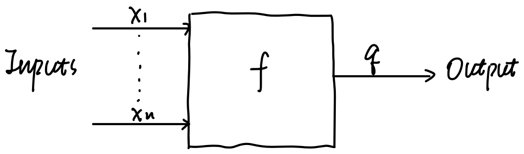
• Production Theory.



$x_i \geq 0 \ \forall i$
$q \geq 0$.

Production Function: $q = f(x_1 \cdots x_n)$.

Assumption: f is continuous and strictly increasing, with $f(\vec{0}) = 0$.

It differs from consumer theory in 2 ways: ① monotonic transformation of f matters.
② no income effect.

Productivity Measures:

①. Average Product of Input i: $AP_i = \frac{q}{x_i}$
②. Marginal Product of Input i: $MP_i = \frac{\partial f}{\partial x_i}$
$\Big\}$ Fixing everything else.

③ Return to Scale (Joint Productivity of Inputs).

$$\begin{cases} CRS: & f(tx) = t f(x) & \forall t > 0 \\ IRS: & f(tx) > t f(x) & \forall t > 1 \\ DRS: & f(tx) < t f(x) & \forall t > 1 \end{cases}$$

Remarks:

① CRS $\Leftrightarrow$ HD1
② IRS, DRS $\not\Rightarrow$ HDk.
③ HDk $\Rightarrow$ IRS if $k > 1$, DRS if $0 < k < 1$.

e.g. $f(x_1, x_2) = x_1^2 + x_2$:

$f(tx_1, tx_2) = t^2 x_1^2 + t x_2 > t f(x_1, x_2) \quad \forall t > 1$, IRS, but not HDk.

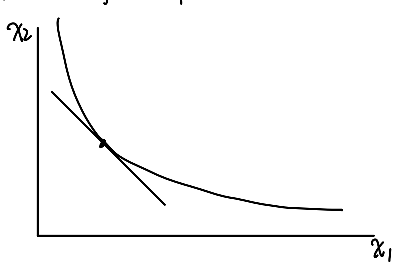
$MP_1 = 2x_1$, $\uparrow$ in x_1; $MP_2 = 1$, constant.

So, IRS $\not\Rightarrow$ MP_i is increasing.

$\uparrow$ Joint $\qquad\qquad$ $\uparrow$ Isolation.

Isoquants

$\{x \in \mathbb{R}_+^n \mid f(x) = q\}$ — similar to indifference curves.



Marginal Rate of Technical Substitution:

$$MRTS_{j,i} = \left|\frac{dx_j}{dx_i}\right|_q = \frac{\partial f/\partial x_i}{\partial f/\partial x_j}\Big|_q = \frac{MP_i}{MP_j}\Big|_q.$$

Proposition: If f is quasi-concave, then it has diminishing MRTS: i.e. $\frac{d\, MRTS_{j,i}}{dx_i} \leq 0$.

Calculus Criterion for Quasi-Concavity.

Bordered Hessian Matrix

$$BH = \begin{bmatrix} 0 & f_1 & f_2 & \cdots & f_n \\ f_1 & f_{11} & f_{12} & \cdots & f_{1n} \\ f_2 & f_{21} & f_{22} & \cdots & f_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ f_n & f_{n1} & f_{n2} & \cdots & f_{nn} \end{bmatrix} \qquad f_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j}$$

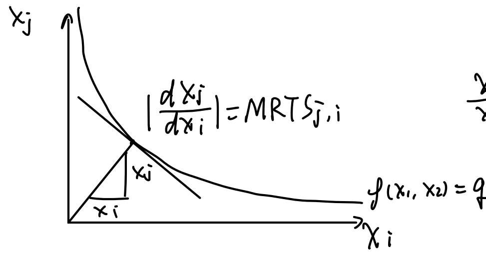
proposition: f is quasi-concave if

$$\begin{vmatrix} 0 & f_1 \\ f_1 & f_{11} \end{vmatrix} = -f_1^2 \le 0 \qquad \begin{vmatrix} 0 & f_1 & f_2 \\ f_1 & f_{11} & f_{12} \\ f_2 & f_{21} & f_{22} \end{vmatrix} \ge 0 \qquad \begin{vmatrix} 0 & f_1 & f_2 & f_3 \\ f_1 & f_{11} & f_{12} & f_{13} \\ f_2 & f_{21} & f_{22} & f_{23} \\ f_3 & f_{31} & f_{32} & f_{33} \end{vmatrix} \le 0 \quad \cdots$$ (alternating the sign).

Remark: MRTS is a good measure of input substitution, but is not unit-free.

The elasticity of Substitution.

$$\sigma_{j,i} = \frac{d(x_j/x_i)}{d\,MRTS_{j,i}} \cdot \frac{MRTS_{j,i}}{x_j/x_i} = \frac{d\ln(x_j/x_i)}{d\ln(MRTS_{j,i})}$$



$$\frac{x_j}{x_i} = g(MRTS_{j,i}).$$

CES Prodution.

$$f(x) = \left(\textstyle\sum_i \alpha_i x_i^{\rho}\right)^{1/\rho}, \quad \alpha_i > 0,\ \textstyle\sum_i \alpha_i = 1,\ \rho < 1$$

$$\frac{\partial f}{\partial x_i} = \frac{1}{\rho}\left(\textstyle\sum_i \alpha_i x_i^{\rho}\right)^{1/\rho - 1} \cdot \alpha_i \rho\, x_i^{\rho-1} = f_i$$

$$f_i = \frac{1}{\rho}\left(\textstyle\sum_i \alpha_i x_i^{\rho}\right)^{1/\rho - 1} \alpha_i \rho\, x_i^{\rho-1}$$

$$\therefore MRTS_{j,i} = \frac{f_i}{f_j} = \frac{\alpha_i}{\alpha_j}\left(\frac{x_j}{x_i}\right)^{1-\rho}$$

$$\therefore \left(\frac{x_j}{x_i}\right)^{1-\rho} = \frac{\alpha_j}{\alpha_i} MRTS_{j,i}$$

$$\ln(x_j/x_i) = \frac{1}{1-\rho}\ln\left(\frac{\alpha_j}{\alpha_i}\right) + \frac{1}{1-\rho}\ln(MRTS_{j,i})$$

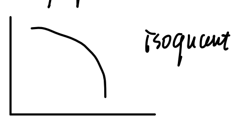
$$\therefore \sigma_{j,i} = \frac{d\ln(x_j/x_i)}{d\ln(MRTS_{j,i})} = \frac{1}{1-\rho} \quad \text{(constant)}.$$

① As $\rho \uparrow$, $\sigma_{j,i} \uparrow$

② $\sigma_{j,i} \to \infty$ as $\rho \to 1$ (perfect substitution, linear prodution)

③ $\sigma_{j,i} \to 0$ as $\rho \to -\infty$ (no substitution, Leontief prodution).

$\rho > 1$: no longer quasi-concave



- **Cost**

Fixed input prices: $w = (w_1, \cdots, w_n)$, $w_i > 0$

Cost function:

$$C(w, q) = \min_x\ w \cdot x \quad \text{s.t. } f(x) \ge q. \quad (\lambda), \quad x \ge 0$$

$$\Downarrow$$

$\bar{x}(w, q)$: conditional input demand.

$C(w, q) = w \cdot \bar{x}(w, q)$: cost function.

- **Properties of Cost Function:**

Suppose that f is continuous and strictly increasing.
Then, the cost function, $C(w,q)$, is
① Continuous in (w,q)
② increasing in q and in w_i
③ HD1 in w
④ concave in w
⑤ if f is strictly quasi-concave and differentiable, then

$$\bar{x}_i = \frac{\partial C}{\partial w_i} \quad \text{(Shepherd's Lemma)}$$

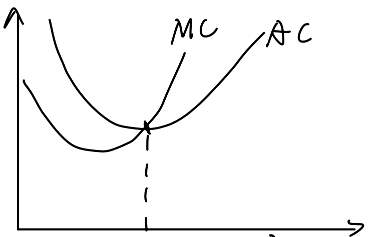
$$\bar{\lambda} = \frac{\partial C}{\partial q} \quad \text{(Marginal cost)}$$

- **The link btw production & cost function:**

Definitions.

Marginal Cost $= MC(q) = \dfrac{\partial C(w,q)}{\partial q}$

Average Cost $= AC(q) = \dfrac{C(w,q)}{q}$



Proposition: AC is increasing if $MC > AC$, decreasing if $MC < AC$

$$\frac{d(AC(q))}{dq} = \frac{C'q - C}{q} = C' - \frac{C}{q} = MC(q) - AC(q)$$

$AC = MC \Rightarrow \min AC.$

Proposition: Suppose $f(x)$ is HDk. with $k>0$. Then, $C(w,q) = \underbrace{C(w,1)}_{K} q^{1/k}$. ☆

proof: By definition, $f(tx) = t^k f(x)$, $k>0$. $t>0$

$$\min_x \; w \cdot x$$
$$\text{s.t. } f(x) \geq q \text{ and } x \geq 0$$
$$\Leftrightarrow \frac{1}{q} f(x) \geq 1.$$

Let $t^k = \frac{1}{q}$. Then, $t = \frac{1}{q^{1/k}} > 0$, and in turn,

$$f\left(\frac{1}{q^{1/k}} x\right) = \frac{1}{q} f(x)$$

Change of Variables. $\hat{x} = \dfrac{x}{q^{1/k}} \Leftrightarrow x = q^{1/k}\hat{x}$

Then, CMP becomes: $\min_{\hat{x}} \; w \cdot q^{1/k} \hat{x}$

$$\text{s.t. } f(\hat{x}) \geq 1 \text{ and } \hat{x} \geq 0$$
$$\Rightarrow \hat{x} = \hat{x}(w)$$
$$\Rightarrow \bar{x}(w,q) = q^{1/k}\hat{x}(w)$$

Note that $\bar{x}(w,1) = \hat{x}(w)$, so $\bar{x}(w,q) = q^{1/k}\bar{x}(w,1)$

$$\therefore \; C(w,q) = w \cdot \bar{x}(w,q)$$
$$= w \cdot \bar{x}(w,1) \cdot q^{1/k}$$
$$= C(w,1) \cdot q^{1/k}.$$

$$MC = \frac{\partial C}{\partial q} = \frac{1}{k} C(w,1)\, q^{\frac{1-k}{k}}.$$

$$MC' = \frac{\partial^2 C}{\partial q^2} = \frac{1-k}{k^2} C(w,1)\, q^{\frac{1-2k}{k}}$$

Hence: $k=1$: CRS $\Rightarrow$ MC is constant (wrt q)
$k>1$: IRS $\Rightarrow$ MC is decreasing
$k<1$: DRS $\Rightarrow$ MC is increasing

e.g. $C(w,q) = \underbrace{C(w,1)}_{K} q^{1/k}$

- $k=1$: CRS $\Leftrightarrow C(w,q) = Kq$ — MC — $MC'=0$
- $k=2$: IRS $\Leftrightarrow C(w,q) = K\sqrt{q}$ — MC ↓ $MC'<0$
- $k=\frac{1}{2}$: DRS $\Leftrightarrow C(w,q) = Kq^2$ — MC ↑ $MC'>0$.

- **Short v.s. long Run Cost:**

Short run: period in which at least one input is fixed.
Long run: all inputs are adjustable.
Which is fixed is based on the application. $\begin{cases} \text{Capital: rental aggreements, time to build.} \\ \text{Labour: wage contracts.} \end{cases}$

Let $x = (x^0, x^1)$, x^1 is fixed.

$$\min_{x^0} w^0 \cdot x^0 + w^1 \cdot x^1$$
$$\text{s.t. } f(x^0, x^1) \geq q.$$
$$\Downarrow$$
$$\hat{x}^0 = \hat{x}^0(w, q, x^1).$$
$$\Downarrow$$

☆ $\underbrace{SC(w, q, x^1)}_{\text{Shortrun cost.}} = \underbrace{w^0 \cdot \hat{x}^0}_{\text{variable cost}} + \underbrace{w^1 \cdot x^1}_{\text{Fixed cost (does not depend on } q\text{).}}$

In general: $\underbrace{C(w, q)}_{\text{Longrun cost.}} = \min_{x^1} SC(w^0, q, x^1)$
$$\Downarrow$$
$$\bar{x}^1 = \bar{x}^1(w, q).$$

Or: Long run: $\min_{x^0, x^1} w \cdot x.$ $\quad$ s.t. $f(x^0, x^1) \geq q.$

Mathematically. $\min_{x,y} g(x,y) \iff \min_{x} \min_{y} g(x,y).$

e.g. Let $SC(x_2) = w_1 q^2/x_2 + w_2 x_2.$

$\min_{x_2} SC(x_2)$: F.O.C. $-\frac{w_1 q^2}{x_2^2} + w_2 = 0.$

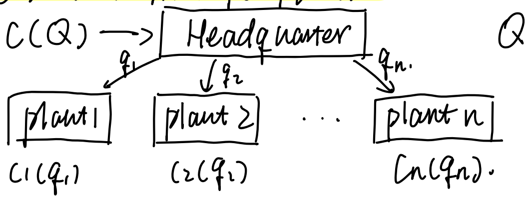
$$\Rightarrow \bar{x}_2 = \sqrt{\frac{w_1}{w_2}} q.$$
$$\Rightarrow SC(\bar{x}_2) = w_1 \cdot q^2 \cdot \frac{1}{q} \sqrt{\frac{w_2}{w_1}} + w_2 \cdot q \cdot \sqrt{\frac{w_1}{w_2}}$$
$$= 2q\sqrt{w_1 w_2}. = C(w, q)$$ constant return to scale.

by Shepherd's lemma: $\bar{x}_1 = \frac{\partial C}{\partial w_1} = \sqrt{\frac{w_2}{w_1}} q.$

Remark: $\frac{\partial SC}{\partial q} = \frac{2w_1}{x_2} q$, increasing in q.

But the underlying production function is CRS, because $\frac{\partial C}{\partial q} = 2\sqrt{w_1 w_2}$, constant in q.
☆ See the proposition above.

- **Cost with multiple plants:**



Q: total production.

$$C(Q) = \min_{q_1 q_2 \cdots q_n} C_1(q_1) + C_2(q_2) + \cdots + C_n(q_n)$$
$$\text{s.t. } q_1 + q_2 + \cdots + q_n = Q$$
$$q_i \geq 0.$$

Substituting for q_n:

$$\min_{q_1 \cdots q_{n-1}} \; C_1(q_1) + \cdots + C_{n-1}(q_{n-1}) + C_n(Q - q_1 - \cdots - q_{n-1}) = \bar{C}$$

F.O.C: $\frac{\partial \bar{C}}{\partial q_1} = C_1'(q_1) - C_n'(q_n) \geq 0 \quad \frac{\partial \bar{C}}{\partial q_1} \cdot q_1 = 0$

$\frac{\partial \bar{C}}{\partial q_2} = C_2'(q_2) - C_n'(q_n) \geq 0 \quad \frac{\partial \bar{C}}{\partial q_2} \cdot q_2 = 0.$

$\vdots$

$\frac{\partial \bar{C}}{\partial q_{n-1}} = C_{n-1}'(q_{n-1}) - C_n'(q_n) \geq 0 \quad \frac{\partial \bar{C}}{\partial q_{n-1}} \cdot q_{n-1} = 0.$

and $q_n = Q - q_1 - \cdots q_{n-1}$, $q_i \geq 0$.

Say if only have two plants:

$MC_1 = C_1'(q_1) = 2q_1$: $\uparrow$. DRS

$MC_2 = C_2'(q) = \frac{1}{2\sqrt{q_1}}$: $\downarrow$. IRS.

When q is small, $MC_1 = 0$, $MC_2 = \infty$.

2 don't do well initially when q is small.

If $q_1 = 0$, $q_2 = Q$, then $C(Q) = \sqrt{Q}$

If $q_1 = Q$, $q_2 = 0$, then $C(Q) = Q^2$.

When $Q > 1$, then $\sqrt{Q} < Q^2$, 2 do well.

Suppose $q_1 . q_2 > 0$, then $\frac{\partial \bar{C}}{\partial q_1} = 0$, $\frac{\partial \bar{C}}{\partial q_2} = 0 \Rightarrow C_1'(q_1) = C_2'(q_2)$

☆ Incomplete or incorrect intuition: to minimize cost, the firm needs to equate marginal cost across plants. Mathematically, we also need to check the SOCs to ensure minimizing.

$$\frac{\partial^2 \bar{C}}{\partial q_i^2} = C_i''(q_i) + C_n''(q_n) \geq 0$$

which may not hold if $C'' \leq 0$.

Results. If $C_i''(q_i) > 0$, i.e. $MC_i' > 0 \Leftrightarrow$ DRS for all i, then, $MC_1(q_1^*) = MC_2(q_2^*) = \cdots = MC_n(q_n^*)$ minimize the total cost. Otherwise: corner solutions.

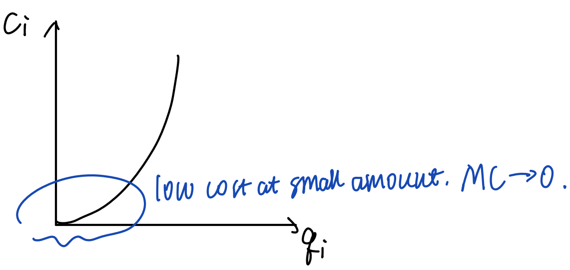
e.g. (Increasing MC, DRS)

Let $C_i(q_i) = q_i^2$ for all $i = 1, 2, \cdots, n$.

$\Rightarrow C_1'(q_1) = C_2'(q_2) = \cdots = C_n'(q_n)$

$\Rightarrow q_1^* = \cdots = q_n^* = \frac{Q}{n}$.

$\Rightarrow C(Q) = n\left(\frac{Q}{n}\right)^2 = \frac{Q^2}{n} \rightarrow 0$ as $n \rightarrow \infty$



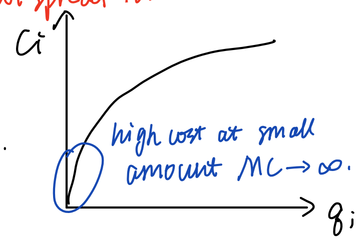
e.g. (Decreasing MR. IRS). for each plant, the performance at the small amount is bad: $MC \to \infty$ as $q \to 0$. therefore, should concentrate large amount in one plant, not spread it.

Let $C_i(q_i) = \sqrt{q_i}$ for all $i = 1, \cdots, n$

$MC_i = \frac{1}{2\sqrt{q_i}}$, $MC_i' < 0 \Leftrightarrow$ IRS.

If $MC_i = MC_j \Rightarrow q_i = q_j = \frac{Q}{n} \Rightarrow C(Q) = n\sqrt{\frac{Q}{n}} = \sqrt{nQ}$, this is maximization.

Therefore, we look at corner solution, that is, $q_i = Q$ for some i. $\Rightarrow C(Q) = \sqrt{Q}$.



Intuition: Decreasing MC implies IRS, which would arise if, for example, learning by doing. Under this technology, it is, therefore, reasonable that the firm would concentrate production in one plant.

Olson (1965): [The Group Size Paradox]. The logic of collective action:

e.g. $C(Q) = \min_{q_1, q_2} C_1(q_1) + C_2(q_2) = \underbrace{q_1^2}_{DRS} + \underbrace{q_2}_{CRS}$

s.t. $q_1 + q_2 = Q$, $q_1 > 0$, $q_2 > 0$.

$\bar{C} = q_1^2 + Q - q_1$

$\frac{\partial \bar{C}}{\partial q_1} = 2q_1 - 1 = 0 \quad q_1^* = \frac{1}{2} \quad \frac{\partial^2 \bar{C}}{\partial q_1^2} = 2 > 0.$

$q_2^* = Q - q_1^* \geq 0$ if $Q \geq \frac{1}{2}$, then $q_1^* = \frac{1}{2}$, $q_2^* = Q - \frac{1}{2}$.

if $Q < \frac{1}{2}$. then $q_1^* = Q$, $q_2^* = 0$

$$\therefore C(Q) = \begin{cases} Q - \frac{1}{4} & \text{if } Q \geq \frac{1}{2} \ (q_1^* = \frac{1}{2},\ q_2^* = Q - \frac{1}{2}) \\ Q^2 & \text{if } Q < \frac{1}{2} \ (q_1^* = Q,\ q_2^* = 0). \end{cases}$$

Total Quantity is small → use DRS plant first.

- **Profit Maximization and Supply.**

Suppose that each firm has no market power and take market prices p as given.

Each firm solves.

$$\max_{q,x} \pi = pq - wx \quad s.t.\ f(x) \geq q.$$

$$\Downarrow$$

$x^* = x^*(p, w)$. the unconditional input demand.

$q^* = f(x^*) = \underline{q^*(p,w)}$ supply.

$x^*(q, w) = \bar{x}(w, q^*(p, w))$

Observation: profit max ⇒ cost min.

Fix q. then. $\max_x \pi \ s.t.\ f(x) \geq q \iff \min_x w \cdot x \ s.t.\ f(x) \geq q$.

proof: immediate from the definition of the problem.

$$\max_{q,x} \pi = pq - w \cdot x \quad s.t.\ f(x) \geq q.$$

So, it is more intuitive to solve profit-max in two steps.

Step 1. fix q, $\max_x \pi \ s.t.\ f(x) \geq q \iff \min_x w \cdot x \ s.t.\ f(x) \geq q. = C(w, q) = w \cdot \bar{x}(w, q)$.

$\Downarrow$ $\bar{x}(w, q)$

Step 2: choose q, so that $\max_q \pi = pq - C(w, q)$.

F.O.C. $\frac{\partial \pi}{\partial q} = p - \underbrace{C_q(w, q)}_{MC(q)} \leq 0$ and $\frac{\partial \pi}{\partial q} q = 0$.

For $q > 0$: $p = MC(q) \Rightarrow q^*(p, w)$

$\Rightarrow x^*(w, q) = \bar{x}(w, q^*(w, p))$.

S.O.C. $\frac{\partial^2 \pi}{\partial q^2} = -C_{qq} \leq 0$

$C_{qq} \geq 0$. Therefore, for a well-defined supply function $q < \infty$, we need DRS.

If CRS: $C(w, q) = C(w, 1) q = cq$.

$\pi = pq - C(w, q)$

$= (p - c) q$.

If $p > c$, then $q^* = +\infty$

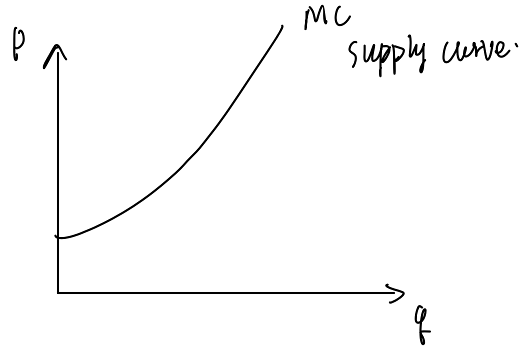
If $p < c$, then $q^* = 0$.

If $p = c$, then, $q^* \in [0, +\infty)$.

If IRS: $C(w, q)$: $MC' < 0$.

$\pi = pq - C(w, q)$.

$\max_q \pi \Rightarrow q^* = +\infty$.



- Profit function and its properties.

$$\Pi(p,w) = \max_q \pi = pq - C(w,q)$$

Suppose q^* is well defined. Then:

① $\Pi(p,w)$ is HD1 in (p,w).

② $\Pi(p,w)$ is convex in (p,w). intuitively: $q = q(p)$ ∴ nonlinear in p.

pf: (p^1, w^1) and (p^2, w^2)

$(p^\alpha, w^\alpha) = (\alpha p^1 + (1-\alpha)p^2, \alpha w^1 + (1-\alpha)w^2)$

WTS: $\alpha\Pi(p^1,w^1) + (1-\alpha)\Pi(p^2,w^2) \geq \Pi(p^\alpha, w^\alpha)$.

③ By the envelope theorem:

$$\begin{cases} \frac{\partial}{\partial p}\Pi(p,w) = q^*(p,w). \\ \frac{\partial}{\partial w_i}\Pi(p,w) = -x_i^*(p,w). \end{cases}$$

$\frac{\partial \Pi}{\partial p} = \frac{\partial \pi}{\partial p}\Big|_{q^*} = q\Big|_{q^*} = q^*$

Hotelling's Lemma: $\frac{\partial \Pi}{\partial w_i} = \frac{\partial \pi}{\partial w_i}\Big|_{q^*} = -\bar{x}_i(w,q)\Big|_{q^*} = -\bar{x}_i(w,q^*) = -x_i^*(p,w)$.

$\left(= -\frac{\partial C}{\partial w_i}, \text{ Shephard's Lemma}\right)$.

④. From convexity: upward-sloping supply.

Hessian: $H = \begin{bmatrix} \frac{\partial^2 \Pi}{\partial p^2} & \\ & \ddots \end{bmatrix} \Rightarrow \frac{\partial^2 \Pi}{\partial p^2} \geq 0. \Rightarrow \frac{\partial q^*}{\partial p} \geq 0.$

Alternatively. F.O.C. $\frac{\partial \pi}{\partial q}\Big|_{q^*} = p - C_q(w, q^*(p,w)) = 0$ for $q^* > 0$.

$$\frac{\partial^2 \pi}{\partial p \partial q}\Big|_{q^*} = 1 - C_{qq}(\cdot)\cdot\frac{\partial q^*}{\partial p} = 0$$

$$\frac{\partial q^*}{\partial p} = \frac{1}{C_{qq}(\cdot)}$$

DRS: $C_{qq}(\cdot) > 0$. ∴ $\frac{\partial q^*}{\partial p} > 0$.

$x_i^* = x_i^*(p_x, p_y, I)$

$X^* = \sum_i x_i^*$

$Q_x^* = \sum_i q_i^*$

Market Clearing P: Demand = Supply.

$X^* = Q_x^* \Rightarrow p_x^*(p_y, I)$.

- General Equilibria

$N = \{1, 2, \cdots, n\}$: the set of consumers (subscript).

$\{1, 2, \cdots, k\}$: the set of goods (superscript)

w_i^j: consumer i's initial endowment of good j.

$w_i = (w_i^1, w_i^2, \cdots, w_i^k) \in \mathbb{R}_+^k$: i's initial endowment.

$w = (w^1, w^2, \cdots, w^k)$: total endowment (supply fixed).

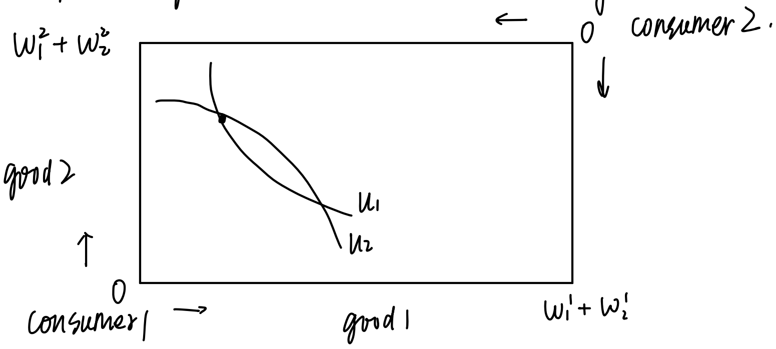
$\mathcal{E} = (u_i, w_i)_{i\in N}$: the exchange economy

Feasible Allocations:

$F(w) = \{x \in \mathbb{R}_+^{nk} \mid \sum_{i\in N} x_i \leq \sum_{i\in N} w_i\}$ where $x_i = (x_i^1, x_i^2, \cdots, x_i^k)$ is the agent i's allocation.

$$\left(\sum_{i\in N} x_i^1, \sum_{i\in N} x_i^2, \cdots, \sum_{i\in N} x_i^k\right) \leq \left(\sum_{i\in N} w_i^1, \sum_{i\in N} w_i^2, \cdots, \sum_{i\in N} w_i^k\right)$$

The Edgeworth Box: a 2×2 Economy



Pareto Efficiency:

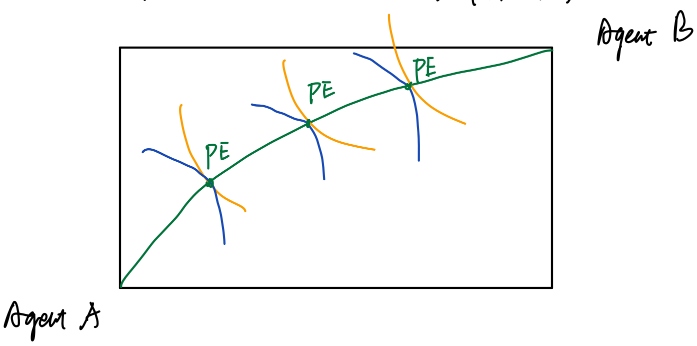
Definition: A feasible allocation, $x \in F(w)$, is P.E. if there is no $y \in F(x)$ such that

- $u_i(y_i) \geq u_i(x_i)$ for all $i \in N$
- $u_j(y_j) > u_j(x_j)$ for some $j \in N$

Example: $N = \{A, B\}$

$u_A = (x_A^1)^{1/2}(x_A^2)^{1/2}$ and $u_B = (x_B^1)^{2/3}(x_B^2)^{1/3}$

$w_A = (1/4, 1/4)$ and $w_B = (3/4, 3/4)$



Since both utilities are strictly concave, P.E. obtains at $MRS_{2,1}^A = MRS_{2,1}^B$

$$\Rightarrow \frac{\frac{1}{2}(x_A^1)^{-\frac{1}{2}}(x_A^2)^{\frac{1}{2}}}{\frac{1}{2}(x_A^1)^{\frac{1}{2}}(x_A^2)^{-\frac{1}{2}}} = \frac{\frac{2}{3}(x_B^1)^{-\frac{1}{3}}(x_B^2)^{\frac{1}{3}}}{\frac{1}{3}(x_B^1)^{\frac{2}{3}}(x_B^2)^{-\frac{2}{3}}}$$

$$\frac{x_A^2}{x_A^1} = 2\frac{x_B^2}{x_B^1}$$

and: $x_A^1 + x_B^1 = 1$, $x_A^2 + x_B^2 = 1$

$$\therefore \quad \frac{x_A^2}{x_A^1} = 2\frac{(1-x_A^2)}{(1-x_A^1)}$$

$$x_A^2 - x_A^1 x_A^2 = 2x_A^1 - 2x_A^1 x_A^2$$

$$x_A^2(1+x_A^1) = 2x_A^1$$

$$x_A^2 = \frac{2x_A^1}{1+x_A^1}$$

So, P.E. set $= \left\{(x_A^1, x_A^2) \in [0,1]^2 : x_A^2 = \frac{2x_A^1}{1+x_A^1}\right\}$.

$u_A = x_A^1 x_A^2$

$MRS_{21}^A = \frac{MU_1}{MU_2} = \frac{x_A^2}{x_A^1}$

☆

Pareto efficiency does not guarantee a higher utility than the initial endowment.

Calculus of Pareto Efficiency:

Theorem: A feasible allocation $\bar{x}$ is P.E. if and only if it solves the following for all $i \in N$:

$$\max_x \; u_i(x_i)$$
$$\text{s.t. } u_j(x_j) \geq u_j(\bar{x}_j), \; j \neq i$$
$$x \in F(w)$$

proof: ($\Rightarrow$): suppose $\bar{x}$ is P.E. but doesnot solves the max problem for some agent j,
Let x' solves that problem.
Then, x' makes no agent worse off than $\bar{x}$ (since x' satisfies the constraints)
but agent j is strictly better off than $\bar{x}_j$, which contradicts $\bar{x}$ being P.E.
Thus $\bar{x}$ as P.E. solves the max problem for all agents.

($\Leftarrow$): suppose $\bar{x}$ solves the max problem for all i but is not P.E.
Then there is a feasible allocation x'' that makes no agent worse off than $\bar{x}$
but some agent j strictly better off, contradicting that $\bar{x}$ solves j's problem.

Another useful approach to P.E.

Let $W = W(u_1, u_2, \cdots, u_n)$ be the social welfare function that is strictly increasing in all u_i's

example: utilitarian welfare function: $W = u_1 + \cdots + u_n$

Rawlsian welfare function: $W = \min\{u_1, \cdots, u_n\}$

Consider:

$$\max_{x} W(u_1(x_1), u_2(x_2), \cdots, u_n(x_n))$$
$$\text{s.t. } x \in F(\omega).$$
$$\Downarrow$$
$$\bar{x} \text{ is P.E.}$$

Theorem: If $\bar{x}$ maximizes welfare, then $\bar{x}$ is P.E. (not the other way around!)

proof: Suppose $\bar{x}$ is not P.E. then, there must be $x' \in F(\omega)$. s.t.

$u_i(x_i') \geq u_i(\bar{x}_i)$ for all i

$u_j(x_j') > u_j(\bar{x}_j)$ for some j.

However since W is strictly increasing,

W at x' > W at $\bar{x}$

$\Rightarrow \Leftarrow$ that $\bar{x}$ is the maximizer.

The correct converse:

Theorem: Let $\bar{x}$ be P.E., with $\bar{x}_i \gg 0$ for all i.
Also, let u_i be continuous, strictly increasing, and strictly concave.
Then, there are some weights $(\alpha_1 \cdots \alpha_n) \in \mathbb{R}^n_{++}$
s.t. $\bar{x}$ maximizes $W = \sum \alpha_i u_i$ subject to feasibility.

proof: omitted.

Theorem If $f(\vec{x})$ is q-concave and HD1 (or less), then $f(x)$ is concave.

Core

Blocking Conditions:

Let $S \subseteq N$ be a coalition of consumers. We say that S blocks (or improves upon) $x \in F(\omega)$ if there is some allocation y such that

① $\sum_{i \in S} y_i \leq \sum_{i \in S} w_i$ feasible for S

② $u_i(y_i) \geq u_i(x_i)$ for all $i \in S$

$u_j(y_j) > u_j(x_j)$ for some $j \in S$

Core

A feasible allocation $x \in F(w)$ is in the core if it cannot be blocked by any coalitions.

Remarks: ① P.E. considers only the grand coalition $S=N$, whereas the Core considers 2^n-1 coalitions.

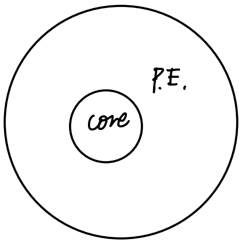
② so, if $x \in$ Core, then x is P.E.

pf: suppose $x \in$ Core is not P.E. then: there is an allocation $y \in F(w)$:

under the grand coalition: $u_i(y_i) \geq u_i(x_i)$ for all $i \in N$

$u_j(y_j) > u_j(x_j)$ for some $j \in N$.

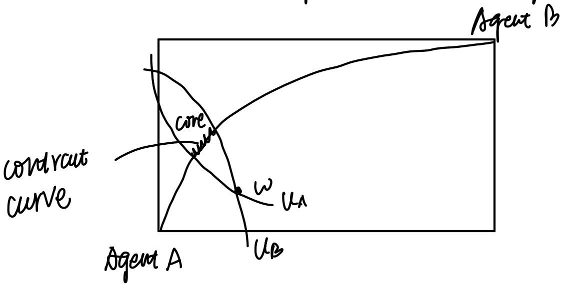
then: N blocks x. $\Rightarrow\Leftarrow$ with $x \in$ Core.



③ An agent cannot be worse off than his/her initial endowment.

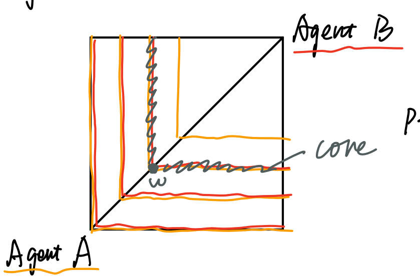
since $S=\{i\}$ would, otherwise, block it.

④ Can be interpreted as voluntary trade.



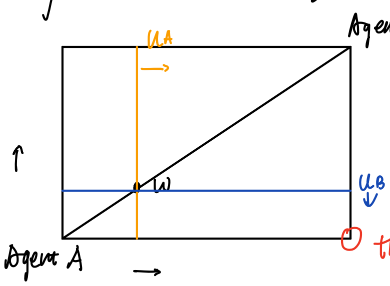
coalitions: $S = \{\{A\}, \{B\}, \{A,B\}\}$.

e.g. $u_A = \min\{x_A^1, x_A^2\}$ $u_B = \max\{x_B^1, x_B^2\}$



P.E. set: the entire box.

e.g. $u_A = x_A^1$ $u_B = x_B^2$



P.E. set = Core = $\{x_A=(1,0), x_B=(0,1)\}$

the only P.E. allocation

Walrasian Equilibrium (General)

Assumption 1: u_i is continuous, strictly increasing, and strictly quasi-concave.

Consider prices: $p = (p_1 \cdots p_k) \gg 0$

Individual Demand:

$$\max_{x_i} u_i(x_i) \quad \text{s.to.} \quad p \cdot x_i \leq \underbrace{p \cdot w_i}_{I_i}$$

price takers

$$\sum_{j=1}^{k} p_j x_i^j \leq \sum_{j=1}^{k} p_j w_i^j$$

$$\Downarrow$$

$$x_i^* = x_i^*(p, p \cdot w_i)$$ Marshallian Demand.

Fact: Under Assumption 1: x_i^* is unique, continuous and HD0 in p.

$$x_i^*(tp, tp \cdot w_i) = x_i^*(p, p \cdot w) \quad \text{for all } t > 0$$

Excess demand for good j:

$$z^j(p) = \underbrace{\sum_{i \in N} x_i^j(p, p \cdot w_i)}_{\text{total demand}} - \underbrace{\sum_{i \in N} w_i^j}_{\text{total supply}}$$

> 0: excess demand

< 0: excess supply

Excess demand vector: $Z(p) = (z^1(p), \cdots, z^k(p))$

Fact: $z(p)$ is continuous and HD0 in p.

Walras' Law: the market value of excess demand is zero.

$$p \cdot Z(p) = 0 . \quad \forall \ p \gg 0$$

proof:
$$p \cdot z(p) = \sum_{j=1}^{k} p_j z^j(p)$$
$$= \sum_{j=1}^{k} p_j \Big(\sum_{i\in N} x_i^j(p, p\cdot w_i) - \sum_{i \in N} w_i^j \Big)$$
$$= \sum_{i\in N} \sum_{j=1}^{k} p_j \big(x_i^j(p, p\cdot w_i) - w_i^j\big)$$
$$= \sum_{i\in N} \underbrace{(p\cdot x_i - p\cdot w_i)}_{=0}$$
$$= 0 \qquad \text{q.e.d.}$$

Remark:

① Since $Z(p)$ HD0 in p, can normalize p. s.t. $\sum p_i = 1$ $\quad t = \frac{1}{\sum p_i} > 0$, $Z\left(\frac{p_1}{\sum p_i}, \frac{p_2}{\sum p_i}, \cdots, \frac{p_k}{\sum p_i}\right)$

or can use p_1 as numerator: $t = \frac{1}{p_1}$, $Z\left(1, \frac{p_2}{p_1}, \cdots, \frac{p_k}{p_1}\right) = Z(p)$

Walras' Law $p \cdot Z(p) = 0$ **implies:**

① If $k=2$, $z^1(p) > 0$, then $z^2(p) < 0$

$\quad p_1 z^1(p) + p_2 z^2(p) = 0$

② If $z^j(p) = 0$ for $j = 1, \cdots, k-1$, then $z^k(p) = 0$

Definition $p^* \in \mathbb{R}^k_+$ is a Walrasian Equilibrium price if $z(p^*) \leq 0$ $(=)$ i.e. $z^j(p^*) \leq 0 \ \forall j$

no excess demand

Theorem: Existence: Under Assumption 1, there is a Walrasian Equilibrium. (Arrow–Debreu, Mckenzie)

proof: using Brouwer's Fixed Point Theorem.

important example showing solution steps:

$u_A = (x_A^1)^{\frac{1}{2}} (x_A^2)^{\frac{1}{2}} \qquad u_B = (x_B^1)^{\frac{2}{3}} (x_B^2)^{\frac{1}{3}}$

$w_A = (\frac{1}{4}, \frac{1}{4}) \qquad w_B = (\frac{3}{4}, \frac{3}{4})$

- At $p = (p_1, p_2)$ the income from initial endowments:

  $I_A = \frac{1}{4}(p_1 + p_2) \quad I_B = \frac{3}{4}(p_1 + p_2)$

- By Walras' Law, it suffices to clear Market 1.

  Marshallian Demands:

$$x_A^1 = \frac{I_A}{2p_1} = \frac{1}{8}\left(1 + \frac{p_2}{p_1}\right)$$
$$x_B^1 = \frac{2 I_B}{3 p_1} = \frac{2}{3} \cdot \frac{3}{4}\left(1 + \frac{p_2}{p_1}\right) = \frac{1}{2}\left(1 + \frac{p_2}{p_1}\right)$$

$z^1(p) \Rightarrow x_A^1 + x_B^1 = \frac{1}{4} + \frac{3}{4}$

$\frac{5}{8}\left(1 + \frac{p_2}{p_1}\right) = 1 \qquad 1 + \frac{p_2}{p_1} = \frac{8}{5}$

$\Rightarrow \frac{p_2^*}{p_1^*} = \frac{3}{5}$

$\therefore\ x_A^{1*} = \frac{1}{5} \qquad x_A^{2*} = \frac{1}{2}\frac{I_1}{p_2} = \frac{1}{2}\cdot\frac{1}{4}\left(\frac{p_1}{p_2}+1\right) = \frac{1}{8}\cdot\frac{8}{3} = \frac{1}{3}$

$x_B^{1*} = \frac{4}{5} \qquad x_B^{2*} = \frac{2}{3}$

$x_A^* = \left(\frac{1}{5}, \frac{1}{3}\right) \qquad x_B^* = \left(\frac{4}{5}, \frac{2}{3}\right)$

$\left(w_A = \left(\frac{1}{4}, \frac{1}{4}\right) \qquad w_B = \left(\frac{3}{4}, \frac{3}{4}\right)\right)$

A: net seller of good 1
net buyer of good 2.

Example: multiple WE

Let $u_A = x_A^1 - \frac{1}{8}(x_A^2)^{-8}$

$u_B = -\frac{1}{8}(x_B^1)^{-8} + x_B^2$

with $w_A = (2, r)$, $w_B = (r, 2)$ where $r = 2^{\frac{8}{9}} - 2^{\frac{1}{9}}$

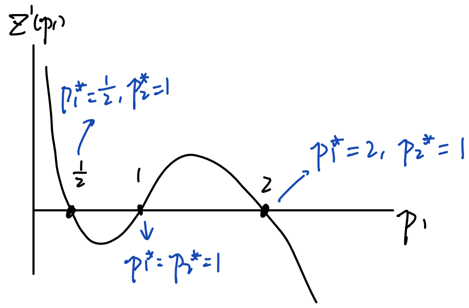
set $p_2 = 1$, then:

$$\max_{x_A^1, x_A^2} u_A = x_A^1 - \frac{1}{8}(x_A^2)^{-8} \quad s.t.\ p_1 x_A^1 + x_A^2 \le 2p_1 + r$$

$\Rightarrow$ Marshallian Demands: $x_A^1 = 2 + \frac{r}{p_1} - \left(\frac{1}{p_1}\right)^{\frac{8}{9}}$, $x_B^1 = \left(\frac{1}{p_1}\right)^{\frac{1}{9}}$

Excess Demand $Z^1(p_1) = x_A^1 + x_B^1 - (2+r)$

$$= r\left(\frac{1}{p_1} - 1\right) - \left(\frac{1}{p_1}\right)^{8/9} + \left(\frac{1}{p_1}\right)^{\frac{1}{9}}$$



Example: non-existence

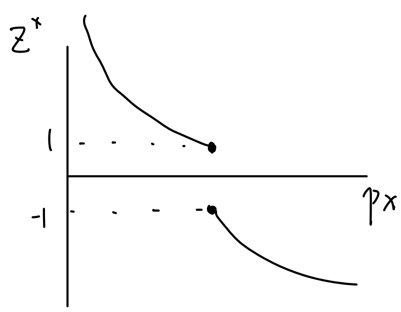
Assumption 1 (u_i continuous, strictly increasing and strictly q-concave $\forall i$) violated.

$u_A = \max\{x_A, y_A\}$, $u_B = (x_B)^{\frac{1}{2}}(y_B)^{\frac{1}{2}}$

$w_A = w_B = (1, 1)$

Marshallian Demands:

$$x_A^* = \begin{cases} 0 & \text{if } p_x/p_y > 1 \\ 0 \text{ or } \frac{I_A}{p_x} & \text{if } p_x/p_y = 1 \\ \frac{I_A}{p_x} & \text{if } p_x/p_y < 1 \end{cases}, \quad x_B^* = \frac{I_B}{2p_x}$$



W.l.o.g. set $p_y = 1$, then $I_A = I_B = p_x + 1$

Excess Demand: $$z^x(p_x) = x_A^* + x_B^* - 2 = \begin{cases} \frac{1}{2p_x} - \frac{3}{2} & \text{if } p_x > 1 \\ \pm 1 & \text{if } p_x = 1 = p_y \\ \frac{3}{2p_x} - \frac{1}{2} & \text{if } p_x < 1 \end{cases}$$

$\therefore$ Free market often peacefully allocate goods.
But is it a "good" allocation?

Intuitively: $\begin{cases} MRS_{21}^A = \frac{p_1}{p_2} \\ MRS_{21}^B = \frac{p_1}{p_2} \end{cases} \Rightarrow MRS_{21}^A = MRS_{21}^B \Rightarrow$ WEA is P.E.

Theorem: If u_i is strictly increasing, then every WEA is in the Core.

proof: Suppose, to the contrary, there is some WEA x^* associated with p^* but $x^* \notin$ Core.
Then there is some coalition $S \subseteq N$ that blocks x^*.
That is, there is an allocation $y \neq x^*$ such that

$$\sum_{i \in S} y_i \le \sum_{i \in S} w_i \quad \text{(feasible)} \quad (*)$$

and: $u_i(y_i) \ge u_i(x_i^*)$ for all $i \in S$ $\quad (**)$

$u_j(y_j) > u_j(x_j^*)$ for some $j \in S$

Since $p^* \in \mathbb{R}^k_{++}$, $(*)$ implies

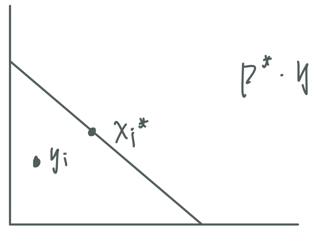
$$p^* \cdot \sum_{i\in S} y_i \le p^* \cdot \sum_{i \in S} w_i$$

Since x^* is a WEA,

$$x_i^* = \arg\max_{x_i} u_i(x_i)$$

$$\text{s.t. } p^* \cdot x_i \le p^* \cdot w_i$$

$$p^* \cdot x_i = p^* \cdot w_i$$



$p^* \cdot y_i < p^* \cdot w_i \Rightarrow u_i(y_i) < u_i(w_i)$

From $(**)$: $p^* \cdot y_i \ge p^* \cdot x_i^* = p^* \cdot w_i$ for all $i \in S$

$p^* \cdot y_j > p^* \cdot x_j^* = p^* \cdot w_j$ for some $j \in S$

then: $p^* \cdot \sum_{i \in S} y_i > p^* \cdot \sum_{i\in S} w_i$, contradiction!

Thus, $x^* \in$ Core

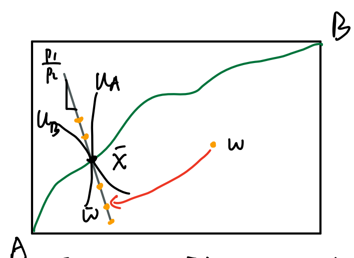
Implications: ① if $\exists$ WEA, Core $\neq \phi$.

② WEA $\subset$ Core $\subset$ P.E., every WEA is also P.E.

$\Downarrow$ indicates that

First Welfare Theorem If u_i is strictly increasing for all i, then every WEA is P.E.

Second Welfare Theorem Suppose u_i is continuous, strictly increasing, and strictly q-concave for all i. If $\bar{x}$ is P.E. then there is some redistribution of initial endowments such that $\bar{x}$ is WEA.



$\bar{x}$ is a WEA under w'.

proof:

Suppose $\bar{x}$ is P.E. Then, $\bar{x}$ is feasible i.e. $\bar{x} \in F(w)$

Redistribute wealth: $w = \bar{x}$. Then, there is a WEA x^* associated with some p^*.

WTS: $x^* = \bar{x}$

Suppose not, i.e. $x^* \neq \bar{x}$

Since x^* is a WEA, for all i:

$u_i(x_i^*) \ge u_i(w_i')$ by UMP

$= u_i(\bar{x}_i)$ by $w_i' = \bar{x}_i$

WTS: $u_i(x_i^*) = u_i(\bar{x}_i)$ for all i.

Suppose not, i.e. $u_j(x_j^*) > u_j(\bar{x}_j)$ for some j

Then $\bar{x}$ cannot be P.E. since x^* is also feasible

(The grand coalition $S = N$ blocks $\bar{x}$ by x^*), A contradiction.

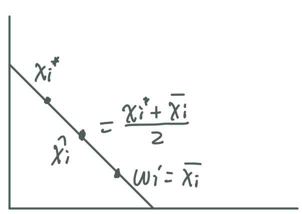
Hence $u_i(x_i^*) = u_i(\bar{x}_i)$ for all i.

$\not\Rightarrow x_i^* = \bar{x}_i$ since they are vectors.

Suppose $x_i^* \neq \bar{x}_i$ for some i.

Since x_i^* maximizes utility,

$$p^* \cdot x_i^* = p^* \cdot w_i' = p^* \cdot \bar{x}_i$$



Take $\hat{x}_i = \frac{x_i^* + \bar{x}_i}{2}$

Note that $p^* \cdot \hat{x}_i = \frac{p^* \cdot x_i^* + p^* \cdot \bar{x}_i}{2} = p^* \cdot w_i'$, so

$\hat{x}_i$ is in the budget set of i.

Moreover, since $x_i^* \neq \bar{x}_i$, u_i is strictly q-concave

$u_i(\hat{x}_i) > u_i(x_i^*)$

this contrasts with that x_i^* maximize utility.

Hence, $x_i^* = \bar{x}_i$ for all i. q.e.d.

Large Economies and Walrasian Equilibrium

Walrasian eq. assumes price-taking consumers. This is meaningful only if there is a large # of them.

Replica Economy

Two types of consumers: A and B.

An r-fold economy, $\mathcal{E}_r$, has r A and r B types.

The same type of agents have the same utility and initial endowments

$A: (u_A, w_A)$

$B: (u_B, w_B)$

Theorem: Equal Treatment of Equals in the Core.

Suppose that u_i is continuous, strictly increasing, and strictly quasi-concave.
If x is an allocation in the core, then every consumer of type t obtains the same allocation: $x_{ti} = x_{tj}$

proof: for simplicity, consider a 2-fold economy.

Let $x = (x_{A1}, x_{A2}, x_{B1}, x_{B2})$ be a core allocation but $x_{A1} \neq x_{A2}$.

W.l.o.g. suppose $u_A(x_{A1}) \geq u_A(x_{A2})$. $u_B(x_{B1}) \geq u_B(x_{B2})$

Let $S = \{A2, B2\}$ and they contemplate a new coalition by themselves.

$\bar{x}_{A2} = \frac{x_{A1}+x_{A2}}{2}$ $\quad \bar{x}_{B2} = \frac{x_{B1}+x_{B2}}{2}$

Since $x_{A1} \neq x_{A2}$ and u_A is strictly q-concave,

$u(\bar{x}_{A2}) > \min\{u(x_{A1}), u(x_{A2})\}$

$u(\bar{x}_{A2}) > u(x_{A2})$

Moreover, $u_B(\bar{x}_{B2}) \geq u_B(x_{B2})$

Finally, $\bar{x}_{A2} + \bar{x}_{B2} = \frac{1}{2}(x_{A1} + x_{A2} + x_{B1} + x_{B2})$

$\leq \frac{1}{2}(2w_A + 2w_B)$

$= w_A + w_B$ Feasible under $S = \{A2, B2\}$

So, $S = \{A2, B2\}$ blocks x, which contradicts with x being in the core

Hence $x_{A1} = x_{A2}$, $x_{B1} = x_{B2}$

Let C_r be the core of an r-fold economy, $\mathcal{E}_r$.

By the equal treatment property,

$(\underbrace{x_A, \cdots, x_A}_{r\text{ times}}, \underbrace{x_B, \cdots, x_B}_{r\text{ times}})$

1-fold: $(x_A, x_B) \in C_1$

2-fold: $(x_A', x_A', x_B', x_B') \to (x_A', x_B') \in C_2$.

$(x_A, x_B) \in C_1 \not\Rightarrow (x_A, x_A, x_B, x_B) \in C_2$ in general

$(x_A', x_A', x_B', x_B') \in C_2 \Rightarrow (x_A', x_B') \in C_1$ always

Lemma: (Shrinking Core)

$C_1 \supseteq C_2 \supseteq \cdots \supseteq C_r \supseteq \cdots$

proof: Suppose $x = (x_A, x_B) \in C_r$, but $x \notin C_{r-1}$

Then there is some coalition in $\mathcal{E}_{r-1}$ that blocks x

But the same coalition could also be formed in $\mathcal{E}_r$,

contradicting with $x \in C_r$.

Then: $x = (x_A, x_B) \in C_r \Rightarrow x = (x_A, x_B) \in C_{r-1}$

$C_{r-1} \supseteq C_r$.

As $r \to +\infty$, C_r shrinks to WEA.

Theorem. (Edgeworth - Debreu - Scarf)

If $x \in C_r$ for every $r = 1, 2, \cdots$, then x is a WEA in $\mathcal{E}_1$.

Equivalently, if x is not a WEA in $\mathcal{E}_1$, then there is an $\hat{r}$-fold economy such that $x \notin C_{\hat{r}}$.

proof: Again, consider two types of consumers A and B.

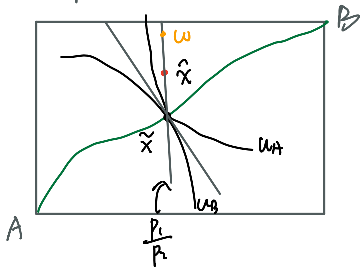
Suppose $\tilde{x} \in C_r \ \forall r$ but $\tilde{x}$ is not a WEA in $\mathcal{E}_1$.

Then, $MRS^A_{2,1}(\tilde{x}_A) = MRS^B_{2,1}(\tilde{x}_B)$

but either $MRS^A_{2,1}(\tilde{x}_A) \neq \frac{p_1}{p_2}$ (otherwise $\tilde{x}$ would be WEA)

or $MRS^B_{2,1}(\tilde{x}_B) \neq \frac{p_1}{p_2}$.

W.l.o.g. suppose $p_1/p_2 > MRS^A_{2,1}$



Then $\exists \hat{x}$ such that $u_A(\hat{x}_A) > u_A(\tilde{x}_A)$

$\hat{x}_A = \frac{1}{r} w_A + \frac{r-1}{r} \tilde{x}_A$ for some $r > 1$.

Consider a coalition S of r type A and $(r-1)$ type B consumers.

Let $\bar{x}_A = \hat{x}_A$, $\bar{x}_B = \tilde{x}_B$ within S.

Clearly, $u_A(\bar{x}_A) > u_A(\tilde{x}_A)$

and $u_B(\bar{x}_B) = u_B(\tilde{x}_B)$

Also $(\bar{x}_A, \bar{x}_B)$ is feasible within S because: $r\bar{x}_A + (r-1)\bar{x}_B = r\hat{x}_A + (r-1)\tilde{x}_B$

$$= r\left[\frac{1}{r} w_A + \frac{r-1}{r}\tilde{x}_A\right] + (r-1)\tilde{x}_B$$

$$= w_A + (r-1)(\tilde{x}_A + \tilde{x}_B)$$

$$\leq w_A + (r-1)(w_A + w_B)$$

$$\therefore r\bar{x}_A + (r-1)\bar{x}_B \leq r w_A + (r-1) w_B$$

Hence, S blocks $\tilde{x} \in C_r$, a contradiction. Hence, $\tilde{x}$ is a WEA in $\mathcal{E}_1$