Topological Spaces

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Topological Spaces.

Def. (Topology). Let X be a space. A topology on X is a set of subsets of X, Υ, following the properties:

(1). $\phi \in \Upsilon$.

(2). $X \in \Upsilon$.

(3). For any collection $\{T_\alpha\}_{\alpha \in I} \subset \Upsilon$, $\bigcup_{\alpha \in I} T_\alpha \in \Upsilon$.

(4). For any $T_1, \cdots, T_m \in \Upsilon$, $\bigcap_{i=1}^{m} T_i \in \Upsilon$.

A topology is a legitimate definition of which sets in X is open. ϕ and X should be considered open; intersection of finitely many open sets is open; union of arbitrarily open sets is open.

Now we look at it first in metric space with focus on $\mathbb{R}^n$.

Def. (Metric Space). A metric space is a set X with a function $d: X \times X \to \mathbb{R}$ such that

(1). $d(x, y) \geq 0 \ \forall x, y \in X$; $d(x, y) = 0$ iff $x = y$.

(2). $d(x, y) = d(y, x)$.

(3). $d(x, z) \leq d(x, y) + d(y, z)$.

Def. (Open ball in metric space) $B_d(x, \varepsilon) = \{y \in X \mid d(x, y) < \varepsilon\}$.

Def. (d-Open). $U \subset X$ is d-open if for every $x \in U$, $\exists \varepsilon > 0$ such that $B_d(x, \varepsilon) \subset U$.

Proposition. (a). Open balls are d-open.

(b). A set is d-open iff it is equal to a union of d-open balls.

(c). Any union of d-open sets is d-open.

(d). Any intersection of finitely many d-open sets is d-open.

(e). Given (X, d_X), (Y, d_Y), $f: X \to Y$ is continuous in the ε-δ sense iff for every d_Y-open $U \subset Y$, $f^{-1}(U)$ is d_X-open in X.

proof. (a). Let $B_d(x, \varepsilon)$ be an open ball. Given $x' \in B_d(x, \varepsilon)$, $x' \in B_d(x', \varepsilon - d(x, x')) \subset B_d(x, \varepsilon)$.

(b). Let A be d-open. Then, for every point $x \in A$, $\exists \varepsilon_x > 0$ s.t. $x \in B_d(x, \varepsilon_x) \subset A$.

Then $A = \bigcup_{x \in A} B_d(x, \varepsilon_x)$.

Let A be a union of d-open balls. Then, for each point, there is a d-open ball around it.

(c). Let A be a union of d-open sets. Then A is a union of d-open balls by (b).

Then, for every point $a \in A$, a is in some open balls in A, thus A is d-open.

(d). Let $A = \bigcap_{i=1}^{n} A_i$, then, for each $a \in A$, $a \in A_i \ \forall i \in \{1, \cdots, n\}$. A_i is d-open.

Then, for each A_i, we can find d-open ball $B_d(a, \varepsilon_i)$ s.t. $a \in B_d(a, \varepsilon_i)$.

Then $a \in \bigcap_{i=1}^{n} B_d(a, \varepsilon_i)$. $a \in B_d(a, \min_i\{\varepsilon_i\}) \subset \bigcap_{i=1}^{n} B_d(a, \varepsilon_i) \subset A$.

Thus A is d-open.

(e) ($\Rightarrow$): We now have $\forall \varepsilon > 0, \exists \delta > 0$, s.t. $d_X(x, y) \leq \delta \Rightarrow d_Y(f(x), f(y)) \leq \varepsilon$.

Let U be d_Y-open in Y. WTS $f^{-1}(U)$ is d_X-open in X.

Let $x \in f^{-1}(U)$. Then $f(x) \in U$. Since U is d_Y-open, $\exists \varepsilon > 0$, $B_{d_Y}(f(x), \varepsilon) \subset U$.

Then, $\exists \delta > 0$, so that for all $x'' \in X$, $d_X(x, x'') < \delta \Rightarrow d_Y(f(x), f(x'')) < \varepsilon$.

Thus, $x \in B_{d_X}(x, \delta)$, and $f(x) \in B_{d_Y}(f(x), \varepsilon) \subset U$. Then $B_{d_X}(x, \delta) \subset f^{-1}(U)$. $f^{-1}(U)$ is d_X-open.

($\Leftarrow$). Let $x \in X$ be such that $f(x) \in U$.

Then, $\exists\ \varepsilon > 0$ s.t. $f(x) \in B_{d_Y}(f(x), \varepsilon) \subset U$.

Since $f^{-1}(B_{d_Y}(f(x), \varepsilon))$ is open in X, $\exists\ \delta > 0$ and $B_{d_X}(x, \delta)$ such that

$$f(B_{d_X}(x, \delta)) \subset B_{d_Y}(f(x), \varepsilon) \subset U.$$

Therefore $d_X(x, x') < \delta \Rightarrow d_Y(f(x), f(x')) < \varepsilon$. Continuous in ε-δ sense.

$\square$

Define Euclidean metric in $\mathbb{R}^n$ as

$$d(\vec{x}, \vec{y}) = \left(\sum_{i=1}^{n} (x_i - y_i)^2\right)^{\frac{1}{2}}$$

Define ρ metric as

$$\rho(\vec{x}, \vec{y}) = \max_i |x_i - y_i|$$

We first prove that ρ is a proper metric.

① $\rho(\vec{x}, \vec{y}) \geq 0$ is trivial

② ($\Rightarrow$): suppose that for some $x_i \neq y_i$, then $\rho > 0$. ($\Leftarrow$) is again trivial.

③
$$\begin{aligned}
\rho(\vec{x}, \vec{y}) + \rho(\vec{y}, \vec{z}) &= \max_i |x_i - y_i| + \max_i |y_i - z_i| \\
&\geq \max_i |x_i - y_i| + |y_i - z_i| \\
&\geq \max_i |x_i - z_i| \\
&= \rho(\vec{x}, \vec{z}).
\end{aligned}$$

Define $B_\rho(x, r) = \{y \in \mathbb{R}^n \mid \rho(x, y) < r\}$, then

$$\begin{aligned}
B_\rho(x, r) &= \{(y_1, \cdots, y_n) \mid |y_i - x_i| < r\ \forall i\} \\
&= \prod_{i=1}^{n} \{y_i \in \mathbb{R} \mid |x_i - y_i| < r\}.
\end{aligned}$$

Then we prove $\rho(\vec{x}, \vec{y}) \leq d(\vec{x}, \vec{y}) \leq \sqrt{n}\,\rho(\vec{x}, \vec{y})$.

$$\begin{aligned}
d(\vec{x}, \vec{y}) &= \left(\sum_{i=1}^{n} |x_i - y_i|^2\right)^{\frac{1}{2}} \\
&\geq \left(\max_i |x_i - y_i|^2\right)^{\frac{1}{2}} \\
&\geq \max_i |x_i - y_i| \\
&= \rho(\vec{x}, \vec{y}).
\end{aligned}$$

$$\begin{aligned}
d(\vec{x}, \vec{y}) &= \left(\sum_{i=1}^{n} (x_i - y_i)^2\right)^{\frac{1}{2}} \\
&\leq \left(\sum_{i=1}^{n} \left(\max_i |x_i - y_i|\right)^2\right)^{\frac{1}{2}} \\
&= \sqrt{n}\,\rho(\vec{x}, \vec{y}).
\end{aligned}$$

Next we prove that $B(\vec{x}, r) \subset B_\rho(\vec{x}, r)$, $B_\rho(\vec{x}, \frac{r}{\sqrt{n}}) \subset B(\vec{x}, r)$.

$\forall \vec{y} \in B(\vec{x}, r)$, $\rho(\vec{x}, \vec{y}) \leq d(\vec{x}, \vec{y}) < r$, so $\vec{y} \in B_\rho(\vec{x}, r)$. $\forall y \in B_\rho(\vec{x}, \frac{r}{\sqrt{n}})$, $d(\vec{x}, \vec{y}) \leq \sqrt{n}\,\rho(\vec{x}, \vec{y}) < r$, so $B_\rho(\vec{x}, \frac{r}{\sqrt{n}}) \subset B(\vec{x}, r)$.

Next we prove that U is ρ-open iff it is open.

($\Leftarrow$): Since U is open, $\forall \vec{x} \in U$, $\exists\ r > 0$ s.t. $B(\vec{x}, r) \subset U$. Then $B_\rho(\vec{x}, \frac{r}{\sqrt{n}}) \subset B(\vec{x}, r) \subset U$.

($\Rightarrow$): Since U is ρ-open, $\forall \vec{x} \in U$, $\exists\ r > 0$ s.t. $B_\rho(\vec{x}, r) \subset U$. Then $B(\vec{x}, r) \subset B_\rho(\vec{x}, r) \subset U$.

Define another matrix for $X \subset \mathbb{R}^n$: $q(\vec{x}, \vec{y}) = \begin{cases} 0 & \text{if } \vec{x} = \vec{y} \\ 1 & \text{if } \vec{x} \neq \vec{y} \end{cases}$

Then, $$B_q(\vec{x}, r) = \begin{cases} \{\vec{x}\} & \text{if } r \le 1 \\ X & \text{if } r > 1 \end{cases}$$

Thus, every set is q-open.

Back to the definition of topology.

For examples:

(1). If (X, d) is a metric space, then the set of all d-open sets in X is a topology.

(2). Discrete topology: $\mathcal{T}_d = \{$all subsets of $X\}$. In (X, q), this topology is induced from $q(\vec{x}, \vec{y}) = \mathbb{1}_{\vec{x} = \vec{y}}$.

(3). Indiscrete topology: $\mathcal{T}_i = \{\phi, X\}$.

(4). Topologies on $X = \{a, b\}$: $\mathcal{T}_1 = \{\phi, \{a\}, \{b\}, X\}$

$\mathcal{T}_2 = \{\phi, X\}$

$\mathcal{T}_3 = \{\phi, \{a\}, X\}$

$\mathcal{T}_4 = \{\phi, \{b\}, X\}$.

(5). Let $(X, \mathcal{T})$ be a metric space. "U is open iff $\forall x \in U, \exists \varepsilon > 0$ s.t. $B_d(x, \varepsilon) \subset U$". This is the metric topology induced by metric d. A different metric may induce a different $\mathcal{T}$.

(d and ρ induce the same topology).

Not every topology comes from a metric. For example, if X is a finite set, then for many metric on X, the induced topology is the discrete topology. This is because if all one-point sets are open, then all subsets are open. To verify, $\forall \vec{x} \in X$, take $r = \min\{d(x, y) \mid y \neq x\}$. Then $B_d(x, r) = \{x\} \subset \{x\}$.

<u>Def</u>. (Finite Complement Topology / Cofinite Topology). Let X be a set.

$$\mathcal{T}_f := \{U \subset X \mid X - U \text{ is finite or } U = \phi\}.$$

<u>Proposition</u>. $\mathcal{T}_f$ is a topology.

proof: (1) $\phi \in \mathcal{T}_f$ (2) $X = X - \phi \in \mathcal{T}_f$

(3). Let $\{U_\alpha\} \subset \mathcal{T}_f$ such that $X - U_\alpha$ is finite.

$$X - (\bigcup_\alpha U_\alpha) = \bigcap_\alpha (X - U_\alpha)$$

is also finite.

(4). Let $\{U_i\}_{i=1}^n \subset \mathcal{T}_f$ such that $X - U_i$ is finite.

$$X - \bigcap_{i=1}^n U_i = \bigcup_{i=1}^n (X - U_i)$$

is also finite.

□

<u>Def</u> (Finer) Let X be a set. Suppose $\mathcal{T}$ and $\mathcal{T}'$ are topologies on X. $\mathcal{T}$ is finer than $\mathcal{T}'$ if $\mathcal{T}' \subset \mathcal{T}$. Strictly finer if $\mathcal{T}' \subsetneq \mathcal{T}$.

<u>Proposition</u>. Let $\mathcal{T}$ be the standard topology on $\mathbb{R}$. Let $\mathcal{T}_f$ be the finite compliment topology. Then $\mathcal{T}_f \subsetneq \mathcal{T}$.

proof: First we prove $\mathcal{T} \neq \mathcal{T}_f$. $(-1, 1) \in \mathcal{T}$, but $(-1, 1) \notin \mathcal{T}_f$ since $\mathbb{R} - (-1, 1)$ is infinite.

Let U be $\mathbb{R} - U = \{a_1, \cdots, a_k\}$. Then, $U = \mathbb{R} - \{a_1, \cdots, a_k\}$. For any $y \in U$, let $r := \{d(y, u_i) \mid i = 1, \cdots, k\}$, then $B_d(y, r) \subset U$. Thus $u \in \mathcal{T}$.

Therefore $\mathcal{T}_f \subsetneq \mathcal{T}$.

□

Then, we revisit the def of a continuous function.

Def. (Continuous function). Let $(X, \mathcal{T})$ and $(Y, \mathcal{U})$ be topological spaces. A function $f: X \to Y$ is continuous if for every open set $U \subset Y$, $f^{-1}(U)$ is open in X.
(i.e. $\forall U \in \mathcal{U}$, $f^{-1}(U) \in \mathcal{T}$). preimage of open set is open.

Def. (Homeomorphism). f is called a homeomorphism if it is bijective and both f and f^{-1} are continuous.
$(X, \mathcal{T})$ and $(Y, \mathcal{U})$ are homeomorphic if $\exists$ a homeomorphism between them.

For example, $X = \{a, b\}$

$$\left.\begin{array}{l} \mathcal{T}_3 = \{\phi, \{a,b\}, \{a\}\} \\ \mathcal{T}_4 = \{\phi, \{a,b\}, \{b\}\} \end{array}\right\rangle \text{incomparable}$$

define $f: X \to X$ by $f(a) = b$, $f(b) = a$, then $f^{-1}(\{b\}) = \{a\}$, so f is continuous.
$g := f^{-1}$, then $g^{-1}(\{a\}) = \{b\}$, so g is continuous.

Basis for a topology.

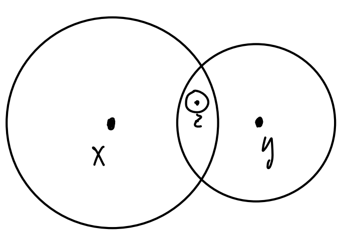
Def. (Basis) Let X be a set, $\mathcal{B}$ a collection of subsets of X. $\mathcal{B}$ is a topological basis if
1. For every $x \in X$, $\exists B \in \mathcal{B}$ such that $x \in B$.
2. For any $B_1, B_2 \in \mathcal{B}$ and any $x \in B_1 \cap B_2$, $\exists B_3 \in \mathcal{B}$ such that $x \in B_3$, $B_3 \subset B_1 \cap B_2$

Given a topological basis $\mathcal{B}$, we obtain a topology $\mathcal{T}$ by declaring $T \in \mathcal{T}$ iff T is a union of sets in $\mathcal{B}$ (i.e. $\forall x \in T$, $\exists B \in \mathcal{B}$ s.t. $x \in B \subset T$).

Def. (Topology generated by $\mathcal{B}$). A subset T of X is said to be open in X ($T \in \mathcal{T}$) if for every $x \in T$, $\exists B \in \mathcal{B}$ such that $x \in B \subset T$. ($\mathcal{T}$ is generated by $\mathcal{B}$).

e.g. (X, d) be a metric space. $B(x, r) \cap B(y, s)$. Let $z \in B(x, r) \cap B(y, s)$.
Take $t = \min(r - d(x, z), s - d(z, y))$. If $w \in B_d(z, t)$, then

$$\begin{aligned} d(w, x) &\le d(w, z) + d(z, x) \\ &< t + d(z, x) \\ &\le r - d(x, z) + d(x, z) \\ &= r. \end{aligned}$$



$$\begin{aligned} d(w, y) &\le d(w, z) + d(z, y) \\ &< t + d(z, y) \\ &\le s - d(z, y) + d(z, y) \\ &= s. \end{aligned}$$

Therefore $B_d(z, t) \subset B(x, r) \cap B(y, s)$.

<u>Theorem</u>: $\mathcal{T}$ is a topology.

pf. First, if U is empty set, then it satisfies the defining condition of openess.
Likewise, $X \in \mathcal{T}$ since for all $x \in X$, $\exists B_x \in \mathcal{B}$ such that $x \in B_x \subset X$.

Then, let $\{T_\alpha\}_{\alpha\in J} \subset \mathcal{T}$. WTS: $T = \bigcup_{\alpha\in J} T_\alpha \in \mathcal{T}$.
Given $x \in T$, $\exists\, \alpha \in J$ such that $x \in T_\alpha$. Since $T_\alpha \in \mathcal{T}$, there is a basis element B such that $x \in B \subset T_\alpha$. Then $x \in B \subset T$, so that $T \in \mathcal{T}$ by definition.

Then, given $T_1 \cdots T_k \in \mathcal{T}$, $x \in T_1 \cap \cdots \cap T_k$. There exists $B_1 \subset T_1, \cdots, B_k \subset T_k$ such that $x \in \bigcap_{i=1}^{k} B_i$
Since by definition $\forall\, x \in \bigcap_{i=1}^{k} B_i$, $\exists\, B' \in \mathcal{B}$ such that $x \in B'$, therefore $\exists\, B_{k+1} \subset \bigcap_{i=1}^{k} B_i$ such that $x \in B_{k+1} \subset B_i \subset T_i\ \forall i$ then $x \in B_{k+1} \subset T_1 \cap \cdots \cap T_k$. Therefore $T_1 \cap \cdots \cap T_k \in \mathcal{T}$.

$\square$

e.g. $X = \mathbb{R}$. $\mathcal{B} = \{$all open intervals (a, b) where $a < b\}$.
$\mathcal{T}$ is the standard topology.
$(a, b) = B\left(\frac{a+b}{2}, \frac{b-a}{2}\right)$

e.g. $X = \mathbb{R}$, $\mathcal{B} = \{(a, b) \mid a < b, a, b \in \mathbb{Q}\}$ gives the same topology as last one. Use the fact that $\mathbb{Q}$ is dense in $\mathbb{R}$, i.e. $\forall x, y \in \mathbb{R}$, $x < y$, $\exists\, z \in \mathbb{Q}$ s.t. $x < z < y$.

On $\mathbb{R}^n$: $d(x, y) = \sqrt{\sum_i (x_i - y_i)^2}$; $\rho(x, y) = \max_i |x_i - y_i|$
Open sets generated by these two are the same notion.

Let $\mathcal{T}$ be the topology from d, $\mathcal{T}'$ from ρ.

<u>Lemma:</u> Suppose d, d' are both metrics on X. Let $\mathcal{T}, \mathcal{T}'$ be the induced topologies.
Suppose $\forall x \in X$, $\forall \varepsilon > 0$, $\exists\, \delta > 0$ such that $B_{d'}(x, \delta) \subset B_d(x, \varepsilon)$, then $\mathcal{T} \subset \mathcal{T}'$.

Proof:
If U is open in $\mathcal{T}$, then for each $x \in U$, $\exists\, \varepsilon > 0$ such that $B_d(x, \varepsilon) \subset U$.
$\exists\, \delta > 0$ such that $B_{d'}(x, \delta) \subset B_d(x, \varepsilon) \subset U$, then U is also open in $\mathcal{T}'$.

q.e.d.

<u>Theorem:</u> Let X be a set. Let $\mathcal{T}$ and $\mathcal{T}'$ be topologies generated by bases $\mathcal{B}$ and $\mathcal{B}'$.
TFAE.
1. $\mathcal{T} \subset \mathcal{T}'$
2. $\forall B \in \mathcal{B}$ is open in $\mathcal{T}'$.
3. $\forall B \in \mathcal{B}$, $x \in B$, $\exists\, B' \in \mathcal{B}'$ s.t. $x \in B' \subset B$.

Proof: $3 \Rightarrow 1$: Given $T \in \mathcal{T}$. WTS $T \in \mathcal{T}'$.
Let $x \in U$. Since $\mathcal{B}$ generates $\mathcal{T}$, $\exists\, B \in \mathcal{B}$ such that $x \in B \subset U$.
by (3): $\exists\, B' \in \mathcal{B}'$, $x \in B' \subset B$. Then $x' \in B' \subset U$. Therefore $U \in \mathcal{T}'$ by definition.

$1 \Rightarrow 3$: We are given $x \in X$ and $B \in \mathcal{B}$ with $x \in B$. Since $B \in \mathcal{T}$ by definition and $\mathcal{T} \subset \mathcal{T}'$ by (1), we have $B \in \mathcal{T}'$. Since $\mathcal{T}'$ is generated by $\mathcal{B}'$, $\exists\, B' \in \mathcal{B}'$ such that $x \in B' \subset B$.

$\square$

Def. (lower limit topology). Let $X = \mathbb{R}$. The lower limit topology on $\mathbb{R}$ is the topology generated by the basis $\{[a,b) \mid a<b\}$.

Def. (ℓ-open). A set $U \subset \mathbb{R}$ is lower limit open (ℓ-open) if $\forall x \in U$, $\exists\ a<b$ such that $a \le x < b$, $[a,b) \subset U$. (i.e. $\exists\ \varepsilon>0$ s.t. $[x, x+\varepsilon) \subset U$).

Subspace topology.

<u>Def</u>. Let $(X, \mathcal{T})$ be a topological space. $Y \subset X$ be a subset. Define a topology on Y: $\mathcal{T}_Y$ the subspace topology by $\mathcal{T}_Y = \{U \cap Y \mid U \in \mathcal{T}\}$. That is, $V \subset Y$ is open in Y iff $V = Y \cap U$ for some open $U \subset X$.

<u>Theorem</u>. The subspace topology is a topology.

proof: ① $\phi = \phi \cap Y$, $\Rightarrow$ ϕ is open in Y.

② $Y = X \cap Y$, $\Rightarrow$ Y is open in Y.

③ Suppose $\{V_\alpha\}_{\alpha \in I}$ are open in Y. For each α: $V_\alpha = U_\alpha \cap Y$ for some U_α open in X.

$$\bigcup_{\alpha \in I} V_\alpha = \bigcup_{\alpha \in I} (U_\alpha \cap Y)$$
$$= \underbrace{\left(\bigcup_{\alpha \in I} U_\alpha\right)}_{\text{open in } X} \cap Y$$

Thus $\bigcup_{\alpha \in I} V_\alpha$ open in Y.

④ Suppose $\{V_i\}_{i=1}^{n}$ open in Y. For each i: $V_i = U_i \cap Y$ for some U_i open in X.

$$\bigcap_{i=1}^{n} V_i = \bigcap_{i=1}^{n} (U_i \cap Y)$$
$$= \left(\bigcap_{i=1}^{n} U_i\right) \cap Y$$

Thus, $\bigcap_{i=1}^{n} V_i$ open in Y.

In topological sense, we define subspace as any subset equipped with the subspace topology.

e.g. $X = \mathbb{R}$, $\mathcal{T}$ standard topology. $Y = [0, 1]$

$[0, 1]$ is open in Y. $A = [0, \frac{1}{2}) = Y \cap (-\frac{1}{2}, \frac{1}{2})$, open in Y.

e.g. $X = \mathbb{R}$, with lower limit topology. $\mathbb{R}_\ell$. $Y = [0, 1]$

$[\frac{1}{2}, 1] = Y \cap [\frac{1}{2}, 2)$ is open in subspace topology.

$\{1\} = Y \cap [1, 2)$ is open in subspace topology.

$\{\frac{1}{2}\}$ is not open since $\nexists$ U open in $\mathbb{R}_\ell$ such that $Y \cap U = \{\frac{1}{2}\}$.

<u>Def</u>. (Basis for the subspace topology of Y). If $\mathcal{B}$ is a basis of X, then $\mathcal{B}_Y = \{B \cap Y \mid B \in \mathcal{B}\}$ is a basis for the subspace topology of Y.

e.g. $\mathcal{B}$ for the subspace topology of $[0, 1] \subset \mathbb{R}$ standard consists of 3 kinds of sets.

(a, b) where $0 < a < b < 1$, $[0, b)$ $0 < b \le 1$, $(a, 1]$ $0 \le a < 1$.

e.g. $S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\} = \{(\cos t, \sin t) \mid t \in \mathbb{R}\}$

$S^n = \{\vec{x} \in \mathbb{R}^{n+1} \mid \|\vec{x}\| = 1\}$

Use the standard topology coming from the standard topology of $\mathbb{R}$.

For $a < b$, define $A_{a,b} := \{(\cos t, \sin t) \mid a < t < b\}$

- If $b - a > 2\pi$, then $A_{a,b} = S^1$.
- If $b - a < 2\pi$, then $A_{a,b}$ = open arc from $(\cos a, \sin a)$ to $(\cos b, \sin b)$.

Claim: $\{A_{a,b}\}$ are a basis for the topology S^1.

proof: $\forall \vec{x} \in S^1$, any $r>0$, $B(\vec{x}, r) \cap S^1 = A_{a,b}$ for some $a<b$.

The $\{A_{a,b}\}$s are intersections of S^1 with a basis for the topology of $\mathbb{R}^2$.

<u>Proposition</u> Let X be a metric space. $Y \subset X$ be a subspace of X.

1. Take the metric topology on X and then the subspace topology on Y, $\mathcal{T}_1$.
2. Consider $d_Y =$ restriction of d on Y, then take the metric topology, $\mathcal{T}_2$.

Then $\mathcal{T}_1 = \mathcal{T}_2$

proof: The basis for the subspace topology $\mathcal{T}_1$ is $\{B_d(x,r) \cap Y \mid x \in X, r>0\}$.

define $B_d^Y(y,r) = \{z \in Y \mid d(y,z) < r, \ y \in Y, r>0\}$

then $B_d^Y(y,r) = B_d(y,r) \cap Y$

the basis for the metric topology on Y $\mathcal{T}_2$ is $\{B_d^Y(y,r) \mid y \in Y, r>0\} = \{B_d(y,r) \cap Y \mid y \in Y, r>0\}$.

Given $B_d^Y(y,r)$ and $x \in B_d^Y(y,r)$, $x \in B_d(y,r) \cap Y \subset B_d^Y(y,r)$, therefore $\mathcal{T}_2 \subset \mathcal{T}_1$.

Conversely, WTS $\mathcal{T}_1 \subset \mathcal{T}_2$, that is, WTS $B_d(x,r) \cap Y$ is open in the metric topology $\mathcal{T}_2$.

Given $y \in B_d(x,r) \cap Y$, $\exists\, \delta>0$ such that $B_d(y,\delta) \subset B_d(x,r)$. Then $B_d(y,\delta) \cap Y \subset B_d(x,r) \cap Y$

Thus $\mathcal{T}_1 \subset \mathcal{T}_2$.

$\Rightarrow \mathcal{T}_1 = \mathcal{T}_2$.

$\square$

Product Topology.

<u>Def</u>. (Product Topology). Let X, Y be topological spaces. The product topology on $X \times Y$ is the topology with basis $\{U \times V \mid U \subset X \text{ open}, V \subset Y \text{ open}\}$.
Given a basis $\mathcal{B}_X$ for the topology of X, and a basis $\mathcal{B}_Y$ for the topology of Y, the sets of the form $\{U \times V \mid U \in \mathcal{B}_X, V \in \mathcal{B}_Y\}$ is a (smaller) basis for the same topology.

e.g. $X = Y = \mathbb{R}$ with standard topology.
The basis for the product topology on $\mathbb{R} \times \mathbb{R}$ is $\{(a,b) \times (c,d)\}$.
Claim: This is just another way of describing the usual topology on $\mathbb{R}^2$.
$\mathcal{T}_{prod} = \mathcal{T}_{stand}$.
proof: We use the fact that $\mathcal{T}_{stand} = \mathcal{T}_{\rho}$.
$\forall \vec{x} \in \mathbb{R}^2$, $\vec{x} \in B_\rho(\vec{x}, r) = (x_1 - r, x_1 + r) \times (x_2 - r, x_2 + r) \subset B(\vec{x}, \sqrt{2} r)$
So $\mathcal{T}_{stand} \subset \mathcal{T}_{\rho}$.
But open rectangles are also open in standard topology. Given any rectangle, given any point in it, there is a square around in it
$\Rightarrow \mathcal{T}_{product} \subset \mathcal{T}_{standard}$. $\square$

<u>Def</u> (Projection Map) Given $X \times Y$. Let $\pi_1 : X \times Y \to X$, $\pi_2 : X \times Y \to Y$ be $\pi_1(x \times y) = x$, $\pi_2(x \times y) = y$.

Claim: π_1 and π_2 are continuous. Indeed. For $U \subset X$ open, $\pi_1^{-1}(U) = U \times Y$ open in product topology.
For $V \subset Y$ open, $\pi_2^{-1}(V) = X \times V$ open in product topology.

We want $U \times Y$ open $\forall U \subset X$ open; $X \times V$ open $\forall V \subset Y$ open, in order that π_1, π_2 is open.
and then $U \times Y \cap X \times V = U \times V$ is open.

The product topology on $X \times Y$ is the coarsest topology for which π_1 and π_2 are continuous.

<u>Lemma</u>. If $A \subset X$ and $B \subset Y$, then the subspace topology on $A \times B$ as the subspace of $X \times Y$ agrees with the product topology coming from the subspace topologies of A and B.
proof: $U \times V$ where $U \subset X$ open and $V \subset Y$ open is the general basic element for $X \times Y$.
Therefore $(U \times V) \cap (A \times B)$ is the general basis element for the subspace topology on $A \times B$.

$$(U \times V) \cap (A \times B) = (U \cap A) \times (V \cap B)$$

Since $U \cap A$ and $V \cap B$ are the general open sets for the subspace topologies on A and B, the set $(U \times V) \cap (A \times B)$ is the general basis element for the product topology coming from the subspace topology of A and B.
The bases agree $\Rightarrow$ topologies are the same.

<u>Lemma</u>. If (X, d_X), (Y, d_Y) are metric spaces, then define a metric ρ on $X \times Y$ as
$\rho(x \times y, x' \times y') = \max\{d_X(x, x'), d_Y(y, y')\}$. This is a metric. Product topology = metric topology.

Closed Sets and Limit Points

Def. (Closed). Let X be a topological space. A set $A \subset X$ is closed iff $X-A$ is open.

e.g. in $\mathbb{R}$: $[a,b]$ closed since $\mathbb{R}-[a,b]=(-\infty,a)\cup(b,+\infty)$ is open.
$(-\infty,a]$ closed since $\mathbb{R}-(-\infty,a]=(a,+\infty)$ is open.

Notes: a). Not every set is open or closed.
b). ϕ, X are both open and closed.
c). In $\mathbb{R}$: $[0,+\infty)$ is open. It is also closed since $\mathbb{R}-[0,+\infty)=(-\infty,0)=\bigcup_{n\in\mathbb{N}}[n,0)$ open
d). Arbitrary intersection of closed sets are closed
$$X-\bigcap_{\alpha\in I}A_\alpha=\bigcup_{\alpha\in I}(X-A_\alpha) \quad \text{open} \Rightarrow \bigcap_{\alpha\in I}A_\alpha \text{ closed.}$$
e). Finite union of closed sets are closed.
$$X-\bigcup_{i=1}^{n}A_i=\bigcap_{i=1}^{n}(X-A_i) \quad \text{open} \Rightarrow \bigcup_{i=1}^{n}A_i \text{ closed.}$$

Def (Limit Point). Let $A\subset X$. A point $x\in X$ is called a limit point of A if for every open set $U\subset X$, $x\in U$, $\exists\, a\in A-\{x\}$ such that $a\in U$. ($a\in U\cap(A-\{x\})\neq\phi$)

e.g. $A=(0,1)\subset\mathbb{R}$. $\frac{1}{2}$, 0 are limit points, -4 is not.
$X=\mathbb{R}$. $A=\mathbb{Z}$. First it is easy to see that non-integers are not limit points.
Second, for $n\in\mathbb{Z}$, it is not a limit point. Given $U=(n-\frac{1}{2},n+\frac{1}{2})$, there is no $a\in\mathbb{Z}-\{n\}$ such that $a\in U=(n-\frac{1}{2},n+\frac{1}{2})$

Theorem. A is closed iff it contains all of its limit points.

proof: ($\Rightarrow$). If A is closed, then $X-A$ is open. Given $x\in X-A$, there exists open set U, $x\in U$ such that $U\cap A=\phi \Rightarrow U\cap(A-\{x\})=\phi$. Thus no points in $X-A$ can be a limit point of A. Therefore A contains all its limit point.

($\Leftarrow$). Suppose A contains all of its limit points. WTS $X-A$ is open.
Let $x\in X-A$. Since x is not a limit point, $\exists\, U_x$ open, $x\in U_x$,
$$U_x\cap(A-\{x\})=\phi \Rightarrow U_x\cap A=\phi \Rightarrow U_x\subset X-A.$$
Therefore $X-A$ is open. A is closed. $\square$

Def. (Closure) For $A\subset X$, the closure of A ($\bar{A}$) is the smallest closed set that contains A.
It is the intersection of all closed sets that contains A.

- $\bar{A}$ is closed.
- If C is any closed set s.t. $A\subset C$, then $\bar{A}\subset C$.

Def. (Interior). The interior of A ($\mathrm{int}(A)$ or A°) is the union of all open sets that are contained in A.

- $\mathrm{int}(A)\subset A\subset\bar{A}$.

Lemma $x\in\bar{A} \Leftrightarrow \forall$ open set U s.t. $x\in U$, $U\cap A\neq\phi$.

proof: (a). if $x\in A$ then obvious.

(b). if $x \notin A$. then, suppose $\exists$ open set U, $x \in U$, $U \cap A = \phi$. Then, $A \subset X - U$. Since $X - U$ is closed, $\bar{A} \subset X - U$. Then $x \notin \bar{A}$. Contradiction. □.

<u>Proposition</u>: $\bar{A} = A \cup \{\text{limit points of } A\}$.

proof. If $x \in \{\text{limit points of } A\}$, then $\forall U \ni x$, U open, $U \cap (A - \{x\}) \neq \phi \Rightarrow U \cap A \neq \phi$. Thus $x \in \bar{A}$

Therefore $A \cup \{\text{limit points of } A\} \subset \bar{A}$.

If $x \in \bar{A}$, then, (a). $x \in A$ is trivial

(b). $x \notin A$. Since $x \in \bar{A}$, then $\forall U \ni x$ open, $U \cap A \neq \phi$. Thus, $\exists a \in A$, $a \in U \cap \{A - \{x\}\}$

Therefore, x is a limit point of A.

Therefore $\bar{A} \subset A \cup \{\text{limit points of } A\}$ □

In general: A is closed $\Leftrightarrow A = \bar{A}$

$x \in \bar{A} \Leftrightarrow \forall$ open set $U \ni x$, $U \cap A \neq \phi$

$\bar{A} = A \cup \{\text{limit points of } A\}$

$\text{int}(A) = X - \overline{(X - A)}$

e.g. in $\mathbb{R}$. $\overline{(a, b)} = [a, b]$.

<u>Def</u> (Boundary). Boundary of A $\text{bd}(A) := \bar{A} - \text{int}(A)$

<u>Lemma</u>. $x \in \text{bd}(A)$ iff every open set $U \ni x$ intersects both A and $X - \bar{A}$.

proof: Let $x \in \bar{A} - \text{int}(A)$. Then $x \in \bar{A}$, thus $\forall U \ni x$ open, $U \cap A \neq \phi$.

Suppose that $\exists U_x \ni x$, $U_x \cap (X - \bar{A}) = \phi$, then $U_x \subset \bar{A}$, and $\bar{A}$ is open. Contradiction. □

e.g. $A = \mathbb{Z} \subset \mathbb{R}$. $\bar{A} = A$. $\text{int}(A) = \phi \Rightarrow \bar{A} = A = \text{bd}(A)$.

$[0, 1] \times \{0\} \subset \mathbb{R}^2$ is closed, $\text{int} = \phi$.

$X = \mathbb{R}$. $A = \mathbb{Q}$. Claim: $\bar{A} = \mathbb{R}$.

proof: WTS $\bar{A} \subset \mathbb{R}$. Let $x \in \bar{A}$. then since $\bar{A}$ is the intersection of all the closed sets in $\mathbb{R}$ containing A. $x \in \mathbb{R}$.

WTS $\mathbb{R} \subset \bar{A}$. Let $x \in \mathbb{R}$. Then, for any $a \in \mathbb{R}$, $b \in \mathbb{R}$, $a < x < b$, $(a, b) \cap A \neq \phi$ by the density of rational numbers. $x \in \bar{A}$. □

<u>Def</u> (Dense) A set A is dense in X if $\bar{A} = X$.

$\Rightarrow \mathbb{Q}$ is dense in $\mathbb{R}$.

601 4

601 4 $\quad$ $15 \times 2652 = 39780$

6012 4 $\quad$ $12 \times 2652 = 31824$

6011 3

Hausdorff Spaces

Motivation:

<u>Def</u> (Convergence of sequence in analysis). Given $x_1, x_2, x_3, \cdots$, $\lim_{n\to\infty} x_n = L$ if $\forall \varepsilon > 0$, $\exists N$ s.t $|x_n - L| \le \varepsilon$ $\forall n \ge N$.

<u>Def</u> (Convergence of sequence in topology). Given $x_1, x_2, x_3, \cdots$, we say the sequence converges to x if for every open set $U \ni x$, $\exists N > 0$ such that $x_n \in U$ for all $n > N$.

In analysis, if $x_n \to x$, such x is unique. However, consider $X = \{a, b, c\}$, $\mathcal{T} = \{\{a,b,c\}, \{a,b\}, \{b,c\}, \{b\}, \phi\}$.
Given a sequence $x_n = b$ $\forall n$, by definition it converges to a, b, and c.

To rule out these strange cases, we introduce Hausdorff space.

<u>Def</u>. A topological space is called Hausdorff if for every $x, y \in X$ with $x \neq y$, there exists open sets U, V such that $x \in U$, $y \in V$, $U \cap V = \phi$.

<u>Proposition</u>. Any metric space is Hausdorff.

proof: Let $x, y \in X$ with $x \neq y$. Let $\varepsilon := \frac{d(x,y)}{2}$.

WTS $B_d(x, \varepsilon) \cap B_d(y, \varepsilon) = \phi$.

If $\exists z \in B_d(x, \varepsilon)$ and $z \in B_d(y, \varepsilon)$, then

$$d(x,y) \le d(x,z) + d(z,y) < 2\varepsilon = d(x,y)$$

Contradiction! $\square$.

<u>Proposition</u>. Let X be Hausdorff. If a sequence $\{x_n\}$ converges to x also to y, then $x = y$.

proof: if $x \neq y$, then we can choose $U \ni x$, $V \ni y$ open such that $U \neq \phi$, $V \neq \phi$, $U \cap V = \phi$.
Since convergence, $\exists N$ such that $\forall n > N$, $x_n \in U$, $x_n \in V$. Contradiction! $\square$.

<u>Lemma</u>. If X is Hausdorff, then all 1-point sets are closed.

proof: Fix $x_0 \in X$. For all y such that $y \neq x_0$, $\exists U_y, V_y$ open such that $x_0 \in U_y$, $y \in V_y$, $U_y \cap V_y = \phi$.
Thus $V_y \subset X - \{x_0\}$. Thus $X - \{x_0\}$ is open, $\{x_0\}$ is closed. $\square$.

<u>Def</u>. (T_1 and T_2). A space is called T_1 if 1-point sets are closed; called T_2 if it is Hausdorff.
$T_2 \Rightarrow T_1$. $T_1 \not\Rightarrow T_2$. Since:

<u>Proposition</u>. If X is infinite with finite complement topology, then X is T_1 but not T_2.

proof. $\mathcal{T}_{finite} = \{U \subset X \mid X - U$ is finite, or $U = \phi\}$.
Let $U = X - \{x_0\}$ for some $x_0 \in X$. Then $X - U = \{x_0\}$ is finite. Thus U is open, $\{x_0\}$ is closed.

Suppose that X is Hausdorff. Then, given arbitrary $x, y \in X$ with $x \neq y$, $\exists U \ni x$, $V \ni y$ open such that $U \cap V = \phi$. Since $X - U$ and $X - V$ is finite, let $A := X - U$, $B := X - V$, then $U = X - A$, $V = X - B$. Then, since $U \cap V = \phi$,

$$(X - A) \cap (X - B) = X - (A \cup B) = \phi$$
$$\Rightarrow X = A \cup B$$

However, $A \cup B$ is finite. Contradiction! $\square$.

<u>Def</u> ($T_{1.5}$) A space is $T_{1.5}$ if for every $x \neq y$, $\exists$ open sets U, V s.t. $x \in U$ and $x \notin V$, $y \in V$ and $y \notin U$.
$T_2 \Rightarrow T_{1.5} \Rightarrow T_1$.

Continuous Functions

Def. (Continuous functions). Given topological spaces X and Y. A function $f: X \to Y$ is continuous if for every $U \subset Y$ open, $f^{-1}(U)$ is open in X.

Facts about continuous functions.

1. Any constant function is continuous.
   $f: X \to Y$, $f(x) = y_0\ \forall x$. Then, $f^{-1}(U) = \begin{cases} X & \text{if } y_0 \in U \\ \phi & \text{otherwise.} \end{cases}$

2. Identity Function $id_X: X \to X$ $\; id_X(x) = x$ is continuous.

3. If $f: X \to Y$, $g: Y \to Z$ both continuous, then $g \circ f: X \to Z$ is continuous.
   for $U \subset Z$ open, $(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U))$.

4. If $A \subset X$ and $f: X \to Y$ is continuous, then $f|_A : A \to Y$ is continuous. For $x \in A$: $f|_A(x) = f(x)$
   For $U \subset Y$ open, $f^{-1}|_A(U) = A \cap f^{-1}(U)$ open in subspace topology.

   inclusion function: $i_A : A \to X$: $i_A(x) = x$ if $x \in A$.

5. If $f: X \to Y$ continuous, $B \subset Y$ is a subspace, $\operatorname{Im}(f) \subset B$, then f is continuous as a function $X \to B$.

6. Let $f: X \to Y$. If X has the discrete topology, then any function $f: X \to Y$ is continuous.
   If X has the indiscrete topology, then very few are (constant function is).

   If $f: X \to Y$ is continuous, and we chose a different topology on X that is finer, $f: X' \to Y$ is still continuous. If we choose a coarser topology on Y, f is still continuous.

<u>Theorem.</u> Let X, Y be topological spaces. TFAE.

(1). f is continuous
(2). $\forall A \subset X$, $f(\bar{A}) \subset \overline{f(A)}$
(3). $\forall$ closed set $B \subset Y$, $f^{-1}(B)$ is closed in X
(4). For each $x \in X$ and each open set $f(x) \in V$, there is an open set $U \in X$ such that $f(U) \subset V$.

proof: (1) $\Rightarrow$ (2) WTS if $x \in \bar{A}$, then $f(x) \in \overline{f(A)}$.
Let V be open in Y such that $f(x) \subset V$. Then $f^{-1}(V)$ is open in X and $x \in f^{-1}(V)$. Since $x \in \bar{A}$, $f^{-1}(V)$ intersects A in some point y. Thus, V intersects $f(A)$ in some point $f(y)$. Thus $f(x) \in \overline{f(A)}$.

(2) $\Rightarrow$ (3) Let B be closed in Y. Let $A := f^{-1}(B) = \{x \mid f(x) \in B\}$. WTS A is closed, that is, $A = \bar{A}$.
WTS $\bar{A} \subset A$.
Since $f(A) = f(f^{-1}(B)) \subset B$, if $x \in \bar{A}$, then
$$f(x) \in f(\bar{A}) \subset \overline{f(A)} \subset \bar{B} = B$$
then $x \in f^{-1}(B) = A$. Thus $\bar{A} \subset A$. A is closed.

(3) $\Rightarrow$ (1). Let $A \subset Y$ be open. Then $Y - A$ is closed. Then $f^{-1}(Y - A) = X - f^{-1}(A)$ is closed, $f^{-1}(A)$ is open. Thus f is continuous.

$(1) \Rightarrow (4)$. Let $x \in X$, V be open in Y such that $f(x) \subset V$. Since f is continuous, $f^{-1}(V)$ is open, and $x \in f^{-1}(V)$. Then, $\exists$ open set $U \subset X$ s.t. $x \in U \subset f^{-1}(V)$. Then $f(U) \subset V$.

$(4) \Rightarrow (1)$. Let $V \subset Y$ be open. Let $x \in X$ be $x \in f^{-1}(V)$. Then $f(x) \in V$. By hypothesis, $\exists$ open set $x \in U_x$, $f(U_x) \subset V$. Thus, $U_x \subset f^{-1}(V)$. Thus $f^{-1}(V) = \bigcup_{x \in f^{-1}(V)} U_x$, it is open. $\square$.

Finite Products

Def (Product topology). Let $X_1, \cdots, X_n$ be topological spaces. The product topology on $X_1 \times \cdots \times X_n$ has the basis $\{U_1 \times \cdots \times U_n \mid U_i \subset X_i \text{ is open}\}$.

Let $\pi_i : X_1 \times \cdots \times X_n \to X_i$ be the projection map. For open $U \subset X_i$, $\pi_i^{-1}(U) = X_1 \times \cdots \times X_{i-1} \times U \times X_{i+1} \times \cdots \times X_n$. open. It is continuous.

Given $f_1 : X \to Y_1, \cdots, f_n : X \to Y_n$, define $f_1 \times \cdots \times f_n : X \to Y_1 \times \cdots \times Y_n$ by $(f_1 \times \cdots \times f_n)(x) = f_1(x) \times \cdots \times f_n(x)$

Theorem. $f_1 \times \cdots \times f_n$ is continuous iff each f_i is continuous.

proof: ($\Rightarrow$). $f_i = \pi_i \circ f = \pi_i(f_1 \times \cdots \times f_n)$

f continuous $\Rightarrow$ f_i continuous.

($\Leftarrow$). Assume that all f_i is open. WTS for any basic open set $U \subset Y_1 \times \cdots \times Y_n$, $f^{-1}(U)$ is open.

$U = U_1 \times \cdots \times U_n$ where $U_i \subset Y_i$ is open.

$$\begin{aligned} f^{-1}(U) &= f^{-1}(U_1 \times \cdots \times U_n) \\ &= \{x \mid f_i(x) = U_i \ \forall i\} \\ &= \bigcap_{i=1}^{n} \{x \mid f_i(x) = U_i\} \\ &= \bigcap_{i=1}^{n} f_i^{-1}(U_i) \quad \text{open.} \end{aligned}$$

$\square$

Claim. If X is a topological space, given $f, g : X \to \mathbb{R}$, we can define $f \pm g : X \to \mathbb{R}$ by $(f \pm g)(x) = f(x) \pm g(x)$, $f \cdot g$. If $g(x) \neq 0$, we can also define f/g.

If f and g are continuous, then so are $f \pm g$, $f \cdot g$, f/g.

proof. for $f+g$: define $+ : \mathbb{R}^2 \to \mathbb{R}$ by $+(x, y) = x + y$, $+$ is continuous.

Given f, g:

$$\begin{tikzcd} X \arrow[r, "f \times g", "\text{ctn}"'] \arrow[rr, bend right, "\text{ctn.}"'] & \mathbb{R} \times \mathbb{R} \arrow[r, "+", "\text{ctn}"'] & \mathbb{R} \end{tikzcd}$$

By this we can also define $-$, $\cdot$.

For division $/ : \mathbb{R} \times (\mathbb{R} - \{0\}) \to \mathbb{R}$: given continuous f, g, $f : X \to \mathbb{R}$, $g : X \to \mathbb{R} - \{0\}$,

$$\begin{tikzcd} X \arrow[r, "f \times g", "\text{ctn}"'] \arrow[rr, bend right, "\text{ctn.}"'] & \mathbb{R} \times (\mathbb{R} - \{0\}) \arrow[r, "/", "\text{ctn}"'] & \mathbb{R} \end{tikzcd}$$

$\square$

The set of continuous functions $X \to \mathbb{R}$ is a commutative ring.

Lemma (The pasting lemma) Let $X = A \cup B$, where A and B are closed in X. $f : A \to Y$, $g : B \to Y$ be continuous. If $f(x) = g(x)$ $\forall x \in A \cup B$, then f and g combine to give a new continuous function $h : X \to Y$

$$h(x) = \begin{cases} f(x) & \text{if } x \in A \\ g(x) & \text{if } x \in B. \end{cases}$$

proof: Let $C \subset Y$ be closed.

$$\begin{aligned} h^{-1}(C) &= (A \cup B) \cap h^{-1}(C) \\ &= (A \cap h^{-1}(C)) \cup (B \cap h^{-1}(C)) \\ &= h^{-1}(f(A) \cap C) \cup h^{-1}(g(B) \cap C) \\ &= f^{-1}(C) \cup g^{-1}(C) \quad \text{is closed} \end{aligned}$$

Thus h is continuous.

$\square$

Q: If $X = A \cup B$ in general, $f|_A$ and $f|_B$ are continuous, is f necessarily continuous?

A: No. Consider on $\mathbb{R}$: $f(x) = \begin{cases} 1, & x > 0 \\ 0, & x \le 0 \end{cases}$ $\quad A = (0, +\infty)$ $\quad B = (-\infty, 0]$.

Theorem. (1) If $X = \bigcup_{i=1}^{n} A_i$ where A_i is closed, $f: X \to Y$ is a function such that $f|_{A_i}$ is continuous for each i, then f is continuous.

(2). If $X = \bigcup_\alpha A_\alpha$, each A_α is open, $f|_{A_\alpha}$ is continuous for each α, then f is continuous.

proof: (1). To prove that f is continuous, WTS $f^{-1}(\text{closed}) = \text{closed}$.

Given $C \subset Y$ closed, $f^{-1}(C) = (f^{-1}(C) \cap A_1) \cup \cdots \cup (f^{-1}(C) \cap A_n)$

$$= f|_{A_1}^{-1}(C) \cup \cdots \cup f|_{A_n}^{-1}(C).$$

Each $f|_{A_i}^{-1}(C) = A_i \cap B_i$ for some $B_i \subset X$ closed, thus $f^{-1}(C)$ is closed.

(2). For U open in Y,

$$\begin{aligned} f^{-1}(U) &= \bigcup_\alpha (f^{-1}(U) \cap A_\alpha) \\ &= \bigcup_\alpha f|_{A_\alpha}^{-1}(U) \end{aligned}$$

Each $f|_{A_\alpha}^{-1}(U) = A_\alpha \cap B_\alpha$ for some B_α open. thus $f^{-1}(U)$ is open.

$\square$.

Infinite Product Topology.

Def (Infinite Products). Given $\{X_\alpha\}_{\alpha\in J}$, J is some index set. Define $\prod_{\alpha\in J} X_\alpha = \{(x_\alpha) \mid x_\alpha \in X_\alpha\}$.

e.g. If $J = \mathbb{Z}_+$, $X_i = \mathbb{R}$, then $\mathbb{R}^\omega = \mathbb{R}\times\mathbb{R}\times\cdots$

$X^\omega = \{$all sequences of elements of $X\}$.

Note: $\mathbb{R}^\infty = \{\vec{x} \mid x_i = 0 \ \forall i \gg 0\}$

$\mathbb{R}^\omega = \prod_{i=1}^{\infty} \mathbb{R}$

Def: (Box and product topology). On $\prod_{\alpha\in J} X_\alpha$:

Option 1: Take as our basis $\{\prod_{\alpha\in J} U_\alpha \mid U_\alpha \subset X_\alpha \text{ open}\}$. In this topology, for $\mathbb{R}^\omega$, $(-1,1)\times(1,3)\times(0,\pi)\times\mathbb{R}\times(0,+\infty)\times\cdots$ would be a basis. This is called box topology.

Option 2: Take as our basis $\{\prod_{\alpha\in J} U_\alpha \mid U_\alpha \subset X_\alpha$ open, $U_\alpha = X_\alpha$ for all but finitely many $\alpha\}$.

e.g. in $\mathbb{R}^\omega$, $(-1,1)\times(1,3)\times(2,+\infty)\times\mathbb{R}\times\mathbb{R}\times\cdots$ is one basis.

The generated topology is called the product topology.

Proposition. Product Topology $\subsetneq$ Box topology.

proof: $\subset$ is obvious.

$\neq$: Claim: in $\mathbb{R}^\omega$, $U = (-1,1)\times(-1,1)\times(-1,1)\times\cdots$ is open in box topology, not in product topology.

Indeed. if it were, then for $\vec{0}\in U$, $\exists\, B = V_1\times\cdots\times V_k\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\cdots$ such that $\vec{0}\in B\subset U$.

However, $(0,0,\ldots,\underset{\substack{\uparrow\\ k}}{0},3,3,\ldots)\in B$, not in U. $B\not\subset U$. Contradiction. $\square$.

Def (Projection map). Given $\prod_{\alpha\in J} X_\alpha$, for each index β, we have a projection map $\pi_\beta : \prod_{\alpha\in J} X_\alpha \to X_\beta$:

$$\pi_\beta(\vec{x}) = x_\beta.$$

e.g. in $\mathbb{R}^\omega$, $\pi_i^{-1}(U) = \mathbb{R}\times\cdots\times U\times\mathbb{R}\times\cdots$ If U is open in $\mathbb{R}$, then $\pi_i^{-1}(U)$ is open in both product and box topology.

$\Rightarrow$ **Theorem** π_β is continuous using either topology.

Theorem. Given functions $f_\alpha : Z \to X_\alpha$, the product function $f := \prod_{\alpha\in J} f_\alpha : Z \to \prod_{\alpha\in J} X_\alpha$ is continuous iff f_α is continuous, using product topology but not the box topology.

proof: $(\Rightarrow)$. $f_\alpha = \pi_\alpha\circ f$.

$(\Leftarrow)$. Suppose f_α is continuous for each α. Let U be open in $\prod_{\alpha\in J} X_\alpha$ using box topology.

WTS $f^{-1}(U)$ is open in Z.

$U = U_1\times\cdots\times U_k\times X_{k+1}\times X_{k+2}\times\cdots$, U_i open in X_i

$$f^{-1}(U) = \{x\in Z \mid f_1(x)\in U_1,\cdots f_k(x)\in U_k, f_{k+1}(x)\in X_{k+1},\cdots\}$$
$$= \bigcap_{i=1}^{k}\{x\in Z \mid f_i(x)\in U_i\}$$
$$= \bigcap_{i=1}^{k} f_i^{-1}(U_i) \quad \text{open}.$$

Therefore f is continuous. $\square$

Homeomorphism & Embedding

Def (Homeomorphism). A map $f: X \to Y$ is called a homeomorphism if (1) f is bijective (2). f is continuous. (3) f^{-1} is continuous.

Def (Homeomorphic) $X \cong Y$ if $\exists$ homeomorphism between them.

Given that f is a bijection, f is continuous means $\forall U \subset Y$ open, $f^{-1}(U)$ is open in X. f^{-1} is continuous means that given V open in X, $(f^{-1})^{-1}(V) = \{y \in Y \mid f^{-1}(y) \in V\} = \{y \in Y \mid y \in f(V)\} = f(V)$ is open in Y. Doing relab le between X and Y is one-to-one.

e.g. ① $(a, b) \cong (c, d)$ by $f(x) = \frac{d-c}{b-a} x + \frac{bc-ad}{b-a}$

② $[0, 1) \cong (0, 1]$ by $f(x) = 1 - x$

③ $(-1, 1) \cong \mathbb{R}$. by $f(x) = \frac{x}{1-x^2}$ or $f(x) = \tan\left(\frac{x}{\pi/2}\right)$

④. $\mathbb{R} \cong (0, +\infty)$ by $f(x) = e^x$. $f^{-1}(x) = \ln(x)$.

⑤ open ball in $\mathbb{R}^n \cong \mathbb{R}^n$

⑥ $[0, 1) \cong [0, +\infty)$ by $f(x) = \frac{x}{1-x^2}$.

Therefore, homeomorphism can lose information on boundedness.

Note that not every continuous (in one direction) bijection is a homeomorphism.
e.g. ① $id: X \to X'$ where X, X' have different topology on X may not be a homeomorphism.
② let $f: \mathbb{R} \to S^1$ be $f(t) = (\cos(2\pi t), \sin(2\pi t))$.
let $g = f|_{[0,1)}$. Then $g: [0,1) \to S^1$ is a bijection. g is continuous, but g^{-1} is not. Take $[0, \frac{1}{4})$ open in the subspace topology of $[0, 1)$. Then $(g^{-1})^{-1}(U) = g(U) =$, not open in subspace topology of S^1, for $\nexists$ open arc containing $(1, 0)$ that is contained in $(g^{-1})^{-1}(U)$.

$\nexists$ homeomorphism between $[0, 1)$ and S^1 . $[0, 1) \not\cong S^1$. $[0, 1)$ not compact, S^1 compact

Def (Embedding). A function $f: X \to Y$ is called an embedding if (1) it is injective (2) it restricts to a homeomorphism $X \to \mathrm{Im}(f)$, where $\mathrm{Im}(f)$ is given the subspace topology.

$$\begin{array}{ccc} f: X & \longrightarrow & Y \quad \text{injective} \\ & {}_{\text{bijection}}\searrow & \cup \\ & & \mathrm{Im}(f) \end{array}$$

e.g. View g from the last example as $g: [0, 1) \to \mathrm{Im}(g) = S^1 \subset \mathbb{R}^2$. It is injective continuous map, but not an embedding. (not a homeomorphism $[0, 1) \to S^1$).

Proposition: Given X, Y, $g: X \to Y$ continuous. Define $f: X \to X \times Y$ by $f(x) = (x, g(x))$. Then this is an embedding of f to $X \times Y$.

proof: First, it is trivial to see that f is injective.

Then, $Im(f) = \{(x, g(x)) \mid x \in X\}$
$= \{(x, y) \mid y = g(x)\}$
$= graph(g)$

Indeed, the inverse function $graph(g) \to f$ is $(x, y) \mapsto x = \pi_1|_{graph(g)}$ is continuous. thus f restricts to a homeomorphism $X \to Im(f)$.

e.g. In $X \times X$: $\Delta = \{(x, x) \mid x \in X\} = graph(id)$.

Often, we want to know whether a space can be embedded into a familiar space. for example, $\mathbb{R}^n$.

Proposition: (a). If $X \cong Y$, X is Hausdorff, then Y is Hausdorff.

(b). If X is Hausdorff, then any subspace of X is Hausdorff.

(c). If X is not Hausdorff and Y is, then X does not embed Y.

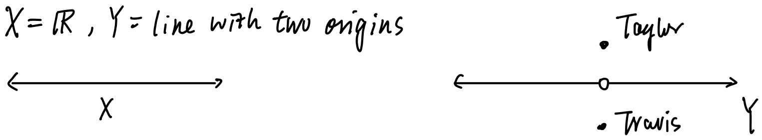
proof: (a). $\exists$ homeomorphism $f: X \to Y$.

Let $y_1, y_2 \in Y$. Then, since f is bijective, $f^{-1}(y_1)$, $f^{-1}(y_2) \in X$.
Since X is Hausdorff, $\exists$ U, V open, $f^{-1}(y_1) \in U$, $f^{-1}(y_2) \in V$, $U \cap V = \phi$.
$y_1 \in f(U)$, $y_2 \in f(V)$, $f(U) \cap f(V) = \phi$, both are open.
Therefore Y is Hausdorff.

(b). Trivial. Just take the subspace topology.

(c). Suppose it does. that is, $\exists$ $f: X \to Im(f) \subset Y$ a homeomorphism.
Since Y is Hausdorff, $Im(X)$ also is. Then, according to (a): $Im(f) \cong X$, $Im(f)$ is Hausdorff, it should be that X is also Hausdorff. Contradiction! $\square$

e.g. Let $X = \mathbb{R}$, Y = line with two origins



with basis ① (a, b) where $ab > 0$
② $(a, 0) \cup \{Taylor\} \cup (0, b)$
③ $(a, 0) \cup \{Travis\} \cup (0, b)$

define $f: \mathbb{R} \to Y$ $f(x) = \begin{cases} x, & x \neq 0 \\ Taylor, & x = 0 \end{cases}$ then $Im(f)$ is Hausdorff.

An application of embedding is knot theory.

Def. (knot) a knot is an embedding of S^1 into $\mathbb{R}^3$

e.g.

Quotient Topology

Def. (Binary Relation) Let X be a set. A binary relation is a function $X \times X \to \{T, F\}$ such that if $(x, y) \mapsto T$, we say $x \sim y$; if $(x, y) \mapsto F$, we say $x \nsim y$.

Def (Equivalence Relation). An equivalence relation is a relation $\sim$ satisfying

(1). $\forall x \in X$, $x \sim x$ (Reflective)

(2). $\forall x, y \in X$, $x \sim y \Rightarrow y \sim x$ (Symmetric).

(3). $\forall x, y, z \in X$, if $x \sim y$, $y \sim z$, then $x \sim z$ (transition).

Def. (Equivalence Classes) An equivalence relation determines a partition of X into equivalence classes. For $x \in X$, $[x] = \{y \mid y \sim x\}$.

If $x \sim y$, then $[x] = [y]$. If $x \nsim y$, then $[x] \cap [y] = \emptyset$.

e.g. $X = \mathbb{Z}$. $x \sim y$ iff $3 \backslash (x - y)$ ($x-y$ is divisible by 3).

$\sim$ is an equivalence relation, since: $x \sim x$: $x - x = 0$, $3 \backslash 0 = 0$

$x \sim y \Rightarrow 3 \backslash (x-y) \Rightarrow 3 \backslash (y - x) \Rightarrow y \sim x$.

$x \sim y \Rightarrow 3 \backslash (x-y)$. $y \sim z \Rightarrow 3 \backslash (y - z)$.

$x - z = x - y + y - z$. $3 \backslash x - z \Rightarrow x \sim z$.

E. Classes: $[0] = \{\cdots -6, -3, 0, 3, 6, \cdots\}$

$[1] = \{\cdots -5, -2, 1, 4, 7, \cdots\}$

$[2] = \{\cdots -4, -1, 2, 5, 8, \cdots\}$.

Def (Quotient Space) $X/\sim$ (or X^*) is the set of equivalent classes of X.

Thus, there is a natural surjective map $p: X \to X/\sim$: $p(x) = [x]$.

Here we first define some general notions.

Def. (Quotient map). Let X, Y be topological spaces, $p: X \to Y$ be a surjective map.

It is a quotient map provided that $U \subset Y$ is open iff $p^{-1}(U)$ is open in X.

Equivalently, it can be defined as $A \subset Y$ is closed iff $p^{-1}(A)$ is closed in X.

Def (Saturated). $C \subset X$ is saturated w.r.t. the surjective map $p: X \to Y$ if $\forall p^{-1}(\{y\})$ such that $p^{-1}(\{y\}) \cap C \neq \emptyset$, $p^{-1}(\{y\}) \subset C$.

In other words, $C \subset X$ is saturated w.r.t. p if it is the complete preimage of some set in Y.

i.e. $\exists U \subset Y$, $C = p^{-1}(U)$.

Proposition. p is a quotient map $\Leftrightarrow$ p is continuous & p maps saturated open sets in X to open sets in Y.

proof: ($\Rightarrow$). Continuity is obvious. U open $\Rightarrow p^{-1}(U)$ open.

$A \subset X$ saturated open. Then $A = p^{-1}(B)$ for some $B \subset Y$. Also $p(A)$ is open in Y

$p(A) = p(p^{-1}(B)) = p(\{x \mid p(x) \in B\}) \subset B$

Also, since p is surjective, $\forall x \in B$, $\exists y \in A$ s.t. $p(y) = x \in p(A)$, $\Rightarrow B \subset P(A)$

Then $B = p(A)$ is open.

($\Leftarrow$). Suppose $V \subset Y$. $p^{-1}(V)$ is open in X. WTS V is open in Y.
First, notice that $p^{-1}(V)$ is automatically saturated w.r.t. p.
Then, $p(p^{-1}(V))$ is open. However, $p(p^{-1}(V)) = V$ since p is surjective onto its image.
continuous $\Rightarrow$ entire Y is p's image.
Thus V is open in Y. $\square$

<u>Def</u>. $f: X \to Y$ is an open (closed map) if $\forall U \subset X$ open (closed), $f(U)$ is open (closed) in Y.

<u>Proposition</u>: Surjective continuous open (closed) maps are quotient maps.
proof: obvious, just use the definition.

<u>Def</u> (Quotient Topology) Let X be a space. A be a set. If $p: X \to A$ is a surjective map, then $\exists$ one topology $\mathcal{T}$ on A relative to which p is a quotient map. $\mathcal{T}$ is the quotient topology of p.

$\mathcal{T} = \{U \subset A \mid p^{-1}(U) \text{ is open in } X\}$.

<u>Proposition</u>: $\mathcal{T}$ is a topology.
proof: ϕ and A are open, $p^{-1}(\phi)$, $p^{-1}(A)$ is open.
Let $\{U_\alpha\}_{\alpha \in J}$ be open sets. Then
$$p^{-1}\left(\bigcup_{\alpha \in J} U_\alpha\right) = \bigcup_{\alpha \in J} p^{-1}(U_\alpha) \quad \text{is open.}$$
Let $\{U_i\}_{i=1}^{n}$ be open sets. Then
$$p^{-1}\left(\bigcap_{i=1}^{n} U_\alpha\right) = \bigcap_{i=1}^{n} p^{-1}(U_\alpha) \quad \text{is open.}$$
$\square$

Back to our main purpose:

Let X be a topological space. $\sim$ be an equivalence relation. $p: X \to X/\sim$: $p(x) = [x]$. Then,

<u>Def</u>. The quotient topology on $X/\sim$ is : $U \subset X/\sim$ is defined to be open if $p^{-1}(U)$ is open.

Therefore, through p: open set on $X/\sim$ $\longleftrightarrow$ saturated open set in X.

Note: A subset $U \subset X/\sim$ is a collection of equivalent classes, and the set $p^{-1}(U)$ is just the union of equivalent classes belonging to U.
Thus, a typical open set of $X/\sim$ is a collection of equivalent classes whose union is an open set of X, according to the definition $\mathcal{T} = \{U \mid p^{-1}(U) \text{ open in } X\}$.

<u>Theorem</u>: Let $p: X \to Y$ be a quotient map. Let $A \subset X$ be a subspace that is saturated wrt p. Let $q: A \to p(A)$ be the map obtained by restricting p to A.
(a). If A is either open or closed in X, then q is a quotient map.
(b). If p is either an open or closed map, then q is a quotient map.

proof: First we check the following equations.

① $q^{-1}(V) = p^{-1}(V)$ if $V \subset p(A)$.

② $p(U \cap A) = p(U) \cap p(A)$ if $U \subset X$.

①: Since $V \subset p(A)$, A is saturated, by def. we have $p^{-1}(V) \subset A$.
Also since $q = p|_A$, we have $p^{-1}(V) = q^{-1}(V) = \{x \in A \mid p(x) \in V\}$.

②: Since $\forall$ $U, A \subset X$, $p(U \cap A) \subset p(U) \cap p(A)$, it suffices to prove that $p(U) \cap p(A) \subset p(U \cap A)$.
Suppose $y = p(u) = p(a)$ for some $u \in U$, $a \in A$. Since A is saturated, $a \in p^{-1}(p(a))$,
$\Rightarrow p^{-1}(p(a)) \subset A$. Then $A \supset p^{-1}(p(a)) = \{x \mid p(x) = p(a)\}$, so that $u \in A$. Then, $y = p(u) \in p(U \cap A)$.

Now suppose A is open or p is open. Given $V \subset p(A)$. Since p is a quotient map, we already have $U \subset A$ open $\Rightarrow q(U) = p|_A(U)$ is open in $p(A)$. Then it suffices to show that given $q^{-1}(V)$ open in A, V is open in $p(A)$.

Suppose A is open. Since $q^{-1}(V)$ is open in A, A is open in X, we have $q^{-1}(V)$ open in X. Since $q^{-1}(V) = p^{-1}(V)$, $p^{-1}(V)$ is also open in X, so that V is open in X by the fact that p is a quotient map. Then, V is open in $p(A)$ since $p(A) \cap V = V$.

Now suppose p is open. Since $q^{-1}(V) = p^{-1}(V)$ and $q^{-1}(V)$ is open in A, we have $p^{-1}(V) = U \cap A$ for some U open in X. Now, $p(p^{-1}(V)) = V$ since p is surjective. Then

$$V = p(p^{-1}(V)) = p(U \cap A) = \underbrace{p(U)}_{\text{open in } Y} \cap p(A)$$

Thus V is open in $p(A)$.

Similar for the case of closedness.

$\square$.

<u>Proposition</u>. Composition of two quotient maps is a quotient map.

proof: Let p, q be two quotient maps, U be open.

$$\begin{aligned}(q \circ p)^{-1}(U) &= \{x \mid q(p(x)) \in U\} \\ &= \{x \mid p(x) \in q^{-1}(U)\} \\ &= \{x \mid p(x) \in \{x \mid q(x) \in U\}\} \\ &= p^{-1}(\{x \mid q(x) \in U\}) \\ &= p^{-1}(q^{-1}(U)), \text{ it is open.}\end{aligned}$$

Let $(q \circ p)^{-1}(U)$ be open, then

$$U = q(p(p^{-1}(q^{-1}(U)))) = q(p((q \circ p)^{-1}(U))) \text{ is also open.}$$

$\square$.

However, the cartesian product of two quotient maps may not be a quotient map. One need local compactedness or both maps being open map.

Theorem Let $p: X \to Y$ be a quotient map. Let Z be a space and let $f: X \to Z$ be a map that is constant on each set $p^{-1}(y)$ $\forall y \in Y$. Then, f induces a map: $\bar{f}: Y \to Z$ such that $\bar{f} \circ p = f$. $\bar{f}$ is continuous iff f is continuous, $\bar{f}$ is a quotient map iff f is a quotient map.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "p"'] & Z \\ Y \arrow[ur, "\bar{f}"'] & \end{tikzcd}$$

proof: $\forall y \in Y$, $f(p^{-1}(y))$ is a one-point set in Z. Denote this point as $\bar{f}(y)$, then we have defined a map $\bar{f}: Y \to Z$ by $\bar{f}(p(x)) = f(x)$, since $\bar{f}(y) = f(p^{-1}(y))$, let $x \in p^{-1}(y)$, then $y = p(x)$, $\bar{f}(y) = f(x) \Rightarrow \bar{f}(p(x)) = f(x)$.

If $\bar{f}$ is continuous, then $f = \bar{f} \circ p$ is continuous.
Conversely, suppose f is continuous. Given $V \subset Z$ open, $f^{-1}(V)$ is open in X. However $f^{-1}(V) = p^{-1}(\bar{f}^{-1}(V))$. Since p is a quotient map, $\bar{f}^{-1}(V)$ is open in Y. Thus $\bar{f}$ is continuous.

If $\bar{f}$ is a quotient map, then $f = \bar{f} \circ p$ is a quotient map.
Conversely, suppose f is a quotient map. Since f is surjective, so is $\bar{f}$.
Let $V \subset Z$. Since $\bar{f}(y) = f(p^{-1}(y))$, f continuous, p^{-1} continuous $\Rightarrow \bar{f}$ continuous.

WTS V is open in Z if $\bar{f}^{-1}(V)$ is open in Y.

Since $\bar{f}^{-1}(V)$ is open in Y, we have $p^{-1}(\bar{f}^{-1}(V))$ is open in X since p is continuous. Since $f^{-1}(V) = p^{-1}(\bar{f}^{-1}(V))$, we have $f^{-1}(V)$ is open in X. Since f is a quotient map, V is open in Z. $\square$

Corollary: Let $f: X \to Z$ be a surjective continuous map. Let $Y = X/\sim$ be the following collection of equivalent classes of X. $X/\sim = \{f^{-1}(z) \mid z \in Z\}$ (in other word, $x \sim y$ in X iff $f(x) = f(y) = z$ for some $z \in Z$). Given $X/\sim$ the quotient topology.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "p"'] & Z \\ X/\sim \arrow[ur, "\bar{f}"'] & \end{tikzcd} \qquad f = \bar{f} \circ p$$

(a). The map f induces a bijective continuous map $\bar{f}: X/\sim \to Z$, which is a homeomorphism iff f is a quotient map.
(b). If Z is Hausdorff, then so is $X/\sim$.

proof: first, check that f is constant on $p^{-1}(y)$ $\forall y \in X/\sim$. For $[x] \in X/\sim$, $x' \sim y'$ if $p(x') = p(y') = [x]$
$p^{-1}([x]) \ni x', y'$. $\Rightarrow f(x') = f(y')$ by definition of the equivalence class here.
Then, by the previous theorem, f induces a continuous map $\bar{f}: X/\sim \to Z$. $\bar{f}$ is bijective since if $\bar{f}([x]) = \bar{f}([y])$, then $f(p^{-1}([x])) = f(p^{-1}([y])) \Rightarrow p^{-1}([x])$ and $p^{-1}([y])$ are in the same equivalence class, therefore $[x] = [y]$.

Suppose $\bar{f}$ is a homeomorphism, then both $\bar{f}$ and p are quotient, which makes $f = \bar{f} \circ p$ quotient.
Conversly, suppose f is a quotient map. By the previous theorem, $\bar{f}$ is a quotient map.
Also, since $\bar{f}$ is bijective, $\bar{f}$ is a homeomorphism.

Suppose Z is Hausdorff. Given $[x], [y] \in X/\sim$, $\bar{f}([x]) \neq \bar{f}([y])$. $\exists$ open $U \ni \bar{f}([x])$, $V \ni \bar{f}([y])$, $U \cap V = \phi$. Since $\bar{f}$ is bijectively continuous, $[x] \in \bar{f}^{-1}(U)$, $[y] \in \bar{f}^{-1}(V)$, $\bar{f}^{-1}(U) \cap \bar{f}^{-1}(V) = \phi$.
$\Rightarrow X/\sim$ Hausdorff. $\square$

e.g. $X = [0,1]$. $0 \sim 1$. $f : [0,1] \to \mathbb{R}^2$: $f(x) = (\cos(2\pi t), \sin(2\pi t))$.
f is a continuous surjective quotient map onto its image S^1. Conclusion: $X/\sim \cong S^1$.

$$\begin{tikzcd} {[0,1]} \arrow[r, "f"] \arrow[d, "p"'] & S^1 \\ {[0,1]/\sim} \arrow[ur, "\bar{f}\ \cong"'] & \end{tikzcd}$$

Generally, suppose given an $\sim$ on X. Figure out something about $X/\sim$. If we can write a function $f : X \to Z$ such that $x \sim x'$ iff $f(x) = f(x')$, then $\bar{f} : X/\sim \to Z$ is injective since $\bar{f}([x]) = f(x)$
Then $X/\sim$ is in bijection with $\mathrm{Im}(f)$. By the previous corollary, if f is continuous, $\bar{f}$ is.
If f is a quotient map, then $\bar{f}$ is a homeomorphism to $\mathrm{Im}(f)$ and an embedding into Z.

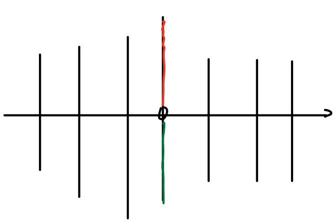
More examples for open & closed maps:
e.g. $\pi_1 : X \times Y \to X$ is open, but $\pi_1 : \mathbb{R}^2 \to \mathbb{R}$ is not closed. Consider $C = \{(x,y) \mid xy = 1\}$, $\pi_1(C) = \mathbb{R} - \{0\}$.
$f : \mathbb{R} \to \mathbb{R}$: $f(x) = x^2$ is closed but not open: $f(\mathbb{R}) = [0, +\infty)$.

e.g. $X = \mathbb{R}$, $\sim$: $x \sim y$ iff $x - y \in \mathbb{Q}$. Then, $X/\sim$ is uncountable.
Suppose countable. $X/\sim = \{[x_1], [x_2], \cdots\}$. $x_i - x_j \notin \mathbb{Q}$ $\forall i, j$. Then, since this is a partition of $\mathbb{R}$. $\mathbb{R} = \bigcup_{i=1}^{n} \{x \mid x \in [x_i]\}$, each $[x_i] = x + \mathbb{Q}$ countable $\Rightarrow \mathbb{R}$ countable. Contradiction!

The topology is the indiscrete topology. The only saturated open sets are ϕ and $\mathbb{R}$.
$\phi = p^{-1}(\phi)$ is trivial.
To see why $\mathbb{R}$ is, if $U \subset \mathbb{R}$ is non-empty saturated and open, then U contains some rational number $r \in \mathbb{Q}$. $r \in U = p^{-1}(V)$ for some $V \subset X/\sim$. $p(r) = [r] = \mathbb{Q} = p(U) \subset V$
However, $\underbrace{\mathbb{Q} = p^{-1}(\mathbb{Q})}_{\text{saturated}} = p^{-1}(p(U)) \subset U = p^{-1}(V)$. $\mathbb{Q} \subset U$. Also U open. $\Rightarrow U = \mathbb{R}$.

$\Rightarrow \mathcal{T}_{\text{quotient}} = \{U \mid p^{-1}(U) \text{ open}\} = \{U \mid p^{-1}(U) = \phi \text{ or } p^{-1}(U) = \mathbb{R}\} = \{\phi, X/\sim\}$
$\Rightarrow X/\sim$ is not Hausdorff even if X is Hausdorff. ☆.

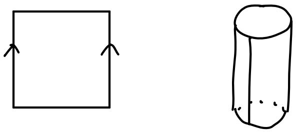
e.g. $X = \mathbb{R}^2 - \{0\}$. $\sim$: $(x,y) \sim (x', y')$ if ① $x = x'$ or ② if $x = x' = 0$. then $\mathrm{sign}(y) = \mathrm{sign}(y')$



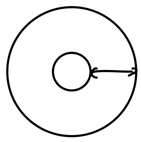
$X/\sim \cong$ line with two origines. $f(x,y) = \begin{cases} x & \text{if } x \neq 0 \\ \text{Taylor} & \text{if } x = 0, y > 0 \\ \text{Travis} & \text{if } x = 0, y < 0. \end{cases}$

e.g. $X=[0,1]^2$. $\sim$: $(0,y)\sim(1,y)$ $\forall y\in[0,1]$.

$X/\sim = I\times I/\sim$, An annulus in $\mathbb{R}^3$. $X/\sim \cong S^1\times[0,1]$ by $(x,y)\mapsto(\cos(2\pi x),\sin(2\pi x),y)$



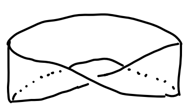
Or: it embeds into $\mathbb{R}^2$ by $(x,y)\mapsto((1+y)\cos(2\pi x),(1+y)\sin(2\pi x))$.



e.g. $X=[0,1]^2$, $\sim$: $(0,y)\sim(1,1-y)$ $\forall y\in[0,1]$

$X/\sim = I\times I/\sim$ is the Möbius strip.

Using cyclical coordinates, this can be embedded into $\mathbb{R}^3$.



e.g. $X=[0,1]^2$. $\sim$: $(0,y)\sim(1,y)$, $(x,0)\sim(x,1)$

$X/\sim = I\times I/\sim \cong ([0,1]/\sim)\times([0,1]/\sim)$

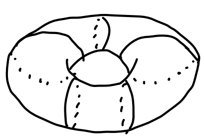
$\cong S^1\times S^1=\{(x,y,z,w)\mid x^2+y^2=1 \text{ and } z^2+w^2=1\}$.

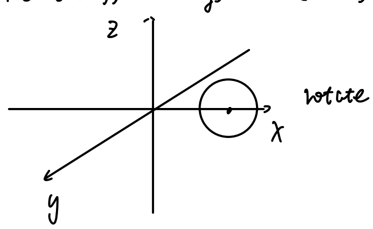
$\subset \mathbb{R}^2\times\mathbb{R}^2=\mathbb{R}^4$

Or we can write an embedding $I\times I/\sim\ \to\mathbb{R}^3$ by

$(\theta,r,z)=(2\pi y,\ 2+\cos(2\pi x),\ \sin(2\pi x))$

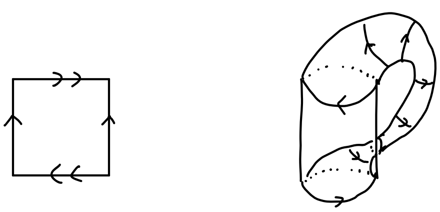
$\Rightarrow (x,y)\mapsto((2+\cos(2\pi x))\cos(2\pi y),\ (2+\cos(2\pi x))\sin(2\pi y),\ \sin(2\pi x))$





e.g. $X=[0,1]^2$. $\sim$: $(0,y)\sim(1,y)$, $(x,0)\sim(1-x,1)$

$X/\sim = I\times I/\sim$, a Klein bottle, K^2.



$\subset \mathbb{R}^3\times\{0\}\subset\mathbb{R}^4$.

We can write down a continuous function $K^2\to\mathbb{R}^3$ but not surjective. However, we can write down an embedding $K^2\to\mathbb{R}^4$ using trig function.

<u>Theorem</u>. $\nexists$ embedding $K^2\to\mathbb{R}^3$.

proof: using tools from algebraic topology.