1. (5 points) G is a group. H is a subgroup of G and N is a normal subgroup of G. H acts on G/N by left multiplication, i.e., for any $h \in H$, $gN \in G/N$,

$$h.gN = (hg)N$$

   Prove that the above action is transitive if and only if $G = HN$.

2. M_2 is the group of isometries on $\mathbb{R}^2$. Let H be the subset of M_2 consisting of all translations and all rotations (around any point).

   (i). (5 points) Prove that H is a subgroup of M_2

   (ii). (5 points) If T is the group of translations on $\mathbb{R}^2$ and R is the group of rotations around origin on $\mathbb{R}^2$, prove that $M_2 = T \rtimes R$.

   (iii). (5 points) What is $[M_2 : H]$, the index of H in M_2? Prove your answer.

3. (5 points) G is a group. $Z(G) = \{g \in G | \forall x \in G, gx = xg\}$. If $[G : Z(G)] = k$, prove that each conjugacy class of G has at most k elements.

4. (5 points) G is a finite group, p is a prime and p divides $|G|$. N is a normal subgroup of G and P is a Sylow p-subgroup of G. Prove that PN/N is a Sylow p-subgroup of G/N.

5. (5 points) Let $\mathbb{Z}$ be the group of integers with addition as composition.

   $G = \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$. $\phi : A_3 \longrightarrow \mathrm{Aut}(G)$ is the homomorphism defined by

$$\phi_\sigma : G \longrightarrow G$$

$$(a_1, a_2, a_3) \mapsto (a_{\sigma(1)}, a_{\sigma(2)}, a_{\sigma(3)})$$

   Find all the elements of finite order in $G \rtimes_\phi A_3$.

6. (5 points) Let $I = (x^2 + 2) \subseteq \mathbb{R}[x]$. Find the multiplicative inverse of

$$2x + 1 + I \in \mathbb{R}[x]/I$$

7. (5 points) R is a ring. Prove that $I = (x)$ is a maximal ideal of $R[x]$ if and only if R is a field.

8. (5 points) Classify groups of order 45 up to isomorphism.