MEASURE AND INTEGRATION

Math 631

Author
Bill Runzhe Zhao
Duke University
2024 fall

Contents

1 Measure Theory and Integration **3**
1.1 Elementary Sets and Measure 3
1.2 Jordan Measure 6
1.3 Lebesgue Measure 7
1.4 Citation 16

1 Measure Theory and Integration

1.1 Elementary Sets and Measure

The motivation to start this study is to lay a foundation to assign a measure to arbitrary sets $E \in \mathbb{R}^d$. The set can be nice set such as intervals, boxes, and polygons, but it can also be any generic or strange set that we wish to study. We want a measure that is well behaved under limits (e.g. Lebesgue measure is well behaved under pointwise limits).

Definition 1.1.1. *Finite Intervals:* $(a, b), (a, b], [a, b), [a, b]$ *for* $a, b \in \mathbb{R}$*. The length is* $b - a$ *and it can be zero.*

Definition 1.1.2. *Box:* $B \in \mathbb{R}^d$ *is a Cartesian product of intervals* $B = I_1 \times ... \times I_d$

Definition 1.1.3. *Elementary Set is any subset of* $\mathbb{R}^d$ *that is a finite union of boxes.*

We denote the set of elementary sets as $\mathcal{E}(\mathbb{R}^d)$. It does not include unbounded sets.

Proposition 1.1.1. *(Boolean closure) The set* $\mathcal{E}(\mathbb{R}^d)$ *is closed taking unions* $(E \cup F)$*, intersects* $(E \cap F)$*, differences* $(E \setminus F)$ *and symmetric differences* $(E \Delta F = (E \setminus F) \cup (F \setminus E))$*, as well as translations:* $E \in \mathcal{E}(\mathbb{R}^d), x \in \mathbb{R}^d \Rightarrow E + x = \{y + x : y \in E\} \in \mathcal{E}(\mathbb{R}^d)$.

Proof. Let $E, F \in \mathcal{E}(\mathbb{R}^d)$, then each of them is a finite union of boxes. $E = \bigcup_{i=1}^n B_i$, $F = \bigcup_{i=1}^m B'_i$.
Denote the collection of $\{B_i\}_{i=1}^n$ and $\{B'_i\}_{i=1}^m$ as a new set of boxes:
$\{C_i\}_{i=1}^{n+m} = \{B_1, ..., B_n, B'_1, ..., B'_m\}$ Then,

$$\begin{aligned} E \cup F &= (\bigcup_{i=1}^n B_i) \cup (\bigcup_{i=1}^m B'_i) \\ &= \bigcup_{i=1}^{n+m} C_i \end{aligned}$$

Thus, $E \cup F$ is a finite union of boxes and is an elementary set.

$$\begin{aligned} E \cap F &= (\bigcup_{i=1}^n B_i) \cap (\bigcup_{i=1}^m B'_i) \\ &= (B_1 \cap (\bigcup_{i=1}^m B'_i)) \cup ... \cup (B_n \cap (\bigcup_{i=1}^m B'_i) \\ &= ((B_1 \cap B'_1) \cup ... \cup (B_1 \cap B'_m)) \cup ... \cup ((B_n \cap B'_1) \cup ... \cup (B_n \cap B'_m)) \\ &= (B_1 \cap B'_1) \cup ... \cup (B_1 \cap B'_m) \cup ... \cup (B_n \cap B'_1) \cup ... \cup (B_n \cap B'_m) \end{aligned}$$

Now we prove that the intersection of two boxes is a box. Suppose $B_i = I_1^i \times ... \times I_d^i$, $B'_j = I_1^{'j} \times ... \times I_d^{'j}$, then $B_i \cap B'_j = (I_1^i \cap I_1^{'j}) \times ... \times (I_d^i \cap I_d^{'j})$. Since the intersection of two intervals is an interval (degenerate or non-degenerate), by definition, $B_i \cap B'_j$ is a box.
Then, by (2), $E \cap F$ is a finite union of $n \times m$ boxes, and it is an elementary set.

Now we look at difference.

$$\begin{aligned}
E \setminus F &= E \cap F^C \\
&= (\bigcup_{i=1}^{n} B_i) \cap (\bigcup_{i=1}^{m} B_i')^C \\
&= (\bigcup_{i=1}^{n} B_i) \cap (\bigcap_{i=1}^{m} B_i^{'C}) \\
&= (B_1 \cap (\bigcap_{i=1}^{m} B_i^{'C})) \cup ... \cup (B_n \cap (\bigcap_{i=1}^{m} B_i^{'C})) \\
&= ((B_1 \cap B_1^{'C}) \cap ... \cap (B_1 \cap B_m^{'C})) \cup ... \cup ((B_n \cap B_1^{'C}) \cap ... \cap (B_n \cap B_m^{'C})) \\
&= ((B_1 \setminus B_1^{'}) \cap ... \cap (B_1 \setminus B_m^{'})) \cup ... \cup ((B_n \setminus B_1^{'}) \cap ... \cap (B_n \setminus B_m^{'}))
\end{aligned}$$

Since B_i and $B_j^{'}$ are boxes, $B_i \setminus B_j^{'}$ are elementary. Then, by the first two parts that we just proved, each $((B_i \setminus B_1^{'}) \cap ... \cap (B_i \setminus B_m^{'}))$ is elementary, and their union is elementary. Thus, $E \setminus F$ is an elementary set.
Next, since E and F are elementary, $E \setminus F$ is elementary and $F \setminus E$ is also elementary. By the part we just proved, their union $(E \setminus F) \cup (F \setminus E)$ is also elementary. Thus, $E \Delta F$ is an elementary set.
Next,

$$\begin{aligned}
E + x &= \bigcup_{i=1}^{n} B_i + x \\
&= \bigcup_{i=1}^{n} (I_1^i + x) \times ... \times (I_d^i + x) \\
&= \bigcup_{i=1}^{n} I_1^{'i} \times ... \times I_d^{'i} \\
&= \bigcup_{i=1}^{b} B'_i
\end{aligned}$$

where we have translation of each interval by x in the second and third equality. Thus, $E + x$ is an elementary set.
Thus, we finished the proof that $\mathcal{E}(\mathbb{R}^d)$ is closed under union, intersection, difference, symmetric difference, and translation. $\square$

Note that this does not form an algebra. Now it is time to give the elementary set a measure.

Lemma 1.1.1. *Let $E \in \mathbb{R}^d$ be an elementary set.*
(1). E can be expressed (partitioned) as the finite union of disjoint boxes, that is, $E = \bigcup_{i=1}^{n} B_i$ for B_i pairwise disjoint.
(2). For any two such partitions $E = \bigcup_{i=1}^{n} B_i = \bigcup_{i=1}^{m} B_i'$, we have $\sum_{i=1}^{n} |B_i| = \sum_{i=1}^{m} |B_i'|$. We denote this value by $m(E) = m^d(E)$, the elementary measure of E. It is independent of partitions.

Proof. Begin with (i).
For the case $d = 1$, $E = \bigcup_{i=1}^n I_i$. Place the $2n$ endpoints of these intervals in increasing order (discarding repititions) and relable them to be $c_1 \le c_2 \le ... \le c_{2n}$.
Let $J_1, ...J_{4n-1}$ be disjoint intervals formed by these endpoints where

$$J_i = \{c_i\} \ for \ 1 \le i \le 2n$$

$$J_i = (c_{i-2n}, c_{i-2n+1}) \ for \ 2n+1 \le i \le 4n-1$$

Basically we have $2n$ endpoints and $2n-1$ open intervals between endpoints by doing this. Then, each I_i can be expressed as some subcollection of $J_1, ..., J_{4n-1}$. Then, $E = \bigcup_{k: J_k \cap E \neq \emptyset} J_k$.
For general cases $d \ge 2$ Now we have $E = \bigcup_{i=1}^n B_i$ where $B_i = I_i^1 \times ... \times I_i^d$. For each m such that $1 \le m \le d$, we apply the $d = 1$ case to get a family of disjoint intervals $\{J_k^m\}_{k=1}^{n_m}$ such that $\bigcup_{i=1}^n I_i^m = \bigcup_{i=1}^{n_m} J_k^m$.
Then, we can get $n_1 \times ... \times n_d$ pairwise disjoint boxes and each of them is represented as $\tilde{B}_{k_1...k_d} = J_{k_1}^1 \times ... \times J_{k_d}^d$ for $(k_1, ..., k_d) \in \{1, ..., n_1\} \times ... \times \{1, ..., n_d\}$.
Then, each B_i is a union of a subcollection of $\{\tilde{B}_{k_1...k_d}\}$, and thus E is expressed as a finite union of pairwise disjoint boxes.

Now let's prove (ii).
First we notice that for any interval I, we have that

$$|I| = \lim_{N \to +\infty} \frac{1}{N} \#(I \cap \frac{1}{N}\mathbb{Z})$$

where $\frac{1}{N}\mathbb{Z} = \{\frac{k}{N} : k \in \mathbb{Z}\}$.
I don't know how to prove this. One way to look at it is to take a sample of rational points in (a, b) with the distance between each adjacent pair to be $\frac{1}{N}$. Another way is through an example where $a = 2$, $b = 4$. Then, $(a, b) \cap \frac{1}{N}\mathbb{Z} = \{z : z \in (aN, bN), z \in \mathbb{Z})\}$. When $n = 1$, $\# = 1$; $n = 2$, $\# = 3$, ... Then the cardinality is equal to $(b-a)N - 1 = 2N - 1$ with each N. Then $\lim_{N \to +\infty} \frac{1}{N}((b-a)N - 1) = b - a$.
By taking Cartesian product,

$$|B| = \lim_{N \to +\infty} \frac{1}{N^d} \#(B \cap \frac{1}{N}\mathbb{Z}^d)$$

For $E = \cup_{i=1}^n B_i$ with pairwise disjoint B_i,

$$\begin{aligned} \frac{1}{N^d} \#(E \cap \frac{1}{N}\mathbb{Z}^d) &= \sum_{i=1}^n \frac{1}{N^d} \#(B_i \cap \frac{1}{N}\mathbb{Z}^d) \\ &\to \sum_{i=1}^n |B_i| \ as \ N \to +\infty \end{aligned}$$

The LHS is independent of partitions, thus $m(E) = \lim_{N \to +\infty} \frac{1}{N^d} \#(E \cap \frac{1}{N}\mathbb{Z}^d)$. $\square$

Now we have the property of the elementary measure:

(i). $m(E) \geq 0$

(ii). Finite additivity: $m(E \cup F) = m(E) + m(F)$ for disjoint $E, F \in \mathcal{E}(\mathbb{R}^d)$. By induction, $m(E_1 \cup ... \cup E_n) = \sum_{i=1}^n m(E_i)$ for disjoint $\{E_i\}$.

(iii). $m(\emptyset) = 0$.

(iv). $m(B) = |B| \; \forall$ box B.

(v). Monotonicity: $E \subset F$, then $m(E) \leq m(F)$.

(vi). Sub-additivity: $m(E \cup F) \leq m(E) + m(F)$ for any $E, F \in \mathcal{E}(\mathbb{R}^d)$. By induction: $m(E_1 \cup ... \cup E_n) \leq \sum_{i=1}^n m(E_i)$.

(vii). $E \in \mathcal{E}(\mathbb{R}^d)$, $x \in \mathbb{R}$, then $m(E + x) = m(E)$.

Proof. (v). $F = E \cup (F \setminus E)$, so $m(F) \overset{\text{(ii)}}{=} m(E) + m(F \setminus E) \overset{\text{(i)}}{\geq} m(E)$.
(vi). Finite sub-additivity: $E \cup F = E \cup (F \setminus E)$, therefore $m(E \cup F) \overset{\text{(ii)}}{=} m(E) + m(F \setminus E) \overset{\text{(v)}}{\leq} m(E) + m(F)$.
(vii). E can be partitioned into finite disjoint boxes $E = \bigcup_{i=1}^n B_i$, where each $B_i = I_i^1 \times ... \times I_i^d$. Then, $E' = E + x = \bigcup_{i=1}^n B_i + x = \bigcup_{i=1}^n I_i^1 \times ... \times I_i^d + x = \bigcup_{i=1}^n (I_i^1 + x) \times ... \times (I_i^1 + x) = \bigcup_{i=1}^n B_i'$. $|B_i| = |B_i'|$. Then, because of disjoint, $m(E) = \sum_{i=1}^n |B_i| = \sum_{i=1}^n |B_i'| = m(E')$. ☐

1.2 Jordan Measure

More advanced sets such as triangle, disk, or rotated boxes can be measured by approaching from without and within by elementary sets.

Definition 1.2.1. *(Jordan Measure) Let $E \in \mathbb{R}^d$ be a bounded set.*
Its Jordan inner measure is $m_J(E) = \sup_{A \subset E,\ A \text{ elementary}} m(A)$.
Its Jordan outer measure is $m^J(E) = \inf_{E \subset B,\ B \text{ elementary}} m(B)$.
If $m_J(E) = m^J(E)$, then E is Jordan measurable.
Let $\mathcal{J}(\mathbb{R}^d)$ be class of Jordan measurable sets. For $E \in \mathcal{J}(\mathbb{R}^d)$, define Jordan measure as $m(E) := m_J(E) = m^J(E)$.

Note that (i). unbounded sets are not Jordan measurable. (ii). The Jordan measure of $E \in \mathcal{E}(\mathbb{R}^d)$ is equal to the elementary measure of E.
Here gives the characterization of Jordan Measure.

Proposition 1.2.1. *Let $E \in \mathbb{R}^d$ be bounded. Then the following are equivalent: (TFAE):*
(1). $E \in \mathcal{J}(\mathbb{R}^d)$.
(2). $\forall \epsilon > 0$, $\exists A, B \in \mathcal{E}(\mathbb{R}^d)$ with $A \subset E \subset B$, such that $m(B \setminus A) < \epsilon$.
(3). $\forall \epsilon > 0$, $\exists A \in \mathcal{E}(\mathbb{R}^d)$ such that $m^J(A \Delta E) < \epsilon$.

Proof. ☐

Proposition 1.2.2. *Let E, $F \in \mathcal{J}(\mathbb{R}^d)$, then,*
(1). $E \cup F$, $E \cap F$, $E \setminus F$, $E \Delta F \in \mathcal{J}(\mathbb{R}^d)$.
(2). $m(E) \geq 0$.
(3). Finite Additivity: If $E \cap F = \emptyset$, then $m(E \cup F) = m(E) + m(F)$.
(4). Monotonicity: If $E \in F$, then $m(E) \leq m(F)$.

(5). Finite subadditivity: $m(E \cup F) \leq m(E) + m(F)$.
(6). Translation Invariabce: $m(E + x) = m(E)$.

Proof. $E \cap F \in \mathcal{J}(\mathbb{R}^d)$:
Using (3) from Proposition 2, let $A, B, C, D \in \mathcal{E}(\mathbb{R}^d)$ with $A \in E \in B$, $C \in E \in D$, such that $m(B \setminus A) < \epsilon$, $m(D \setminus C) < \epsilon$. Then we claim that $m((B \cap D)\ (A \cap C)) \leq 2\epsilon$.

$$\begin{aligned}(B \cap D) \setminus (A \cap C) &= (B \cap D) \cap (A \cap C)^C \\ &= B \cap D \cap (A^C \cup C^C) \\ &= (B \cap D \cap A^C) \cup (B \cap D \cap C^C) \\ &\subset (B \cap A^C) \cup (D \cap C^C) \\ &= (B \setminus A) \cup (D \setminus C)\end{aligned}$$

Then, $m((B \cap D) \setminus (A \cap C)) \leq m((B \setminus A) \cup (D \setminus C)) \leq 2\epsilon$.
Since $A \cap C \subset E \cap F \subset B \cap D$, we have $E \cap F$ is also Jordan measurable. ☐

There are sets that are not Jordan measurable. For example, $E = [0, 1] \cap \mathcal{Q}$ is not Jordan measurable. This is because E contains no open interval, any elementary set $A \in E$ can only be a finite union of singletons, therefore, $m_J(E) = 0$. However, we can prove that $m_J(E) = 1$.
Similarly, there exists open and bounded sets that are not Jordan measurable, and there exists compact sets that are not Jordan measurable. So, we need Lebegue measure.

1.3 Lebesgue Measure

Definition 1.3.1. *The lebesgue outer measure (exterior measure) of $E \in \mathbb{R}^d$ is*

$$m^*(E) := \inf\{\sum_{i=1}^{\infty} |B_i| : B_1, B_2, ...boxes, E \subset \bigcup_{i=1}^{\infty} B_i\}$$

Note that $m^*(E) \leq m^J(E)$ since it is the infimum over a bigger set.
For $E = [0, 1] \cap \mathcal{Q} = \{q_1, q_2, ...\}$, by taking boxes $B_i = \{q_i\}$, we get the Lebesgue measure $m^*(E) \leq \sum_{i=1}^{\infty} |B_i| = \sum_{i=1}^{\infty} 0 = 0$.

Similarly, $m^*(E) = 0$ for any countable E, just take $\{x_i\}$ as boxes. Or, take boxes $B_i = (q_i - \frac{\epsilon}{2^i}, q_i + \frac{\epsilon}{2^i})$, then $m^*(E) \leq m(\bigcup_{i=1}^{\infty} B_i) \leq \sum_{i=1}^{\infty} |B_i| = 2\epsilon$. Since ϵ is arbitrary, we have $m^*(E) = 0$.

Definition 1.3.2. *Lebesgue Measurability: A set $E \in \mathbb{R}^d$ is Lebesgue measurable if $\forall \epsilon > 0$, $\exists$ an open set $U \in \mathbb{R}^d$, $E \subseteq U$, such that $m^*(U \setminus E) < \epsilon$.*

We denote the class of all Lebesgue measurable sets by $\mathcal{L}(\mathbb{R}^d)$. for $E \in \mathcal{L}(\mathbb{R}^d)$, its Lebesgue measure is $m(E) := m^*(E)$. Now we gives the properties of the Lebesgue outer measure (the outer measure axioms).

Proposition 1.3.1. *(The outer measure axioms)*

1. *$m^*(\emptyset) = 0$.*

2. *Monotonicity: if $E \subset F \subseteq \mathbb{R}^d$, then $m^*(E) \leq m^*(F)$.*

3. *σ-subadditivity: If $E_1, E_2, ... \subset \mathbb{R}^d$ is a countable sequence of sets, then, $m^*(\bigcup_{n=1}^{\infty} E_i) \leq \sum_{i=1}^{\infty} m^*(E_i)$.*

Proof. (3).
We know that,

$$m^*(E_i) = \inf\{\sum_{j=1}^{\infty} |B_j^i| : B_1^i, B_2^i, ...boxes, E_i \subset \bigcup_{j=1}^{\infty} B_j^i\}$$

Then, for each i, there exists $B_1^i, B_2^i, ...$ to be boxes, such that $E_i \subset \bigcup_{j=1}^{\infty} B_j^i$ and

$$\sum_{j=1}^{\infty} |B_j^i| \leq m^*(E_i) + \frac{\epsilon}{2^i}$$

Since $E_i \subset \bigcup_{j=1}^{\infty} B_j^i$, we have $\bigcup_{i=1}^{\infty} E_i \subset \bigcup_{i=1}^{\infty} \bigcup_{j=1}^{\infty} B_j^i$, then we have

$$\begin{aligned}
m^*(\bigcup_{i=1}^{\infty} E_i) &\leq m^*(\bigcup_{i=1}^{\infty} \bigcup_{j=1}^{\infty} B_j^i) \text{ by monotonicity} \\
&\leq \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |B_j^i| \text{ by definition of Lebesgue measure: infimum} \\
&= \sum_{i=1}^{\infty} (\sum_{j=1}^{\infty} |B_j^i|) \text{ by Tonelli's Theorem for series} \\
&= \sum_{i=1}^{\infty} (m^*(E_i) + \frac{\epsilon}{2^i}) \\
&= \sum_{i=1}^{\infty} m^*(E_i) + \epsilon
\end{aligned}$$

Since ϵ is arbitrary, we then have $m^*(\bigcup_{n=1}^{\infty} E_i) \leq \sum_{i=1}^{\infty} m^*(E_i)$. $\square$

Also, (3) and (1) together give the finite subadditivity property by just let $\emptyset = E_{k+1} = E_{k+2} = ...$, that is, $m^*(\bigcup_{i=1}^{k} E_i) \leq \sum_{i=1}^{k} m^*(E_i)$.

Lemma 1.3.1. *Finite additivity for separated sets: Let $E, F \subset \mathbb{R}^d$ be such that $dist(E, F) := \inf\{|x - y| : x \in E, y \in F\} > 0$, then $m^*(E \cup F) = m^*(E) + m^*(F)$.*

Proof. First, we prove $\leq$. This is natual from σ-additivity: $m^*(E \cup F) \leq m^*(E) + m^*(F)$.
Next, we prove $\geq$. Without loss of generality, assume that $m^*(E \cup F) < +\infty$.
Let $\epsilon > 0$ be arbitrary, then, by definition there exists a countable collection of boxes $B_1, B_2, ...$ such that,

$$E \cup F \subset \bigcup_{i=1}^{\infty} B_i$$

$$\sum_{i=1}^{\infty} |B_i| \leq m^*(E) + \epsilon$$

Fix $\delta \in (0, dist(E, F))$. By subdividing these boxes into finer boxes B_i', we may assume that $diam(B_i') < \delta$. Then, some of these boxes have intersection with E while others have intersection with F.

Let $I = \{i : B_i' \cap E \neq \emptyset\}$, $J = \{j : B_j' \cap F \neq \emptyset\}$. Then $B_i' \cap B_j' = \emptyset$ cause otherwise we would have a box with diameter bigger than δ.
Then, $m^*(E) \leq \sum_{i \in I} |B_i'|$, $m^*(F) \leq \sum_{j \in J} |B_j'|$.

$$\begin{aligned} m^*(E) + m^*(F) &\leq \sum_{i \in I \cup J} |B_i'| \\ &\leq \sum_{i=1}^{\infty} |B_i| \\ &\leq m^*(E \cup F) + \epsilon \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, we have $m^*(E) + m^*(F) \leq m^*(E \cup F)$.
Now that we have $\leq$ and $\geq$, we have $=$. □

Lemma 1.3.2. *Outer measurability for elementary sets: Let $E \in \mathcal{E}(\mathbb{R}^d)$, then $m^*(E) = m(E)$, the elementary measure.*

Proof. First we prove $\leq$. We already know that $m^*(E) \leq m^J(E) = m(E)$, thus $\leq$ holds.
Next we prove $\geq$.
Consider first the case where E is closed, then, E is compact. Then we can use the Heine-Borel Theorem which states that any covering of a compact set by a collection of open sets contains a finite subcovering.
Take a covering of E by boxes: $E \subset \bigcup_{i=1}^{\infty} B_i$ such that $\sum_{i=1}^{\infty} |B_i| \leq m^*(E) + \epsilon$. For each box B_i, find an open box B_i' such that $B_i \subset B_i'$ and $|B_i'| \leq |B_i| + \frac{\epsilon}{2^i}$.
Then, $\sum_{i=1}^{\infty} |B_i'| \leq \sum_{i=1}^{\infty} |B_i| + \epsilon \leq m^*(E) + 2\epsilon$.
Using Heine-Borel Theory, there is a finite N such that $E \subset \bigcup_{i=1}^{N} B_i'$. Then,

$$\begin{aligned} m(E) &\leq \sum_{i=1}^{N} |B_i| \\ &\leq \sum_{i=1}^{\infty} |B_i| \\ &\leq m^*(E) + 2\epsilon \end{aligned}$$

Since $\epsilon > 0$ arbitrary, we have $\geq$.
Now consider the case where E is not closed. Then, write E as a finite union of disjoint boxes $E = \bigcup_{i=1}^{k} Q_i$, which need not be closed.
Let $\epsilon > 0$ be arbitrary, and for each $j \in \{1, ..., k\}$, find a closed sub-box $Q_j' \subset Q_j$ such that $|Q_j'| \geq |Q_j| - \frac{\epsilon}{k}$. Then, by the previous discussion and finite additivity of elementary

measure, we have

$$\begin{aligned}
m^*(\bigcup_{j=1}^k Q'_j) &= m(\bigcup_{j=1}^k Q'_j) \\
&= \sum_{j=1}^k m(Q'_j) \\
&\geq \sum_{j=1}^k m(Q_j) - \epsilon \\
&= m(E) - \epsilon
\end{aligned}$$

Also, $\bigcup_{j=1}^k Q'_j \subset E$, so by monotonicity, we have

$$\begin{aligned}
m^*(E) &\geq m^*(\bigcup_{j=1}^k Q'_j) \\
&\geq m(E) - \epsilon
\end{aligned}$$

Then $m(E) \leq m^*(E) + \epsilon$. Since $\epsilon > 0$ arbitrary, $\geq$ holds. □

Lemma 1.3.3. *(Outer measure of countable unions of almost disjoint boxes): Let $E = \bigcup_{i=1}^\infty B_i$ be a countable union of almost disjoint boxes, then $m^*(E) = \sum_{i=1}^\infty |B_i|$. Almost disjoint means that $B_i^o \cap B_j^o = \emptyset \ \forall i \neq j$ (topological interior doesn't intersect).*

Proof. From countable sub-additivity and **Lemma 1.3.2**,

$$m^*(E) \leq \sum_{i=1}^\infty m^*(B_i) = \sum_{i=1}^\infty |B_i|$$

Therefore, it suffices to show that

$$m^*(E) \geq \sum_{i=1}^\infty |B_i|$$

Notice that for each $N \in \mathbb{N}$,

$$E \supset \bigcup_{i=1}^N B_i$$

Then,

$$\begin{aligned}
m^*(E) &\geq m^*(\bigcup_{i=1}^N B_i) \\
&= m(\bigcup_{i=1}^N B_i) \\
&= \sum_{i=1}^N |B_i|
\end{aligned}$$

Let $N \to \infty$, we have $m^*(E) \geq \sum_{i=1}^\infty |B_i|$ Therefore we conclude the proof. □

From this lemma we have a corollary.

Corollary 1.3.1. *If* $E = \bigcup_{i=1}^{\infty} B_i = \bigcup_{i=1}^{\infty} B_i'$, $(B_i)_{i\in\mathbb{N}}$ *and* $(B_i')_{i\in\mathbb{N}}$ *are almost disjoint boxes, then* $\sum_{i=1}^{\infty} |B_i| = \sum_{i=1}^{\infty} |B_i'|$.

Lemma 1.3.4. *An open set* $U \subseteq \mathbb{R}^d$ *is the countable union of almost disjoint boxes. (in fact, the countable union of almost disjoint closed boxes).*

Proof. For $n \in \mathbb{Z}$, let $\mathcal{Q}_n$ be the collection of all closed cubes of the form

$$[\frac{k_1}{2^n}, \frac{k_1+1}{2^n}] \times ... \times [\frac{k_d}{2^n}, \frac{k_d+1}{2^n}] \text{ for some integers } k_1, ..., k_d$$

Define $\mathcal{Q}_{\geq 0} := \bigcup_{n=1}^{\infty} \mathcal{Q}_n$ to be the union of all dyadic cubes of side length ≤ 1. Notice that $\mathcal{Q}_{\geq 0}$ has a tree structure, that is, for each $Q \in \mathcal{Q}_n$, $\exists! Q' \in \mathcal{Q}_{n-1}$ such that $Q \subset Q'$.
Given these, we have the dyadic nesting property: $\forall Q_1, Q_2 \in \mathcal{Q}_{\geq 0}$ with $Q_1^o \cap Q_2^o \neq \emptyset$, either $Q_1 \subseteq Q_2$ or $Q_2 \subseteq Q_1$.
Since U is open, $\forall x \in U$, $\exists$ open ball $B(x, r) \subset U$. Therefore, $\exists$ closed $Q \in \mathcal{Q}_{\geq 0}$ such that $x \in Q \subseteq E$. Then, let $Q_U = \{Q \subset \mathcal{Q}_{\geq 0} : Q \subseteq U\}$. Then,

$$U = \bigcup_{Q \in \mathcal{Q}_U} Q \text{ with } \mathcal{Q}_U \text{ being countable}$$

To get almost disjoint subcollection, take $\mathcal{Q}_U^* \subseteq \mathcal{Q}_U$ to be a subcollection of maximal elements with respect to set inclusion, which means that they are not contained in any other cube in $\mathcal{Q}_U$.

$$\mathcal{Q}_U^* := \{Q \in \mathcal{Q}_{\geq 0} : Q \subseteq U, Q' \nsubseteq U \text{ for any } Q' \in \mathcal{Q}_{\geq 0} \text{ and } Q' \supset Q\}$$

First we see that if $Q \subseteq U$ then $Q \subseteq \mathcal{Q}_U$, then $\mathcal{Q}_U \subseteq \mathcal{Q}_U^*$. Together with the definition of $\mathcal{Q}_U^*$, we see that $\mathcal{Q}_U^* = \mathcal{Q}_U$. Second, by dyadic nesting property, every cube in $\mathcal{Q}$ is contained in exactly one maximal cube in $\mathcal{Q}^*$, and that any two such maximal cubes in $\mathcal{Q}^*$ are almost disjoint. Thus, $U = \bigcup_{Q \in \mathcal{Q}^8} Q$ are almost disjoint, and also countable. ☐

Lemma 1.3.5. *(Outer regularity): For any* $E \subseteq \mathbb{R}^d$,

$$m^*(E) = \inf_{E \subset U, U \text{ } open} m^*(U)$$

Proof. $(\leq)$: it is easy to see from monotonicity that $\forall U \supset E$, $m^*(E) \leq m*(U)$, thus

$$m^*(E) \leq \inf_{E \subset U, U \text{ } open} m^*(U)$$

Therefore it suffices to prove that

$$m^*(E) \geq \inf_{E \subset U, U \text{ } open} m^*(U)$$

By definition of the outer Lebesgue measure,

$$m^*(E) = \inf\{\sum_{i=1}^{\infty} |B_i| : E \subset \bigcup_{i=1}^{\infty} B_i, B_1, B_2, ...boxes\}$$

Then, $\forall \epsilon > 0$, $\exists B_1', B_2'...$ such that

$$\sum_{i=1}^{\infty} |B_i'| \leq m^*(E) + \epsilon$$

Enlarge each box B_i' to be an open box $B_i' \subset B_i''$ such that

$$|B_i''| \leq |B_i'| + \frac{\epsilon}{2^i}$$

Thus, $E \subset \bigcup_{i=1}^{\infty} B_i''$ where $\bigcup_{i=1}^{\infty} B_i''$ is open.

$$\begin{aligned}\sum_{i=1}^{\infty} |B_i''| &\leq \sum_{i=1}^{\infty} |B_i'| + \epsilon \\ &= m^*(E) + 2\epsilon\end{aligned}$$

Since $\bigcup_{i=1}^{\infty} B_i''$ is open, by countable sub-additivity and the definition of infimum,

$$\inf_{E \subset U, U \ open} m^*(U) \leq m^*(\bigcup_{i=1}^{\infty} B_i'') \leq \sum_{i=1}^{\infty} |B_i''| \leq m^*(E) + 2\epsilon$$

Since $\epsilon > 0$ is arbitrary, we have

$$m^*(E) \geq \inf_{E \subset U, U \ open} m^*(U)$$

□

There are plenty of Lebesgue measurable sets, as we can see from the following proposition.

Proposition 1.3.2. *(Existence of Lebesgue measurable sets). Let $E \subseteq \mathbb{R}^d$, then $E \subset \mathcal{L}(\mathbb{R}^d)$ if*

1. *E is open.*
2. *E is closed.*
3. *E is a null set, i.e. $m^*(E) = 0$.*
4. *$E = \emptyset$.*
5. *if $E \in \mathcal{L}(\mathbb{R}^d)$, then $\mathbb{R}^d \setminus E \in \mathcal{L}(\mathbb{R}^d)$.*
6. *$E = \bigcup_{i=1}^{\infty} E_i \in \mathcal{L}(\mathbb{R}^d)$ where $E_i \in \mathcal{L}(\mathbb{R}^d)$.*
7. *$E = \bigcap_{i=1}^{\infty} E_i \in \mathcal{L}(\mathbb{R}^d)$ where $E_i \in \mathcal{L}(\mathbb{R}^d)$.*

Proof. (1) is immediate from definition. By **Lemma 1.3.4**, write E as $E = \bigcup_{i=1}^{\infty} B_i$ where B_i are disjoint boxes. Expand each B_i to be an open box $B_i' \supset B_i$ such that $\forall \epsilon > 0$,

$$|B_i'| \leq |B_i| + \frac{\epsilon}{2^i}$$

Then, by σ-additivity and Lemma 1.3.3,

$$m^*(\bigcup_{i=1}^{\infty} B_i') \leq \sum_{i=1}^{\infty} |B_i'| \leq m(E) + \epsilon$$

Therefore

$$m^*(\bigcup_{i=1}^{\infty} B_i' \setminus E) \leq \epsilon$$

Thus, we found an open set $\bigcup_{i=1}^{\infty} B_i' \supset E$, such that $m^*(\bigcup_{i=1}^{\infty} B_i' \setminus E) \leq \epsilon$.
(3) and (4) are immediate.
Since $E_i \in \mathcal{L}(\mathbb{R}^d)$, $\exists$ open set E_i' such that $E_i \subset E_i'$, $m^*(E_i') \leq m^*(E_i) + \frac{\epsilon}{2^i}$. Then, $\bigcup_{i=1}^{\infty} E_i \subset \bigcup_{i=1}^{\infty} E_i'$ where $\bigcup_{i=1}^{\infty} E_i'$ is open. Since

$$\begin{aligned}
\bigcup_{i=1}^{\infty} E_i' \setminus \bigcup_{i=1}^{\infty} E_i &= \bigcup_{i=1}^{\infty} E_i' \cap (\bigcap_{i=1}^{\infty} E_i^C) \\
&= \bigcup_{i=1}^{\infty} (E_i' \cap (\bigcap_{i=1}^{\infty} E_i^C)) \\
&= \bigcup_{i=1}^{\infty} (E_i' \cap (\bigcap_{i=1}^{\infty} E_i^C)) \\
&\subset \bigcup_{i=1}^{\infty} (E_i' \cap E_i^C) \\
&= \bigcup_{i=1}^{\infty} (E_i' \setminus E_i)
\end{aligned}$$

Then, by monotonicity and σ-additivity,

$$\begin{aligned}
m^*(\bigcup_{i=1}^{\infty} E_i' \setminus \bigcup_{i=1}^{\infty} E_i) &\leq m^*(\bigcup_{i=1}^{\infty} (E_i' \setminus E_i)) \\
&\leq \sum_{i=1}^{\infty} m^*(E_i' \setminus E_i) \\
&= \sum_{i=1}^{\infty} m^*(E_i') - m^*(E_i) \\
&\leq \epsilon \qquad (1)
\end{aligned}$$

Therefore, $\bigcup_{i=1}^{\infty} E_i \in \mathcal{L}(\mathbb{R}^d)$.
(2). First, we can express each closed set as $E = \bigcup_{i=1}^{\infty}$ for E_n closed and bounded (for example, $E_n = \overline{B(0,n)} \cap E$ for $n = 1, 2, ...$). Then by (6), it suffices to verify the claim when E is closed and bounded, hense compact.
By Lemma 1.3.5, $\exists U \supset E$ open, such that $m^*(U) \leq m^*(E) + \epsilon$. Therefore, it suffices to show that $m^*(U \setminus E) \leq \epsilon$.
If w have finite additivity for m^*, then we have $m^*(U \setminus E) + m^*(E) = m^*(U) \leq m^*(E) + \epsilon$ and then $m^*(U \setminus E) \leq \epsilon$. But we don't have it, so we should instead fo the following.
Since $U \setminus E$ is also open, by Lemma 1.3.4, $U \setminus E = \bigcup_{i=1}^{\infty} Q_i$ where Q_i are almost disjoint closed boxes. Then by Lemma 1.3.3, $m^*(U \setminus E) = \sum_{i=1}^{\infty} |Q_i|$.

We truncate the sum: for any finite $N \in \mathbb{N}$, $\bigcup_{i=1}^N Q_i$ is closed and disjoint from E. From **Exercise 1.2.4** (Let $E, F \subset \mathbb{R}^d$ be disjoint closed sets, with at least one of E, F being compact. Then $dist(E, F) > 0$), since E is compact and $\bigcup_{i=1}^N Q_i$ is closed, we have $dist(E, \bigcup_{i=1}^N Q_i) > 0$. Then by Lemma 1.3.1,

$$\begin{aligned} m^*(\bigcup_{i=1}^N Q_i) + m^*(E) &= m^*(E \cup \bigcup_{i=1}^N Q_i) \\ &\leq m^*(U) \\ &\leq m^*(E) + \epsilon \end{aligned}$$

$$\sum_{i=1}^N |Q_i| = m^*(\bigcup_{i=1}^N Q_i) \leq \epsilon$$

Let $N \to \infty$,

$$m^*(U \setminus E) = \sum_{i=1}^\infty |Q_i| = m^*(\bigcup_{i=1}^\infty Q_i) \leq \epsilon$$

Therefore $E \in \mathcal{L}(\mathbb{R}^d)$.
(5). Since $E \in \mathcal{L}(\mathbb{R}^d)$, for every $n \in \mathbb{N}$, $\exists U_n \supset E$ such that $m^*(U_n \setminus E) < \frac{1}{n}$.
Let $F_n := U_n^C$, then $(\mathbb{R}^d \setminus E) \supset F_n$ for all n. Since

$$(\mathbb{R}^d \setminus E) \setminus F_n = (\mathbb{R}^d \setminus E) \cap F_n^C = (\mathbb{R}^d \setminus E) \cap U_n = U_n \setminus E$$

we have

$$m^*((\mathbb{R}^d \setminus E) \setminus F_n) < \frac{1}{n}$$

Let $F := \bigcup_{i=1}^\infty F_n$, then $(\mathbb{R}^d \setminus E) \supset F$. From monotonicity, we have

$$m^*((\mathbb{R}^d \setminus E) \setminus F) \leq m^*((\mathbb{R}^d \setminus E) \setminus F_n) < \frac{1}{n} \; \forall n \in \mathbb{N}$$

Taking $n \to \infty$, we have $m^*((\mathbb{R}^d \setminus E) \setminus F) = 0$, thus $(\mathbb{R}^d \setminus E) \setminus F$ is a null set, and is Lebesgue measurable. Therefore, $\mathbb{R}^d \setminus E$ is the union of this null set and F. Since by definition $F = \bigcup_{i=1}^\infty U_n^C$ where U_n^C is closed, F is Lebesgue measurable. Therefore, by (6), $\mathbb{R}^d \setminus E \in \mathcal{L}(\mathbb{R}^d)$.
(7). Since $E_i \in \mathcal{L}(\mathbb{R}^d)$, we have $E_i^C \in \mathcal{L}(\mathbb{R}^d)$ and $\bigcup_{i=1}^\infty E_i^C \in \mathcal{L}(\mathbb{R}^d)$. Therefore $(\bigcap_{i=1}^\infty E_i)^C \in \mathcal{L}(\mathbb{R}^d)$ and $\bigcap_{i=1}^\infty E_i \in \mathcal{L}(\mathbb{R}^d)$. □

For $E \in \mathcal{L}(\mathbb{R}^d)$, its Lebesgue measure is defined to be $m(E) := m^*(E)$, and it has the following properties, which is significantly better than Lebesgue outer measure.

Proposition 1.3.3. *(The measure axioms)*

1. $m(\emptyset) = 0$.

2. *(σ-additivity) For a countable sequence of disjoint sets $E_1, E_2 ... \in \mathcal{L}(\mathbb{R}^d)$,*

$$m(\bigcup_{i=1}^\infty E_i) = \sum_{i=1}^\infty m(E_n)$$

Proof. (1). is trivial.
(2). *Case 1.* E_n is compact.
Then, by **Exercise 1.2.4**, $dist(E_i, E_j) > 0$, and

$$m(\bigcup_{i=1}^{N} E_i) = \sum_{i=1}^{N} m(E_i)$$

By monotonicity,

$$m(\bigcup_{i=1}^{\infty} E_i) \geq m(\bigcup_{i=1}^{N} E_i) = \sum_{i=1}^{N} m(E_i)$$

Let $N \to \infty$,

$$m(\bigcup_{i=1}^{\infty} E_i) \geq \sum_{i=1}^{\infty} m(E_i)$$

Also from σ-subadditivity,

$$m(\bigcup_{i=1}^{\infty} E_i) \leq \sum_{i=1}^{\infty} m(E_i)$$

Therefore we have

$$m(\bigcup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} m(E_i)$$

Case 2. E_n is not compact but bounded.
For each E_n, it can be written as the union of a compact set U_n and a set with outer measure $\frac{\epsilon}{2^n}$. Thus,

$$m(E_n) \leq m(U_n) + \frac{\epsilon}{2^n}$$

$$\sum_{n=1}^{\infty} m(E_n) \leq \sum_{n=1}^{\infty} m(U_n) + \epsilon$$

We just showed that for compact set,

$$\sum_{n=1}^{\infty} m(U_n) = m(\bigcup_{i=1}^{\infty} U_n)$$

and by monotonicity,

$$m(\bigcup_{i=1}^{\infty} U_n) \leq m(\bigcup_{i=1}^{\infty} E_n)$$

Thus,

$$\sum_{n=1}^{\infty} m(E_n) \leq m(\bigcup_{i=1}^{\infty} E_n) + \epsilon$$

Since $\epsilon > 0$ arbitrary, we have

$$\sum_{n=1}^{\infty} m(E_n) \leq m(\bigcup_{i=1}^{\infty} E_n)$$

Also from σ-subadditivity, we have

$$\sum_{n=1}^{\infty} m(E_n) \geq m(\bigcup_{i=1}^{\infty} E_n)$$

Thus

$$\sum_{n=1}^{\infty} m(E_n) = m(\bigcup_{i=1}^{\infty} E_n)$$

Case 3. E_n is not compact and not closed.
Decompose $\mathbb{R}^d$ into annulis, for $m = 1, 2, ...,$

$$A_m := \{x \in \mathcal{R} : m - 1 \leq |x| \leq m\}$$

Then, each E_n can be written as $E_n = \bigcup_{m=1}^{\infty} E_n \cap A_m$ for $E_n \cap A_m$ bounded, measurable, and disjoint.
Then, by previous argument,

$$m(E_n) = \sum_{m=1}^{\infty} m(E_n \cap A_m)$$

Also, for $E_n \cap A_m$ bounded, measurable, and disjoint,

$$\bigcup_{n=1}^{\infty} E_n = \bigcup_{n=1}^{\infty} \bigcup_{m=1}^{\infty} E_n \cap A_m$$

Then

$$m(\bigcup_{n=1}^{\infty} E_n) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} m(E_n \cap A_m) = \sum_{n=1}^{\infty} m(E_n)$$

□

Of course, there are non-measurable sets in $\mathbb{R}^d$.

1.4 Citation

This is a citation[**?**].

References