1. Real Numbers

1.2 The set of real numbers

Cardinality: $|A| = |B|$ if there exists a bijection $f : A \to B$

1. A is finite if empty or $|A| = |\{1, ..., n\}|$ for some n; otherwise, infinite
2. A is countably infinite if $|A| = |N|$
3. A is countable if finite or countably infinite

Examples: $|\mathbb{Z}| = |\mathbb{N}| = |\mathbb{Q}|$ — coutably infinite; $\mathbb{R}$, $\{x_n\}$ where $x_n = 0/1$ — uncountable

An ordered set is a set S along with a binary relation < satisfying:

1. (Trichotomy property) For any $x, y \in S$, exactly one of $x < y$, $x = y$, or $y < x$ holds
2. (Transitive property) If $x < y$, and $y < z$, then $x < z$

Let $E \subseteq S$, where S is an ordered set

1. upper bound: $b \in S$ s.t. $x \leq b$ for all $x \in E$
2. lower bound: $b \in S$ s.t. $x \geq b$ for all $x \in E$
3. least upper bound $\sup$: $b_0 \leq b$ for all upper bounds b
4. greatest lower bound $\inf$: $b_0 \geq b$ for all lower bounds b

LUB property: An ordered set S has the least-upper-bound property if every non-empty subset $E \subset S$ that is bounded above has a least upper bound, that is, $\sup E \in S$ exists.

Characterizaion of $\mathbb{R}$: There is a unique ordered field $\mathbb{R}$ with the least upper bound property that contains $\mathbb{Q}$.

Archimedean Property of $\mathbb{R}$: If $x, y \in \mathbb{R}$ and $x > 0$, then there exists an $n \in \mathbb{N}$ s.t. $nx > y$.

"$\mathbb{Q}$ is dense in $\mathbb{R}$": If $x, y \in \mathbb{R}$ and $x < y$, then there exists an $r \in \mathbb{Q}$ s.t. $x < r < y$.

If $\sup A \in A$, A has a maximum: $\max A = \sup A$

If $\inf A \in A$, A has a minimum: $\min A = \inf A$

1.3 Absolute value and bounded functions

Absolute values

1. $|x| \leq y$ iff $-y \leq x \leq y$
2. $-|x| \leq x \leq |x|$

Triangle inequality: $|x + y| \leq |x| + |y|$ (proof combining 1 and 2)

Reverse triangle inequality: $||x| - |y|| \leq |x - y| \quad (\leq |x| + |-y| = |x| + |y|)$