Consumer Theory

- A finite sets of goods:

  $1, 2, \cdots, n$

  $x^i = (x^i_1, x^i_2, \cdots, x^i_n)$

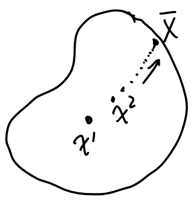
  $x^i \in X \subseteq \mathbb{R}^n$ i.e. $x^i_j \geq 0$

  consumption set

- Assumptions on Consumption Set X: must specify

  1). X is a closed set (including its limit points).

  Take a convergent sequence $x^1, x^2 \cdots x^n \cdots \in X$, $\lim_{n \to \infty} x^n = \bar{x}$, $\bar{x} \in X$.

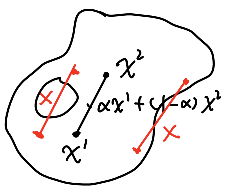


  2). X is a convex set (No gaps).

  $\forall\ x^1, x^2 \in X$ and $\alpha \in [0, 1]$

  $\alpha x^1 + (1-\alpha) x^2 \in X$

  $\sum \alpha_i x^i \quad \sum \alpha_i = 1, \alpha_i \geq 0$



  3). $\vec{0} \in X$ (voluntary consumption).

  We assume: $X = \mathbb{R}^n_+$.

- **Axiom of Choice:**

  Let $\succsim$ be a binary relation.

  $x^1 \succsim x^2$: x^1 is at least as good as x^2 (weakly preferred to)

  Axiom 1: Completedness.

  $\forall\ x^1, x^2 \in X$: $x^1 \succsim x^2$ or $x^2 \succsim x^1$

  Can always compare two bundles. Allow for indifference.

  Axiom 2: Transitivity.

  $\forall\ x^1, x^2, x^3 \in X$: if $x^1 \succsim x^2$, $x^2 \succsim x^3$, then $x^1 \succsim x^3$.

- Definition: Axiom 1 and Axiom 2 : $\succsim$ is a preference relation.
  Consumer can rank all bundles from the best to the worst.
  $x^1 \succsim x^2 \succsim x^3 \succsim \cdots\cdots$ no circles.

  Strict preference: $x^1 \succ x^2 \Leftrightarrow x^1 \succsim x^2$ and $x^2 \not\succsim x^1$
  Indifference: $x^1 \sim x^2 \Leftrightarrow x^1 \succsim x^2$ and $x^2 \succsim x^1$

- If X is a finite or countbly infinite consumption set, then there exists a scoring rule $s: X \to \mathbb{R}$ such that $x^i \succsim x^j \Leftrightarrow s(x^i) \geq s(x^j)$.

  completeness + transitivity = preference relation
  $\Downarrow$ (may or may not) $\quad x^1 \succsim x^2 \succsim \cdots \succsim x^n$
  Utility function $u: X \to \mathbb{R}$.
  if: $x^1 \succsim x^2 \Leftrightarrow u(x^1) \geq u(x^2)$, $\forall x^1, x^2 \in \mathbb{R}^n$.

  If X is uncountbly infinite, then we need Axiom 3 for the existence of such rule.

Axiom 3: Continuity.
  Better set: $B(x) = \{y \in X \mid y \succsim x\}$.
  Worse set: $W(x) = \{y \in X \mid y \precsim x\}$.
  Then, $\succsim$ is continuous if both sets $B(x)$ and $W(x)$ are closed.
  if $y^{(n)} \succsim x$, then $\lim_{n\to\infty} y^{(n)} \succsim x$. There is no sudden jump in preference.

- Definition: A utility function: a real-valued function $u: X \to \mathbb{R}$ represents preference if:
  $x^1 \succsim x^2 \Leftrightarrow u(x^1) \geq u(x^2)$.

- Theorem: Existence of Utility:
  A binary relation $\succsim$ on $X \in \mathbb{R}^n_+$ is complete, transitive and continuous
  iff ($\Leftrightarrow$)
  there is a continuous utility function that represents $\succsim$.
  Without Axiom 3, we cannot guarantee utility exists.

Counterexample: Lexicographic Preference: (Dictionary)

$$(x_1, x_2) \succsim (y_1, y_2) \text{ if } \begin{cases} x_1 > y_1 \\ x_1 = y_1 \text{ and } x_2 \geq y_2. \end{cases}$$
$\succ \qquad\qquad >$

i) completeness: ✓
  $x_1 > y_1$ and $x_2 \gtreqless y_2$: $(x_1, x_2) \succsim (y_1, y_2)$
  $x_1 = y_1$ and $x_2 \geq y_2$: $(x_1, x_2) \succsim (y_1, y_2)$
  $<$: $(y_1, y_2) \succsim (x_1, x_2)$
  $x_1 < y_1$ and $x_2 \gtreqless y_2$: $(y_1, y_2) \succsim (x_1, x_2)$.

ii): transitivity: ✓

$(x_1, x_2) \succsim (y_1, y_2)$ if $\begin{cases} x_1 > y_1 \\ x_1 = y_1 \text{ and } x_2 \geq y_2 \end{cases}$

$(y_1, y_2) \succsim (z_1, z_2)$ if $\begin{cases} y_1 > z_1 \\ y_1 = z_1 \text{ and } y_2 \geq z_2 \end{cases}$

if $x_1 > y_1$ and $y_1 \geq z_1$

then $x_1 > z_1$

$(x_1, x_2) \succsim (z_1, z_2)$

if $x_1 = y_1$ and $x_2 \geq y_2$:

if $y_1 > z_1$:

then $x_1 > z_1$

$(x_1, x_2) \succsim (z_1, z_2)$

if $y_1 = z_1$ and $y_2 \geq z_2$:

then $x_1 = z_1$ and $x_2 \geq z_2$

$(x_1, x_2) \succsim (z_1, z_2)$.

iii). not continuous.

Take $x^{(n)} = (\frac{1}{n}, 0)$, $y = (0, 1)$

$x^{(n)} \succsim y \quad \forall n$

$x^{(n)} \longrightarrow \bar{x} = (0, 0)$ as $n \to \infty$

$\bar{x} \precsim y$

$\therefore B(y)$ is not closed. Lexicographic preference is not continuous.

Show that it has no utility preference:

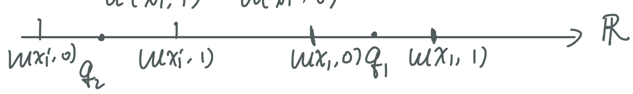
Assume that it has. $u: X \to \mathbb{R}$ trick is here!

Take $(x_1, 1), (x_1, 0), x_1 \in \mathbb{R}$

then $(x_1, 1) \succ (x_1, 0)$

$\Updownarrow$

$u(x_1, 1) > u(x_1, 0)$



☆ using the def of utility, from $\succsim$ to $u(\cdot) \geq u(\cdot)$. (strict). $u \in \mathbb{R}$.

Form some contradiction regarding $\mathbb{R}$!

We can find a rational number $q_1 \in (u(x_1, 0), u(x_1, 1))$

Take $(x_1', 1), (x_1', 0)$, $x_1' < x_1$

then: $(x_1', 0) \prec (x_1', 1) \prec (x_1, 0) \prec (x_1, 1)$

$\Updownarrow \qquad\qquad \Updownarrow$

$u(x_1', 0) < u(x_1', 0) < u(x_1, 0) < u(x_1, 1)$

we can find a rational number $q_2 \in (u(x_1, 0), u(x_2, 0))$.

Thus, we constructed a 1-1 relation from $\mathbb{R}$ to $\mathbb{Q}$

However: $\mathbb{Q}$ is countable, $\mathbb{R}$ is uncountable, there is no 1-1, between them

contradiction.

$\therefore u$ doesnot exists for Lexicographic preference.

- Result: If $u(x)$ represents some preference, then, so does $v(x) = h(u(x))$, where $h: \mathbb{R} \to \mathbb{R}$, strictly increasing.

  proof: $x_1 \succsim x_2 \Leftrightarrow u(x_1) \geq u(x_2)$

  $h(u(x_1)) \geq h(u(x_2))$

$V(x_1) \geq V(x_2) \iff x_1 \succsim x_2$.

$\rightarrow$ Job of utility function: order the bundle, not mark them!

To have a more trackable theory, we need more.

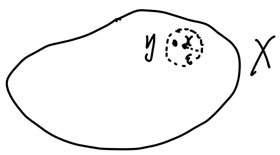
Axiom 4' (Local non-Satiation):

$\forall x \in X$ and $\forall \varepsilon > 0$, there exists $y \in X$, s.t.

$$\|y - x\| < \varepsilon$$

and $$y \succ x$$

It will ensure the use of one's income.
doesn't tell where y is.



Axiom 4. motonomicity (tell where y is).

If $x^1 \geq x^2$, then $x^1 \succsim x^2$ (more is better).

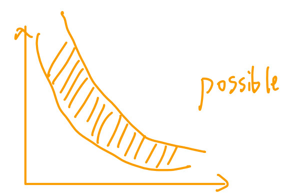
$u(x^1) \geq u(x^2)$

Notation: $x^1 \geq x^2$ if $x^1_i \geq x^2_i \ \forall i = 1, \cdots, n$

$x^1 > x^2$ if $x^1_i \geq x^2_i \ \forall i$ and $x^1_j > x^2_j$ for some j.

$x^1 \gg x^2$ if $x^1_i > x^2_i \ \forall i$.

It does not rule out indifference!



Axiom 5: Convexity.

If $x^1 \succsim x^2$, then: $tx^1 + (1-t)x^2 \succsim x^2$ for all $t \in [0,1]$

$u(x^1) \geq u(x^2), \Rightarrow u(tx^1 + (1-t)x^2) \geq \min\{u(x^1), u(x^2)\}$.

Strict convexity: $x^1 \succsim x^2$ and $x^1 \neq x^2 \Rightarrow tx^1 + (1-t)x^2 \succ x^2$ for all $t \in (0,1)$

$x^1 \sim x^2$ and $x^1 \neq x^2 \Rightarrow tx^1 + (1-t)x^2 \succ x^2 \sim x^1$

monotonicity $\Rightarrow$ increasing $u(x)$.

convexity $\Rightarrow$ quasi-concave $u(x)$.

Convex preference $\iff$ quasi-concave utility.

- **Definition: quasi-concave function.**

$f: X \to \mathbb{R}$ is quasi-concave if $\forall x^1, x^2 \in X$ and $t \in [0,1]$,

$$f(tx^1 + (1-t)x^2) \geq \min\{f(x^1), f(x^2)\}.$$

$\iff$ if $f(x^1) \geq f(x^2) \Rightarrow f(tx^1 + (1-t)x^2) \geq f(x^2)$.

strictly q-concave: $>$ for $x^1 \neq x^2$ $t \in (0,1)$.

- **Theorem** f is q-concave if and only if it's better-than set $\{x \in X \mid f(x) \geq k\}$ is convex for all $k \in \mathbb{R}$.

proof: $\Rightarrow$ suppose f is q-concave. Then, take $x^1, x^2 \in X$ such that $f(x^1) \geq k$, $f(x^2) \geq k$.

Then: $f(tx^1 + (1-t)x^2) \geq \min\{f(x^1), f(x^2)\} \geq k$ definition.

This implies that $tx^1 + (1-t)x^2 \in \{x \in X \mid f(x) \geq k\}$

So the better than set is convex.

$x^1 \in B$, $x^2 \in B$ know.

convex comb. of $x^1, x^2 \in B$ want to show

$\Downarrow$

B convex

$\Leftarrow$: suppose that the set is convex. $\{x \in X \mid f(x) \geq k\}$. Take x^1, x^2 from this set. $f(x^1) \geq k$, $f(x^2) \geq k$

Then, $tx^1 + (1-t)x^2$ also is in it, which means $f(tx^1+(1-t)x^2) \geq k$.

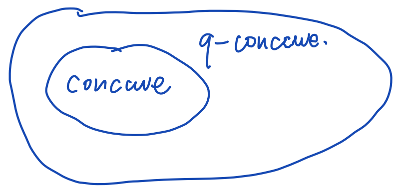
So, $f(tx^1+(1-t)x^2) \geq k = \min\{f(x^1), f(x^2)\}$. q.e.d.

Concave function:

$f: X \to \mathbb{R}$ is concave if $\forall x^1, x^2$ and $t \in [0,1]$:

$f(tx^1+(1-t)x^2) \geq tf(x^1) + (1-t)f(x^2) \geq \min\{f(x_1), f(x_2)\}$

Therefore:



- Proposition: $f: X \to \mathbb{R}$ is q-concave and $h: \mathbb{R} \to \mathbb{R}$ is a strictly increasing function, then $h[f(x)]$ is q-concave.

- Relationships:

Completeness + Transitivity $\Rightarrow$ can rank bundle

+ continuity $\Rightarrow$ there is an (continuous) utility

{ monotonicity $\Rightarrow$ u is increasing

or

convexity $\Rightarrow$ utility is q-concave.

- **The value of one more unit of good**

- Marginal utility: $MU_i = \frac{\partial u}{\partial x_i}$: unreliable. e.g. $V = 1000u$, V & u represent the same preference

$$\frac{\partial V}{\partial x_i} = 1000 \frac{\partial u}{\partial x_i}$$

- Marginal Rate of Substitution: MRS

How much a consumer is willing to trade good i with good j to stay at the same utility level.

$u(x_1, x_2, \cdots, x_n) = u_0$.

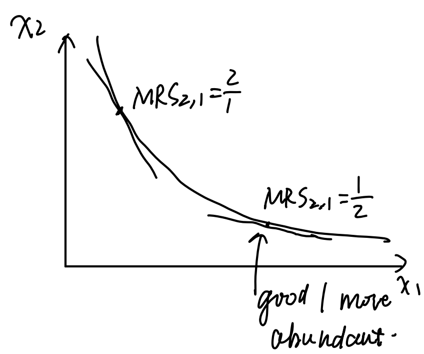
Fix $x_3, \cdots, x_n, u_0$,

$u(x_1, x_2(x_1), \cdots, x_n) = u_0$ $\quad \left(\frac{dx_3}{dx_1} = \frac{dx_4}{dx_1} = \cdots = \frac{dx_n}{dx_1} = 0\right)$.

$\frac{\partial u}{\partial x_1} + \frac{\partial u}{\partial x_2}\frac{dx_2}{dx_1} = 0$ differentiate both sides by x_1

$$\left.\frac{dx_2}{dx_1}\right|_{u_0} = -\frac{\partial u/\partial x_1}{\partial u/\partial x_2} = -\frac{MU_1}{MU_2}$$

$\therefore MRS_{2,1} = \left|\frac{dx_2}{dx_1}\right|_{u_0}$ depends on u_0 and the current consumption bundle.



e.g. $u = x_1 x_2$

Method 1: $MRS_{2,1} = -\frac{\partial u/\partial x_1}{\partial u/\partial x_2} = \left.\frac{x_2}{x_1}\right|_{u_0} = \frac{u_0/x_1}{x_1} = \frac{u_0}{x_1^2}$

Method 2: $x_2 = \frac{u_0}{x_1}$

$MRS_{2,1} = \left|\frac{dx_2}{dx_1}\right|_{u_0} = \frac{u_0}{x_1^2}$ $\qquad \frac{d\,MRS_{2,1}}{dx_1} < 0$ diminishing MRS.

- Proposition: If preferences are convex, or equivalently, utility function is q-concave, then there is diminishing MRS.

- Proposition: Suppose there are 2 goods. Then:

  convex preference $\Leftrightarrow$ diminishing MRS

  proof?

Show that $u(x)$ and $h(u(x))$ has the same $MRS_{j,i}$, where $h'>0$.

$$MRS_{ji} = \frac{du/dx_i}{du/dx_j} = \frac{h' \cdot \partial u/\partial x_i}{h' \cdot \partial u/\partial x_j}$$

- Utility Maximization
  - Set up:
    ① Consumption bundle $x \in X = \mathbb{R}^n_+$
    ② Fixed price vector $p = (p_1, \cdots, p_n) \gg 0$
    ③ Fixed income $I > 0$.

  - The budget (or feasible) set:

$$B = \{x \in X \mid p \cdot x \le I\}$$
$$\sum_{i=1}^{n} p_i x_i \le I.$$

Result: B is non-empty, convex, closed and bounded.

non-empty: $x=0 \in B$

convex: $x', x'' \in B \Rightarrow tx' + (1-t)x'' \in B$

compact $\Leftrightarrow$ $\begin{cases} \text{closed}: x^1, \cdots, x^n \in B, x^{(n)} \to \bar{x} \Rightarrow \bar{x} \in B \\ \text{bounded}: x^i_j \le M < \infty, \forall j \end{cases}$

$x^i_j = I/p_j$. $M = \max\{I/p_1, \cdots, I/p_n\}$.

- Utility Maximization Problem. UMP.

$$\max_{x \in \mathbb{R}^n_+} u(x)$$
$$\text{s.t. } p \cdot x \le I$$
$$x \ge 0 \ (x_i \ge 0, \forall i).$$

A non-empty compact set has a min and max.

$[\ \ ]$, compact $(\ \ ]$: not closed, bounded.

A: compact set, $f: A \to \mathbb{R}$ continuous. then the set of f is compact.

- If u is continuous and B is compact, then there is a max:

$$\max_x u(x)$$
$$\text{s.t. } p \cdot x \le I$$
$$x \ge 0$$

EXISTENCE

$$\Downarrow$$

$$x^* = x^*(p, I) = (x_1^*(p,I), \cdots, x_n^*(p,I)).$$

Marshallian (Ordinary) Demand.

By definition, $u(x^*) \ge u(x)$, $\forall x \in \mathbb{R}^n_+$ s.t. $p \cdot x \le I$.

- $x^*(p, I)$ is unique if $u(x)$ is strictly quasi-concave.

e.g. If $u(x_1, x_2) = x_1 + x_2$, x^* not unique.

proof: Suppose that it is not unique. That is,

$x^{*\prime} \ne x^{*\prime\prime}$, then $u(x^{*\prime}) = u(x^{*\prime\prime})$

$u(tx^{*\prime} + (1-t)x^{*\prime\prime}) > \min\{u(x^{*\prime}), u(x^{*\prime\prime})\}$. (using the q-concavity to find another maximizer)

$= u(x^{*\prime})$.

Now we need to show $tx^{*\prime} + (1-t)x^{*\prime\prime}$ is also feasible.

Know that $\begin{cases} p \cdot x^{*\prime} \le I \\ p \cdot x^{*\prime\prime} \le I \end{cases}$

$\therefore \begin{cases} t\,p \cdot x^{*\prime} \le tI \\ (1-t)\,p \cdot x^{*\prime\prime} \le (1-t)I \end{cases} \Rightarrow p \cdot [tx^{*\prime} + (1-t)x^{*\prime\prime}] \le tI + (1-t)I = I$

not $x^{*\prime}$ and $x^{*\prime\prime}$ anymore!

Then, the feasible $tx^{*\prime} + (1-t)x^{*\prime\prime}$ is the maximizer. Contradiction!

- Homogeneous Functions:

  $f: \mathbb{R}^n \to \mathbb{R}$ is HDk where $k \in \mathbb{R}$ if:

  $$f(tx) = t^k f(x) \quad \forall x, \forall t > 0.$$

- Euler's Theorem:

  Suppose f is twice continuously differentiable, then f is HDk iff

  $$\sum_i \frac{\partial f}{\partial x_i} x_i = k f(x), \quad \forall x.$$

- Properties of Marshallian Demand.

  Suppose $u(x)$ is continuous.

① $x^*(p, I)$ is HD0 in (p, I)

$x^*(tp, tI) = x^*(p, I)$, $\forall t > 0$. it is immune to pure inflation

proof: since (p, I) appears in the budget set. We are focusing on it.

$$\begin{array}{ll} \max\ u(x) & \max\ u(x) \\ \text{s.t. } tp \cdot x \le tI \quad \Leftrightarrow & \text{s.t. } p \cdot x \le I \\ x \geq 0 & x \geq 0 \end{array}$$

Suppose that $x_i^*(p, I)$ is twice-differentiable, then:

$$\frac{\partial x_i^*}{\partial p_1} p_1 + \frac{\partial x_i^*}{\partial p_2} p_2 + \cdots + \frac{\partial x_i^*}{\partial p_n} p_n + \frac{\partial x_i^*}{\partial I} I = 0$$

② If $u(x)$ is strictly increasing in some x_i, then the budget constraint is binding.

$$p \cdot x^* = I$$

proof: if $p \cdot x^* < I$, then the consumer would increase x^* to $x^* + \varepsilon$, $\varepsilon > 0$ to be strictly better off, contradicting to x^* be the maximizer. (Local non-satiation).

(See it Later in Lagrange.)

③ If $\succsim$ is convex, or equivalently, $u(x)$ is q-concave then the set of x^* is convex.

That is, if $\hat{x}^*$ and $\hat{\hat{x}}^*$ solves UMP, then so must $t\hat{x}^* + (1-t)\hat{\hat{x}}^*$.

proof: $\because \succsim$ is convex

$\therefore t\hat{x}^* + (1-t)\hat{\hat{x}}^* \succsim \hat{x}^*$

$t\hat{x}^* + (1-t)\hat{\hat{x}}^* \succsim \hat{\hat{x}}^*$

$u(t\hat{x}^* + (1-t)\hat{\hat{x}}^*) \geq \min\{u(\hat{x}^*), u(\hat{\hat{x}}^*)\}$.

Say that $\hat{x}^* \succsim \hat{\hat{x}}^*$

that is, $u(\hat{x}^*) \geq u(\hat{\hat{x}}^*)$

Then: $u(t\hat{x}^* + (1-t)\hat{\hat{x}}^*) \geq u(\hat{\hat{x}}^*)$

If $u(t\hat{x}^* + (1-t)\hat{\hat{x}}^*) > u(\hat{\hat{x}}^*)$, then $\hat{\hat{x}}^*$ doesnot solve UMP

Therefore, $u(t\hat{x}^* + (1-t)\hat{\hat{x}}^*) = u(\hat{\hat{x}}^*)$, also solves UMP.

- Finding Marshallian Demand.

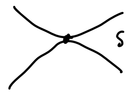
  If $u(x)$ is differentiable.

$$\begin{array}{lll} \max\limits_{x \in \mathbb{R}^n_+} u(x) & & \max\limits_{x} u(x) \\ \text{s.t. } p \cdot x \le I & \Rightarrow & \text{s.t. } I - p \cdot x \geq 0 \quad (\lambda) \\ x \geq 0 & & x \geq 0 \end{array}$$

Turn an constrained to an unconstrained problem Via Lagrange method.

$\mathcal{L}(x,\lambda) = u(x) + \lambda[I - p\cdot x]$

$\min_{\lambda} \max_{x} \mathcal{L}(x,\lambda)$ s.t. $x \geq 0$, $\lambda \geq 0$.


saddle points.

- Kuhn-Tucker FOC (necessary condition for optimality).

$$\begin{cases} \dfrac{\partial \mathcal{L}}{\partial x_i} = \dfrac{\partial u}{\partial x_i} - \lambda p_i \leq 0 & \dfrac{\partial \mathcal{L}}{\partial x_i} x_i = 0 \\ \dfrac{\partial \mathcal{L}}{\partial \lambda} = I - p\cdot x \geq 0 & \dfrac{\partial \mathcal{L}}{\partial \lambda}\cdot \lambda = 0 \\ x \geq 0 & \lambda \geq 0 \end{cases}$$

☆: All evaluated at the solution (x^*, λ^*).

- Result: If $\frac{\partial u}{\partial x_i} > 0$ for some i, then $\lambda^* > 0$, B is binding, $p\cdot x^* = I$.

proof: $\underset{+}{\dfrac{\partial u}{\partial x_i^*}} \leq \underset{+}{\lambda^* p_i}$

Then, $\lambda^* > 0$.

Then, $\frac{\partial \mathcal{L}}{\partial \lambda^*} = 0 \iff I - p\cdot x^* = 0$

- Result: If $x_1^* > 0$ and $x_2^* > 0$. then:

$$\begin{cases} \dfrac{\partial \mathcal{L}}{\partial x_1^*} = \dfrac{\partial u}{\partial x_1^*} - \lambda^* p_1 = 0 \\ \dfrac{\partial \mathcal{L}}{\partial x_2^*} = \dfrac{\partial u}{\partial x_2^*} - \lambda^* p_2 = 0 \end{cases} \Rightarrow MRS_{2,1} = \frac{\partial u/\partial x_1^*}{\partial u/\partial x_2^*} = \frac{p_1}{p_2}.$$

- Exercises: derive the Marshallian demand of $u(x_1, x_2) = x_1 + x_2$
  $u(x_1, x_2) = x_1 + \ln x_2$

- Indirect Utility:

$$V(p, I) = \max_{x} u(x) \quad \text{s.t. } t\, p\cdot x \leq t I, \; x \geq 0$$

$$\Downarrow$$

$$x^*(p, I)$$

The max utility under (p, I) is
$V(p,I) = u(x^*(p,I))$

- Properties of Indirect Utility.

suppose $u(x)$ is continuous (x^*, V exists)

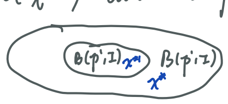
① $V(p,I)$ is HD0 in (p,I).

$V(tp, tI) = u(x^*(tp, tI)) = u(x^*(p,I)) = V(p,I)$. $\forall t > 0$.

② V is decreasing in p_i, increasing in I.

proof: suppose that $p' > p \Rightarrow B(p', I) \subset B(p, I)$.

$u(x^{*\prime})$ under $B(p', I)$ can't be larger than $u(x^*)$ under $B(p, I)$.



Suppose that $x^{*\prime} \in B' \subset B$, $x^* \in B$

assume that $u(x^{*\prime}) \geq u(x^*)$.

By the definition of x^* that $\max_{\text{s.t. } B} u(x)$.

However, we already have $u(x^*) \geq u(x)$, $\forall x$ in B

Therefore, contradiction! $V(p', I) = u(x^{*'}) \leq u(x^*) = V(p, I)$.

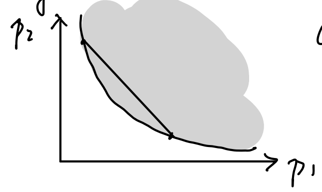
for I, $I\uparrow$, B expands. $I\downarrow$: B shrinks (to B')

③. V is quasi-convex in (p, I).

Definition: $f: A \to \mathbb{R}$ is quasi-convex if $-f$ is quasi-concave, i.e.

$f(tx' + (1-t)x^2) \leq \max\{f(x'), f(x^2)\}$.

Or equivalently, the worse-than set $\{x \in A \mid f(x) \leq k\}$ is convex for all $k \in \mathbb{R}$.



Consumer prefers more extreme price pairs.

V is q-convex regardless of the shape of $u(x)$, q-concave or not.

proof: take (p', I'), (p^2, I^2) and (p^t, I^t)

$$\begin{cases} p^t = tp' + (1-t)p^2 \\ I^t = tI' + (1-t)I^2 \end{cases} \quad \forall t \in [0, 1]$$

WTS: $V(p^t, I^t) \leq \max\{V(p', I'), V(p^2, I^2)\}$

It suffices to show that the budget set shrinks.

$B(p^t, I^t) \subseteq B(p', I')$ or $B(p^t, I^t) \subseteq B(p^2, I^2)$.

That is, if $x \in B(p^t, I^t)$, then $x \in B(p', I')$

or

if $x \in B(p^t, I^t)$, then $x \in B(p^2, I^2)$.

Suppose not. that is,

$x \in B(p^t, I^t)$ but $x \notin B(p', I')$ and $x \notin B(p^2, I^2)$.

Then: $[tp' + (1-t)p^2] \cdot x \leq tI' + (1-t)I^2$

$t \cdot p' \cdot x > I' \cdot t$

$(1-t) \cdot p^2 \cdot x > I^2 \cdot (1-t)$. Add up, contradict to!

Then, the original statement is true.

$V(p^t, I^t) \leq \max\{V(p', I'), V(p^2, I^2)\}$

④. $V(p, I)$ is continuous in (p, I).

proof. Berge's Theorem of maximization.

e.g. $u(x_1, x_2) = x_1 + x_2$:

$$x_1^* = \begin{cases} 0, & p_1 > p_2 \\ [0, I/p_1], & p_1 = p_2 \\ I/p_1, & p_1 < p_2 \end{cases} \qquad x_2^* = \begin{cases} I/p_2, & p_1 > p_2 \\ [0, I/p_2], & p_1 = p_2 \\ 0, & p_1 < p_2 \end{cases} \qquad V = x_1^* + x_2^* = \frac{I}{\min\{p_1, p_2\}}.$$ Continuous.

$(p_1, p_2) = (1-\varepsilon, 1)$: $x_1^* = I/(1-\varepsilon)$, $x_2^* = 0$ $\quad V = I/(1-\varepsilon)$

$= (1+\varepsilon, 1)$: $\underbrace{x_1^* = 0, \ x_2^* = I}_{\text{Big change}}$ $\quad \underbrace{V = I}_{\text{not affected}}$.

Proposition: If $u(x)$ is also strictly quasi-concave so that Marshallian demand, x^*, is unique and thus, a function of (p, I), then x^* is also continuous.

- Differentiable Utility.

- **The Envelop Theorem** (provide short cut to comparative static)

  Consider the following optimization:

  $$M(a) = \max_{x} f(x, a) \quad s.t.\ x \geq 0$$
  $$\Downarrow$$
  $$x^*(a) \text{ — unique solution}$$

  Want to find $M'(a)$

Direct Method: $M(a) = f(x^*(a), a)$

$$M'(a) = \underbrace{f_x(x^*(a), a)}_{=0 \text{ by Kuhn-Tucker condition}} + f_a(x^*(a), a)$$
$$= f_a(x^*(a), a)$$

plugging in $x^*(a)$ in $f(x,a)$, then differenciating w.r.t. a.

Envelop Method: Ignore the fact that $x^*(a)$ also depend on a. Treat x in $f(x,a)$ as a parameter. First, take a partial wrt. a. Then, evaluate at $x = x^*(a)$.

$M'(a) = f_a(x, a)\big|_{x^*(a)} = f_a(x^*(a), a)$. first differentiating $f(x,a)$ wrt. a, then plug in $x^*(a)$.

In our context:

$$V(a) = \max_{x_1 \cdots x_n} f(x_1, x_2, \cdots, x_n, a)$$
$$\Downarrow$$
$$x_1^*(a), x_2^*(a) \cdots, x_n^*(a) \leftarrow \begin{cases} \dfrac{\partial f(x^*, a)}{\partial x_1} = 0 \\ \vdots \\ \dfrac{\partial f(x^*, a)}{\partial x_n} = 0 \end{cases}$$

$$V(a) = f(\underbrace{x_1^*(a), \cdots, x_n^*(a)}_{\text{indirect}}, \underbrace{a}_{\text{direct}})$$

$$V'(a) = \underbrace{\frac{\partial f}{\partial x_1}}_{=0} \frac{dx_1^*(a)}{da} + \cdots + \underbrace{\frac{\partial f}{\partial x_n}}_{=0} \frac{dx_n^*(a)}{da} + \frac{\partial f}{\partial a}\bigg|_{x_1^*, \cdots, x_n^*}$$

$$= \frac{\partial f}{\partial a}\bigg|_{x_1^*(a), \cdots, x_n^*(a)}.$$

$a \rightarrow x_i^*(a) \longrightarrow V$: indirect

$a \longrightarrow V$: direct, dominating now.

- **Roy's identity**: From indirect utility to Marshallian Demand.

  Suppose $u(x)$ is differentiable, increasing, and strictly q-concave in some good i. Then:

  $$x_i^*(p, I) = -\frac{\partial v/\partial p_i}{\partial v/\partial I}$$

proof: $V(p, I) = \max_{x} u(x)$
$\quad s.t.\ p \cdot x \leq I$
$\quad x \geq 0$

$$\Downarrow$$
$$x^*(p, I).$$

$$\Leftrightarrow V(p, I) = \min_{\lambda} \max_{x} \mathcal{L} = u(x) + \lambda[I - p \cdot x]$$
$$s.t.\ x \geq 0,\ \lambda \geq 0.$$

By the Envelope Theorem:

- $\dfrac{\partial V}{\partial I} = \dfrac{\partial \mathcal{L}}{\partial I}\bigg|_{x^*, \lambda^*} = \lambda^*$ (shadow price of the constraint I, measures its importance).

(Aside: $\frac{\partial u}{\partial x_i} - \lambda p_i \le 0$, $\frac{\partial \mathcal{L}}{\partial x_i} x_i = 0$
$\quad + \quad + +$ $\quad =0 \quad >0$

Thus, $p \cdot x^* = I$, Binding B. $\lambda^* > 0$)

If we do it by Direct Method,

$V(p, I) = u(x^*(p, I))$

$\frac{\partial V}{\partial I} = \frac{\partial u}{\partial x_1}\frac{\partial x_1^*}{\partial I} + \dots + \frac{\partial u}{\partial x_n}\frac{\partial x_n^*}{\partial I}$

Long and complicated.

- $\frac{\partial V}{\partial p_i} = \frac{\partial \mathcal{L}}{\partial p_i}\Big|_{x^*, \lambda^*} = -\lambda^* x_i^*$

Thus: $x_i^* = x_i^*(p, I) = -\frac{\partial V/\partial p_i}{\partial V/\partial I}$.

• The Dual Problem: Expenditure Minimization.

UMP (primal)

$V(p, I) = \max_{x} u(x)$

s.t. $p \cdot x \le I$

$x \ge 0$

$\Downarrow$

$x^*(p, I)$ – Marshallian Demand
ordinary "
uncompensated "

EMP (dual)

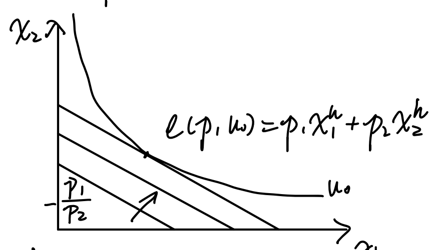
$e(p, u_0) = \min_{x} p \cdot x$

s.t. $u(x) \ge u_0$

$x \ge 0$

$\Downarrow$

$x^h(p, u_0)$ – Hicksian Demand
compensated demand.



objective function: iso-expenditure line

• Properties of Hicksian Demand and Expenditure

① Hicksian Demand is HD0 while expenditure is HD1 in p. that is,

$x^h(tp, u_0) = x^h(p, u_0)$, $\forall t \in (0, 1]$: $x^h(p, u_0)$ also solves EMP under (tp, u_0) ← use this in proof.

$e(tp, u_0) = t\, e(p, u_0)$, $\forall t \in (0, 1]$: $\Delta\%\uparrow$ in all $p_i \to \Delta\%\uparrow$ in e.

proof: let $\hat{p} = tp$, and $\hat{x}$ be the corresponding Hicksian demand. x^h be the Hicksian demand under p.

This means:

① minimizing: $\hat{p} \cdot \hat{x} \le \hat{p} \cdot x$, $\forall x$ ② "feasible": $u(\hat{x}) \ge u_0$

①: $t \cdot p \cdot \hat{x} \le t \cdot p \cdot x^h$

$p \cdot \hat{x} \le p \cdot x^h$

Suppose that x^h does not solve EMP under p

If $\hat{x} \ne x^h$, then $p \cdot \hat{x} < p \cdot x^h$

Then, since $u(\hat{x}) \ge u_0$, x^h cannot solve the EMP. not optimal under p.

then, there is a better $\hat{x}$ under p, contradicting that x^h solves EMP under p.

"feasible"

Then: $\hat{x} = x^h$

$\hat{x} = x^h(tp, I) = x^h(p, I)$ then, x^h also solves EMP under tp.

Then, $e(tp, u_0) = tp \cdot x^h(tp, u_0) = t\underbrace{p \cdot x^h(p, u_0)}_{x^h \text{ solves EMP under } (p, u_0)} = t \cdot e(p, u_0)$

Also, x^h solves EMP under (tp, u_0).

easier way to illustrate: suppose, that $x^h(p,u_0)$ does not solve EMP under (tp, u_0).
That is, there is an $\hat{x} \neq x^h$, that:

$tp\cdot\hat{x} \le tp\cdot x^h$ and $u(\hat{x}) \ge u_0$.

$p\cdot\hat{x} \le p\cdot x^h$ and $u(\hat{x}) \ge u_0$.

if $\hat{x} \neq x^h$, then

$p\cdot\hat{x} < p\cdot x^h$ and $u(\hat{x}) \ge u_0$.

$\hat{x}$ cannot solve EMP under (p, u_0). Contradictory!

Hence, $x^h(p,u_0)$ must also solve EMP under (tp, u_0).

② $e(p,u_0)$ is continuous in (p,u_0). So is $x^h(p,u_0)$ if $u(x)$ is strictly quasi-concave and thus, x^h is unique.

③ $e(p,u_0)$ is increasing in u_0

proof: $e(p,u_0) = \min_x p\cdot x$ s.t. $u(x) \ge u_0$

$u(x) \ge u_0$

$= \max_\lambda \min_x \mathcal{L}(x) = p\cdot x + \lambda[u(x) - u_0]$

Envelop Theorem: $\frac{\partial e}{\partial u_0} = \frac{\partial \mathcal{L}}{\partial u_0}\Big|_{x^*,\lambda^*} = -\lambda^*$

since $\frac{\partial \mathcal{L}}{\partial x_i} = p_i + \lambda\frac{\partial u}{\partial x_i} \ge 0$, $\underbrace{\frac{\partial \mathcal{L}}{\partial x_i}}_{=0}\cdot \underbrace{x_i}_{>0} = 0$

$p_i + \lambda\frac{\partial u}{\partial x_i} = 0$

$+ \quad +$

$\therefore \lambda^* < 0$

$\frac{\partial \mathcal{L}}{\partial \lambda} = 0$: Binding

$\Rightarrow \frac{\partial e}{\partial u_0} = -\lambda^* > 0$

④. $e(p,u_0)$ is increasing in each p_i. (proved by Envelope Theorem)

⑤. $e(p,u_0)$ is concave in p.

proof: WTS: $te(p^1,u_0) + (1-t)e(p^2,u_0) \le e(p^t,u_0)$.

where $p^t = tp^1 + (1-t)p^2$, $t\in[0,1]$.

Take p^1, p^2, $p^t = tp^1 + (1-t)p^2$, $t\in[0,1]$

Let x^1, x^2, x^t be the corresponding Hicksian Demand.

By definition of the EMP solution under each case,

$p^1\cdot x^1 \le p^1\cdot x$, $\forall x$

$p^2\cdot x^2 \le p^2\cdot x$, $\forall x$.

$$\begin{cases} tp^1\cdot x^1 \le t\cdot p^1\cdot x, & \forall x \\ (1-t)p^2\cdot x^2 \le (1-t)p^2\cdot x, & \forall x \end{cases}$$

$\therefore tp^1\cdot x^1 + (1-t)p^2\cdot x^2 \le [tp^1 + (1-t)p^2]x$, $\forall x$ including x^t

$te(p^1,u_0) + (1-t)e(p^2,u_0) \le e(p^t,u_0)$

Meaning: All else equal, if p_i doubles, then expenditure less than doubles. (concave)
But if all prices double, then $e(tp,u_0) = te(p,u_0)$

Suppose e is twice continuously differentiable in p:

Then, e is concave iff its Hessian Matrix is negative semi-definite

$$H=\begin{bmatrix} e_{11} & e_{12} & \cdots & e_{1n} \\ e_{21} & e_{22} & \cdots & e_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ e_{n1} & e_{n2} & \cdots & e_{nn} \end{bmatrix} \qquad e_{ij}=\frac{\partial}{\partial p_j}\frac{\partial e}{\partial p_i} = \frac{\partial^2 e}{\partial p_j \partial p_i}$$

Young's Theorem: if e is twice diff. then $e_{ij}=e_{ji}$

Thus, H is symmetric.

• Definition. Negative semi-definite:

$$H=\begin{bmatrix} e_{11} & e_{12} & \cdots & e_{1n} \\ e_{21} & e_{22} & \cdots & e_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ e_{n1} & e_{n2} & \cdots & e_{nn} \end{bmatrix}$$

Leading principle minors must alternate the sign.

$H_1=e_{11}\leq 0$

$H_2=\begin{vmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{vmatrix} \geq 0$

In particular, $e_{ii}\leq 0$, $i=1,\cdots,n$

$H_3 \leq 0$

etc.

⑥ Shephard's Lemma.

Suppose $u(x)$ is differentiable and strictly quasi-concave. Then,

$$x_i^h=\frac{\partial e}{\partial p_i}$$

proof: $e(p,I)=\min\limits_{x} \; p\cdot x$

s.t. $u(x)\geq u_0$.

$x\geq 0$

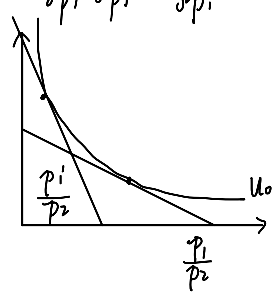
$$=\max_{\lambda}\min_{x} \mathcal{L}(x)=p\cdot x+\lambda[u(x)-u_0]$$

Envelop Theorem: $\frac{\partial e}{\partial p_i}=\frac{\partial \mathcal{L}}{\partial p_i}\Big|_{x_i^*,\lambda^*}=x_i^*(p,I)$,

Combining ⑤ ⑥:

- Hicksian demand is downward sloping in p_i.

$$\frac{\partial x^h}{\partial p_i}=\frac{\partial}{\partial p_i}\frac{\partial e}{\partial p_i}=\frac{\partial^2 e}{\partial p_i^2}\leq 0$$



Budget is not an issue here!

- Cross-price effects are equal.

$$\frac{\partial x_i^h}{\partial p_j}=\frac{\partial}{\partial p_j}\frac{\partial e}{\partial p_i}=\frac{\partial}{\partial p_i}\frac{\partial e}{\partial p_j}=\frac{\partial x_j^h}{\partial p_i}$$

$$\therefore \frac{\partial x_i^h}{\partial p_j}=\frac{\partial x_j^h}{\partial p_i}$$

Some Insights from Euler's equation:

If $f(x)$ is HD1, then $\frac{\partial f(x)}{\partial x_i}$ is HD0.

$\Downarrow$

$e(p,I)$ is HD1, by Shephard's Lemma, $x_i^h = \frac{\partial e}{\partial p_i}$ is HD0 (it must be)

More over, by Euler's equation: $\sum_{i=1}^{n} \frac{\partial x_i^h}{\partial p_i} \cdot p_i = 0$. $\quad \left(\sum_{i=1}^{n} \frac{\partial f(x_i)}{\partial x_i} \cdot x_i = k \cdot f(x). \right)$

- Substitution Matrix

$$S = \begin{bmatrix} \frac{\partial x_1^h}{\partial p_1} & \frac{\partial x_1^h}{\partial p_2} & \cdots & \frac{\partial x_1^h}{\partial p_n} \\ \frac{\partial x_2^h}{\partial x_1} & \frac{\partial x_2^h}{\partial x_2} & \cdots & \frac{\partial x_2^h}{\partial p_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial x_n^h}{\partial p_1} & \frac{\partial x_n^h}{\partial p_2} & \cdots & \frac{\partial x_n^h}{\partial p_n} \end{bmatrix}$$

$$S \cdot \begin{bmatrix} p_1 \\ \vdots \\ p_n \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{n} \frac{\partial x_1^h}{\partial p_i} p_i \\ \vdots \\ \sum_{i=1}^{n} \frac{\partial x_n^h}{\partial p_i} p_i \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

Properties: ① Symmetric: $\frac{\partial x_i^h}{\partial p_j} = \frac{\partial x_j^h}{\partial p_i}$

② Negative semi-definite

③ $S \cdot p = 0$

- e.g. let $S = \begin{bmatrix} a & b \\ 2 & -\frac{1}{2} \end{bmatrix}$ $\quad p = \begin{bmatrix} 8 \\ p \end{bmatrix}$

By symmetry: $b = 2$

$$S = \begin{bmatrix} a & 2 \\ 2 & -\frac{1}{2} \end{bmatrix}$$

By negative semi-definite: $a \le 0$.

$a(-\frac{1}{2}) - 2 \times 2 \geq 0$, $\quad a \le -8$

By HD0: $\begin{cases} 8a + 2p = 0 \\ 8 \times 2 - \frac{1}{2} p = 0 \end{cases} \Rightarrow \begin{cases} a = -8 \\ p = 32. \end{cases}$

$V(p,I) = \max_x u(x)$

$\quad$ s.t. $p \cdot x \le I = e(p,u_0)$

$\quad \Downarrow$

$\quad x^*(p,I)$ $\quad$ $e(p,u_0)$

$\quad ||$

$\quad x^h(p,u_0)$

$\quad$ solve this UMP when $I = e(p,u_0)$

$V(p,e(p,u_0)) = u_0$

$e(p,u_0) = \min_x p \cdot x$

$\quad$ s.t. $u(x) \geq u_0 = u(x^*(p,I)) = V(p,I)$

$\quad \Downarrow$

$\quad x^h(p,u_0)$ $\quad$ $V(p,I)$

$\quad ||$

$\quad x^*(p,I)$ solves this EMP when $u_0 = V(p,I)$

$e(p,v(p,I)) = I$

• The Duality Theorem.

Suppose u is continuous, strictly increasing in at least one good.

$$e(p, v(p,I)) = I \qquad v(p, e(p,u_0)) = u_0$$

$$x^h(p, v(p,I)) = x^*(p,I) \qquad x^*(p, e(p,u_0)) = x^h(p,u_0)$$

$$\forall\ p, I, u_0.$$

proof. (UMP $\Rightarrow$ EMP): suppose that $x^*(p,I)$ solves UMP but x^* does not solve EMP subject to $u(x) \geq u(x^*) = v(p,I)$.

Let $x' \neq x^*$ be a solution to EMP. Then, $u(x') \geq u(x^*)$ (feasibility)

$p \cdot x' < p \cdot x^*$ (optimality).

Because it is a strict inequality, there exist another $x'' > x'$ $(x'' = x + (\varepsilon, \cdots, \varepsilon), \varepsilon > 0)$

$p \cdot x' < p \cdot x'' < p \cdot x^* \ (\leq I)$ <u>feasible under UMP</u>

By monotonicity, $u(x'') > u(x^*)$

$u(x'') > u(x') \geq u(x^*)$. <u>$x''$ better than x^* under UMP</u>

Then, x^* cannot be utility maximizing under (p,I) because x'' is feasible and yields a strictly higher utility. A contradiction!

Then, x^* solves EMP s.t. $u(x) \geq u(x^*) = v(e,p)$

(EMP $\Rightarrow$ UMP) Suppose x^h solves EMP but does not solve UMP s.t. $p \cdot x \leq I = p \cdot x^h$.

Let x' be a solution to UMP. then,

$p \cdot x' = I$, $u(x') > u(x^h) \cdot (\geq u_0)$

Since $I > 0$, $u(x)$ is continuous. There exists a $t \in [0,1]$, such that

$p \cdot tx' < I = p \cdot x^h$, $u(tx') > u(x^h)$. $(\geq u_0)$

Then, x^h does not solve EMP because tx' costs less and yields higher utility. A contradiction!

Thus, x^h solves UMP s.t. $p \cdot x \leq I = e(p,u_0)$

• Income and Substitution Effect of Price Changes.

using: $x_i^h(p,u_0) = x_i^*(p, e(p,u_0))$

$$\frac{\partial x_i^h(p,u_0)}{\partial p_j} = \frac{\partial x_i^*(p,I)}{\partial p_j} + \frac{\partial x_i^*(p,I)}{\partial I} \cdot \frac{\partial e(p,u_0)}{\partial p_j} \quad = x_j^h(p, v(p,I)) = x_j^*(p,I).$$

$$\therefore \frac{\partial x_i^*(p,I)}{\partial p_j} = \frac{\partial x_i^h}{\partial p_j}\bigg|_{u_0 = v(p,I)} - x_j^*(p,I)\frac{\partial x_i^*(p,I)}{\partial I} \quad \text{(Slutsky Equation)}$$

using: $x_i^*(p,I) = x_i^h(p, v(p,I))$

$$\frac{\partial x_i^*(p,I)}{\partial p_j} = \frac{\partial x_i^h}{\partial p_j} + \frac{\partial x_i^h}{\partial v} \cdot \frac{\partial v(p,I)}{\partial p_i}$$

$x_j^* = -\dfrac{\partial v/\partial p_j}{\partial v/\partial I}$ Roy's Identity

$\partial v/\partial p_j = -x_j^* \cdot \partial v/\partial I$

$$= \frac{\partial x_i^h}{\partial p_j} - x_j^* \frac{\partial x_i^h}{\partial v} \cdot \frac{\partial v}{\partial I}$$

$$= \frac{\partial x_i^h}{\partial p_j} - x_j^* \frac{\partial x_i^h(p, v(p,I))}{\partial I}$$

$$= \frac{\partial x_i^h}{\partial p_i}\bigg|_{u_0 = v(p,I)} - x_j^* \frac{\partial x_i^*(p,I)}{\partial I}.$$

when $i=j$:

$$\underbrace{\frac{\partial x_i^*}{\partial p_i}}_{\text{Total Effect}} = \underbrace{\left.\frac{\partial x_i^h}{\partial p_i}\right|_{u_0=v(p,I)}}_{\text{substitution Effect.}} \underbrace{- x_i^* \frac{\partial x_i^*}{\partial I}}_{\text{Income effect.}}$$

$$\frac{\partial x_i^h}{\partial p_i} = \frac{\partial}{\partial p_i}\frac{\partial e(p,u_0)}{\partial p_i}$$
$$= \frac{\partial^2 e(p,u_0)}{\partial p_i^2} \leq 0$$

- Definition: Normal Good: $\frac{\partial x_i^*}{\partial I} \geq 0$

  Inferior Good: $\frac{\partial x_i^*}{\partial I} \leq 0$

  depending on preference $(u(x))$ and (p, I) not on quality of goods.

- Conclusions:

  ① If Normal, then $\frac{\partial x_i^*}{\partial p_i} < 0$

  ② If $\frac{\partial x_i^*}{\partial p_i} > 0$ (Giffen good). then Inferior. $\frac{\partial x_i^*}{\partial I} < 0$.

- Notes: Slutsky is useful in small changes $\Delta p_i \to 0$ to proving results and determining signs.

- e.g. $u = x_1^{1/2} x_2^{1/2}$

  Marshallion Demand: $x_1^* = \frac{I}{2p_1}$, $x_2^* = \frac{I}{2p_2}$

$$V = x_1^{*\frac{1}{2}} x_2^{*\frac{1}{2}} = \frac{I}{2\sqrt{p_1 p_2}} = u_0 \quad \text{(by Duality)}$$

Duality: $e = I = 2\sqrt{p_1 p_2}\cdot u_0 \qquad (e(p, v(p,I)) = I)$

$$x_1^h = \frac{e}{2p_1} = \sqrt{\frac{p_2}{p_1}}\cdot u_0 \qquad p_1^{-\frac{1}{2}}$$

When solving x^h, keep u_0!

$$x_2^h = \frac{e}{2p_2} = \sqrt{\frac{p_1}{p_2}}\cdot u_0$$

$$\frac{\partial x_1^*}{\partial p_1} = -\frac{I}{2p_1^2} < 0 \quad \frac{\partial x_2^*}{\partial p_2} = \frac{I}{-2p_2^2} < 0. \text{ Normal Goods.}$$

Substitution Effect $\left.\frac{\partial x_1^h}{\partial p_1}\right|_{u_0=v(p,I)} = -\frac{\sqrt{p_2}}{2p_1\sqrt{p_1}}\cdot V = -\frac{\sqrt{p_2}}{2p_1\sqrt{p_1}}\cdot\frac{I}{2\sqrt{p_1p_2}} = -\frac{I}{4p_1^2}$

Income Effect: $\frac{\partial x_1^*}{\partial p_1} - \left.\frac{\partial x_1^h}{\partial p_1}\right|_{u_0=v(p,I)} = -\frac{I}{2p_1^2} + \frac{I}{4p_1^2} = -\frac{I}{4p_1^2} < 0$

In this case, $SE = IE$

Elasticity of Demand:

A unit-free measure of the magnitude change of demand.

$y=f(x)$:

$$\varepsilon_{y,x} \approx \frac{\%\Delta y}{\%\Delta x} = \frac{\Delta y/y}{\Delta x/x} \quad \Delta\to 0 \quad \varepsilon_{y,x} = \frac{dy/y}{dx/x} = \frac{dy}{dx}\cdot\frac{x}{y} \quad \text{or: } \varepsilon_{y,x} = \frac{d\ln y}{d\ln x}$$

→ estimating $y = ax+b$: a is slope. estimating $\ln y = a\ln x + b$: a is elasticity.

Applications.

① Price Elasticity of demand:

$$\varepsilon_i = \frac{\partial x_i^*}{\partial p_i}\cdot\frac{p_i}{x_i^*}$$

② Cross Price Elasticity of demand:

$$\varepsilon_{ij} = \frac{\partial x_i^*}{\partial p_j}\cdot\frac{p_j}{x_i^*}$$

③ Income Elasticity

$$\varepsilon_{x_i,I} = \eta_i = \frac{\partial x_i^*}{\partial I}\cdot\frac{I}{x_i^*}$$

Result: let $s_i = \frac{p_i x_i^*}{I}$ be the income share of good i.

Then, $\sum_{i=1}^{n} s_i\cdot\varepsilon_{x_i,I} = 1$, $\sum_{j\neq i}^{n} s_j\cdot\varepsilon_{j,i} = -s_i$

proof: from budget constraint:

$$\sum_{i=1}^{n} p_i x_i^*(p, I) = I$$

$$\sum_{i=1}^{n} p_i \frac{\partial x_i^*}{\partial I} = 1$$

$$\sum_{i=1}^{n} p_i\cdot\frac{x_i^*}{I}\cdot\varepsilon_{x_i,I} = 1$$

$$\therefore \sum_{i=1}^{n} s_i\cdot\varepsilon_{x_i,I} = 1$$

$$\sum_{j=1}^{n} p_j\cdot x_j^* = I$$

$$p_i x_i^* + \sum_{j\neq i}^{n} p_j x_j^* = I$$

take derivative wrt p_i:

$$x_i^* + \sum_{j\neq i}^{n} p_j\cdot\frac{\partial x_j^*}{\partial p_i} = 0$$

$$\sum_{j\neq i}^{n} p_j\cdot\varepsilon_{j,i}\cdot\frac{x_j^*}{p_i} = -x_i^*$$

$$\frac{1}{p_i}\sum_{j\neq i}^{n} p_j x_j^* \varepsilon_{j,i} = -x_i^*$$

$$\sum_{j\neq i}^{n} \frac{p_j x_j^*}{I}\varepsilon_{j,i} = -\frac{p_i x_i^*}{I}$$

$$\therefore \sum_{j\neq i}^{n} s_j\cdot\varepsilon_{j,i} = -s_i$$

· Measurement of Consumer welfare.

Assume $p_i^1 > p_i^0 \Rightarrow u_1 < u_0$

Case 0 : $(p_i^0, p_{-i}^0, I) \rightarrow u_0 = v(p^0, I)$

Case 1 : $(p_i^1, p_{-i}^0, I) \rightarrow u_1 = v(p^1, I)$

$e(p_i^1, p_{-i}^0, u) - e(p_i^0, p_{-i}^0, u)$:

① $u = u_0$: Compensating Variation : CV

$$CV = e(p_i^1, p_{-i}^0, u_0) - \underbrace{e(p_i^0, p_{-i}^0, u_0)}_{I}$$

$$= \int_{p_i^0}^{p_i^1} \frac{\partial}{\partial p_i} e(p_i, p_{-i}, u_0)\, dp_i$$

$$= \int_{p_i^0}^{p_i^1} x_i^h(p_i, p_{-i}, u_0)\, dp_i$$

$$x_i^h(p, v(p, I)) = x_i^*(p, I)$$

$$\frac{\partial x_i^h}{\partial u} \cdot \underbrace{\frac{\partial v}{\partial I}}_{(+)} = \frac{\partial x_i^*}{\partial I}$$

$$\therefore \frac{\partial x_i^h}{\partial u} \overset{\text{sign}}{=} \frac{\partial x_i^*}{\partial I}$$

Assume normal, then: $\frac{\partial x_i^*}{\partial I} > 0$, $\frac{\partial x_i^h}{\partial u} > 0$.

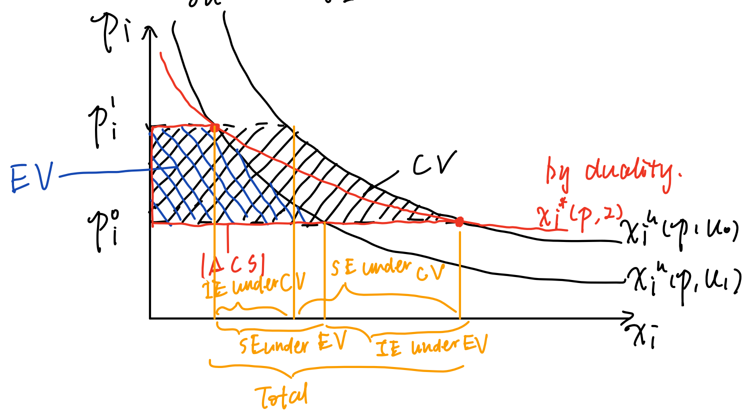
as $u \downarrow$, $x_i^h \downarrow$

② $u = u_1$: Equivalence Variation : EV

$$EV = e(p_i^1, p_{-i}^0, u_1) - e(p_i^0, p_{-i}^0, u_1)$$

$$= \int_{p_i^0}^{p_i^1} \frac{\partial}{\partial p_i} e(p_i, p_{-i}, u_1)\, dp_i$$

$$= \int_{p_i^0}^{p_i^1} x_i^h(p_i, p_{-i}, u_1)\, dp_i$$



Consumer Surplus :

$$CS = \int_{p_i}^{\infty} x_i^*(p, I)\, dp_i$$

$$\Delta CS = \int_{p_i^0}^{p_i^1} x_i^*(p, I)\, dp_i$$

← It is preferred because we observe x^* in the data.
How good it is depends on how small the income effect is.
Slutsky Equation:

$$\frac{\partial x_i^*}{\partial p_i} = \left.\frac{\partial x_i^h}{\partial p_i}\right|_{u=U} - x_i^* \frac{\partial x_i^*}{\partial I}$$

$\therefore$ If normal, then : $EV \le |\Delta CS| \le CV$

If inferior, then : $CV \le |\Delta CS| \le EV$

Proposition: Suppose utility is quasi-linear. $u(x) = x_0 + \phi(x_1 \cdots x_k)$
where $\partial\phi/\partial x_i > 0$ $\forall i = 1, \cdots, k$ and ϕ is strictly concave.
Also suppose $x_0^* > 0$, $x_i^* > 0$. then,
① $x_i^*(p, I) = x_i^h(p, u) = x_i(p)$, $\forall i = 1 \cdots k$. (No income effect).
② $CV_i = \Delta CS_i = EV_i$

pf: Consumer solves:

$$\max_x u(x) = x_0 + \phi(x_1 \cdots x_k)$$

$$\text{s.t. } p_0 x_0 + \sum_{i=1}^{k} p_i x_i \le I \leftarrow = \text{ because } \partial u/\partial x_i > 0$$

$$x_0, x_i \ge 0$$

$$\Updownarrow$$

$$\max_{x_1 \cdots x_k} \frac{I - \sum_{i=1}^{k} p_i x_i}{p_0} + \phi(x_1 \cdots x_k)$$

$$\text{FOC: } \frac{\partial}{\partial x_i}(\cdot) = -\frac{p_i}{p_0} + \frac{\partial \phi}{\partial x_i} = 0 \quad , i = 1 \cdots k, \text{ if } x_i^* > 0$$

$\therefore x_i^*(p, I) = x_i^*(p) > 0$ I does not affect x_i^*, no income effect

$$x_0^*(p, I) = \frac{I - \sum_{i=1}^{k} p_i x_i^*}{p_0} > 0 \qquad i = 1, \cdots, k.$$

$$V(p,I) = x_0^* + \phi(x_1^*(p), \cdots, x_k^*(p))$$
$$= \frac{I - \sum_{i=1}^{k} p_i x_i^*(p)}{p_0} + \phi(x_1^*(p), \cdots, x_k^*(p)) \quad i = 1, \cdots, k$$
$$= \frac{I}{p_0} - \frac{1}{p_0}\sum_{i=1}^{k} p_i x_i^*(p) + \phi(x_1^*(p), \cdots, x_k^*(p)), \quad i = 1, \cdots, k$$
$$= \frac{I}{p_0} + h(p) \qquad , \frac{\partial h}{\partial p_i} < 0, \quad i = 0, \cdots, k$$

By Roy's Identity:
$$x_i^* = -\frac{\partial V/\partial p_i}{\partial V/\partial I} = -p_0 \cdot \frac{\partial h(p)}{\partial p_i} \qquad , i = 1, \cdots, k$$

By duality:
$$u = \frac{e}{p_0} + h(p)$$
$$e = p_0(u - h(p))$$

By Shepherd's Lemma:
$$x_i^h = \frac{\partial e}{\partial p_i} = -p_0 \frac{\partial h(p)}{\partial p_i}$$

$$\therefore x_i^*(p,I) = x_i^h(p,I) = x_i(p), \quad \forall i = 1, \cdots, k$$

· Integrability:

Preference $\succsim$ + Axioms (comp. trans. cont.)

$\Updownarrow$

continuous utility : $u(x): \mathbb{R}_+^n \to \mathbb{R}$.

$\Rightarrow$ demand. $x^*(p,I)$. We only observe this. Is this "rational"? i.e. does it come from some utility maximization?

Theorem: integrability:

A continuous differentiable function $x(p,I)$ is the demand generated by some strictly increasing and strictly quasi-concave utility function

$\Updownarrow$ iff:

1) $p \cdot x(p,I) = I$ (Budget Balancedness, BB)

2) the substitution or Slutsky matrix
$$S_{i,j} = \left[\frac{\partial x_i}{\partial p_j} + x_j \frac{\partial x_i}{\partial I}\right]$$
is symmetric ($S_{i,j} = S_{j,i}$) and negative semi-definite.

Aside: $\frac{\partial x_i^*}{\partial p_i} = \frac{\partial x_i^h}{\partial p_i} - x_i^* \frac{\partial x_i^*}{\partial I}$

$\therefore S_{i,j} = \left[\frac{\partial x_i^h}{\partial p_j}\right]$ if $x = x^*$

Proposition:

If $x(p,I)$ satisfies BB and its Slutsky matrix is symmetric, then $x(p,I)$ is HD0 in (p,I).

Given info:

$p \cdot x(p,I) = I$

$\sum_j p_j x_j(p,I) = I$

$S_{i,j} = \left[\frac{\partial x_i}{\partial p_j} + x_j \frac{\partial x_i}{\partial I}\right]$

WTS:

$x(p,I) = (x_1(p,I), \cdots, x_n(p,I))$

$x_i(tp, tI) = t^0 x_i(p,I)$

$\sum_j \frac{\partial x_i}{\partial p_j} p_j + \frac{\partial x_i}{\partial I} I = 0$

See HD, partial derivative: think of Euler.

proof. By BB:

$p\cdot x(p,I)=I \Rightarrow p_i x_i + \sum_{j\neq i} p_j x_j = I$

Differentiate both sides wrt p_i and I:

$\sum_j p_j \frac{\partial x_j}{\partial p_i} = -x_i \quad \left(x_i + \frac{\partial x_i}{\partial p_i} + \sum_{j\neq i} p_j x_j = 0\right)$

$\sum_j p_j \frac{\partial x_j}{\partial I} = 1$

WTS: $x_i(p,I)$ is HD0: $\Omega = \sum_j \frac{\partial x_i}{\partial p_j} p_j + \frac{\partial x_i}{\partial I} I = 0$.

$\Omega = \sum_j \frac{\partial x_i}{\partial p_j} p_j + \frac{\partial x_i}{\partial I} \sum_j p_j x_j$

$= \sum_j p_j \underbrace{\left(\frac{\partial x_i}{\partial p_j} + x_j \frac{\partial x_i}{\partial I}\right)}_{S_{i,j}}$

$S_{i,j} = S_{j,i}$ by symmetry.

$\therefore \Omega = \sum_j p_j \left(\frac{\partial x_j}{\partial p_i} + x_i \frac{\partial x_j}{\partial I}\right)$

$= \sum_j p_j \frac{\partial x_j}{\partial p_i} + x_i \sum_j p_j \frac{\partial x_j}{\partial I}$

$= -x_i + x_i$

$= 0$

e.g. $x_1 = \frac{\alpha_1 I}{p_1}$ $\quad x_2 = \frac{\alpha_2 I}{p_2}$

$\alpha_1, \alpha_2 > 0, \quad \alpha_1 + \alpha_2 = 1$

① $p_1 x_1 + p_2 x_2 = I$ ✓ BB

② $S_{11} = \frac{\partial x_1}{\partial p_1} + x_1 \frac{\partial x_1}{\partial I}$

$= -\frac{\alpha_1 I}{p_1^2} + \frac{\alpha_1 I}{p_1} \cdot \frac{\alpha_1}{p_1}$

$= \frac{\alpha_1 I}{p_1^2}(\alpha_1 - 1)$

$= -\frac{\alpha_1 \alpha_2 I}{p_1^2}$

$S_{12} = \frac{\partial x_1}{\partial p_2} + x_2 \frac{\partial x_1}{\partial I}$

$= 0 + \frac{\alpha_2 I}{p_2} \cdot \frac{\alpha_1}{p_1}$

$= \frac{\alpha_1 \alpha_2}{p_1 p_2} I$

$S_{21} = \frac{\partial x_2}{\partial p_1} + x_1 \frac{\partial x_2}{\partial I}$

$= 0 + \frac{\alpha_1 I}{p_1} \cdot \frac{\alpha_2}{p_2}$

$= \frac{\alpha_1 \alpha_2}{p_1 p_2} I$

$S_{22} = \frac{\partial x_2}{\partial p_2} + x_2 \frac{\partial x_2}{\partial I}$

$= -\frac{\alpha_2 I}{p_2^2} + \frac{\alpha_2 I}{p_2} \cdot \frac{\alpha_2}{p_2}$

$= \frac{\alpha_2 I}{p_2^2}(1 - \alpha_2)$

$= -\frac{\alpha_1 \alpha_2 I}{p_2^2}$

$\therefore$ symmetric

$S_{11} \leq 0$. $S_{11}S_{22} - S_{12}S_{21} = \frac{\alpha_1^2 \alpha_2^2 I^2}{p_1^2 p_2^2} - \frac{\alpha_1^2 \alpha_2^2 I^2}{p_1^2 p_2^2} = 0 \geq 0$

$\therefore$ negative semi-definite

$\therefore$ This demand is generated by some strictly increasing and quasi-concave utility function.

- **Revealed Preference.**

Assumption: a unique bundle x^i at p^i

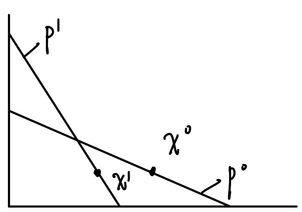
Weak Axiom of Revealed Preference: the basic axiom of rational choice from discrete data.

A consumer's choice satisfies WARP if for every distinct bundles $x^0 \neq x^1$ where x^0 is chosen at p^0 and x^1 is chosen at p^1, we have the following:

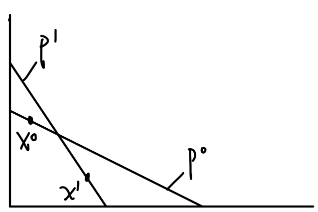
$p^0 \cdot x^1 \leq p^0 \cdot x^0 \quad \Rightarrow \quad p^1 \cdot x^0 > p^1 \cdot x^1$

(x^1 is feasible at p^0 but choose x^0) $\quad$ (x^0 should not be feasible at p^1)

$x^0 \ RP \ x^1 \quad\quad x^1 \ \cancel{RP} \ x^0$

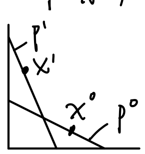


Satisfies WARP



not. $\left.\begin{array}{l} x^0 \ RP \ x^1 \\ x^1 \ RP \ x^0 \end{array}\right\}$ $\begin{array}{l} u(x^0) \geq u(x^1) \\ u(x^1) \geq u(x^0) \end{array}$ $\therefore$ contradict rational choice.

If $x^0 \ \cancel{RP} \ x^1$, $x^1 \ \cancel{RP} \ x^0$, nothing to check for inconsistency. Automatically satisfies WARP.



WARP is not enough to find a $u(x)$.

WARP + BB $\Rightarrow x(p, I)$ HD0.

proof: (p°, I°) and $(p', I') = (tp^\circ, tI^\circ)$

WTS: $x^\circ(p^\circ, I^\circ) = x'(p', I')$. $(= x'(tp^\circ, tI^\circ) = x'(p^\circ, I^\circ))$

Known: $p' \cdot x' = I'$, $p^\circ \cdot x^\circ = I^\circ$

Need: $\begin{cases} p^\circ \cdot x' \le p^\circ \cdot x^\circ \quad ① \\ p' \cdot x^\circ \le p' \cdot x' \end{cases}$ so, no WARP!

Suppose $x^\circ \ne x'$.

$p' \cdot x' = I'$

$tp^\circ \cdot x' = tI^\circ$

$p^\circ \cdot x' = I^\circ = p^\circ \cdot x^\circ$

$\therefore$ ① holds. If WARP, x° should not be feasible under p'.

$p' \cdot x^\circ > p' \cdot x'$

However: $p^\circ \cdot x^\circ = I^\circ$

$\frac{1}{t} p' \cdot x^\circ = \frac{1}{t} I'$

$p' \cdot x^\circ = I' = p' \cdot x'$

$p' \cdot x^\circ \le p' \cdot x'$

contradiction!

Then, $x^\circ = x'$, HD0.

WARP + BB $\Rightarrow$ S is negative semi-definite.

$$S = \left[\frac{\partial x_i}{\partial p_j} + x_j \frac{\partial x_i}{\partial I} \right]$$

WARP + BB $\Rightarrow$ S is symmetric? No if > 2 goods

Yes if $n=2$ goods.

proof: WTS: $\frac{\partial x_1}{\partial p_2} + x_2 \frac{\partial x_1}{\partial I} = \frac{\partial x_2}{\partial p_1} + x_1 \frac{\partial x_2}{\partial I}$

Known: $x(p, I)$ HD0. $p_1 x_1 + p_2 x_2 = I$

Differenciate wrt. I: $p_1 \frac{\partial x_1}{\partial I} + p_2 \frac{\partial x_2}{\partial I} = 1$

Differenciate wrt p_2: $p_1 \frac{\partial x_1}{\partial p_2} + x_2 + p_2 \frac{\partial x_2}{\partial p_2} = 0$. $(*)$

Since $x(p, I)$ is HD0:

$$p_1 \frac{\partial x_1}{\partial p_1} + p_2 \frac{\partial x_1}{\partial p_2} + I \frac{\partial x_1}{\partial I} = 0 \qquad p_1 \frac{\partial x_2}{\partial p_1} + p_2 \frac{\partial x_2}{\partial p_2} + I \frac{\partial x_2}{\partial I} = 0. \ (**)$$

$$\frac{x_2}{p_1}\left[p_1 \frac{\partial x_1}{\partial I} + p_2 \frac{\partial x_2}{\partial I} = 1\right]$$

$$x_2 \frac{\partial x_1}{\partial I} + \frac{p_2 x_2}{p_1} \frac{\partial x_2}{\partial I} = \frac{x_2}{p_1}$$

$$x_2 \frac{\partial x_1}{\partial I} + \frac{I - p_1 x_1}{p_1} \frac{\partial x_2}{\partial I} = \frac{x_2}{p_1}$$

$$x_2 \frac{\partial x_1}{\partial I} = \frac{x_2}{p_1} - \frac{I}{p_1} \frac{\partial x_2}{\partial I} + x_1 \frac{\partial x_2}{\partial I}$$

plug $(*)$ into $(**)$:

$$p_1 \frac{\partial x_2}{\partial p_1} - x_2 - p_1 \frac{\partial x_1}{\partial p_2} + I \frac{\partial x_2}{\partial I} = 0.$$

divide by p_1:

$$\frac{\partial x_2}{\partial p_1} - \frac{x_2}{p_1} - \frac{\partial x_1}{\partial p_2} + \frac{I}{p_1} \frac{\partial x_2}{\partial I} = 0.$$

$$\frac{\partial x_2}{\partial p_1} = \frac{\partial x_1}{\partial p_2} + \frac{x_2}{p_1} - \frac{I}{p_1} \frac{\partial x_2}{\partial I}$$

$$\therefore \frac{\partial x_1}{\partial p_2} + x_2 \frac{\partial x_1}{\partial I} = \frac{\partial x_1}{\partial p_2} + \frac{x_2}{p_1} - \frac{I}{p_1} \frac{\partial x_2}{\partial I} + x_1 \frac{\partial x_2}{\partial I}$$

$$= \frac{\partial x_2}{\partial p_1} + x_1 \frac{\partial x_2}{\partial I}$$

WARP + BB + $n=2$ => Slutsky Matrix is negative semi-definite, symmetric
=> by Integrability Theorem, there is a strictly increasing and strictly q-concave utility function that rationalizes the consumption data.

If $n>2$, then WARP + BB $\not\Rightarrow$ symmetric Slutsky Matrix.
WARP + BB $\not\Rightarrow$ transitive preference. Need:

Strong Axiom of Revealed Preference

If, for every distinct sequence of bundles, $x^0, x^1 \cdots x^k$, $x^0 \; RP \; x^1, x^1 \; RP \; x^2, \cdots, x^{k-1} \; RP \; x^k$, then: $x^1 \; RP \; x^k$. Then it satisfies SARP.

① It assumes away non-transitivity or preference cycles.
For only $n=2$ two bundles $x^0 \neq x^1$: WARP $\Leftrightarrow$ SARP.

② It checks the consistency among 3, 4, ⋯ bundles. so, stronger than WARP.

SARP + BB => $\exists$ strictly increasing and strictly q-concave utility function that rationalize the consumption data.
$\Updownarrow$
rationality.

$$\max_{x} u(x) \quad s.t. \; p^0 \cdot x \le p^0 \cdot x^0 \longrightarrow x^*(p_0, \underset{p^0 \cdot x^0}{\underset{\uparrow}{I}}) = x_0.$$

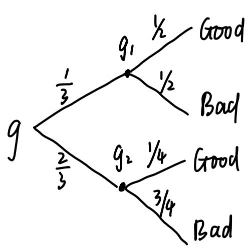
SARP + BB => Slutsky Matrix $\left[\frac{\partial x_i}{\partial p_j} + x_j \frac{\partial x_i}{\partial I}\right]$ is symmetric.
WARP + BB =>. ∼∼∼∼ ∼∼∼∼∼∼∼ negative semi-definite.
SARP => WARP, by integrability theorem, => $\exists$ $u(x)$ strictly increasing & strictly quasi-concave.
(If only $n=2$ goods: WARP + BB => $u(x)$).

GARP: allows for indifferences. Africa's Theorem

Weakness of the revealed preference theorem:
①. actual consumption data often involves violation of SARP (or GARP).
②. there may be different utilities that rationalize the data but gives contradicting rankings of off-sample bundles.

• Uncertainty:

$A = \{a_1 \cdots\cdots a_n\}$: a set of outcomes.
$g = (p_1 \circ a_1 \cdots p_n \circ a_n)$: a gamble. $\Sigma_i p_i = 1$
outcome a_i with prob. p_i
$g = (1 \circ a_i)$: a degenerate gamble
$g = (p_1 \circ g_1 \cdots p_n \circ g_n)$: a compound gamble.



$g = (\frac{1}{3} \circ g_1, \frac{2}{3} \circ g_2)$
$g' = (\underset{\frac{1}{3}\times\frac{1}{2}+\frac{2}{3}\times\frac{1}{4}}{\uparrow} \circ \text{Good}, \underset{\frac{1}{3}\times\frac{1}{2}+\frac{2}{3}\times\frac{3}{4}}{\uparrow} \circ \text{Bad})$

All compound gambles can be simplified to a simple gamble by multiplication of probabilities.

Axiom of Choice under Uncertainty.

Axiom G1: (Completeness).
For any $g_1, g_2 \in G$: $g_1 \succsim g_2$ or $g_2 \succsim g_1$.

Axiom G2: (Transitivity):
For $g_1, g_2, g_3 \in G$: if $g_1 \succsim g_2$, $g_2 \succsim g_3$, then $g_1 \succsim g_3$.

G1 & G2: can order any finite set of gambles

$$a_1 \succsim a_2 \succsim \cdots \succsim a_n$$

$g=(1\circ a_1)$ $g=(1\circ a_2)$ $g=(1\circ a_n)$

Axiom G3: (continuity):

For every $g\in G$: there is an $\alpha\in[0,1]$ such that
$g \sim (\alpha\circ a_1, (1-\alpha)\circ a_n)$

$\rightarrow$. How $u(a_i)$ is determined.
$a_i \sim (u(a_i)\circ a_1, (1-u(a_i))\circ a_n)$

Axiom G4: (monotonicity): $\forall \alpha,\beta\in[0,1]$:

$(\alpha\circ a_1, (1-\alpha) a_n) \succsim (\beta\circ a_1, (1-\beta)\circ a_n) \iff \alpha \geq \beta$

It rules out $a_1 \sim a_n$. Then $a_1 \succ a_n$.

Theorem: Axiom G1 to G4 guarantee the existence of utility representation: $\exists\, u: G\rightarrow \mathbb{R}$, such that
$g \succsim g' \iff u(g)\geq u(g')$

Corollary: If $u(g)$ represents $\succsim$ on gambles, so does $\phi(u)$, where ϕ is a strictly increasing function.

Definition: Let $g=(p_1\circ a_1 \cdots p_n\circ a_n)$. We say that $u(g)$ is von Neumann-Morgenstein utility if it has the following Expected Utility Property:

$$u(g)=\sum_i p_i u(a_i).$$

Axiom 5: (Independence): If $g\succsim g'$, then for any $g''\in G$: $\alpha\in[0,1]$:
$(\alpha\circ g, (1-\alpha)\circ g'') \succsim (\alpha\circ g', (1-\alpha)\circ g'')$.

Theorem: Under G1 – G5, there exists a VN-M utility.

Remark: The VN-M is at the expence of: ① Axiom G5. ② It is not immune to any monotonic transformation.
e.g. $[u(g)]^2=[\sum_i p_i u(a_i)]^2 \neq \sum_i p_i u^2(a_i)$.

Proposition: Suppose $u(\cdot)$ is a VN-M utility that represents preferences $\succsim$. Then,
$v(\cdot)$ is also a VN-M utility for $\succsim$ $\iff$ $v(g)=\alpha+\beta u(g)$ where $\beta>0$ and $\alpha\in\mathbb{R}$.

proof: Let $u(g)$ be the VN-M utility for $\succsim$.

$(\Rightarrow)$: suppose $v(g)$ is also a VN-M utility for $\succsim$

WTS: $v(g)=\alpha+\beta u(g)$

Take a gamble: $g=(p_1\circ a_1 \cdots p_n\circ a_n)$.

$u(g)=\sum_i p_i u(a_i)$

$v(g)=\sum_i p_i v(a_i)$

Since u & v represents the same preferences.

$v(g)=\phi(u(g))$, where ϕ is strictly increasing.

$\therefore v(g)=\sum_i p_i v(a_i)=\sum_i p_i \phi(u(a_i))$

Moreover, $v(g)=\phi(u(g))=\phi(\sum_i p_i u(a_i))$

let $x_i=u(a_i)$.

Then, $\sum_i p_i \phi(x_i)=\phi(\sum_i p_i x_i)$ both convex ($\geq$) and concave ($\leq$).

$\therefore \phi$ is linear.

$\therefore v(g)=\phi(u(g))=\alpha+\beta u(g)$.

($\Leftarrow$): Suppose that $V(g) = \alpha + \beta u(g)$ represents $\succsim$, where $\beta > 0$.

WTS: $V(g) = \sum_i p_i (V(a_i))$

Since $u(g)$ is vN-M. $u(g) = \sum_i p_i u(a_i)$

$$V(g) = \alpha + \sum_i p_i \beta u(a_i)$$
$$= \sum_i p_i \alpha + \sum_i p_i \beta u(a_i)$$
$$= \sum_i p_i (\alpha + \beta u(a_i))$$
$$= \sum_i p_i V(a_i)$$

Therefore, $V(g)$ is vN-M.

Since u represents $\succsim$:

$$g_1 \succsim g_2 \Leftrightarrow u(g_1) \geq u(g_2)$$
$$\Leftrightarrow \alpha + \beta u(g_1) \geq \alpha + \beta u(g_2) \quad \text{since } \beta > 0$$
$$\Leftrightarrow V(g_1) \geq V(g_2)$$

• Risk Aversion.

$g = (p_1 \circ w_1 \cdots p_n \circ w_n)$ $\quad w_i$: Monetary outcomes.

u: vN-M.

u differentiable, $u' > 0$.

Definition. For gamble g: the agent is said to be

- risk-averse: $u(\sum_i p_i w_i) > \sum_i p_i u(w_i)$
- risk-neutral: $u(\sum_i p_i w_i) = \sum_i p_i u(w_i)$
- risk-loving: $u(\sum_i p_i w_i) < \sum_i p_i u(w_i)$.

Agent can be risk-averse to g and risk-loving to g'
$g \neq g'$.

Proposition. For all gambles: the agent is:

- risk-averse $\Leftrightarrow u'' < 0$ strictly concave
- risk-neutral $\Leftrightarrow u'' = 0$ linear
- risk-loving $\Leftrightarrow u'' > 0$ strictly convex.

proof: by definition of them.

Concepts Definition:

① **Certainty Equivalence (CE).** The sure money that makes the agent indifferent.

$\sum_i p_i u(w_i) = u(CE)$ $\qquad$ lowest $w_i \leq CE \leq$ highest w_i

② **Risk Premium.** The amount of money individual is willing to pay / receive to avoid uncertainty.

$P = \sum_i p_i w_i - CE$.

$u(CE) = u(\sum_i p_i w_i - P) = \sum_i p_i u(w_i)$.

- If risk-averse: $u(\sum_i p_i w_i) > u(\sum_i p_i w_i - P)$, $P > 0$.
- If risk-neutral: $u(\sum_i p_i w_i) = u(\sum_i p_i w_i - P)$, $P = 0$. $\quad$ since $u' > 0$.
- If risk-loving: $u(\sum_i p_i w_i) < u(\sum_i p_i w_i - P)$, $P < 0$.

Degree of Risk Aversion:

Arrow-Pratt Measure of Absolute Risk Aversion

$$r_a(w) = -\frac{u''(w)}{u'(w)}$$

averse: $r_a(w) > 0$. $\quad$ neutral: $r_a(w) = 0$ $\quad$ loving: $r_a(w) < 0$

It is invariant to affine transformation

proof: Let $v(w) = \alpha + \beta u(w)$

$$r_a(w)\big|_v = -\frac{\beta u''(w)}{\beta u'(w)} = r_a(w)\big|_u .$$

Proposition More concave $\Leftrightarrow$ more risk-averse

Let u and v be two VN-M utilities, $u'>0$, $v'>0$. Then,

$$-\frac{u''(w)}{u'(w)} > -\frac{v''(w)}{v'(w)} \iff \exists\, h(\cdot),\ h'>0,\ h''<0,\ u(w) = h(v(w)).$$

proof: ($\Leftarrow$): Given $u(w) = h(v(w))$.

$$u'(w) = h'(v(w))\,v'(w) = h'v'$$

$$u''(w) = h''(u(w))\,v'(w)\,v'(w) + h'(v(w))\,v''(w) = h''(v')^2 + h'v''.$$

$$\therefore\ -\frac{u''(w)}{u'(w)} = -\frac{h''(v')^2 + h'v''}{h'v'} = -\underbrace{\frac{h''v'}{h'}}_{+} - \frac{v''}{v'}$$

$$\therefore\ -\frac{u''(w)}{u'(w)} = -\frac{v''(w)}{v'(w)}.$$

($\Rightarrow$): define $h(w) = u(v^{-1}(w))$.

$$u(v^{-1}(w)) = h(w).$$

$$\underset{+}{u'(v^{-1}(w))}\ \underbrace{v^{-1}(w)'}_{+} = h'(w).$$

$$\therefore\ h'(w) > 0.$$

$$\underset{-}{u''(v^{-1}(w))}\underset{+}{(v^{-1}(w)')^2} + \underset{-}{u'(v^{-1}(w))}\ \underbrace{v^{-1}(w)''}_{+} = h''.$$

$$\therefore\ h'' < 0.$$

$$\therefore\ \exists\, h,\ h'>0,\ h''<0,\ u(v^{-1}(w)) = h(w)$$

$$u(w) = h(v(w)).$$

$$v(v^{-1}(x)) = x$$

$$\underset{+}{v'(v^{-1}(x))}\ \underbrace{v^{-1}(x)'}_{+} = 1$$

$$\underset{-}{v''(v^{-1}(x))}\underset{+}{(v^{-1}(x)')^2} + \underset{+}{v'(v^{-1}(x))}\ \underbrace{v^{-1}(x)''}_{+} = 0.$$

Proposition. Comparing CEs.

Suppose $-\frac{u''(w)}{u'(w)} > -\frac{v''(w)}{v'(w)}$ then $CE_u < CE_v$ $\left(\sum_i p_i w_i - CE_u > \sum_i p_i w_i - CE_v\right)$.

proof: Take any gambles $g = (p_1 \circ a_1, \cdots, p_n \circ a_n)$

$$u(CE_u) = \sum_i p_i u(w_i) \qquad v(CE_v) = \sum_i p_i v(w_i)$$

By the previous proposition, $\exists\, h,\ h'>0,\ h''<0,\ u(w) = h(v(w))$.

$$\begin{aligned}
\therefore\ u(CE_u) &= \textstyle\sum_i p_i u(w_i) \\
&= \textstyle\sum_i p_i h(v(w_i)) \\
&< h\left(\textstyle\sum_i p_i v(w_i)\right) \quad \text{Jensen's Inequality.} \\
&= h(v(CE_v)). \\
&= u(CE_v)
\end{aligned}$$

$\because\ u' > 0$

$\therefore\ CE_u < CE_v.$ q.e.d.

e.g. Demand for insurance.

Consider a Risk-Averse agent.

W_0: initial wealth.

p: prob. of an accident

L: loss in an accident

q: insurance coverage.

πq: insurance premium. $\pi \in (0,1)$.

Q: how much q will agent buy?

Write down gamble $g = (p \circ (W_0 - L - \pi q + q), (1-p) \circ (W_0 - \pi q))$.

$$\max_q U(q) = p\, u(W_0 - L - \pi q + q) + (1-p) u(W_0 - \pi q).$$

$$\frac{\partial U}{\partial q} = p\, u'(W_0 - L - \pi q + q)(1-\pi) - \pi(1-p) u'(W_0 - \pi q) = 0.$$

Suppose the insurance premium is fair: $\pi = p$

$$p(1-p) u'(W_0 - L - \pi q + q) = p(1-p) u'(W_0 - \pi q),$$

since $u' > 0$, $q^* = L$.

Check: SOC: $\frac{\partial^2 U}{\partial q^2} < 0$ so q^* is maximizing.

So, under fair insurance, agent buys full coverage.

Fair insurance ($\pi = p$) is guaranteed by perfectly competitive insurance market.

Assuming risk-neutral firms: the expected profit:

linear profit. $\Pi = p(\pi q - q) + (1-p)\pi q.$

$= (\pi - p) q.$

zero profit condition: $\Pi = 0 \Rightarrow \pi^* = p$

If not perfectly competitive: $\Pi > 0 \Rightarrow \pi^* > p$.

From the FOC. $p\, u'(W_0 - L - \pi q + q)(1-\pi) = \pi(1-p) u'(W_0 - \pi q)$.

$\pi > p$

$1 - p > 1 - \pi$

$\therefore u'(W_0 - L - \pi q + q) > u'(W_0 - \pi q)$.

since $u'' < 0$: $W_0 - L - \pi q + q < W_0 - \pi q$

$q^* < L$ under covered.

- Consumer side frictions:
  ① Moral Hazard (Hidden Action)
  ② Adverse Selection (Hidden Information)

- Constant Absolute Risk Averse Utility.

$$-\frac{u''(w)}{u'(w)}=r$$

Let $v(w)=u'(w)$. Then:

$$-\frac{v'(w)}{v(w)}=r \iff -\frac{dv/dw}{v}=r$$

$$\frac{dv}{v}=-r\,dw$$

$$\int\frac{dv}{v}=-\int r\,dw$$

$$\ln v=-rw+c$$

$$\therefore v(w)=e^{-rw}\cdot e^{c}=e^{-rw}\cdot k.$$

$$\therefore u'(w)=e^{-rw}\cdot k$$

$$\therefore u(w)=-\frac{k}{r}e^{-rw}+\text{constant}.$$

Since u is immune to linear transformations, set $k=1$, constant $=\frac{1}{r}$

$$u(w)=\frac{1-e^{-rw}}{r}$$

$$\lim_{r\to 0}u(w)=\lim_{r\to 0}\frac{1}{r}-\frac{e^{-rw}}{r}=\lim_{r\to 0}w\frac{e^{-rw}}{1}=w.\quad \text{(As expected)}.$$

Arrow-Pratt measure comes from:

Consider the following gamble:

$g=(\frac{1}{2}\circ(w_0+h),\frac{1}{2}\circ(w_0-h))$ (equally likely to win & lose h).

$\therefore \mathbb{E}[g]=w_0$

Let $p(h)$ be the risk premium: $CE=w_0-p(h)$.

By definition: $u(CE)=u(g)$

$$u(w_0-p(h))=\frac{1}{2}u(w_0+h)+\frac{1}{2}u(w_0-h).\quad (*)$$

$\therefore p(0)=0.$

To find $p'(0)$, differentiate $(*)$ wrt h:

$$u'(w_0-p(h))(-p'(h))=\frac{1}{2}u'(w_0+h)-\frac{1}{2}u'(w_0-h)$$

$$h=0\Rightarrow p'(h)=0.$$

To find $p''(0)$, further differentiate $(*)$ wrt h:

$$u''(w_0-p(h))(-p'(h))+u'(w_0-p(h))(-p''(h))=\frac{1}{2}u''(w_0+h)+\frac{1}{2}u''(w_0-h).$$

$$h=0\Rightarrow p''(0)=-\frac{u''(w_0)}{u'(w_0)}$$

By Taylor's expansion: $p(h)=p(0)+p'(0)(h-0)+\frac{p''(0)}{2!}(h-0)^2+O(h^3)$

$$\approx \underbrace{-\frac{u''(w_0)}{u'(w_0)}}_{r_a(w_0)}\frac{h^2}{2}$$

For small gambles, the risk premium is proportional to $r_a(w)$.
However, $r_a(w)$ is used as a measure of risk aversion for all gambles, big or small. This result gives an idea where it comes from. It also reveals that for very small gambles, the risk premium is proportional to $r_a(w)$.

• In reality, the risk is such that you win / lose h percentage of your initial wealth.

$g = (\frac{1}{2} \circ (W_0 + hW_0), \frac{1}{2} \circ (W_0 - hW_0))$

• Arrow-Pratt measure of Relative Risk Aversion

$\Gamma_r(W) = -\frac{u''(W)}{u'(W)} W$

$u(W_0 - p(h) W_0) = \frac{1}{2} u(W_0 + hW_0) + \frac{1}{2} u(W_0 - hW_0)$.

• Constant Relative Risk Aversion Utility (CRRA)

$$-\frac{u''(W)}{u'(W)} W = \rho$$

$$u(W) = \begin{cases} \frac{W^{1-\rho} - 1}{1-\rho} & \text{for } \rho \in (0,1) \\ \ln W & \text{for } \rho = 1 \end{cases}$$

Observations: $\Gamma_r(W) = \Gamma_a(W) \cdot W$

① if CARA, then $\Gamma_r(W) = r \cdot W$, increasing relative risk aversion in wealth.

As $W\uparrow$, $g = (\frac{1}{2} \circ (W + \underline{100}), \frac{1}{2} \circ (W - \underline{100})) \sim (\frac{1}{2} \circ (W_0 + 100), \frac{1}{2} \circ (W_0 - 100))$

the stake ↓ in relative terms, but still indifference, then it must be that $\Gamma_r \uparrow$

② if CRRA, then $\Gamma_a(W) = \frac{\rho}{W}$, decreasing absolute risk aversion in wealth.

As $W\uparrow$, $g = (\frac{1}{2} \circ (W + 1\% W), \frac{1}{2} \circ (W - 1\% W)) \sim (\frac{1}{2} \circ (W_0 + 1\% W_0), \frac{1}{2} \circ (W_0 - 1\% W_0))$.

the stake ↑ in absolute terms, but still indifference, then it must be that $\Gamma_a \downarrow$

• Critiques of Expected Utility Theory.

① Allais Paradox

② Ellsberg Paradox

③ The Rabin Paradox