mathbb{R}^d \to \mathbb{C})$. $f \sim g$ *if the* L^1 *distance*

$$d_{L^1}(f, g) := ||f - g||_{L^1(\mathbb{R}^d)} = 0$$

That is, $\{x \in \mathbb{R}^d : f(x) \neq g(x)\}$ *is a null set.*

Here we record another definition of distance.

Definition 1.4.9. *(Supremum distance or infinite norm distance). Let* $f, g \in D$*, then*

$$d_\infty(f, g) = \sup_{x \in D} |f(x) - g(x)|$$

It measures how close two functions are uniformly.

If $d_\infty(f, f_n) \to 0$ as $n \to \infty$, then $f_n \to f$ uniformly.

Proposition 1.4.3. *Let $f : \mathbb{R}^d \to [0, +\infty]$ be measurable.*

1. *If $\int_{\mathbb{R}^d} f(x)\,dx < \infty$, then f is finite almost everywhere. Give a counterexample to show that the converse statement is false.*

2. *$\int_{\mathbb{R}^d} f(x)\,dx = 0$ if and only if f is zero almost everywhere.*

Proof. (1). We prove the inverse statement. Suppose f is infinite at some non-trivial measure support, then $\forall \lambda > 0$, $m(\{x \in \mathbb{R}^d : f(x) \geq \lambda\}) > 0$.
Using the Markov's Inequality, we have

$$\int_{\mathbb{R}^d} f \geq \lambda m(\{x \in \mathbb{R}^d : f(x) \geq \lambda\}) \ \forall \lambda > 0$$

Since λ can be arbitrarily large, we have $\int_{\mathbb{R}^d} f = \infty$. Then, if $\int_{\mathbb{R}^d} f < \infty$ then f is finite almost everywhere.

(2) First we prove $\Leftarrow$. Since $f = 0$ almost everywhere, $m(\{x \in \mathbb{R}^d : f(x) > 0\}) = 0$. Divide the domain into two disjoint measurable sets $\mathbb{R}^d = \{x \in \mathbb{R}^d : f(x) > 0\} \cup \{x \in \mathbb{R}^d : f(x) = 0\}$. Define a simple function $h = \sup_{x \in \{f>0\}} f \times \mathbf{1}_{\{f>0\}} + 0 \times \mathbf{1}_{\{f=0\}}$. Clearly $h \geq f$. Since f is measurable,

$$\begin{aligned}
\int_{\mathbb{R}^d} f &= \inf_{h \geq f, h \text{ simple}} \text{Simp} \int_{\mathbb{R}^d} h \\
&\leq \text{Simp} \int_{\mathbb{R}^d} \big(\sup_{x \in \{f>0\}} f \times \mathbf{1}_{\{f>0\}} + 0 \times \mathbf{1}_{\{f=0\}} \big) \\
&= \sup_{x \in \{f>0\}} f \times m(\{f > 0\}) + 0 \times m(\{f = 0\}) \\
&= 0
\end{aligned}$$

Also because of non-negativity, we have $\int_{\mathbb{R}^d} f = 0$.

Now we prove $\Rightarrow$. Since Markov's Inequality,

$$\begin{aligned}
m(\{x \in \mathbb{R}^d : f(x) \geq \lambda\}) &\leq \frac{1}{\lambda} \int_{\mathbb{R}^d} f \\
&= 0
\end{aligned} \tag{6}$$

This is true for arbitrary $\lambda > 0$. Therefore we conclude that $m(\{x \in \mathbb{R}^d : f(x) > 0\}) = 0$ is a null set. Therefore, $f = 0$ almost everywhere. $\square$

We also record another basic inequality.

Lemma 1.4.4. *(Triangle inequality). Let $f \in L^1(\mathbb{R}^d \to \mathbb{C}$, then*

$$\left| \int_{\mathbb{R}^d} f(x) dx \right| \leq \int_{\mathbb{R}^d} |f(x)| dx$$

Proof. If f is real valued, then by definition

$$\begin{aligned} |\int_{\mathbb{R}^d} f| &= |\int_{\mathbb{R}^d} f_+ - \int_{\mathbb{R}^d} f_-| \\ &\leq |\int_{\mathbb{R}^d} f_+| + |\int_{\mathbb{R}^d} f_-| \\ &= \int_{\mathbb{R}^d} |f_+| + \int_{\mathbb{R}^d} |f_-| \\ &= \int_{\mathbb{R}^d} |f| \end{aligned} \tag{7}$$

If f is complex valued, then we use the fact that $\forall z \in \mathbb{C}$, $z = |z|e^{i\theta}$ for some $\theta \in (-\pi, \pi]$. Then,

$$|\int_{\mathbb{R}^d} f| = e^{i\theta} \int_{\mathbb{R}^d} f = \int_{\mathbb{R}^d} e^{i\theta} f$$

Taking real parts of both sides, we get

$$|\int_{\mathbb{R}^d} f| = \int_{\mathbb{R}^d} Re(e^{i\theta} f)$$

Since

$$Re(e^{i\theta} f) \leq |e^{i\theta} f| = |f|$$

we have

$$|\int_{\mathbb{R}^d} f| \leq \int_{\mathbb{R}^d} |f|$$

□

1.4.5 Littlewood's Three Principles

Littlewood's Three Principles gives informal heuristics about the basic intuition of Lebesgue measure theory.

1. Measurable sets are "almost open".
2. Absolutely integrable functions are "almost continuous".
3. Pointwise convergent sequences of f_n are "almost uniformly convergent".

Here we see an instance of the second principle. The following theorem says that simple dunctions, step functions, and continuous and compactly supported functions are dense subsets of $L^1(\mathbb{R}^d)$ w.r.t. $L^1(\mathbb{R}^d)$ semi-metric.

Theorem 1.4.4. *(Approximation of L^1 functions). Let $f \in L^1(\mathbb{R}^d)$, $\epsilon > 0$. Then,*

1. *$\exists$ simple function $g \in L^1(\mathbb{R}^d)$, such that $||f - g||_{L^1(\mathbb{R}^d)} \leq \epsilon$.*
2. *$\exists$ step function $g \in L^1(\mathbb{R}^d)$, such that $||f-g||_{L^1(\mathbb{R}^d)} \leq \epsilon$. (step function $g = \sum_{i=1}^N c_i \mathbf{1}_{B_i}$ where B_i are boxes).*
3. *$\exists$ $g \in C_c(\mathbb{R}^d)$, such that $||f - g||_{L^1(\mathbb{R}^d)} \leq \epsilon$.*
   ($C_c(\mathbb{R}^d) := \{g : \mathbb{R}^d \to \mathbb{C} : g$ is continuous and compactly supported$\}$]. Compactly supported means that $\{x : g(x) \neq 0\}$ has compact closure, that is, is contained in a ball).

Proof. (1). When f is unsigned, by definition of Lebesgue integral,

$$\int_{\mathbb{R}^d} f = \sup_{0\le g\le f, g \text{ simple}} \text{Simp} \int_{\mathbb{R}^d} g = \sup_{0\le g\le f, g \text{ simple}} \int_{\mathbb{R}^d} g$$

Then $\exists g$ simple, such that

$$\int_{\mathbb{R}^d} g \ge \int_{\mathbb{R}^d} f - \epsilon$$

Also since

$$\int_{\mathbb{R}^d} f \ge \int_{\mathbb{R}^d} g$$

we have by linearity

$$\int_{\mathbb{R}^d} |f-g| \le \int_{\mathbb{R}^d} (f-g) = \int_{\mathbb{R}^d} f - \int_{\mathbb{R}^d} g \le \epsilon$$

(2). It suffices to consider f simple, since by (1), for general $f \in L^1(\mathbb{R}^d)$, it is within ϵ distance with a simple function, so we can then apply the Triangle Inequality.
Let $f \in L^1(\mathbb{R}^d)$ be simple,

$$f = \sum_{i=1}^N c_i \mathbf{1}_{E_i}$$

We approximate each $\mathbf{1}_{E_i}$ by a step function g_i to have

$$||f - \sum_{i=1}^N c_i g_i||_{L^1(\mathbb{R}^d)} = ||\sum_{i=1}^N c_i(\mathbf{1}_{E_i} - g_i)||_{L^1(\mathbb{R}^d)} \le \sum_{i=1}^N |c_i|\epsilon$$

So, it suffices to consider $f = \mathbf{1}_{E_i}$ for E_i measurable, and approximate it using an elementary set A_i.
By **Exercise 1.2.16.** in Textbook, E_i differs from an elementary set by a set of arbitrarily small Lebesgue outer measure. Then, $\exists A_i$ such that

$$\epsilon \ge m(E_i \Delta A_i) = \int_{\mathbb{R}^d} |\mathbf{1}_{E_i} - \mathbf{1}_{A_i}| = ||(\mathbf{1}_{E_i} - \mathbf{1}_{A_i}||_{L^1(\mathbb{R}^d)}$$

Thus we have

$$||\sum_{i=1}^N c_i(\mathbf{1}_{E_i} - g_i)||_{L^1(\mathbb{R}^d)} = \sum_{i=1}^N |c_i|||(\mathbf{1}_{E_i} - \mathbf{1}_{A_i}||_{L^1(\mathbb{R}^d)} \le \sum_{i=1}^N |c_i|\epsilon$$

Therefore we have the claim (2).

(3). Again, by (1), (2), linearity, and triangle inequality, it suffices to show (iii) for $f = \mathbf{1}_B$ for box B. Set the continuous and compactly supported function as

$$g(x) = \max\{1 - R dist(x, B), 0\} \text{ for } R \text{ large enough}$$

so that

$$\int_{\mathbb{R}^d} |f-g| \le \epsilon$$

and then we have the claim.

$\square$

Before we see the instances of the third principle, we need first define local uniformity.

Definition 1.4.10. *(Local Uniformity). $f_n : \mathbb{R}^d \to \mathbb{C}$ converges to $f : \mathbb{R}^d \to \mathbb{C}$ locally uniformly if $\forall E \subset \mathbb{R}^d$ bounded, $f_n \to f$ uniformly on E.*

For example,
(1). $f)n(x) = \frac{x}{n}$, $n = 1, 2, ...$, $f_n \to f$ locally but not globally uniformly.
(2). $\sum_{i=1}^{N} \frac{x^n}{n!} \to e^x$ locally but not globally uniformly.
(3). $f_n(x) = \begin{cases} \frac{1}{nx}\mathbf{1}_{x>0} & x \neq 0 \\ 0 & x = 0 \end{cases}$, $f_n \to f$ pointwise, but neither locally nor uniformly.

Theorem 1.4.5. *(Egorov). Let $f_n : \mathbb{R}^d \to \mathbb{C}$ converge pointwise a.e. to $f : \mathbb{R}^d \to \mathbb{C}$. f_n and f are measurable. Then, $\forall \epsilon > 0$, $\exists A \subset \mathbb{R}^d$ measurable with $m(A) < \epsilon$ and $f_n \to f$ locally uniformly on $\mathbb{R}^d \setminus A$.*

Proof. We may assume $f_n \to f$ pointwise everywhere by including $\{x : f_n(x) \not\to f(x)\}$ inside A.
thus, $\forall x \in \mathbb{R}^d$, $\forall m > 0$, $\exists N(x, m) \in \mathbb{N}$ (this means that the choice of N depends on x and m) such that

$$|f_n(x) - f(x)| \leq \frac{1}{m} \ \forall n > N(x, m)$$

Write this set-theoretically, we have for each m,

$$E_{N,m} = \{x \in \mathbb{R}^d : |f_n(x) - f(x)| > \frac{1}{m} \text{ for some } n \geq N\}$$

as our "bad set", and

$$\bigcap_{N \in \mathbb{N}} E_{N,m} = \emptyset \ \forall m > 0$$

It is clear that $E_{N,m}$ is Lebesgue measurable and monotonic decreasing in N. Applying the downward monotone convergence theorem, for each m we have

$$m(E_{N,m} \cap B(0, R)) \to m(\emptyset) = 0 \ \forall R \in (0, +\infty)$$

That is, $\forall m \geq 1$, $\exists N_m$ such that

$$m(E_{N,m} \cap B(0, m)) \leq \frac{\epsilon}{2^m} \ \forall N \geq N_m$$

(This holds when $N = N_m$).
Let $A = \bigcup_{m=1}^{\infty} E_{N_m,m} \cap B(0, m)$, then

$$m(A) \leq \sum_{m=1}^{\infty} m(E_{N_m,m} \cap B(0, m)) = \epsilon$$

Thus $\forall \delta > 0$, $R \in (0, +\infty)$, take $m > \max\{R, \frac{1}{\delta}\}$. we have for any $x \in B(0, R) \setminus A$, $x \neq E_{N_m,m}$. Then, $|f_n(x) - f(x)| \leq \frac{1}{m}$. Now we have the choice of N_m not depending on x. Therefore $f_n \to f$ uniformly on $B(0, R) \setminus A$. Since every bounded set in $\mathbb{R}^d$ is in such a ball, we have the claim. $\square$

Remark. This is not true if we don't have local uniformity. For example, "travelling bump" $f_n = \mathbf{1}_{n,n+1}$ converges to 0 pointwise, but not uniformly, not even if we delete any set of finite measure.

Now we look at another version of Littlewood's second principle (absolute integrable functions are almost continuous).

Theorem 1.4.6. *(Lusin's Theorem). Let $f : \mathbb{R}^d \to \mathbb{C}$ be absolutely integrable. Then $\forall \epsilon > 0$, $\exists E \subset \mathbb{R}^d$ with $m(E) \leq \epsilon$ such that the restriction of f to $\mathbb{R}^d \setminus E$.*

Remark. We need the restriction of f rather than f itself. For example, $f = \mathbf{1}_{\mathbb{Q}\cap[0,1]}$ is not continuous on $\mathbb{R} \setminus E$ for $E \subset [0, 1]$ with finite measure, but it is continuous on $\mathbb{R} \setminus E$ if we take $E := \mathbb{Q}$.

Proof. We use Egorov's Theorem and the third version of Littlewood's second principle in this proof.
By the density of $C_c(\mathbb{R}^d)$ in $L^1(\mathbb{R}^d)$, let $\epsilon > 0$, then $\forall n \in \mathbb{N}$, $\exists f_n \in C_c(\mathbb{R}^d)$ such that $||f_n - f||_{L^1(\mathbb{R}^d)} \leq \frac{\epsilon}{4^n}$.
By Markov's Inequality, for

$$E_n = \{x \in \mathbb{R}^d : |f_n(x) - f(x)| > \frac{1}{2^{n-1}}\}$$

we have

$$m(E_n) \leq 2^{n-1}||f_n - f||_{L^1(\mathbb{R}^d)} \leq \frac{\epsilon}{2^{n+1}}$$

Let $E := \bigcup_{n=1}^{\infty} E_n$, then it is measurable and

$$m(E) \leq \sum_{i=1}^{\infty} m(E_n) = \frac{\epsilon}{2}$$

and $f_n \to f$ uniformly on $\mathbb{R}^d \setminus E$ by Egorov's theorem.
Since the uniform limit of continuous functions is continuous, we have f is continuous on $\mathbb{R}^d \setminus E$. ∎

Proposition 1.4.4. *(Littlewood-like principles). The following facts are not, strictly speaking, instances of any of Littlewood's three principles, but are in a similar spirit.*

1. *(Absolutely integrable functions almost have bounded support) Let $f : \mathbb{R}^d \to \mathbb{C}$ be an absolutely integrable function, and let $\varepsilon > 0$. Then there exists a ball $B(0, R)$ outside of which f has an L^1 norm of at most ε, or in other words that*

$$\int_{\mathbb{R}^d \setminus B(0,R)} |f(x)| \, dx \leq \varepsilon.$$

2. *(Measurable functions are almost locally bounded) Let $f : \mathbb{R}^d \to \mathbb{C}$ be a measurable function supported on a set of finite measure, and let $\varepsilon > 0$. Then there exists a measurable set $E \subset \mathbb{R}^d$ of measure at most ε outside of which f is locally bounded, or in other words that for every $R > 0$ there exists $M < \infty$ such that $|f(x)| \leq M$ for all $x \in B(0, R) \setminus E$.*

Proof. (1). Since f is absolutely integrable, there exists a continuous, compactly supported g such that $||f-g||_{L^1(\mathbb{R}^d)} \leq \epsilon$. Since g is compactly supported, $\exists R > 0$ such that $B(0,R)$ contains the support of g. Then, $g(x) = 0 \; \forall x \in \mathbb{R}^d \setminus B(0,R)$. Therefore, by linearity and non-negativity of Lebesgue integral,

$$\begin{aligned} ||f-g||_{L^1(\mathbb{R}^d)} &= \int_{\mathbb{R}^d} |f-g| \\ &= \int_{\mathbb{R}^d \setminus B(0,R)} |f-g| + \int_{B(0,R)} |f-g| \\ &= \int_{\mathbb{R}^d \setminus B(0,R)} |f| + \int_{B(0,R)} |f-g| \\ &\geq \int_{\mathbb{R}^d \setminus B(0,R)} |f| \end{aligned}$$

Since the LHS $\leq \epsilon$, we have $\int_{\mathbb{R}^d \setminus B(0,R)} |f| \leq \epsilon$.

(2). **Exercise 1.3.23** says that the hypothesis in Lusin's theorem can be modified to be that f being measurable and finite everywhere (or a.e.). Then, there exists a measurable set $E \subset \mathbb{R}^d$ with $m(E) \leq \epsilon$, such that f is continuous on $\mathbb{R}^d \setminus E$.

We can enlarge E to an open set F such that $E \subset F$, $m(F) \leq m(E) + \epsilon \leq 2\epsilon$. Then $\mathbb{R}^d \setminus F \subset \mathbb{R}^d \setminus E$, and f is continuous on $\mathbb{R}^d \setminus F$.

$\forall R > 0$, $B(0,R) \setminus F = B(0,R) \cap (\mathbb{R}^d \setminus F)$ and we can see that it is closed and bounded by $B(0,R)$ itself. (Here the ball should be a closed ball and contains its boundary. Then, $B(0,R) \setminus F$ is compact. Since $B(0,R) \setminus F \subset \mathbb{R}^d \setminus F$, f is continuous on $B(0,R) \setminus F$, and it is uniformly continuous because of restricting to compact sub-support. Since a uniformly continuous function is bounded, $\exists M > 0$, $|f| \leq M$. $\square$

1.5 Abstract Measure Spaces

Now we study the measure and integration on a general space $\mathcal{X}$. Generally, a measurable space is specified by the followings
(1). a set $\mathcal{X}$.
(2). $\mathcal{B}$ a collection of subsets of $\mathcal{X}$ that are "measuable".
(3). a mapping $\mu : \mathcal{B} \to [0, +\infty]$ as a measure.

Definition 1.5.1. *(Boolean Algebra). Let $\mathcal{X}$ be a set. A Boolean Algebra on $\mathcal{X}$ is a collection os subsets of $\mathcal{X}$ such that*
(1). $\emptyset \in \mathcal{B}$.
(2). $E \in \mathcal{B} \Rightarrow \mathcal{X} \setminus E \in \mathcal{B}$.
(3). (Finite Unions). $E, F \in \mathcal{B} \Rightarrow E \cup F \in \mathcal{B}$.

Generalize the (3) of Boolean Algebra we have σ-algebra.

Definition 1.5.2. *(σ-algebra). Let $\mathcal{X}$ be a set. A σ-algebra on $\mathcal{X}$ is a collection os subsets of $\mathcal{X}$ such that*
(1). $\emptyset \in \mathcal{B}$.
(2). $E \in \mathcal{B} \Rightarrow \mathcal{X} \setminus E \in \mathcal{B}$.
(3). (Countable Unions). $E_1, E_2, ... \in \mathcal{B} \Rightarrow \bigcup_{i=1}^{\infty} E_i \in \mathcal{B}$.

The pair $(\mathcal{X}, \mathcal{B})$ of a set $\mathcal{X}$ together with a σ-algebra on that set is a *measurable space*.

For σ-algebras $\mathcal{B}$ and $\mathcal{B}'$ with $\mathcal{B} \subseteq \mathcal{B}'$, we say that $\mathcal{B}'$ *refines* $\mathcal{B}$ or is a *refinement* of $\mathcal{B}$. Or we say $\mathcal{B}$ is *coarser* than $\mathcal{B}'$ or is a *coersening* of $\mathcal{B}'$.

Here we give some examples of σ-algebras.
(1). $\mathcal{L}(\mathbb{R}^d)$ is a σ-algebra on $\mathbb{R}^d$, but $\mathcal{J}(\mathbb{R}^d)$ and $\mathcal{E}(\mathbb{R}^d)$ are not.
(2). Trivial σ-algebra: $\{\emptyset, \mathcal{X}\}$
(3). Discrete σ-algebra: $\{E : E \subset \mathcal{X}\} = 2^{\mathcal{X}}$ as power sets.
(4). Null algebra: $\mathcal{N}(\mathbb{R}^d) = \{E \subset \mathbb{R}^d : m(E) = 0 \text{ or } m(\mathbb{R}^d \setminus E) = 0\}$.
(5). Atomic σ-algebra: given a partition of $\mathcal{X}$ into disjoint sets $(A_\alpha)_{\alpha \in I}$ as "atoms", let $\mathcal{B} = \{\bigcup_{\alpha \in J} A_\alpha : J \subseteq I\}$.

Proposition 1.5.1. *(Intersection of σ-algebras). The intersection*

$$\bigwedge_{\alpha \in I} \mathcal{B}_\alpha := \bigcap_{\alpha \in I} \mathcal{B}_\alpha$$

of an arbitrary (and possibly infinite or uncountable) number of σ-algebras $\mathcal{B}_\alpha$ is again a σ-algebra, and is the finest σ-algebra that is coarser than all of the $\mathcal{B}_\alpha$.

Proof. Here we prove by verifying the definition.
(i). Since $\emptyset \in \mathcal{B}_\alpha \ \forall \alpha \in I$, we have $\emptyset \in \bigwedge_{\alpha \in I} \mathcal{B}_\alpha$.
(ii). If $E \in \bigwedge_{\alpha \in I} \mathcal{B}_\alpha$, then $E \in \mathcal{B}_\alpha \ \forall \alpha \in I$. Since each $\mathcal{B}_\alpha$ is a σ-algebra, we have $E^C \in \mathcal{B}_\alpha \ \forall \alpha \in I$. Therefore, $E^C \in \bigwedge_{\alpha \in I} \mathcal{B}_\alpha$.
(iii). If $E_1, E_2, ... \in \bigwedge_{\alpha \in I} \mathcal{B}_\alpha$, then $E_1, E_2, ... \in \mathcal{B}_\alpha \ \forall \alpha \in I$. Then $\bigcup_{n=1}^{\infty} E_n \in \mathcal{B}_\alpha \ \forall \alpha \in I$. Then $\bigcup_{n=1}^{\infty} E_n \in \mathcal{B}_\alpha \in \bigwedge_{\alpha \in I} \mathcal{B}_\alpha$.
Therefore, $\bigwedge_{\alpha \in I} \mathcal{B}_\alpha$ verifies the definition, and it is a σ-algebra.
Suppose that another σ-algebra $\mathcal{B}'$ is coarser than all of the $\mathcal{B}_\alpha$, that is, $\mathcal{B}' \subset \mathcal{B}_\alpha \ \forall \alpha \in I$.

Then, $\mathcal{B}' \subset \bigwedge_{\alpha \in I} \mathcal{B}_\alpha$. This way, $\bigwedge_{\alpha \in I} \mathcal{B}_\alpha$ is finer than any σ-algebra that is coarser than all of the $\mathcal{B}_\alpha$. □

Definition 1.5.3. *(Generation of σ-algebras). For a family of sets $\mathcal{F} \in 2^{\mathcal{X}}$, define $\langle \mathcal{F} \rangle$, the σ-algebra generated by (F), as the intersection of all σ-algebras that contains $\mathcal{F}$.*

Remark. If $\mathcal{F}$ is a family of sets in $\mathcal{X}$. $P(E)$ is a property of sets $E \subset \mathcal{X}$ which obeys the following axioms
(1). $P(\emptyset)$ is true.
(2). $P(E)$ is true for all $E \in \mathcal{F}$.
(3). If $P(E)$ is true for some $E \in \mathcal{X}$, then $P(\mathcal{X} \setminus E)$ is true also.
(4) If $E_1, E_2, ... \in \mathcal{X}$ are such that $P(E_n)$ is true for all n, then $P(\bigcup_{n=1}^{\infty})$ is true also.
Then we conclude that $P(E)$ is true for all $E \in \langle \mathcal{F} \rangle$.

Definition 1.5.4. *(Borel σ-algebra). The Borel σ-algebra $\mathcal{B}[\mathcal{X}]$ on a topolitical space $(\mathcal{X}, \mathcal{T})$ is the σ-agebra generated by $\mathcal{T}$, the class of open sets in $\mathcal{X}$. Elements of $\mathcal{B}[\mathcal{X}]$ are called Borel measurable.*

Proposition 1.5.2. *(Generation of Borel σ-algebras). The Borel σ-algebra $\mathcal{B}[\mathbb{R}^d]$ of a Euclidean set is generated by any of the following collections of sets:*

(i) The open subsets of $\mathbb{R}^d$.

(ii) The closed subsets of $\mathbb{R}^d$.

(iii) The compact subsets of $\mathbb{R}^d$.

(iv) The open balls of $\mathbb{R}^d$.

(v) The boxes in $\mathbb{R}^d$.

(vi) The elementary sets in $\mathbb{R}^d$.

Proof. First, by definition, $\mathcal{B}[\mathbb{R}^d]$ is the σ-algebra generated by the open subsets of $\mathbb{R}^d$. This means that $\mathcal{B}[\mathbb{R}^d]$ is the intersection of all the σ-algebras that contains the open subsets of $\mathbb{R}^d$. To prove that $\mathcal{B}[\mathbb{R}^d]$ is generated by another family of sets is to prove that, every σ-algebra that contains the open subsets of $\mathbb{R}^d$ also contains that family of sets, and vice versa. This logic can be generalized to show that $\mathcal{B}[\mathbb{R}^d]$ can be generated by other family of sets from (ii) to (vi).
(i)-(ii). Let $\mathcal{B}$ be any σ-algebra containing all the open subsets of $\mathbb{R}^d$. Let E be any closed set in $\mathbb{R}^d$. Then E^C is open, and $E^C \in \mathcal{B}$. Then, $E = (E^C)^C \in \mathcal{B}$.
Let $\mathcal{B}$ be any σ-algebra containing all the closed subsets of $\mathbb{R}^d$. Let E be any open set in $\mathbb{R}^d$. Then E^C is closed, and $E^C \in \mathcal{B}$. Then, $E = (E^C)^C \in \mathcal{B}$.
(ii)-(iii). Let $\mathcal{B}$ be any σ-algebra containing all the closed subsets of $\mathbb{R}^d$. Let E be any compact set in $\mathbb{R}^d$. Then, by Heine-Borel Theorem, E is closed and $E \in \mathcal{B}$.
Let $\mathcal{B}$ be any σ-algebra containing all the compact subsets of $\mathbb{R}^d$. Let E be any closed set in $\mathbb{R}^d$. Then, $\forall n \in \mathbb{N}$, $F \cap \overline{B(0,n)}$ is bounded and closed and thus compact. Then $E \cap \overline{B(0,n)} \in \mathcal{B}$. By countable unions, $\bigcup_{n=1}^{\infty} E \cap \overline{B(0,n)} = E \in \mathcal{B}$.
(iii)-(iv). Let $\mathcal{B}$ be any σ-algebra containing all the compact subsets of $\mathbb{R}^d$. Let E be any open balls in $\mathbb{R}^d$. By **Lemma 1.2.11**, E can be written as the countable union of almost

disjoint closed boxes $E = \bigcup_{n=1}^{\infty} B_n$. Since B_n $\forall n \in \mathbb{N}$ is closed and bounded, it is compact, then $B_n \in \mathcal{B}$ $\forall n \in \mathbb{N}$. By countable unions, $E = \bigcup_{n=1}^{\infty} B_n \in \mathcal{B}$.
Let $\mathcal{B}$ be any σ-algebra containing all the open balls of $\mathbb{R}^d$. Let E be any compact set in $\mathbb{R}^d$. Then F^C is open. Since every open set can be written as a countable union of open balls, by countable unions, $F^C \in \mathcal{B}$, and then $F = (F^C)^C \in \mathcal{B}$.
(iv)-(v). Let $\mathcal{B}$ be any σ-algebra containing all the open balls of $\mathbb{R}^d$. Let B be any box in $\mathbb{R}^d$. Then, B can be written as the intersection of an open box B' and a closed box B'', $B = B' \cap B''$. Since $B^C = B'^C \cup B''^C$ and B'^C is closed and B''^C is open, which means that each of them is in $\mathcal{B}$ as we have shown in (i) and (ii), by countable unions, $B^C \in \mathcal{B}$. Therefore, $B = (B^C)^C \in \mathcal{B}$.
Let $\mathcal{B}$ be any σ-algebra containing all the boxes of $\mathbb{R}^d$. Let E be any open ball in $\mathbb{R}^d$. Then by **Lemma. 1.2.11**, $E = \bigcup_{n=1}^{\infty}$ where $B_1, B_2, ...$ are almost disjoint boxes. Since $B_1, B_2, ... \in \mathcal{B}$, by countable unions, $E \in \mathcal{B}$.
(v)-(vi). Let $\mathcal{B}$ be any σ-algebra containing all the boxes of $\mathbb{R}^d$. Let E be any elementary set in $\mathbb{R}^d$. Then, by definition, $E = \bigcup_{i=1}^{n} B_i$ where B_i are boxes. Then, by countable unions, $E \in \mathcal{B}$.
Let $\mathcal{B}$ be any σ-algebra containing all the elementary sets in $\mathbb{R}^d$. Let B be any box in $\mathbb{R}^d$. Then B itself is an elementary set and then $B \in \mathcal{B}$. □

Proposition 1.5.3. *Let E be a Borel measurable subset of $\mathbb{R}^{d_1+d_2}$.*

(i) For any $x_1 \in \mathbb{R}^{d_1}$, the slice

$$\{x_2 \in \mathbb{R}^{d_2} : (x_1, x_2) \in E\}$$

is a Borel measurable subset of $\mathbb{R}^{d_2}$. Similarly, for every $x_2 \in \mathbb{R}^{d_2}$, the slice

$$\{x_1 \in \mathbb{R}^{d_1} : (x_1, x_2) \in E\}$$

is a Borel measurable subset of $\mathbb{R}^{d_1}$.

(ii) Give a counterexample to show that this claim is not *true if "Borel" is replaced with "Lebesgue" throughout.*
(Hint: the Cartesian product of any set with a point is a null set, even if the first set was not measurable.)

Proof. The proof uses **Remark 1.4.15**.
(i). Let $\mathcal{F}$ be the family of open sets in $\mathbb{R}^{d_1+d_2}$, let $E \subset \mathbb{R}^{d_1+d_2}$, then $P(E) : \forall x_1 \in \mathbb{R}^{d_1}, \{x_2 \in \mathbb{R}^{d_2} : (x_1, x_2) \in E\} \in \mathcal{B}[\mathbb{R}^{d_2}]$ is a property of set E.
(i). we verify that $P(\emptyset) : \forall x_1 \in \mathbb{R}^{d_1}, \{x_2 \in \mathbb{R}^{d_2} : (x_1, x_2) \in \emptyset\} \in \mathcal{B}[\mathbb{R}^{d_2}]$ is true. Since $\{x_2 \in \mathbb{R}^{d_2} : (x_1, x_2) \in \emptyset\} = \emptyset \in \mathcal{B}[\mathbb{R}^{d_2}]$ by definition of σ-algebra, $P(\emptyset)$ is true.
(ii). we verify that $P(E)$ is true for all $E \in \mathcal{F}$. Since $E \in \mathcal{F}$ is an open set, $\{x_2 \in \mathbb{R}^{d_2} : (x_1, x_2) \in E\}$ is an open set in $\mathbb{R}^{d_2}$, then $\{x_2 \in \mathbb{R}^{d_2} : (x_1, x_2) \in E\} \in \mathcal{B}[\mathbb{R}^{d_2}]$ by the definition of Borel σ-algebra. Thus $P(E)$ is true for all $E \in \mathcal{F}$.
(iii). We verity that if $P(E)$ is true for some $E \in \mathbb{R}^{d_1+d_2}$, then $P(\mathbb{R}^{d_1+d_2} \setminus E)$ is true also. Since $\{x_2 \in \mathbb{R}^{d_2} : (x_1, x_2) \in E\} \in \mathcal{B}[\mathbb{R}^{d_2}]$ is true for some $E \in \mathbb{R}^{d_1+d_2}$, $\{x_2 \in \mathbb{R}^{d_2} : (x_1, x_2) \in \mathbb{R}^{d_1+d_2} \setminus E\} = \mathbb{R}^{d_2} \setminus \{x_2 \in \mathbb{R}^{d_2} : (x_1, x_2) \in E\} \in \mathcal{B}[\mathbb{R}^{d_2}]$ by the definition of σ-algebra.
(iv). we verify that if $E_1, E_2, ... \in \mathbb{R}^{d_1+d_2}$ are such that $P(E_n)$ is true for all n, then

$P(\bigcup_{n=1}^\infty E_n)$ is true also. Since $\{x_2 \in \mathbb{R}^{d_2} : (x_1,x_2) \in E_n\} \in \mathcal{B}[\mathbb{R}^{d_2}]$ is true for all n, then $\{x_2 \in \mathbb{R}^{d_2} : (x_1,x_2) \in \bigcup_{n=1}^\infty E_n\} = \bigcup_{n=1}^\infty \{x_2 \in \mathbb{R}^{d_2} : (x_1,x_2) \in E_n\} \in \mathcal{B}[\mathbb{R}^{d_2}]$ by the definition of σ-algebra.
Thus, $P(E) : \{x_2 \in \mathbb{R}^{d_2} : (x_1,x_2) \in E\} \in \mathcal{B}[\mathbb{R}^{d_2}]$ is true for all $E \in \langle \mathcal{F} \rangle = \mathcal{B}[\mathbb{R}^{d_1+d_2}]$.

Let $\mathcal{F}$ be the family of open sets in $\mathbb{R}^{d_1+d_2}$, let $E \in \mathbb{R}^{d_1+d_2}$, then $P(E) : \forall x_2 \in \mathbb{R}^{d_2}, \{x_1 \in \mathbb{R}^{d_1} : (x_1,x_2) \in E\} \in \mathcal{B}[\mathbb{R}^{d_1}]$ is a property of sets E.
(i). we verify that $P(\emptyset) : \forall x_2 \in \mathbb{R}^{d_2}, \{x_1 \in \mathbb{R}^{d_1} : (x_1,x_2) \in \emptyset\} \in \mathcal{B}[\mathbb{R}^{d_1}]$ is true. Since $\{x_1 \in \mathbb{R}^{d_1} : (x_1,x_2) \in \emptyset\} = \emptyset \in \mathcal{B}[\mathbb{R}^{d_1}]$ by definition of σ-algebra, $P(\emptyset)$ is true.
(ii). we verify that $P(E)$ is true for all $E \in \mathcal{F}$. Since $E \in \mathcal{F}$ is an open set, $\{x_1 \in \mathbb{R}^{d_1} : (x_1,x_2) \in E\}$ is an open set in $\mathbb{R}^{d_1}$, then $\{x_1 \in \mathbb{R}^{d_1} : (x_1,x_2) \in E\} \in \mathcal{B}[\mathbb{R}^{d_1}]$ by the definition of Borel σ-algebra. Thus $P(E)$ is true for all $E \in \mathcal{F}$.
(iii). We verity that if $P(E)$ is true for some $E \in \mathbb{R}^{d_1+d_2}$, then $P(\mathbb{R}^{d_1+d_2} \setminus E)$ is true also. Since $\{x_1 \in \mathbb{R}^{d_1} : (x_1,x_2) \in E\} \in \mathcal{B}[\mathbb{R}^{d_1}]$ is true for some $E \in \mathbb{R}^{d_1+d_2}$, $\{x_1 \in \mathbb{R}^{d_1} : (x_1,x_2) \in \mathbb{R}^{d_1+d_2} \setminus E\} = \mathbb{R}^{d_1} \setminus \{x_1 \in \mathbb{R}^{d_1} : (x_1,x_2) \in E\} \in \mathcal{B}[\mathbb{R}^{d}]$ by the definition of σ-algebra.
(iv). we verify that if $E_1, E_2, ... \in \mathbb{R}^{d_1+d_2}$ are such that $P(E_n)$ is true for all n, then $P(\bigcup_{n=1}^\infty E_n)$ is true also. Since $\{x_1 \in \mathbb{R}^{d_1} : (x_1,x_2) \in E_n\} \in \mathcal{B}[\mathbb{R}^{d_1}]\}$ is true for all n, then $\{x_1 \in \mathbb{R}^{d_1} : (x_1,x_2) \in \bigcup_{n=1}^\infty E_n\} = \bigcup_{n=1}^\infty \{x_1 \in \mathbb{R}^{d_1} : (x_1,x_2) \in E_n\} \in \mathcal{B}[\mathbb{R}^{d_1}]$ by the definition of σ-algebra.
Thus, $P(E) : \{x_1 \in \mathbb{R}^{d_1} : (x_1,x_2) \in E\} \in \mathcal{B}[\mathbb{R}^{d_1}]$ is true for all $E \in \langle \mathcal{F} \rangle = \mathcal{B}[\mathbb{R}^{d_1+d_2}]$.

(ii)First, we construct a set F such that $F \notin \mathcal{L}(\mathbb{R})$, following **Proposition 1.2.18**.
$(\mathbb{Q},+)$ is a subgroup of $(\mathbb{R},+)$, and it partitions $\mathbb{R}$ into disjoint cosets $x+\mathbb{Q}$ for $x \in \mathbb{R}$. Then, this creates a quotient group $\mathbb{R}/\mathbb{Q} = \{x+\mathbb{Q} : x \in \mathbb{R}\}$. Each coset $C = x+\mathbb{Q} \in \mathbb{R}/\mathbb{Q}$ is dense in $\mathbb{R}$, so it has non-empty intersection with $[0,1]$. By axiom of choice, select $x_c \in C \cap [0,1]$ for each $C \in \mathbb{R}/\mathbb{Q}$ and put them together into a set $F = \{x_c : x_c \in C \cap [0,1]\ \forall C \in \mathbb{R}/\mathbb{Q}\}$. We have already prove that $F \notin \mathcal{L}(\mathbb{R})$.
Since $\{0\} \times F$ is a null set, it is Lebesgue measurable and has measure zero. However, after slicing we are only left with F, which is not Lebesgue measurable. □

Definition 1.5.5. *(Measure Space). Let $(\mathcal{X},\mathcal{B})$ be a measurable space. A measure on $(\mathcal{X},\mathcal{B})$ is a map $\mu : \mathcal{B} \to [0,+\infty]$ that obeys*
(1). $\mu(\emptyset) = 0$.
(2). (σ-additivity): $E_1, E_2, ... \in \mathcal{B}$ are disjoint measuable sets, then

$$\mu(\bigcup_{n=1}^\infty E_n) = \sum_{n=1}^\infty \mu(E_n)$$

Such a triple $(\mathcal{X},\mathcal{B},\mu)$ is called a measure space.

Here we give some examples of measure.
(1). Zero measure: $\mu(E) = 0\ \forall E \in \mathcal{B}$.
(2). Dirac measure: $\delta_x(E) := \mathbf{1}_E(x)$ for a given point $x \in \mathcal{X}$. It is a "point mass" for $E \in \mathcal{B}$.
(3). Counting measure: #:

$$\#(E) := \begin{cases} \text{cardinality of } E & \text{when finite} \\ +\infty & \text{else} \end{cases}$$

(4). Countable combination of measure μ_i

$$\mu = \sum_{i=1}^{\infty} c_i\mu_i \text{ for } c_i \in [0, +\infty]$$

where $(c_i\mu_i)(E) = c_i\mu_i(E)$, $\mu(E) = \sum_{i=1}^{\infty} c_i\mu_i(E)$.
Note that for countable $\mathcal{X}$, $\# = \sum_{x\in\mathcal{X}} \delta_x$, because $(\sum_{x\in\mathcal{X}} \delta_x)(E) = \sum_{x\in\mathcal{X}} \delta_x(E) = \sum_{x\in\mathcal{X}} \mathbf{1}_E(x) = \#E$
(5). Let $f : \mathbb{R}^d \to [0, +\infty]$ be measurable. For $f \in L^1(\mathbb{R}^d)$, set $\mu(E) = \int_E f dm$. For σ-additivity, we need to verify that for $E_1, E_2, ... \in \mathcal{L}(\mathbb{R}^d)$ disjoint,

$$\mu(\bigcup_{n=1}^{\infty} E_n) = \int_{\bigcup_{n=1}^{\infty} E_n} f dm = \sum_{n=1}^{\infty} \int_{E_n} f\mathbf{1}_{E_i} dm$$

Notice that

$$\int_{\bigcup_{n=1}^{\infty} E_n} f dm = \int_{\mathbb{R}^d} \sum_{i=1}^{\infty} f\mathbf{1}_{E_i} dm$$

We can exchange the sum according to the monotone convergence theorem later.

$$LHS = \sum_{i=1}^{\infty} \int_{\mathbb{R}^d} f\mathbf{1}_{E_i} dm \tag{8}$$

1.6 Differentiation Theorems

1.6.1 Lebesgue Differentiation Theorem

We want to show that

Theorem 1.6.1. *Let $f : \mathbb{R}^d \to \mathbb{C}$ be absolutely integrable. Then for almost every $x \in \mathbb{R}^d$,*

$$\frac{1}{m(B(x,r))} \int_{B(x,r)} |f(y) - f(x)| dy \to 0$$

and hense

$$\lim_{r\to 0} \frac{1}{m(B(x,r))} \int_{B(x,r)} f(y) dy = f(x)$$

as $r \to 0$, where $B(x,r) := \{y \in \mathbb{R}^d : |x-y| < r\}$. We say x is a Lebesgue point of f.

This result is a convergence theorem, because there are

1. an assertion that for all functions f in a given class (in this case, the class of absolutely integrable dunctions $f : \mathbb{R}^d \to \mathbb{C}$).
2. a certain sequence of linear expressions $T_r f$ (in this case, $T_r f(x) = \frac{1}{m(B(x,r))} \int_{B(x,r)} f(y) dy$)
.

To prove such convergence theorem, we use *dense argument*:

1. establish claim for "dense" class of "nice" functions. By "dense" we means that a general function f in the original class can be approximated to arbitrary accuracy in a suitable sense by a function in the nice subclasses.

2. establish a maximal inequality to control errors.

Frist we see an example of using *dense argument*:

Proposition 1.6.1. *(Translation is continuous in L^1). For $f : \mathbb{R}^d \to \mathbb{C}$, $h \in \mathbb{R}^d$, set the shift function $f_h : \mathbb{R}^d \to \mathbb{C}$ to be $f_h(x) := f(x-h)$. If $f \in L^1(\mathbb{R}^d)$, then $||f_h - f||_{L^1(\mathbb{R}^d)} \to 0$ as $h \to 0$.*

Proof. First verify this claim for a dense subclass of f. Consider first the case $f \in C_c(\mathbb{R}^d$. Then $f_h \to f$ uniformly. Then

$$\begin{aligned} ||f - f_h||_{L^1} &= \int_{\mathbb{R}^d} |f - f_h| \\ &\leq \int_{Supp(f) \cup Supp(f_h)} ||f - f_h||_{L^\infty} \\ &\leq ||f - f_h||_{L^\infty}(m(Supp(f)) + m(Supp(f_h))) \\ &= ||f - f_h||_{L^\infty} 2m(Supp(f_h)) \\ &\to 0 \end{aligned}$$

Now, for general $f \in L^1(\mathbb{R}^d)$, let $g \in C_c(\mathbb{R}^d)$ such that $||f - g||_{L^1} < \epsilon$ where $\epsilon > 0$ is fixed. By triangle ineuality,

$$\begin{aligned} ||f - f_h||_{L^1} &\leq ||g - g_h||_{L^1} + ||f - g||_{L^1} + ||f_h - g_h||_{L^1} \\ &= ||g - g_h||_{L^1} + ||f - g||_{L^1} + ||(f - g)_h||_{L^1} \\ &= ||g - g_h||_{L^1} + 2||f - g||_{L^1} \\ &< ||g - g_h||_{L^1} + 2\epsilon \\ &\to 2\epsilon \end{aligned} \tag{9}$$

Then $||f - f_h||_{L^1} \to 0$ as $h \to 0$. □

To prove that Theorem 1.6.1. holds, we use the density argument. The first step which is the densesubclass case is eady.

Proposition 1.6.2. *Theorem 1.6.1. holds whenever f is continuous.*

Proof. ddd. □

The quantitative estimate needed is the following.

Theorem 1.6.2. *(Hardy-Littlewood Maximal Inequality). For $f : \mathbb{R}^d \to \mathbb{C}$ in $L^1(\mathbb{R}^d)$, and the Hardy-Littlewood maximal function*

$$Mf(x) := \sup_{r>0} \frac{1}{m(B(x,r))} \int_{B(x,r)} |f|, \; x \in \mathbb{R}^d$$

for any $\lambda > 0$, we have

$$m(\{x \in \mathbb{R}^d : Mf(x) \geq \lambda\}) \leq \frac{C_d}{\lambda} ||f||_{L^1(\mathbb{R}^d)}$$

for some constant $C_d > 0$ depending only on d.

To prove the Hardy-Littlewood Maximal Inequality, we need the following Lemma.

Lemma 1.6.1. *(Vitali's covering lemma). For any finite collection of open balls $\mathcal{B} = \{B_1, B_2, ..., B_n\}$, $B_i \subset \mathbb{R}^d$, there is a sub-collection $\mathcal{B}' = \{B_1', B_2', ..., B_m'\}$ of pairwise disjoint balls, such that*

$$\bigcup_{i=1}^{n} B_i \subset \bigcup_{i=1}^{m} 3B_i'$$

where $3B_i$ is the ball with the same center but 3 times diameter of B_i. By finite additivity,

$$m(\bigcup_{i=1}^{n} B_i) \leq 3^d \sum_{i=1}^{m} m(B_i')$$

Proof. Here we prove this lemma using a greedy algorithm.
Take B_1' to be the largest ball among $\mathcal{B}$. Then we do the following infuction. Now having chosen $\{B_1', B_2', ..., B_k'\}$. If the remaining balls each has non-empty intersections with $\bigcup_{i=1}^{k} B_i'$, then stop. Otherwise, take the B_{k+1}' to be the largest among $\mathcal{B} \setminus \{B_1', B_2', ..., B_k'\}$ that is disjoint from $\bigcup_{j=1}^{k} B_j'$.
Therefore, we have

1. we must stop at $\leq n$ rounds.
2. ending collection $\mathcal{B}' = \{B_1', ..., B_m'\}$ must be pairwise disjoint.

Then it remains to show that each ball $B_i \in \mathcal{B}$ is covered by the triples $3B_j'$ of the subcollection. That is, fix arbitrary B_i where $1 \leq i \leq n$, it is enough to show that $B_i \subset \bigcup_{j=1}^{m} 3B_j'$. First, notice that $B_i \cap \bigcup_{j=1}^{m} B_i' \neq \emptyset$, since otherwise the algorithm won't stop at $\mathcal{B}' = \{B_1', ..., B_m'\}$.
Then, let $j_0 := \min\{j : B_j' \cap B_i \neq \emptyset\}$. Then, B_{j_0}' is the first ball in the subcollection $\mathcal{B}'$ that has non-empty intersection with B_i, and we also have $B_i \cap (\bigcup_{j=1}^{j_0-1} B_j') = \emptyset$.
Then, $m(B_i) \leq m(B_{j_0}')$ since otherwise B_i would have been chosen. Therefore $diam(B_i) \leq diam(B_{j_0}')$, so B_i cannot intersect with both B_{j_0}' and $(3B_{j_0}')^C$. Therefore, $\forall i \in \{1, ..., n\}$,

$$B_i \subset 3B_{j_0}' \subset \bigcup_{j=1}^{m} 3B_j'$$

Therefore

$$\bigcup_{i=1}^{n} B_i \subset \bigcup_{j=1}^{m} 3B_j'$$

$\square$

1.7 Citation

This is a citation[**?**].

References