Parameterized Surfaces

$$\vec{r}(u,v) = x(u,v)\hat{i} + y(u,v)\hat{j} + z(u,v)\hat{k}$$

① $\vec{r}(u,v) = \langle u+v,\ u-2v,\ 3+u-v \rangle$

Do $P(4,-5,1)$ & $Q(0,4,6)$ lie on the surface?

P:

(i) $u+v=4$

(ii) $u-2v=-5$

(iii) $3+u-v=1$

$3+2u=5$

$\boxed{u=1,\ v=3}$

verify all 3 satisfied

(i) ✓
(ii) ✓
(iii) ✓

Q:

(i) $u+v=0$

(ii) $u-2v=4$

(iii) $3+u-v=6$

$3+2u=6$

$u=\frac{3}{2}$

$v=-3/2$ by (i)

check

$\frac{3}{2}+(-2)(-\frac{3}{2}) \overset{?}{=} 4$

$\frac{3}{2}+3 \neq 4$

Contradiction.

Q not on surface!.

How can I visualize this? $\vec{r}(u,v) = \langle u+v,\ u-2v,\ 3+u-v \rangle$

fix u:

$u=0$ $\quad \vec{r}(0,v) = \langle v, -2v, 3-v \rangle$

$u=1$ $\quad \vec{r}(1,v) = \langle 1+v, 1-2v, 4-v \rangle$

$u=2$ $\quad \vec{r}(2,v) = \langle 2+v, 2-2v, 5-v \rangle$

} Traces out plane

⑤ $\vec{r}(s,t) = \langle s\cos t,\ s\sin t,\ s \rangle$

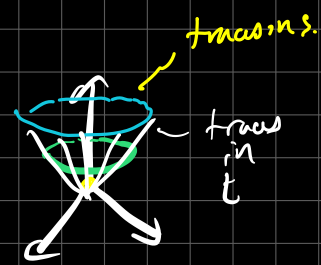
height depends on s.

Circle in xy - radius depends on s.

for $s=0$ $\quad \vec{r}(0,t) = \langle 0,0,0 \rangle$

$s=1$ $\quad \vec{r}(1,t) = \langle \cos t, \sin t, 1 \rangle$

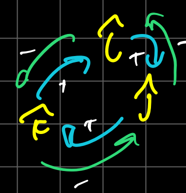
$\vec{r}(2,t) = \langle 2\cos t, 2\sin t, 2 \rangle$



Cone

(19) Plane through origin containing $\underline{\hat{i}-\hat{j}}$ and $\underline{\hat{j}-\hat{k}}$

find a param. of the plane.



$$\vec{n} = (\hat{i}-\hat{j}) \times (\hat{j}-\hat{k}) = \hat{i}\times\hat{j} - \hat{i}\times\hat{k} - \hat{j}\times\hat{j} + \hat{j}\times\hat{k}$$
$$= \hat{k} - (-\hat{j}) + \vec{0} + \hat{i}$$
$$= \hat{i} + \hat{j} + \hat{k}$$

normal vector : $\langle 1,1,1 \rangle$ point $(0,0,0)$

$$1(x-0) + 1(y-0) + 1(z-0) = 0$$
$$x+y+z=0 \rightarrow z = -x-y$$

now param.

option (1) $\vec{r}(x,y) = \langle x, y, -x-y \rangle$

option (2) let $x=u$, $y=v$

option (3) $x = -y-z$ $\vec{r}(y,z) = \langle -y-z, y, z \rangle$

⋮

advice: if $z = f(x,y)$ and you have to param.

let $x=x$, $y=y$, $z=f(x,y)$.

$$\vec{r}(x,y) = \langle x, y, f(x,y) \rangle$$

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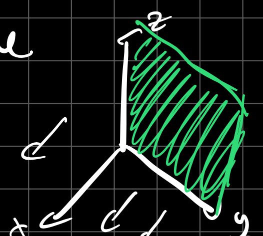
(21) find parametric representation of the part of $4x^2-4y^2-z^2=4$

that lies in front of yz plane



$\boxed{x>0}$ b/c of this solve for x

$4x^2-4y^2-z^2=4$

$$4x^2 = 4+4y^2+z^2 \Rightarrow x^2 = 1+y^2+\tfrac{1}{4}z^2 \Rightarrow x = \sqrt{1+y^2+\tfrac{1}{4}z^2}$$

$$\vec{r}(y,z) = \left\langle \sqrt{1+y^2+\tfrac{1}{4}z^2},\ y,\ z \right\rangle.$$

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(32)

$$x = 2\cos\theta + r\cos(\theta/2) \qquad -\tfrac{1}{2} \le r \le \tfrac{1}{2}$$
$$y = 2\sin\theta + r\sin(\theta/2) \qquad 0 \le \theta \le 2\pi.$$
$$z = r\sin(\theta/2)$$

use a grapher to view this. (see zoom video). @ 1:02

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(34)

$x = u^2+1$ find tangent plane to surface at given point.

$y = v^3+1$

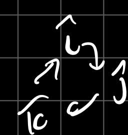
$z = u+v$ need point & normal vector.

point: $(5,2,3)$ need to find.

We have: $\vec{r}(u,v) = \langle u^2+1,\ v^3+1,\ u+v \rangle$

$$\begin{cases} \vec{r}_u(u,v) = \langle 2u, 0, 1 \rangle \\ \vec{r}_v(u,v) = \langle 0, 3v^2, 1 \rangle \end{cases}$$

think $\vec{r},\ \vec{r}\,'(t)$



$$\vec{r}_u \times \vec{r}_v = (2u\hat{i} + \hat{k}) \times (3v^2\hat{j} + \hat{k})$$
$$= 2uv^2\hat{k} + 2u(-\hat{j}) + 3v^2(-\hat{i})$$
$$= -3v^2\hat{i} - 2u\hat{j} + 2uv^2\hat{k}$$

We are at $(5,2,3)$

$x = u^2+1$ $\Rightarrow$ $5 = u^2+1 \to u = \pm 2$

and: $y = v^3+1$ $\qquad 2 = v^3+1 \to v = \pm 1$

$z = u+v$ $\qquad 3 = u+v \to u = 2, v = 1$

$(\vec r_u \times \vec r_v)(2,1) = \underbrace{-3\hat i - 4\hat j + 4\hat k}_{\text{normal vector.}}$

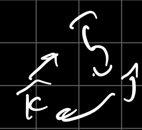
plane: $-3(x-5) - 4(y-2) + 4(z-3) = 0.$

Area of a surface.

Recall: what was arc length? $\displaystyle\int_a^b \underline{\|\vec r\,'(t)\|}\,dt$ when C is traced out by $\vec r(t)$, $a \le t \le b$

now have $\vec r(u,v)$

Area of surface $\displaystyle\iint_D \|\vec r_u \times \vec r_v\|\,dA$



if let $\boxed{z = f(x,y) \quad \vec r = \langle x, y, f(x,y)\rangle}$

$\vec r_x = \langle 1, 0, f_x\rangle$

$\vec r_y = \langle 0, 1, f_y\rangle$

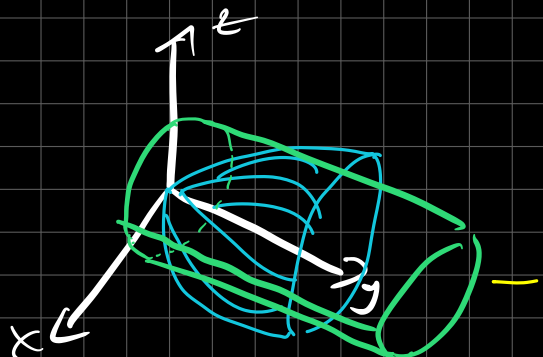
$\vec r_x \times \vec r_y = (\hat i + f_x \hat k) \times (\hat j + f_y \hat k)$

$= \hat k + f_y(-\hat j) + f_x(-\hat i)$

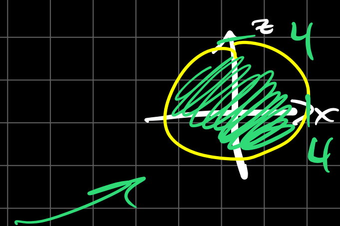
$= -f_x \hat i - f_y \hat j + \hat k$

$\|\vec r_x \times \vec r_y\| = \sqrt{f_x^2 + f_y^2 + 1}$

(47) find surface area of part of paraboloid $y = x^2 + z^2$ lies within $x^2 + z^2 = 16$



restrict x & z $\quad x^2 + z^2 \le 16$



need to param. $y = x^2 + z^2$, $x^2 + z^2 \leq 16$

r, θ

$x = r\cos\theta$

$z = r\sin\theta$

$x^2 + z^2 = r^2$

two options

(A) $\vec{r}(x,z) = \langle x, x^2+z^2, z \rangle$

(B) $\vec{r}(r,\theta) = \langle r\cos\theta, r^2, r\sin\theta \rangle$

use (A) $\vec{r}_x = \langle 1, 2x, 0 \rangle$ $\vec{r}_z = \langle 0, 2z, 1 \rangle$

$$\vec{r}_x \times \vec{r}_z = (\hat{\imath} + 2x\hat{\jmath}) \times (2z\hat{\jmath} + \hat{k}) = 2z\hat{k} - \hat{\jmath} + 2x\hat{\imath}$$

$$= 2x\hat{\imath} - \hat{\jmath} + 2z\hat{k}$$

$$\|\vec{r}_x \times \vec{r}_z\| = \sqrt{(2x)^2 + (1)^2 + (2z)^2}$$

$$= \sqrt{4x^2 + 4z^2 + 1}$$

could have just said:

$y = f(x,z)$

$$S.A. = \iint_D \sqrt{f_x^2 + f_z^2 + 1}\, dA$$

$$SA = \iint_D \sqrt{4x^2 + 4z^2 + 1}\, dA$$

$0 \leq r \leq 4$

$0 \leq \theta \leq 2\pi$

$x = r\cos\theta$

$z = r\sin\theta$

$x^2 + z^2 = r^2$

$$= \int_0^{2\pi} \int_0^4 \sqrt{4r^2+1}\; r\, dr\, d\theta$$

simple double integral.