Connectedness

Def. (Separation). Let X be a topological space. A separation of X is a pair of open sets U and V such that
(a). $U \neq \phi$, $V \neq \phi$
(b). $U \cup V = X$
(c). $U \cap V = \phi$

i.e. $V = X - U$, $U = X - V$.
U and V are called clopen

Def. (Connectedness). A space is called disconnected if $\exists$ a separation. Connected if $\nexists$ a separation.

X is connected if the only clopen sets are ϕ, X.

Connected space is somehow indivisible.

Theorem. $\mathbb{R}$ with the standard topology is connected, so are all the intervals and rays.

Any set with discrete topology is disconnected. Any set with indiscrete topology is connected.

$Y = [0, 1] \cup [2, 3] \subset \mathbb{R}$ is disconnected since separated by $Y \cap (-\infty, \frac{3}{2})$ and $(\frac{3}{2}, +\infty) \cap Y$.

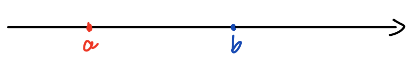
$\mathbb{Q} \subset \mathbb{R}$ is disconnected in many ways. For example, $U = \mathbb{Q} \cap (-\infty, \pi)$, $V = \mathbb{Q} \cap (\pi, +\infty)$.

Def. (Convex). A subset $Y \subset \mathbb{R}$ is convex if given $a, b \in Y$, $a < b$, and $c \in \mathbb{R}$ such that $a < c < b$, $\Rightarrow c \in Y$. i.e. if $a < b \in Y$, then $[a, b] \subset Y$.

Basic property of $\mathbb{R}$.
- If $x < y$, then $\exists z$ s.t. $x < z < y$.
- Least upper bound property. Let $A \subset \mathbb{R}$. A number b is called an upper bound for A if $\forall a \in A$, $a \leq b$. A is called bounded above if it has an upper bound in $\mathbb{R}$.
  This property says that, if $A \subset \mathbb{R}$ is a non-empty set that is bounded above, then it has a least upper bound. It may or may not be in A.

Theorem. Any convex subset of $\mathbb{R}$ is connected.

proof: Suppose $Y = A \cup B$, both open in subspace topology, non empty, $A \cap B = \phi$.
Choose $a \in A$, $b \in B$. WLOG $a < b$.



Let $A_0 := A \cap [a,b]$ $\quad B_0 := B \cap [a,b]$

Then $A_0 \cup B_0 = [a,b]$, $A_0 \cap B_0 = \phi$. A_0, B_0 open in subspace topology.

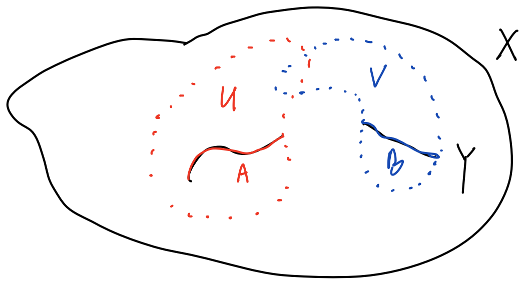
Let $c = \sup A_0$. then $a \le c \le b$. according to def. Then, since $A_0 \subset [a,b]$, we have $c \in [a,b]$. WTS: $c \notin A_0$, $c \notin B_0$.

If $c \in B_0$, then $c \neq a$. Since B_0 is open in subspace topology on $[a,b]$, then $\exists\, d < c$ s.t. $(d, c] \subset B_0$. Since $c = \sup A_0$, d is not an upper bound of A_0.
So, $\exists\, e \in A_0$ with $d < e \le c$. Then $e \in A_0 \cap B_0$. $\Rightarrow\Leftarrow$.

If $c \in A_0$, then $c \neq b$. Since A_0 is open in subspace topology on $[a,b]$, then $\exists\, [c, d) \subset A_0$. Then $\exists$ some f with $c < f < d$. $f \in A_0$. Then it contradict with c being upper bound. $\Rightarrow\Leftarrow$

$\square$

Let $Y \subset X$. A separation of Y is a pair of non-empty open sets $A, B \subset Y$ s.t. $A \cup B = Y$, $A \cap B = \phi$
There exists $A = Y \cap U$, $B = Y \cap V$ where U & V open in X.



U, V can intersect outside Y in X.

<u>Lemma</u>. A does not contain any limit point of B. B does not contain any limit point of A.

proof. If A & B are separation, $B \subset V$, $A \subset U$, then $A \cap V = \phi$, $B \cap U = \phi$.
So points in B are not limit point of A. (limit points of A are in U, $U \cap B = \phi$).

e.g. $\mathbb{R}$: For any a, $(-\infty, a)$ and $[a, +\infty)$ are both open $\Rightarrow$ separation
$\mathbb{R}_{\text{finite}}$: No two open sets are disjoint $\Rightarrow$ no separation.

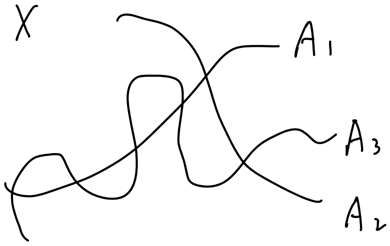
<u>Lemma</u>. If X is a space, Y is a connected subspace, and C, D are a separation of X, then either $Y \subset C$ or $Y \subset D$.

proof: Look at $A = C \cap Y$, $B = D \cap Y$. Both are open in subspace topology.
$A \cup B = Y$, $A \cap B = \phi$. However, Y is connected. Then, either $Y \subset A$ or $Y \subset B$. $\square$

<u>Lemma</u>. Suppose $\{A_\alpha\}$ are subspaces such that for any α, β, $A_\alpha \cap A_\beta \neq \phi$.
Then, $\bigcup_\alpha A_\alpha$ is connected.

proof:

Let $A = \bigcup_\alpha A_\alpha$.
Suppose $A = C \cup D$ where $C \cap D = \phi$, open in subspace topology.
Then, $C = A \cap U$, $D = A \cap V$ for some open U, V.



Each A_α is either in C or in D. If $A_\alpha \subset C$, $A_\beta \subset D$, then $A_\alpha \cap A_\beta = \phi$.
Contradiction!
Therefore either all A_α are in C or all A_α are in D. A is connected.

$\square$.

<u>Theorem</u>. If X is connected, $f: X \to Y$ is continuous, then $f(X)$ is connected in the subspace topology. $f(X) = \mathrm{im}(f)$

proof: Suppose $\mathrm{im}(f) = C \cup D$, $C \cap D = \phi$. C and D are both open in the subspace topology.
$C = \mathrm{im}(f) \cap U$, $D = \mathrm{im}(f) \cap V$. U, V are open in Y, $\mathrm{im}(f) \cap U \cap V = \phi$.
$A = f^{-1}(C) = f^{-1}(U)$, $B = f^{-1}(D) = f^{-1}(V)$ both open
$A \cup B = X$ since $f^{-1}(C) \cup f^{-1}(D) = \{x \mid f(x) \in C \text{ or } f(x) \in D\} = \{x \mid f(x) = \mathrm{im}(f)\} = X$
$A \cap B = \phi$ since $C \cap D = \phi$.
Since X is connected, either $A = X$ or $B = X$.
Therefore either $\mathrm{im}(f) = C$ or $\mathrm{im}(f) = D$, since $f(A) = f(X) = \mathrm{im}(f) \subset C$. $\square$

$(f^{-1}(C) = \{x \mid f(x) \in \mathrm{im}(f) \text{ and } f(x) \in U\}$
$= f^{-1}(\mathrm{im}(f)) \cap f^{-1}(U)$
$= X \cap f^{-1}(U)$
$= f^{-1}(U))$

<u>Corollary</u>. If $X \cong Y$, then X is connected iff Y is connected.

proof:

$$X \underset{g}{\overset{f}{\rightleftarrows}} Y \qquad g = f^{-1} \text{ both continuous} \qquad f \text{ is a homeomorphism.}$$

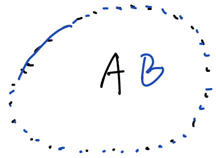
$\mathrm{im}(f) = Y$, $\mathrm{im}(g) = X$.
If X is connected, then Y is. If Y is connected, then X is. $\square$

<u>Corollary</u> (Intermediate Value Theorem).
Suppose X is connected. $f: X \to \mathbb{R}$ is continuous. Suppose $a, b \in X$ s.t. $f(a) < f(b)$.
Then $\forall r$ with $f(a) < r < f(b)$, $\exists c \in X$ s.t. $f(c) = r$. (meaning that $\forall r \in (f(a), f(b))$, $r \in \mathrm{im}(f)$).

proof: from previous, $\mathrm{im}(f)$ is a connected subspace of $\mathbb{R}$. $f(a), f(b) \in \mathrm{im}(f)$.
If $r \notin \mathrm{im}(f)$, then $\mathrm{im}(f) = (\mathrm{im}(f) \cap (-\infty, r)) \cup (\mathrm{im}(f) \cap (r, +\infty))$.
Contradiction!
Then, $r \in \mathrm{im}(f)$, $\exists c \in X$ s.t. $f(c) = r$

$\square$

<u>Lemma</u>. If X is a space, A is a connected subspace and $A \subset B \subset \bar{A}$ (i.e. $B = A \cup \{$some or all of its limit points$\}$), then B is connected.

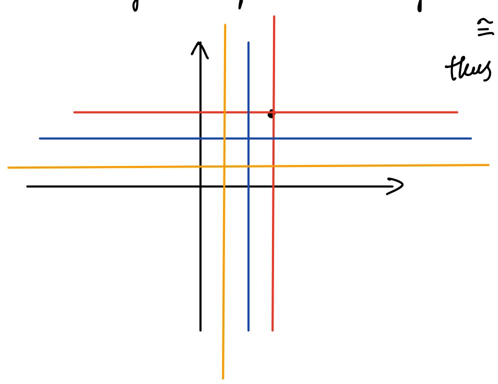


proof: Suppose $B = C \cup D$, $C \cap D = \phi$. C, D both open in subspace topology.
$\exists$ U, V open, s.t. $C = B \cap U$, $D = B \cap V$. $B \cap U \cap V = \phi$.
Then $A \cap U$ and $A \cap V$ forms a partition of A. Then either $A \subset A \cap U$ or $A \subset A \cap V$.
Suppose $A \subset A \cap U$ WLOG. Then, $A \subset U$. Then, since $A \subset (A \cap U) \subset (B \cap U)$, we have $A \cap V = \phi$.
Let $b \in B$. If $b \in V$, then: (a) if $b \in A$, then $B \cap U \cap V \neq \phi$, $\Rightarrow\Leftarrow$. (b). if b is a limit point of A, then, since V is open, $\exists$ open set $b \in U_b \subset V$. Also since limit point, $\exists\, a \in A$, $a \in U_b \subset V$. Then $A \cap V \neq \phi$. $\Rightarrow\Leftarrow$.
Then, $b \notin V$. Thus, for all $b \in B$, $b \in B \cap U$. Then $B \subset B \cap U = C$
Thus B is connected.

$\square$

Theorem: If X and Y are connected, then so is $X \times Y$.
Hence if $X_1 \cdots X_n$ are connected, so is $X_1 \times \cdots \times X_n$.

proof: given $x \times y \in X \times Y$. let $T_{x \times y} = \underbrace{\{x\} \times Y}_{\cong Y \text{ thus connected}} \cup \underbrace{X \times \{y\}}_{\cong X \text{ thus connected}}$.



$\Rightarrow T_{x \times y}$ is connected because it is a union of connected sets with non trivial intersection.

$X \times Y = \bigcup_{x \times y \in X \times Y} T_{x \times y}$, and $\forall\, x \times y$ and $z \times w$, $T_{x \times y} \cap T_{z \times w} = \{x \times w, z \times y\} \neq \phi$

by previous lemma, $X \times Y$ is connected.

E.g. $\mathbb{R}^n$ is connected.
Any $(a_1, b_1) \times \cdots \times (a_n, b_n)$ is connected, allowing $a_i = -\infty$, $b_i = +\infty$.

Theorem. If $n \geq 2$, then $\forall\, \vec{a} \in \mathbb{R}^n$, $\mathbb{R}^n - \{\vec{a}\}$ is connected.

proof: Let $U = \{x \in \mathbb{R}^n \mid x_n > a_n\}$, $V = \{x \in \mathbb{R}^n \mid x_n < a_n\}$.
By construction, U and V are connected. $U = \mathbb{R}^{n-1} \times (a_n, +\infty)$. $V = \mathbb{R}^{n-1} \times (-\infty, a_n)$.
Let $U' = U \cup \underbrace{(\{\vec{x} \mid x_n = a_n\} - \{a\})}_{\text{some of } U\text{'s limit points}}$, $V' = V \cup \underbrace{(\{\vec{x} \mid x_n = a_n\} - \{a\})}_{\text{some of } V\text{'s limit points.}}$

Therefore U', V' are connected.
$\mathbb{R}^n - \{\vec{a}\} = U' \cup V'$. $U' \cap V' \neq \phi$. $\Rightarrow \mathbb{R}^n - \{\vec{a}\}$ is connected. $\square$.

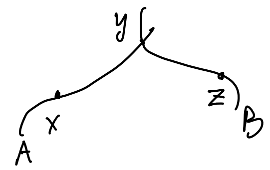
~~Corollary.~~ for $n \geq 2$: $\mathbb{R}^n - \{\text{points}\}$ are connected.

We define an equivalence relation: $x \underset{c}{\sim} y$ iff $\exists$ connected subspace $A \subset X$ s.t. $x \in A$, $y \in A$.

Claim: this is an equivalence relation.

① $x \underset{c}{\sim} x$. ② $x \underset{c}{\sim} y \Rightarrow y \underset{c}{\sim} x$

③ if $x \underset{c}{\sim} y$, $y \underset{c}{\sim} z$, then $x \underset{c}{\sim} z$



X is connected if $\forall x, y \in X$, $x \underset{c}{\sim} y$.
The equivalent classes are called the <u>components of X</u>.

If X is connected and $f: X \to Y$ is continuous, then $\mathrm{im}(f) = f(X)$ is connected.
If X is connected, then any quotient space of X is connected, since p as a quotient map is continuous.

For any $n \geq 1$, S^n is connected, $\mathbb{R}P^n$ is connected.

<u>Def</u>. (Path) Let X be a topological space, $x, y \in X$. A path from x to y is a continuous function $f: [a, b] \to X$ s.t. $f(a) = x$, $f(b) = y$.

$x \underset{p}{\sim} y$ if $\exists$ a path from x to y. $\underset{p}{\sim}$ is also an equivalent relation.

<u>Def</u>. (path connected). X is path-connected if for all $x, y \in X$, there exists a path from x to y.

If $f: [a, b] \to X$ is a path from x to y, for any $c < d \in \mathbb{R}$, $\exists$ map: $[c, d] \xrightarrow[\cong]{\varphi} [a, b] \to X$.
$f \circ \varphi$ is also a path.

If $f: [0, 1] \to X$ is a path from x to y, $g: [0, 1]$ is a path from y to z, then

$$(f * g)(t) = \begin{cases} f(2t), & t \in [0, \frac{1}{2}] \\ g(2t-1), & t \in [\frac{1}{2}, 1] \end{cases}$$

is a path from x to z.

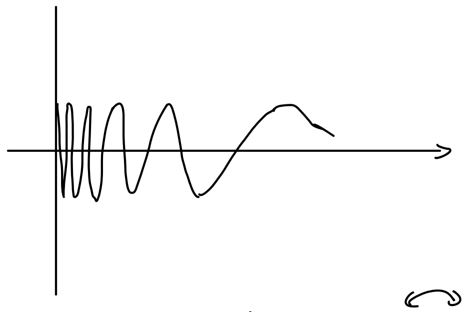
The equivalent classes are called the <u>path components.</u>

Theorem. If X is path connected, it is connected. Each path component is contained in a component.

proof: If $x \underset{p}{\sim} y$, then $\exists f:[a,b] \to X$ s.t. $f(a)=x$, $f(b)=y$. $\mathrm{im}(f)$ is a connected subspace containing x, y.

Suppose X is path connected. If $X = U \cup V$, U and V are open, $U \cap V = \phi$ and $x \in U$, $y \in V$, $f:[a,b] \to X$ is a path from x to y, then $f^{-1}(U)$ and $f^{-1}(V)$ are a separation of $[a,b]$. $\Rightarrow\Leftarrow$. $\square$.

Important Example: Topologist's sine curve. $C = \{(x,y) \in \mathbb{R}^2 \mid x>0,\ y=\sin(\frac{1}{x})\}$.



C is path-connected and connected. $C \cong (0,+\infty)$.

C is not closed

$\bar{C} = C \cup (\{0\} \times [-1,1])$ is a closed subset of $\mathbb{R}^2$ and is connected. (Lemma above).

limit points of C.

Theorem: $\bar{C}$ is not path connected.

proof: Suppose $\exists f:[a,b] \to \bar{C}$ were a path from $(0,0)$ to some (x,y), $x>0$.

Then $f^{-1}(\{0\} \times [-1,1])$ is a closed subset of $[a,b]$, so it has a maximum value c.

closed

Restrict f to $f:[c,d] \to \bar{C}$, then, $f(c) \in \{0\} \times [-1,1]$, $f(t) \in C\ \ \forall t>c$.

Reparameterize so $c=0$, $d=1$. $f(t) = (x(t), y(t))$ where $y(t) = \sin\left(\frac{1}{x(t)}\right)$.

Let $t_n = \frac{2}{\pi(2n+1)}$, then $t_n \to 0$ as $n \to \infty$. $y(t_n) = (-1)^n$. Then $\lim_{n\to\infty} t_n = 0$.

$\lim_{n\to\infty} y(t_n)$ doesn't exists. So, f not continuous, $\Rightarrow\Leftarrow$. $\square$.

Theorem. For $n \geq 2$, $\mathbb{R}^n - \{\text{one point}\}$ is path connected.

proof. Let $x, y \in \mathbb{R}^n - \{0\}$, we could take a straight line path $f(t) = tx + (1-t)y$, $t \in [0,1]$. $\Rightarrow \mathbb{R}^n$ is path connected.

If $y = \lambda x$ where $\lambda < 0$, then instead choose some $z \notin \mathrm{Span}(x)$

Choose a path $x \to z$ and another $z \to y$. $\square$

We can similarly prove that when $n \geq 2$, $\mathbb{R}^n - \{\text{finite points}\}$ is path connected.

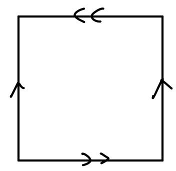
<u>Theorem</u>. If $f: X \to Y$ is continuous, X is path-connected, then $f(X)$ is path-connected.
proof: given $y_1, y_2 \in im(f)$, choose x_1, x_2 with $f(x_1) = y_1$, $f(x_2) = y_2$. Choose a path $g: x_1 \to x_2$
then $f \circ g$ is a path from y_1 to y_2. $\square$

If $X \simeq Y$, then X is path connected iff Y is path connected.

Any quotient of a path connected space is path connected.

S^n, $\mathbb{R}P^n$ are path connected.

For example, $K = [0,1]^2/\sim$ is path connected.



<u>Theorem</u>. Any product of path connected spaces is path connected in the product topology.
proof. Given two sequences $\vec{x} = (x_\alpha)$, $\vec{y} = (y_\alpha)$.
Choose $f_\alpha : [0,1] \to X_\alpha$, $f_\alpha(0) = x_\alpha$, $f_\alpha(1) = y_\alpha$ a path
$f = \prod_\alpha f_\alpha : [0,1] \to \prod_\alpha X_\alpha$ is continuous.

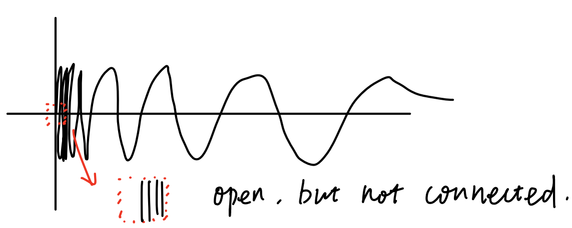
<u>Claim</u>. $\mathbb{R}^\omega$ is not connected in the box topology.
proof: Let $B = \{\vec{x} \in \mathbb{R}^\omega \mid \text{For some } R>0, |x_i| < R \ \forall i\}$ = set of bounded sequence.
Claim: B is both open and closed. (Easy. Just think about it in box topology).
Given $\vec{x} \in \mathbb{R}^\omega$. Let $U_{\vec{x}} = (x_1 - 1, x_1 + 1) \times (x_2 - 1, x_2 + 1) \times (x_3 - 1, x_3 + 1) \times \cdots$
If $\vec{x}$ is bounded, then every sequence in $U_{\vec{x}}$ is bounded. $U_{\vec{x}} \subset B$.
If $\vec{x}$ is unbounded, then every sequence in $U_{\vec{x}}$ is unbounded. $U_{\vec{x}} \subset \mathbb{R}^\omega - B$.

Therefore we found a separation of $\mathbb{R}^\omega$: B and $\mathbb{R}^\omega - B$. $\square$.

<u>Def</u>. (Locally (path) connected). A space X is called locally path connected if $\forall x \in X$, every open set $U \ni x$, $\exists$ (path) connected open set V with $x \in V \subset U$.

<u>Proposition</u>. (Any open sets in) $\mathbb{R}^n$ is locally connected and locally path connected.
proof: open balls of any radius are path connected.

Claim: Topologist's sine curve is connected, but not locally connected.



<u>Theorem</u>. If X is locally path connected, then components and path components agree.

LPC + C $\Rightarrow$ PC

LC $\Leftrightarrow$ components are open
LPC $\Leftrightarrow$ path components are open.

Compactness

Def. (Covering). Let X be a topological space. An open covering $\mathcal{F}$ of X is a collection of open sets whose union is all of X.

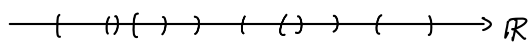
e.g. $\mathcal{F} = \{X\}$
$\mathcal{F} = \{\text{all open sets}\}$
X is a metric space. $\mathcal{F} = \{\text{all balls of radius } \frac{1}{3}\}$

Def. (Subcovering). Given an open covering $\mathcal{F}$, a subcovering is a subset of $\mathcal{F}$ that is still a covering.

Def. (Compact). X is called compact if every open covering has a finite subcovering.

e.g. of non-compact spaces:

1. $X = \mathbb{R}$. with standard topology.
   $\mathcal{F} = \{(x, x+1) \mid x \in \mathbb{R}\}$.



Every point is in some $(x-\frac{1}{2}, x+\frac{1}{2})$. But there is no finite subcovering.
If there were, then $\{(x_1, x_1+1), (x_2, x_2+1), \cdots, (x_n, x_n+1)\}$. Chouse $y = \max\{x_1, \cdots, x_n\} + 10000$
y is not in any of these sets.

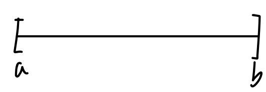
or. $\mathcal{F} = \{(-R, R) \mid R \in \mathbb{R}\}$ also has no finite subcovering.

2. $X = [0, 1]$
   $\mathcal{F} = \{(\frac{1}{n}, 1] \mid n \in \mathbb{Z}_+\}$ is open in X, and is a covering. But it has no finite subcovering.

e.g. X = finite set. $\mathcal{T}$ = any topology on X. Then $(X, \mathcal{T})$ is compact.

Theorem. Any closed interval $[a, b]$ $\forall a \le b$ is compact.

proof: Given an open covering $\mathcal{F}$ of $[a, b]$.
Let $S = \{x \in [a, b] \mid [a, x] \text{ is contained in a finite subcollection of } \mathcal{F}\}$
WTS: $b \in S$



Let $s = \sup S$. then $s \le b$.
Claim 1: $s \in S$. Indeed. if $s = a$, then we are done.
Otherwise, if $s > a$. choose $U \in \mathcal{F}$ s.t. $s \in U$. Then $\exists\, \varepsilon > 0$, s.t. $(s-\varepsilon, s] \subset U$.
$s - \varepsilon$ is not an upper bound of S. Then $\exists\, x$ with $s - \varepsilon < x \le s$, $x \in S$.
Therefore, $[a, s]$ is contained by fintely many sets in $\mathcal{F}$, so $s \in S$.

Claim 2: $s=b$. Indeed, suppose otherwise $s<b$.

We now know that $[a,s] \subset U_1 \cup \cdots \cup U_n$, $U_i \in \mathcal{F}$.

Suppose $s \in U_i$. For some ε, $[s, s+\varepsilon) \subset U_i$. Then, $[a, s+\frac{\varepsilon}{2}) \subset U_1 \cup \cdots \cup U_i$

Contradiction, since now $s+\frac{\varepsilon}{2} \in S$. s is not an upper bound.

Therefore, $s=b$.

Therefore, $[a,b]$ is compact. $\square$

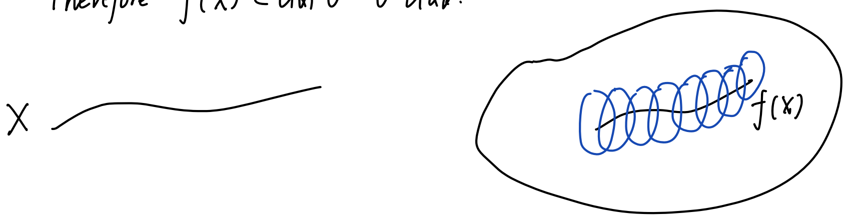
<u>Theorem</u> If X is compact, $f: X \to Y$ is continuous, then $f(X)$ is compact in the subspace topology of Y.

proof. Given some collection of open sets $\{U_\alpha\}$ in Y such that $f(X) \subset \bigcup_\alpha U_\alpha$,

WTS: $\exists$ finite subcollection that covers $f(X)$.

Since $\{f^{-1}(U_\alpha)\}$ is an open covering of X, since compact, $X = f^{-1}(U_{\alpha_1}) \cup \cdots \cup f^{-1}(U_{\alpha_k})$

Therefore $f(X) \subset U_{\alpha_1} \cup \cdots \cup U_{\alpha_k}$.



$\square$

<u>Corollary</u>: If $X \cong Y$, then X is compact iff Y is compact.

<u>Corollary</u>: Any quotient of a compact space is compact. quotient maps are surjective & continuous.

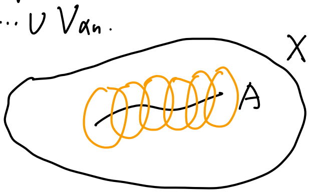
<u>Theorem</u>: If X is compact, then any closed set $A \subset X$ is also compact.

proof: given some collection $\{V_\alpha\}$ of open sets such that $A \subset \bigcup_\alpha V_\alpha$.

$X-A$ is open, then $\bigcup_\alpha V_\alpha \cup (X-A)$ is an open covering of X.

Since X is compact, there is a finite subcovering $\{V_{\alpha_1}, \cdots, V_{\alpha_n}, X-A\}$

$\Rightarrow A \subset V_{\alpha_1} \cup \cdots \cup V_{\alpha_n}$.



$\square$

<u>Theorem</u>. If X is Hausdorff, $A \subset X$ is compact, then A is closed.

proof: WTS: $X-A$ is open. For every $x \in X-A$, $\exists$ open set U s.t. $x \in U \subset X-A$.

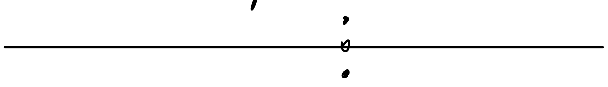
For each $a \in A$, $\exists$ open sets U_a and V_a s.t. $x \in U_a$, $a \in V_a$, $U_a \cap V_a = \phi$.

Since A is compact, $\exists \{V_{a_i}\}_{i=1}^n$ that covers A. Then, $x \in \bigcap_{i=1}^n U_{a_i} \subset X-A$.

Thus $X-A$ is open, A is closed.

$\square$

Counterexample of non-Hausdorff space:
Line with two origins.



Let $C = \{\text{Travis}\} \cup (0,1]$, then $C \cong [0,1]$, thus compact.
But $\{\text{Taylor}\} \subset \overline{C}$, so not closed.

<u>Theorem</u>. Let $f: X \to Y$ be continuous. X is compact. Y is Hausdorff. Then, f is a closed map.
proof: If $C \subset X$ is closed, then by lemma, it is compact. Then $f(C)$ is compact.
Since Y is Hausdorff, it is closed. $\square$.

<u>Theorem</u>. Further, if f is surjective, it is a quotient map.
if f is injective, it is an embedding.
if f is bijective, it is a homeomorphism.

A counterexample: $f: [0,1) \to S^1 \subset \mathbb{R}^2$: $f(t) = (\cos(2\pi t), \sin(2\pi t))$ is a bijection onto S^1, but since $[0,1)$ is not compact, it is not a homeomorphism.

An example: suppose $(X, \mathcal{T})$ is compact Hausdorff. If $\mathcal{T}' \supset \mathcal{T}$, then $id: (X,\mathcal{T}) \to (X, \mathcal{T}')$ is homeomorphism, since $(X, \mathcal{T}')$ is Hausdorff also, and id is bijective.

Now let's look at a specific case: metric space (X, d).

<u>Def.</u> (X, d) is bounded if $\exists N > 0$ s.t. $d(x,y) < N \ \forall x, y \in X$.

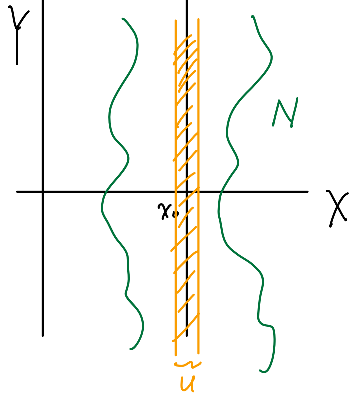
<u>Proposition</u>. If (X,d) is compact in the metric topology, then it is bounded.
proof: fix $x_0 \in X$. We look at the open sets $\{B_d(x_0, r) \mid r > 0\}$. Since this is an open covering, we have a finite subcovering $X = \bigcup_{i=1}^{n} B_d(x_0, r_i) = B_d(x, R)$, $R = \max\{r_1 \cdots r_n\}$.
Then $d(x,y) < 2R \ \forall x, y$. Bounded. $\square$.

This tells us that compact set can't wander off to infinity.

Given any metric space (X, d), define $\bar{d}(x,y) = \min\{d(x,y), 1\}$
Claim: $\bar{d}$ is a new metric on X, but gives the same topology.
$B_d(x, r) = B_{\bar{d}}(x, r)$ if $r \le 1$.
Also, $(X, \bar{d})$ is bounded.

Before proving the Heine-Borel Theorem, we prove the followings:

<u>Lemma</u>. (Tube lemma). Assume Y is compact. For any $x_0 \in X$, any open set $N \in X \times Y$ s.t. $\{x_0\} \times Y \subset N$, $\exists$ open set $U \subset X$ s.t. $x_0 \in U$, $U \times Y \subset N$.



proof: For each $y \in Y$, $\exists$ open $U_y \subset X$, $V_y \subset Y$ s.t. $(x_0, y) \in U_y \times V_y \subset N$
$\{x_0\} \times Y \cong Y$, compact. Take a finite subcover $V_{y_1}, \cdots, V_{y_n}$, then
$\{x_0\} \times Y \subset (U_{y_1} \times V_{y_1}) \cup \cdots \cup (U_{y_n} \times V_{y_n}) \subset N$
Let $U = U_{y_1} \cap \cdots \cap U_{y_n}$, then $\{x_0\} \times Y \subset U \times Y \subset N$. $\square$.

<u>Theorem:</u> If X, Y are compact, then so is $X \times Y$.
proof: Given an open covering $\mathcal{F}$ of $X \times Y$.
Claim 1: $\forall x_0 \in X$, we can cover $x_0 \times Y$ with finitely many open sets from $\mathcal{F}$.
Indeed, since $\{x_0\} \times Y \cong Y$, it is compact, it should be covered by some finite subcovering of $\mathcal{F}$.
Then, by the tube lemma, we can find $U_x \in X$, such that $U_x \times Y$ is in this finite union of sets from $\mathcal{F}$.
The $\{U_x\}_{x \in X}$ covers X. Choose finitely many of them s.t. $U_{x_1} \cup \cdots \cup U_{x_k} = X$,
Then, $(U_{x_1} \times Y) \cup \cdots \cup (U_{x_k} \times Y)$ covers $X \times Y$. $\square$

<u>Corollary</u>. If $X_1 \cdots X_n$ are compact, then $X_1 \times \cdots \times X_n$ is compact.

<u>Theorem</u>. For $A \subset \mathbb{R}^n$ with standard topology. A is compact iff it is closed and bounded.
(using either Euclidean d or $\rho := \max|x-y|$). <u>(Heine-Borel Theorem)</u>.

proof: by the previous theorem, $[a_1, b_1] \times \cdots \times [a_n, b_n]$ is compact.
If $C \subset \mathbb{R}^n$ is compact, then, since $\mathbb{R}^n$ is Hausdorff, it is closed. Also it is bounded since metric space.
Conversely, if C is bounded, then $\exists R > 0$ with $\rho(0, x) \le R$ $\forall x \in C$.
$C \subset [-R, R] \times \cdots \times [-R, R] =: D$
Since D is compact, if C is closed, then C is also compact. $\square$

Note: C is bounded with $d \iff C$ is bounded with ρ, since $\rho(x, y) \le d(x, y) \le \sqrt{n}\, \rho(x, y)$.

Corollary: S^n is compact. (closed and bounded in $\mathbb{R}^{n+1}$).
$\mathbb{R}P^n$ is compact since it is a quotient of S^n.
Polygond surfaces are compact, since it is also a quotient of a compact space.

If X is compact, $f: X \to \mathbb{R}$ is continuous, $f(X)$ is closed and bounded. So, $\exists A, B \in \mathbb{R}$, s.t. $A < f(x) < B$ $\forall x \in X$.
$f(X)$ is closed $\Rightarrow$ let $a = \inf f(X)$, $b = \sup f(X)$. $a, b \in \overline{f(X)} = f(X)$ since $f(X)$ is closed (contains its limit points).
Then, $\exists x_{min}, x_{max} \in X$, s.t. $f(x_{min}) = a$, $f(x_{max}) = b$. <u>(Extreme Value Theorem)</u>.

Consequences of Heine-Borel Theorem

- $S^n \subset \mathbb{R}^{n+1}$ is compact, since it is closed and bounded.
- $\mathbb{R}P^n$ is compact, since it is a quotient of S^n.

More about metric space:

<u>Def.</u> (Distance from x to A). Let (X, d) be a metric space. Let $A \subset X$ non empty.
$\forall x \in X$, define the distance from x to A to be
$$d(x, A) = \inf\{d(x, a) \mid a \in A\}.$$

<u>Proposition.</u> Fix A. $d(x, A)$ is continuous w.r.t. x.

proof: Given $x, y \in X$. For each $a \in A$:
$$d(x, A) \le d(x, a) \le d(x, y) + d(y, a)$$
Thus $d(x, A) - d(x, y) \le \inf d(y, a) = d(y, A)$
$$d(x, A) - d(y, A) \le d(x, y)$$
Exchange x, y, $|d(x, A) - d(y, A)| \le d(x, y)$.
Therefore, $\forall \varepsilon > 0$, $d(x, y) < \varepsilon \Rightarrow |d(x, A) - d(y, A)| < \varepsilon$. Continuous.

$\square$

<u>Def.</u> (Diameter). The diameter of a bounded subset $A \subset (X, d)$ is $\sup\{d(a_1, a_2) \mid a_1, a_2 \in A\}$.

<u>Lemma</u> (The Lebesgue number lemma). Let $\mathcal{F}$ be an open covering of (X, d). If X is compact, there is a $\delta > 0$ such that for each subset of X having diameter less than δ, there exists an element of $\mathcal{F}$ containing it.
δ is called a Lebesgue number for the covering $\mathcal{F}$.

proof: If $X \in \mathcal{F}$, then any positive number is a Lebesgue number, since X contains any subset.
Assume not. Choose a finite subcovering $\{A_1, \dots, A_n\}$. For each i, let $C_i := X - A_i$,
define $f: X \to \mathbb{R}$: $f(x) = \frac{1}{n} \sum_{i=1}^{n} d(x, C_i)$

WTS $f(x) > 0$ $\forall x \in X$. Given $x \in X$, choose i s.t. $x \in A_i$. Then choose $\varepsilon > 0$ s.t. $B_d(x, \varepsilon) \subset A_i$.
Then $d(x, C_i) \geq \varepsilon$, so that $f(x) \geq \frac{1}{n}\varepsilon > 0$.

Since f is continuous on a compact space X, it has a minimum value δ.
WTS δ is our required Lebesgue number.
Let $B \subset X$ such that its diameter is less than δ. Choose $x_0 \in B$, then
$$B \subset B_d(x_0, \delta).$$
Now, $\delta \le f(x_0) \le d(x_0, C_m)$ where $d(x_0, C_m)$ is the largest of $d(x_0, C_i)$.
Then, $B_d(x_0, \delta) \subset A_m$ where $A_m = X - C_m \in \mathcal{F}$.

$\square$

<u>Def.</u> (Uniformly Continuous). $f: (X, d_X) \to (Y, d_Y)$ is uniformly continuous if given $\varepsilon > 0$, $\exists\, \delta > 0$ s.t for every pair of $x_0, x_1 \in X$, $d_X(x_0, x_1) < \delta \Rightarrow d_Y(f(x_0), f(x_1)) < \varepsilon$.

<u>Theorem</u> (Uniform Convergence Theorem). Let $f: (X, d_X) \to (Y, d_Y)$ be continuous. X is compact. Then f is uniformly continuous.

proof: Given $\varepsilon > 0$. Take the open covering of Y by balls $B_{d_Y}(y, \frac{\varepsilon}{2})$. Then, Let $\mathcal{F}$ be $\{f^{-1}(B_{d_Y}(y, \frac{\varepsilon}{2}))\}$ an open covering of X. Choose δ to be a Lebesgue number for the covering $\mathcal{F}$. Then, if x_1, x_2 are two points of X such that $d_X(x_1, x_2) < \delta$, then the two point set $\{x_1, x_2\}$ has diameter less than δ. so $\exists\, y \in Y$ s.t. $\{f(x_1), f(x_2)\} \subset B_{d_Y}(y, \frac{\varepsilon}{2})$, $d(f(x_1), f(x_2)) < \varepsilon$. $\square$

Then we prove the uncountability of $\mathbb{R}$ involving no expantion, just the order property.

<u>Lemma</u>. If X is compact, $C_1, C_2, \cdots$ are non-empty closed sets. $C_1 \supset C_2 \supset C_3 \supset \cdots$
Then $\bigcap_{i \in \mathbb{Z}_+} C_i \neq \phi$.

proof. Let $U_n = X - C_n$, then $\bigcup_{n \in \mathbb{Z}_+} U_n = X - \bigcap_{n \in \mathbb{Z}_+} C_n$, $U_1 \subset U_2 \subset U_3 \subset \cdots$
If $\bigcap_{n \in \mathbb{Z}_+} C_n$ is empty, then $\{U_n\}$ are open coverings of X.

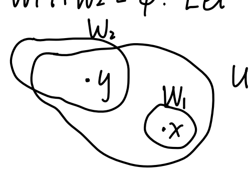
Since X is compact, $\exists$ a finite subcovering. Then, $X_n = U_n$ for some n.
Then C_n is empty. $\Rightarrow \Leftarrow$. $\square$

Cf. Closedness is important. $(0,1) \supset (0, \frac{1}{2}) \supset (0, \frac{1}{3}) \supset \cdots$, but intersection is empty.
Compactness: in $\mathbb{R}$. let $C_n = [n, +\infty)$, $C_1 \supset C_2 \supset \cdots$, intersection is empty.

Def. (Countable Infinite). A set is countably infinite if it is a bijection of $\mathbb{N}$.
S is countable if it is either finite or countably infinite.
Equivalently, S is countable if $\exists$ surjective $\mathbb{N} \mapsto S$.

Def. (Isolated point). Let X be a topological space. $x \in X$ is an isolated point if $\{x\}$ is open.

<u>Theorem</u> If X is a compact Hausdorff space with no isolated points, then X is uncountable.
proof: Step 1:
Claim. For any non-empty open set $U \subset X$ and any point x, there is a non-empty open set V s.t. $V \subset U$, $x \notin \overline{V}$.
Indeed, since U is not one-point set, choose $y \in U - \{x\}$. Choose W_1, W_2 open such that $x \in W_1$, $y \in W_2$, $W_1 \cap W_2 = \phi$. Let $V = W_2 \cap U$ open in U. Then $x \notin \overline{V}$.



Step 2:
Suppose we have a surjective $f: \mathbb{N} \to X$. Let $x_n = f(n)$.
By step 1, $\exists$ open set V_1 s.t. $x_1 \notin \overline{V_1}$ (Take $U = X$).
By the same idea, $\exists$ open set $V_2 \subset V_1$ s.t. $x_2 \notin \overline{V_2}$
$\cdots\cdots$
Therefore we can find a sequence of open sets $V_1 \supset V_2 \supset V_3 \supset \cdots$ s.t. $x_n \notin \overline{V_n}$.
$\overline{V_1} \supset \overline{V_2} \supset \overline{V_3} \supset \cdots$

So there must be some $z \in \bigcap_{n \in \mathbb{N}} \overline{V_n}$ by previous lemma.

Then $z = x_i$ for some i. But $x_i \notin \overline{V_i}$. $\Rightarrow\Leftarrow$. $\square$

<u>Corollary</u>: Every closed interval in $\mathbb{R}$ is uncountable.

<u>Def</u> (Limit point compact). A space X is called limit point compact if every infinite subset of X has a limit point.

Cf. $\mathbb{Z} \subset \mathbb{R}$.

for each $x \in X - A$, $\exists$ open U_x s.t. $U_x \cap (A - \{x\}) = \phi$ $\Rightarrow x \in U_x \subset X - A$

$\Rightarrow X - A = \bigcup_{x \in X-A} U_x$, open

$\Rightarrow A = X - \bigcup_{x \in X-A} U_x$, closed

<u>Proposition</u>. If X is compact, then it is limit point compact.

proof. Let A be an infinite subset of X.

Suppose it has no limit points. Then $A = \overline{A}$. A is closed. Then A is compact.

Also, for each $a \in A$, $\exists$ U_a open, s.t. $U_a \cap A = \{a\}$

Recall limit point: if x is a limit point of A, then $\forall$ open $U \ni x$, $\exists$ $z \in A$ s.t. $U \cap (A - \{x\}) = z$.

Then, $\{U_a\}$ covers A. $\exists$ a finite subcover $\{U_{a_i}\}_{i=1}^{n}$. Then, since $U_{a_i} \cap A = \{a_i\}$, $A = \{a_i\}_{i=1}^{n}$, A is finite. Contradiction!

Given a sequence $(x_1, x_2, x_3, \cdots)$. A subsequence is $(x_{i_1}, x_{i_2}, x_{i_3}, \cdots)$ for some $i_1 < i_2 < i_3 < \cdots$.

<u>Def</u>. (Sequential Compactness). X is sequentially compact if every sequence has a convergent subsequence.

<u>Theorem</u> If X is a metric space with metric topology, then compactness, limit point compactness, and sequential compactness.

proof: By the previous theorem, compactness $\Rightarrow$ limit point compactness.

Suppose that it is limit point compact. Take a sequence $(x_1, x_2, x_3, \cdots)$ as an infinit subset of X. Since it has a limit point, let it be x. Let $\varepsilon > 0$ be given. Since x is a limit point, choose x_{i_1} from it so that $x_{i_1} \in B_d(x, \frac{\varepsilon}{2})$.

...

choose x_{i_n} from it so that $x_{i_n} \in B_d(x, \frac{\varepsilon}{2^n})$.

Therefore we found a subsequence $(x_{i_1}, x_{i_2}, \cdots, x_{i_n}, \cdots)$ that converges to x.

Therefore it is sequentially compact.

my proof.

Suppose that it is not limit point compact. Then, let A be an infinite subset of X such that it has no limit points. Then $\forall\, a \in A$ $\exists$ open $B_d(a, \varepsilon_a) \ni a$, $\varepsilon_a > 0$ s.t. $B_d(a, \varepsilon_a) \cap A = \{a\}$, $B_d(a, \varepsilon_a) \cap (A - \{a\}) = \phi$. Let $\varepsilon = \inf_{m \in A - \{n\}} \inf_{n \in A} d(m, n) - \varepsilon_m$

Then $\varepsilon > 0$, since by previous deduction $d(m, n) - \varepsilon_m > 0$. Then, choose any infinite sequence $(x_1, x_2, \cdots, x_n, \cdots)$ and any subsequence $(x_{i_1}, x_{i_2}, \cdots)$. then $\forall\, j, k \in \mathbb{N}$, $d(x_{i_j}, x_{i_k}) > \varepsilon$. not Cauchy. Therefore not convergence.

Therefore not sequentially compact.

Thus sequentially compact $\Rightarrow$ limit point compact. $\square$

Textbook proof of sequential compact $\Rightarrow$ compact:

Step 1: WTS that if X is sequentially compact, then the Lebesgue number lemma holds for X.

Let $\mathcal{A}$ be an open covering of X. Assume that $\nexists\, \delta > 0$ s.t. each set of diameter less than δ has an element in $\mathcal{A}$ containing it. Want a contradiction.

$\forall n \in \mathbb{N}_+$, $\exists\, C_n$ with diameter less than $\frac{1}{n}$ not contained in any element of $\mathcal{A}$.

Choose $x_n \in C_n$. By hypothesis, some $\{x_{n_i}\} \subset \{x_n\}$ converges (say to a).

Then, $a \in A$ for some $A \in \mathcal{A}$. Since A is open, we may choose $\varepsilon > 0$ s.t. $B(a, \varepsilon) \subset A$.

If i is large enough that $\frac{1}{n_i} < \frac{\varepsilon}{2}$, then $C_{n_i} \subset B(x_{n_i}, \frac{\varepsilon}{2})$.

If i is also large enough that $d(x_{n_i}, a) < \frac{\varepsilon}{2}$, then $C_{n_i} \subset B(x_{n_i}, \frac{\varepsilon}{2}) \subset B(a, \varepsilon) \subset A$.

This contradict with $\nexists\, A \subset \mathcal{A}$ s.t. $C_{n_i} \subset A$.

Step 2. WTS that if X is sequentially compact, then given $\varepsilon > 0$, $\exists$ a finite covering of X by open ε-balls.

By contradiction, suppose that $\exists\, \varepsilon > 0$, s.t. X cannot be covered by finitely many ε-balls. Construct $\{x_n\}$ as follows: choose $x_1 \in X$, then $B(x_1, \varepsilon)$ is not all of X; choose $x_2 \in X - B(x_1, \varepsilon)$, $\cdots$, choose $x_n \in X - (B(x_1, \varepsilon) \cup \cdots \cup B(x_n, \varepsilon))$ using the fact that these balls do not cover X. Then, by construction, $d(x_{n+1}, x_i) \geq \varepsilon$ for $i = 1, \cdots, n$.

Then, $\{x_n\}$ does not have any convergent subsequence. Contradiction.

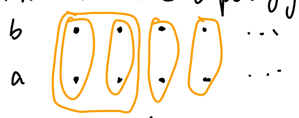
Step 3: WTS X is compact.

Let $\mathcal{A}$ be an open covering. Then, $\mathcal{A}$ has a Lebesgue number $\delta > 0$. Let $\varepsilon = \frac{\delta}{3}$, then we can find a finite covering of X by ε-balls. Each of these balls has diameter $\leq \frac{2}{3}\delta$, so it lies in an element of $\mathcal{A}$. Choosing one such element of $\mathcal{A}$ for each of these balls, we find a finite subcovering.

$\square$

e.g. $X = \mathbb{N}$ with discrete topology. $Y = \{a, b\}$ with indiscrete topology.

$X \times Y$:



Non-Hausdorff. Limit point compact, not sequential compact.

$I = [0, 1]$.

Fundamental Groups

<u>Def.</u> (Homotopy) Let X, Y be topological spaces. $f, g: X \to Y$ be two different continuous maps.
A homotopy between f and g is a continuous map $F: X \times I \to Y$ such that
$F(x, 0) = f(x)$, $F(x, 1) = g(x)$, $\forall x \in X$.

For each $t \in I$, $f_t: X \to Y$ $f_t(x) = F(x, t)$ is continuous.

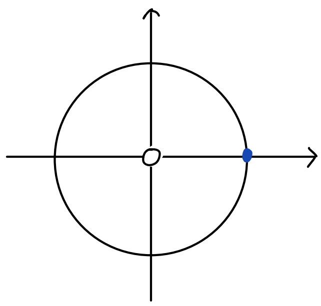
F is a continuous interpolation between f and g.

<u>Def.</u> f and g are homotopic $f \simeq g$ if $\exists$ a homotopy from f to g.

e.g. if $Y = \mathbb{R}^n$. any continuous $f, g: X \to Y$ are homotopic,
define $F(x, t) = (1-t) f(x) + t g(x)$. Also continuous. $t \in I$.

e.g. $X = S^1$. $Y = \mathbb{R}^2 - \{\vec{0}\}$. $f: S^1 \to \mathbb{R}^2 - \{\vec{0}\}$ inclusion $f(\vec{x}) = \vec{x}$; $g: S^1 \to \mathbb{R}^2 - \{\vec{0}\}$ constant $g(\vec{x}) = (1, 0)$.
there is no homotopy btw f and g.

$F(x, t) = (1-t) f(x) + t g(x)$
does not work.



<u>Def.</u> (Simply Connected) X is simply connected if it is path connected, and all continuous maps $S^1 \to X$ are homotopic.

$\Rightarrow$ $\mathbb{R}^2 - \{\vec{0}\}$ is not simply connected. (above f and g are not homotopic)

<u>Def.</u> (Homotopy relative to A). Let X and Y be topological spaces. Let $A \subset X$ (can be ϕ).
f, g are continuous map $X \to Y$. $f|_A = g|_A$. A homotopy relative to A (rel. A) is a continuous map $F: X \times I \to Y$ s.t. $F(x, 0) = f(x)$, $F(x, 1) = g(x)$ $\forall x \in X$.
Also $F(a, t) = f(a) = g(a)$ $\forall a \in A$, $t \in [0, 1]$.

If such homotopy exists, we write $f \underset{A}{\simeq} g$.

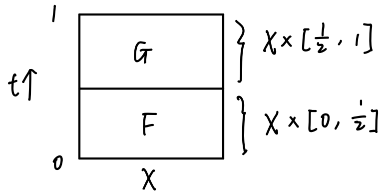
e.g. $X = \{a\}$. Y.
A function $\{a\} \to Y$ is just a choice of point in Y. A homotopy is a path between points.

<u>Proposition</u> Homotopy (rel. A) is an equivalence relationship on the set of continuous functions $X \to Y$.
proof. ① Reflexivity. $f \underset{A}{\simeq} f$. define $F(x, t) = f(x)$.
② Symmetry. given F, a homotopy $f \to g$. define $G(x, t) := F(x, 1-t)$. $g \to f$.
③ Transitivity. If $f \underset{A}{\simeq} g$ via F, $g \underset{A}{\simeq} h$ via G, define $H: X \times I \to Y$ by

$$H(x,t) = \begin{cases} F(x, 2t) & , t\in[0,\frac{1}{2}] \\ G(x, 2t-1) & , t\in[\frac{1}{2},1] \end{cases}$$

$\forall a\in A,\ H(a,t) = \begin{cases} F(a,2t) = f(a) = g(a) & , t\in[0,\frac{1}{2}] \\ G(a, 2t-1) = g(a) = h(a) & , t\in[\frac{1}{2},1] \end{cases}$

then $H(a,t) = f(a) = h(a)$.



F, G overlap on the overlap.

Then $f \underset{A}{\simeq} h$.

<u>Def</u>: $[X, Y]$ is the set of homotopy classes of continuous maps $X\to Y$.

e.g. $[X, \mathbb{R}^n]$ is a one point set. Only one class because all continuous functions are homotopic.

<u>Lemma</u>. If $f_0, f_1 : X\to Y$ are homotopic, $f_0 \underset{A}{\simeq} f_1$, and $g: Y\to Z$ continuous, then $g\circ f_0 \underset{A}{\simeq} g\circ f_1$

If $h: W\to X$ is continuous, then $f_0\circ h \underset{h^{-1}(A)}{\simeq} f_1\circ h$.

proof: just define a homotopy $G = g\circ F(x,t)$

$X\times I \xrightarrow{F} Y \xrightarrow{g} Z$.

Also, $W\times I \xrightarrow{h\times id} X\times I \xrightarrow{F} Y$, define homotopy $H(x,t) = F(h\times id(w,t))$

$= F(h(w), t)$

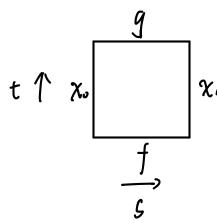
$\square$

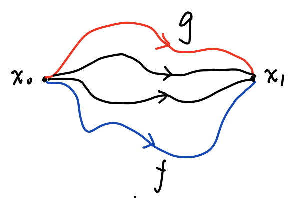
<u>Def</u>. (Path homotopy). If X is a space, $x_0, x_1\in X$. A path $x_0 \rightsquigarrow x_1$ is a continuous function $f: [0,1]\to X$, $f(0) = x_0$, $f(1) = x_1$.

If f, g are paths $x_0\rightsquigarrow x_1$, we say a path homotopy is a homotopy rel. $\{0,1\}$.

$F: \underset{s}{[0,1]}\times \underset{t}{[0,1]} \to X$ s.t. $F(s,0) = f(s)\ \forall s$, $F(s,1) = g(s)\ \forall s$ (time)

$F(0,t) = x_0\ \forall t$, $F(1,t) = x_1\ \forall t$ (state/position).





$f \underset{p}{\simeq} g$ if $\exists$ a path homotopy btw f and g.

e.g. In $\mathbb{R}^n$. Is it true that any two paths from x_0 to x_1 are path homotopic?

in $\mathbb{R}^2 - \{0\}$, $x_0 = (-1,0)$, $x_1 = (1,0)$, $\nexists$ homotopy btw x_0 and x_1

<u>Def</u>. (Loop). Let $x_0\in X$. A loop based at x_0 is a path $x_0\rightsquigarrow x_0$ i.e. $f: [0,1]\to X$ continuous, $f(0) = x_0$, $f(1) = x_0$

<u>Def</u>. (Fundamental Group) $\pi_1(X, x_0)$ is the set of path-homotopic classes of loop based at x_0.

It's the fundamental group of X based at x_0.

<u>Def</u>. (Coextension $*$) Given a path $f: x_0\rightsquigarrow x_1$, $g: x_1\rightsquigarrow x_2$, we define $f*g: x_0\rightsquigarrow x_2$ by

$$f*g = \begin{cases} f(2s) & , s\in[0,\frac{1}{2}] \\ g(2s-1) & , s\in[\frac{1}{2},1] \end{cases}$$

Lemma. If $f_0 \underset{P}{\simeq} f_1 : x_0 \rightsquigarrow x_1$, $g_0 \underset{P}{\simeq} g_1 : x_1 \rightsquigarrow x_2$, then $f_0 * g_0 \underset{P}{\simeq} f_1 * g_1$.

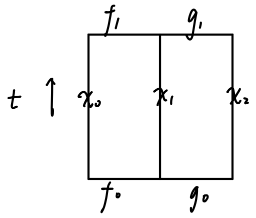
proof: if F is a homotopy $f_0 \to f_1$, G is a homotopy $g_0 \to g_1$, define

$$F * G(s,t) = \begin{cases} F(2s, t), & s \in [0, \frac{1}{2}] \\ G(2s-1, t), & s \in [\frac{1}{2}, 1] \end{cases}$$

so that $F(0,t) = x_0$, $F(1,t) = G(0,t) = x_1$, $G(1,t) = x_2$

$F(2s, 0) = f_0(2s)$ $F(2s, 1) = f_1(2s)$

$G(2s-1, 0) = g_0(2s-1)$, $G(2s-1, 1) = g_1(2s-1)$.



$\square$

Corollary. $*$ gives a well defined binary operation on $\pi_1(X, x_0)$. $[f] * [g] = [f * g]$.

Proposition (Reparameterization). If $f: x_0 \rightsquigarrow x_1$ is a path, $\varphi: [0,1] \to [0,1]$ is a continuous function s.t. $\varphi(0) = 0$, $\varphi(1) = 1$, then $f \simeq f \circ \varphi$.

proof: notice that $\varphi \underset{\{0,1\}}{\simeq} id_{[0,1]}$ via homotopy $G(s,t) = (1-t)\varphi(s) + ts$.

apply lemma: $f = f \circ id \simeq f \circ \varphi$.

$\square$

Def. ($\bar{f}$) If $f: x_0 \rightsquigarrow x_1$ is a path then $\bar{f}: x_1 \rightsquigarrow x_0 : \bar{f}(s) = f(1-s)$ is also a path.

Def. (e) Define e_x as the constant path at x: $e_x(s) = x \quad \forall s \in [0,1]$

Then, $e_{x_0} * f \simeq f \simeq f * e_{x_1}$, $f: x_0 \rightsquigarrow x_1$ path.

Def. (Group): A group is a set G and a binary operation $*: G \times G \to G$ satisfying.

① $\forall a, b, c \in G$, $(a * b) * c = a * (b * c)$. Associativity.

② $\exists$ a distinguishable $e \in G$: $e * a = a = a * e$. Identity element.

③ $\forall a \in G$, $\exists \bar{a} \in G$ s.t. $\bar{a} * a = e$, $a * \bar{a} = e$. Inverse.

Def (Commutative) G is commutative or abelian if $\forall a, b \in G$, $a * b = b * a$.

e.g. • $(\mathbb{R}, +)$, $e = 0$

- $(\mathbb{Z}, +)$, $e = 0$
- $(\mathbb{R} - \{0\}, \cdot)$, • $(\mathbb{Q} - \{0\}, \cdot)$
- $G = \{A_{n \times n} \mid \text{invertible}\}$, $*$ = matrix multiplication

  $(AB)C = A(BC)$.

  $I_n = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$ identity.

- $(\mathbb{R}^n, +)$

Theorem $(\pi_1(X, x_0), *)$ is a group. $*$ is defined as $[f]*[g] = [f*g]$

proof:

$$(f*g)(s) = \begin{cases} f(2s) & s\in[0,\frac{1}{2}] \\ g(2s-1) & s\in[\frac{1}{2},1] \end{cases}$$

First we show associativity of paths, not just loops.

Let f be a path $x_0 \rightsquigarrow x_1$, g be a path from $x_1 \rightsquigarrow x_2$. $h: x_2 \rightsquigarrow x_3$.

define $\forall s\in[0,1]$

$$(f*g)*h(s) = \begin{cases} (f*g)(2s) & , s\in[0,\frac{1}{2}] \\ h(2s-1) & , s\in[\frac{1}{2},1] \end{cases}$$

$$= \begin{cases} f(4s) & s\in[0,\frac{1}{4}] \\ g(4s-1) & s\in[\frac{1}{4},\frac{1}{2}] \\ h(2s-1) & s\in[\frac{1}{2},1] \end{cases}$$

$$f*(g*h)(s) = \begin{cases} f(2s) & , s\in[0,\frac{1}{2}] \\ (g*h)(2s-1) & , s\in[\frac{1}{2},1] \end{cases}$$

$$= \begin{cases} f(2s) & s\in[0,\frac{1}{2}] \\ g(4s-2) & s\in[\frac{1}{2},\frac{3}{4}] \\ h(4s-3) & s\in[\frac{3}{4},1] \end{cases}$$

$\Rightarrow f*(g*h) \simeq (f*g)*h$. so, they are homotopic.

By the property of equivalence relation, $[f*(g*h)] = [f*(g*h)]$

Since $[f*(g*h)] = [f]*[g*h] = [f]*([g]*[h])$

$[(f*g)*h] = [f*g]*[h] = ([f]*[g])*[h]$,

we know that $[f]*([g]*[h]) = ([f]*[g])*[h]$.

Then, the constant path at x_0 is the identity element e_{x_0} (in $\pi(X, x_0)$: $[e_{x_0}]$)

$[e_{x_0}]*[f] = [f]*[e_{x_0}] = [f]$.

More generally, if f is a path from x_0 to x_1, $e_{x_0}*f \underset{p}{\simeq} f$, $f \underset{p}{\simeq} f*e_{x_1}$

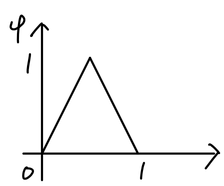
If f is a loop, then $[e_{x_0}]*[f] = [f]*[e_{x_0}] = [f]$.

Inverse: If f is a path $x_0 \rightsquigarrow x_1$, then $\bar{f}$ is a path from x_1 to x_0: $\bar{f}(s) = f(1-s)$.

Claim: $f*\bar{f} \underset{p}{\simeq} e_{x_0}$. $\bar{f}*f \underset{p}{\simeq} e_{x_1}$

Think about
$$f*\bar{f}(s) = \begin{cases} f(2s) & , s\in[0,\frac{1}{2}] \\ \bar{f}(2s-1) & , s\in[\frac{1}{2},1] \end{cases} = \begin{cases} f(2s) & , s\in[0,\frac{1}{2}] \\ f(2-2s) & , s\in[\frac{1}{2},1] \end{cases}$$

then $f*\bar{f} = f\circ\varphi$ where $\varphi = \begin{cases} 2s & s\in[0,\frac{1}{2}] \\ 2-2s & s\in[\frac{1}{2},1] \end{cases}$



$\varphi: [0,1] \to [0,1]$ continuous, $\varphi(0) = \varphi(1) = 0$ is a path from 0 to 0.

$\varphi \simeq$ constant path at 0. $= e_0$

Therefore, $f*\bar{f} = f\circ\varphi \underset{p}{\simeq}$ constant path at $x_0 = e_{x_0}$

Then, $[f]*[\bar{f}] = [e_{x_0}]$. Same way: $[\bar{f}]*[f] = [e_{x_0}]$

Define $\overline{[f]} := [\bar{f}]$ since if $f\simeq g$, then $\bar{f}\simeq\bar{g}$. Then, $\overline{[f]}*[f] = [f]*\overline{[f]} = [e_{x_0}]$ $\square$

Def: (Group homomorphism) Suppose $(G, *)$ and $(H, \cdot)$ are groups. A function $\varphi: G \to H$ is called a group homomorphism if for all $g_1, g_2 \in G$, $\varphi(g_1 * g_2) = \varphi(g_1) \cdot \varphi(g_2)$

If so, $\varphi(e_G) = e_H$. $\varphi(\bar{g}) = \overline{\varphi(g)}$ automatically.

Def: (Group Isomorphism) $\varphi: G \to H$ is called a group isomorphism if it is a group homomorphism, and $\exists$ a group homomorphism $\psi: H \to G$ s.t. $\psi \circ \varphi = id_G$, $\varphi \circ \psi = id_H$.

Fact: (a) If φ is a group homomorphism and is bijective, then φ^{-1} is also a group homomorphism.

$\forall h_1, h_2 \in H$, $\varphi^{-1}(h_1 \cdot h_2) = \varphi^{-1}(h_1) * \varphi^{-1}(h_2)$

since $\varphi(\varphi^{-1}(h_1) * \varphi^{-1}(h_2)) = \varphi(\varphi^{-1}(h_1)) \cdot \varphi(\varphi^{-1}(h_2)) = h_1 \cdot h_2$.

then $\varphi^{-1}(h_1) * \varphi^{-1}(h_2) = \varphi^{-1}(h_1 \cdot h_2)$.

(b) A bijective group homomorphism is a group isomorphism.

since $\varphi: G \to H$, $\varphi^{-1}: H \to G$ both group homomorphism.

$\varphi^{-1} \circ \varphi = id_G$. $\varphi \circ \varphi^{-1} = id_H$.

G and H are isomorphic if $\exists$ an isomorphism. $G \cong H$ They have the same cardinality.

For example: $(\{\cdots -2, -1, 0, 1, 2, \cdots\}, +)$. $(\{\cdots$ -two, -one, zero, one, two $\cdots\}$, addition$)$ are isomorphic.

Theorem. Suppose $k: X \to Y$ is a continuous map. Suppose $x_0 \in X$, $k(x_0) = y_0$. Define a map $k_*: \pi_1(X, x_0) \to \pi_1(Y, y_0)$ by $k_*([f]) = [k \circ f]$

(a). this is well-defined.

(b) k_* is a group homomorphism, it is called the <u>induced homomorphism</u> of k.

proof: (a). if $f \simeq_p f'$, then $k \circ f \simeq_p k \circ f'$. therefore they are in the same class.

(b). $k \circ (f * g)(s) = \begin{cases} k(f(2s)), & s \in [0, \frac{1}{2}] \\ k(g(2s-1)), & s \in [\frac{1}{2}, 1] \end{cases} = \begin{cases} (k \circ f)(2s), & s \in [0, \frac{1}{2}] \\ (k \circ g)(2s-1), & s \in [\frac{1}{2}, 1] \end{cases} = (k \circ f) * (k \circ g)(s)$ by definition.

so $k \circ (f * g) = (k \circ f) * (k \circ g)$. Thus,

$k_*([f * g]) = [k \circ (f * g)] = [(k \circ f) * (k \circ g)] = [k \circ f] * [k \circ g] = k_*([f]) * k_*([g])$ $\square$

Theorem: Properties of k_*:

(a). $(id_X)_* = id_{\pi_1(X, x_0)}$

proof: id_X: $id_X(x_0) = x_0$ let $f: x_0 \rightsquigarrow x_0$ a loop. $[f] \in \pi_1(X, x_0)$.

$(id_X)_*([f]) = [id_X \circ f] = [f]$

Thus $(id_X)_* = id_{\pi_1(X, x_0)}$.

(b). If $k: X \to Y$, $l: Y \to Z$ are continuous. then $(l \circ k)_* = l_* \circ k_*$.

proof: $(l \circ k)_*([f]) = [(l \circ k) \circ f]$

$= [l \circ (k \circ f)]$

$= l_*([k \circ f])$

$= l_*(k_*([f]))$
$= (l_* \circ k_*)([f]).$

$\cong$ < homeomorphism in topological space / isomorphism in groups. $\square$

Corollary. If $X \cong Y$ by a homeomorphism taking x_0 to y_0, then $\pi_1(X, x_0) \cong \pi_1(Y, y_0)$.

proof: if $k: X \to Y$ continuous. $k^{-1}: Y \to X$ continuous, then

$$\pi_1(X, x_0) \underset{(k^{-1})_*}{\overset{k_*}{\rightleftarrows}} \pi_1(Y, y_0)$$

$(k^{-1} \circ k) = id_X$, then $(k^{-1} \circ k)_* = id_{\pi_1(X, x_0)}$ by Property (a).
$(k \circ k^{-1}) = id_Y$, then $(k \circ k^{-1})_* = id_{\pi_1(Y, y_0)}$
$(k^{-1}_*) \circ (k_*) = (k^{-1} \circ k)_* = id_{\pi_1(X, x_0)}$ by Property (b) and (a).
$(k_*) \circ (k^{-1}_*) = (k \circ k^{-1})_* = id_{\pi_2(Y, y_0)}$
Therefore $k^{-1}_* = (k_*)^{-1}$ by definition, $\pi_1(X, x_0) \cong \pi_1(Y, y_0)$.
$k_* = (k^{-1}_*)^{-1}$

$\square$

π_1 is a functor from the category of pointed topological spaces to the category of groups.

Proposition. If x_0 and x_1 are in the same path component of X, then $\pi_1(X, x_0) \cong \pi_1(X, x_1)$.

proof: Let h be a path between x_0 and x_1.
Define $A_h: \pi_1(X, x_0) \to \pi_1(X, x_1)$ by $A_h([f]) = [\bar{h} * f * h]$, $[f] \in \pi_1(X, x_0)$.
↑ starts from x_1 to x_0 back to x_1.

If $f \underset{p}{\simeq} f'$, then $\bar{h} * f * h \underset{p}{\simeq} \bar{h} * f' * h$, so it is well defined.

Claim: A_h is a group homomorphism. Indeed, given $[f], [g] \in \pi_1(X, x_0)$

$$\begin{aligned} A_h([f] * [g]) &= A_h([f * g]) \\ &= [\bar{h} * f * g * h] \\ &= [\bar{h} * f] * [g * h] \\ &= [\bar{h} * f] * \underbrace{[h * \bar{h}]}_{= [e_{x_0}]} * [g * h] \\ &= [\bar{h} * f * h] * [\bar{h} * g * h] \\ &= A_h([f]) * A_h([g]). \end{aligned}$$

Claim. A_h and $A_{\bar{h}}$ are inverses. Indeed: $A_{\bar{h}}: \pi_1(X, x_1) \to \pi_1(X, x_0)$

$$\begin{aligned} A_{\bar{h}}(A_h([f])) &= [h * A_h([f]) * \bar{h}] \\ &= [h * [\bar{h} * f * h] * \bar{h}] \\ &= [h * [\bar{h} * f] * [h] * \bar{h}] \\ &= [h] * [\bar{h} * f] * [h] * [\bar{h}] \\ &= [h * \bar{h}] * [f] * [h * \bar{h}] \\ &= [f]. \end{aligned}$$

$A_{\bar{h}}$ is also a group homomorphism.
Then, $A_{\bar{h}} \circ A_h = id_{\pi_1(X, x_0)}$, $A_h \circ A_{\bar{h}} = id_{\pi_1(X, x_1)}$.
$\pi_1(X, x_0) \cong \pi_1(X, x_1)$.

$\square$

Given X, x_0, $\pi_1(X, x_0)$ is a group of homotopy classes of loops based at x_0.
This is a topological variant: if $k: X \to Y$ is a homeomorphism, then $\pi_1(X, x_0)$ and $\pi_1(Y, k(x_0))$ are isomorphic groups.
More generally, any continuous map $k: X \to Y$ induces a homomorphism $k_*: \pi_1(X, x_0) \to \pi_1(Y, y_0)$ where $y_0 = k(x_0)$.
If x_0 and x_1 are in the same path component of X, then $\pi_1(X, x_0) \cong \pi_1(X, x_1)$. The identification may depend on the choice of path.
If X is path connected, then this is true $\forall x_0, x_1$.

Def. (Simply Connected). X is simply connected if ① X is path connected. ② $\pi_1(X, x_0) = \{[e_{x_0}]\}$.

e.g. $\mathbb{R}^n$ is simply connected.

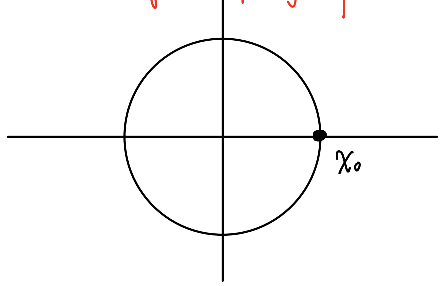
Def. (Convex) A subset $A \subset \mathbb{R}^n$ is convex if $\forall \vec{x}, \vec{y} \in A$, the line segment from $\vec{x}$ to $\vec{y}$ is in A. $\{(1-t)\vec{x} + t\vec{y} \mid t \in [0,1]\} \subset A$.

Proposition: Any convex subset of $\mathbb{R}^n$ is simply connected.
proof: Let $A \subset \mathbb{R}^n$ be convex. Any loop $f: [0,1] \to A$ with $f(0) = f(1) = \vec{x}_0$ is path homotopic to a constant map e_{x_0} by $F: [0,1] \times [0,1] \to A$

$$F(s,t) = (1-t) f(s) + t\vec{x}_0$$

by convexity, $\{F(s,t) \mid s \in [0,1],\ t \in [0,1]\} \subset A$.
Therefore $\pi_1(X, x_0) = \{[e_{x_0}]\}$. □.

Consider $S^1 \subset \mathbb{R}^2$, $x_0 = (1, 0)$. The vigorous proof of this part is the next.



Theorem. $\pi_1(S^1, x_0) \cong (\mathbb{Z}, +)$.
? Recall: let $p: \mathbb{R} \to S^1$ be $p(t) = (\cos(2\pi t), \sin(2\pi t))$. Then $p^{-1}(x_0) = \mathbb{Z}$.

For any continuous path $f: [0,1] \to S^1$, and any choice of t_0 such that $p(t_0) = f(0)$,
Claim: $\exists$ a unique continuous function $\tilde{f}: [0,1] \to \mathbb{R}$ s.t. $p \circ \tilde{f} = f$, $\tilde{f}(0) = t_0$.

If f and g are paths in S^1 with $f(0) = g(0)$, $f(1) = g(1)$, and f & g are path homotopic, then $\tilde{f}$ and $\tilde{g}$ are also path homotopic.

Given a path homotopy $F: [0,1] \times [0,1] \to S^1$. we can find a path homotopy $\tilde{F}: [0,1] \times [0,1] \to \mathbb{R}$ s.t. $p \circ \tilde{F} = F$.

If f is a loop based at $x_0 = (1, 0)$, take $t_0 \in \mathbb{Z}$, $t_0 = p^{-1}(x_0)$

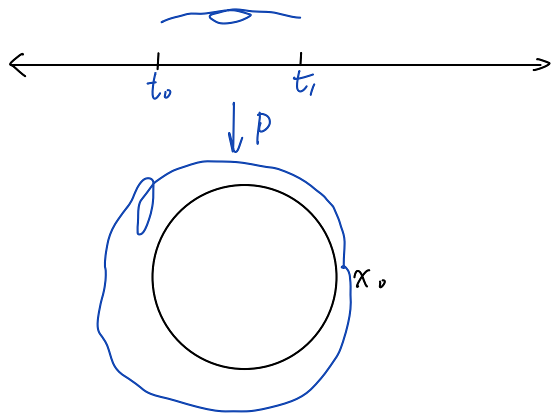
Take path $\tilde{f}: [0,1] \to \mathbb{R}$ such that $t_1 = \tilde{f}(1) \in p^{-1}(x_0) = \mathbb{Z}$.

Then, define the winding number of f to be $t_1 - t_0 = w(f)$

$w(f) = t_1 - t_0$ (independent of the choice of t_0)

$= \frac{1}{2\pi}$ (net change of θ)

$=$ ending point of $\tilde{f}$ − initial point of $\tilde{f}$.



If $f \underset{p}{\simeq} g$, then $\tilde{f} \underset{p}{\simeq} \tilde{g}$, so $\tilde{f} \underset{p}{\simeq} \tilde{g}$ has the same initial and ending points.
so $w(f) = w(g)$.
So, winding number depends only on the element of $\pi_1(S^1, x_0)$.

Therefore, $w : \pi_1(S^1, x_0) \to \mathbb{Z}$.

Claim. (a) $w(f * g) = w(f) + w(g)$
(b) $w(\bar{f}) = -w(f)$.

$\Rightarrow$ w is a group homomorphism.

Claim: If $w(f) = w(g)$, then $f \underset{p}{\simeq} g$.
proof: take $\tilde{f}$ and $\tilde{g} : [0,1] \to \mathbb{R}$ with $p \circ \tilde{f} = f$, $p \circ \tilde{g} = g$, $\tilde{f}(0) = \tilde{g}(0) = t_0$
s.t. $p(t_0) = f(0)$
Hence $\tilde{f}(1) = \tilde{g}(1) = t_1$ since $w(f) = w(g)$.
Let $\tilde{F}(s,t) = (1-t)\tilde{f}(s) + t\tilde{g}(s)$, a path homotopy btw $\tilde{f}$ and $\tilde{g}$.
Define $F = p \circ \tilde{F}$, then $f \underset{p}{\simeq} g$. So, w is injective. □

Define a loop $f_n : [0,1] \to S^1$ by $f_n(t) = (\cos(2\pi n t), \sin(2\pi n t))$. then $w(f_n) = n$.

e.g. 2. $\mathbb{R}^2 - \{\vec{0}\}$, $x_0 = (1,0)$, the same argument works using $[0,+\infty) \times \mathbb{R}$ sim. conn.
$p : (0,+\infty) \times \mathbb{R} \to \mathbb{R}^2 - \{0\}$
$p(r,t) = (r\cos(2\pi t), r\sin(2\pi t))$ $\quad p^{-1}((1,0)) = \{1\} \times \{2k\}$
We could start by taking $f \simeq \frac{f}{\|f\|}$, then use the same argument

Fact: $\mathbb{R}^n - \{0\} \cong S^{n-1} \times (0,+\infty)$ because $\vec{x} \mapsto \left(\frac{\vec{x}}{\|\vec{x}\|}, \|\vec{x}\|\right)$

Theorem Let X, Y be spaces. $x_0 \in X$, $y_0 \in Y$. Then $\pi_1(X \times Y, x_0 \times y_0) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0)$
Then: $[f \times g] \mapsto [f] \times [g]$

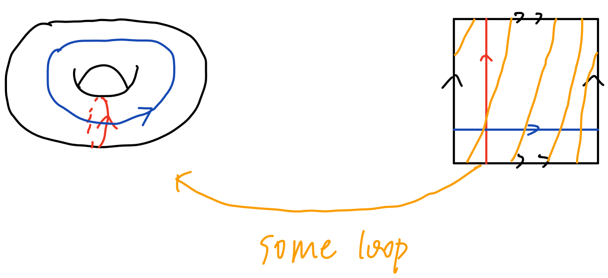
Def (Product of Group). $(G, *)$. $(H, \cdot)$. On $G \times H$, define $(g_1, h_1) \circledast (g_2, h_2) = (g_1 * g_2, h_1 \cdot h_2)$.

Def A loop $X \times Y$ based at $x_0 \times y_0$ is (f, g) where $f: [0,1] \to X$, $g: [0,1] \to Y$ are loops.

p_1, p_2: projections onto X and Y: $p_1: X \times Y \to X$, $p_2: X \times Y \to Y$.
$p_{1*}: \pi_1(X \times Y, x \times y) \to \pi_1(X, x)$
$[f] \times [g] \to [f]$

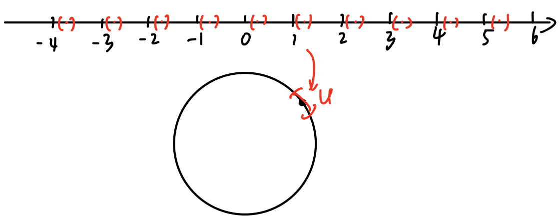
In the previous example, $\mathbb{R}^2 - \{0\} \cong S^1 \times (0, +\infty)$.
Then, $\pi'(\mathbb{R}^2 - \{0\}) \cong \underbrace{\pi_1(S^1)}_{\mathbb{Z}} \times \underbrace{\pi_1((0, +\infty))}_{\{[e]\}}$

Also, $\pi'(S^1 \times S^1) \cong \mathbb{Z} \times \mathbb{Z} = \mathbb{Z}^2$



Now we have
$\pi_1(S^1, x_0) \cong \mathbb{Z}$
$\pi_1(\mathbb{R}^2 - \{0\}) \cong \mathbb{Z}$
$\pi_1(X \times Y, x_0 \times y_0) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0)$.
$\pi_1(S^1 \times S^1) \cong \mathbb{Z} \times \mathbb{Z}$.
How to compute the fundamental groups?

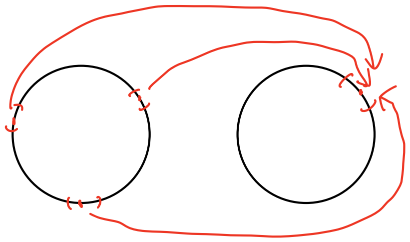
Def. (Covering Map). A function $p: Y \to X$ is called a covering map if
(a). p is surjective and continuous
(b). For every $x \in X$, there is an open set $U \subset X$ s.t. $x \in U$ and $p^{-1}(U) = \bigcup V_\alpha$ where $\{V_\alpha\}$ are open and disjoint, and $p|_{V_\alpha}: V_\alpha \to U$ is a homeomorphism. (bijective; $p|_{V_\alpha}$ continuous; $p|_{V_\alpha}^{-1}$ continuous).
Y is called the covering space of X.
e.g. $p: \mathbb{R} \to S^1$, $p(t) = (\cos(2\pi t), \sin(2\pi t))$



$p^{-1}(U)$ = union of disjoint open intervals, each homeomorphic to U.

e.g. $Y = X \times Z$. Z = any set with discrete topology.
$p: Y \to X$: $p(x,z) = x$ is a covering map. $x \in U$, $p^{-1}(U)$ open, $p^{-1}(U) = \bigcup_{z \in Z} \{U \times z\}$

e.g. Let $k \in \mathbb{Z} - \{0\}$. $f_k : S^1 \to S^1$ $f_k(\cos\theta, \sin\theta) = (\cos(k\theta), \sin(k\theta))$
for $k = 3$.



<u>Def</u>. (Lift). If $p: Y \to X$ is a covering map, $f: Z \to X$ is continuous, a lift of f is a continuous function $\tilde{f}: Z \to Y$ such that $p \circ \tilde{f} = f$.

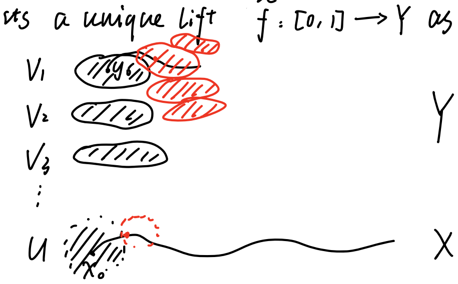
$$\begin{tikzcd} & Y \arrow[d, "p"] \\ Z \arrow[ur, dashed, "\tilde{f}"] \arrow[r, "f"'] & X \end{tikzcd}$$

e.g.

$$\begin{tikzcd} & \mathbb{R} \arrow[d, "p"] \\ {[a,b]} \arrow[ur, dashed, "\tilde{f}"] \arrow[r, "f"'] & S^1 \end{tikzcd}$$

$p(t) = (\cos(2\pi t), \sin(2\pi t))$.

<u>Lemma</u> Suppose $p: Y \to X$ is a covering map.
(a). Given any path $f: [0,1] \to X$ with $f(0) = x_0$, and given a choice of $y_0 \in p^{-1}(x_0)$, there exists a unique lift $\tilde{f}: [0,1] \to Y$ as a path in Y with $\tilde{f}(0) = y_0$ and $p \circ \tilde{f} = f$.



By compactness of $[0,1]$, this terminates Y.

$$\begin{tikzcd} & Y \arrow[d, "p"] \\ {[0,1]} \arrow[ur, dashed, "\tilde{f}"] \arrow[r, "f"'] & X \end{tikzcd}$$

(b). If f and g are paths with $f(0) = g(0)$, $f(1) = g(1)$, $f \simeq_p g$, also $\tilde{f}$ and $\tilde{g}$ are lifts starting at the same y_0, then $\tilde{f} \simeq_p \tilde{g}$. In particular, $\tilde{f}(1) = \tilde{g}(1)$.

proof: (a). Cover X by open sets U, each of which is evenly covered by p. Find a subdivision of $[0,1]$, say $s_0, \cdots, s_i$, such that for each i the set $f([s_i, s_{i+1}]) \subset U$ (use the Lebesgue number lemma). We define $\tilde{f}$ step by step.
First, define $\tilde{f}(0) = y_0$. Then, supposing $\tilde{f}(s)$ is defined for $0 \le s \le s_i$, we define $\tilde{f}$ on $[s_i, s_{i+1}]$ as follows. The set $f([s_i, s_{i+1}])$ lies in some open set U that is covered by p evenly. Let $\{V_\alpha\}$ be a partition of $p^{-1}(U)$ into slices; each V_α is matched homeomorphically onto U by p. Now $\tilde{f}(s_i)$ lies in one of these sets, say in V_0. Define $\tilde{f}(s)$ for $s \in [s_i, s_{i+1}]$ by

$$\tilde{f}(s) = (p|_{V_0})^{-1}(f(s))$$

Because $p|_{V_0}: V_0 \to U$ is a homeomorphism, $\tilde{f}$ will be continuous on $[s_i, s_{i+1}]$. Continuing this way, we define $\tilde{f}$ on all of $[0,1]$. Continuity of $\tilde{f}$ follows from the pasting lemma. $p \circ \tilde{f} = f$ follows from the definition of $\tilde{f}$.

The uniqueness of $\tilde{f}$ is also proven step by step. Suppose that $\tilde{\tilde{f}}$ is another lift of f beginning at y_0. Then $\tilde{\tilde{f}}(0) = y_0 = \tilde{f}(0)$. Suppose that $\tilde{\tilde{f}}(s) = \tilde{f}(s)$ for all $s \in [0, s_i]$. Let V_0 be as in the previous paragraph. Then for $s \in [s_i, s_{i+1}]$, $\tilde{f}(s)$ is defined as $(p|_{V_0})^{-1}(f(s))$. What can $\tilde{\tilde{f}}(s)$ equal? Since $\tilde{\tilde{f}}$ is a lift of f, it must carry the interval $[s_i, s_{i+1}]$ into the set $p^{-1}(U) = \bigcup V_\alpha$. The slices $\{V_\alpha\}$ are open and disjoint. Because $\tilde{\tilde{f}}([s_i, s_{i+1}])$ is connected, it must lie entirely in one of the sets V_α. Because $\tilde{\tilde{f}}(s_i) = \tilde{f}(s_i)$, which is in V_0, $\tilde{\tilde{f}}$ must carry all of $[s_i, s_{i+1}]$ into the set V_0. Thus, for $s \in [s_i, s_{i+1}]$, $\tilde{\tilde{f}}(s)$ must equal some point y of V_0 lying in $p^{-1}(f(s))$. But there is only one such point y, namely, $(p|_{V_0})^{-1}(f(s))$. Hence

$$\tilde{\tilde{f}}(s) = \tilde{f}(s) \quad \text{for } s \in [s_i, s_{i+1}].$$

1b). First we prove the following lemma.

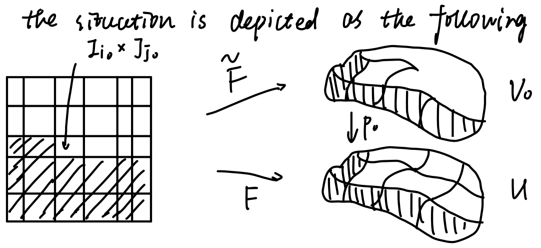
$$\begin{tikzcd} & Y \arrow[d, "p"] \\ {[0,1]} \arrow[ur, dashed, "\tilde{f}"] \arrow[r, "f"'] & X \end{tikzcd}$$

Same, choose $y_0 \in p^{-1}(x_0)$. Let the map $F: I \times I \to X$ be continuous, with $F(0,0) = x_0$. There is a unique lift of F to a continuous map $\tilde{F}: I \times I \to Y$ such that $\tilde{F}(0,0) = y_0$. If F is a path homotopy, then $\tilde{F}$ is a path homotopy.

Indeed. given F, we first define $\tilde{F}(0,0) = y_0$. Next, we use 1a) to extend $\tilde{F}$ to $0 \times I$ and $I \times 0$, then we extend $\tilde{F}$ to all of $I \times I$ as follows:

Choose subdivisions
$$s_0 < s_1 < \cdots < s_m$$
$$t_0 < t_1 < \cdots < t_n$$
of I fine enough that each rectangle $I_i \times J_j = [s_{i-1}, s_i] \times [t_{j-1}, t_j]$ is mapped by F into an open set of X that is evenly covered by p (Lebesgue number lemma). We define the lift $\tilde{F}$ step by step, beginning with the rectangle $I_1 \times J_1$, continuing with the other rectangles $I_i \times J_1$ in the "bottom row", then with the rectangles $I_i \times J_2$ in the next row, and so on.

In general, given i_0 and j_0, assume that $\tilde{F}$ is defined on the set A which is the union of $0 \times I$ and $I \times 0$ and all the rectangles "previous" to $I_{i_0} \times J_{j_0}$ ($I_i \times J_j$ for which $j < j_0$ and for which $j = j_0$ and $i < i_0$). Assume also that $\tilde{F}$ is a continuous lift of $F|_A$. We define $\tilde{F}$ on $I_{i_0} \times J_{j_0}$. Choose an open set U of X that is evenly covered by p and contains the set $F(I_{i_0} \times J_{j_0})$. Let $\{V_\alpha\}$ be a partition of $p^{-1}(U)$ into slices; each of V_α is mapped homeomorphically onto U by p. Now $\tilde{F}$ is already defined on the set $C = A \cap (I_{i_0} \times J_{j_0})$. This set is the union of the left and the bottom edges of the rectangle $I_{i_0} \times J_{j_0}$, so it is connected. Therefore, $\tilde{F}(C)$ is connected and must lie entirely within one of the sets V_α. Suppose it lies in V_0, then the situation is depicted as the following graph.



Let $p_0 : V_0 \to U$, $P_0 = P|_{V_0}$. Since $\tilde{F}$ is a lift of $F|_A$, we know that for $x \in C$,

$$p_0(\tilde{F}(x)) = p(\tilde{F}(x)) = F(x).$$

so that $\tilde{F}(x) = p_0^{-1}(F(x))$. Hence, we may extend $\tilde{F}$ by defining

$$\tilde{F}(x) = p_0^{-1}(F(x))$$

for $x \in I_{i_0} \times J_{j_0}$. By the pasting lemma, it is continuous. Continuing this way, we define $\tilde{F}$ on all of I^2.

To check uniqueness, note that at each step of the construction of $\tilde{F}$, as we extend $\tilde{F}$ first to the bottom and left edges of I^2, and then to the rectangles $I_i \times J_j$, one by one, there is only one way to extend $\tilde{F}$ continuously. Thus, once the value of $\tilde{F}$ at $(0,0)$ is specified, $\tilde{F}$ is completely determined.

Now suppose F is a path homotopy. WTS $\tilde{F}$ is also a path homotopy.

The map F carries $0 \times I$ into a single point x_0 of X. Because $\tilde{F}$ is a lift of F, it carries $0 \times I$ to $p^{-1}(x_0)$. But this set has discrete topology as a subspace of Y. Since $0 \times I$ is connected and $\tilde{F}$ is continuous, $\tilde{F}(0 \times I)$ is connected and thus must equal a one-point set. Similarly, $\tilde{F}(1 \times I)$ must be a one-point set.

Thus $\tilde{F}$ is a path homotopy.

Now we prove (b).

Let $F : I \times I \to X$ be a path homotopy btw f and g. Then $F(0,0) = x_0$.

Let $\tilde{F} : I \times I \to Y$ be the lift of F to Y such that $\tilde{F}(0,0) = y_0$. By the above lemma $\tilde{F}$ is a homotopy, so that $\tilde{F}(0 \times I) = \{y_0\}$, $\tilde{F}(1 \times I) = \{y_1\}$ (one-point)

The restriction $\tilde{F}|_{I \times 0}$ of $\tilde{F}$ to the bottom edge of $I \times I$ is a path on Y beginning at y_0 that is a lift of $F|_{I \times 0}$. By uniqueness of path liftings, we must have $\tilde{F}(s,0) = \tilde{f}(s)$. Similarly, $\tilde{F}|_{I \times 1}$ is a path on Y that is a lifting of $F|_{I \times 1}$, and it begins at y_0 because $\tilde{F}(0 \times I) = \{y_0\}$. By uniqueness of path liftings, $\tilde{F}(s,1) = \tilde{g}(s)$. Therefore, both $\tilde{f}$ and $\tilde{g}$ ends at y_1, $\tilde{F}$ is a path homotopy btw them. □

<u>Def</u> (Lift correspondence). Let $p : Y \to X$ be a covering map. Let $x_0 \in X$. Choose $y_0 \in p^{-1}(x_0)$. Given an element $[f] \in \pi_1(X, x_0)$. Let $\tilde{f}$ be the lift of f to a path in Y that begins at y_0. Let $L([f])$ denote the endpoint $\tilde{f}(1)$ of $\tilde{f}$. Then $L : \pi_1(X, x_0) \to p^{-1}(x_0)$ is a well-defined set map, depending on the choice of y_0, derived from the covering map p.

<u>Theorem</u>. If $p : Y \to X$ is a covering map, and Y is simply connected, then $\pi_1(X, x_0)$ is in bijection with $p^{-1}(x_0)$ by $L : \pi_1(X, x_0) \to p^{-1}(x_0)$. $L([f]) = \tilde{f}(1)$.

proof: Fix $y_0 \in p^{-1}(x_0)$. Given a loop $f : [0,1] \to X$, $f(0) = f(1) = x_0$.

Take a path (lift) $\tilde{f} : [0,1] \to Y$ with $\tilde{f}(0) = y_0$, $p(\tilde{f}(1)) = f(1) = x_0 \Rightarrow \tilde{f}(1) \in p^{-1}(x_0)$.

By (b) of the previous lemma, this only depends on $[f] \in \pi_1(X, x_0)$

Now we see why $L : \pi_1(X, x_0) \to p^{-1}(x_0)$ is a bijection.

Surjective: For every $y \in p^{-1}(x_0)$, since Y is simply connected, it is path connected. So choose path $g : y_0 \rightsquigarrow y$. Let $f = p \circ g$, then $L([f]) \overset{\text{def}}{=} g(1) = y$. g is a lift of f

① f continuous. ② $(p \circ g)(0) = p(g(0)) = p(y_0) = x_0$. $(p \circ g)(1) = p(g(1)) = p(y) = x_0$. $\Rightarrow f$ a loop.

Injective: If $L([f]) = L([g])$, then $\tilde{f}$ and $\tilde{g}$ are both paths in Y: $y_0 \rightsquigarrow y := \tilde{f}(1) = \tilde{g}(1)$.
Because Y is simply connected, $\tilde{f} \underset{p}{\simeq} \tilde{g}$ by path homo. $\tilde{F}$ $\Rightarrow$ $f \underset{p}{\simeq} g$ by $p \circ \tilde{F}$, so $[f] = [g]$

□

Theorem. $\pi_1(S^1, x_0) \cong (\mathbb{Z}, +)$.

proof. Let $p: \mathbb{R} \to S^1$: $p(t) = (\cos(2\pi t), \sin(2\pi t))$.
Let $t_0 = 0$. let $x_0 = p(t_0) = (1, 0)$. then $t_0 \in p^{-1}(x_0) = \mathbb{Z}$.
Since $\mathbb{R}$ is simply connected, the lift correspondence $L: \pi_1(S^1, x_0) \to \mathbb{Z}$ is bijective.
WTS L is a homomorphism.
Given $[f], [g] \in \pi_1(S^1, x_0)$. Let $\tilde{f}, \tilde{g}$ be their lifts to paths on $\mathbb{R}$ beginning at 0
let $n = \tilde{f}(1)$, $m = \tilde{g}(1)$, then $L([f]) = n$, $L([g]) = m$. Let $\tilde{\tilde{g}}$ be the path $\tilde{\tilde{g}}(s) = n + \tilde{g}(s)$ on $\mathbb{R}$.
Because $(p \circ \tilde{\tilde{g}})(s) = p(n + \tilde{g}(s)) = p(\tilde{g}(s)) = p \circ \tilde{g}(s) = g(s)$, the path $\tilde{\tilde{g}}$ is the lift of g, with $\tilde{\tilde{g}}(0) = n$; it begins at n. Then the product $\tilde{f} * \tilde{\tilde{g}}$ is defined

$$\tilde{f} * \tilde{\tilde{g}}(s) = \begin{cases} \tilde{f}(2s) & s \in [0, \frac{1}{2}] \quad 0 \rightsquigarrow n \\ \tilde{\tilde{g}}(2s-1) & s \in [\frac{1}{2}, 1] \quad n \rightsquigarrow n+m \end{cases}$$

and it is the lifting of $f * g$ that begins at 0, since

$$p \circ (\tilde{f} * \tilde{\tilde{g}}) = (p \circ \tilde{f}) * (p \circ \tilde{\tilde{g}}) = f * g.$$

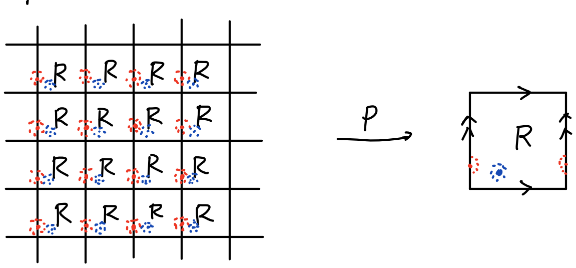
The endpoint of this path is $\tilde{\tilde{g}}(1) = n + m$. Then, by definition,

$$\begin{aligned} L([f] * [g]) = L([f * g]) &= \text{endpoint of } \tilde{f} * \tilde{\tilde{g}} \\ &= n + m \\ &= L([f]) + L([g]). \end{aligned}$$

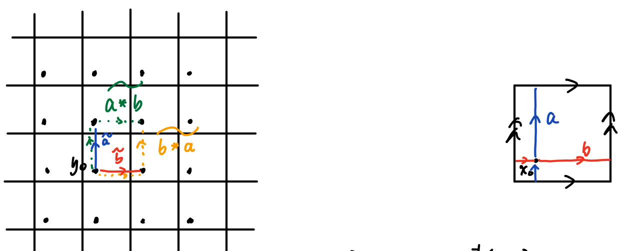
□

e.g. $p: \mathbb{R}^2 \to S^1 \times S^1 = T^2$
$(s, t) \mapsto (\cos(2\pi s), \sin(2\pi s)) \times (\cos(2\pi t), \sin(2\pi t))$
$T^2 = [0,1]^2/\sim$, $(x, 0) \sim (x, 1)$ $\forall x$, $(0, y) \sim (1, y)$, $\forall y$.

$Y = \mathbb{R}^2$ $\qquad\qquad\qquad\qquad$ $X = T^2$



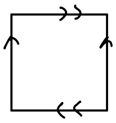
send every square to a square.
$p(x, y) = [(x \bmod 1, y \bmod 1)]$ is a covering map

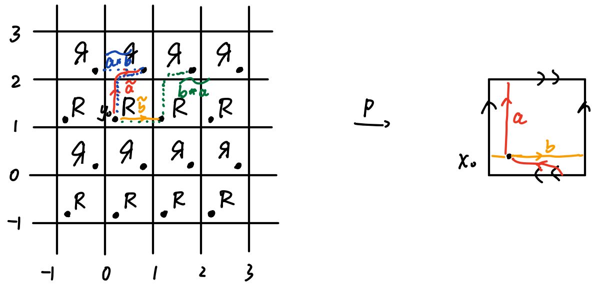


$\pi_1(T^2, x_0) \cong p^{-1}(x_0)$ $\qquad$ $[a * b] = [b * a] \Rightarrow [a] * [b] = [b] * [a]$
$\mathbb{Z}^2$

e.g. Klein bottle: $[0,1]^2/\sim$

$(0,y)\sim(1,y)\ \forall y,\ (x,0)\sim(1-x,1)\ \forall x.$





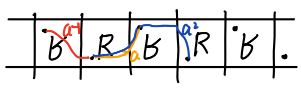
In $\pi_1(K,x_0)$, $[a*b]\neq[b*a]$, so it is not abelian.

$[a]*[b]\neq[b]*[a]$

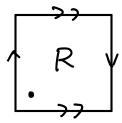
But, $[a]*[b]=[b]*[\bar a]$ since $\bar a*b=b*a$.

Any element of $\pi_1(K^2,x_0)$ can be written (uniquely) as a^ib^j, when $ab=ba^{-1}$

e.g. Möbius strip



$\mathbb{R}\times[0,1]$



$\pi_1(Mob)\cong\mathbb{Z}$

We should figure out a simply connected cover to do the calculation.

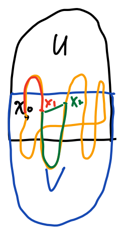
We saw that $f_n:S^1\to S^1$, $(\cos\theta,\sin\theta)\mapsto(\cos n\theta,\sin n\theta)$ is an n-to-1 covering map.

"Galois correspondence": If X is path connected, then $\{$path connected covering spaces of $X\}\leftrightarrow$ subgroups of $\pi_1(X,x_0)$.

<u>Theorem.</u> If $X=U\cup V$, both open, each is simply connected. $U\cap V$ is non-empty and path-connected. then X is simply connected.

More generally, if only assume $X=U\cup V$ both open, $U\cap V$ path connected, then $\exists$ a way to determine $\pi_1(X,x_0)$ from $\pi_1(U,x_0)$, $\pi_1(V,x_0)$, and $\pi_1(U\cup V,x_0)$. (Seifer-von Kumpen Theorem).

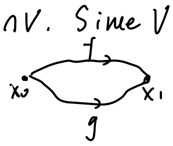
proof: Take $x_0\in U\cap V$.



Take a loop f based at x_0. By compactness of $[0,1]$, we can write $f=f_1*\cdots*f_n$ where each f_i is a path in U or in V from x_{i-1} to x_i.

If $f_i\subset V$, then choose some $g_i: x_{i-1}\rightsquigarrow x_i$ in $U\cap V$. Since V is simply connected, $f_i\underset{p}{\simeq}g_i$.

(Fact: if X is simply connected, then given



. $f\underset{p}{\simeq}f*e_{x_1}\underset{p}{\simeq}f*(\bar g*g)$

$\underset{p}{\simeq}(f*\bar g)*g$

$\underset{p}{\simeq}e_{x_0}*g$

$\underset{p}{\simeq}g$.

Then $f\underset{p}{\simeq}$ a loop in U.

Since U is simply connected, $f\simeq e$. Then X is simply connected.

Theorem. For $n \geq 3$, $\mathbb{R}^n - \{\text{point}\}$ is simply connected.

proof: Take the point $\vec{0}$.

Let $U = \mathbb{R}^n - \{\text{non-negative } z\text{-axis}\}$

$= \mathbb{R}^n - \{(0, \cdots, 0, z) \mid z \geq 0\}$. open

$V = \mathbb{R}^n - \{\text{non-positive } z\text{-axis}\}$

$= \mathbb{R}^n - \{(0, \cdots, 0, z) \mid z \leq 0\}$. open

U is simply connected since it is star-convex with center $\vec{y} = (0, \cdots, 0, -1)$, any loop $f \underset{p}{\simeq} e_{y_0}$.

Similar for V, also simply connected $((0, \cdots, 0, 1))$.

$U \cap V = \mathbb{R}^n - \{(0, \cdots, 0, z) \mid z \in \mathbb{R}\} = \underbrace{(\mathbb{R}^{n-1} - \{0\})}_{\text{path connected for } n-1 \geq 2} \times \mathbb{R}$ is path connected

$U \cup V = \mathbb{R}^n - \{\vec{0}\}$.

Therefore $\mathbb{R}^n - \{\text{point}\}$ is simply connected.

Theorem $\mathbb{R}^2 \not\cong \mathbb{R}^n$ for $n > 2$.

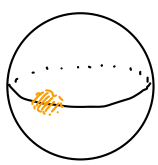
proof: If $\exists$ a homeomorphism $f: \mathbb{R}^2 \to \mathbb{R}^n$, then $\mathbb{R}^2 - \{0\} \cong \mathbb{R}^n - \{\overset{\text{one point}}{f(0)}\}$. But $\mathbb{R}^2 - \{0\}$ is not simply connected. $\mathbb{R}^n - \{f(0)\}$ is. Contradiction! $\square$.

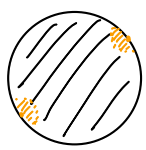
$\mathbb{R}^n - \{0\} \cong S^{n-1} \times \mathbb{R}_+$ by homeomorphism $\vec{x} \longmapsto \left(\frac{\vec{x}}{\|x\|}, \|x\|\right)$. $\pi_1(\mathbb{R}_+) = \{[e]\}$

Then S^n is simply connected for $n \geq 2$

$p: S^n \to \mathbb{R}P^n$

S^2

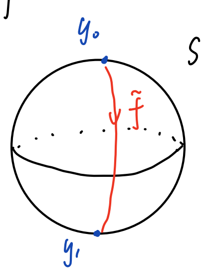




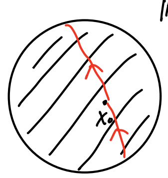
$x \sim -x$ for $x \in S^n$

This is a 2-to-1 covering map. Since S^n is simply connected, for $n \geq 2$, $\pi_1(\mathbb{R}P^n)$ is a group of two elements e, a, where $a^2 = e$. $\cong \mathbb{Z}/\sim$

S^2



$\mathbb{R}P^2$



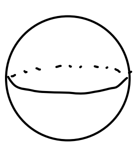
Look at this loop that does not end at y_0 in the covering space. This is the non-trivial element of $\pi_1(\mathbb{R}P^n)$.

Corollary. $S^1 \cong \mathbb{R}P^1$, $S^n \not\cong \mathbb{R}P^n$ for $n \geq 2$.

proof: for $n \geq 2$, $\pi_1(S^n)$ is trivial, $\pi_1(\mathbb{R}P^n)$ is non trivial.

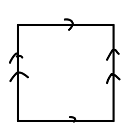
We know for closed surfaces:

S^2



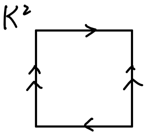
$\pi_1(S^2) = \{[e]\}$

T^2



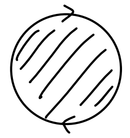
$\pi_1(T^2) \cong \mathbb{Z} \times \mathbb{Z}$ infinite group

K^2



$\pi_1(K^2)$ non abelian infinite group.

$\mathbb{R}P^2$



$\pi_1(\mathbb{R}P^2)$ has 2 elements.

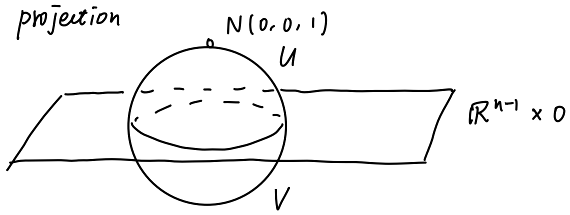
....

π_1 completely classifies compact surfaces.

For $n \geq 2$, S^n is simply connected.

<u>Proposition:</u> $S^n - \{point\} \cong \mathbb{R}^n$.

proof: by Stereographic projection



For any point $\vec{x} \in S^n - \{N\} \subset \mathbb{R}^{n+1}$, $\exists$ unique line L_x through $\vec{x}$ and N

Define $f(x) = L_x \cap (\mathbb{R}^{n-1} \times 0) \in \mathbb{R}^n$ $\quad f(x) = \frac{1}{1-x_{n+1}}(x_1, \cdots, x_n)$.

f is a homeomorphism $S^n - \{N\} \to \mathbb{R}^n$

$\square$

<u>Def</u> (Retraction). Given $A \subset X$. A retraction is a continuous function $r: X \to A$ such that $r|_A = id_A$.

A is the retract of X.

e.g. $r: \mathbb{R} \to \{x_0\}$. $r(x) = x_0 \ \forall x \in \mathbb{R}$.

$r: \mathbb{R}^n \to S^{n-1}$. $r(x) = \frac{x}{\|x\|} \quad \forall x \in \mathbb{R}^n$

e.g. $X = [a,b]$. $A = \{a,b\}$

There is no retraction $f: X \to A$.

proof: suppose there were. Then f is continuous. Since X is connected, $Im(f) \subset A$ is also connected. $Im(f) \subset \{a,b\}$.

$f(a) = f|_A(a) = id_A(a) = a$

$f(b) = f|_A(b) = id_A(b) = b$

Therefore $Im(f) = \{a,b\}$, not connected. contradiction!

<u>Lemma.</u> Suppose $A \subset X$. Choose $x_0 \in A$. Suppose $r: X \to A$ is a retraction, then the induced map $r_*: \pi_1(X, x_0) \to \pi_1(A, x_0)$ is surjective.

proof: $r: X \to A$. $r|_A = id_A$. Let $i: A \to X$ be inclusion.

then $r \circ i = id_A$.

$$\pi_1(A, x_0) \xrightarrow{i_*} \pi_1(X, x_0) \xrightarrow{r_*} \pi_1(A, x_0)$$

then $r_* \circ i_* = id_{\pi_1(A, x_0)}$

For any $[f] \in \pi_1(A, x_0)$, $[f] = r_*(i_*([f]))$

so $[f] \in Im(r_*)$. $\exists [g] \in \pi_1(X, x_0)$ s.t. $r_*([g]) = [f]$

$\square$

<u>Proposition</u> Also, i_* is injective, since $r \circ i = id_A$, $r_* \circ i_* = id_{\pi_1(A, x_0)}$

if $[f] \neq [g]$ in $\pi_1(A, x_0)$ but $i_*([f]) = i_*([g])$ in $\pi_1(X, x_0)$, then

$r_*(i_*([f])) = r_*(i_*[g])$, $id_{\pi_1(A, x_0)}([f]) = id_{\pi_1(A, x_0)}([g])$

$\Rightarrow [f] = [g]$. Contradiction. $\square$

Corollary: if X is simply connected, A is not, then there is no retraction.

proof: If there were retraction $r: X \to A$, then the induced map $r_*: \pi_1(X, x_0) \to \pi_1(A, x_0)$ is surjective. But $\pi_1(X, x_0)$ is trivial, $\pi_1(A, x_0)$ not trivial. So contradiction. $\square$

Denote $D^2 = \{\vec{x} \in \mathbb{R}^2 \mid \|x\| \le 1\}$ $S^1 = \{\vec{x} \in \mathbb{R}^2 \mid \|x\| = 1\}$.

Theorem: there is no retraction $f: D^2 \to S^1$.

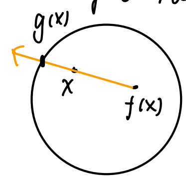
proof: D^2 is simply connected, S^1 is not. $\square$

Theorem (Brower fixed point theorem in $\mathbb{R}^2$) Any continuous function: $D^2 \to D^2$ has a fixed point, i.e., $\exists x \in D^2$, $f(x) = x$.

proof: non-constructive.

Suppose $f: D^2 \to D^2$ has no fixed points. $\forall x \in D^2$, $f(x) \neq x$.

Define $g: D^2 \to D^2$ by drawing a ray from $f(x)$ to x, it hits the boundary at $g(x)$.



Let $v = x - f(x)$. Want $\|f(x) + tv\| = 1$

$$\|f(x)\|^2 + 2t\, f(x)\cdot v + t^2\|v\|^2 - 1 = 0$$

$$\|v\|^2 t^2 + 2 f(x)\cdot v\, t + \underbrace{\|f(x)\|^2 - 1} = 0$$

$$t_* = \frac{-2f(x)\cdot v \pm \sqrt{4(f(x)\cdot v)^2 - 4\|v\|^2(\|f(x)\|^2 - 1)}}{2\|v\|^2}$$

$g(x) = f(x) + t_* v$, depending continuously on x and $f(x)$, thus continuous.

If $x \in S^1$, then $g(x) = x$. $g|_{S^1} = id_{S^1}$. g is a retraction $D^2 \to S^1$.

But no retraction exists from D^2 to S^1. Contradiction. $\square$

There are other topological invariant apart from π_1.

$\pi_1(X, x_0)$: homotopy classes of maps: $I \to X$, $0,1 \to x_0$.

it is also homotopy classes of maps: $S^1 \to X$, $s_0 \to x_0$.

Define $\pi_n(X, x_0)$ as the set of homotopy classes of maps: $S^n \to X$, $s_0 \to x_0$. s_0 is a fixed point in S^n.

"nth homotopy group of X" = set of homotopy classes of maps $I^n \to X$, $\partial I^n \to x_0$.

If $r: X \to A$ is a retraction, then $r_*: \pi_n(X, x_0) \to \pi_n(A, x_0)$ is surjective.

$\pi_i(D^n) = \{0\}$, $\pi_i(\mathbb{R}^n) = \{0\}$ $\forall i \ge 1$

Theorem. $\pi_n(S^n) \cong \mathbb{Z}$

Corollary: $\nexists$ no retraction $D^n \to S^{n-1}$

Corollary: Brower fixed point theorem: Any continuous $f: D^n \to D^n$ has a fixed point.