New York University — Courant Institute, MATH-GA.2450-001

Complex Variables I

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Disclaimer:

These are lecture notes for the course *Complex Variables I (MATH-GA.2450-001)*, given at New York University in Fall 2022.

These notes are preliminary and may contain typos. If you see any mistakes or think that the presentation is unclear and could be improved, please send an email to: maximilian.nitzschner@cims.nyu.edu. All comments and suggestions are appreciated.

Contents

0. Motivation

This course will investigate the analysis of *complex variables*. Let us sketch some aspects that motivate its study.

0.1. Holomorphic functions

Real analysis is concerned with developing tools for differentiation and integration of certain real functions $f : \mathbb{R}^m \to \mathbb{R}^n$. In this course we will study *complex functions* $f : \mathbb{C} \to \mathbb{C}$, and define a similar notion as for real differentiability, namely that of *holomorphic functions*.

This notion will turn out to exhibit striking features, which the notion of real differentiability lacks. For instance, we will show:

- Every holomorphic function admits arbitrarily many complex derivatives.
- Every holomorphic function is *analytic*, meaning it can be developed locally into a uniformly converging Taylor series.

$\rightsquigarrow$ holomorphic functions as a *natural* class of functions as object of study (Traditional names of complex analysis are *Theory of Functions*, *Théorie des Fonctions*, *Funktionentheorie*,..., often omitting reference to complex variables in the name).

0.2. Applications in real analysis

Complex analysis may also be seen as a versatile tool for problems in real analysis. We will see in this course

- ... how to calculate real integrals using *residue calculus*, which may be hard or impossible to calculate without it, such as

$$\int_{-\infty}^{\infty} \frac{\sin^2(x)}{x^2} \mathrm{d}x = \frac{\pi}{2}, \qquad \int_0^{2\pi} \frac{1}{a + \sin(\theta)} \mathrm{d}\theta = \frac{2\pi}{\sqrt{a^2 - 1}}, a > 0.$$

- ... an elementary proof of the *fundamental theorem of algebra*, stating that every non-constant polynomial p of degree $n \geq 1$ with real coefficients can be factorized as $p(X) = \gamma \prod_{j=i}^{n} (X - \alpha_j)$, with $\gamma, \alpha_j \in \mathbb{C}$.

1. Complex numbers

(Reference: Marsden-Hoffman, Basic complex analysis, 3rd Ed., Sections 1.1–1.3)

Starting Point: We will take from Real analysis the existence and uniqueness of a totally ordered, complete field, the *real numbers* denoted by $\mathbb{R}$.

1.1. Definition of complex numbers

Historically (16th century): The equation

$$x^2 + 1 = 0 \tag{1.1}$$

has no solutions over $\mathbb{R}$. Idea to remedy this: Introduce *ad hoc* a quantity $i(=\sqrt{-1})$, the *imaginary unit*, fulfilling

$$i^2 = -1 \tag{1.2}$$

and define *complex numbers* as expressions of the form $x + yi$ with $x, y \in \mathbb{R}$. We then treat i as a symbol and require the usual rules of arithmetics to hold: Two such complex numbers $x + yi$ and $x' + y'i$ can be added and multiplied

$$(x + yi) + (x' + y'i) = (x + x') + (y + y')i, \tag{1.3}$$

$$(x + yi) \cdot (x' + y'i) = xx' + xy'i + x'yi + x'y'i^2 = (xx' - yy') + (xy' + x'y)i. \tag{1.4}$$

A real number x is then identified with the expression $x + 0 \cdot i$, and $0 + yi$ for some real numbers y is abbreviated as yi and called *purely imaginary*. Clearly, $0(= 0 + 0i)$ is the only complex number that is both real and purely imaginary.

If $x + yi \neq 0$ (meaning either $x \neq 0$ or $y \neq 0$), one also has

$$(x + yi) \cdot \left(\frac{x}{x^2 + y^2} - \frac{y}{x^2 + y^2} i \right) = 1, \tag{1.5}$$

meaning that every non-zero number has a multiplicative inverse.

Let us be slightly more precise in describing the structure of the complex numbers.

Definition 1.1. A *field* is a non-empty set F together with two binary operations

$$+ : \begin{cases} F \times F \to F \\ (u, v) \mapsto u + v \end{cases} \qquad \text{and} \qquad \cdot : \begin{cases} F \times F \to F \\ (u, v) \mapsto u \cdot v \end{cases} \tag{1.6}$$

such that

(i) $(F, +)$ is an *abelian group*, meaning that (i.1) For every $u, v, w \in F$: $(u + v) + w = u + (v + w)$ (associativity of $+$).

(i.2) There is an element $0_F \in F$ such that $u + 0_F = u$ for every $u \in F$ (neutral element for $+$).

(i.3) For every $u \in F$, there is an element $(-u) \in F$ fulfilling $u + (-u) = 0_F$ (inverse element for $+$).

(i.4) For every $u, v \in F$: $u + v = v + u$ (commutativity of $+$).

(ii) $(F \setminus \{0_F\}, \cdot)$ is an abelian group[1], meaning that

(ii.1) For every $u, v, w \in F \setminus \{0_F\}$: $(u \cdot v) \cdot w = u \cdot (v \cdot w)$ (associativity of $\cdot$).

(ii.2) There is an element $1_F \in F$ such that $u \cdot 1_F = u$ for every $u \in F \setminus \{0_F\}$ (neutral element for $\cdot$).

(ii.3) For every $u \in F \setminus \{0_F\}$, there is an element $u^{-1} \in F$ fulfilling $u \cdot (u^{-1}) = 1_F$ (inverse element for $\cdot$).

(ii.4) For every $u, v \in F \setminus \{0_F\}$: $u \cdot v = v \cdot u$ (commutativity of $\cdot$).

(iii) The distributive law holds: For every $u, v, w \in F$, one has $(u + v) \cdot w = u \cdot v + u \cdot w$.

Classical examples of fields include $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ for p prime, the field $\mathbb{R}(X)$ of rational functions in one variable over $\mathbb{R}$, ...

We shall now see that $\mathbb{R}^2$ can be turned into a field by choosing an appropriate multiplication, and this will in fact be the field $\mathbb{C}$ of complex numbers, with the rules (1.2)–(1.4).

Theorem 1.2. *We define on $\mathbb{R}^2$ the addition and multiplication*

$$(x, y) + (x', y') = (x + x', y + y'), \qquad (x, y) \cdot (x', y') = (xx' - yy', xy' + yx'). \tag{1.7}$$

Then $\mathbb{C} := \mathbb{R}^2$ with these operations is a field with neutral element $0_{\mathbb{C}} = (0, 0)$ for $+$ and neutral element $1_{\mathbb{C}} = (1, 0)$ for $\cdot$. The field $\mathbb{R}$ is contained in $\mathbb{C}$ as a subfield via the map $x \mapsto (x, 0)$. With $i := (0, 1)$, every number $z = (x, y) \in \mathbb{C}$ has a unique representation as $z = x + y \cdot i$ with $x, y \in \mathbb{R}$.

Proof. Since $\mathbb{R}^2$ is an $\mathbb{R}$-vector space with $(0, 0)$ as zero element, part (i) of the field axioms follows. For (ii.1), consider $z = (x, y)$, $z' = (x', y')$ and $z'' = (x'', y'')$, then

$$\begin{aligned}
(z \cdot z') \cdot z'' &= (xx' - yy', xy' + x'y) \cdot (x'', y'') \\
&= (xx'x'' - yy'x'' - xy'y'' - x'yy'', xx'y'' - yy'y'' + xy'x'' + x'yx'') \\
&= (x, y) \cdot (x'x'' - y'y'', x'y'' + x''y') \\
&= z \cdot (z' \cdot z'').
\end{aligned}$$

For (ii.2), we calculate

$$(x, y) \cdot (1, 0) = (x \cdot 1 - y \cdot 0, x \cdot 0 + y \cdot 1) = (x, y),$$

[1] By (iii) below, one has $0_F \cdot u = (0_F + 0_F) \cdot u = 0_F \cdot u + 0_F \cdot u$, so in fact $0_F \cdot u = 0_F$ by (i.3). For this reason (ii.1), (ii.2) and (iii.4) hold for every $u, v, w \in F$.

so indeed, $(1,0) = 1_{\mathbb{C}}$. For (ii.3), looking at (1.5), we calculate for $z = (x,y) \neq (0,0)$:

$$(x,y) \cdot \left(\frac{x}{x^2+y^2}, -\frac{y}{x^2+y^2} \right) = \left(\frac{x^2 - y(-y)}{x^2+y^2}, \frac{x(-y)+yx}{x^2+y^2} \right) = (1,0) = 1_{\mathbb{C}},$$

so $(\frac{x}{x^2+y^2}, -\frac{y}{x^2+y^2}) = z^{-1}$ is an inverse of z for $\cdot$. Finally, we need to check (ii.4) and (iii), so let again $z = (x,y)$, $z' = (x',y')$ and $z'' = (x'',y'')$, then

$$\begin{aligned} z \cdot z' &= (x,y) \cdot (x',y') = (xx' - yy', xy' + yx') \\ &= (x'x - y'y, y'x + x'y) = (x',y') \cdot (x,y) = z' \cdot z, \\ (z+z') \cdot z'' &= (x+x', y+y') \cdot (x'',y'') \\ &= (xx'' + x'x'' - yy'' - y'y'', xy'' + x'y'' + yx'' + y'x'') \\ &= (x,y) \cdot (x'',y'') + (x',y') \cdot (x'',y'') = z \cdot z'' + z' \cdot z''. \end{aligned}$$

Consider the map $\varphi : \mathbb{R} \to \mathbb{C}$, $\varphi(x) = (x,0)$. This map fulfills for $x, x' \in \mathbb{R}$:

$$\begin{aligned} \varphi(x+x') &= (x+x',0) = (x,0) + (x',0) = \varphi(x) + \varphi(x'), \\ \varphi(x \cdot x') &= (x \cdot x', 0) = (x,0) \cdot (x',0) = \varphi(x,0) \cdot \varphi(x',0) \\ \varphi(1) &= (1,0) = 1_{\mathbb{C}}, \end{aligned}$$

and since $\varphi(x) = (0,0)$ entails that $x = 0$, it is an injective homomorphism of fields[2], therefore $\mathbb{R}$ can be identified with the subfield $\varphi(\mathbb{R}) = \{(x,0); x \in \mathbb{R}\} \subseteq \mathbb{C}$.

Finally, we note that $z = (x,y)$ can be written as

$$(x,y) = (x,0) + (0,y) = \underbrace{(x,0)}_{=\varphi(x)} + \underbrace{(y,0)}_{=\varphi(y)} \cdot \underbrace{(0,1)}_{=i} = x + y \cdot i,$$

where in the last line we identified by abuse of notation x,y with their respective images under φ. Uniqueness of this representation is immediate, since $x + y \cdot i = a + b \cdot i$ translates to $(x,y) = (a,b)$ in $\mathbb{R}^2$. $\square$

Some remarks are in order.

Remark 1.3. (i) The above Theorem justifies the notation $x + yi$ for a complex number $z = (x,y)$, which we will mostly use. In the notation $z = x + yi$ we implicitly understand that x, y are real numbers.

(ii) The inverse of $z = x + yi \neq 0$ can be intuitively calculated as follows:

$$\frac{1}{z} = \frac{1}{x+yi} \cdot \frac{x-yi}{x-yi} = \frac{x-yi}{x^2+y^2}. \tag{1.8}$$

This also allows for a simple calculation of quotients w/z for $z \neq 0$.

(iii) The field $\mathbb{C}$ cannot be *ordered*: Assume to the contrary, it was ordered, then either $i > 0$ or $i < 0$. In the first case, $i^3 = i^2 \cdot i = -i < 0$, a contradiction (since the product of positive numbers is positive), whereas in the second case $(-i) > 0$, and with $(-i)^3 = -(-i) < 0$ we obtain again a contradiction.

(iv) None of the $\mathbb{R}^n$, $n \geq 3$ can be turned into a field.

[2] In fact, every homomorphism $\psi : K \to L$ between fields is injective.

1.2. First properties of complex numbers

We will now turn to some elementary properties of the complex numbers $\mathbb{C}$ that we introduced in the previous section.

Definition 1.4. Let $z = x + yi \in \mathbb{C}$. We call x its *real part* and y its *imaginary part*, denoted

$$\mathrm{Re}(z) = x, \qquad \mathrm{Im}(z) = y, \tag{1.9}$$

respectively (recall that the representation $z = x + yi$ with $x, y \in \mathbb{R}$ is unique. Moreover, we define the *modulus* of z by

$$|z| = \sqrt{x^2 + y^2} \; (\in \mathbb{R}_0^+ = [0, \infty)). \tag{1.10}$$

Furthermore, we call the number

$$\overline{z} = x - yi \tag{1.11}$$

the *complex conjugate* of z.

Recall that geometrically, $\mathbb{C} = \mathbb{R}^2$, so we may visualize $z = (x, y)$ as a point in the plane. With this, the modulus is simply the (Euclidean) distance of z to the origin, and complex conjugation can be understood as reflection at the real axis. Adding and subtracting complex numbers will simply correspond to known vector addition / subtraction.

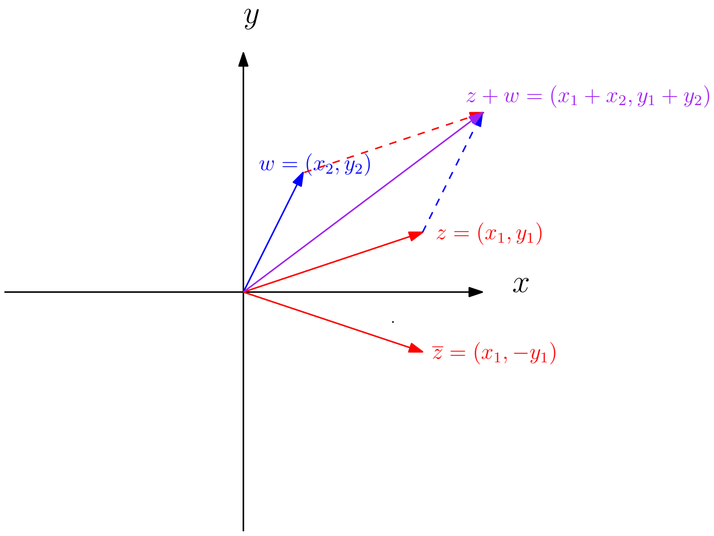

Figure 1.1.: Addition of two complex numbers and complex conjugation.

We discuss geometric interpretations of other operations below with the introduction of polar coordinates.

Proposition 1.5. *The following properties hold for $z, w \in \mathbb{C}$:*

(i) $\overline{z+w} = \overline{z} + \overline{w}$ *and* $\overline{zw} = \overline{z} \cdot \overline{w}$.

(ii) $|z|^2 = z \cdot \overline{z} = |\overline{z}|^2$.

(iii) $z = \overline{z}$ *if and only if* $z \in \mathbb{R}$.

(iv) $\overline{z/w} = \overline{z}/\overline{w}$ *for* $w \neq 0$.

(v) $\mathrm{Re}(z) = \frac{1}{2}(z + \overline{z})$ *and* $\mathrm{Im}(z) = \frac{1}{2i}(z - \overline{z})$.

(vi) $\overline{\overline{z}} = z$.

Proof. Let $z = x + yi$ and $w = a + bi$ for the entire proof, and so $\overline{z} = x - yi$ and $w = a - bi$. One has

$$\begin{aligned} \overline{z+w} &= \overline{(x+a) + (y+b)i} = x + a - (y+b)i = \overline{z} - \overline{w}, \\ \overline{z \cdot w} &= \overline{(xa - yb) + (xb + ya)i} = (xa - yb) - (xb - ya)i = \overline{z} \cdot \overline{w}, \end{aligned}$$

establishing (i). For (ii), calculate

$$z \cdot \overline{z} = (x + yi) \cdot (x - yi) = x^2 + y^2 = |z|^2,$$

and clearly $|\overline{z}|^2 = x^2 + (-y)^2 = |z|^2$. Item (iii) is immediate from the definition ($x + yi = x - yi \;\Leftrightarrow\; y = 0$). For (iv), note that by (i)

$$\overline{w} \cdot \overline{\left(\frac{z}{w}\right)} = \overline{w \cdot \frac{z}{w}} = \overline{z},$$

and the claim follows by division by $\overline{w}(\neq 0)$. Items (v) and (vi) follow by insertion of the definition, for instance:

$$\frac{z + \overline{z}}{2} = \frac{x + yi + x - yi}{2} = x = \mathrm{Re}(z).$$

□

We will need some additional properties of the modulus of a complex number.

Proposition 1.6. *The following properties hold for $z, w, z_1, ..., z_n, w_1, ..., w_n \in \mathbb{C}$:*

(i) $|z| \geq 0$ *and* $|z| = 0$ *if and only if* $z = 0$.

(ii) $-|z| \leq \mathrm{Re}(z) \leq |z|$ *and* $-|z| \leq \mathrm{Im}(z) \leq |z|$.

(iii) $|z \cdot w| = |z||w|$ *and* $|\frac{z}{w}| = \frac{|z|}{|w|}$ *if* $w \neq 0$.

(iv) (Triangle inequalities) $|z + w| \leq |z| + |w|$ *and* $|z - w| \geq ||z| - |w||$.

(v) (Cauchy-Schwarz inequality)

$$\left|\sum_{j=1}^{n} \overline{z_j} w_j\right| \leq \sqrt{\sum_{j=1}^{n} |z_j|^2} \sqrt{\sum_{j=1}^{n} |w_j|^2} \tag{1.12}$$

Proof. Items (i) and (ii) are clear from the definition. For item (iii), consider using Proposition 1.5, (ii) and (i)

$$|z \cdot w|^2 = zw\overline{zw} = z\overline{z} \cdot w\overline{w} = |z|^2|w|^2,$$

and we may take the square root (in $\mathbb{R}$, since every modulus is nonnegative). Moreover, we see that $|w| \cdot |\frac{z}{w}| = |w \cdot \frac{z}{w}| = |z|$, so the second assertion in (ii) follows as well. For (iv), expand first (using again Proposition 1.5, (ii) and (i))

$$|z+w|^2 = (z+w)(\overline{z}+\overline{w}) = |z|^2 + |w|^2 + z\overline{w} + w\overline{z} = |z|^2 + |w|^2 + 2\mathrm{Re}(z\overline{w}),$$

upon use of Proposition 1.5, (v), in the last step. Now by (iii), $\mathrm{Re}(z\overline{w}) \leq |z\overline{w}| = |z||w|$, so in fact:

$$|z+w|^2 \leq (|z|+|w|)^2,$$

and the first part of (iv) follows by taking the square root in $\mathbb{R}$. The second triangle inequality follows from the first, since we can replace z by $z-w$ to obtain:

$$|z| = |z-w+w| \leq |z-w| + |w| \qquad \Rightarrow \qquad |z-w| \geq |z| - |w|.$$

Interchanging the roles of z and w yields that also

$$|z-w| \geq |w| - |z|,$$

and the claim follows. Finally, for (v), we use a proof that works for general inner products in Hilbert spaces: The claim is trivial if all w_j, $j = 1, ..., n$ are zero, so we assume that this is not the case. Consider on $\mathbb{C}^n$ the form $\langle \cdot, \cdot \rangle : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$, defined by ($Z = (z_1, ..., z_n)^\top, W = (w_1, ..., w_n)^\top \in \mathbb{C}^n$):

$$\langle Z, W \rangle = \left\langle \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix}, \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix} \right\rangle = \sum_{j=1}^{n} \overline{z_j} w_j. \tag{1.13}$$

This form is positive definite (meaning $\langle Z, Z \rangle \geq 0$ for any $Z \in \mathbb{C}^n$ and $\langle Z, Z \rangle = 0$ if and only if $Z = 0$), sesquilinear (meaning $\langle Z + \alpha Z', W \rangle = \langle Z, W \rangle + \overline{\alpha}\langle Z', W \rangle$ and $\langle Z, W + \beta W' \rangle = \langle Z, W \rangle + \beta \langle Z, W' \rangle$ for any $Z, Z', W, W' \in \mathbb{C}^n$ and $\alpha, \beta \in \mathbb{C}$) and Hermitian (meaning $\langle Z, W \rangle = \overline{\langle W, Z \rangle}$). We immediately see that for $\alpha \in \mathbb{C}$:

$$\begin{aligned} 0 \leq \langle Z - \alpha W, Z - \alpha W \rangle &= \langle Z, Z \rangle + |\alpha|^2 \langle W, W \rangle - \overline{\alpha}\langle W, Z \rangle - \alpha \langle Z, W \rangle \\ &= \langle Z, Z \rangle + |\alpha|^2 \langle W, W \rangle - 2\mathrm{Re}(\alpha \langle Z, W \rangle) \end{aligned}$$

Choosing $\alpha = \frac{\langle W,Z\rangle}{\langle W,W\rangle}$, we find

$$0 \leq \langle Z, Z\rangle + \frac{|\langle Z, W\rangle|^2}{\langle W, W\rangle} - 2\mathrm{Re}\left(\frac{|\langle Z, W\rangle|^2}{\langle W, W\rangle}\right) = \langle Z, Z\rangle - \frac{|\langle Z, W\rangle|^2}{\langle W, W\rangle}.$$

Rearranging and using $Z = (z_1, \ldots, z_n)^\top, W = (w_1, \ldots, w_n)^\top \in \mathbb{C}^n$ gives the claim. $\square$

Remark 1.7. Of course, since $|\overline{z}| = |z|$ for any complex number $z \in \mathbb{C}$, we could have written the Cauchy-Schwarz inequality (1.12) as

$$\left|\sum_{j=1}^n z_j w_j\right| \leq \sqrt{\sum_{j=1}^n |z_j|^2}\sqrt{\sum_{j=1}^n |w_j|^2}. \tag{1.14}$$

1.3. Polar coordinates

We have already established that we can visualize $z = (x, y) = x + yi \in \mathbb{C}$ as an arrow in the *complex plane*. This idea allows us very naturally to introduce polar coordinates, namely to write

$$x + yi = (r\cos(\theta)) + (r\sin(\theta)) \cdot i, \tag{1.15}$$

where $r = |z| = \sqrt{x^2 + y^2}$ is the modulus of z and θ is the called the *argument* of z, denoted by $\arg(z) = \theta$. If $z \neq 0$, we can unambiguously define this number if we restrict to a (half-open) interval of length 2π, for instance $(-\pi, \pi]$.

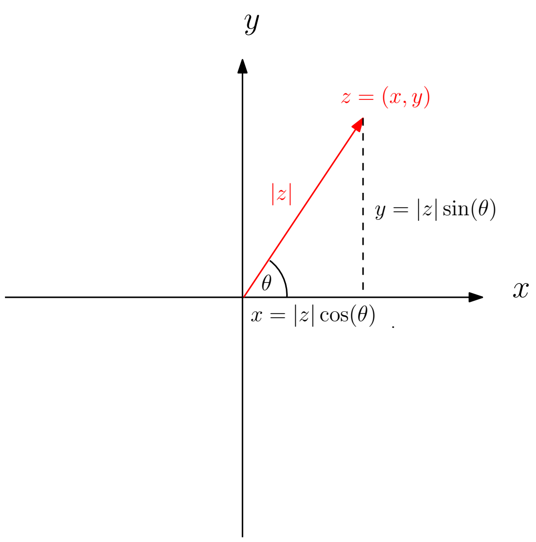

Figure 1.2.: Visualization of polar coordinates.

Definition 1.8. Let $z \in \mathbb{C} \setminus \{0\}$. We denote the unique real number $\theta \in (-\pi, \pi]$ that fulfills $\frac{z}{|z|} = (\cos(\theta) + \sin(\theta) \cdot i)$ as the *principal value of the argument* of z, denoted by $\mathrm{Arg}(z) = \theta$.

So while the principal value of the argument is a real number in $(-\pi, \pi]$, the argument can be thought of as a countable set of possible values in $\mathbb{R}$ that θ can attain to make (1.15) hold, i.e. for $z \in \mathbb{C} \setminus \{0\}$:

$$\arg(z) = \{\mathrm{Arg}(z) + 2\pi n; n \in \mathbb{Z}\} = \mathrm{Arg}(z) + 2\pi\mathbb{Z}. \tag{1.16}$$

We will now investigate the product of complex numbers in polar coordinates. Let $z, w \in \mathbb{C} \setminus \{0\}$, with

$$z = r(\cos(\theta) + \sin(\theta)i), \qquad w = s(\cos(\psi) + \sin(\psi)i).$$

By applying the definition of the multiplication of complex numbers, we readily obtain

$$\begin{aligned} z \cdot w &= rs\left[(\cos(\theta)\cos(\psi) - \sin(\theta)\sin(\psi)) + (\cos(\theta)\sin(\psi) + \sin(\theta)\cos(\psi)) \cdot i\right] \\ &= rs\left[\cos(\theta + \psi) + \sin(\theta + \psi) \cdot i\right], \end{aligned} \tag{1.17}$$

having used the *trigonometric addition rules* from real analysis. To summarize, upon multiplication of two complex numbers

- the moduli are multiplied,
- the arguments are added.

More precisely

$$|z \cdot w| = |z||w|, \qquad \arg(z \cdot w) = \arg(z) + \arg(w). \tag{1.18}$$

The latter equality is an equality between subsets (!) of $\mathbb{R}$, since we have established that $\arg(z)$ attains infinitely many values. We may also reformulate this in terms of the principal value of the argument:

$$\mathrm{Arg}(z \cdot w) = \mathrm{Arg}(z) + \mathrm{Arg}(w) \qquad \mathrm{mod}\ 2\pi, \tag{1.19}$$

where we understand to shift the sum of the arguments back into the interval $(-\pi, \pi]$ if needed.

Example 1.9. Let $z = i$ and $w = -1 + i$. Then $|z| = 1$, $|w| = \sqrt{2}$, whereas

$$\begin{aligned} \mathrm{Arg}(z) = \tfrac{\pi}{2} \qquad &\Rightarrow \qquad \arg(z) = \{\tfrac{\pi}{2} + 2n\pi; n \in \mathbb{Z}\}, \\ \mathrm{Arg}(w) = \tfrac{3\pi}{4} \qquad &\Rightarrow \qquad \arg(w) = \{\tfrac{3\pi}{4} + 2n\pi; n \in \mathbb{Z}\}. \end{aligned}$$

We have $z \cdot w = i(-1 + i) = -1 - i$. In polar coordinates:

$$\begin{aligned} \mathrm{Arg}(z \cdot w) &= (\tfrac{\pi}{2} + \tfrac{3\pi}{4}) \ \mathrm{mod}\ 2\pi = \tfrac{5\pi}{4} \ \mathrm{mod}\ 2\pi = -\tfrac{3\pi}{4} (\in (-\pi, \pi]) \\ \arg(z \cdot w) &= \{-\tfrac{3\pi}{4} + 2n\pi; n \in \mathbb{Z}\} = \{..., -\tfrac{11\pi}{4}, -\tfrac{3\pi}{4}, \tfrac{5\pi}{4}, \tfrac{13\pi}{4}, ...\}, \end{aligned}$$

and of course $|z \cdot w| = 1 \cdot \sqrt{2}$, and indeed

$$-1 - i = \sqrt{2}\left[\cos(-\tfrac{3\pi}{4}) + \sin(-\tfrac{3\pi}{4})i\right].$$

End of Lecture 1

The strength of polar coordinates comes with a simple formula for the nth power and nth roots (n a positive integer) of a complex number.

Proposition 1.10. *If $z \in \mathbb{C} \setminus \{0\}$ with polar coordinate representation $z = r(\cos(\theta) + \sin(\theta)i)$, then one has for n a positive integer (De Moivre's formula):*

$$z^n = r^n(\cos(n\theta) + \sin(n\theta)i). \tag{1.20}$$

Moreover, the equation $w^n = z$ has exactly n solutions in $\mathbb{C}$, which are given by

$$w \in \left\{ r^{\frac{1}{n}} \left(\cos(\tfrac{\theta}{n} + \tfrac{2\pi k}{n}) + \sin(\tfrac{\theta}{n} + \tfrac{2\pi k}{n})i \right) ; k \in \{0, 1, ..., n-1\} \right\}. \tag{1.21}$$

Proof. Equation (1.20) follows by iterating (1.17) (with $\psi = \theta$ and $r = s$). For (1.21), we define for $k \in \{0, ..., n-1\}$ the complex number $w_k = r^{\frac{1}{n}} \left(\cos(\frac{\theta}{n} + \frac{2\pi k}{n}) + \sin(\frac{\theta}{n} + \frac{2\pi k}{n})i \right)$, then by de Moivre's formula (1.20), we see

$$w_k^n = r(\cos(\theta + 2\pi k) + \sin(\theta + 2\pi k)i) = r(\cos(\theta) + \sin(\theta)i) = z.$$

So the $w_0, ..., w_{n-1}$ are indeed solutions. To see that these are pairwise distinct, assume that for some $k, k' \in \{0, ..., n-1\}$,

$$\cos(\tfrac{\theta}{n} + \tfrac{2\pi k}{n}) = \cos(\tfrac{\theta}{n} + \tfrac{2\pi k'}{n}), \sin(\tfrac{\theta}{n} + \tfrac{2\pi k}{n}) = \sin(\tfrac{\theta}{n} + \tfrac{2\pi k'}{n}),$$

then necessarily $\frac{2\pi(k-k')}{n} \in 2\pi\mathbb{Z}$, but since $|k - k'| < n$, we must have $k = k'$. Since the polynomial $X^n - z \in \mathbb{C}[X]$ cannot have more than n zeros[3], we have found all solutions.

□

Let us comment on this result.

Remark 1.11. (i) De Moivre's formula can be applied to find useful trigonometric identities. For instance, in the case $n = 3$, take

$$\begin{aligned} \cos(3\theta) + \sin(3\theta)i &= (\cos(\theta) + \sin(\theta)i)^3 \\ &= \cos^3(\theta) - 3\cos(\theta)\sin^2(\theta) + (3\cos^2(\theta)\sin(\theta) - \sin^3(\theta))i \end{aligned}$$

and taking real and imaginary part, respectively, we find

$$\cos(3\theta) = \cos^3(\theta) - 3\cos(\theta)\sin^2(\theta), \tag{1.22}$$
$$\sin(3\theta) = -\sin^3(\theta) + 3\sin(\theta)\cos^2(\theta). \tag{1.23}$$

We will see later (using the complex exponential) an even easier route to infer trigonometric identities using complex numbers.

[3]This is true for any polynomial of degree n over a field F. Indeed, if α is a zero of the polynomial $f(X) \in F[X] \setminus \{0\}$, one has $(X - \alpha) | f(X)$, so $f(X) = (X - \alpha)g(X)$, and g has a degree smaller than f by one unit.

(ii) Since we have established that $\mathbb{C}$ is not ordered, one should treat all roots of a complex numbers in an equal manner (in $\mathbb{R}$, we could speak of the nonnegative root of $x \geq 0$. There is no equivalent of such notion in $\mathbb{C}$). For this reason, expressions like $\sqrt{-1}$ should be avoided (it could stand for either i or $-i$).

(iii) The role of complex conjugation of a complex number z in polar coordinates is also very simple: If $z = (r\cos(\theta)+\sin(\theta)i)$, then $\overline{z} = r(\cos(\theta)-\sin(\theta)i) = r(\cos(-\theta)+\sin(-\theta)i)$ by the symmetry properties of cos and sin, so we have established that $\operatorname{Arg}(\overline{z}) = -\operatorname{Arg}(z)$ if $z \in \mathbb{C} \setminus \mathbb{R}$ (and of course the argument remains unchanged if z is real).

1.4. Elementary Complex functions

We briefly introduce some elementary complex functions, motivated by their real analogues.

Polynomials and rational functions

The most basic functions we will encounter are *polynomials*

$$P : \mathbb{C} \to \mathbb{C}, \qquad z \mapsto a_n z^n + \ldots + a_0, \tag{1.24}$$

where $n \in \{0, 1, \ldots\}$, $a_n \in \mathbb{C} \setminus \{0\}$, a_0,...,$a_{n-1} \in \mathbb{C}$. We have already established that polynomials of the form $P(z) = a_n z^n + a_0$ have exactly n zeros if $a_n, a_0 \neq 0$, and we will later see that in fact, all polynomials of degree $n \geq 1$ have *exactly* n zeros (counting multiplicities). An important result to keep in mind is:

Proposition 1.12. *If z is a zero of the polynomial function P with $a_0, \ldots, a_n \in \mathbb{R}$, then $\overline{z}$ is also a zero of P.*

Proof. Simply use

$$0 = P(z) = a_n z^n + \ldots + a_0 \qquad \Rightarrow \qquad 0 = \overline{a_n z^n + \ldots + a_0} = a_n \overline{z}^n + \ldots + a_n$$

by Proposition 1.5, (i). ☐

We will also encounter *rational functions*

$$f : \mathbb{C} \setminus \{z_1, \ldots, z_n\} \to \mathbb{C}, \qquad z \mapsto \frac{P(z)}{Q(z)}, \tag{1.25}$$

where P and Q are polynomials without common zeros, and $z_1, \ldots, z_n$ are the zeros of Q. The zeros of Q are called *poles* of f. We will study poles of rational functions in much greater detail later in the course.

A particular class of rational function is given by *linear fractions* or *Möbius transforms*: For $a, b, c, d \in \mathbb{C}$ with $ad - bc \neq 0$, these are defined by

$$f : \mathbb{C} \setminus \{-\tfrac{d}{c}\} \to \mathbb{C}, \qquad z \mapsto \frac{az + b}{cz + d}. \tag{1.26}$$

Möbius transforms have a special importance in geometry, since these are exactly the maps that transform (generalized) circles[4] into (generalized) circles.

[4]Meaning: Either a circle or a straight line.

Exponential function, trigonometric functions

Recall from real analysis the notion of the exponential function

$$\exp : \mathbb{R} \to \mathbb{R}, \qquad x \mapsto \exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}. \tag{1.27}$$

Recall that it exhibits the crucial property $\exp(u+v) = \exp(u)\exp(v)$ for every $u, v \in \mathbb{R}$.

We insert for $y \in \mathbb{R}$ the expression $y \cdot i$ *formally* in the real power series and rearrange:

$$\begin{aligned} \exp(yi) &= 1 + \frac{yi}{1!} + \frac{(yi)^2}{2!} + \frac{(yi)^3}{3!} + \frac{(yi)^4}{4!} + \frac{(yi)^5}{5!} + \dots \\ &= 1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \dots + \left(y - \frac{y^3}{3!} + \frac{y^5}{5!} - \dots\right) i \\ &= \cos(y) + \sin(y) \cdot i, \end{aligned} \tag{1.28}$$

where we used the real series expansions of sin and cos

$$\begin{aligned} \cos(y) &= \sum_{n=0}^{\infty} \frac{(-1)^n y^{2n}}{(2n)!}, \\ \sin(y) &= \sum_{n=0}^{\infty} \frac{(-1)^n y^{2n+1}}{(2n+1)!}. \end{aligned} \tag{1.29}$$

Equation (1.28) is the famous *Euler formula.*

WARNING: We have not proved anything here, since we have not established any notion of convergence in $\mathbb{C}$ yet! The above discussion was merely to motivate the following definition.[5]

Definition 1.13. We define the *complex exponential function* as

$$\exp : \mathbb{C} \to \mathbb{C}, \qquad z = x + yi \mapsto \exp(x)(\cos(y) + \sin(y)i). \tag{1.30}$$

Of course this definition is motivated by the aim to retain the equality $\exp(z+w) = \exp(z)\exp(w)$ also for $z, w \in \mathbb{C}$ (and the definition handles the case $z \in \mathbb{R}$, $w \in i\mathbb{R}$). We collect some properties of the complex exponential.

Proposition 1.14. *The following properties hold for the complex exponential:*

(i) *For all* $z, w \in \mathbb{C}$*, one has* $\exp(z+w) = \exp(z)\exp(w)$.

(ii) $\exp(\mathbb{C}) = \mathbb{C} \setminus \{0\}$.

(iii) $\exp(yi) = \cos(y) + \sin(y)i$ *and* $|\exp(x+yi)| = \exp(x)$ *for* $x, y \in \mathbb{R}$.

[5] In fact, another approach would be to first discuss power series in $\mathbb{C}$ and introduce exp, sin and cos only through those. Ultimately, both approaches lead to the same functions, but we can in fact define these functions relying on their *real* counterparts only!

(iv) $\exp(z + 2\pi k i) = \exp(z)$ *for* $z \in \mathbb{C}$ *and* $k \in \mathbb{Z}$*, and* $\exp(z) = 1$ *if and only if* $z \in 2\pi i\mathbb{Z}$*.*

(v) $\overline{\exp(x + yi)} = \exp(x - yi)$ *for* $x, y \in \mathbb{R}$*, i.e.* $\overline{\exp(z)} = \exp(\overline{z})$ *for* $z \in \mathbb{C}$*.*

Proof. For (i), we let $z = x + yi$ and $w = a + bi$. A direct calculation involving the standard (real) trigonometric identities gives

$$\begin{aligned}\exp(z)\exp(w) &= \exp(x)(\cos(y) + \sin(y)i)(\exp(a)\cos(b) + \sin(b)i) \\ &= \exp(x + a)[(\cos(y)\cos(b) - \sin(y)\sin(b)) + (\cos(y)\sin(b) + \sin(y)\cos(b))i] \\ &= \exp(x + a)(\cos(y + b) + \sin(y + b)i) = \exp(z + w).\end{aligned}$$

For (ii), we note that $\exp(\mathbb{R}) = \mathbb{R}^+$ and $(\cos, \sin)(\mathbb{R}) = S^1 = \{(x, y) \in \mathbb{R}^2; x^2 + y^2 = 1\}$, so for every $w = |w|\frac{w}{|w|}$ with $w \neq 0$, there is $z = x + yi$ with $e^x = |w|$ and $(\cos(y), \sin(y)) = \frac{w}{|w|}$ (viewing $\mathbb{C}$ as $\mathbb{R}^2$). For (iii), the first part is immediate since $\exp(0) = 1$, and the second part follows since

$$|\exp(x)\exp(yi)| = |\exp(x)|\underbrace{|\cos(y) + \sin(y)i|}_{=\sqrt{\cos^2(y)+\sin^2(y)}=1} = \exp(x).$$

For the first part of (iv), use that sin and cos are 2π-periodic, for its second part, the nontrivial direction follows by writing out (with $z = x + yi$):

$$\exp(z) = \exp(x)(\cos(y) + \sin(y)i) = 1,$$

so looking at the modulus yields $x = 0$, whereas $(\cos(y), \sin(y)) = (1, 0)$ yields that $y \in 2\pi\mathbb{Z}$. Finally, (v) follows from the definition and Remark (1.11), (iii). ☐

Particular values of the complex exponential are $\exp(\frac{\pi}{2}i) = i$, $\exp(\pi i) = -1$, $\exp(\frac{3\pi}{2}i) = -i$ and $\exp(2\pi i) = 1$. We will now turn to the trigonometric functions on $\mathbb{C}$.

Definition 1.15. We define the *complex sine* and *cosine functions* as

$$\begin{aligned}\sin : \mathbb{C} \to \mathbb{C}, \qquad \sin(z) &= \frac{1}{2i}(\exp(iz) - \exp(-iz)), \\ \cos : \mathbb{C} \to \mathbb{C}, \qquad \cos(z) &= \frac{1}{2}(\exp(iz) + \exp(-iz)).\end{aligned} \tag{1.31}$$

These definitions are motivated by the fact that inserting $z = y \in \mathbb{R}$ on the right-hand side and using the definition of the complex exp produces exactly $\sin(y)$ and $\cos(y)$. We collect some properties of these functions:

Proposition 1.16. *The following properties of the complex trigonometric functions hold:*

(i) $\sin^2(z) + \cos^2(z) = 1$ *for* $z \in \mathbb{C}$*.*

(ii) (Euler formula) $\exp(iz) = \cos(z) + \sin(z) \cdot i$ *for* $z \in \mathbb{C}$*.*

(iii) $\sin(z+w) = \sin(z)\cos(w)+\sin(w)\cos(z)$ *and* $\cos(z+w) = \cos(z)\cos(w)-\sin(z)\sin(w)$ *for* $z, w \in \mathbb{C}$*.*

Proof. For (i):

$$\begin{aligned}\sin^2(z)+\cos^2(z) &= \left(\frac{\exp(iz)-\exp(-iz)}{2i}\right)^2+\left(\frac{\exp(iz)+\exp(-iz)}{2}\right)^2\\ &= \frac{-\exp(2iz)+2-\exp(-2iz)+\exp(2iz)+2+\exp(-2iz)}{4}=1.\end{aligned}$$

For (ii):

$$\cos(z)+\sin(z)\cdot i=\frac{\exp(iz)+\exp(-iz)}{2}+\frac{\exp(iz)-\exp(-iz)}{2}=\exp(iz).$$

For (iii), we only prove the second claim:

$$\begin{aligned}&\frac{\exp(iz)+\exp(-iz)}{2}\frac{\exp(iw)+\exp(-iw)}{2}-\frac{\exp(iz)-\exp(-iz)}{2i}\frac{\exp(iw)-\exp(-iw)}{2i}\\ &=\frac{\exp(i(z+w))+\exp(-i(z+w))+\exp(i(z-w))+\exp(i(w-z))}{4}\\ &+\frac{\exp(i(z+w))-\exp(i(z-w))-\exp(i(w-z))+\exp(-i(z+w))}{4}\\ &=\frac{\exp(i(z+w))+\exp(-i(z+w))}{2},\end{aligned}$$

from which the claim follows. □

Logarithms

We have seen in Proposition 1.14, (ii), that $\exp$ maps $\mathbb{C}$ to $\mathbb{C}\setminus\{0\}$. We are interested in the inverse operation, but due to the periodic nature of $\exp$ in the complex plane, some care is required.

Definition 1.17. Let $w\in\mathbb{C}\setminus\{0\}$. A solution z of the equation $\exp(z)=w$ is called a *logarithm of* w, denoted $z=\log(w)$.

As we have seen previously with the argument of a complex number, the logarithm will be multi-valued. In order to define it as a proper function, we will need to restrict its possible range to a strip of width 2π in the imaginary direction (i.e. choose a *branch* of the logarithm). More precisely, we have the following proposition

Proposition 1.18. *Every $w\in\mathbb{C}\setminus\{0\}$ has countably many logarithms, which are given by*

$$\log(w)=\underbrace{\log(|w|)}_{\textit{real}\ \log}+i\cdot\arg(w)=\{\log(|w|)+i\cdot\operatorname{Arg}(w)+2\pi in;n\in\mathbb{Z}\}\tag{1.32}$$

(recall that $\operatorname{Arg}(w)\in(-\pi,\pi]$*).*

Proof. First, let z be an element of the above set. Then

$$\exp(z) = \exp(\log(|w|) + i\mathrm{Arg}(w) + 2\pi i n) = |w| \exp(i\mathrm{Arg}(w)) \underbrace{\exp(2\pi i n)}_{=1}.$$

Now by Euler's formula, $\exp(i\mathrm{Arg}(w)) = \cos(\mathrm{Arg}(w)) + \sin(\mathrm{Arg}(w))i = \frac{w}{|w|}$ by the Definition 1.8 of the principal value of the argument. So, the elements of the set in question are indeed logarithms. Conversely, assume that z is a logarithm of $w = |w|(\cos(\theta) + \sin(\theta)i)$ (with $\theta = \mathrm{Arg}(w)$), then writing $z = x + yi$, we see that

$$|\exp(z)| = \exp(x)$$

by Proposition 1.14, (iii), and so $x = \log(|w|)$. On the other hand,

$$\exp(z) = \exp(x)(\cos(y) + \sin(y) \cdot i) = |w|(\cos(\theta) + \sin(\theta)i).$$

This means that $y = \theta + 2\pi n$, $n \in \mathbb{Z}$, so we have indeed found every possible solution. ☐

To remedy the fact that we have infinitely many logarithms, we thus need to restrict to a certain *branch* of length 2π.

Definition 1.19. For $w \in \mathbb{C} \setminus \{0\}$, we set

$$\mathrm{Log}(w) = \log(|w|) + i\mathrm{Arg}(w), \tag{1.33}$$

with $-\pi < \mathrm{Arg}(w) \leq \pi$, the *principal value of the logarithm of* w.

The function $\mathrm{Log} : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ that attains values in $\mathbb{R} \times (-\pi, \pi] \subseteq \mathbb{C}$ is then called the *principal branch of the logarithm.* At this point, one should recall that the choice $\mathrm{Arg} \in (-\pi, \pi]$ was somewhat arbitrary, and one might have chosen $\mathrm{Arg} \in (y_0, y_0 + 2\pi]$ for any $y_0 \in \mathbb{R}$. Naturally, there exists a proper function $\mathrm{Log} : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ whose range is

$$A_{y_0} = \{x + iy; y \in (y_0, y_0 + 2\pi]\} = \mathbb{R} \times (y_0, y_0 + 2\pi], \tag{1.34}$$

which is the *branch of the logarithm lying in* A_{y_0}.

Proposition 1.20. *The restriction* $\exp|_A$ *of the exponential function to the set* $A = A_{-\pi} = \{z = x + yi; -\pi < y \leq \pi\} \subseteq \mathbb{C}$ *gives a bijection from* A *to* $\mathbb{C} \setminus \{0\}$, *with inverse given by* $w \mapsto \mathrm{Log}(w)$.

Proof. For the injectivity of the restricted exponential, let $\exp(z) = \exp(w)$ for $z, w \in A$, then $\exp(z - w) = 1$, so by Proposition 1.14, (iv), we see that $z - w \in 2\pi i\mathbb{Z}$, but since $|\mathrm{Im}(z) - \mathrm{Im}(w)| < 2\pi$, indeed $z = w$. The surjectiviy of $\exp|_A$ is also clear (in the proof of Proposition 1.14, (ii), we actually only needed $(\cos, \sin)((-\pi, \pi]) = S^1$). The function Log maps $\mathbb{C} \setminus \{0\}$ to A, and is the inverse of $\exp|_A$ by definition. ☐

Remark 1.21. (i) We stress again that due to periodicity, one cannot hope for an inverse of $\exp : \mathbb{C} \to \mathbb{C}$ (in the same way as, for instance, $\sin : \mathbb{R} \to \mathbb{R}$ has no inverse). So care is required even for 'natural' looking expressions: In general, $\mathrm{Log}(\exp(z)) \neq z$ (unless $\mathrm{Im}(z) \in (-\pi, \pi]$), for instance

$$\mathrm{Log}(\exp(2 + 3\pi i)) = \mathrm{Log}(-\exp(2)) = 2 + \pi i.$$

(ii) In the same way, the rules for logarithms must be adapted to reflect this issue: For instance, it is true that for $z, w \in \mathbb{C} \setminus \{0\}$:

$$\log(zw) = \log(z) + \log(w) \tag{1.35}$$

(an equality between sets!), but of course one only has

$$\operatorname{Log}(zw) - \operatorname{Log}(z) - \operatorname{Log}(w) \in 2\pi i\mathbb{Z}. \tag{1.36}$$

2. Continuity

(Reference: Marsden-Hoffman, Basic complex analysis, 3rd Ed., Section 1.4)

2.1. Complex sequences and series

We will turn to the question of continuity of complex functions. For this, we need to recall some elementary toplogical notions for $\mathbb{C}$ and for complex sequences.

Definition 2.1. A sequence $(z_n)_{n\in\mathbb{N}}$ of complex numbers is *bounded*, if there is $C \geq 0$ with

$$|z_n| \leq C \qquad \text{for all } n \in \mathbb{N}. \tag{2.1}$$

It is called *convergent* if there is $z \in \mathbb{C}$ such that for every $\varepsilon > 0$, there is an $N = N(\varepsilon) \in \mathbb{N}$ with

$$|z_n - z| < \varepsilon \qquad \text{for all } n \geq N. \tag{2.2}$$

We say that $(z_n)_{n\in\mathbb{N}}$ is *convergent* and call z the *limit* of $(z_n)_{n\in\mathbb{N}}$, denoted $z = \lim_{n\to\infty} z_n$.

End of Lecture 2

The limit has the usual properties from real analysis (viewing $\mathbb{C}$ as $\mathbb{R}^2$, $|\cdot|$ is simply the Euclidean norm), and also behaves well with respect to complex conjugation and taking real or imaginary part.

Proposition 2.2. *(i) The limit of a convergent complex sequence $(z_n)_{n\in\mathbb{N}}$ is unique.*

(ii) Convergent sequences are bounded.

(iii) If $\lim_{n\to\infty} z_n = z$ and $\lim_{n\to\infty} w_n = w$, then

$$\lim_{n\to\infty}(z_n + w_n) = z + w, \qquad \lim_{n\to\infty}(z_n \cdot w_n) = z \cdot w.$$

(iv) If $\lim_{n\to\infty} z_n = z \neq 0$, then also $\lim_{n\to\infty} \frac{1}{z_n} = \frac{1}{z}$.

(v) If $\lim_{n\to\infty} z_n = z$, then also

$$\lim_{n\to\infty} \overline{z_n} = \overline{z}, \qquad \lim_{n\to\infty} \mathrm{Re}(z_n) = \mathrm{Re}(z),$$
$$\lim_{n\to\infty} |z_n| = |z|, \qquad \lim_{n\to\infty} \mathrm{Im}(z_n) = \mathrm{Im}(z).$$

(vi) If for a sequence $(z_n)_{n\in\mathbb{N}}$, one has $\lim_{n\to\infty} \mathrm{Re}(z_n) = \mathrm{Re}(z)$ and $\lim_{n\to\infty} \mathrm{Im}(z_n) = \mathrm{Im}(z)$, then also $\lim_{n\to\infty} z_n = z$.

Proof. The claims (i)–(iv) are as in the real variable case: For (i), assume $\lim_{n\to\infty} z_n = z$ and $\lim_{n\to\infty} z_n = z'$, then for a given $\varepsilon > 0$, there exist $N, N' \in \mathbb{N}$ with

$$|z_n - z| < \varepsilon \text{ for } n \geq N, \qquad |z_n - z'| < \varepsilon \text{ for } n \geq N',$$

so for $n \geq \max\{N, N'\}$, one has

$$|z - z'| \leq |z_n - z| + |z_n - z'| < 2\varepsilon,$$

and since $\varepsilon > 0$ was arbitrary, we must have $z = z'$. For (ii), let $\lim_{n\to\infty} z_n = z$ and choose $\varepsilon = 1$, then there exists $N \in \mathbb{N}$ with

$$|z_n - z| < 1 \text{ for } n \geq N,$$

so indeed by the triangle inequality, $|z_n| \leq |z - z_n| + |z|$ holds for all $n \in \mathbb{N}$ and therefore

$$|z_n| \leq \max\{1 + |z|, |z_1|, ..., |z_{N-1}|\} =: C.$$

For (iii), the additivity property follows by the triangle inequality, and for the multiplicativity, we first use that by (ii) the sequence $(z_n)_{n\in\mathbb{N}}$ is bounded (we call this bound C). Then, for every $n \in \mathbb{N}$:

$$|z_n w_n - zw| \leq |z_n||w_n - w| + |w||z_n - z|.$$

For a given $\varepsilon > 0$, there exist $N, N' \in \mathbb{N}$ such that

$$|z_n - z| \leq \frac{\varepsilon}{2(1 + |w|)} \text{ for } n \geq N, \qquad |w_n - w| \leq \frac{\varepsilon}{2(1 + C)} \text{ for } n \geq N'$$

so for $n \geq \max\{N, N'\}$, one has

$$|z_n w_n - zw| \leq \frac{\varepsilon C}{2(1 + C)} + \frac{\varepsilon |w|}{2(1 + |w|)} \leq \varepsilon.$$

For (iv), either one can adapt the proof of the real case, or use

$$\lim_{n\to\infty} \frac{1}{z_n} = \lim_{n\to\infty} \left(\frac{1}{|z_n|^2} \cdot \overline{z_n} \right) = \frac{1}{|z|^2} \cdot \overline{z} = \frac{1}{z} \tag{2.3}$$

where we employed (iii), (v) and the fact that $(|z_n|)_{n\in\mathbb{N}}$ converges as a sequence of real numbers (so $\lim_{n\to\infty} \frac{1}{|z_n|^2} = \frac{1}{|z|^2}$ can be used). For (v), use that

$$|\overline{z}_n - \overline{z}| = |\overline{z_n - z}| = |z_n - z|,$$

to show that $\lim_{n\to\infty} \overline{z_n} = \overline{z}$, and the representations $\mathrm{Re}(w) = \frac{w+\overline{w}}{2}$ and $\mathrm{Im}(w) = \frac{w-\overline{w}}{2i}$ together with (iii) to show the claims about the convergence of real and imaginary part. Finally, for the absolute value, we use the reverse triangle inequality 1.6, (iv) to write

$$||z_n| - |z|| \leq |z - z_n|,$$

and the claim follows. For (vi), we use

$$|z_n - z| \leq |\mathrm{Re}(z_n) - \mathrm{Re}(z)| + |\mathrm{Im}(z_n) - \mathrm{Im}(z)|.$$

□

We will now introduce Cauchy sequences and prove the completness of $\mathbb{C}$, assuming the completeness of $\mathbb{R}$.

Definition 2.3. A sequence of complex numbers $(z_n)_{n\in\mathbb{N}}$ is called a *Cauchy sequence*, if for every $\varepsilon > 0$, there is an $N = N(\varepsilon) \in \mathbb{N}$ with

$$|z_n - z_m| < \varepsilon \qquad \text{for all } n, m \geq N. \tag{2.4}$$

Proposition 2.4. *A sequence of complex numbers $(z_n)_{n\in\mathbb{N}}$ is a Cauchy sequence if and only if it converges. This means, $\mathbb{C}$ is complete*[1].

Proof. If $(z_n)_{n\in\mathbb{N}}$ converges to a limit z, let $\varepsilon > 0$ and take $N \in \mathbb{N}$ such that for every $n \geq N$, one has $|z_n - z| < \frac{\varepsilon}{2}$. By the triangle inequality,

$$|z_n - z_m| \leq |z_n - z| + |z_m - z| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \qquad \text{for all } n, m \geq N.$$

So $(z_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. On the other hand, if $(z_n)_{n\in\mathbb{N}}$ is a Cauchy sequence then (since $|\mathrm{Re}(w)| \leq |w|$ and $|\mathrm{Im}(w)| \leq |w|$) the sequences $(\mathrm{Re}(z_n))_{n\in\mathbb{N}}$ and $(\mathrm{Im}(z_n))_{n\in\mathbb{N}}$ are Cauchy sequences in $\mathbb{R}$. But since $\mathbb{R}$ is complete, these sequences converge, and by Proposition 2.2 above, $(z_n)_{n\in\mathbb{N}}$ converges as well. $\square$

As in the case of real numbers, we have the notion of a series.

Definition 2.5. An *infinite series* $\sum_{\nu=1}^{\infty} z_\nu$ with $z_\nu \in \mathbb{C}$ is understood as the sequence $(S_n)_{n\in\mathbb{N}}$ of partial sums, defined as

$$S_n = \sum_{\nu=1}^{n} z_\nu. \tag{2.5}$$

If $(S_n)_{n\in\mathbb{N}}$ converges and has limit $S \in \mathbb{C}$, we say the series is *convergent* and write $S = \sum_{\nu=1}^{\infty} z_\nu$.

Remark 2.6. (i) An example of a convergent series is the *geometric series* $\sum_{\nu=0}^{\infty} z^\nu$ for $|z| < 1$, since $\sum_{\nu=0}^{n} z^\nu = \frac{1-z^{n+1}}{1-z}$, and one has

$$\sum_{\nu=0}^{\infty} z^\nu = \frac{1}{1-z}. \tag{2.6}$$

(ii) Complex power series are series of the form $\sum_{\nu=0}^{\infty} a_\nu z^\nu$ with $(a_\nu)_{\nu\in\mathbb{N}} \subseteq \mathbb{C}$. These will play a crucial role in later chapters.

(iii) One has like in the real case the easy properties:

- If $\sum_{\nu=1}^{\infty} z_\nu$ converges, then necessarily $(z_n)_{\nu\in\mathbb{N}}$ converges to zero (this follows since $(S_n)_{n\in\mathbb{N}}$ is a Cauchy sequence, and so $|S_n - S_{n-1}| = |z_n|$ (for $n \geq 1$) has to converge to zero).
- If $\sum_{\nu=1}^{\infty} |z_\nu|$ converges (we say the series *converges absolutely*), then so does $\sum_{\nu=1}^{\infty} z_\nu$. Indeed, $(S_n)_{n\in\mathbb{N}}$ can easily be shown to be a Cauchy sequence.

[1] In any given metric space (M, d), any convergent sequence is always a Cauchy sequence, adapting the proof below. The space (M, d) is called *complete*, if every Cauchy sequence converges. A typical example of a space which is not complete is $\mathbb{Q}$ with $d(x, y) = |x - y|$.

2.2. Topology of $\mathbb{C}$, continuous functions

In order to proceed with the definition of continuous and holomorphic (complex differentiable) functions, we need to introduce / recall elementary notions from set topology. Notice that the notions of open, closed, compact, connected, ... sets are *exactly* the same as for $\mathbb{R}^n$ (in our case, simply $n = 2$).

Definition 2.7. (i) Let $r > 0$ and $z_0 \in \mathbb{C}$. We call $D(z_0, r) = \{z \in \mathbb{C}; |z - z_0| < r\}$ the *(open) r disk around* z_0. We call $\dot{D}(z_0, r) = D(z_0, r) \setminus \{z_0\}$ the *deleted (open) r disk around* z_0.

(ii) Let $A \subseteq \mathbb{C}$. A point $z_0 \in A$ is called an *interior point* of A, if there is $r > 0$ such that $D(z_0, r) \subseteq A$.

(iii) A set $A \subseteq \mathbb{C}$ is called *open* if every point $z_0 \in A$ is an interior point of A.

(iv) A set $A \subseteq \mathbb{C}$ is called *closed* if $A^c = \mathbb{C} \setminus A$ is open.

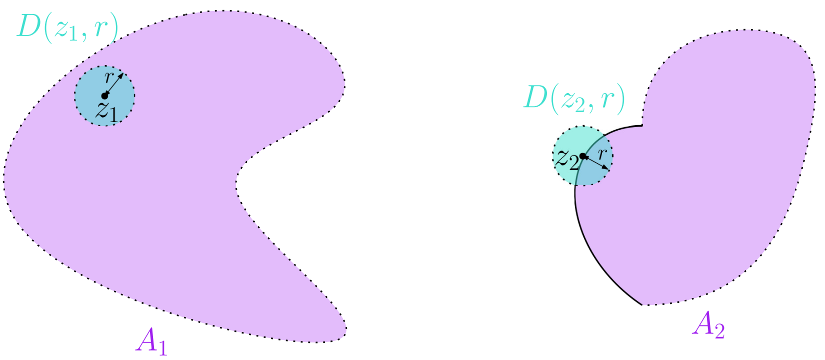

Figure 2.1.: The set A_1 is open, since every point of A_1 is an interior point (here, z_1 is shown with an open disk around z_1 contained in A_1). The set A_2 is not open, since the point z_2 is not an interior point (for every $\varepsilon > 0$, $D(z_2, \varepsilon) \cap {A_2}^c \neq \emptyset$).

The following statements are immediate from the definition.

Proposition 2.8. *(i) $\emptyset$ and $\mathbb{C}$ are both closed and open.*

(ii) The union of arbitrarily many open sets is open. The intersection of arbitrarily many closed sets is closed.

(iii) The intersection of finitely many open sets is open. The union of finitely many closed sets is closed.

The previous definition naturally leads to the following definition.

Definition 2.9. Let $A \subseteq \mathbb{C}$ be a set. We define the *interior* of A by

$$\mathring{A} = \bigcup_{U \subseteq A, U \text{ open}} U, \tag{2.7}$$

and the *closure* of A by

$$\overline{A} = \bigcap_{C \supseteq A, C \text{ closed}} C. \tag{2.8}$$

Finally, the *boundary* of A is defined by

$$\partial A = \overline{A} \setminus \mathring{A}. \tag{2.9}$$

We will also need the notions of bounded and compact subsets of $\mathbb{C}$, that we now introduce.

Definition 2.10. (i) A subset $A \subseteq \mathbb{C}$ is called *bounded*, if there is $C \geq 0$ such that for every $z \in A$, one has $|z| \leq C$.

(ii) We say that $A \subseteq \mathbb{C}$ is *compact*, if every open cover of A admits a finite subcover, that is if for any collection $\{U_\alpha\}_{\alpha \in \mathscr{A}}$ with $A \subseteq \bigcup_{\alpha \in \mathscr{A}} U_\alpha$ and U_α open, there are finitely many $\alpha_1, ..., \alpha_n \in \mathscr{A}$ such that $A \subseteq \bigcup_{j=1}^n U_{\alpha_j}$.

We refer to standard texts in topology for the following important result:

Proposition 2.11. *The following conditions for a subset $A \subseteq \mathbb{C}$ are equivalent:*

(i) A is compact.

(ii) A is bounded and closed.

(iii) Every sequence $(z_n)_{n \in \mathbb{N}} \subseteq A$ has a convergent subsequence, which converges to a point in A.

We will now turn to functions mapping a subset of $\mathbb{C}$ into $\mathbb{C}$ and define limits and continuity. Not surprisingly, the continuity properties will be the same as if we viewed such function simply as maps from a subset of $\mathbb{R}^2$ into $\mathbb{R}^2$.

Definition 2.12. Let $A \subseteq \mathbb{C}$ be a subset and $f : A \to \mathbb{C}$ a function[2]. Let $z_0 \in \mathbb{C}$ be an *accumulation point* of A (i.e. infinitely many elements of A lie in every deleted r disk around z_0). We say that f has a *limit* w in z_0, denoted by

$$\lim_{z \to z_0} f(z) = w, \tag{2.10}$$

if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$|f(z) - w| < \varepsilon, \qquad \text{for all } z \in \dot{D}(z_0, \delta) \cap A. \tag{2.11}$$

Remark 2.13. Limits are, if they exist, unique, which can be proven as in the case of complex sequences (see Proposition 2.2, (i)).

[2]When using the notation $f : A \to \mathbb{C}$, we always implicitly understand the set A to be nonempty.

With the definition of limits at our disposal, we can now define continuity.

Definition 2.14. Let $A \subseteq \mathbb{C}$ and $f : A \to \mathbb{C}$ a function. We say that f is *continuous* in $z_0 \in A$ if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$|f(z) - f(z_0)| < \varepsilon, \qquad \text{for all } z \in D(z_0, \delta) \cap A. \tag{2.12}$$

We say that f is continuous on A, if it is continuous in every $z_0 \in A$.

Note that the definition includes the case where z_0 is an *isolated point* of A (not an accumulation point): In this case, every function will be continuous in z_0.

Lemma 2.15. *Let $A \subseteq \mathbb{C}$ and $f : A \to \mathbb{C}$ a function and $z_0 \in A$ an accumulation point of A. Then, the following statements are equivalent:*

(i) *f is continuous in z_0.*

(ii) $\lim_{z \to z_0} f(z) = f(z_0)$.

(iii) *For every sequence $(z_n)_{n\in\mathbb{N}} \subseteq A$ with $\lim_{n\to\infty} z_n = z_0$, one has $\lim_{n\to\infty} f(z_n) = f(z_0)$.*

Proof. The direction (i) ⇒ (ii) is clear.
For the direction (ii) ⇒ (iii), let $\varepsilon > 0$ and suppose $(z_n)_{n\in\mathbb{N}} \subseteq A$ converges to z_0. There exists $\delta > 0$ such that $|f(z) - f(z_0)| < \varepsilon$, provided $|z - z_0| < \delta$. Then choose $N \in \mathbb{N}$ (depending on this δ) such that $|z_n - z_0| < \delta$ for every $n \geq N$. We have shown: $|f(z_n) - f(z_0)| < \varepsilon$ if $n \geq N$, and so $\lim_{n\to\infty} f(z_n) = f(z_0)$.
For the direction (iii) ⇒ (i), we argue by contradiction. Suppose f was not continuous in z_0, then (since it is an accumulation point) there exists some $\varepsilon > 0$ such that we find for every $n \in \mathbb{N}$ a point $z_n \in D(z_0, \frac{1}{n}) \cap A$ with $|f(z_n) - f(z)| \geq \varepsilon$. The sequence $(z_n)_{n\in\mathbb{N}} \subseteq A$ clearly converges to z_0, but $(f(z_n))_{n\in\mathbb{N}}$ does not converge to $f(z)$, giving the desired contradiction. □

We collect some standard facts about continuity.

Proposition 2.16. *Let $A \subseteq \mathbb{C}$ and $f, g : A \to \mathbb{C}$ two functions that are continuous in $z_0 \in A$.*

(i) *The functions $f + g$ and $f \cdot g$, defined as*

$$f + g : \begin{cases} A \to \mathbb{C} \\ z \mapsto f(z) + g(z) \end{cases} \qquad \textit{and } f \cdot g : \begin{cases} A \to \mathbb{C} \\ z \mapsto f(z)g(z) \end{cases} \tag{2.13}$$

are continuous in z_0.

(ii) *The functions $\mathrm{Re} f : z \mapsto \mathrm{Re}(f(z))$, $\mathrm{Im} f : z \mapsto \mathrm{Im}(f(z))$, $\overline{f} : z \mapsto \overline{f(z)}$ and $|f| : z \mapsto |f(z)|$ (all defined on A) are continuous in z_0.*

(iii) *If $g(z_0) \neq 0$, there is $\delta > 0$ such that $g(z) \neq 0$ for $z \in A \cap D(z_0, \delta)$ and the function $\frac{f}{g}$, defined as*

$$\frac{f}{g} : \begin{cases} A \cap D(z_0, \delta) \to \mathbb{C} \\ z \mapsto \frac{f(z)}{g(z)} \end{cases} \tag{2.14}$$

is continuous in z_0.

(iv) *If $h : B \to \mathbb{C}$ is a function defined on $B \supseteq f(A)$, that is continuous in $f(z_0)$, the composition $h \circ f : A \to \mathbb{C}$ is continuous in z_0 as well.*

Proof. We only consider the case where $z_0 \in A$ is an accumulation point of A (otherwise, there is nothing to show). All proofs follow from Lemma 2.15 using the properties for limits of complex sequences that we proved in Proposition 2.2. For instance for (i): Consider a sequence $(z_n)_{n\in\mathbb{N}} \subseteq A$ with $\lim_{n\to\infty} z_n = z$. By the continuity of f and g in z_0, we have that $\lim_{n\to\infty}(f(z_n)) = f(z_0)$ and $\lim_{n\to\infty}(g(z_n)) = g(z_0)$, so we obtain

$$\lim_{n\to\infty}(f(z_n) + g(z_n)) = f(z_0) + g(z_0), \qquad \lim_{n\to\infty}(f(z_n)g(z_n)) = f(z_0)g(z_0).$$

Since $(z_n)_{n\in\mathbb{N}}$ was arbitrary, the claim follows. The items (ii), (iv) and the second part of (iii) follow similarly. For the first part of (iii), we simply choose $\varepsilon = \frac{|g(z_0)|}{2} > 0$, and find $\delta > 0$ so that for $z \in A \cap D(z_0, \delta)$, $|g(z) - g(z_0)| < \varepsilon$, which entails that for these z one has $|g(z)| \geq |g(z_0) - |g(z) - g(z_0)|| \geq \frac{|g(z_0)|}{2} > 0$. □

Remark 2.17. (i) Polynomials are continuous on $\mathbb{C}$ and rational functions are continuous away from their poles. The functions exp, sin and cos are also continuous on $\mathbb{C}$.

(ii) The functions defined on $\mathbb{C} \setminus \{0\}$, $z \mapsto \mathrm{Arg}(z)$ and $z \mapsto \mathrm{Log}(z)$ are not continuous. Consider for instance the sequence $z_n = \exp(\theta_n i)$, where $\theta_n = \begin{cases} \pi - \frac{1}{n}, & \text{odd } n \\ -\pi + \frac{1}{n}, & \text{even } n. \end{cases}$

Clearly, $\lim_{n\to\infty} z_n = -1$, but $\mathrm{Arg}(z_n) = \theta_n$ does not converge. The same happens for Log (see (1.33)). Nevertheless both functions are continuous when restricted to $\mathbb{C}_- = \mathbb{C}\setminus\{(x, 0); x \leq 0\}$.

(iii) Let $A \subseteq \mathbb{C}$ and $f : A \to \mathbb{C}$ be continuous. If $K \subseteq A$ is compact, then $f(K)$ is compact as well. Moreover, f attains its maximum an minimum on K, meaning that there is a point $\widehat{z} \in K$ with $|f(\widehat{z})| = \sup_{z\in K} |f(z)|$ (and similar for the minimum).

The last topological notion we will frequently use is that of connectedness and path-connectedness, which we now introduce.

Definition 2.18. Let $A \subseteq \mathbb{C}$.

(i) A is called *connected* if it cannot be written as a disjoint union of two non-empty, relatively open subsets A_1 and A_2.[3]

(ii) A is called *path-connected* if for every two points $z, w \in A$, one can find a continuous map $\gamma : [0, 1] \to A$ with $\gamma(0) = z$ and $\gamma(1) = w$.

(iii) A is called a *domain* if it is both open and connected.

One has the following topological facts, which we state without proof:

Proposition 2.19. *(i) Every path-connected set $A \subseteq \mathbb{C}$ is also connected.*

[3] $B \subseteq A$ is *relatively open* if there exists an open set $U \subseteq \mathbb{C}$ with $B = U \cap A$.

(ii) *If A is a domain, it is also path-connected. In fact, the path $\gamma : [0,1] \to A$ can be taken to be continuously differentiable (as a curve in $\mathbb{R}^2$, that is, componentwise continuously differentiable).*

(iii) *The image of a (path-)connected set A under a continuous map is again (path-)connected.*

Proof. See, for instance, *Marsden-Hoffman, Basic complex analysis, 3rd Ed.*, Propositions 1.4.14, 1.4.15 and 1.4.16. □

2.3. Extended complex plane $\overline{\mathbb{C}}$, Riemann sphere

We conclude this chapter by introducing a concept that will be convenient, namely the *point at infinity.* We designate the symbol ∞ (which is not in $\mathbb{C}$) as an additional point and join it to $\mathbb{C}$. This is called the *extended complex plane*

$$\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}. \tag{2.15}$$

On $\overline{\mathbb{C}}$, we introduce the following operations (where $z \in \mathbb{C}$)

$$\begin{aligned} z + \infty &= \infty, \\ z \cdot \infty &= \infty, \qquad \text{for } z \neq 0, \\ \infty + \infty &= \infty, \\ \infty \cdot \infty &= \infty, \\ \frac{z}{\infty} &= 0, \\ \frac{z}{0} &= \infty, \qquad \text{for } z \neq 0. \end{aligned} \tag{2.16}$$

We also want to explain what it means to be "close to ∞":

Definition 2.20. A subset $A \subseteq \overline{\mathbb{C}}$ is called *open*, if the following holds

(i) $A \cap \mathbb{C}$ is open.

(ii) If $\infty \in A$, there is an $K > 0$ with $D(\infty, 1/K) = \{z \in \mathbb{C}; |z| > K\} \cup \{\infty\} \subseteq A$.

Limits involving ∞ are then defined accordingly:

- For a sequence $(z_n)_{n\in\mathbb{N}} \subseteq \overline{\mathbb{C}}$, $\lim_{n\to\infty} z_n = \infty$ means that for every $K > 0$, there is an $N \in \mathbb{N}$ such that $|z_n| > K$ for every $n \geq N$.
- For a function f defined on a set containing some $\dot{D}(\infty, 1/K) = \{z \in \mathbb{C}; |z| > K\}$, we write $\lim_{z\to\infty} f(z) = w$ if for any $\varepsilon > 0$, there exists $K > 0$ such that $|f(z) - w| < \varepsilon$ if $|z| > K$.
- Finally, for a function f defined on $A \subseteq \overline{\mathbb{C}}$ with accumulation point $z_0 \in \mathbb{C}$, we write $\lim_{z\to z_0} f(z) = \infty$ if for every $K > 0$, there exists a $\delta > 0$ such that $|f(z)| > K$ for all $z \in \dot{D}(z_0, \delta) \cap A$.

A geometric model to visualize $\overline{\mathbb{C}}$ is given by the *Riemann sphere*. This is the sphere

$$S^2 = \{(x, y, s) \in \mathbb{R}^3\,;\, x^2 + y^2 + s^2 = 1\} = \{(z, s) \in \mathbb{C} \times \mathbb{R}\,;\, |z|^2 + s^2 = 1\}, \tag{2.17}$$

in which the xy-plane is supposed to represent the complex plane.

If we connect the north pole $N = (0, 1)$ (the first coordinate is the complex number $0 + 0i$) with a point $(w, 0) \in \mathbb{C} \times \{0\}$ by a straight line, it intersects S^2 in exactly two points, N and $\tau(w)$. We call the inverse function $\sigma : S^2 \setminus \{N\} \to \mathbb{C}$ the the *stereographic projection.* If move the second intersection point (z, s) of the line with the sphere closer to N, we see that the corresponding point $(\sigma(z, s), 0)$ moves further away from the origin, i.e. $|\sigma(z, s)|$ becomes larger and larger. We thus naturally identify N (or rather its image) with the point at infinity.

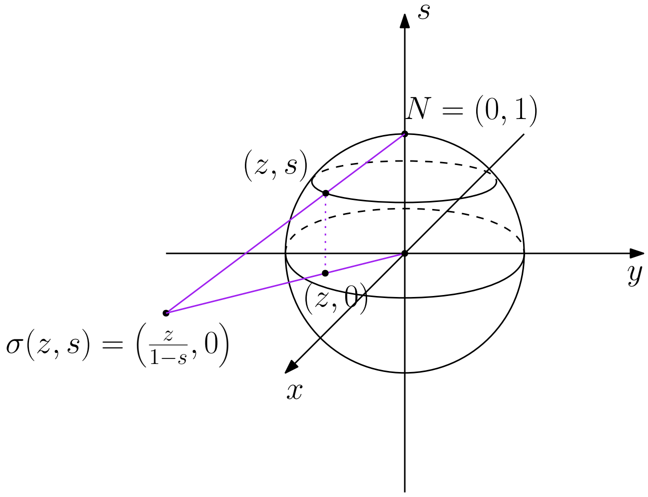

Figure 2.2.: The Riemann sphere together with the image of a point $(z, s) \in S^2 \setminus \{N\}$ in $\mathbb{C}$.

With a little algebra, we have the stereographic projection (now including $N = (0, 1)$)

$$\sigma : S^2 \to \overline{\mathbb{C}}, \qquad (z, s) \mapsto \begin{cases} \frac{z}{1-s}, & \text{for } (z, s) \neq N, \\ \infty, & \text{for } (z, s) = N. \end{cases} \tag{2.18}$$

and its inverse

$$\tau : \overline{\mathbb{C}} \to S^2, \qquad z \mapsto \begin{cases} \left(\frac{2z}{|z|^2+1}, \frac{|z|^2-1}{|z|^2+1}\right), & \text{for } z \in \mathbb{C}, \\ N, & \text{for } z = \infty. \end{cases} \tag{2.19}$$

The sets S^2 with the induced topology from $\mathbb{R}^3$ and $\overline{\mathbb{C}}$ with the topology declared in Definition 2.20 above[4] are in fact *homeomorphic*, with homeomorphism given by σ.

End of Lecture 3

[4] It can be shown that the subsets of $\overline{\mathbb{C}}$ that we declared to be open, in fact are the open sets of a topology, and $\overline{\mathbb{C}}$ is compact in this topology, see ⇝ *Exercises*.

3. Holomorphic functions

(Reference: Marsden-Hoffman, Basic complex analysis, 3rd Ed., Sections 1.5 – 1.6)

3.1. Differentiation of complex functions

In this chapter, we introduce the central concept of this lecture, which is that of a *holomorphic* function. We will see later, that the term *analytic* function (which in real analysis designates functions that are locally given by a convergent power series) can in fact be used synonymously.

Definition 3.1. Let $A \subseteq \mathbb{C}$ be an open set. A function $f : A \to \mathbb{C}$ is said to be *differentiable (in the complex sense)* at $z_0 \in A$, if the limit

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \tag{3.1}$$

exists in $\mathbb{C}$. A different notation is $\frac{\mathrm{d}}{\mathrm{d}z} f(z_0) = f'(z_0)$. We call $f'(z_0) \in \mathbb{C}$ the *complex derivative* of f at z_0. If f is differentiable for all $z_0 \in A$, we say that f is *holomorphic* (or *analytic*) on A (or simply holomorphic / analytic). We say that f is *holomorphic at* z_0, if there is an open r disk $D(z_0, r) \subseteq A$ such that $f|_{D(z_0,r)}$ is holomorphic. A function that is defined on $\mathbb{C}$ and holomorphic is called *entire*.

Note that one can write equivalently

$$f'(z_0) = \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h}. \tag{3.2}$$

Let us give some very basic examples.

Example 3.2. (i) Constant functions $f \equiv c \in \mathbb{C}$ are entire with $f'(z_0) = 0$ for every $z_0 \in \mathbb{C}$.

(ii) The function $f : \mathbb{C} \to \mathbb{C}$, $z \mapsto z^n$ for integer $n \geq 1$ is entire. Indeed, using the binomial theorem, we see that for any $z_0 \in \mathbb{C}$:

$$\begin{aligned} \lim_{h \to 0} \frac{(z_0 + h)^n - z_0^n}{h} &= \lim_{h \to 0} \frac{1}{h} \left(\sum_{k=0}^{n} \binom{n}{k} z_0^{n-k} h^k - z_0^n \right) \\ &= \lim_{h \to 0} \left(n z_0^{n-1} + h \sum_{k=2}^{n} \binom{n}{k} z_0^{n-k} h^{k-2} \right) = n z_0^{n-1}. \end{aligned} \tag{3.3}$$

(iii) The function $\mathrm{Re} : \mathbb{C} \to \mathbb{C}$, $z \mapsto \mathrm{Re}(z)$ is not differentiable (in the complex sense) in any point $z_0 \in \mathbb{C}$. Indeed, consider for $z_0 \in \mathbb{C}$, $h \neq 0$

$$\frac{\mathrm{Re}(z_0 + h) - \mathrm{Re}(z_0)}{h} = \frac{1}{h} \mathrm{Re}(h) = \begin{cases} 1, & \text{for } h \in \mathbb{R}, \\ 0, & \text{for } h \in \mathbb{R}i. \end{cases} \tag{3.4}$$

(iv) In the same manner, $z \mapsto \mathrm{Im}(z)$ and $z \mapsto \overline{z}$ are not differentiable (in the complex sense) in any point $z_0 \in \mathbb{C}$. The function $z \mapsto |z|^2$ is only differentiable (in the complex sense) in $z_0 = 0$ (it is not holomorphic in $z_0 = 0$).

We derive some elementary properties of the complex derivative, that are analogues of their real counterparts.

Lemma 3.3. *Let $A \subseteq \mathbb{C}$ be open, $z_0 \in A$ and $f : A \to \mathbb{C}$ a function. The following are equivalent:*

(i) f is differentiable in the complex sense in z_0 and $f'(z_0) = w$.

(ii) There exists a function $\psi : A \to \mathbb{C}$, continuous in z_0 with

$$f(z) = f(z_0) + \psi(z)(z - z_0), \qquad \text{and } \psi(z_0) = w. \tag{3.5}$$

Proof. Assume that f is differentiable in z_0 with $w = f'(z_0)$. We set

$$\psi(z) = \begin{cases} \frac{f(z)-f(z_0)}{z-z_0}, & \text{for } z \in A \setminus \{z_0\} \\ w, & \text{for } z = z_0. \end{cases} \tag{3.6}$$

Since $\lim_{z\to z_0} \psi(z) = f'(z_0) = w = \psi(z_0)$, the function ψ is continuous. On the other hand, suppose that a function ψ with the desired properties exists, then

$$\frac{f(z) - f(z_0)}{z - z_0} = \psi(z), \tag{3.7}$$

and taking $z \to z_0$ on both sides yields by the continuity of ψ that $f'(z_0)$ exists and equals $\psi(z_0) = w$. ☐

Proposition 3.4. *Suppose that $A \subseteq \mathbb{C}$ is open and f is differentiable (in the complex sense) in $z_0 \in A$. Then f is continuous in z_0.*

Proof. Note that $\lim_{z\to z_0}(z - z_0) = 0$. By the previous lemma, we can write $f(z) = f(z_0) + \psi(z)(z - z_0)$ with ψ continuous in z_0 and $\psi(z_0) = f'(z_0)$. Therefore:

$$\begin{aligned} \lim_{z\to z_0} f(z) &= f(z_0) + \lim_{z\to z_0} (\psi(z) \cdot (z - z_0)) \\ &= f(z_0) + \underbrace{\lim_{z\to z_0} \psi(z)}_{=f'(z_0)} \cdot \underbrace{\lim_{z\to z_0} (z - z_0)}_{=0} = f(z_0), \end{aligned} \tag{3.8}$$

where we used the rules for sums and products of continuous functions from Proposition 2.16, (i). ☐

Proposition 3.5. *Suppose that $A \subseteq \mathbb{C}$ is open and f, g are differentiable (in the complex sense) in $z_0 \in A$. Then the following statements hold.*

(i) $f + g$ is differentiable in z_0 and has derivative

$$(f + g)'(z_0) = f'(z_0) + g'(z_0). \tag{3.9}$$

(ii) *$f \cdot g$ is differentiable in z_0 and has derivative*

$$(f \cdot g)'(z_0) = f'(z_0)g(z_0) + f(z_0)g'(z_0). \tag{3.10}$$

(iii) *If $g(z_0) \neq 0$, then $\frac{f}{g}$ (which can be defined defined on $D(z_0, \delta)$ for some $\delta > 0$) is differentiable in z_0 and has derivative*

$$\left(\frac{f}{g}\right)'(z_0) = \frac{f'(z_0)g(z_0) - f(z_0)g'(z_0)}{g(z_0)^2}. \tag{3.11}$$

(iv) *If $h : B \to \mathbb{C}$ is a function defined on an open set $B \supseteq f(A)$, that is differentiable in $f(z_0)$, the composition $h \circ f$ is differentiable in z_0 and has derivative*

$$(h \circ f)'(z_0) = h'(f(z_0))f'(z_0). \tag{3.12}$$

Proof. The proofs are as for their real counterparts. We only prove (iv): By Lemma 3.3, there is a function $\psi : B \to \mathbb{C}$, continuous in $f(z_0)$ with

$$h(w) - h(f(z_0)) = \psi(w)(w - f(z_0)), \qquad \text{and} \qquad \psi(f(z_0)) = h'(f(z_0)).$$

Choosing $w = f(z)$ yields (for $z \neq z_0$):

$$(*) \qquad \frac{h(f(z)) - h(f(z_0))}{z - z_0} = \psi(f(z))\frac{f(z) - f(z_0)}{z - z_0}.$$

Now again by Lemma 3.3, there is a function $\phi : A \to \mathbb{C}$, continuous in z_0 with

$$f(z) - f(z_0) = \phi(z)(z - z_0) \qquad \text{and} \qquad \phi(z_0) = f'(z_0).$$

We insert this into $(*)$ and find:

$$\lim_{z\to z_0} \frac{h(f(z)) - h(f(z_0))}{z - z_0} = \lim_{z\to z_0} \left(\psi(f(z))\phi(z)\right) = \psi(f(z_0))\phi(z_0) = h'(f(z_0))f'(z_0).$$

□

Example 3.6. Polynomial functions $P : \mathbb{C} \to \mathbb{C}, z \mapsto \sum_{\nu=0}^{n} a_\nu z^\nu$ are entire and have derivative

$$P'(z) = \sum_{\nu=1}^{n} \nu a_\nu z^{\nu-1}. \tag{3.13}$$

Rational functions $f : \mathbb{C} \setminus \{z_1, ..., z_n\}$ with $f(z) = \frac{P(z)}{Q(z)}$, where $z_1, ..., z_n$ are the zeros of Q are holomorphic, and their derivative can be calculated by the quotient rule (Proposition 3.5, (iii)).

3.2. The Cauchy-Riemann equations

We introduced $\mathbb{C}$ as $\mathbb{R}^2$ with an additional multiplication structure. From real analysis, one has a notion of *(total) differentiability* of a function $f : \mathbb{R}^2 \to \mathbb{R}^2$. Naturally, the question arises, how these two notions are related. We already saw that the function $\mathrm{Re} : \mathbb{C} \to \mathbb{C}$, $z \mapsto \mathrm{Re}(z)$ is not differentiable in the complex sense, even though the same (!) function $(x, y) \mapsto (x, 0)$ viewed as a map from $\mathbb{R}^2 \to \mathbb{R}^2$ is totally differentiable. We shall see that the complex structure forces certain relations between the (real) partial derivatives, if a function is to be differentiable in the complex sense.

Let $A \subseteq \mathbb{C}$ be an open set and $f : A \to \mathbb{C}$. Recall that $z = x + yi = (x, y)$. We define the functions $u, v : \mathbb{R}^2 \to \mathbb{R}$

$$u(x,y) = \mathrm{Re}(f(x,y)), \qquad v(x,y) = \mathrm{Im}(f(x,y)), \tag{3.14}$$

meaning that $f(x,y) = u(x,y) + v(x,y)i$. To stress the viewpoint as a map from a subset of $\mathbb{R}^2$ into $\mathbb{R}^2$, we may use the column vector notation

$$f : A \to \mathbb{R}^2, \qquad \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix}. \tag{3.15}$$

We recall the notion of the *Jacobi matrix* or *Jacobian* $Df(x,y)$, given by the partial derivatives

$$Df(x,y) = \begin{pmatrix} \frac{\partial u(x,y)}{\partial x} & \frac{\partial u(x,y)}{\partial y} \\ \frac{\partial v(x,y)}{\partial x} & \frac{\partial v(x,y)}{\partial y} \end{pmatrix}. \tag{3.16}$$

We also recall that the function $f : A \to \mathbb{R}^2$ is called *totally differentiable* in $(x_0, y_0) \in A$, if there is matrix $M \in \mathbb{R}^{2\times 2}$ such that for every $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$\frac{\left\| f(x,y) - f(x_0,y_0) - M \cdot \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix} \right\|}{\left\| \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix} \right\|} < \varepsilon, \qquad \text{for } \left\| \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix} \right\| < \delta, \tag{3.17}$$

see also Definition A.5 in the Appendix. Here $\|\cdot\|$ denotes the Euclidean norm[1] on $\mathbb{R}^2$. In this case, the functions u and v admit partial derivatives in (x_0, y_0) and M is necessarily given by the Jacobi matrix

$$M = Df(x_0, y_0), \tag{3.18}$$

see also Theorem A.6 in the Appendix. We can now formulate the fundamental connection between real and complex differentiability.

Theorem 3.7. *Let $A \subseteq \mathbb{C}$ be open, $z_0 \in A$ and $f : A \to \mathbb{C}$ a function with decomposition $f(x,y) = u(x,y) + v(x,y)i$ and $u(x,y), v(x,y) \in \mathbb{R}$. The following are equivalent:*

[1]This is of course nothing else but the modulus of the corresponding complex number, but we insist here on only speaking about the real differentiability, and write $\|\cdot\|$ for clarity.

(i) f *is differentiable in* z_0 *in the complex sense.*

(ii) f *is totally differentiable in* (x_0, y_0) *in the real sense and at* (x_0, y_0)*, the partial derivatives fulfill the* Cauchy-Riemann equations

$$\begin{aligned} \frac{\partial u}{\partial x}(x_0, y_0) &= \frac{\partial v}{\partial y}(x_0, y_0), \\ \frac{\partial u}{\partial y}(x_0, y_0) &= -\frac{\partial v}{\partial x}(x_0, y_0). \end{aligned} \tag{3.19}$$

Moreover, the complex derivative $f'(z_0)$ *can be expressed as follows:*

$$\begin{aligned} f'(z_0) &= \frac{\partial u}{\partial x}(x_0, y_0) + i\frac{\partial v}{\partial x}(x_0, y_0) = \frac{\partial f}{\partial x}(x_0, y_0) \\ if'(z_0) &= i\frac{\partial v}{\partial y}(x_0, y_0) + \frac{\partial u}{\partial y}(x_0, y_0) = \frac{\partial f}{\partial y}(x_0, y_0). \end{aligned} \tag{3.20}$$

Proof. Let us first assume that f is differentiable in the complex sense in z_0 with $f'(z_0) = w$. Then, by definition

$$\lim_{h \to 0} \left| \frac{f(z_0 + h) - f(z_0)}{h} - w \right| = \lim_{h \to 0} \left| \frac{f(z_0 + h) - f(z_0) - w \cdot h}{h} \right| = 0, \tag{3.21}$$

and so we see that

$$\lim_{h \to 0} \frac{|f(z_0 + h) - f(z_0) - w \cdot h|}{|h|} = 0. \tag{3.22}$$

We write $w = a + bi$ with $a, b \in \mathbb{R}$ and $h = k + \ell i$ with $k, \ell \in \mathbb{R}$, so

$$w \cdot h = ak - b\ell + (a\ell + bk)i.$$

We now view the statement (3.22) as a statement about convergence in $\mathbb{R}^2$, and thus we can write

$$\lim_{(k,\ell) \to (0,0)} \frac{\left\| f(x_0 + k, y_0 + \ell) - f(x_0, y_0) - \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \cdot \begin{pmatrix} k \\ \ell \end{pmatrix} \right\|}{\left\| \begin{pmatrix} k \\ \ell \end{pmatrix} \right\|} = 0. \tag{3.23}$$

The above equation means that f is totally differentiable in $(x_0, y_0) \in \mathbb{R}^2$ and

$$Df(x_0, y_0) = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \tag{3.24}$$

is the corresponding Jacobi matrix. Thus, we must have

$$\begin{aligned} \frac{\partial u}{\partial x}(x_0, y_0) &= a = \frac{\partial v}{\partial y}(x_0, y_0) \\ \frac{\partial u}{\partial y}(x_0, y_0) &= -b = -\frac{\partial v}{\partial x}(x_0, y_0). \end{aligned} \tag{3.25}$$

Conversely, assume that f is totally differentiable in (x_0, y_0) in the real sense, with partial derivatives of u and v fulfilling the Cauchy-Riemann equations (3.19). By defining a and b by (3.25), the total differentiability of f in the real sense in (x_0, y_0) is exactly the statement (3.23). Setting $w = a + bi$ and $h = k + \ell i$, we recover (3.22), and thus f is differentiable in the complex sense in z_0.

Finally, we note that since $w = f'(z_0) = a + bi$, the formulas (3.25) readily imply (3.20). □

Corollary 3.8. *Let $A \subseteq \mathbb{C}$ be open and $f : A \to \mathbb{C}$ a function.*

(i) If all partial derivatives $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x}$, $\frac{\partial v}{\partial y}$ exist, are continuous on A and fulfill the Cauchy-Riemann equations (3.19), f is holomorphic.

(ii) Assume additionally that A is connected. If f is holomorphic and attains only real values, it must be constant.

Proof. For (i), note that if all partial derivatives exist and are continuous, f is totally differentiable in the real sense (see also Theorem A.7 in the Appendix). Therefore, Theorem 3.7 applies and we obtain that f is holomorphic. For (ii), assume that f only attains real values, so $v \equiv 0$. By the Cauchy-Riemann equations (3.19), we find that

$$\frac{\partial u}{\partial x}(x,y) = \frac{\partial u}{\partial y}(x,y) = \frac{\partial v}{\partial x}(x,y) = \frac{\partial v}{\partial y}(x,y) = 0$$

for every $(x, y) \in A$. The only differentiable functions with gradient identical to zero are constants, so indeed u and v must be constant. □

Let us demonstrate the Cauchy-Riemann equations with a simple example.

Example 3.9. Consider the function $f : \mathbb{C} \to \mathbb{C}$, $z \mapsto z^2$. For this function, we have

$$u(x,y) = \mathrm{Re}((x+yi)^2) = x^2 - y^2, \qquad v(x,y) = \mathrm{Im}((x+yi)^2) = 2xy.$$

We already know that f is entire by Example 3.2, (ii), so the Cauchy-Riemann equations should be fulfilled at every $(x_0, y_0) \in \mathbb{R}^2$, and indeed:

$$\begin{aligned}\frac{\partial u}{\partial x}(x_0,y_0) &= 2x_0 = \frac{\partial v}{\partial y}(x_0,y_0)\\ \frac{\partial u}{\partial y}(x_0,y_0) &= -2y_0 = -\frac{\partial v}{\partial x}(x_0,y_0).\end{aligned}$$

In fact, one can observe that in the above example both u and v are harmonic functions: We recall that a function $g : A \to \mathbb{R}$ defined on an open subset $A \subseteq \mathbb{R}^2$, is called *harmonic function* if it is twice continuously differentiable and

$$\Delta g(x,y) = \frac{\partial^2 g}{\partial x^2}(x,y) + \frac{\partial^2 g}{\partial y^2}(x,y) = 0, \qquad \text{for } (x,y) \in A. \tag{3.26}$$

This is not a coincidence, but pertains to all holomorphic functions.

Corollary 3.10. *Let $A \subseteq \mathbb{C}$ be open and $f : A \to \mathbb{C}$ be holomorphic. Assuming that $u = \mathrm{Re}(f)$ and $v = \mathrm{Im}(f)$ are both twice continuously differentiable in the real sense*[2]*, they are harmonic functions.*

Proof. By the Cauchy-Riemann equations (3.19), we see that for any $(x_0, y_0) \in A$, one has

$$\Delta u(x_0, y_0) = \frac{\partial^2 u}{\partial x^2}(x_0, y_0) + \frac{\partial^2 u}{\partial y^2}(x_0, y_0) = \frac{\partial^2 v}{\partial x \partial y}(x_0, y_0) - \frac{\partial^2 v}{\partial y \partial x}(x_0, y_0) = 0, \tag{3.27}$$

using Schwarz' theorem on the interchangability of the order of partial derivatives for twice continuously differentiable functions in the last step (see also Theorem A.4 in the Appendix). The proof that $\Delta v(x_0, y_0) = 0$ for any $(x_0, y_0) \in A$ works analogously. □

The preceeding result can be used to test whether a given function $u : A \to \mathbb{R}$ defined on $A \subseteq \mathbb{R}^2$ open can be the real part of a holomorphic function. If u and v are harmonic and fulfill the Cauchy-Riemann equations (3.19), we say that v is a *conjugate harmonic function* of u. We will come back to the study of harmonic functions in two dimensions later.

Let us now discuss the differentiability properties of the exponential and trigonometric functions introduced in section 1.4.

Proposition 3.11. *The exponential function* $\exp : \mathbb{C} \to \mathbb{C}$, $z \mapsto \exp(z)$ *is entire and fulfills*

$$\exp'(z) = \exp(z) \qquad \textit{for } z \in \mathbb{C}. \tag{3.28}$$

Proof. By definition $f(z) = f(x, y) = \exp(x)(\cos(y) + i \sin(y))$, so we can read off: $u(x, y) = \exp(x)\cos(y)$ and $v(x, y) = \exp(x)\sin(y)$. These functions are smooth (infinitely differentiable in the sense of real variables), and they fulfill

$$\begin{aligned} \frac{\partial u}{\partial x}(x, y) &= \exp(x)\cos(y) = \frac{\partial v}{\partial y}(x, y), \\ \frac{\partial u}{\partial y}(x, y) &= -\exp(x)\sin(y) = -\frac{\partial v}{\partial x}(x, y). \end{aligned}$$

So exp is entire by Corollary 3.8. We then use (3.20) to find:

$$\exp'(z) = \frac{\partial u}{\partial x}(x, y) + i\frac{\partial v}{\partial x}(x, y) = \exp(x)(\cos(y) + i \sin(y)) = \exp(z).$$

□

Corollary 3.12. *The trigonometric functions* $\sin : \mathbb{C} \to \mathbb{C}$ *and* $\cos : \mathbb{C} \to \mathbb{C}$ *are entire and fulfill*

$$\sin'(z) = \cos(z), \qquad \cos'(z) = -\sin(z), \qquad \textit{for } z \in \mathbb{C}. \tag{3.29}$$

[2] We will later see that this assumption is redundant: In fact, every holomorphic function will be infinitely differentiable, so u and v are smooth functions.

Proof. This follows directly from the definitions and the sum, product and chain rules for complex derivatives. Indeed:

$$\sin'(z) = \frac{1}{2i}\left(\frac{\mathrm{d}}{\mathrm{d}z}\exp(iz) - \frac{\mathrm{d}}{\mathrm{d}z}\exp(-iz)\right) = \frac{1}{2}(\exp(iz) + \exp(-iz)) = \cos(z),$$

and similarly for $\cos'(z)$. □

We finally discuss the principal value of the complex logarithm.

Proposition 3.13. *Let* $\mathbb{C}_- = \mathbb{C} \setminus \{x + yi \in \mathbb{C}\,;\, y = 0, x \leq 0\}$. *Then, the restriction of* Log, *defined in* (1.33), *is holomorphic, with*

$$\mathrm{Log}'(z) = \frac{1}{z}, \qquad \textit{for } z \in \mathbb{C}_-. \tag{3.30}$$

Proof. The proof uses either the inverse function theorem, or the Cauchy-Riemann equations in polar coordinates and is given as an ⇝ *Exercise*. □

End of Lecture 4

4. Contour integrals and Cauchy's theorem

(Reference: Marsden-Hoffman, Basic complex analysis, 3rd Ed., Sections 2.1 – 2.4)

4.1. Contour integrals

Definition 4.1. Let $I = [a, b] \subseteq \mathbb{R}$ with $a < b$ be a closed interval and $\gamma : I \to \mathbb{C}$ a map. We denote $\gamma = \gamma_1 + \gamma_2 \cdot i$ with $\gamma_1(t), \gamma_2(t) \in \mathbb{R}$ for every $t \in [a, b]$. We say that γ is *continuous* / *differentiable*[1] / *continuously differentiable (C^1)*, if this holds for both γ_1 and γ_2. If γ is differentiable at $t \in I$, we set

$$\gamma'(t) = \gamma_1'(t) + i\gamma_2'(t). \tag{4.1}$$

We say that a continuous map γ is *piecewise continuously differentiable (piecewise C^1)* if there exists a partition $a = a_0 < a_1 < \dots < a_n = b$ of I such that $\gamma|_{[a_{j-1}, a_j]}$ is continuously differentiable for every $1 \leq j \leq n$.

Convention: Without specification, the term *curve* will always mean a continuous, piecewise C^1 map.

Lemma 4.2. *(i) Sums and products of differentiable functions $\gamma, \delta : I \to \mathbb{C}$ are again differentiable and for $t \in [a, b]$,*

$$(\gamma + \delta)'(t) = \gamma'(t) + \delta'(t), \tag{4.2}$$

$$(\gamma \cdot \delta)'(t) = \gamma'(t)\delta(t) + \gamma(t)\delta'(t). \tag{4.3}$$

(ii) Let $A \subseteq \mathbb{C}$ be open and $\gamma : I \to \mathbb{C}$ differentiable with $\gamma(I) \subseteq A$. For $f : A \to \mathbb{C}$ holomorphic, the function

$$\delta : I \to \mathbb{C}, \qquad t \mapsto f(\gamma(t)) \tag{4.4}$$

is differentiable and has derivative in $t \in I$

$$\delta'(t) = f'(\gamma(t)) \cdot \gamma'(t). \tag{4.5}$$

Proof. Part (i) works as in the case for real variables. For part (ii) we write $f = u + v \cdot i$ and $\gamma = \gamma_1 + \gamma_2 \cdot i$, then

$$\delta(t) = u(\gamma_1(t), \gamma_2(t)) + iv(\gamma_1(t), \gamma_2(t)). \tag{4.6}$$

[1]Recall: $f : [a, b] \to \mathbb{R}$ is differentiable in $[a, b]$ if it is differentiable in (a, b), and the limits $\lim_{x \downarrow a} \frac{f(x) - f(a)}{x - a}$ and $\lim_{x \uparrow b} \frac{f(x) - f(b)}{x - b}$ exist.

This is differentiable in the real sense (as a map from $I \to \mathbb{R}^2 = \mathbb{C}$) by the chain rule from real analysis (see also Theorem A.8 in the Appendix), so we find

$$\begin{aligned}\delta'(t) &= \frac{\partial u}{\partial x}(\gamma_1(t), \gamma_2(t))\gamma_1'(t) + \frac{\partial u}{\partial y}(\gamma_1(t), \gamma_2(t))\gamma_2'(t) \\ &\quad + i\left(\frac{\partial v}{\partial x}(\gamma_1(t), \gamma_2(t))\gamma_1'(t) + \frac{\partial v}{\partial y}(\gamma_1(t), \gamma_2(t))\gamma_2'(t)\right) \\ &\overset{(3.19)}{=} \left(\frac{\partial u}{\partial x}(\gamma_1(t), \gamma_2(t)) + i\frac{\partial v}{\partial x}(\gamma_1(t), \gamma_2(t))\right)(\gamma_1'(t) + i\gamma_2'(t)) \\ &\overset{(3.20)}{=} f'(\gamma(t)) \cdot \gamma'(t).\end{aligned} \tag{4.7}$$

□

We now define two types of integrals: The integral of a curve $\gamma : I \to \mathbb{C}$ and the integral of a continuous complex function *along* a curve. The second integral type will rely on the first.

Definition 4.3. Let $I = [a, b] \subseteq \mathbb{R}$ with $a < b$ and $\gamma : I \to \mathbb{C}$ continuous with $\gamma = \gamma_1 + i\gamma_2$ (so γ_1, γ_2 are continuous). We define the integrals of γ:

$$\int_a^b \gamma(t)\mathrm{d}t = \int_a^b \gamma_1(t)\mathrm{d}t + i\int_a^b \gamma_2(t)\mathrm{d}t, \quad \text{and} \quad \int_b^a \gamma(t)\mathrm{d}t = -\int_a^b \gamma(t)\mathrm{d}t. \tag{4.8}$$

In other words, the integral is obtained by integrating componentwise the vector-valued function $\gamma : I \to \mathbb{R}^2$. The continuity of γ ensures the continuity of γ_1 and γ_2 on the bounded and closed (hence compact) set $I = [a, b]$, so all integrals are well-defined. We list some elementary properties of this integral.

Lemma 4.4. *Let $I = [a, b] \subseteq \mathbb{R}$ with $a < b$ and $\gamma, \delta : I \to \mathbb{C}$ continuous.*

(i) For $a < c < b$, one has

$$\int_a^b \gamma(t)\mathrm{d}t = \int_a^c \gamma(t)\mathrm{d}t + \int_c^b \gamma(t)\mathrm{d}t. \tag{4.9}$$

(ii) For all $\lambda \in \mathbb{C}$:

$$\int_a^b (\lambda\gamma(t) + \delta(t))\mathrm{d}t = \lambda\int_a^b \gamma(t)\mathrm{d}t + \int_a^b \delta(t)\mathrm{d}t. \tag{4.10}$$

(iii) The real and imaginary parts of the integral fulfill

$$\mathrm{Re}\left(\int_a^b \gamma(t)\mathrm{d}t\right) = \int_a^b \mathrm{Re}(\gamma(t))\mathrm{d}t, \quad \mathrm{Im}\left(\int_a^b \gamma(t)\mathrm{d}t\right) = \int_a^b \mathrm{Im}(\gamma(t))\mathrm{d}t. \tag{4.11}$$

(iv) The map $t \mapsto \int_a^t \gamma(x)\mathrm{d}x$ is differentiable on I and has the derivative

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_a^t \gamma(x)\mathrm{d}x\right) = \gamma(t). \tag{4.12}$$

(v) If $\gamma : I \to \mathbb{C}$ is C^1, then

$$\int_a^b \gamma'(t)\mathrm{d}t = \gamma(b) - \gamma(a). \tag{4.13}$$

(vi) The modulus of the integral fulfills:

$$\left|\int_a^b \gamma(t)\mathrm{d}t\right| \leq \int_a^b |\gamma(t)|\mathrm{d}t. \tag{4.14}$$

Proof. Items (i) through (iii) follow immediately from the definition of the integral and the respective properties of the real integrals.

For item (iv) and (v), we can apply the Fundamental Theorem of Calculus componentwise.

For item (vi), we only need to consider the case where the left-hand side is strictly positive (the integral of a real, non-negative function is non-negative, so the claim is fulfilled when the left-hand side is zero). In this case, we can write

$$\int_a^b \gamma(t)\mathrm{d}t = r\exp(i\varphi), \qquad r > 0, \varphi \in \mathbb{R}, \tag{4.15}$$

From here, we see that

$$\begin{aligned} r = \exp(-i\varphi)\int_a^b \gamma(t)\mathrm{d}t &\overset{(4.10)}{=} \int_a^b \exp(-i\varphi)\gamma(t)\mathrm{d}t \\ = \int_a^b \mathrm{Re}(\exp(-i\varphi)\gamma(t))\mathrm{d}t &+ i\int_a^b \mathrm{Im}(\exp(-i\varphi)\gamma(t))\mathrm{d}t \end{aligned}. \tag{4.16}$$

We then take the real part of the above equation and obtain

$$r = \int_a^b \mathrm{Re}(\exp(-i\varphi)\gamma(t))\mathrm{d}t \leq \int_a^b |\exp(-i\varphi)\gamma(t)|\mathrm{d}t. \tag{4.17}$$

The claim follows since $\left|\int_a^b \gamma(t)\mathrm{d}t\right| = r$ by the polar representation (4.15). □

We will now move to integrals of continuous complex functions over a curve γ.

Definition 4.5. Let $A \subseteq \mathbb{C}$ be open and $f : A \to \mathbb{C}$ continuous. Let $I = [a, b] \subseteq \mathbb{R}$ with $a < b$ and $\gamma : I \to \mathbb{C}$ a piecewise C^1 curve with $\gamma(I) \subseteq A$. We define the *integral of f along γ* as

$$\int_\gamma f(z)\mathrm{d}z = \sum_{j=1}^{n} \int_{a_{j-1}}^{a_j} f(\gamma(t)) \cdot \gamma'(t)\mathrm{d}t, \tag{4.18}$$

where $a = a_0 < a_1 < \ldots < a_n = b$ is a partition such that $\gamma|_{[a_{j-a}, a_j]}$ is C^1 for every $1 \leq j \leq n$.

Remark 4.6. (i) Since $t \mapsto \gamma'(t)$ is continuous on $[a_{j-1}, a_j]$ and $t \mapsto f(\gamma(t))$ is continuous as composition of continuous maps, the product $t \mapsto f(\gamma(t)) \cdot \gamma'(t)$ itself is a continuous map from $[a_{j-1}, a_j]$ to $\mathbb{C}$, and the integrals on the right-hand side of (4.18) are defined as in Definition 4.3.

(ii) It is easy to show that the definition does not depend on the choice of the partition: If $a = \widetilde{a}_0 < \widetilde{a}_1 < \ldots < \widetilde{a}_m = b$ is another partition of $[a, b]$ such that $\gamma|_{[\widetilde{a}_{j-1}, \widetilde{a}_j]}$ is C^1 for every $1 \le j \le m$, we can choose a common refinement $a = \mathfrak{z}_0 < \ldots < \mathfrak{z}_k = b$ (such that every $\mathfrak{z}_j$ corresponds to some a_ℓ or $\widetilde{a}_\ell$) and use (4.9) to show that both $\sum_{j=1}^{n} \int_{a_{j-1}}^{a_j} f(\gamma(t)) \cdot \gamma'(t)\mathrm{d}t$ and $\sum_{j=1}^{m} \int_{\widetilde{a}_{j-1}}^{\widetilde{a}_j} f(\gamma(t)) \cdot \gamma'(t)\mathrm{d}t$ are equal to $\sum_{j=1}^{k} \int_{\mathfrak{z}_{j-1}}^{\mathfrak{z}_j} f(\gamma(t)) \cdot \gamma'(t)\mathrm{d}t$.

(iii) If we write $f(z) = u(x, y) + iv(x, y)$ with real functions u and v, we can calculate the complex contour integral as

$$\begin{aligned} \int_\gamma f(z)\mathrm{d}z = &\int_a^b (u(\gamma_1(t), \gamma_2(t))\gamma_1'(t) - v(\gamma_1(t), \gamma_2(t))\gamma_2'(t))\mathrm{d}t \\ &+ i \int_a^b (u(\gamma_1(t), \gamma_2(t))\gamma_2'(t) + v(\gamma_1(t), \gamma_2(t))\gamma_1'(t))\mathrm{d}t. \end{aligned} \tag{4.19}$$

Now that we have defined the integral of f along the curve γ, we should first make sure that the integral is invariant under reparametrizations. For instance, a full unit circle around the origin, traversed in a counter-clockwise direction may be parametrized by the curves

$$\gamma : \begin{cases} [0, 1] \mapsto \mathbb{C}, \\ t \mapsto \exp(2\pi i t), \end{cases} \qquad \delta : \begin{cases} [0, 2\pi] \mapsto \mathbb{C}, \\ t \mapsto \exp(it). \end{cases} \tag{4.20}$$

Naturally, we should expect that the integrals $\int_\gamma f(z)\mathrm{d}z$ and $\int_\delta f(z)\mathrm{d}z$ for some continuous function f defined on $A \subseteq \mathbb{C}$ with $\gamma([0, 1]) = \delta([0, 2\pi]) = S^1 \subseteq A$ should coincide. This is in fact the case, as we shall now see.

Proposition 4.7. *Let $A \subseteq \mathbb{C}$ be open, $f : A \to \mathbb{C}$ continuous. Let*

- *$I = [a, b] \subseteq \mathbb{R}$ with $a < b$ and $\gamma : I \to \mathbb{C}$ piecewise C^1 with $\gamma(I) \subseteq A$*
- *$\widetilde{I} = [\widetilde{a}, \widetilde{b}] \subseteq \mathbb{R}$ with $\widetilde{a} < \widetilde{b}$ and $\widetilde{\gamma} : \widetilde{I} \to \mathbb{C}$ piecewise C^1.*

Assume that there exists a C^1 function $\tau : I \to \widetilde{I}$ with $\tau'(t) > 0$ for every $t \in [a, b]$, fulfilling $\tau(a) = \widetilde{a}, \tau(b) = \widetilde{b}$ and $\gamma = \widetilde{\gamma} \circ \tau$. Then it holds that $\widetilde{\gamma}(\widetilde{I}) = \gamma(I)$ and we have

$$\int_{\widetilde{\gamma}} f(z)\mathrm{d}z = \int_\gamma f(z)\mathrm{d}z. \tag{4.21}$$

Proof. Since τ is C^1 with posititive derivative, it is an increasing and continuous function, so since $\tau(a) = \widetilde{a}$ and $\tau(b) = \widetilde{b}$, we see that $\tau(I) = \widetilde{I}$.

By breaking up $I = [a, b]$ into subintervals (if necessary), we assume without loss of generality that γ itself is C^1. Using the chain rule, we see that

$$\gamma'(t) = \frac{\mathrm{d}}{\mathrm{d}t}\widetilde{\gamma}(\tau(t)) = \widetilde{\gamma}'(\tau(t))\tau'(t). \tag{4.22}$$

Using the substitution rule, we see that

$$\begin{aligned}\int_a^b f(\gamma(t)) \cdot \gamma'(t)\mathrm{d}t &= \int_a^b f(\widetilde{\gamma}(\tau(t))) \cdot \widetilde{\gamma}'(\tau(t))\tau'(t)\mathrm{d}t \\ &= \int_{\widetilde{a}}^{\widetilde{b}} f(\widetilde{\gamma}(s))\widetilde{\gamma}'(s)\mathrm{d}s,\end{aligned} \tag{4.23}$$

which is exactly the claim. ☐

We call a function τ as in the above Proposition a *reparametrization* of the curve γ, and so we have demonstrated that the complex contour integral is invariant under reparametrizations.

Definition 4.8. (i) Let $I = [a, b] \subseteq \mathbb{R}$ with $a < b$ and $\gamma : I \to \mathbb{C}$ a continuous curve. The map

$$-\gamma : I \to \mathbb{C}, \qquad t \mapsto \gamma(a + b - t) \tag{4.24}$$

is called the *opposite curve* of γ.

(ii) Let $I_1 = [a, b] \subseteq \mathbb{R}$, $I_2 = [b, c] \subseteq \mathbb{R}$ with $a < b < c$ and $\gamma_1 : I_1 \to \mathbb{C}$, $\gamma_2 : I_2 \to \mathbb{C}$ two continuous curves with $\gamma_1(b) = \gamma_2(b)$. Let $I = I_1 \cup I_2 = [a, c]$. The map

$$\gamma_1 + \gamma_2 : I \to \mathbb{C}, \qquad t \mapsto \begin{cases} \gamma_1(t), & t \in [a, b], \\ \gamma_2(t), & t \in [b, c], \end{cases} \tag{4.25}$$

is called the *join* or *union* or *sum* of the curves γ_1 and γ_2 (also denoted $\gamma_1 * \gamma_2$)[2].

The above definitions coincide with their intuitive meanings. Indeed, the opposite curve $-\gamma$ has the same image as γ, but is traversed in an opposite direction. The join of γ_1 and γ_2 is a curve starting at $\gamma_1(a)$, moving to $\gamma_1(b) = \gamma_2(b)$ via the map γ_1 and then to $\gamma_2(c)$ via the map γ_2. Inductively, one can define the join of $\gamma_1, \gamma_2, ..., \gamma_n$.

Proposition 4.9. *Let $A \subseteq \mathbb{C}$ and $f, g : A \to \mathbb{C}$ continuous. Let $I = [a, b] \subseteq \mathbb{R}$ with $a < b$ and let $\gamma : I \to \mathbb{C}$ be a piecewise C^1 curve with $\gamma(I) \subseteq A$.*

(i) For every $\lambda \in \mathbb{C}$, one has

$$\int_\gamma (\lambda f(z) + g(z))\mathrm{d}z = \lambda \cdot \int_\gamma f(z)\mathrm{d}z + \int_\gamma g(z)\mathrm{d}z. \tag{4.26}$$

(ii) For the integral along $-\gamma$, it holds that

$$\int_{-\gamma} f(z)\mathrm{d}z = -\int_\gamma f(z)\mathrm{d}z. \tag{4.27}$$

[2] By slight abuse of notation, we also define the join of $\gamma_1 : I_1 = [a, b] \to \mathbb{C}$ and $\gamma_2 : I_2 = [c, d] \to \mathbb{C}$ if $\gamma_1(b) = \gamma_2(c)$ as the join of γ_1 and $\widetilde{\gamma}_2 : \widetilde{I}_2 := [b, b + d - c] \to \mathbb{C}$, where $\widetilde{\gamma}_2(t) = \gamma_2(t + c - b)$ is a reparametrization of γ_2, i.e. $\gamma_1 + \gamma_2 := \gamma_1 + \widetilde{\gamma}_2$, which maps $I = I_1 \cup \widetilde{I}_2 = [a, b + d - c]$ into $\mathbb{C}$

(iii) *Suppose $I' = [b, c]$ with $b < c$ and $\delta : I' \to \mathbb{C}$ is another piecewise C^1 curve with $\delta(I') \subseteq A$ and $\delta(b) = \gamma(b)$. Then*

$$\int_{\gamma+\delta} f(z)\mathrm{d}z = \int_{\gamma} f(z)\mathrm{d}z + \int_{\delta} f(z)\mathrm{d}z. \tag{4.28}$$

Proof. Item (i) follows readily from (4.10) (applied to the continuous functions $t \mapsto f(\gamma(t))\gamma'(t)$ and $t \mapsto g(\gamma(t))\gamma'(t)$). For item (ii), note that $-\gamma$ is given on $[a, b]$ by $t \mapsto \gamma(a+b-t)$, and this function is C^1 when restricting to intervals $[\widetilde{a}_{j-1}, \widetilde{a}_j]$, $j = 1, \ldots, n$, with $a = \widetilde{a}_0 < \ldots < \widetilde{a}_n = b$, and we set $\widetilde{a}_j = a + b - a_{n-j}$ (with $j = 0, \ldots, n$). Therefore

$$\begin{aligned}
\int_{-\gamma} f(z)\mathrm{d}z &= -\sum_{j=1}^{n} \int_{\widetilde{a}_{j-1}}^{\widetilde{a}_j} f(\gamma(a+b-t))\gamma'(a+b-t)\mathrm{d}t \\
&\overset{s=a+b-t}{=} \sum_{j=1}^{n} \int_{a+b-\widetilde{a}_{j-1}}^{a+b-\widetilde{a}_j} f(\gamma(s))\gamma'(s)\mathrm{d}s \\
&= -\sum_{j=1}^{n} \int_{a_{n-j}}^{a_{n+1-j}} f(\gamma(s))\gamma'(s)\mathrm{d}s = -\int_{\gamma} f(z)\mathrm{d}z.
\end{aligned} \tag{4.29}$$

Part (iii) follows directly from the definition of the join of two curves and the definition of contour integrals. □

We demonstrate the evaluation of some complex contour integrals by some examples.

Example 4.10. (i) Let us start with a very simple example which will play a fundamental role in the rest of this course: Let $z_0 \in \mathbb{C}$, and consider the integral

$$\int_{\gamma} \frac{1}{z - z_0}\mathrm{d}z, \tag{4.30}$$

where γ is the full circle with radius $R > 0$, centered in z_0, traveresed in a counter-clockwise sense. A simple parametrization for γ is

$$\gamma : \begin{cases} [0, 2\pi] \to \mathbb{C} \\ t \mapsto R\exp(it) + z_0. \end{cases} \tag{4.31}$$

We therefore calculate

$$\int_{\gamma} \frac{1}{z - z_0}\mathrm{d}z = \int_0^{2\pi} \underbrace{\frac{1}{R\exp(it)}}_{=f(\gamma(t))} \underbrace{i \cdot R\exp(it)}_{=\gamma'(t)} \mathrm{d}t = 2\pi i. \tag{4.32}$$

(ii) Let us compute

$$\int_{\gamma} z^3\mathrm{d}z \tag{4.33}$$

where γ is the line segment between 0 and $1+i$. A simple parameterization for γ is

$$\gamma : \begin{cases} [0,1] \to \mathbb{C} \\ t \mapsto (1+i)t. \end{cases} \tag{4.34}$$

We therefore calculate

$$\int_\gamma z^3 \mathrm{d}z = \int_0^1 \underbrace{(1+i)^3 t^3}_{=f(\gamma(t))} \cdot \underbrace{(1+i)}_{=\gamma'(t)} \mathrm{d}t = -4 \int_0^1 t^3 \mathrm{d}t = -1. \tag{4.35}$$

We will need another concept for complex contour integrals, that of arc length. We will introduce arc length, integrals with respect to arc length and a standard bound on complex contour integrals, involving the arc length.

Definition 4.11. Let $I = [a,b] \subseteq \mathbb{R}$ with $a < b$ and γ be a piecewise C^1 curve, such that $\gamma|_{[a_{j-1},a_j]}$ for $j = 1, ..., n$ are C^1, where $a = a_0 < a_1 < ... < a_n = b$. Then the quantity

$$\ell(\gamma) = \sum_{j=1}^{n} \int_{a_{j-1}}^{a_j} |\gamma'(t)| \mathrm{d}t \tag{4.36}$$

is called the *arc length* of γ.

Remark 4.12. (i) The arc length is well defined and finite: Indeed, since the function $\gamma_j : [a_{j-1}, a_j] \to \mathbb{C}, t \mapsto \gamma'(t)$ is continuous and the modulus function $|\cdot| : \mathbb{C} \to \mathbb{R}, z \mapsto |z|$ is continuous, their composition $t \mapsto |\gamma'(t)|$ is a continuous map from a compact interval into $\mathbb{R}$, so the claim follows.

(ii) The arc length is invariant under reparametrization.

We can also define integrals with respect to arc length:

Definition 4.13. Let $A \subseteq \mathbb{C}$ be open and $f : A \to \mathbb{C}$ continuous. Let $I = [a,b] \subseteq \mathbb{R}$ with $a < b$ and $\gamma : I \to \mathbb{C}$ a piecewise C^1 curve with $\gamma(I) \subseteq A$. We define the *integral of f with respect to the length of γ* as

$$\int_\gamma f(z) \mathrm{d}s = \int_\gamma f(z) |\mathrm{d}z| = \sum_{j=1}^{n} \int_{a_{j-1}}^{a_j} f(\gamma(t)) |\gamma'(t)| \mathrm{d}t, \tag{4.37}$$

where $a = a_0 < a_1 < ... < a_n = b$ is a partition such that $\gamma|_{[a_{j-1},a_j]}$ is C^1 for every $1 \leq j \leq n$.

This integral is again invariant under reparametrization. By definition,

$$\ell(\gamma) = \int_\gamma \mathrm{d}s. \tag{4.38}$$

We will now present a standard bound on complex contour integrals.

Proposition 4.14. *Let $A \subseteq \mathbb{C}$ be open and $f : A \to \mathbb{C}$ continuous. Let $I = [a,b] \subseteq \mathbb{R}$ with $a < b$ and $\gamma : I \to \mathbb{C}$ a piecewise C^1 curve with $\gamma(I) \subseteq A$. Then, one has*

$$\left|\int_\gamma f(z)\mathrm{d}z\right| \leq \int_\gamma |f(z)|\mathrm{d}s. \tag{4.39}$$

In particular, if $\sup_{z\in\gamma(I)} |f(z)| \leq M \in [0,\infty)$, *then*

$$\left|\int_\gamma f(z)\mathrm{d}z\right| \leq M\ell(\gamma). \tag{4.40}$$

Proof. We first prove (4.39). As usual consider a partition $a = a_0 < \ldots < a_n = b$ such that $\gamma|_{[a_{j-1},a_j]}$, $j = 1,\ldots,n$ are all C^1. We use the definition of the contour integral and apply the bound (4.14) to obtain

$$\begin{aligned}
\left|\int_\gamma f(z)\mathrm{d}z\right| &= \left|\sum_{j=1}^{n}\int_{a_{j-1}}^{a_j} f(\gamma(t))\gamma'(t)\mathrm{d}t\right| \\
&\leq \sum_{j=1}^{n}\left|\int_{a_{j-1}}^{a_j} f(\gamma(t))\gamma'(t)\mathrm{d}t\right| \\
&\overset{(4.14)}{\leq} \sum_{j=1}^{n}\int_{a_{j-1}}^{a_j} |f(\gamma(t))||\gamma'(t)|\mathrm{d}t = \int_\gamma |f(z)|\mathrm{d}s.
\end{aligned} \tag{4.41}$$

We now show (4.40). First note that since γ is continuous and I is compact, the image $\gamma(I)$ is compact as well[3], and the continuous function f attains its maximum on compact sets, the expression $\sup_{z\in\gamma(I)} |f(z)|$ is finite. Moreover, we see that

$$\left|\int_\gamma f(z)\mathrm{d}z\right| \overset{(4.39)}{\leq} \int_\gamma \underbrace{|f(z)|}_{\leq M}\mathrm{d}s \leq M\int_\gamma \mathrm{d}s \overset{(4.38)}{=} M\ell(\gamma). \tag{4.42}$$

□

We will now show the *Fundamental Theorem of Calculus for Contour Integrals.*

Theorem 4.15. *Let $A \subseteq \mathbb{C}$ be open, $I = [a,b] \subseteq \mathbb{R}$ with $a < b$ and $\gamma : I \to \mathbb{C}$ be a piecewise C^1 curve with $\gamma(I) \subseteq A$. Assume that $f : A \to \mathbb{C}$ is continuous and has a* primitive *F (this means that $F : A \to \mathbb{C}$ is holomorphic and fulfills $F'(z) = f(z)$ for every $z \in A$). Then,*

$$\int_\gamma f(z)\mathrm{d}z = F(\gamma(b)) - F(\gamma(a)). \tag{4.43}$$

In particular, if γ is a closed curve ($\gamma(a) = \gamma(b)$), then

$$\int_\gamma f(z)\mathrm{d}z = 0. \tag{4.44}$$

[3] This is a general property of continuous images of compact sets, see also Remark 2.17, (iii).

Proof. As always, let $a = a_0 < \ldots < a_n = b$ be a partition such that $\gamma_j = \gamma|_{[a_{j-1},a_j]}$, $j = 1, \ldots, n$, are C^1. Then

$$\begin{aligned}\int_\gamma f(z)\mathrm{d}z &= \sum_{j=1}^n \int_{\gamma_j} f(z)\mathrm{d}z = \sum_{j=1}^n \int_{a_{j-1}}^{a_j} \underbrace{f(\gamma(t))}_{=F'(\gamma(t))} \gamma'(t)\mathrm{d}t \\ &\overset{(4.5)}{=} \sum_{j=1}^n \int_{a_{j-1}}^{a_j} \frac{\mathrm{d}}{\mathrm{d}t} F(\gamma(t))\mathrm{d}t \overset{(4.13)}{=} \sum_{j=1}^n (F(\gamma(a_j) - F(\gamma(a_{j-1})) \\ &= F(\gamma(b)) - F(\gamma(a)).\end{aligned} \tag{4.45}$$

The claim (4.44) is immediate. □

Corollary 4.16. *Let $A \subseteq \mathbb{C}$ be a domain and $f : A \to \mathbb{C}$ a holomorphic function with $f'(z) = 0$ for every $z \in A$. Then f is constant.*

Proof. Fix $z_0 \in A$ and let $z \in A$ be another point. By Proposition 2.19, (ii), there exists a piecewise C^1 curve γ connecting z_0 and z. Since f is a primitive of f', we then have

$$f(z) - f(z_0) = \int_\gamma f'(w)\mathrm{d}w = 0, \tag{4.46}$$

so f is constant. □

Of course we could have obtained the last statement also by using the Cauchy-Riemann equations.

End of Lecture 5

We have seen that the existence of a primitive allows the calculation of complex contour integrals in a very easy way, and the value of such an integral is manifestly independent of the geometry of the path γ, and only depends on the endpoints $\gamma(a)$ and $\gamma(b)$. The next result shows that the path-independence of complex contour integrals is in fact equivalent to the existence of a primitive.

Theorem 4.17. *Suppose $A \subseteq \mathbb{C}$ is a domain and $f : A \to \mathbb{C}$ is continuous. The following are equivalent:*

(i) Integrals over f are path independent, i.e. if $z_0, z_1 \in A$ and $\gamma_0 : I_0 = [a_0, b_0] \to \mathbb{C}$ and $\gamma_1 : I_1 = [a_1, b_1] \to \mathbb{C}$ (with $a_j < b_j$, $j = 0, 1$) are piecewise C^1 curves with $\gamma_0(I_0), \gamma_1(I_1) \subseteq A$ fulfilling $\gamma_0(a_0) = \gamma_1(a_1) = z_0$ and $\gamma_0(b_0) = \gamma_1(b_1) = z_1$, then

$$\int_{\gamma_0} f(z)\mathrm{d}z = \int_{\gamma_1} f(z)\mathrm{d}z. \tag{4.47}$$

(ii) Integrals over f along closed curves are zero, i.e. if $\gamma : I = [a, b] \to \mathbb{C}$ (with $a < b$) is a piecewise C^1 curve with $\gamma(I) \subseteq A$ and $\gamma(a) = \gamma(b)$, then

$$\int_\gamma f(z)\mathrm{d}z = 0. \tag{4.48}$$

(iii) There is a primitive of f on A, i.e. there exists $F : A \to \mathbb{C}$ holomorphic with

$$F'(z) = f(z), \qquad \text{for } z \in A. \tag{4.49}$$

Proof. We first prove the equivalence of (i) and (ii).

Let $\gamma : I \to \mathbb{C}$ be a closed piecewise C^1 curve with $\gamma(I) \subseteq A$. If γ is constant, $\int_\gamma f(z)\mathrm{d}z = 0$, so assume γ is not constant. Then there exist $a < c < b$ and $\gamma(c) \neq \gamma(a) = \gamma(b)$. Consider

$$\gamma_1 : [a,c] \to \mathbb{C}, t \mapsto \gamma(t), \qquad \gamma_2 : [c,b] \to \mathbb{C}, t \mapsto \gamma(t). \tag{4.50}$$

It follows that $\gamma = \gamma_1 + \gamma_2$ and therefore

$$\int_\gamma f(z)\mathrm{d}z = \int_{\gamma_1} f(z)\mathrm{d}z + \int_{\gamma_2} f(z)\mathrm{d}z \overset{(4.27)}{=} \int_{\gamma_1} f(z)\mathrm{d}z - \int_{-\gamma_2} f(z)\mathrm{d}z \overset{(4.47)}{=} 0,$$

since $\gamma_1(a) = (-\gamma_2)(c)$ and $\gamma_1(c) = (-\gamma_2)(b)$. This shows that (i) implies (ii).

Now assume that $\gamma_0 : I_0 \to \mathbb{C}$ and $\gamma_1 : I_1 \to \mathbb{C}$ are two piecewise C^1 curves with $\gamma_0(I_0), \gamma_1(I_1) \subseteq A$ and $\gamma_0(a_0) = \gamma_1(a_1)$ and $\gamma_0(b_0) = \gamma_1(b_1)$. Then $\gamma_0 + (-\gamma_1)$ is a piecewise C^1 closed curve, so

$$0 \overset{(4.48)}{=} \int_{\gamma_0} f(z)\mathrm{d}z + \int_{-\gamma_1} f(z)\mathrm{d}z \overset{(4.27)}{=} \int_{\gamma_0} f(z)\mathrm{d}z - \int_{\gamma_1} f(z)\mathrm{d}z,$$

and so (ii) implies (i).

The fact that (iii) implies (ii) was already shown in Theorem 4.15, so we are left with showing that (i) implies (iii). So let $f : A \to \mathbb{C}$ be continuous and let $z_0 \in A$. Assuming that (i) holds, we define the function

$$F : \begin{cases} A \to \mathbb{C}, \\ z \mapsto F(z) = \int_{\gamma_z} f(w)\mathrm{d}w, \end{cases} \tag{4.51}$$

where $\gamma_z : [a,b] \to \mathbb{C}$, with $a < b$ is *any* piecewise C^1 curve with $\gamma_z(a) = z_0$ and $\gamma_z(b) = z$. This function is well defined, since

- domains are path-connected, and for $z_0, z \in A$ there is always a (piecewise) C^1 path connecting z_0 to z (see Proposition 2.19),
- the integral is finite by the standard bound (4.40) for any choice of the curve γ_z, since piecewise C^1 curves have finite length,
- the value of the integral is independent of the path γ_z we use, by the assumption (4.47).

We now need to show that F is holomorphic with $F' = f$ on A. Let $\varepsilon > 0$ and take for $z \in A$ a $\delta > 0$ small enough such that $|f(z) - f(w)| < \varepsilon$ for every $w \in D(z,\delta)$ where also $D(z,\delta) \subseteq A$ (since A is open and f is continuous in z). We then let $\gamma_w = \gamma_z + \gamma_{[z,w]}$ where

$$\gamma_{[z,w]} : [0,1] \to \mathbb{C}, \qquad t \mapsto tw + (1-t)z.$$

Note that $\gamma_{[z,w]}([0,1]) \subseteq D(z,\delta)$ and $\gamma_w = \gamma_z + \gamma_{[z,w]}$ is a piecewise C^1 curve connecting z_0 and w with image in A. For $w \in \dot{D}(z,\delta)$, we therefore have

$$F(w) - F(z) = \int_{\gamma_z + \gamma_{[z,w]}} f(u)\mathrm{d}u - \int_{\gamma_z} f(u)\mathrm{d}u$$
$$= \int_{\gamma_{[z,w]}} f(u)\mathrm{d}u.$$

We therefore obtain that

$$\left|\frac{F(w)-F(z)}{w-z} - f(z)\right| = \left|\int_0^1 \frac{f(tw+(1-t)z)\cdot(w-z)}{w-z}\mathrm{d}t - f(z)\right|$$
$$\overset{(4.14)}{\leq} \int_0^1 |f(tw+(1-t)z) - f(z)|\mathrm{d}t < \varepsilon.$$

This means exactly that $\lim_{w\to z} \frac{F(w)-F(z)}{w-z} = f(z)$, and so indeed $F'(z) = f(z)$ for every $z \in A$. $\square$

4.2. Cauchy's integral theorem

We have seen that the existence of a primitive of a continuous function f is equivalent to integrals of f along closed curves being zero. On $\mathbb{C}$, a function that is merely continuous need not have a primitive, for instance it holds that

$$\int_{\partial D(0,1)} \overline{z}\mathrm{d}z = 2\pi i (\neq 0). \tag{4.52}$$

As we shall see, the situation is rather different if we assume f to be holomorphic, rather than just continuous. We aim at answering the following question:

Are there domains on which holomorphic functions always have a primitive?

In this section, we will first prove that integrating a holomorphic function $f : A \to \mathbb{C}$ along the boundary of a triangle contained in A yields zero, which is a version of *Goursat's Lemma*. This is the main technical result on which the rest of the entire chapter, and in a way the rest of the entire course is based.

With the help of this fundamental result, we show

(I) that holomorphic functions on *star-shaped domains* always have a primitive.

(II) that holomorphic functions on *simply connected domains* always have a primitive.

Definition 4.18. For $z_1, z_2, z_3 \in \mathbb{C}$, we define the *triangle spanned by* z_1, z_2, z_3 by

$$\triangle(z_1,z_2,z_3) = \{\lambda_1 z_1 + \lambda_2 z_2 + \lambda_3 z_3\,;\, \lambda_1,\lambda_2,\lambda_3 \geq 0, \lambda_1+\lambda_2+\lambda_3 = 1\}. \tag{4.53}$$

We also define the piecewise C^1 curve

$$\gamma_{\partial\triangle(z_1,z_2,z_3)} : [0,3] \to \mathbb{C}, \qquad \gamma_{\partial\triangle(z_1,z_2,z_3)}(t) = \begin{cases} z_1 + t(z_2 - z_1), & \text{for } t \in [0,1] \\ z_2 + (t-1)(z_3 - z_2), & \text{for } t \in [1,2], \\ z_3 + (t-2)(z_1 - z_3), & \text{for } t \in [2,3]. \end{cases} \tag{4.54}$$

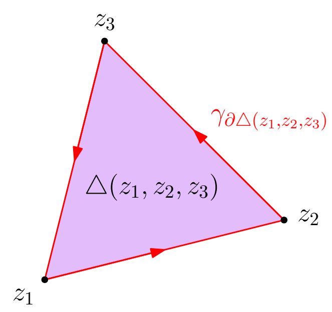

Figure 4.1.: Triangle $\triangle(z_1, z_2, z_3)$ and its boundary curve $\gamma_{\partial\triangle(z_1,z_2,z_3)}$.

The image of the curve $\gamma_{\partial\triangle(z_1,z_2,z_3)}$ forms the boundary of the triangle $\triangle(z_1, z_2, z_3)$, i.e. one has that $\gamma_{\partial\triangle(z_1,z_2,z_3)}([0,3]) = \partial \triangle (z_1, z_2, z_3)$.

We can now state the fundamental *Lemma of Goursat*.

Lemma 4.19. *Let $A \subseteq \mathbb{C}$ be open and $f : A \to \mathbb{C}$ a holomorphic function. Assume that $z_1, z_2, z_3 \in A$ are such that $\triangle(z_1, z_2, z_3) \subseteq A$. Then*

$$\int_{\gamma_{\partial\triangle(z_1,z_2,z_3)}} f(z)\mathrm{d}z = 0. \tag{4.55}$$

Remark 4.20. (i) We stress that the entire closed triangle $\triangle(z_1, z_2, z_3)$ must be contained in A, and not only its boundary. For instance $\triangle(-1-i, 1-i, i)$ is *not* contained in $\mathbb{C} \setminus \{0\}$.

(ii) Goursat's original proof uses rectangles rather than triangles (see, for instance, *Marsden-Hoffman, Basic complex analysis, 3rd Ed.*, Proposition 2.3.1 for a proof using rectangles). The proof using triangles is due to Pringsheim.

Proof of Lemma 4.19. The cases where two or more of z_1, z_2 and z_3 coincide or one of the points is on the line connecting the other two are trivial. We will therefore assume that z_1, z_2 and z_3 are pairwise distinct and do not lie on a line, and that $\triangle(z_1, z_2, z_3)$ is traversed in a counter-clockwise direction (otherwise consider $-\gamma_{\partial\triangle(z_1,z_2,z_3)}$).

We abbreviate $\triangle(z_1, z_2, z_3)$ by $\triangle$ and $\gamma_{\partial\triangle(z_1,z_2,z_3)}$ by $\gamma_{\partial\triangle}$. We join the centers of the segments between each of the three points and subsequently obtain four triangles $\triangle_1, \triangle_2, \triangle_3, \triangle_4 \subseteq$

$\triangle$. There boundaries are traversed in a counterclockwise direction, and the respective piecewise C^1 curves parametrizing the boundaries are denoted by $\gamma_{\partial\triangle_1}, \gamma_{\partial\triangle_2}, \gamma_{\partial\triangle_3}, \gamma_{\partial\triangle_4}$.

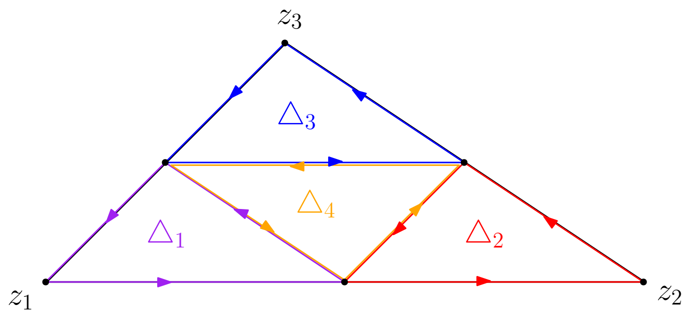

Figure 4.2.: The triangles $\triangle_1, \triangle_2, \triangle_3, \triangle_4 \subseteq \triangle$.

One has that

$$\int_{\gamma_{\partial\triangle}} f(z)\mathrm{d}z = \int_{\gamma_{\partial\triangle_1}} f(z)\mathrm{d}z + \int_{\gamma_{\partial\triangle_2}} f(z)\mathrm{d}z + \int_{\gamma_{\partial\triangle_3}} f(z)\mathrm{d}z + \int_{\gamma_{\partial\triangle_4}} f(z)\mathrm{d}z \tag{4.56}$$

since the parts of the integrals corresponding to integrals in opposite directions along the edges of $\triangle_4$ cancel by (4.27). Using the triangle inequality, we find

$$\left|\int_{\gamma_{\partial\triangle}} f(z)\mathrm{d}z\right| \leq \left|\int_{\gamma_{\partial\triangle_1}} f(z)\mathrm{d}z\right| + \ldots + \left|\int_{\gamma_{\partial\triangle_4}} f(z)\mathrm{d}z\right|. \tag{4.57}$$

Let $\triangle^{(1)} \in \{\triangle_1, \triangle_2, \triangle_3, \triangle_4\}$ be the triangle (with piecewise C^1 boundary curve $\gamma_{\partial\triangle^{(1)}}$ traversed in a counterclockwise direction) such that $\left|\int_{\gamma_{\partial\triangle^{(1)}}} f(z)\mathrm{d}z\right|$ is maximal among the $\left|\int_{\gamma_{\partial\triangle_j}} f(z)\mathrm{d}z\right|$, $j = 1, 2, 3, 4$ (choose the smallest j if it is not unique). Then we find

$$\left|\int_{\gamma_{\partial\triangle}} f(z)\mathrm{d}z\right| \leq 4 \cdot \left|\int_{\gamma_{\partial\triangle^{(1)}}} f(z)\mathrm{d}z\right|. \tag{4.58}$$

We can iterate this construction to find a sequence of closed triangles $\triangle =: \triangle^{(0)} \supseteq \triangle^{(1)} \supseteq \triangle^{(2)} \supseteq \triangle^{(3)} \supseteq \ldots$, and we have in general

$$\triangle^{(n+1)} \subseteq \triangle^{(n)}, \qquad \left|\int_{\gamma_{\partial\triangle}} f(z)\mathrm{d}z\right| \leq 4^n \cdot \left|\int_{\gamma_{\partial\triangle^{(n)}}} f(z)\mathrm{d}z\right|, n \in \mathbb{N}. \tag{4.59}$$

Since the $(\triangle^{(n)})_{n=0}^{\infty}$ are all closed and bounded, they are compact (see Proposition 2.11), and there exists a point $z_0 \in \bigcap_{n=0}^{\infty} \triangle^{(n)}$ (see Problem set 3). Moreover, for a given $\delta > 0$ there exists $N = N(\delta) \in \mathbb{N}$ with

$$\triangle^{(n)} \subseteq D(z_0, \delta), \qquad \text{for } n \geq N, \tag{4.60}$$

since the diameter of $\triangle^{(n)}$ is converging to 0 as $n \to \infty$ and $z_0 \in \triangle^{(n)}$. Note that f is differentiable in the complex sense in z_0, so by Lemma 3.3, there is a function $\phi : A \to \mathbb{C}$, continuous in z_0 with

$$f(z) = f(z_0) + (z - z_0)\phi(z), \qquad \text{and } \phi(z_0) = f'(z_0).$$

Note that for $\varepsilon > 0$, there exists $\delta > 0$ and

$$|\phi(z) - f'(z_0)| < \varepsilon \qquad \text{for } z \in D(z_0, \delta) \cap A. \tag{4.61}$$

For $n \geq N(\delta)$, $\triangle^{(n)} \subseteq D(z_0, \delta)$, and so we write

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + (\phi(z) - f'(z_0))(z - z_0) \tag{4.62}$$

and integrate over $\gamma_{\partial\triangle^{(n)}}$:

$$\begin{aligned}
&\int_{\gamma_{\partial\triangle^{(n)}}} f(z)\mathrm{d}z \\
&\overset{(4.62)}{=} \int_{\gamma_{\partial\triangle^{(n)}}} f(z_0)\mathrm{d}z + \int_{\gamma_{\partial\triangle^{(n)}}} f'(z_0)(z - z_0)\mathrm{d}z + \int_{\gamma_{\partial\triangle^{(n)}}} (\phi(z) - f'(z_0))(z - z_0)\mathrm{d}z \\
&= f(z_0)\int_{\gamma_{\partial\triangle^{(n)}}} \mathrm{d}z + f'(z_0)\int_{\gamma_{\partial\triangle^{(n)}}} (z - z_0)\mathrm{d}z + \int_{\gamma_{\partial\triangle^{(n)}}} (\phi(z) - f'(z_0))(z - z_0)\mathrm{d}z.
\end{aligned} \tag{4.63}$$

The first two integrands have primitives $z \mapsto z$ and $z \mapsto \frac{1}{2}(z - z_0)^2$ on A, and so by (4.44) of Theorem 4.15 their integrals over the closed curve $\gamma_{\partial\triangle^{(n)}}$ vanish. Equation (4.63) reduces to

$$\int_{\gamma_{\partial\triangle^{(n)}}} f(z)\mathrm{d}z = \int_{\gamma_{\partial\triangle^{(n)}}} (\phi(z) - f'(z_0))(z - z_0)\mathrm{d}z. \tag{4.64}$$

Since $\triangle^{(n)} \subseteq D(z_0, \delta)$, for $z \in \partial\triangle^{(n)}$ one has $|\phi(z) - f'(z_0)| < \varepsilon$ by (4.61) for $n \geq N$. Moreover, for $z \in \partial\triangle^{(n)}$ one has $|z - z_0| \leq \ell(\gamma_{\partial\triangle^{(n)}})$, since $z_0 \in \triangle^{(n)}$. By (4.40), we find

$$\left|\int_{\gamma_{\partial\triangle^{(n)}}} f(z)\mathrm{d}z\right| \leq \varepsilon \cdot \ell(\gamma_{\partial\triangle^{(n)}})^2 \leq \frac{\varepsilon}{4^n} \cdot \ell(\gamma_{\partial\triangle})^2, \tag{4.65}$$

where we used $\ell(\gamma_{\partial\triangle^{(n+1)}}) = \frac{1}{2}\ell(\gamma_{\partial\triangle^{(n)}})$ for all $n \in \mathbb{N} \cup \{0\}$ by construction. Combining (4.59) and (4.65) yield

$$\left|\int_{\gamma_{\partial\triangle}} f(z)\mathrm{d}z\right| \leq 4^n \cdot \frac{\varepsilon}{4^n}\ell(\gamma_{\partial\triangle})^2 = \varepsilon\ell(\gamma_{\partial\triangle})^2. \tag{4.66}$$

Since $\varepsilon > 0$ was arbitrary, the claim follows. □

Next we introduce star-shaped domains. The first major result will then be that every holomorphic function defined on a star-shaped domain has a primitive.

Definition 4.21. An open set $A \subseteq \mathbb{C}$ is called *star-shaped domain* if there is a point $z_\star \in A$ such that for any $z \in A$, also the connecting path between $z_\star$ and z is contained in A, that is

$$\{\lambda z + (1-\lambda)z_\star\,;\,\lambda \in [0,1]\} \subseteq A. \tag{4.67}$$

The point $z_* \in A$ is called a *star center* of A.

Note that star-shaped domains are clearly connected (so the name domain is justified).

Example 4.22.
(i) $\mathbb{C}$ is a star-shaped domain, every point $z \in \mathbb{C}$ is a star center.

(ii) Non-empty, convex, open subsets A of $\mathbb{C}$ are star-shaped domains, every point $z \in A$ is a star center.

(iii) $\mathbb{C}_- = \mathbb{C} \setminus \{x \in \mathbb{R}\,;\, x \leq 0\}$ is star shaped. Every point on the positive real line is a star center.

(iv) $\mathbb{C} \setminus \{0\}$, annuli $D(z_0,R) \setminus \overline{D(z_0,r)}$, where $0 < r < R$ and deleted disks $\dot{D}(z_0,R)$ where $R > 0$ are *not* star-shaped domains.

We can now state *Cauchy's integral theorem for star-shaped domains.*

Theorem 4.23. *Let $A \subseteq \mathbb{C}$ be a star-shaped domain and $f : A \to \mathbb{C}$ holomorphic. Then the following statements hold:*

(i) There is a primitive of f on A, i.e. there exists $F : A \to \mathbb{C}$ holomorphic with

$$F'(z) = f(z), \qquad \text{for } z \in A. \tag{4.68}$$

(ii) Integrals over f are path independent, i.e. for a piecewise C^1 curve $\gamma : I = [a,b] \to A$ (where $a < b$) with $\gamma(I) \subseteq A$, the integral $\int_\gamma f(z)\mathrm{d}z$ depends only on $\gamma(b)$ and $\gamma(a)$.

(iii) Integrals over f along closed curves are zero, i.e. if $\gamma : I = [a,b] \to \mathbb{C}$ (where $a < b$) is a piecewise C^1 curve with $\gamma(I) \subseteq A$ and $\gamma(a) = \gamma(b)$, then $\int_\gamma f(z)\mathrm{d}z = 0$.

Proof. We have already proved in Theorem 4.17 that the three statements are equivalent, so we only need to establish (i).

Let $z_\star \in A$ be a star center and denote for $z \in A$ the straight line connecting $z_\star$ and z by $\gamma_{[z_\star,z]} : I = [0,1] \to \mathbb{C}$. Clearly, $\gamma_{[z_\star,z]}(I) \subseteq A$, since A is star-shaped. We define

$$F(z) = \int_{\gamma_{[z_\star,z]}} f(w)\mathrm{d}w. \tag{4.69}$$

We claim that

$$\lim_{w\to z} \left| \frac{F(w)-F(z)}{w-z} - f(z) \right| = 0, \qquad \text{for all } z \in A. \tag{4.70}$$

First we note that for $\delta > 0$ small enough $\triangle(z_\star, w, z) \subseteq A$ for every $w \in D(z,\delta)$. Indeed, since A is open, $D(z,\delta) \subseteq A$ for some $\delta > 0$, and so for $w \in D(z,\delta)$, every point on the line

$[z, w]$ connecting z and w is in A. Since A is star-shaped, the line connecting $z_\star$ and any given point on $[z, w]$ is contained in A, and so the closed triangle is contained in A.

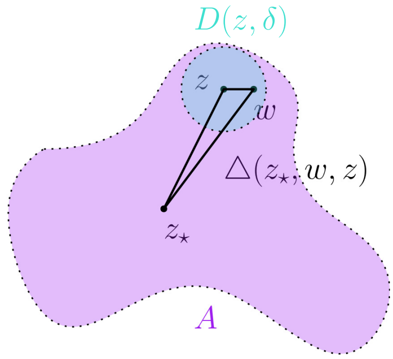

Figure 4.3.: Star-shaped domain A with star center $z_\star$ and closed triangle $\triangle(z_\star, w, z)$ contained in A.

By Goursat's Lemma 4.19 we therefore obtain

$$0 = \int_{\gamma_{\partial\triangle(z_\star,w,z)}} f(u)\mathrm{d}u = \int_{\gamma_{[z_\star,w]}} f(u)\mathrm{d}u + \int_{\gamma_{[w,z]}} f(u)\mathrm{d}u + \int_{\gamma_{[z,z_\star]}} f(u)\mathrm{d}u, \tag{4.71}$$

where $\gamma_{\partial\triangle(z_\star,w,z)}$ is the piecewise C^1 curve traversing the boundary of $\triangle(z_\star, w, z)$ and $\gamma_{[z_1,z_2]}$ stands for a parametrization of the straight line connecting z_1 and z_2. Inserting the definition of F gives (using (4.27))

$$F(w) - F(z) = \int_{\gamma_{[z,w]}} f(u)\mathrm{d}u, \tag{4.72}$$

and so for $w \neq z$:

$$\frac{F(w) - F(z)}{w - z} - f(z) = \frac{1}{w - z} \int_{\gamma_{[z,w]}} (f(u) - f(z))\mathrm{d}u. \tag{4.73}$$

Let $\varepsilon > 0$. Since f is continuous in z,we find that $|f(u) - f(z)| < \varepsilon$ for every $u \in D(z, \delta') \cap A$ for some $\delta' > 0$. Letting $\delta'' = \min\{\delta, \delta'\}$, we infer that for $w \in \dot{D}(z, \delta'') \subseteq A$ one has

$$\left|\frac{F(w) - F(z)}{w - z} - f(z)\right| = \frac{1}{|w - z|} \left|\int_{\gamma_{[z,w]}} (f(u) - f(z))\mathrm{d}u\right| \overset{(4.40)}{<} \frac{\varepsilon \cdot |w - z|}{|w - z|} \leq \varepsilon. \tag{4.74}$$

This means that F is differentiable in the complex sense in $z \in A$ and it holds that $F'(z) = f(z)$. $\square$

The integral theorem can be used to simplify significantly the calculation of certain path integrals by deformation. We exemplify this by the following calculation.

Example 4.24. Let $\xi \in \mathbb{R}$. We consider the function $f : \mathbb{R} \to \mathbb{R}$, $x \mapsto \exp(-\pi x^2)$ and we aim at calculating its *Fourier transform*[4]

$$\widehat{f}(\xi) = \int_{-\infty}^{\infty} \exp(-\pi x^2) \cdot \exp(-2\pi i x \xi) \mathrm{d}x. \tag{4.75}$$

Note that

$$\exp(-\pi x^2) \exp(-2\pi i x \xi) = \exp(-\pi(x + i\xi)^2) \exp(-\pi \xi^2). \tag{4.76}$$

Also, we recall from Real Analysis that

$$\int_{-\infty}^{\infty} \exp(-\pi x^2) \mathrm{d}x = 1. \tag{4.77}$$

We claim that

$$\int_{-\infty}^{\infty} \exp(-\pi(x + i\xi)^2) \mathrm{d}x = 1, \qquad \text{for } \xi \in \mathbb{R}. \tag{4.78}$$

We will show this claim using Cauchy's integral theorem. Consider the rectangle for $\xi, R > 0$:

$$Q_R = \{z \in \mathbb{C};\, |\mathrm{Re}(z)| \leq R, 0 \leq \mathrm{Im}(z) \leq \xi\}. \tag{4.79}$$

Its boundary when traversed in a counterclockwise fashion can be parametrized as $\gamma_{\partial Q_R} = \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4$, where

$$\begin{aligned}
\gamma_1 &: [0,1] \to \mathbb{C}, & t &\mapsto 2Rt - R,\\
\gamma_2 &: [1,2] \to \mathbb{C}, & t &\mapsto R + i(t-1)\xi,\\
\gamma_3 &: [2,3] \to \mathbb{C}, & t &\mapsto R - 2R(t-2) + i\xi,\\
\gamma_4 &: [3,4] \to \mathbb{C}, & t &\mapsto -R + i(4-t)\xi.
\end{aligned} \tag{4.80}$$

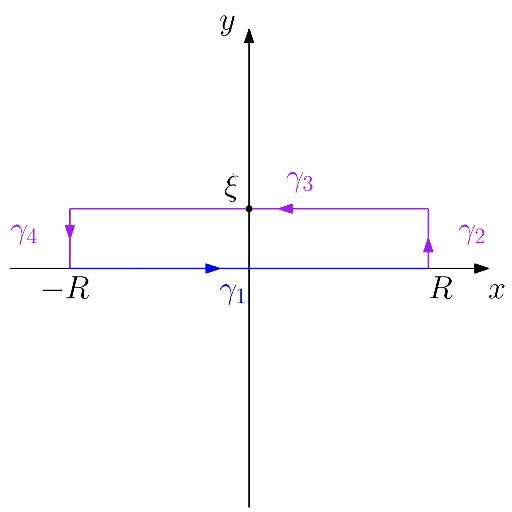

Figure 4.4.: Paths γ_1, γ_2, γ_3 and γ_4.

[4] There are other possible definitions of the Fourier transform.

The function $f : z \mapsto \exp(-\pi z^2)$ is holomorphic on the star-shaped domain $\mathbb{C}$ which contains $\gamma_{\partial Q_R}$, so by Cauchy's Integral Theorem 4.23, (iii), we have

$$0 = \int_{\gamma_{\partial Q_R}} f(z)\mathrm{d}z = \sum_{j=1}^{4} \int_{\gamma_j} f(z)\mathrm{d}z, \tag{4.81}$$

and rearranging terms, we find

$$\begin{aligned} \int_{-R}^{R} \exp(-\pi(x+i\xi)^2)\mathrm{d}x = -\int_{\gamma_3} f(z)\mathrm{d}z &= \int_{\gamma_1} f(z)\mathrm{d}z + \underbrace{\int_{\gamma_2} f(z)\mathrm{d}z}_{=:I_2} + \underbrace{\int_{\gamma_4} f(z)\mathrm{d}z}_{=:I_4} \\ &= \int_{-R}^{R} \exp(-\pi x^2)\mathrm{d}x + I_2 + I_4. \end{aligned} \tag{4.82}$$

We know bound the terms I_2 and I_4 using (4.40):

$$|I_2| \leq \max_{z \in \gamma_2([1,2])} |f(z)|\ell(\gamma_2) \leq \xi \cdot \exp(-\pi(R^2 - \xi^2)) \to 0, \qquad \text{as } R \to \infty, \tag{4.83}$$

and similarly

$$|I_4| \leq \max_{z \in \gamma_4([3,4])} |f(z)|\ell(\gamma_4) \leq \xi \cdot \exp(-\pi(R^2 - \xi^2)) \to 0, \qquad \text{as } R \to \infty, \tag{4.84}$$

and so we infer from (4.82)

$$\int_{-\infty}^{\infty} \exp(-(\pi + i\xi)^2)\mathrm{d}x = \lim_{R\to\infty} \int_{-R}^{R} \exp(-\pi x^2)\mathrm{d}x \overset{(4.77)}{=} 1, \tag{4.85}$$

and similarly for $\xi < 0$. This proves (4.78), and therefore we obtain that

$$\widehat{f}(\xi) = \exp(-\pi\xi^2). \tag{4.86}$$

End of Lecture 6

4.3. Homotopies and simply connected domains

We have demonstrated that Cauchy's theorem can be used advantageously by 'deforming the contour'. In this section, we give a precise definition of a continuous deformation of a contour, namely *homotopy*. This notion will be used to define a more general class of domains than star-shaped domains on which every holomorphic function has a primitive, namely the *simply connected domains*.

Definition 4.25. Let $A \subseteq \mathbb{C}$ be open and $\gamma_0, \gamma_1 : I = [0,1] \to \mathbb{C}$ two continuous curves with $\gamma_0(I), \gamma_1(I) \subseteq A$ and

$$\gamma_0(0) = \gamma_1(0), \qquad \gamma_0(1) = \gamma_1(1). \tag{4.87}$$

We say that γ_0 and γ_1 are *homotopic with fixed endpoints* in A if there is a continuous function $H : [0,1] \times [0,1] \to \mathbb{C}$ with $H([0,1]^2) \subseteq A$ fulfilling

(i) $H(0,t) = \gamma_0(t)$ and $H(1,t) = \gamma_1(t)$ for every $t \in [0,1]$,

(ii) For every $s \in [0,1]$, it holds that

$$H(s,0) = \gamma_0(0) = \gamma_1(0), \qquad H(s,1) = \gamma_0(1) = \gamma_1(1). \tag{4.88}$$

The function H is called a *homotopy* between γ_0 and γ_1.

The intuitive meaning of the definition is that as s ranges from 0 to 1, the curve $H(0,\cdot) = \gamma_0$ is deformed in a continuous manner into $H(1,\cdot) = \gamma_1$. For instance, consider the curves $\gamma_0, \gamma_1 : I \to \mathbb{C}$, given by

$$\gamma_0(t) = 1 + (i-1)t, \qquad \gamma_1(t) = \exp(i\tfrac{\pi t}{2}), \qquad \text{for } t \in [0,1]$$

with $\gamma_0(0) = \gamma_1(0) = 1$ and $\gamma_0(1) = \gamma_1(1) = i$. A homotopy between these curves with fixed endpoints in $\mathbb{C}$ is given by

$$H(s,t) = s\gamma_1(t) + (1-s)\gamma_0(t), \qquad \text{for } s,t \in [0,1]^2. \tag{4.89}$$

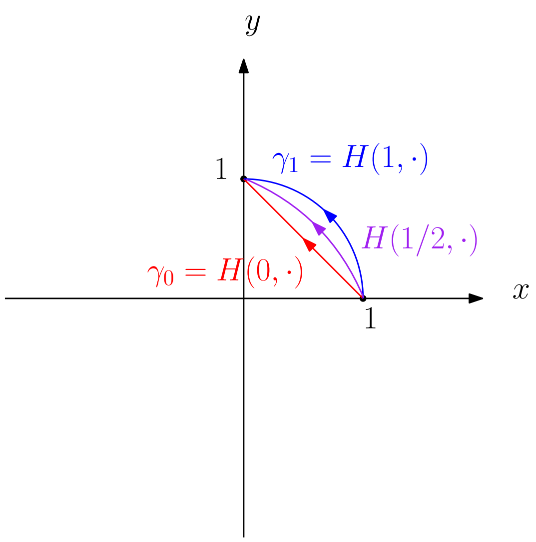

Figure 4.5.: Homotopy between γ_0 and γ_1.

Remark 4.26. (i) Homotopy with fixed endpoints is an equivalence relation.

(ii) The fact that γ_0, γ_1 are defined on $I = [0,1]$ is no restriction, since every continuous curve $\gamma : \widetilde{I} = [a,b] \to \mathbb{C}$ with $a < b$ can be reparametrized to be defined on $I = [0,1]$.

(iii) The construction (4.89) works in general if the domain A in which the curves γ_0 and γ_1 are defined is convex.

(iv) In the definition of homotopy we purposefully do not rule out the case of $\gamma_0(0) = \gamma_1(0) = \gamma_0(1) = \gamma_1(1)$. This special case gives homotopy of closed curves fixed at a point[5].

Definition 4.27. (i) Let $A \subseteq \mathbb{C}$ be open and $\gamma : I = [0,1] \to \mathbb{C}$ with $\gamma(I) \subseteq A$ a closed continuous curve with $z_0 = \gamma(0) = \gamma(1)$. We say that γ is *null-homotopic* (in A) if it is homotopic (in A) to the constant curve $\gamma_{\{z_0\}} : [0,1] \to \mathbb{C}$ with $\gamma_{\{z_0\}}(t) = z_0$ for all $t \in [0,1]$.

(ii) A given domain A is *simply connected* if every closed curve in A is null-homotopic.

Intuitively, a simply connected domain is a connected open set with no holes.

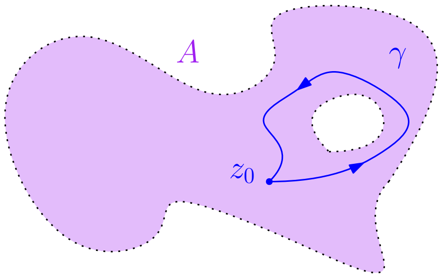

Figure 4.6.: The domain A is *not* simply connected, since the closed curve γ with start- and endpoint z_0 is not null-homotopic.

Remark 4.28. (i) A domain A is simply connected if and only if every two continuous curves $\gamma_0, \gamma_1 : I = [0,1] \to \mathbb{C}$ in A with $\gamma_0(0) = \gamma_1(0)$ and $\gamma_0(1) = \gamma_1(1)$ are homotopic with fixed endpoints in A.

(ii) Convex domains are simply connected by Remark 4.26, (iii). More generally, star-shaped domains are simply connected.

Proof. We only prove item (i). Clearly, if every two continuous curves in A with the same start- and endpoint are homotopic, every closed curve is null-homotopic in A.

Conversely, suppose that every closed curve is null-homotopic. Let $\gamma_0(0) = \gamma_1(0) = z_0$ and $\gamma_0(1) = \gamma_1(1) = z_1$ be two continuous curves and consider the closed curve

$$\delta(t) = (\gamma_0 + (-\gamma_1))(t) = \begin{cases} \gamma_0(2t), & 0 \le t \le \frac{1}{2}, \\ \gamma_1(2-2t), & \frac{1}{2} \le t \le 1. \end{cases}$$

[5]More generally, one could also define (free) homotopy of two closed curves γ_0 and γ_1 in A by replacing (4.88) in the definition of homotopy by the condition $H(s,0) = H(s,1)$ for every $s \in [0,1]$. In this case, the start- and endpoints of γ_0 and γ_1 do not need to coincide.

This curve is homotopic to $\gamma_{\{z_0\}}$, call this homotopy H (so $H(0,\cdot) = \delta$, $H(1,t) = z_0$ for every $t \in [0,1]$). We define the map

$$\phi : [0,1] \to \partial([0,1] \times [0,1]), \qquad s \mapsto \begin{cases} (4s, 0), & 0 \le s \le \frac{1}{4}, \\ (1, 2s - \frac{1}{2}), & \frac{1}{4} \le s \le \frac{3}{4} \\ (4 - 4s, 1) & \frac{3}{4} \le s \le 1. \end{cases}$$

Then the map $\widetilde{H} : [0,1] \times [0,1] \to \mathbb{C}$, defined by

$$\widetilde{H}(s,t) = H((0, \tfrac{t}{2}) + (1-t)\phi(s))$$

is a homotopy between γ_0 and γ_1 in A as can be readily verified. □

We will prove a version of Cauchy's integral theorem on simply connected domains. We need two lemmas as preparation.

Lemma 4.29. *Let $A \subseteq \mathbb{C}$ be open and γ_0 and γ_1 be homotopic C^1 curves with fixed endpoints in A. Then there exists a homotopy $H : [0,1] \times [0,1] \to \mathbb{C}$ between γ_0 and γ_1 in A (i.e. with $H([0,1]^2) \subseteq A$) with $H \in C^1([0,1] \times [0,1]; \mathbb{C})$. We say that H is a* C^1-homotopy *between γ_0 and γ_1.*

The proof of the above lemma is technical and relies on mollification of continuous functions. It is sketched in Appendix A.2. The next preparation shows that a compact non-empty set contained in an open set has a positive distance to the boundary of the open set.

Lemma 4.30. *Let $K \subseteq A \subseteq \mathbb{C}$ with A open and K compact and non-empty. There is a number $r > 0$ such that*

$$d(K, \partial A) = \inf\{|z - w|\, ;\, z \in K, w \in \partial A\} > r. \tag{4.90}$$

Proof. For every $z \in K$, take $r_z > 0$ with $D(z, 2r_z) \subseteq A$. Clearly $\{D(z, r_z)\}_{z \in K}$ is an open cover of K. By compactness, there is a finite subcover

$$K \subseteq \bigcup_{\nu=1}^{N} D(z_\nu, r_\nu), \qquad r_\nu = r_{z_\nu}. \tag{4.91}$$

Set $\rho = \min_{\nu=1,\dots,N} r_\nu$. For every $z \in K$, one has a $\nu \in \{1, \dots, N\}$ with $|z - z_\nu| < r_\nu$. For $w \in D(z, \rho)$, one has

$$|w - z_\nu| \le |w - z| + |z - z_\nu| < \rho + r_\nu \le 2r_\nu. \tag{4.92}$$

So $w \in D(z_\nu, 2r_\nu) \subseteq A$, and so $D(z, \rho) \subseteq A$. Setting $r = \frac{\rho}{2}$, we find that for $z \in K$ and $w \in \partial A$, $|z - w| \ge \rho > \frac{\rho}{2} = r$. □

The following is *Cauchy's integral theorem for simply connected domains.*

Theorem 4.31. *Let $A \subseteq \mathbb{C}$ be a simply connected domain and $f : A \to \mathbb{C}$ holomorphic. Then the following statements hold:*

(i) *There is a primitive of f on A, i.e. there exists $F : A \to \mathbb{C}$ holomorphic with*

$$F'(z) = f(z), \qquad \text{for } z \in A. \tag{4.93}$$

(ii) *Integrals over f are homotopy invariant, i.e. for homotopic piecewise C^1 curves $\gamma_0, \gamma_1 : I = [0,1] \to A$ with $\gamma(I) \subseteq A$ fulfilling $\gamma_0(0) = \gamma_1(0)$ and $\gamma_0(1) = \gamma_1(1)$, it holds that*

$$\int_{\gamma_0} f(z)\mathrm{d}z = \int_{\gamma_1} f(z)\mathrm{d}z. \tag{4.94}$$

(iii) *Integrals over f along closed curves are zero, i.e. if $\gamma : I = [0,1] \to \mathbb{C}$ is a piecewise C^1 curve with $\gamma(I) \subseteq A$ and $\gamma(0) = \gamma(1)$, then $\int_\gamma f(z)\mathrm{d}z = 0$.*

Proof. We have already proved in Theorem 4.17 that the three statements are equivalent, so we only need to establish (ii).[6]

We first assume that γ_0 and γ_1 are homotopic C^1 curves and let $H \in C^1([0,1]^2;\mathbb{C})$ be a C^1 homotopy in A between γ_0 and γ_1, which exists by Lemma 4.29. Note that

$$K = H([0,1]^2) \subseteq A \tag{4.95}$$

is a compact set. There is $r > 0$ with $d(K, \partial A) > r$, see Lemma 4.30, and finitely many open disks $D(z_j, r)$, $1 \le i \le n$ with $z_j \in K$ and $\overline{D(z_j, r)} \subseteq A$ cover K. Indeed, $K \subseteq \bigcup_{z\in K} D(z,r)$, so $\{D(z,r)\}_{z\in K}$ is an open cover of the compact set K that must admit a finite open subcover, and since $d(K, \partial A) > r$, we must have $\overline{D(z,r)} \subseteq A$ for every $z \in K$.

We define $\gamma_s(t) := H(s,t)$ for $(s,t) \in [0,1]^2$, so the curves $\gamma_s : [0,1] \to \mathbb{C}, t \mapsto \gamma_s(t)$ are C^1 curves. We then define

$$F : \begin{cases} [0,1] \to \mathbb{C}, \\ s \mapsto \int_{\gamma_s} f(z)\mathrm{d}z. \end{cases} \tag{4.96}$$

We will now show that F is constant on $[0,1]$. Define

$$\widetilde{I} = \{s \in [0,1]\,;\, F(s) = F(0)\}. \tag{4.97}$$

We show that $\widetilde{I}$ is non-empty, closed and open (in the induced topology of $[0,1]$). Since $[0,1]$ is connected, we must have $\widetilde{I} = [0,1]$.

- Since $0 \in \widetilde{I}$, $\widetilde{I} \neq \emptyset$.
- The map
$$s \mapsto F(s) = \int_0^1 f(\gamma_s(t))\gamma_s'(t)\mathrm{d}t$$
is continuous (since the map $(s,t) \mapsto f(\gamma_s(t)) \cdot \gamma_s'(t)$ is uniformly continuous, see ⇝ *Exercises*). Since $\widetilde{I}$ is the inverse image of the closed set $\{F(0)\}$ under this map, it is also closed.

[6] By Remark 4.28, (i), note that for A simply connected, we obtain part (i) of Theorem 4.17, since two piecewise C^1 curves with the same start- and endpoint are homotopic.

► We show that $\widetilde{I}$ is open: Let $s_0 \in \widetilde{I}$, we show that there exists $\delta > 0$ such that $(s_0 - \delta, s_0 + \delta) \cap [0,1] \subseteq \widetilde{I}$. There exists $1 \le J \le n$ such that

$$\gamma_{s_0}([0,1]) \subseteq \bigcup_{j=1}^{J} D(z_j, r) \tag{4.98}$$

and for $0 = t_0 < t_1 < \ldots < t_J = 1$, one has

$$\gamma_{s_0}([t_{j-1}, t_j]) \subseteq D(z_j, r), \qquad 1 \le j \le J. \tag{4.99}$$

Note that

$$\gamma_{s_0}(t_j) \subseteq D(z_j, r) \cap D(z_{j+1}, r). \qquad \text{for } 1 \le j \le J-1. \tag{4.100}$$

Since the $D(z_j, r)$ are open for every $1 \le j \le J$, there is a $\delta > 0$ such that (4.99) and (4.100) hold[7] for s_0 replaced by $s \in [0,1]$ with $|s - s_0| < \delta$. Consider for $1 \le j \le J$ the closed (piecewise C^1) curves

$$\Gamma_j = \gamma_s|_{[t_{j-1}, t_j]} + \zeta_j - \gamma_{s_0}|_{[t_{j-1}, t_j]} - \zeta_{j-1}, \tag{4.101}$$

where ζ_j denotes the straight line segment between $\gamma_s(t_j)$ and $\gamma_{s_0}(t_j)$. By convention, ζ_0 is the constant curve in the point $\gamma_{s_0}(0) = \gamma_s(0)$ and ζ_J is the constant curve in the point $\gamma_{s_0}(1) = \gamma_s(1)$.

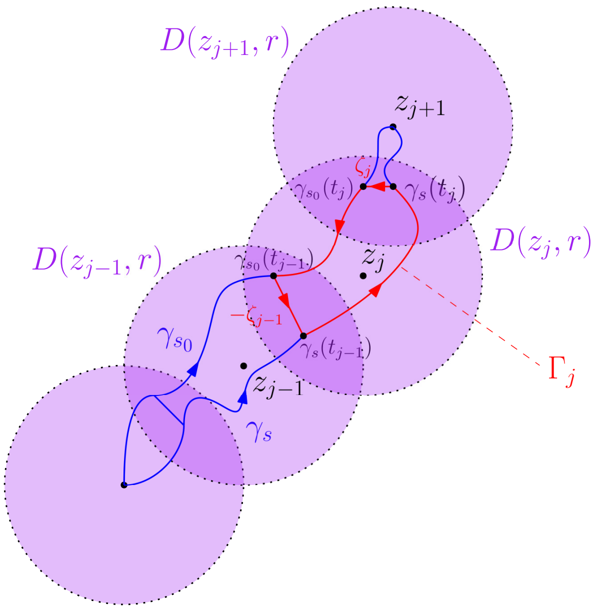

Figure 4.7.: Elements of the construction of Γ_j.

[7]Here, one has to use that H is uniformly continous (as a continuous function on the compact set $[0,1] \times [0,1]$).

By construction, the image of Γ_j is contained in $D(z_j, r)$ which is a star-shaped domain since it is convex. We therefore apply Cauchy's integral theorem for star-shaped domains 4.23, (iii), and infer that

$$\int_{\Gamma_j} f(z)\mathrm{d}z = 0, \qquad \text{for } 1 \leq j \leq J. \tag{4.102}$$

Therefore, we see that

$$F(s) - F(s_0) = \int_{\gamma_s} f(z)\mathrm{d}z - \int_{\gamma_{s_0}} f(z)\mathrm{d}z = \sum_{j=1}^{J} \int_{\Gamma_j} f(z)\mathrm{d}z = 0. \tag{4.103}$$

This shows that $\widetilde{I}$ is open and concludes the proof in the case of C^1 curves γ_0 and γ_1. Now assume γ_0 is only piecewise C^1, with $0 = t_0 < t_1 < \dots < t_K = 1$ and $\gamma_0|_{[t_{k-1}, t_k]} \in C^1([t_{k-1}, t_k]; \mathbb{C})$. One can replace γ_0 in the disks[8] $D(\gamma_0(t_k), R_k) \subseteq A$, $1 \leq k \leq K-1$ by a C^1 curve ζ_k staying in $D(\gamma_0(t_k), R_k)$ with

$$\zeta_k(\pm 1) = \gamma_0(t_k \pm \varepsilon), \qquad \zeta_k'(\pm 1) = \gamma_0'(t_k \pm \varepsilon) \qquad \text{for } 1 \leq k \leq K-1.$$

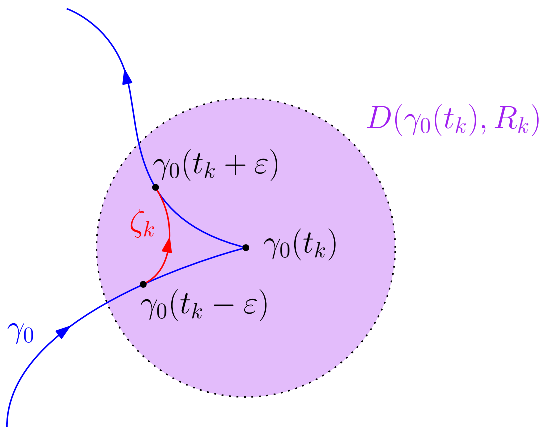

Figure 4.8.: Replacing the curve $\gamma_0|_{[t_k-\varepsilon, t_k+\varepsilon]}$ by ζ_k.

Applying again Cauchy's integral theorem for star-shaped domains 4.23, (ii), we see that

$$\int_{\zeta_k} f(z)\mathrm{d}z = \int_{\gamma|_{[t_k-\varepsilon, t_k+\varepsilon]}} f(z)\mathrm{d}z \qquad \text{for } 1 \leq k \leq K-1.$$

The curve $\widetilde{\gamma}_0$ obtained by replacing all $\gamma_0|_{[t_k-\varepsilon, t_k+\varepsilon]}$ by ζ_k is then a C^1 curve and fulfills

$$\int_{\widetilde{\gamma}_0} f(z)\mathrm{d}z = \int_{\gamma_0} f(z)\mathrm{d}z. \tag{4.104}$$

[8] Such disks exist since A is open.

Applying the same construction to γ_1 yields a C^1 curve $\widetilde{\gamma}_1$ with $\int_{\widetilde{\gamma}_1} f(z)\mathrm{d}z = \int_{\gamma_1} f(z)\mathrm{d}z$ and we can apply the first part of the theorem. □

The theorem above shows that if $A \subseteq \mathbb{C}$ is a simply connected domain, then every holomorphic function $f : A \to \mathbb{C}$ has a primitive on A. In fact, the converse is also true and can be shown using conformal mappings and the *Riemann mapping theorem*. In other words, simply connected domains are *exactly* the domains, in which every holomorphic function has a primitive.

End of Lecture 6.5

4.4. Cauchy's integral formula

We will now state and prove *Cauchy's integral formula*, which will follow using Cauchy's integral theorem 4.23 for star-shaped domains. This powerful formula will be instrumental in proving that every holomorphic function is differentiable in the complex sense infinitely often.

Theorem 4.32. *Let $A \subseteq \mathbb{C}$ be open and $f : A \to \mathbb{C}$ holomorphic. Let $r > 0$ and $z_0 \in A$ such that*

$$\overline{D(z_0, r)} = \{z \in \mathbb{C}\,;\, |z - z_0| \leq r\} \subseteq A. \tag{4.105}$$

Then for all $z \in D(z_0, r)$, Cauchy's integral formula *holds:*

$$f(z) = \frac{1}{2\pi i} \int_{\partial D(z_0,r)} \frac{f(w)}{w - z} \mathrm{d}w. \tag{4.106}$$

Here $\partial D(z_0, r)$ is any parametrization of the curve running along the boundary of $D(z_0, r)$ in a counterclockwise direction, for instance $\gamma_{\partial D(z_0,r)}(t) = z_0 + r\exp(2\pi i t)$, $t \in [0, 1]$.

Proof. Let $z \in D(z_0, r)$ be fixed and z_1 be one of the intersections of the line connecting z to z_0 with $\partial D(z_0, r)$ (if $z = z_0$, any point on $\partial D(z_0, r)$ will work). Furthermore, let $\varepsilon > 0$ be chosen such that $\overline{D(z, \varepsilon)} \subseteq D(z_0, r)$. We define the closed piecewise C^1 curve γ_1 starting in z_1, moving in a counterclockwise fashion along the large half-circle, the first part of the diameter, the smaller half-circle in a clockwise fashion and finally the second part of the diameter. The piecewiese closed C^1 curve γ_2 is defined in a similar fashion along the other part of the circle. Finally, we let ζ_ε be a parametrization of $\partial D(z, \varepsilon)$ in a counterclockwise fashion. The images of both curves γ_1 and γ_2 are in (two different!) star-shaped domains, obtained by deleting from $D(z_0, r + \delta)(\subseteq A)$ for some $\delta > 0$ (such a δ exists by Lemma 4.30) a half-line at z orthogonal to the line connecting z_1 and z_0, on which the function

$$w \mapsto \frac{f(w)}{w - z} \in \mathbb{C} \tag{4.107}$$

is holomorphic.

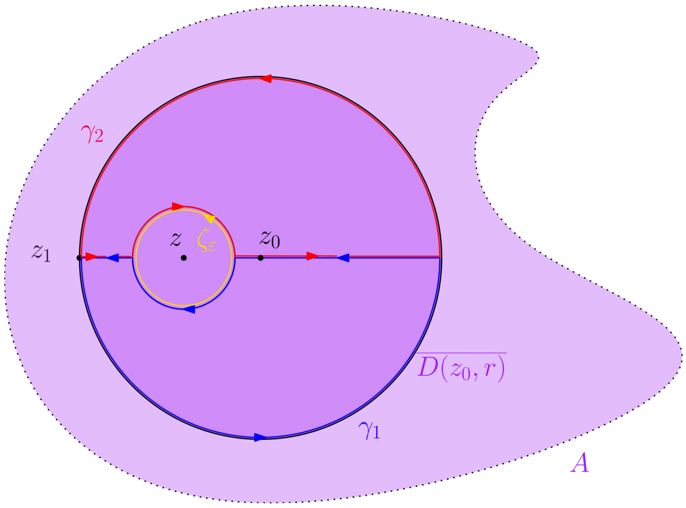

Figure 4.9.: Construction of the curves γ_1, γ_2 and ζ_ε.

Therefore Cauchy's integral theorem for star-shaped domains applies and we obtain

$$0 = \int_{\gamma_1} \frac{f(w)}{w-z}\mathrm{d}w + \int_{\gamma_2} \frac{f(w)}{w-z}\mathrm{d}w = \int_{\partial D(z_0,r)} \frac{f(w)}{w-z}\mathrm{d}w + \int_{-\zeta_\varepsilon} \frac{f(w)}{w-z}\mathrm{d}w. \tag{4.108}$$

We therefore obtain

$$\int_{\partial D(z_0,r)} \frac{f(w)}{w-z}\mathrm{d}w = f(z)\int_{\zeta_\varepsilon} \frac{1}{w-z}\mathrm{d}w + \int_{\zeta_\varepsilon} \frac{f(w)-f(z)}{w-z}\mathrm{d}w. \tag{4.109}$$

The first integral in (4.109) is easily calculated by using the parametrization $\zeta_\varepsilon(t) = z + \varepsilon\exp(2\pi i t)$, $t \in [0,1]$:

$$\int_{\zeta_\varepsilon} \frac{1}{w-z}\mathrm{d}w = \int_0^1 \frac{2\pi i\varepsilon\exp(2\pi i t)}{\varepsilon\exp(2\pi i t)}\mathrm{d}t = 2\pi i. \tag{4.110}$$

For the second integral in (4.109), use that f is continuous in $w = z$ (since it is holomorphic), so for a given $\varepsilon' > 0$, choosing $\delta > 0$ small enough, one has

$$|f(w) - f(z)| < \varepsilon', \qquad \text{for } w \in D(z,\delta) \cap A. \tag{4.111}$$

Choose $\varepsilon \in (0,\delta)$, then all points $w = z + \varepsilon\exp(2\pi i t)$ on the circle $\partial D(z,\varepsilon)$ fulfill

$$|w - z| = |\varepsilon\exp(2\pi i t)| = \varepsilon < \delta. \tag{4.112}$$

Therefore, we obtain

$$\left|\int_{\zeta_\varepsilon} \frac{f(w)-f(z)}{w-z}\mathrm{d}w\right| \overset{(4.40)}{\leq} \ell(\zeta_\varepsilon)\cdot\frac{\varepsilon'}{\varepsilon} = 2\pi\varepsilon'. \tag{4.113}$$

Since this is true for every $\varepsilon' > 0$, we obtain that

$$\left| \int_{\partial D(z_0,r)} \frac{f(w)}{w-z} \mathrm{d}w - 2\pi i f(z) \right| \le 2\pi\varepsilon' \qquad \text{for } \varepsilon' > 0, \tag{4.114}$$

and so the claim follows by letting ε' tend to zero. □

The Cauchy integral formula allows us to calculate the values of a holomorphic function in the interior of a disk from their values on the boundary. In fact, by interchanging differentiation and integration, we can show similar formulas for the derivatives of a holomorphic function. The formal justification is the following *Leibniz rule.*

Lemma 4.33. *Let $a < b \in \mathbb{R}$ and $A \subseteq \mathbb{C}$ open. Consider a continuous function $f : [a,b] \times A \to \mathbb{C}$, such that $f(t,\cdot)$ is holomorphic for every fixed $t \in [a,b]$, and $(t,z) \mapsto \frac{\partial f}{\partial z}(t,z)$ is continuous on $[a,b] \times A$. Then the function $g : A \to \mathbb{C}$, defined by*

$$g(z) = \int_a^b f(t,z)\mathrm{d}t \tag{4.115}$$

is holomorphic and one has

$$g'(z) = \int_a^b \frac{\partial f}{\partial z}(t,z)\mathrm{d}t. \tag{4.116}$$

Proof. We first show the respective claim over the set of real numbers. Let $a < b \in \mathbb{R}$ and $c < d \in \mathbb{R}$ and $f_{\mathbb{R}} : [a,b] \times [c,d] \to \mathbb{R}$ continuous. Furthermore assume that for fixed $t \in [a,b]$, $f(t,\cdot)$ is continuously differentiable. We will show the

Claim: The function $g_{\mathbb{R}} : [a,b] \to \mathbb{R}$ is continuously differentiable in (c,d) and for $x \in (c,d)$, one has

$$g'_{\mathbb{R}}(x) = \int_a^b \frac{\partial f_{\mathbb{R}}(t,x)}{\partial x}\mathrm{d}t.$$

Proof of Claim: We consider the expression for $x_0 \in (c,d)$:

$$\frac{g_{\mathbb{R}}(x) - g_{\mathbb{R}}(x_0)}{x - x_0} = \int_a^b \frac{f_{\mathbb{R}}(t,x) - f_{\mathbb{R}}(t,x_0)}{x - x_0}\mathrm{d}t. \tag{4.117}$$

By the mean value theorem, there is a ξ between x and x_0 such that

$$\frac{f_{\mathbb{R}}(t,x) - f_{\mathbb{R}}(t,x_0)}{x - x_0} = \left(\frac{\partial f_{\mathbb{R}}}{\partial x}\right)(t,\xi). \tag{4.118}$$

Note that ξ may depend on t. Let $\varepsilon > 0$. By uniform continuity of $\partial f_{\mathbb{R}}/\partial x$ on the compact set $[a,b] \times [c,d]$, there exists a $\delta > 0$ such that

$$\left| \left(\frac{\partial f_{\mathbb{R}}}{\partial x}\right)(t_1,x_1) - \left(\frac{\partial f_{\mathbb{R}}}{\partial x}\right)(t_2,x_2) \right| < \varepsilon \qquad \text{for } |x_1 - x_2| < \delta, |t_1 - t_2| < \delta. \tag{4.119}$$

We then obtain that

$$\left| \left(\frac{\partial f_{\mathbb{R}}}{\partial x}\right)(t,\xi) - \left(\frac{\partial f_{\mathbb{R}}}{\partial x}\right)(t,x_0) \right| < \varepsilon \qquad \text{for } |x - x_0| < \delta, \tag{4.120}$$

and δ does not depend on t. Finally we see that

$$\left|\frac{g_{\mathbb{R}}(x)-g_{\mathbb{R}}(x_0)}{x-x_0}-\int_a^b\left(\frac{\partial f_{\mathbb{R}}}{\partial x}\right)(t,x_0)\mathrm{d}t\right|\leq\varepsilon(b-a)\qquad\text{for } |x-x_0|<\delta. \tag{4.121}$$

The *Claim* follows by letting ε go to zero.

Now since for every $t\in[a,b]$ the function $f(t,\cdot)$ is differentiable in the complex sense, we obtain by (3.20)

$$\int_a^b\frac{\partial f}{\partial z}(t,z)\mathrm{d}t=\int_a^b\frac{\partial f(t,x,y)}{\partial x}\mathrm{d}t,\ \text{ and }\ \int_a^b i\left(\frac{\partial f}{\partial z}\right)(t,z)\mathrm{d}t=\int_a^b\frac{\partial f}{\partial y}(t,x,y)\mathrm{d}t. \tag{4.122}$$

We see that the function g is admits continuous real derivatives with respective to x and y, therefore it is totally differentiable in the real sense. By writing $f(t,x,y)=u(t,x,y)+v(t,x,y)i$ with $u,v\in\mathbb{R}$, one can verify the Cauchy-Riemann equations using (4.122) for g and conclude that g is holomorphic: Indeed,

$$\begin{aligned}\int_a^b\frac{\partial f}{\partial z}(t,z)\mathrm{d}t&=\int_a^b\frac{\partial f}{\partial x}(t,x,y)\mathrm{d}t=\int_a^b\frac{\partial u}{\partial x}(t,x,y)\mathrm{d}t+i\int_a^b\frac{\partial v}{\partial x}(t,x,y)\\(-i)\int_a^b i\left(\frac{\partial f}{\partial z}\right)(t,z)\mathrm{d}t&=(-i)\int_a^b\frac{\partial f}{\partial y}(t,x,y)\mathrm{d}t=\int_a^b\frac{\partial v}{\partial y}(t,x,y)\mathrm{d}t-i\int_a^b\frac{\partial u}{\partial y}(t,x,y)\mathrm{d}t.\end{aligned} \tag{4.123}$$

Finally, we see that

$$g'(z)\overset{(3.20)}{=}\frac{\partial g}{\partial x}(x,y)=\int_a^b\frac{\partial f}{\partial x}(t,x,y)\mathrm{d}t\overset{(4.122)}{=}\int_a^b\frac{\partial f}{\partial z}(t,z)\mathrm{d}t. \tag{4.124}$$

□

Theorem 4.34. *Let $A\subseteq\mathbb{C}$ open and $f:A\to\mathbb{C}$ holomorphic. Then the follwing statements hold:*

(i) f has arbitrarily many complex derivatives on A.

(ii) Let $r>0$ and $z_0\in A$ such that $\overline{D(z_0,r)}\subseteq A$. Then for every $n\in\mathbb{N}$ and $z\in D(z_0,r)$, one has the Generalized version of Cauchy's integral formula*:*

$$f^{(n)}(z)=\frac{n!}{2\pi i}\int_{\partial D(z_0,r)}\frac{f(w)}{(w-z)^{n+1}}\mathrm{d}w, \tag{4.125}$$

where $\partial D(z_0,r)$ is a parametrization of the boundary of $D(z_0,r)$ as in Theorem 4.32.

Proof. Assume we are in the situation of (ii). We apply Cauchy's integral formula (4.106) and find that

$$f(z)=\frac{1}{2\pi i}\int_{\partial D(z_0,r)}\frac{f(w)}{w-z}\mathrm{d}w\qquad\text{for } z\in D(z_0,r). \tag{4.126}$$

Now the function $z \mapsto \frac{f(w)}{w-z}$ is continuously differentiable (note that $w - z$ is non-zero for $w \in \partial D(z_0, r)$) with

$$\frac{\partial}{\partial z}\left(\frac{f(w)}{w-z}\right) = \frac{f(w)}{(w-z)^2}. \tag{4.127}$$

So by Leibniz' rule (4.116), we can differentiate (4.126) with respect to z (after inserting a parametrization of $\partial D(z_0, r)$) and we obtain

$$f'(z) = \frac{1}{2\pi i}\int_{\partial D(z_0,r)} \frac{\partial}{\partial z}\left(\frac{f(w)}{w-z}\right) \mathrm{d}w = \frac{1}{2\pi i}\int_{\partial D(z_0,r)} \frac{f(w)}{(w-z)^2}\mathrm{d}w, \qquad \text{for } z \in D(z_0, r). \tag{4.128}$$

More precisely, write can write

$$\begin{aligned} f(z) &= \frac{1}{2\pi i}\int_0^1 \frac{f(z_0 + r\exp(2\pi i t))}{z_0 + r\exp(2\pi i t) - z} \cdot 2\pi i r\exp(2\pi i t)\mathrm{d}t = \int_0^1 g(t,z)\mathrm{d}t, \\ g(t,z) &= \frac{f(z_0 + r\exp(2\pi i t))}{z_0 + r\exp(2\pi i t) - z} r\exp(2\pi i t) \end{aligned} \tag{4.129}$$

and use the Leibniz rule (4.116) (since $g : [0,1] \times D(z_0, r) \to \mathbb{C}$ is continuous and $g(t, \cdot)$ is holomorphic with continuous derivative).

The claim now follows by induction on $n \in \mathbb{N}$. Indeed, the case $n = 1$ is (4.128), and assuming we have alredy proved the claim for $n \in \mathbb{N}$, one has

$$\begin{aligned} f^{(n+1)}(z) &= \frac{\mathrm{d}}{\mathrm{d}z} f^{(n)}(z) \\ &= \frac{\mathrm{d}}{\mathrm{d}z}\frac{n!}{2\pi i}\int_{\partial D(z_0,r)} \frac{f(w)}{(w-z)^{n+1}}\mathrm{d}w \\ &\overset{(4.116)}{=} \frac{(n+1)\cdot n!}{2\pi i}\frac{f(w)}{(w-z)^{n+2}}\mathrm{d}w. \end{aligned} \tag{4.130}$$

The claim (i) now follows since for $z_0 \in A$, there is an $r > 0$ such that $\overline{D(z_0, r)} \subseteq A$, and $f|_{D(z_0,r)}$ has arbitrarily many derivatives. □

4.5. Consequences of Cauchy's integral formulas

In this section, we state and prove some applications of the Cauchy integral formulas (4.106) and (4.125). In particular, we will prove *Liouville's Theorem* and infer from it the *Fundamental Theorem of Algebra.*

The first ingredient for Liouville's theorem are the following *Cauchy inequalities*:

Proposition 4.35. *Let $z_0 \in \mathbb{C}$, $r > 0$ and $f : D(z_0, r) \to \mathbb{C}$ holomorphic. If there is a number $M > 0$ with $|f(z)| \leq M$ for all $z \in D(z_0, r)$, then for every $n \in \mathbb{N}$, one has*

$$|f^{(n)}(z_0)| \leq M\frac{n!}{r^n}. \tag{4.131}$$

Proof. Take the generalized version of Cauchy's integral formula with $z = z_0$ and $\overline{D(z_0, \rho)} \subseteq D(z_0, r)$ for any $0 < \rho < r$. Note that for $w \in \partial D(z_0, \rho)$, one has $|z_0 - w| = \rho$, and so

$$\left|\frac{f(w)}{(w - z_0)^{n+1}}\right| \le \frac{M}{\rho^{n+1}}, \qquad \text{for } w \in \partial D(z_0, \rho). \tag{4.132}$$

We therefore have

$$\begin{aligned} |f^{(n)}(z_0)| &\overset{(4.125)}{=} \left| \frac{n!}{2\pi i} \int_{\partial D(z_0,\rho)} \frac{f(w)}{(w - z_0)^{n+1}} \mathrm{d}w \right| \\ &\overset{(4.40)}{\le} \frac{n!}{2\pi} \frac{M}{\rho^{n+1}} \underbrace{\ell(\gamma_{\partial D(z_0,r)})}_{=2\pi\rho} = n! \frac{M}{\rho^n}. \end{aligned} \tag{4.133}$$

Since this is true for every $\rho \in (0, r)$, the claim follows by letting $\rho \uparrow r$. □

We now show as an easy consequence *Liouville's Theorem.*

Theorem 4.36. *A bounded*[9] *entire function $f : \mathbb{C} \to \mathbb{C}$ is constant.*

Proof. Let f be bounded, i.e. take $C > 0$ with $|f| \le C$. Then for any $z_0 \in \mathbb{C}$, $r > 0$ the conditions of Proposition 4.35 are fulfilled and Cauchy's inequality (4.131) yields

$$|f'(z_0)| \le \frac{C}{r}, \qquad \text{for } z_0 \in \mathbb{C}, r > 0. \tag{4.134}$$

Letting r tend to infinity, we see that for every $z_0 \in \mathbb{C}$, $|f'(z_0)| = 0$, so $f' = 0$ on the connected set $\mathbb{C}$. Therefore, f must be constant. □

A striking application of Liouville's Theorem is in the proof of the *Fundamental Theorem of Algebra*, that we present now.

Theorem 4.37. *Let $a_0, ..., a_n \in \mathbb{C}$, with $n \in \mathbb{N}$ and $a_n \neq 0$. Then the polynomial*

$$P : \mathbb{C} \to \mathbb{C}, \qquad z \mapsto \sum_{\nu=0}^{n} a_\nu z^n \tag{4.135}$$

has a zero, i.e. there exists a point $z_0 \in \mathbb{C}$ with $P(z_0) = 0$.

Proof. Assume that $P(z) \neq 0$ for all $z \in \mathbb{C}$. Then the function $z \mapsto \frac{1}{P(z)}$ would be an entire function. Now consider for $z \neq 0$:

$$P(z) = z^n \sum_{\nu=0}^{n} a_\nu z^{\nu - n} = z^n \left(a_n + \frac{a_{n-1}}{z^{n-1}} + ... + \frac{a_0}{z^n} \right). \tag{4.136}$$

Clearly, $\lim_{z\to\infty}(\frac{a_{n-1}}{z} + ... + \frac{a_0}{z^n}) = 0$, so there exists a $K > 0$ with

$$\left| \frac{a_{n-1}}{z} + ... + \frac{a_0}{z^n} \right| < \frac{|a_n|}{2}, \qquad \text{for } |z| > K. \tag{4.137}$$

[9] This means that there exists some $C > 0$ such that $|f(z)| \le C$ for every $z \in \mathbb{C}$.

By the inverse triangle inequality, we see that

$$|P(z)| = |z|^n \left| a_n + \frac{a_{n-1}}{z} + \ldots + \frac{a_0}{z^n} \right| \geq |z|^n \frac{|a_n|}{2}. \tag{4.138}$$

Since $a_n \neq 0$ and $n \geq 1$, we see that $|P(z)| \to \infty$ as $z \to \infty$, and so for $M > 0$, there is $R > 0$ with

$$|P(z)| \geq M \qquad \text{for } |z| > R, \tag{4.139}$$

so

$$\left| \frac{1}{P(z)} \right| \leq \frac{1}{M} \qquad \text{for } |z| > R. \tag{4.140}$$

But by assumption, $\frac{1}{P}$ is entire, so it is continuous, and is therefore bounded on the compact set $\overline{D(0,R)}$, see Remark 2.17, (iii). In total, $\frac{1}{P}$ would be a bounded entire function, so by Liouville's Theorem 4.36, $\frac{1}{P}$ would have to be constant, which is a contradiction. So P must have a zero. □

As another application of Cauchy's Integral Formula (more precisely, Theorem 4.34, (i)) we show *Morera's Theorem*, which is in a way a converse to Goursat's Lemma 4.19.

Theorem 4.38. *Let $A \subseteq \mathbb{C}$ open and $f : A \to \mathbb{C}$ a continuous function. Assume that for every $\triangle(z_1, z_2, z_3) \subseteq A$, one has that*

$$\int_{\gamma_{\partial\triangle(z_1,z_2,z_3)}} f(z)\mathrm{d}z = 0. \tag{4.141}$$

Then f is holomorphic on A.

Proof. For every $z_0 \in A$, there exists $\varepsilon > 0$ such that $D(z_0, \varepsilon) \subseteq A$. It suffices to show that $f|_{D(z_0,\varepsilon)}$ is holomorphic. We define (as in (4.69)) the function on the star-shaped domain $D(z_0, \varepsilon)$:

$$F(z) = \int_{\gamma_{[z_0,z]}} f(w)\mathrm{d}w. \tag{4.142}$$

As in the proof of Cauchy's Theorem 4.23 for star-shaped domains, F is a primitive of $f|_{D(z_0,\varepsilon)}$, meaning it is holomorphic and fulfills $F'(z) = f(z)$ for every $z \in D(z_0, \varepsilon)$. By Theorem 4.34, (i), F has infinitely many complex derivatives and in particular, f is holomorphic. □

5. Maximum modulus theorem and harmonic functions

(Reference: Marsden-Hoffman, Basic complex analysis, 3rd Ed., Section 2.5)

In this chapter we discuss an important consequence of Cauchy's integral formula, the *maximum modulus theorem*, as well as *harmonic functions* in two dimensions, motivated by the fact that the real and imaginary parts of a holomorphic functions are harmonic.

5.1. Maximum modulus theorem

The following proposition is immediate from the Cauchy integral formula (4.106). This is the *mean value property* of holomorphic functions.

Proposition 5.1. *Let $A \subseteq \mathbb{C}$ be open and $f : A \to \mathbb{C}$ holomorphic. Let $r > 0$ and $z_0 \in A$ such that $\overline{D(z_0, r)} \subseteq A$. Then*

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + r\exp(it))\mathrm{d}t. \tag{5.1}$$

Proof. This follows immediately from Cauchy's integral formula (4.106). Indeed,

$$\begin{aligned} f(z_0) &= \frac{1}{2\pi i} \int_{\gamma_{\partial D(z_0,r)}} \frac{f(w)}{w - z_0}\mathrm{d}z \\ &= \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(z_0 + r\exp(it))}{r\exp(it)} ri\exp(it)\mathrm{d}t \\ &= \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + r\exp(it))\mathrm{d}t. \end{aligned} \tag{5.2}$$

□

End of Lecture 7

So the value of a holomorphic function at a point agrees with its "circular average" around that point. We use this fact to show the *local maximum modulus principle*:

Theorem 5.2. *Let $A \subseteq \mathbb{C}$ be open and $f : A \to \mathbb{C}$ holomorphic. Assume that for $r > 0$ and $z_0 \in A$ with $D(z_0, r) \subseteq A$, one has that*

$$|f(z_0)| \geq |f(z)|, \qquad \text{for } z \in D(z_0, r). \tag{5.3}$$

Then f is constant on $D(z_0, r)$.

Proof. Let $\delta \in (0, r)$, then by the mean value property (5.1) and (4.14), we obtain

$$|f(z_0)| = \left| \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + \delta \exp(it)) \mathrm{d}t \right| \leq \frac{1}{2\pi} \int_0^{2\pi} |f(z_0 + \delta \exp(it))| \mathrm{d}t. \tag{5.4}$$

Since by assumption, for all $t \in [0, 2\pi]$ one has $|f(z_0 + \delta \exp(it))| \leq |f(z_0)|$, it follows that

$$\frac{1}{2\pi} \int_0^{2\pi} |f(z_0 + \delta \exp(it)| \mathrm{d}t \leq \frac{|f(z_0)|}{2\pi} \int_0^{2\pi} \mathrm{d}t = |f(z_0)|. \tag{5.5}$$

By combining (5.4) and (5.5), we see that

$$\int_0^{2\pi} \underbrace{(|f(z_0)| - |f(z_0 + \delta \exp(it))|)}_{\geq 0} \mathrm{d}t = 0, \tag{5.6}$$

and the integrand is continuous and non-negative, so we conclude that $|f(z_0)| = |f(z)|$ for $z \in \partial D(z_0, \delta)$. But since $D(z_0, r) = \{z_0\} \cup \bigcup_{\delta \in (0,r)} \partial D(z_0, \delta)$, we obtain that $|f|$ has the constant value $|f(z_0)|$ on all of $D(z_0, r)$. Since $f|_{D(z_0,r)}$ is a holomorphic function with constant modulus on the connected set $D(z_0, r)$, it must be constant itself (see $\rightsquigarrow$ Exercises). $\square$

So a holomorphic function cannot attain a local maximum in its modulus, unless it is constant in a neighborhood of the point where the maximum is attained.

Remark 5.3. (i) Note a holomorphic function can attain a minimum in its modulus at zeros, without being zero in a neighborhood.

(ii) However, assume that a holomorphic function $f : A \to \mathbb{C}$ does not have zeros in $D(z_0, r) \subseteq A$ and $|f(z_0)| \leq |f(z)|$ for all $z \in D(z_0, r)$ (i.e. z_0 is a local minimum in modulus of f). We can apply the maximum modulus principle to the holomorphic function $\frac{1}{f}$ and obtain that f must be constant in $D(z_0, r)$.

We next show a *global version* of the *maximum modulus principle.*

Theorem 5.4. *Let A be a bounded domain and $f : \overline{A} \to \mathbb{C}$ a continuous function such that the restriction $f|_A$ is holomorphic. If $|f|$ attains its maximum in A, then f is constant on $\overline{A}$.*

Proof. Set $M = \sup_{z \in \overline{A}} |f(z)|$. Since $\overline{A}$ is closed and bounded, it is compact by Proposition 2.11 so M is finite (and there is a point $\widehat{z} \in \overline{A}$ such that $|f(\widehat{z})| = M$), see Remark 2.17, (iii). Assume now that there exists a point $z_0 \in A$ such that $|f(z_0)| = M$. We define

$$B = \{z \in A\,;\, |f(z)| = M\}. \tag{5.7}$$

Note that B is non-empty since $z_0 \in B$ and closed (indeed $B = |f|_A|^{-1}(\{M\})$ is the inverse image of a closed set under the continuous map $|f|_A|$). We now argue that A is also open. Suppose $z \in B$ and take some $\varepsilon > 0$ such that $D(z, \varepsilon) \subseteq A$. Since $|f|$ attains a local maximum in z, we must have that $|f(w)| = M$ for all $w \in D(z, \varepsilon)$ by the local maximum modulus principle (Proposition 5.1). The latter means that $D(z, \varepsilon) \subseteq B$, so B is open. Therefore, both B and $A \setminus B$ are open (relative in A), and since A is connected and B is nonempty, we must have $A = B$. Again since A is connected, $|f|$ can only be constant if f is constant. Finally, we see that f is also constant on $\overline{A}$ by continuity. $\square$

Corollary 5.5. *Under the same hypotheses as Theorem 5.4, one has*

$$\max_{z\in\overline{A}} |f(z)| = \max_{z\in\partial A} |f(z)|. \tag{5.8}$$

Proof. We already know that the maximum is attained in $\overline{A}$. If it is attained in A, $|f|$ is constant on $\overline{A}$, but then (5.8) is trivially true. Otherwise, $|f|$ does not attain its maximum in A, but in $\partial A = \overline{A} \setminus A$, so (5.8) follows again. □

All assumptions in the global version of the maximum modulus theorem and its corollary are necessary:

- Take the unbounded domain $A = \{x+iy\,;\, x, y > 0\}$, then $f : \overline{A} \to \mathbb{C}$, $z \mapsto \exp(-iz^2)$ is holomorphic on A and fulfills $|f(z)| = 1$ for every $z \in \partial A$, but $f(r(1+i)) = \exp(2r^2) \to \infty$ for $r \to \infty$.

- Take the disconnected set $A = D(0,1) \cup D(3,1)$ and define on $\overline{A} = \overline{D(0,1)} \cup \overline{D(3,1)}$ the function (holomorphic in A)

$$f(z) = \begin{cases} 1, & z \in D(0,1) \\ 2, & z \in D(3,1) \end{cases}$$

  Then $|f|$ attains its maximum in $3 \in D(3,1)$ but f is not constant[1].

5.2. Harmonic functions, Poisson formula

We already argued in Corollary 3.10 that the real and imaginary parts u, v of a holomorphic function $f = u + iv : A \to \mathbb{C}$ ($A \subseteq \mathbb{C}$ open) are necessarily harmonic due to the Cauchy-Riemann equations, if they are twice continuously differentiable. In fact the latter part of the statement is redundant, since we already showed (see Theorem 4.34, (i)) that f has arbitrarily many complex derivatives on A, and so u, v are smooth functions. We show that the opposite is also true.

Proposition 5.6. *Let $A \subseteq \mathbb{C}$ be a domain and $u : A \to \mathbb{R}$ twice continuously differentiable and harmonic.*

(i) u is smooth (infinitely differentiable) and for $z_0 \in A$, there exists $r > 0$ and a holomorphic function $f : D(z_0, r) \to \mathbb{C}$ such that $u|_{D(z_0,r)} = \mathrm{Re}(f)$.

(ii) If A is simply connected, then there is a holomorphic function $f : A \to \mathbb{C}$ such that $u = \mathrm{Re}(f)$.

Proof. We first prove (ii). The function $g : A \to \mathbb{C}$, defined by

$$g(z) = \frac{\partial u}{\partial x}(x,y) - i\frac{\partial u}{\partial y}(x,y) =: U(x,y) + iV(x,y) \tag{5.9}$$

[1] The function f is however constant on the connected components of A

is holomorphic. Indeed, all partial derivatives $\frac{\partial U}{\partial x}, \frac{\partial U}{\partial y}, \frac{\partial V}{\partial x}, \frac{\partial V}{\partial y}$ are continuous (since u is twice continuously differentiable), and one has that

$$\frac{\partial U}{\partial x}(x,y) = \frac{\partial^2 u}{\partial x^2}(x,y) = -\frac{\partial^2 u}{\partial y^2}(x,y) = \frac{\partial V}{\partial y}(x,y), \qquad (x,y) \in A, \tag{5.10}$$

since that $\Delta u(x,y) = 0$. Moreover, by Schwarz' theorem we have

$$\frac{\partial U}{\partial y}(x,y) = \frac{\partial^2 u}{\partial y \partial x}(x,y) = \frac{\partial^2 u}{\partial x \partial y}(x,y) = -\frac{\partial V}{\partial x}(x,y), \qquad (x,y) \in A. \tag{5.11}$$

So by Corollary 3.8, (i), the function g is holomorphic. Since A is simply connected, by Cauchy's integral theorem 4.31 for simply connected domains, g has a primitive f on A, so $f'(z) = g(z)$. Writing $f = \widetilde{u} + i\widetilde{v}$ gives

$$f'(z) \overset{(3.20)}{=} \frac{\partial \widetilde{u}}{\partial x}(x,y) - i\frac{\partial \widetilde{u}}{\partial y}(x,y), \qquad (x,y) \in A, \tag{5.12}$$

and so $\frac{\partial}{\partial x}(u - \widetilde{u}) = \frac{\partial}{\partial y}(u - \widetilde{u}) = 0$ on A, so that $u - \widetilde{u} = const$. By subtracting this constant, we obtain that $u = \mathrm{Re}(f)$ on A.

We now show (i). Since A is open, we can find $r > 0$ such that $D(z_0, r) \subseteq A$. Since $D(z_0, r)$ is simply connected, we restrict u to $D(z_0, r)$ to find by (ii) a holomorphic function $f : D(z_0, r) \to \mathbb{C}$ with $u|_{D(z_0,r)} = \mathrm{Re}(f)$. Finally, since f has arbitrarily many derivatives by Theorem 4.34, (i), $u|_{D(z_0,r)}$ has arbitrarily many real derivatives in $D(z_0, r)$. Finally, $z_0 \in A$ was arbitrary, so u is smooth on all of A. □

We say that the function v is a *harmonic conjugate* of u. So in other words, on a simply connected domain, every harmonic function has a harmonic conjugate. We now show that the mean value property (that we obtained for holomorphic functions) holds for harmonic functions.

Proposition 5.7. *Let $A \subseteq \mathbb{C}$ be open and $u : A \to \mathbb{C}$ harmonic. Let $r > 0$ and $z_0 = (x_0, y_0) \in A$ such that $\overline{D(z_0, r)} \subseteq A$. Then*

$$u(x_0, y_0) = \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + r\exp(it))\mathrm{d}t. \tag{5.13}$$

Proof. We apply Proposition 5.6 and find f holomorphic on $D(z_0, r)$ with $u = \mathrm{Re}(f)$. So by Proposition 5.1, we have

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + r\exp(it))\mathrm{d}t. \tag{5.14}$$

The result follows by taking the real part of the above equation and using (4.11). □

With the mean value property, we can obtain the maximum principle in the exactly same fashion as for the modulus of a holomorphic function. We formulate both the local and global versions in one theorem.

Theorem 5.8. *Let $A \subseteq \mathbb{R}^2$ be open and $u : A \to \mathbb{R}$ harmonic.*

(i) Assume that for $r > 0$ and $z_0 \in A$ with $D((x_0, y_0), r) \subseteq A$, one has either

$$u(x_0, y_0) \geq u(x, y), \qquad \text{for } (x, y) \in D((x_0, y_0), r). \tag{5.15}$$

or

$$u(x_0, y_0) \leq u(x, y), \qquad \text{for } (x, y) \in D((x_0, y_0), r). \tag{5.16}$$

Then u is constant on $D((x_0, y_0), r)$.

(ii) Suppose additionally that A is connected and bounded, and u extends to a continuous function on $\overline{A}$. If u attains its minimum or its maximum in A, then u is constant on $\overline{A}$. Moreover,

$$\begin{aligned} \max_{(x,y)\in\overline{A}} u(x, y) &= \max_{(x,y)\in\partial A} u(x, y), \\ \min_{(x,y)\in\overline{A}} u(x, y) &= \min_{(x,y)\in\partial A} u(x, y). \end{aligned} \tag{5.17}$$

Proof. For the statements abound local or global maxima, we repeat the proof of Theorem 5.2, Theorem 5.4 and Corollary 5.5, using the mean value property for harmonic functions (5.13). For the statements about minima, apply the respective statement about maxima to the harmonic function $-u$. □

Consider now a bounded domain $A \subseteq \mathbb{R}^2$ and suppose that we are given continuous functions $g : A \to \mathbb{R}$ and $\xi : \partial A \to \mathbb{R}$. We say that a function $u : \overline{A} \to \mathbb{R}$ that is twice continuously differentiable in the real sense in A fulfills the *Poisson equation* with boundary condition ξ, if

$$\begin{aligned} \Delta u(x, y) &= g(x, y), && \text{for } (x, y) \in A, \\ u(x, y) &= \xi(x, y), && \text{for } (x, y) \in \partial A. \end{aligned} \tag{5.18}$$

The special case where $g \equiv 0$ is called the *Dirichlet problem.* Proving the existence of a solution to these problems is non-trivial and depends on the regularity of the boundary (this involves methods from the theory of partial differential equations, such as Perron's method, or alternatively can be studied using Brownian motion). It is however very easy to show that a solution to this problem must be unique.

Proposition 5.9. *The solution u to (5.18) is unique.*

Proof. Let u_1 and u_2 be two solutions to (5.18). Then $u = u_2 - u_1$ is a harmonic function and $u|_{\partial A} = 0$. Applying the maximum principle in the form (5.17) shows us that

$$0 = \min_{(x,y)\in\partial A} u(x, y) = \min_{(x,y)\in\overline{A}} u(x, y) \leq \max_{(x,y)\in\overline{A}} u(x, y) = \max_{(x,y)\in\partial A} u(x, y) = 0, \tag{5.19}$$

so $u \equiv 0$ on $\overline{A}$. □

In the case of $A = D(0, r)$, one can in fact give an explicit solution to the Dirichlet problem (5.18) (where $g \equiv 0$) relying on Cauchy's integral formula.

Theorem 5.10. *Let $r > 0$ and $u : \overline{D(0,r)} \to \mathbb{R}$ a continuous function harmonic in $D(0,r)$. For $\rho \in (0,r)$ and $\theta \in (-\pi,\pi]$, one has* Poisson's formula

$$u(\rho\exp(i\theta)) = \frac{r^2-\rho^2}{2\pi}\int_0^{2\pi}\frac{u(r\exp(it))}{r^2-2r\rho\cos(\theta-t)+\rho^2}\mathrm{d}t, \tag{5.20}$$

or equivalently

$$u(z) = \frac{1}{2\pi}\int_0^{2\pi} u(r\exp(it))\frac{r^2-|z|^2}{|r\exp(it)-z|^2}\mathrm{d}t. \tag{5.21}$$

Proof. Since $D(0,r)$ is simply connected and u is harmonic on $D(0,r)$, we can use Proposition 5.6 to find a holomorphic function $f : D(0,r) \to \mathbb{C}$ with $u = \mathrm{Re}(f)$. Using Cauchy's integral formula for $0 < s < r$, we obtain

$$f(z) = \frac{1}{2\pi i}\int_{\partial D(0,s)}\frac{f(w)}{w-z}\mathrm{d}w, \qquad |z| < s. \tag{5.22}$$

We set $\widetilde{z} = \frac{s^2}{\overline{z}}$ (the *reflection* of z on the circle $\partial D(0,s)$) and we obtain

$$f(z) = \frac{1}{2\pi i}\int_{\partial D(0,s)} f(w)\left(\frac{1}{w-z}-\frac{1}{w-\widetilde{z}}\right)\mathrm{d}w, \qquad |z| < s, \tag{5.23}$$

since the function $w \mapsto \frac{f(w)}{w-\widetilde{z}}$ is holomorphic in $D(0,s+\varepsilon)$ for some $\varepsilon > 0$. We can then rewrite (since $|w| = s$ for $w \in \partial D(0,s)$):

$$\begin{aligned}\frac{1}{w-z}-\frac{1}{w-\widetilde{z}} &= \frac{1}{w-z}-\frac{1}{w-|w|^2/\overline{z}} = \frac{1}{w-z}-\frac{\overline{z}}{w(\overline{z}-\overline{w})}\\ &= \frac{|w|^2-|z|^2}{w|w-z|^2}.\end{aligned}$$

It follows that

$$f(z) = \frac{1}{2\pi i}\int_{\partial D(0,s)}\frac{f(w)(|w|^2-|z|^2)}{w|w-z|^2}\mathrm{d}w. \tag{5.24}$$

Inserting the standard parametrization $\gamma_{\partial D(0,s)} : [0,2\pi] \to \mathbb{C}$, $t \mapsto s\exp(it)$ and writing $z = \rho\exp(i\theta)$ (for $\rho = |z| \in [0,s)$), we obtain

$$f(\rho\exp(i\theta)) = \frac{1}{2\pi}\int_0^{2\pi}\frac{f(s\exp(it))(s^2-\rho^2)}{|s\exp(it)-\rho\exp(i\theta)|^2}\mathrm{d}t. \tag{5.25}$$

Taking the real part in (5.25), we obtain

$$u(\rho\exp(i\theta)) = \frac{1}{2\pi}\int_0^{2\pi}\frac{u(s\exp(it))(s^2-\rho^2)}{s^2+\rho^2-2s\rho\cos(\theta-t)}\mathrm{d}t. \tag{5.26}$$

Keep ρ and $t \in [0,2\pi]$ fixed, and use that as $s \to r$, the one has

$$\frac{u(s\exp(it))(s^2-\rho^2)}{s^2+\rho^2-2s\rho\cos(\theta-t)} \to \frac{u(r\exp(it))(r^2-\rho^2)}{r^2+\rho^2-2r\rho\cos(\theta-t)} \tag{5.27}$$

uniformly in $t \in [0, 2\pi]$. More precisely, the function

$$[\tfrac{r+\rho}{2}, r] \times [0, 2\pi] \to \mathbb{R}, \qquad (s,t) \mapsto \frac{u(s\exp(it))(s^2-\rho^2)}{s^2+\rho^2-2s\rho\cos(\theta-t)} \tag{5.28}$$

is continuous and since $[\frac{r+\rho}{2}, r] \times [0, 2\pi]$ is compact, it is also uniformly continuous and one has that

$$\lim_{s\to r} \sup_{t\in[0,2\pi]} \left| \frac{u(s\exp(it))(s^2-\rho^2)}{s^2+\rho^2-2s\rho\cos(\theta-t)} - \frac{u(r\exp(it))(r^2-\rho^2)}{r^2+\rho^2-2r\rho\cos(\theta-t)} \right| = 0. \tag{5.29}$$

Therefore, it holds that (as $s \to r$)

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{u(s\exp(it))(s^2-r^2)}{s^2+\rho^2-2s\rho\cos(\theta-t)}\mathrm{d}t \to \frac{1}{2\pi}\int_0^{2\pi} \frac{u(r\exp(it))(r^2-\rho^2)}{r^2+\rho^2-2r\rho\cos(\theta-t)}\mathrm{d}t, \tag{5.30}$$

which shows the claim. □

Theorem 5.10 gives a formula to show how u is given in $D(0,r)$ *if* u solves the Dirichlet problem. Conversely, we can simply argue that inserting the boundary values ξ on $\partial D(0,r)$ we do in fact obtain a solution of the Dirichlet problem.

Proposition 5.11. *Let $r > 0$ and $\xi : \partial D(0,r) \to \mathbb{R}$ be continuous. Then the function u given by*

$$u(z) = \begin{cases} \frac{1}{2\pi}\int_0^{2\pi} \xi(r\exp(it))\frac{r^2-|z|^2}{|r\exp(it)-z|^2}\mathrm{d}t, & z \in D(0,r), \\ \xi(z), & z \in \partial D(0,r) \end{cases} \tag{5.31}$$

is harmonic in $D(0,r)$ and continuous on $\overline{D(0,r)}$.

Proof. We first note that

$$\frac{r^2-|z|^2}{|r\exp(it)-z|^2} = \mathrm{Re}\left(\frac{r\exp(it)+z}{r\exp(it)-z}\right) \tag{5.32}$$

for $z \in D(0,r)$. Therefore we see that since $(t,z) \mapsto \xi(r\exp(it))\frac{r\exp(it)+z}{r\exp(it)-z}$ (defined on $[0,2\pi] \times D(0,\rho)$, with $\rho \in (0,r)$ arbitrary) is continuous and holomorphic for fixed t, we can apply Leibniz' rule (Lemma 4.33) to see that u the real part of a function that is holomorphic on every $D(0,\rho)$, $\rho \in (0,r)$, and so it is harmonic on $D(0,r)$. We need to show that u is continuous on the boundary. First note that $\frac{1}{2\pi}\int_0^{2\pi}\frac{r^2-|z|^2}{|r\exp(it)-z|^2}\mathrm{d}t = 1$ from (5.21) applied with $u \equiv 1$ for $|z| < r$. Now again for $|z| < r$,

$$|u(z) - \xi(r\exp(i\theta))| \le \frac{1}{2\pi}\int_0^{2\pi} |\xi(r\exp(it)) - \xi(r\exp(i\theta))|\frac{r^2-|z|^2}{|r\exp(it)-z|^2}\mathrm{d}t. \tag{5.33}$$

Without loss of generality, let $\theta \in (0, 2\pi)$. For a given $\varepsilon > 0$, there exists a $\delta > 0$ (smaller than θ) such that if $t \in (\theta - \delta, \theta + \delta)$ one has $|\xi(r\exp(it)) - \xi(r\exp(i\theta))| < \varepsilon$, and so

$$\begin{aligned}|u(z) - \xi(r\exp(i\theta))| \leq{}& \frac{1}{2\pi}\int_{\theta-\delta}^{\theta+\delta} \varepsilon \frac{r^2 - |z|^2}{|r\exp(it) - z|^2}\mathrm{d}t \\ &+ \frac{\sup_{z\in\partial D(0,r)}|\xi(z)|}{\pi}(2\pi - 2\delta) \sup_{t\in[0,\theta-\delta]\cup[\theta+\delta,2\pi]} \frac{r^2 - |z|^2}{|r\exp(it) - z|^2}\end{aligned} \tag{5.34}$$

The first summand is bounded from above by ε (by bounding the integral by an integral on $[0, 2\pi]$ and using the nonnegativity of the integrand). As $z \to r\exp(i\theta)$, the second summand in the above equation converges to zero. In total, we see that as $z \to r\exp(i\theta)$, $u(z) \to \xi(r\exp(i\theta))$. The case where $\theta = 0$ can be treated similarly. In total, we see that u is continuous on $\partial D(0, r)$.

End of Lecture 8

□

6. Power Series and Taylor's theorem

(Reference: Marsden-Hoffman, Basic complex analysis, 3rd Ed., Sections 3.1–3.2)

6.1. Sequences and series of holomorphic functions

We start by introducing the important notion of uniform convergence of sequences of complex functions. The results of section 2.1 on series and sequences are taken as a foundation without further mention.

Definition 6.1. Let $A \subseteq \mathbb{C}$ and $(f_n)_{n\in\mathbb{N}}$ a sequence of functions $f_n : A \to \mathbb{C}$.

(i) We say that $(f_n)_{n\in\mathbb{N}}$ *converges uniformly on* A to some function $f : A \to \mathbb{C}$, if for every $\varepsilon > 0$, there is an $N = N(\varepsilon) \in \mathbb{N}$ such that

$$|f_n(z) - f(z)| < \varepsilon \qquad \text{for all } n \geq N, z \in A. \tag{6.1}$$

In this case, we write $f_n \rightrightarrows f$ (on A) as $n \to \infty$

(ii) Similarly, the series $\sum_{n=1}^{\infty} f_n(z)$ is said to *converge uniformly on* A if the sequence of functions $(S_n)_{n\in\mathbb{N}}$ of functions $S_n : A \to \mathbb{C}$, defined by

$$S_n(z) = \sum_{\nu=1}^{n} f_\nu(z), \tag{6.2}$$

converges uniformly.

Note that uniform convergence implies pointwise convergence $\lim_{n\to\infty} f_n(z) = f(z)$, but not vice versa.

The next theorem is called the *Cauchy criterion* for uniform convergence.

Theorem 6.2. *Let $A \subseteq \mathbb{C}$. Consider a sequence $(f_n)_{n\in\mathbb{N}}$ of functions $f_n : A \to \mathbb{C}$.*

(i) The sequence $(f_n)_{n\in\mathbb{N}}$ converges uniformly if and only if for every $\varepsilon > 0$, there is an $N = N(\varepsilon) \in \mathbb{N}$ such that

$$|f_n(z) - f_m(z)| < \varepsilon, \qquad \text{for all } n, m \geq N, z \in A. \tag{6.3}$$

(ii) The series $\sum_{n=1}^{\infty} f_n(z)$ converges uniformly if and only if for every $\varepsilon > 0$, there is an $N = N(\varepsilon) \in \mathbb{N}$ such that

$$\left| \sum_{\nu=n+1}^{n+p} f_\nu(z) \right| < \varepsilon, \qquad \text{for all } n \geq N, p \in \mathbb{N}, z \in A. \tag{6.4}$$

Proof. It is clear that (ii) follows from (i), applied to partial sums. So we only need to prove (i).

($\Rightarrow$) Assume that $f_n \rightrightarrows f$ on A. For $\varepsilon > 0$, let $N = N(\varepsilon) \in \mathbb{N}$ such that $|f_n(z) - f(z)| < \varepsilon/2$ for all $n \geq N$ and all $z \in A$. Then, for $z \in A$,

$$|f_n(z) - f_m(z)| \leq |f_n(z) - f(z)| + |f_m(z) - f(z)| < \varepsilon, \qquad \text{for all } n, m \geq N. \tag{6.5}$$

($\Leftarrow$) Let $f(z) = \lim_{n\to\infty} f_n(z)$ (which exists since for every fixed $z \in A$, $(f_n(z))_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{C}$, and so converges). For $\varepsilon > 0$, choose $N = N(\varepsilon) \in \mathbb{N}$ such that

$$|f_n(z) - f_m(z)| < \frac{\varepsilon}{2}, \qquad \text{for all } n, m \geq N. \tag{6.6}$$

Now for $z \in A$, we find $M = M(\varepsilon, z) \in \mathbb{N}$ large enough (depending also on z) such that

$$|f_m(z) - f(z)| < \frac{\varepsilon}{2}, \qquad \text{for all } m \geq M. \tag{6.7}$$

Therefore, we find that for every $n \geq N$ and $z \in A$, one has

$$|f_n(z) - f(z)| \leq \underbrace{|f_n(z) - f_m(z)|}_{<\frac{\varepsilon}{2}, \text{ by (6.6)}} + \underbrace{|f_m(z) - f(z)|}_{<\frac{\varepsilon}{2}, \text{ by (6.7)}} < \varepsilon. \tag{6.8}$$

(In the intermediate step, one has to choose $m \geq \max\{M(z,\varepsilon), N\}$). □

An important property of uniform convergence is that it preserves continuity.

Lemma 6.3. *Let $A \subseteq \mathbb{C}$. Consider a sequence $(f_n)_{n\in\mathbb{N}}$ of continuous functions $f_n : A \to \mathbb{C}$.*

(i) Assume that $f_n \rightrightarrows f$ on A as $n \to \infty$ for some function $f : A \to \mathbb{C}$. Then f is continuous.

(ii) If $\sum_{n=1}^{\infty} f_n(z) =: g(z)$ converges uniformly on A, then g is continuous.

Proof. Again, we only need to show (i). Consider $z_0 \in A$. Let $\varepsilon > 0$ and choose first $N \in \mathbb{N}$, then $\delta > 0$ such that $|f_N(z) - f_N(z_0)| < \varepsilon/3$ for $z \in D(z_0, \delta) \cap A$ (by continuity), $|f_N(z_0) - f(z_0)| < \varepsilon/3$, $|f_N(z) - f(z)| < \varepsilon/3$ (for all $z \in A$) by uniform convergence, then

$$|f(z) - f(z_0)| \leq |f(z) - f_N(z)| + |f_N(z) - f_N(z_0)| + |f_N(z_0) - f(z_0)| < \varepsilon, \tag{6.9}$$

which holds when $z \in D(z_0, \delta) \cap A$. □

We now give an important sufficient condition for the uniform convergence of a series of functions, the *Weierstrass M test.*

Theorem 6.4. *Let $A \subseteq \mathbb{C}$ and let $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions $f_n : A \to \mathbb{C}$. Suppose that there is a sequence of non-negative reals $(M_n)_{n\in\mathbb{N}}$ with*

(i) $|f_n(z)| \leq M_n$ for all $z \in A$, $n \in \mathbb{N}$.

(ii) $\sum_{n=1}^{\infty} M_n$ converges.

Then the series $\sum_{n=1}^{\infty} f_n(z)$ *converges* uniformly absolutely *on* A*, meaning that* $\sum_{n=1}^{\infty} |f_n(z)|$ *converges uniformly on* A*. Moreover,* $\sum_{n=1}^{\infty} f_n(z)$ *converges uniformly on* A*.*[1]

Proof. Since $\sum_{n=1}^{\infty} M_n$ converges, the sequence of partial sums is a Cauchy sequence, so for $\varepsilon > 0$ there exists $N = N(\varepsilon) \in \mathbb{N}$ such that $\sum_{\nu=n+1}^{n+p} M_\nu < \varepsilon$ for $n \geq N$, $p \in \mathbb{N}$. Therefore, one has

$$\left| \sum_{\nu=n+1}^{n+p} f_\nu(z) \right| \leq \sum_{\nu=n+1}^{n+p} |f_\nu(z)| \leq \sum_{\nu=n+1}^{n+p} M_\nu < \varepsilon, \qquad \text{for all } n \geq N, p \in \mathbb{N}, z \in A. \tag{6.10}$$

Both statements therefore follow by applying the Cauchy criterion (see (6.4)), Theorem 6.4, (ii), either to the sequence $(|f_n|)_{n\in\mathbb{N}})$ or to $(f_n)_{n\in\mathbb{N}}$. ☐

Example 6.5. For fixed $0 \leq q < 1$, $\sum_{n=0}^{\infty} z^n$ converges uniformly absolutely on $\overline{D(0,q)}$, since $|z^n| \leq q^n$ for $|z| \leq q$ and $\sum_{n=0}^{\infty} q^n < \infty$, with $M_n := q^n \geq 0$, so Theorem 6.4 applies.

Weierstrass' M test will be instrumental in the study of power series in the next section. We will now show the *Weierstrass approximation theorem* 6.7. We need the following preparation.

Proposition 6.6. *Let* $A \subseteq \mathbb{C}$ *open,* $\gamma : [a,b] \to \mathbb{C}$*,* $a < b$*, a piecewise* C^1 *curve with* $\gamma([a,b]) \subseteq A$*, and* $(f_n)_{n\in\mathbb{N}}$ *a sequence of continuous functions* $f_n : A \to \mathbb{C}$ *with* $f_n \rightrightarrows f$ *on every compact subset* $K \subseteq A$*, as* $n \to \infty$ *for some function* $f : A \to \mathbb{C}$*. Then*

$$\lim_{n\to\infty} \int_\gamma f_n(z)\mathrm{d}z = \int_\gamma f(z)\mathrm{d}z. \tag{6.11}$$

Moreover, if $\sum_{n=1}^{\infty} f_n(z)$ *converges uniformly on every compact subset* $K \subseteq A$*, as* $n \to \infty$*, one has*

$$\sum_{n=1}^{\infty} \int_\gamma f_n(z)\mathrm{d}z = \int_\gamma \left(\sum_{n=1}^{\infty} f_n(z) \right) \mathrm{d}z. \tag{6.12}$$

Proof. First we note that f is continuous on A by Lemma 6.3, (i) (and the fact that for every $z \in A$, there is $\varepsilon > 0$ with $\overline{D(z,\varepsilon)} \subseteq A$, and the closed disk is compact). If $\ell(\gamma) = 0$ all integrals are zero, so assume $\ell(\gamma) > 0$. Since γ is continuous, also $\gamma([a,b]) \subseteq A$ is compact, and so by uniform convergence, for a given $\varepsilon > 0$ there exists $N \in \mathbb{N}$ sich that

$$|f_n(z) - f(z)| < \frac{\varepsilon}{\ell(\gamma)}, \qquad \text{for } n \geq N, z \in \gamma([a,b]). \tag{6.13}$$

Now we use the standard bound (4.40) to infer

$$\left| \int_\gamma f_n(z)\mathrm{d}z - \int_\gamma f(z)\mathrm{d}z \right| \leq \sup_{z\in\gamma([a,b])} |f_n(z) - f(z)| \ell(\gamma) \overset{(6.13)}{\leq} \varepsilon, \qquad \text{for } n \geq N. \tag{6.14}$$

This completes the proof of (6.11). The proof of (6.12) follows by applying the same reasoning to partial sums. ☐

[1] Every series that converges uniformly absolutely on some A also converges uniformly on A, as can be seen by Cauchy's criterion in Theorem 6.2, (ii).

Theorem 6.7. *Let $A \subseteq \mathbb{C}$ be open and let $(f_n)_{n\in\mathbb{N}}$ be a sequence of holomorphic functions $f_n : A \to \mathbb{C}$.*

(i) *Assume that $f_n \rightrightarrows f$ on every compact $K \subseteq A$ for some $f : A \to \mathbb{C}$. Then f is holomorphic and it holds that*

$$\begin{aligned} f_n'(z) &\to f'(z), \qquad \text{for } z \in A, \text{ as } n \to \infty,, \\ f_n' &\rightrightarrows f' \qquad \text{on every compact } K \subseteq A, \text{ as } n \to \infty. \end{aligned} \tag{6.15}$$

(ii) *If $g(z) = \sum_{n=1}^{\infty} f_n(z)$ converges uniformly on every compact $K \subseteq A$, then g is holomorphic and*

$$g'(z) = \sum_{n=1}^{\infty} f_n'(z), \qquad \text{for } z \in A, \tag{6.16}$$

and the convergence is uniform on every compact $K \subseteq A$.

Proof. Again we only need to show (i), since (ii) follows by considering partial sums. To show that f is holomorphic, consider $z_0 \in A$ and take $\varepsilon > 0$ such that $\overline{D(z_0,\varepsilon)} \subseteq A$. We know by Lemma 6.3 that f is continuous on $\overline{D(z_0,\varepsilon)}$. Consider a triangle $\triangle(z_1,z_2,z_3) \subseteq D(z_0,\varepsilon)$. By Proposition 6.6 and Cauchy's Integral Theorem 4.23 for star-shaped domains, we now obtain

$$\int_{\gamma_{\partial\triangle(z_1,z_2,z_3)}} f(z)\mathrm{d}z \overset{(6.11)}{=} \lim_{n\to\infty} \int_{\gamma_{\partial\triangle(z_1,z_2,z_3)}} f_n(z)\mathrm{d}z = 0. \tag{6.17}$$

By Morera's Theorem 4.38, we see that f is holomorphic on $D(z_0,\varepsilon)$, and since z_0 was arbitrary, f is holomorphic on A. We turn to the proof of (6.15). Let $K \subseteq A$ compact and let $r > 0$ as in Lemma 4.30. We can cover K by finitely many open disks $D(z_j,r)$, $j = 1,\ldots,M$ and choose $\rho > r$ so that $\overline{D(z_j,\rho)} \subseteq A$ for every $j = 1,\ldots,M$. Now let $z \in K$, then $z \in D(z_{j_\star},r)$ for some $j_\star \in \{1,\ldots,M\}$. We apply Cauchy's Integral formula for derivatives (4.125) to see that

$$f_n'(z) = \frac{1}{2\pi i}\int_{\partial D(z_{j_\star},\rho)} \frac{f_n(w)}{(w-z)^2}\mathrm{d}w, \qquad f'(z) = \frac{1}{2\pi i}\int_{\partial D(z_{j_\star},\rho)} \frac{f(w)}{(w-z)^2}\mathrm{d}w \tag{6.18}$$

By assumption, f_n converges uniformly to f on every compact subset contained in A, in particular on all $\partial D(z_j,\rho)$, $j = 1,\ldots,M$ so we see that for $\varepsilon > 0$ there exist an $N = N(\varepsilon) \in \mathbb{N}$ such that $|f_n(w) - f(w)| < \varepsilon$ for all $n \geq N$ and $w \in \partial D(z_j,\rho)$, $j = 1,\ldots,M$. Now note that since $z \in D(z_{j_\star},r)$ and $w \in \partial D(z_{j_\star},\rho)$, we find $|w-z| \geq \rho - r$ by the inverse triangle inequality. Therefore we obtain that

$$|f_n'(z) - f'(z)| = \left|\frac{1}{2\pi i}\int_{\partial D(z_{j_\star},\rho)} \frac{f_n(w)-f(w)}{(w-z)^2}\mathrm{d}w\right| \leq \frac{1}{2\pi}\frac{\varepsilon\ell(\gamma_{\partial D(z_{j_\star},\rho)})}{(\rho-r)^2} = \frac{\varepsilon\rho}{(\rho-r)^2} \tag{6.19}$$

for every $n \geq N$. Note that ρ, r are independent of $z \in K$, so we obtain the desired uniform convergence on K. Note that this implies pointwise convergence on K, and the pointwise convergence on A follows, since every $z \in A$ is contained in some $\overline{D(z,\varepsilon)}$.

□

6.2. Power series

Definition 6.8. An infinite series of the form

$$\sum_{n=0}^{\infty} a_n(z-z_0)^n, \qquad \text{where } a_n \in \mathbb{C} \tag{6.20}$$

is called a *power series* with coefficients $(a_n)_{n\in\mathbb{N}_0}$ around $z_0 \in \mathbb{C}$.

Lemma 6.9. *If the power series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ is convergent for some $z = z_1 \in \mathbb{C}$ with $|z_1 - z_0| = r$, then it converges for every $z \in D(z_0, r)$. Moreover, the power series converges uniformly absolutely in $\overline{D(z_0,\rho)}$, for every $\rho \in [0, r)$.*

Proof. There is nothing to show for $r = 0$, so assume $r > 0$. For $z = z_1$ the series is convergent, which implies in particular that its terms are bounded. There is a $c > 0$ with $|a_n|r^n = |a_n(z_1 - z_0)^n| \leq c$ for all $n \in \mathbb{N}_0$, or

$$|a_n| \leq \frac{c}{r^n}, \qquad \text{for } n \in \mathbb{N}_0. \tag{6.21}$$

Now for $|z - z_0| \leq \rho < r$, we see that

$$|a_n(z-z_0)^n| = |a_n||z-z_0|^n \leq c\left(\frac{\rho}{r}\right)^n. \tag{6.22}$$

The claim follows by the Weierstrass M test (Theorem 6.4) for $M_n = \left(\frac{\rho}{r}\right)^n$, since $\frac{\rho}{r} \in [0,1)$. $\square$

The preceding result motivates the following theorem.

Theorem 6.10. *For every given power series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ there exists a unique $R \in [0,\infty) \cup \{\infty\}$ such that*

(i) The series converges uniformly absolutely on every $\overline{D(z_0,\rho)}$ if $0 \leq \rho < R$.

(ii) For $|z - z_0| > R$, the series diverges.

(iii) The formula of Cauchy-Hadamard holds:

$$R = \frac{1}{\limsup_{n\to\infty} \sqrt[n]{|a_n|}}. \tag{6.23}$$

(with the convention $\frac{1}{0} = \infty$ and $\frac{1}{\infty} = 0$).

(iv) If the limit

$$\lim_{n\to\infty} \frac{|a_n|}{|a_{n+1}|} \tag{6.24}$$

exists, it equals R.

The number R is called the radius of convergence *of the power series.*

Proof. By Lemma 6.9, the existence of R, as well as (i) and (ii) follow. Indeed, we can set

$$R = \sup\{|z - z_0| \geq 0\,;\, \sum_{n=0}^{\infty} a_n (z - z_0)^n \text{ converges for } z \in \mathbb{C}\} \tag{6.25}$$

(which may be infinite). To show (iii), we denote by $0 \leq \widetilde{R} \leq \infty$ the expression on the right-hand side of (6.23) and choose $\rho \in (0, \widetilde{R})$. Then

$$\limsup_{n\to\infty} \sqrt[n]{|a_n|} = \frac{1}{\widetilde{R}} < \frac{1}{\rho}, \tag{6.26}$$

so there is an $N \in \mathbb{N}$ with $\sqrt[n]{|a_n|} < \frac{1}{\rho}$ for all $n \geq N$, so

$$|a_n (z - z_0)^n| \leq \left(\frac{|z - z_0|}{\rho}\right)^n. \tag{6.27}$$

The series $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ thus converges if $\frac{|z-z_0|}{\rho} < 1$, so for $|z - z_0| < \rho$. Since $\rho \in (0, \widetilde{R})$ was arbitrary, we see that it converges for $0 \leq |z - z_0| < \widetilde{R}$, which means that $\widetilde{R} \leq R$. Now let $0 \leq \widetilde{R} < \infty$, and choose $\rho \in (\widetilde{R}, \infty)$. Then

$$\frac{1}{\rho} < \frac{1}{\widetilde{R}} = \limsup_{n\to\infty} \sqrt[n]{|a_n|}. \tag{6.28}$$

Therefore, there exists a sequence $(n_k)_{k\in\mathbb{N}}$ of natural numbers with

$$\frac{1}{\rho} < |a_{n_k}|^{1/n_k} \qquad \text{for } k \in \mathbb{N}. \tag{6.29}$$

This means $\rho^{-n_k} < |a_{n_k}|$, so for $|z - z_0| > \rho$ one has

$$|a_{n_k} (z - z_0)^{n_k}| > 1 \qquad \text{for } k \in \mathbb{N}. \tag{6.30}$$

The members of the series $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ do not converge to zero and so the series must diverge. Again $\rho \in (\widetilde{R}, \infty)$ was arbitrary, so $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ diverges for $|z - z_0| > \widetilde{R}$, which entails that $R \leq \widetilde{R}$. The claim (iv) follows similarly. ☐

We say that the set $\{z \in \mathbb{C}\,;\, |z - z_0| = R\}$ is the *circle of convergence* of the power series $\sum_{n=0}^{\infty} a_n (z - z_0)^n$. We will now show that a power series is holomorphic inside its circle of convergence.

Theorem 6.11. *A power series $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ defines a holomorphic function*

$$f : D(z_0, R) \to \mathbb{C}, \qquad f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \tag{6.31}$$

where R is the radius of convergence. Moreover, one has

$$f'(z) = \sum_{n=1}^{\infty} n a_n (z - z_0)^{n-1}, \tag{6.32}$$

and the power series on the right-hand side of this equation has again R as its radius of convergence. The coefficients a_n, $n \in \mathbb{N}_0$, are given by the formula

$$a_n = \frac{f^{(n)}(z_0)}{n!} = \frac{1}{2\pi i} \int_{\partial D(z_0,r)} \frac{f(w)}{(w-z_0)^{n+1}} \mathrm{d}w \tag{6.33}$$

for any $r \in (0, R)$.

Proof. The fact that f is holomorphic and equation (6.32) follows by Theorem 6.10, (i), and Theorem 6.7, (ii). Indeed, assume that $K \subseteq D(z_0, R)$ is compact, then $K \subseteq \overline{D(0,\rho)}$ for some $\rho < R$, and thus by the former theorem, the series converges uniformly on K. We now show that the power series representing f' has R as its radius of convergence. Indeed we write for $z \neq z_0$: $\sum_{n=1}^{\infty} na_n(z-z_0)^{n-1} = \frac{1}{z-z_0} \sum_{n=0}^{\infty} na_n(z-z_0)^n$. Note that the multiplication by $\frac{1}{z-z_0}$ does not change whether the power series converges or diverges, so we can apply the Cauchy-Hadamard formula (6.23):

$$\frac{1}{\limsup_{n\to\infty} |na_n|^{\frac{1}{n}}} = \frac{1}{\limsup_{n\to\infty} n^{\frac{1}{n}} |a_n|^{\frac{1}{n}}} = R. \tag{6.34}$$

Finally, by iterating this procedure we see that for $k \in \mathbb{N}$, we have

$$f^{(k)}(z) = \sum_{n=k}^{\infty} n \cdot (n-1) \cdot \ldots \cdot (n-k+1) a_n (z-z_0)^{n-k}. \tag{6.35}$$

Inserting $z = z_0$ in this formula shows that $f^{(k)}(z_0) = k! a_k$, and this yields (6.33), in view of the generalized version of Cauchy's integral formula (4.125). ☐

Remark 6.12. (i) The above theorem shows that if a function f is given by a power series (6.31), we must necessarily have

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z-z_0)^n. \tag{6.36}$$

(ii) It is also clear that a function f given by a power series (6.31) must have a primitive (since it is a holomorphic function on the star-shaped domain $D(z_0, R)$), and the above Theorem can be used to find a primitive. Indeed, one readily verifies that

$$F : D(z_0, R) \to \mathbb{C}, \qquad F(z) = \sum_{n=0}^{\infty} \frac{a_n}{n+1} (z-z_0)^{n+1} \tag{6.37}$$

has f as its derivative (and since taking the derivative does not change the radius of convergence, F has the same radius of convergence as f).

6.3. Taylor's Theorem

In the previous chapter we saw that a function given by a convergent power series is holomorphic and is given by the *Taylor series* (6.36). In this section we show the converse: Any holomorphic function can be locally developed into a Taylor series. This feature has some fundamental consequences for holomorphic functions, which we explore.

Theorem 6.13. *Let $z_0 \in \mathbb{C}$, $r > 0$ and $f : D(z_0, r) \to \mathbb{C}$ holomorphic. Then for every $z \in D(z_0, r)$ one has*

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n \tag{6.38}$$

(in particular the series on the right-hand side of (6.38) *converges).*

Proof. Let $|z| < r$ and define $h(z) = f(z_0 + z)$, then h is holomorphic on $D(0, r)$. For $\rho \in (0, r)$, the Cauchy integral formula (4.106) yields

$$h(z) = \frac{1}{2\pi i} \int_{\partial D(0,\rho)} \frac{h(w)}{w - z} \mathrm{d}w, \qquad z \in D(0, \rho). \tag{6.39}$$

Note that for $\zeta \neq 1$ and $n \in \mathbb{N}$ one has the formula

$$1 + \zeta + \ldots + \zeta^{n-1} = \frac{1 - \zeta^n}{1 - \zeta} \qquad \Rightarrow \qquad \frac{1}{1 - \zeta} = 1 + \zeta + \ldots + \zeta^{n-1} + \frac{\zeta^n}{1 - \zeta} \tag{6.40}$$

We use this for $\zeta = \frac{z}{w}$ with $w \in \partial D(0, \rho)$ and $z \in D(0, \rho)$ (so that $|\zeta| < 1$), giving

$$\begin{aligned} \frac{1}{w - z} &= \frac{1}{w} \cdot \frac{1}{1 - \frac{z}{w}} \\ &= \frac{1}{w} \cdot \left(1 + \frac{z}{w} + \ldots + \left(\frac{z}{w}\right)^{n-1} + \frac{\left(\frac{z}{w}\right)^n}{1 - \frac{z}{w}} \right) \\ &= \frac{1}{w} + \frac{z}{w^2} + \ldots + \frac{z^{n-1}}{w^n} + \frac{z^n}{w^n} \frac{1}{w - z}. \end{aligned} \tag{6.41}$$

Therefore we find

$$h(z) = \sum_{\nu=0}^{n-1} z^\nu \left(\frac{1}{2\pi i} \int_{\partial D(0,\rho)} \frac{h(w)}{w^{\nu+1}} \mathrm{d}w \right) + \frac{z^n}{2\pi i} \int_{\partial D(0,\rho)} \frac{h(w)}{w^n(w - z)} \mathrm{d}w. \tag{6.42}$$

We see that by the generalized Cauchy integral formula (4.125),

$$\frac{1}{2\pi i} \int_{\partial D(0,\rho)} \frac{h(w)}{w^{\nu+1}} = \frac{h^{(\nu)}(0)}{\nu!} \tag{6.43}$$

for $\nu = 0, 1, \ldots, n - 1$. In other words

$$h(z) = \sum_{\nu=0}^{n-1} \frac{h^{(\nu)}(0)}{\nu!} z^\nu + \frac{z^n}{2\pi i} \int_{\partial D(0,\rho)} \frac{h(w)}{w^n(w - z)} \mathrm{d}w. \tag{6.44}$$

Note now that for $w \in \partial D(0,\rho)$ we have that for some $\widetilde{\rho} \in (0,\rho)$ with $|z| \leq \widetilde{\rho}$, $|w-z| \geq \rho - \widetilde{\rho}$, so setting $M = \max_{w \in \partial D(0,\rho)} |h(w)|$ we find that

$$\begin{aligned} \left| \frac{z^n}{2\pi i} \int_{\partial D(0,\rho)} \frac{h(w)}{w^n(w-z)} \mathrm{d}w \right| &\overset{(4.40)}{\leq} \frac{\widetilde{\rho}^n}{2\pi} \frac{M}{\rho^n(\rho-\widetilde{\rho})} \ell(\gamma_{\partial D(0,\rho)}) \\ &= \left(\frac{\widetilde{\rho}}{\rho}\right)^n \frac{M\rho}{\rho-\widetilde{\rho}} \to 0, \text{ as } n \to \infty. \end{aligned} \tag{6.45}$$

We find that

$$h(z) = \sum_{\nu=0}^{\infty} \frac{h^{(\nu)}(0)}{\nu!} z^\nu, \qquad \text{for } z \in \mathbb{C}, |z| \leq \widetilde{\rho}. \tag{6.46}$$

Since $0 < \widetilde{\rho} < \rho < r$ is arbitrary, we see that the claim holds. □

With this we obtain the following equivalence.

Corollary 6.14. *Let $A \subseteq \mathbb{C}$ be open and $f : A \to \mathbb{C}$ a map. The following are equivalent.*

(i) f is holomorphic on A.

(ii) f is analytic *on A in the sense that for every $z_0 \in A$ there is $\varepsilon > 0$ with $D(z_0,\varepsilon) \subseteq A$ and a power series*

$$\sum_{n=0}^{\infty} a_n(z-z_0)^n, \tag{6.47}$$

converges on $D(z_0,\varepsilon)$ and is equal to $f|_{D(z_0,\varepsilon)}$.

Proof. Assume that f is holomorphic. For $z_0 \in A$ there is $\varepsilon > 0$ with $D(z_0,\varepsilon) \subseteq A$. Then (ii) follows from Taylor's Theorem 6.13.

Conversely suppose the statements in item (ii) hold. For a fixed $z_0 \in A$ and $\varepsilon > 0$ with $D(z_0,\varepsilon) \subseteq A$, $f|_{D(z_0,\varepsilon)}$ is holomorphic by Theorem 6.11. □

This Corollary justifies that the terms "holomorphic" and "analytic" may be used interchangeably. Every holomorphic function can therefore locally be expressed as a Taylor series.

End of Lecture 9

Example 6.15. (i) Consider the exponential function $\exp$ that we introduced in (1.30). Since $\exp$ is holomorphic on every $D(0,r)$, $r > 0$ and fulfills $\exp^{(n)}(0) = \exp(0) = 1$ for every $n \in \mathbb{N}$, we see that

$$\exp(z) = \sum_{n=0}^{\infty} \frac{1}{n!} z^n \tag{6.48}$$

for all $z \in \mathbb{C}$. We could alternatively have defined $\exp$ in this way and concluded $\exp' = \exp$ by Theorem 6.11 and the fact that the radius of convergence of the power series on the right-hand side in (6.48) is ∞.

(ii) Similarly as in the real case, we also have

$$\begin{aligned}\sin(z) &= \sum_{n=0}^{\infty}(-1)^n \frac{z^{2n+1}}{(2n+1)!}, \\ \cos(z) &= \sum_{n=0}^{\infty}(-1)^n \frac{z^{2n}}{(2n)!},\end{aligned} \tag{6.49}$$

which are both valid for all $z \in \mathbb{C}$.

(iii) The principal value of the logarithm Log on $\mathbb{C}_-$ has the Taylor series

$$\mathrm{Log}(z) = \mathrm{Log}(z_0) + \sum_{\nu=1}^{\infty} \frac{(-1)^{\nu-1}}{z_0^{\nu}\nu}(z-z_0)^{\nu}, \qquad \text{for } z \in D(z_0, r) \subseteq \mathbb{C}_-. \tag{6.50}$$

Note that by the Cauchy-Hadamard formula (6.23), the power series on the right-hand side of (6.50) is $|z_0|$. If however $\mathrm{Re}(z_0) < 0$, then $D(z_0, |z_0|) \nsubseteq \mathbb{C}_-$. So the radius of convergence of the Taylor series representing a holomorphic function need not be contained in the domain of this function!

We give some computation rules for power series.

1. *Identity of power series*

   If two power series $\sum_{n=0}^{\infty} a_n z^n$ and $\sum_{n=0}^{\infty} b_n z^n$ converge on some open disk around 0 and represent the same function, then $a_n = b_n$ for all $n \in \mathbb{N}_0$ (This is just another version of Remark 6.12, (i)).

2. *Sum of power series*

   If two power series $\sum_{n=0}^{\infty} a_n z^n$ and $\sum_{n=0}^{\infty} b_n z^n$ have radii of convergence $R_1, R_2 > 0$, then $\sum_{n=0}^{\infty}(a_n + b_n)z^n$ has radius of convergence at least $\min\{R_1, R_2\}$ and for $z \in D(0, \min\{R_1, R_2\})$ one has

$$\sum_{n=0}^{\infty}(a_n+b_n)z^n = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=0}^{\infty} b_n z^n. \tag{6.51}$$

3. *Multiplication of power series*

   If two power series $\sum_{n=0}^{\infty} a_n z^n$ and $\sum_{n=0}^{\infty} b_n z^n$ have radii of convergence $R_1, R_2 > 0$, then one has the *Cauchy Product formula*

$$\begin{aligned}\left(\sum_{n=0}^{\infty} a_n z^n\right) \cdot \left(\sum_{n=0}^{\infty} b_n z^n\right) &= \sum_{n=0}^{\infty} c_n z^n, \qquad |z| < \min\{R_1, R_2\} \\ c_n &= \sum_{\nu=0}^{n} a_\nu b_{n-\nu}.\end{aligned} \tag{6.52}$$

This follows from Taylor's theorem 6.13. Indeed, the functions f and g, defined on $D(0, \min\{R_1, R_2\})$ by $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ are holomorphic, and so is $f \cdot g$. We see that

$$(f \cdot g)(z) = \sum_{n=0}^{\infty} \frac{(f \cdot g)^{(n)}(0)}{n!} z^n. \tag{6.53}$$

The claim then follows by using that $(f \cdot g)^{(n)}(0) = \sum_{\nu=0}^{n} \binom{n}{\nu} f^{(\nu)}(0) g^{(n-\nu)}(0) = n! \sum_{\nu=0}^{n} a_\nu b_{n-\nu}$ (using (6.33) in the last step).

4. *Multiplicative inversion of power series*

   Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a power series with radius of convergence $R > 0$ and $a_0 \neq 0$. Then there exists $\varepsilon > 0$ such that $f(z) \neq 0$ for all $z \in D(0, \varepsilon)$, and $g = \frac{1}{f}$ is holomorphic in $D(0, \varepsilon)$. Developing g into a Taylor series (using Taylor's theorem 6.13), $g(z) = \sum_{n=0}^{\infty} b_n z^n$ holds for $z \in D(0, \varepsilon)$, and since

$$f(z) \cdot g(z) = 1 = 1 + 0 \cdot z + 0 \cdot z^2 + \ldots \tag{6.54}$$

   we can combine items 1. and 3. to see that the coefficients $b_0, b_1, b_2, \ldots$ are fulfill

$$\begin{aligned} \sum_{\nu=0}^{n} a_\nu b_{n-\nu} &= \begin{cases} 1, & n = 0, \\ 0, & n \geq 1 \end{cases} \qquad \text{i.e.} \\ a_0 b_0 &= 1, \\ a_0 b_1 + a_1 b_0 &= 0 \\ a_0 b_2 + a_1 b_1 + a_2 b_0 &= 0, \\ &\vdots \end{aligned} \tag{6.55}$$

5. *Sequences of power series*

   Assume that the power series

$$f_j(z) = \sum_{n=0}^{\infty} c_{jn} z^n, \qquad j \in \mathbb{N}_0, \tag{6.56}$$

   all converge in $D(0, R)$, $R > 0$ and assume that $|f_n(z)| \leq M_n$ for some sequence $(M_n)_{n \geq 0}$ of non-negative real numbers such that $\sum_{n=0}^{\infty} M_n$ converges. Then the function $F = \sum_{j=0}^{n} f_j$ is holomorphic on $D(0, R)$ and one has

$$F(z) = \sum_{n=0}^{\infty} \left(\sum_{j=0}^{\infty} c_{jn} \right) z^n. \tag{6.57}$$

   Indeed, this follows from the Weierstrass M-test (Theorem 6.4) and Weierstrass approximation Theorem 6.4.

6. *Insertion of a power series into another one*

   Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=1}^{\infty} b_n z^n$ (i.e. $g(0) = 0$) with positive radius of convergence. Then $f \circ g$ has a positive radius of convergence and for $f(g(z)) = \sum_{n=0}^{\infty} c_n z^n$ one has

$$\begin{aligned} c_0 &= f(g(0)) = f(0) = a_0 \\ c_1 &= f'(g(0))g'(0) = a_1 b_1 \\ c_2 &= \frac{f''(g(0))g'(0)^2 + f'(g(0))g''(0)}{2} = a_2 b_1^2 + a_1 b_2 \\ &\vdots \end{aligned} \tag{6.58}$$

   This is the same as one obtains by naively substituting the representation of g into f.

By the first and last item it is also possible to find the Taylor series of the inverse function of a holomorphic function (since $f(f^{-1}(z)) = f^{-1}(f(z)) = z$).

Example 6.16. Let us compute the first few terms of the Taylor series centered in $z_0 = 0$ of $f(z) = \exp(-z^2)\sin(z)$, which is an entire function.

$$\begin{aligned} \exp(-z^2) &= \sum_{n=0}^{+\infty} (-1)^n \frac{z^{2n}}{n!} = 1 - z^2 + \frac{z^4}{2} - \frac{z^6}{6} + \dots, \\ \sin(z) &= \sum_{n=0}^{+\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} = z - \frac{z^3}{6} + \frac{z^5}{120} - \dots \end{aligned} \tag{6.59}$$

Hence,

$$\exp(-z^2)\sin(z) = z - \frac{7}{6}z^3 + \frac{27}{40}z^5 + \dots \tag{6.60}$$

6.4. Mapping properties of holomorphic functions

We have established that holomorphic functions can always be represented locally as a power series. This can be used as a very powerful tool to obtain a number of mapping properties of holomorphic functions. We first show the *isolation of zeros* for a holomorphic function that is not constant zero. We first recall the following definition.

Definition 6.17. Let $B \subseteq A \subseteq \mathbb{C}$. A point $z \in A$ is called an *accumulation point of B in A* if for every $\varepsilon > 0$, there exists a point $w \in B$ with $0 < |w - z| < \varepsilon$.

Proposition 6.18. *Let $A \subseteq \mathbb{C}$ be a domain and $f : A \to \mathbb{C}$ holomorphic and not identically equal to zero. Then the set of zeros*

$$N(f) = \{z \in A\,;\, f(z) = 0\} \tag{6.61}$$

does not have an accumulation point in A.

Proof. Assume that there is an accumulation point of $z_0 \in A$ in $N(f)$, and therefore a sequence $(z_n)_{n\in\mathbb{N}} \subseteq N(f) \setminus \{z_0\}$ with $z_0 = \lim_{n\to\infty} z_n$. There exists an $r > 0$ such that

$$f(z) = \sum_{\nu=0}^{\infty} a_\nu (z - z_0)^\nu, \qquad z \in D(0, r). \tag{6.62}$$

Since f is continuous, we must have $a_0 = f(z_0) = \lim_{n\to\infty} f(z_n) = 0$. This argument can now be repeated for the power series

$$\frac{f(z)}{z - z_0} = \sum_{\nu=0}^{\infty} a_{\nu+1}(z - z_0)^\nu \tag{6.63}$$

(the power series on the right hand side converges on $D(z_0, r)$ and represents the function $z \mapsto \frac{f(z)}{z-z_0}$ on $\dot{D}(z_0, r)$), we then infer that $a_1 = 0$ as well. By iteration, we find that $a_\nu = 0$ for all $\nu \in \mathbb{N}_0$. We thus see that $f(z) = 0$ for $z \in D(0, r)$. Now consider

$$U = \{z \in A\,;\, z \text{ is an accumulation point of } N(f)\} \subseteq A. \tag{6.64}$$

By assumption, $z_0 \in U$, so $U \neq \emptyset$. Furthermore, U is open as we have just seen. We show that U is also closed (relative to A): Indeed, let $z_1 \in A \setminus U$. We show that $D(z_1, \varepsilon) \subseteq A \setminus U$ for some $\varepsilon > 0$. Indeed, otherwise for every $\varepsilon > 0$, there is a point $w \in \dot{D}(z_1, \varepsilon) \cap U$, and since w is an accumulation point of $N(f)$, there exists an element $\xi \in N(f)$ with $0 < |w - \xi| < \min\{|w - z_1|, \varepsilon - |w - z_1|\}$. We infer that $0 < |z_1 - \xi| < \varepsilon$, which is a contradiction.

We thus see that $A \setminus U$ is open, and so U is closed (relative to A). Since A is connected, we see that $U = A$. But by continuity, we would infer that $f \equiv 0$ on A, which cannot be true. In total, the initial assumption must have been false, and $N(f)$ cannot have an accumulation point in A. ☐

Note that while the set of zeros does not have any accumulation points in A, it may of course have accumulation points in $\mathbb{C}$. For instance, the holomorphic function

$$f : \mathbb{C} \setminus \{0\} \to \mathbb{C}, \qquad z \mapsto \sin(\tfrac{2\pi}{z}) \tag{6.65}$$

has a zero set $N(f)$ that contains $\{\frac{1}{n}\,;\, n \in \mathbb{N}\}$. The latter set has the accumulation point 0, but of course $0 \notin \mathbb{C} \setminus \{0\}$.

An extremely useful application of this isolation of zeros is the following *Identity Theorem for holomorphic functions*:

Theorem 6.19. *Let $A \subseteq \mathbb{C}$ be a domain and $f, g : A \to \mathbb{C}$ two holomorphic functions. The following are equivalent:*

(i) $f = g$.

(ii) The coincidence set

$$\{z \in A\,;\, f(z) = g(z)\} \tag{6.66}$$

has an accumulation point in A.

(iii) *There exists a point* $z_0 \in A$ *such that* $f^{(n)}(z_0) = g^{(n)}(z_0)$ *for all* $n \in \mathbb{N}_0$.

Proof. The equivalence of (i) and (ii) follows immediately from Proposition 6.18, applied to the function $f - g$. Since (i) trivially implies (iii), we only need to show that (iii) implies (ii). By Taylor's theorem 6.13, the functions f and g agree on $D(z_0, r)$ for some $r > 0$, and (ii) follows. □

Let us comment on this remarkable property of holomorphic functions.

Remark 6.20. (i) The identity theorem asserts that for a domain $A \subseteq \mathbb{C}$, the values of a holomorphic function $f : A \to \mathbb{C}$ are already determined by the values on a *very small* subset, for instance a small piece of a line or a small disk. There is a strong "global" rigidity in the values of a complex function! The situation in $\mathbb{R}$ is of course completely different: There mere existence of smooth functions with compact support (i.e. vanishing outside a compact subset) shows the failure of the identity theorem.

(ii) It is crucial for the validity of the identity theorem that A be connected. Indeed, if A was only supposed to be open, one has

$$f : D(0,1) \cup D(3,1) \to \mathbb{C}, \qquad z \mapsto \begin{cases} 1, & z \in D(0,1), \\ 0, & z \in D(3,1), \end{cases} \tag{6.67}$$

which is holomorphic and identical to zero on $D(3, 1)$, but of course not identical to zero on A.

(iii) Holomorphic extensions are unique in the following sense: Let $A \subseteq \mathbb{C}$ a domain and $M \subseteq A$ a set that contains at least one accumulation point in A. Let $f : M \to \mathbb{C}$ a map. If a holomorphic function $\widetilde{f} : A \to \mathbb{C}$ exists with $\widetilde{f}|_M = f$, then it is unique with this property. In particular, the real functions $\exp, \sin, \cos : \mathbb{R} \to \mathbb{R}$ have *unique* extensions to $\mathbb{C}$.

We will now show the *open mapping theorem*.

Theorem 6.21. *Let* $A \subseteq \mathbb{C}$ *be a domain and* $f : A \to \mathbb{C}$ *a non-constant holomorphic function. Then* $f(A)$ *is also a domain.*

Proof. Since f is continuous and A is connected, $f(A)$ is connected as well. We need to show that $f(A)$ is open. Let $z_0 \in A$ and consider $w_0 = f(z_0)$. Choose $r > 0$ small enough such that $\overline{D(z_0, r)} \subseteq A$ and f does not attain the value w_0 on $\partial D(z_0, r)$. This is possible since the zeros of the function $z \mapsto f(z) - w_0$ are isolated by Proposition 6.18. Therefore

$$\delta = \min_{z \in \partial D(z_0, r)} |f(z) - w_0| > 0. \tag{6.68}$$

Thus, for $|w - w_0| < \frac{\delta}{2}$, we see that for $z \in \partial D(z_0, r)$:

$$|f(z) - w| \geq \big| |f(z) - w_0| - |w - w_0| \big| > \delta - \frac{\delta}{2} = \frac{\delta}{2}. \tag{6.69}$$

Since $|f(z_0) - w| < \frac{\delta}{2}$, we see that the function $z \mapsto f(z) - w$ must have a zero in $D(z_0, r)$ by Remark 5.3. This in turn implies that f attains w in $D(z_0, r)$, and therefore $D(w_0, \frac{\delta}{2}) \subseteq f(A)$. □

The open mapping theorem gives an alternative proof for the fact that a holomorphic function $f : A \to \mathbb{C}$ on a domain A must be constant if one of $\mathrm{Re}(f)$, $\mathrm{Im}(f)$ or $|f|$ is constant: Indeed, if one of these hold, one can see easily that the $f(A)$ is not a domain, and so f must be constant.

7. Singularities and Laurent decomposition

(Reference: Marsden-Hoffman, Basic complex analysis, 3rd Ed., Section 3.3)

7.1. Singularities

We recall the definition of the deleted r-disk ($r > 0$) around a point $z_0 \in \mathbb{C}$,

$$\dot{D}(z_0, r) = D(z_0, r) \setminus \{z_0\} = \{z \in \mathbb{C}\,;\, 0 < |z - z_0| < r\}. \tag{7.1}$$

Definition 7.1. Let $A \subseteq \mathbb{C}$ open and $f : A \to \mathbb{C}$ holomorphic. Assume that $z_0 \in \mathbb{C} \setminus A$, but $\dot{D}(z_0, r) \subseteq A$ for some $r > 0$. Then z_0 is called a *(isolated) singularity* of f.

Note that $A \cup \{z_0\} = A \cup D(z_0, r)$ is open. In what follows, we only consider isolated singularities. An example of a non-isolated singularity would be the point 0 for the function

$$f : \mathbb{C} \setminus (\{k^{-1}\,;\, k \in \mathbb{Z} \setminus \{0\}\} \cup \{0\}) \to \mathbb{C}, \qquad z \mapsto \sin(\tfrac{\pi}{z})^{-1}, \tag{7.2}$$

since $z_k = \frac{1}{k} \to 0$ as $k \to \infty$ (the z_k themselves are for $k \in \mathbb{Z} \setminus \{0\}$ isolated singularities). From now on we will say “singularity” instead of “isolated singularity”. We will see three types of singularities:

- removable singularities,
- poles,
- essential singularities.

We will study these singularities step-by-step.

Definition 7.2. Let $A \subseteq \mathbb{C}$ open, $f : A \to \mathbb{C}$ holomorphic and z_0 a singularity of f. Then z_0 is called *removable* if f can be extended to a holomorphic function on $A \cup \{z_0\}$, i.e. if there is a holomorphic function $\widetilde{f} : A \cup \{z_0\} \to \mathbb{C}$ with $\widetilde{f}|_A = f$.

Note that $\widetilde{f}$ is continuous in z_0 and is therefore uniquely determined by f. We will simply write f instead of $\widetilde{f}$ (if it exists).

Example 7.3. (i) The function $f : \mathbb{C} \setminus \{1\} \to \mathbb{C}$, $z \mapsto \frac{z^2-1}{z-1}$ has a removable singularity at $z_0 = 1$, with holomorphic extension $f(z) = z + 1$.

(ii) The function $g : \mathbb{C} \setminus \{0\} \to \mathbb{C}$, $z \mapsto \frac{\sin(z)}{z}$ has a removable singularity at 0. This follows from Taylor's theorem, since $g(z) = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} \mp \ldots$ for all $z \in \mathbb{C}$.

The following is the *Riemann removability condition.*

Theorem 7.4. *Let $A \subseteq \mathbb{C}$ open, $f : A \to \mathbb{C}$ holomorphic and z_0 a singularity of f. The following are equivalent.*

(i) z_0 is removable.

(ii) There exists a $\delta > 0$ with $\dot{D}(z_0, \delta) \subseteq A$ such that f is bounded on $\dot{D}(z_0, \delta)$.

Proof. Assume that z_0 is removable and let $\widetilde{f}$ be the holomorphic extension. There exists a $\delta > 0$ such that

$$|\widetilde{f}(z) - \widetilde{f}(z_0)| < 1, \qquad \text{for } z \in D(z_0, \delta). \tag{7.3}$$

For $z \in \dot{D}(z_0, \delta)$, one has $\widetilde{f}(z) = f(z)$, and hence

$$\begin{aligned} |f(z)| &\leq |f(z) - \widetilde{f}(z_0)| + |\widetilde{f}(z_0)| \\ &\overset{(7.3)}{=} |\widetilde{f}(z) - \widetilde{f}(z_0)| + |\widetilde{f}(z_0)| \\ &\leq 1 + |\widetilde{f}(z_0)|, \end{aligned} \tag{7.4}$$

so we obtain the boundedness of f on $\dot{D}(z_0, \delta)$.

Conversely, suppose that f is bounded on $\dot{D}(z_0, \delta)$ for some $\delta > 0$. Define the function $g : D(z_0, \delta) \to \mathbb{C}$ by

$$g(z) = \begin{cases} (z - z_0)^2 f(z), & \text{for } z \neq z_0, \\ 0, & \text{for } z = z_0. \end{cases} \tag{7.5}$$

This function is holomorphic on $\dot{D}(z_0, \delta)$ and since f is bounded on the same set,

$$\lim_{z \to z_0} \frac{g(z) - g(z_0)}{z - z_0} = \lim_{z \to z_0} (z - z_0) f(z) = 0. \tag{7.6}$$

So g is also differentiable in the complex sense in $z = z_0$ with derivative $g'(z_0) = 0$. By Taylor's Theorem 6.13, g has the Taylor series on $D(z_0, \delta)$

$$g(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \qquad a_n = \frac{g^{(n)}(z_0)}{n!}. \tag{7.7}$$

Since $g(z_0) = g'(z_0) = 0$, we see that $a_0 = a_1 = 0$, so we set

$$h(z) = \sum_{n=2}^{\infty} a_n (z - z_0)^{n-2}, \qquad z \in D(z_0, \delta). \tag{7.8}$$

By construction, h is holomorphic on $D(z_0, \delta)$ it holds that $h(z) = f(z)$ for all $z \in \dot{D}(z_0, \delta)$ because

$$h(z) = \frac{1}{(z - z_0)^2} \sum_{n=2}^{\infty} a_n (z - z_0)^n = \frac{g(z)}{(z - z_0)^2} = f(z), \qquad \text{for } z \in \dot{D}(z_0, \delta). \tag{7.9}$$

We can therefore set

$$\widetilde{f}(z) = \begin{cases} h(z), & \text{for } z \in D(z_0, \delta), \\ f(z), & \text{for } z \notin D(z_0, \delta), \end{cases} \tag{7.10}$$

to obtain a holomorphic extension of f to $A \cup \{z_0\}$. □

End of Lecture 10

We next turn to the discussion of poles.

Definition 7.5. Let $A \subseteq \mathbb{C}$ open, $f : A \to \mathbb{C}$ holomorphic and z_0 a singularity of f. We say that z_0 is a *pole* of f, if there exists a $\delta > 0$ with $D(z_0, \delta) \subseteq A \cup \{z_0\}$ and a holomorphic function $g : D(z_0, \delta) \to \mathbb{C}$ with $g(z_0) \neq 0$ and $m \in \mathbb{N}$ with

$$f(z) = \frac{g(z)}{(z - z_0)^m}, \qquad \text{for } z \in \dot{D}(z_0, \delta). \tag{7.11}$$

The number m is called the *pole order* ∞-ord$(f; z_0)$ of f in z_0. If $m = 1$, z_0 is called a *simple pole* of f.

Remark 7.6. Importantly, the order of a pole is well-defined. Indeed, suppose that we have two holomorphic functions $g, h : D(z_0, \delta_1) \to \mathbb{C}$ and $h : D(z_0, \delta_1) \to \mathbb{C}$ with $\delta_1, \delta_2 > 0$ and $g(z_0) \neq 0 \neq h(z_0)$ as well as $m_1, m_2 \in \mathbb{N}$ with

$$\frac{g(z)}{(z - z_0)^{m_1}} = f(z) = \frac{h(z)}{(z - z_0)^{m_2}}, \qquad z \in \dot{D}(z_0, \min\{\delta_1, \delta_2\}). \tag{7.12}$$

If $m_1 - m_2 > 0$, we find $(z - z_0)^{m_1 - m_2} = \frac{g(z)}{h(z)}$ in a small deleted disk around z_0 (since $h(z_0) \neq 0$, $\frac{1}{h}$ exists in such a deleted disk). Letting $z \to z_0$, we find that $g(z_0) = 0$, but this was ruled out. Similarly, $m_2 - m_1 > 0$ can be ruled out.

Example 7.7. The function $f : \mathbb{C} \setminus \{0\} \to \mathbb{C}$, $z \mapsto \frac{\exp(z)}{z^m}$ for $m \in \mathbb{N}$ is holomorphic and has a pole of order m in $z_0 = 0$.

The following theorem gives a characterization of poles, similarly as the characterization of removable singularities earlier in Theorem 7.4.

Theorem 7.8. *Let $A \subseteq \mathbb{C}$ open, $f : A \to \mathbb{C}$ holomorphic and z_0 a singularity of f. The following are equivalent.*

(i) *z_0 is a pole of f.*

(ii) $\lim_{z \to z_0} |f(z)| = \infty$.

Proof. Assume that z_0 is a pole of f, and let

$$f(z) = \frac{g(z)}{(z - z_0)^m}, \qquad \text{for } z \in \dot{D}(z_0, \delta), \tag{7.13}$$

with g, m and δ as in the definition. Since $g(z_0) \neq 0$, the continuity of g in z_0 and $m \geq 1$ imply

$$\lim_{z \to z_0} |f(z)| = \lim_{z \to z_0} \frac{|g(z)|}{|z - z_0|^m} = \infty. \tag{7.14}$$

Conversely, assume that $\lim_{z\to z_0} |f(z)| = \infty$. There exists a deleted δ-disk $\dot{D}(z_0,\delta) \subseteq A$ on which f has no zero. Therefore $\frac{1}{f}$ is holomorphic and bounded on this disk. By Theorem 7.4, $\frac{1}{f}$ can be extended to a holomorphic function $\widetilde{f}$ on $D(z_0,\delta)$, fulfilling $\widetilde{f}(z_0) = 0$. By Taylor's Theorem 6.13, we have

$$\widetilde{f}(z) = \sum_{\nu=m}^{\infty} a_\nu (z-z_0)^\nu, \qquad a_\nu \in \mathbb{C}, \tag{7.15}$$

where $m \in \mathbb{N}$ and $a_m \in \mathbb{C} \setminus \{0\}$. We can therefore write

$$\widetilde{f}(z) = (z-z_0)^m h(z), \tag{7.16}$$

where $h : D(z_0,\delta) \to \mathbb{C}$ is holomorphic and does not have zeros in $D(z_0,\delta)$ and fulfills $h(z_0) = a_m \neq 0$. We find that

$$f(z) = \frac{1}{\widetilde{f}(z)} = \frac{\frac{1}{h(z)}}{(z-z_0)^m} = \frac{g(z)}{(z-z_0)^m}, \tag{7.17}$$

where $g = \frac{1}{h}$ is holomorphic on $D(z_0,\delta)$ and fulfills $g(z_0) \neq 0$. □

We finally move to the third class of singularities, the essential singularities.

Definition 7.9. Let $A \subseteq \mathbb{C}$ open, $f : A \to \mathbb{C}$ holomorphic and z_0 a singularity of f. If z_0 is neither a pole, nor a removable singularity, it is called a *essential singularity*.

Example 7.10. The function $f : \mathbb{C}\setminus\{0\} \to \mathbb{C}$, $z \mapsto \exp(\frac{1}{z})$ has an essential singularity in $z_0 = 0$. Indeed, $(\frac{1}{n})_{n\in\mathbb{N}}$ and $(\frac{1}{in})_{n\in\mathbb{N}}$ both converge to 0, but

$$\begin{aligned} f(\tfrac{1}{n}) &= \exp(n) \to \infty, \qquad \text{as } n \to \infty \\ \big|f(\tfrac{1}{in})\big| &= |\exp(in)| = 1 \to 1, \qquad \text{as } n \to \infty. \end{aligned} \tag{7.18}$$

To understand the behavior of a holomorphic function near essential singularities, we now prove the *Theorem of Casorati-Weierstrass.*

Theorem 7.11. *Let $A \subseteq \mathbb{C}$ be open and $f : A \to \mathbb{C}$ holomorphic. Let z_0 be an essential singularity of f and assume that $\dot{D}(z_0,\delta) \subseteq A$, $\delta > 0$. For every $w \in \mathbb{C}$, and every $\varepsilon > 0$, there exists a $z \in \dot{D}(z_0,\delta)$ such that $|f(z) - w| < \varepsilon$.*

Proof. Assume to the contrary that there exists $w \in \mathbb{C}$ and $\varepsilon > 0$ such that

$$|f(z) - w| \geq \varepsilon, \qquad \text{for all } z \in \dot{D}(z_0,\delta). \tag{7.19}$$

Therefore the function g, defined on $\dot{D}(z_0,\delta)$ by

$$g(z) = \frac{1}{f(z) - w} \tag{7.20}$$

is holomorphic, bounded and has no zeros. By the Riemann removability condition (Theorem 7.4), we can extend g to a holomorphic function on $D(z_0,\delta)$.

- **Case I:** $g(z_0) \neq 0$. Then $f(z) = \frac{1}{g(z)} + w$ holds on all of $D(z_0, \delta)$, making f holomorphic on $D(z_0, \delta)$. In other words, z_0 is a removable singularity of f, a contradiction.

- **Case II:** $g(z_0) = 0$. Since g is not identically zero on $D(z_0, \delta)$, we can write

$$g(z) = (z - z_0)^m h(z), \qquad m \in \mathbb{N}, \tag{7.21}$$

with some holomorphic function $h : D(z_0, \delta) \to \mathbb{C}$ with $h(z_0) \neq 0$. This follows from Taylor's theorem 6.13 and the identity theorem 6.19. For $z \in \dot{D}(z_0, \delta)$ we have

$$f(z) = \frac{1}{g(z)} + w = \frac{1}{h(z)(z - z_0)^m} + w = \frac{G(z)}{(z - z_0)^m}, \tag{7.22}$$

where we defined $G(z) = \frac{1}{h(z)} + w(z - z_0)^m$ for $z \in D(z_0, \delta)$. Now $G(z_0) = \frac{1}{h(z_0)} \neq 0$, so f has a pole in z_0, which is a contradiction.

□

Remark 7.12. (i) The converse to the Theorem of Casorati-Weierstrass is true as well: If the image of every deleted disk around z_0 in $\mathbb{C}$ is dense in $\mathbb{C}$, then z_0 cannot be removable or a pole by Theorems 7.4 and 7.8.

(ii) The *Theorem of Picard* refines the theorem of Casorati-Weierstrass: It asserts that under the same asssumptions, for every $\delta > 0$ with $\dot{D}(z_0, \delta) \subseteq A$, there exists some $c(z_0, \delta) \in \mathbb{C}$ such that $\mathbb{C} \setminus \{c(z_0, \delta)\} \subseteq f(\dot{D}(z_0, \delta))$. This means that $f(\dot{D}(z_0, \delta))$ is not only dense, but is always all of $\mathbb{C}$ except for possibly one point.

End of Lecture 10.5

7.2. Laurent Decomposition

We will now see how to find a series expansion of holomorphic functions around an isolated singularity, the *Laurent expansion.* This will also give us a helpful characterization of the type of singularity under consideration.

To simplify notation, we introduce for $0 \leq r < R \leq \infty$ the *annulus*

$$\mathscr{A}_r^R(z_0) = \{z \in \mathbb{C}\,;\, r < |z - z_0| < R\}. \tag{7.23}$$

Note that $\mathscr{A}_0^R(z_0) = \dot{D}(z_0, R)$. We restrict our attention to the case $z_0 = 0$ and write $\mathscr{A}_r^R = \mathscr{A}_r^R(0)$.

We will now prove the *Laurent decomposition theorem.*

Theorem 7.13. *Let $0 \leq r < R \leq \infty$ and let $f : \mathscr{A}_r^R \to \mathbb{C}$ be holomorphic. Then f has the decomposition*

$$f(z) = g(z) + h(\tfrac{1}{z}), \qquad \textit{for } z \in \mathscr{A}_r^R \tag{7.24}$$

and holomorphic functions

$$g : D(0,R) \to \mathbb{C}, \qquad h : D(0, \tfrac{1}{r}) \to \mathbb{C}, \ \textit{with}\ h(0) = 0. \tag{7.25}$$

This decomposition is unique. The function $z \mapsto h(\frac{1}{z})$ *is the* principal part *of the* Laurent decomposition (7.24).

We need a technical lemma as a preparation for the proof of the Laurent decomposition.

Lemma 7.14. *Let* $0 \le r < \rho < P < R \le \infty$ *and assume that* $G : \mathscr{A}_r^R \to \mathbb{C}$ *is holomorphic. Then one has*

$$\int_{\partial D(0,\rho)} G(z)\mathrm{d}z = \int_{\partial D(0,P)} G(z)\mathrm{d}z. \tag{7.26}$$

Proof. Consider the notation as in the following picture.

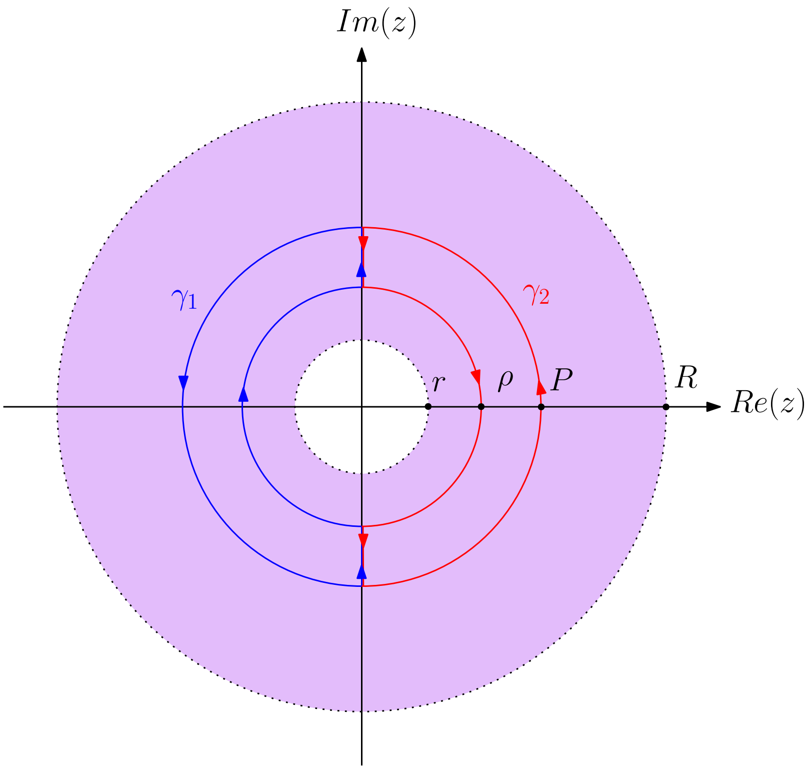

Figure 7.1.: The curves γ_1 and γ_2 contained in the annulus $\mathscr{A}_r^R$.

We see that γ_1 and γ_2 are are both closed and contained in simply connected domains $\mathscr{A}_r^R \setminus [0,\infty)$ and $\mathscr{A}_r^R \setminus (-\infty, 0]$ respectively. Thus,

$$\int_{\gamma_1} G(z)\mathrm{d}z + \int_{\gamma_2} G(z)\mathrm{d}z = 0, \tag{7.27}$$

which implies the result. $\square$

Proof of Theorem 7.13. We first show the uniqueness. Assume that

$$f(z) = g(z) + h(\tfrac{1}{z}) = \widetilde{g}(z) + \widetilde{h}(\tfrac{1}{z}), \qquad \text{for all } z \in \mathscr{A}_r^R, \tag{7.28}$$

with g, h or $\widetilde{g}, \widetilde{h}$ as required in the theorem. We set $G = g - \widetilde{g}$ and $H = h - \widetilde{h}$, so $G(z) = -H(\frac{1}{z})$ for all $z \in \mathscr{A}_r^R$. We define the function $F : \mathbb{C} \to \mathbb{C}$ by

$$F(z) = \begin{cases} G(z), & |z| < R, \\ -H(\frac{1}{z}), & |z| > r, \end{cases} \tag{7.29}$$

which is well defined and entire. Let $\rho \in (r, R)$, then on the compact set $\overline{D(0,\rho)}$, the continuous function $F(z) = G(z)$ is bounded. The continuous function H is also bounded on the compact set $\overline{D(0,\frac{1}{\rho})}$, therefore $F(z) = -H(\frac{1}{z})$ is bounded on $|z| \le \rho$. In total, F is bounded and entire, and must be constant by Liouville's Theorem 4.36. Furthermore,

$$\lim_{z\to\infty} |F(z)| = \lim_{z\to\infty} \left|H(\tfrac{1}{z})\right| = 0, \tag{7.30}$$

since $h(0) = \widetilde{h}(0) = 0$ and $H = h - \widetilde{h}$. It follows that F is identically equal to zero on $\mathbb{C}$, and so G and H must be both identically equal to zero.

We turn to the existence of the Laurent decomposition. By the uniqueness property, it suffices to show the existence on every annulus $\mathscr{A}_\rho^P$ (for $r < \rho < P < R$). Fix $z \in \mathscr{A}_\rho^P$. We define the function $G : \mathscr{A}_r^R \to \mathbb{C}$ by setting

$$G(w) = \begin{cases} \frac{f(w)-f(z)}{w-z}, & w \neq z, \\ f'(z), & w = z, \end{cases} \tag{7.31}$$

which is continuous on $\mathscr{A}_r^R$ and holomorphic on $\mathscr{A}_r^R \setminus \{z\}$. By the Riemann removability condition 7.4, we see that G is in fact holomorphic on $\mathscr{A}_\rho^P$, and we find that (using Lemma 7.14)

$$\begin{aligned} \int_{\partial D(0,P)} \frac{f(w)}{w-z}\mathrm{d}w - f(z)\int_{\partial D(0,P)} \frac{\mathrm{d}w}{w-z} &= \int_{\partial D(0,P)} G(w)\mathrm{d}w = \int_{\partial D(0,\rho)} G(w)\mathrm{d}w \\ &= \int_{\partial D(0,\rho)} \frac{f(w)}{w-z}\mathrm{d}w - f(z)\int_{\partial D(0,\rho)} \frac{\mathrm{d}w}{w-z}. \end{aligned} \tag{7.32}$$

Note that $w \mapsto \frac{1}{w-z}$ is holomorphic for $w \neq z$, and since $|z| > \rho$ we find that $\int_{\partial D(0,\rho)} \frac{\mathrm{d}w}{w-z} = 0$ by Cauchy's inegral theorem. Moreover, by Cauchy's integral formula we have (since $|z| < P$) that $\int_{\partial D(0,P)} \frac{\mathrm{d}w}{w-z} = 2\pi i$. Inserting this into (7.32) yields

$$\begin{aligned} f(z) &= \frac{1}{2\pi i}\left(\int_{\partial D(0,P)} \frac{f(w)}{w-z}\mathrm{d}w - \int_{\partial D(0,\rho)} \frac{f(w)}{w-z}\mathrm{d}w\right) = g(z) + h(\tfrac{1}{z}), \\ g(z) &= \frac{1}{2\pi i}\int_{\partial D(0,P)} \frac{f(w)}{w-z}\mathrm{d}w, \qquad \text{for } |z| < P, \\ h(z) &= -\frac{1}{2\pi i}\int_{\partial D(0,\rho)} \frac{f(w)}{w-\frac{1}{z}}\mathrm{d}w, \qquad \text{for } 0 < |z| < \frac{1}{\rho}. \end{aligned} \tag{7.33}$$

As in the proof of the generalized Cauchy integral formula 4.34 one can see that g can be extended to a holomorphic function $D(0,P)$, whereas h can be extended to a holomorphic function on $\dot{D}(0,\frac{1}{\rho})$. We show that h can be extended to a holomorphic function by $h(0)=0$. Indeed for $|w|=\rho$ and $|z|<\frac{1}{\rho}$

$$|w-\tfrac{1}{z}| \geq ||\tfrac{1}{z}|-|w|| = |\tfrac{1}{z}-\rho| > 0. \tag{7.34}$$

Using the standard bound for integrals we have

$$|h(z)| \leq \frac{1}{2\pi}\frac{\max_{|w|=\rho}|f(w)|}{|\frac{1}{z}|-\rho}\cdot 2\pi\rho \to 0, \qquad \text{as } z\to 0. \tag{7.35}$$

In particular, h is bounded for z near 0, and by the Riemann removability condition 7.4, there is a removable singularity in $z=0$, and by continuity $h(0)=0$. □

From this theorem we can now infer the the *Laurent expansion theorem.*

Theorem 7.15. *Let $z_0\in\mathbb{C}$ and $0\leq r<R\leq\infty$ and $f:\mathscr{A}_r^R(z_0)\to\mathbb{C}$ holomorphic. Then f has a unique* Laurent expansion

$$f(z)=\sum_{n=0}^{\infty}a_n(z-z_0)^n+\sum_{n=1}^{\infty}\frac{b_n}{(z-z_0)^n}, \tag{7.36}$$

and both series on the right hand side of the equation converge absolutely on $\mathscr{A}_r^R(z_0)$ and uniformly absolutely on every compact subset of $\mathscr{A}_r^R(z_0)$. Moreover, one has the representation formulas for $\rho\in(r,R)$:

$$\begin{aligned} a_n &= \frac{1}{2\pi i}\int_{\partial D(z_0,\rho)}\frac{f(w)}{(w-z_0)^{n+1}}\mathrm{d}w, \qquad n\in\mathbb{N}_0,\\ b_n &= \frac{1}{2\pi i}\int_{\partial D(z_0,\rho)} f(w)(w-z_0)^{n-1}\mathrm{d}w, \qquad n\in\mathbb{N}. \end{aligned} \tag{7.37}$$

Proof. We may assume without loss of generality that $z_0=0$. By the Laurent decomposition theorem 7.13, we have $f(z)=g(z)+h(\frac{1}{z})$ and we see let

$$\begin{aligned} g(z) &= \sum_{n=0}^{\infty}a_nz^n, \qquad \text{for all } z\in D(0,R),\\ h(z) &= \sum_{n=1}^{\infty}b_nz^n, \qquad \text{for all } z\in D(0,\tfrac{1}{r}) \end{aligned} \tag{7.38}$$

be the Taylor series of g and h (note that $h(0)=0$). The uniqueness then follows from the uniqueness of the Laurent decomposition. The claims on convergence follow from Theorem 6.10, (i). We show the representation formula (7.37). Let $n\in\mathbb{N}_0$. Note that by (6.33), we see that

$$a_n=\frac{1}{2\pi i}\int_{\partial D(0,\rho)}\frac{g(w)}{w^{n+1}}\mathrm{d}w, \qquad \rho\in(0,R). \tag{7.39}$$

Now let $\rho > r$. Note that $w \mapsto \frac{1}{w}$ maps $\partial D(0,\rho)$ to $D(0,\frac{1}{\rho})$ and reverts the direction of the parametrization of this circle, so

$$\int_{\partial D(0,\rho)} \frac{h(\frac{1}{w})}{w^{n+1}}\mathrm{d}w = -\int_{\partial D(0,\frac{1}{\rho})} \frac{h(\zeta)}{(\frac{1}{\zeta})^{n+1}}\mathrm{d}\left(\frac{1}{\zeta}\right) = \int_{\partial D(0,\frac{1}{\rho})} h(\zeta)\zeta^{n-1}\mathrm{d}\zeta. \tag{7.40}$$

The last integral is zero by the Cauchy integral theorem 4.23, since $\frac{1}{\rho} < \frac{1}{r}$ and $\zeta \mapsto h(\zeta)\zeta^{n-1}$ is holomorphic for $n \geq 0$.[1] In total, we obtain the formula (6.33) for a_n. For b_n we can argue analogously. □

Example 7.16. Consider the function

$$f : \mathbb{C} \setminus \{1,3\} \to \mathbb{C}, \qquad z \mapsto \frac{2}{z^2 - 4z + 3}. \tag{7.41}$$

We want to calculate its Laurent series in the annulus $\mathscr{A}_1^3$. For this, we write

$$f(z) = \frac{1}{1-z} + \frac{1}{z-3}, \tag{7.42}$$

and see that for $|z| > 1$ one has

$$\frac{1}{1-z} = -\frac{1}{z}\frac{1}{1-\frac{1}{z}} = -\frac{1}{z}\sum_{n=0}^{\infty}\frac{1}{z^n} = -\sum_{n=0}^{\infty}\frac{1}{z^{n+1}}, \tag{7.43}$$

whereas for $|z| < 3$ one has

$$\frac{1}{z-3} = -\frac{1}{3}\frac{1}{1-\frac{z}{3}} = -\frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{z}{3}\right)^n = -\sum_{n=0}^{\infty}\frac{z^n}{3^{n+1}}. \tag{7.44}$$

Thus the Laurent series of f in $\mathscr{A}_1^3$ is given by

$$f(z) = \underbrace{-\sum_{n=0}^{\infty}\frac{z^n}{3^{n+1}}}_{=g(z)} \underbrace{-\sum_{n=1}^{\infty}\frac{1}{z^n}}_{=h\left(\frac{1}{z}\right)}. \tag{7.45}$$

The following Theorem gives a characterization of (isolated) singularities in terms of Laurent series.

Theorem 7.17. *Let $A \subseteq \mathbb{C}$ be open, $f : A \to \mathbb{C}$ holomorphic and z_0 a singularity of f. Let $R > 0$ be such that $\mathscr{A}_0^R(z_0) = \dot{D}(z_0,R) \subseteq A$, and consider the Laurent series (7.36) of f on $\dot{D}(z_0,R)$, which exists by Theorem 7.15. In this situation, one has*

(i) z_0 is removable if and only if $b_n = 0$ for all $n \geq 1$.

[1] For $n = 0$, use that $h(0) = 0$ and use the Riemann removability condition 7.4.

(ii) z_0 *is a pole of order* $m \geq 1$ *if and only if* $b_n = 0$ *for all* $n > m$ *and* $b_m \neq 0$.

(iii) z_0 *is essential if and only if* $b_n \neq 0$ *for infinitely many* $n \geq 1$.

Proof. See ⇝ *Exercises*. □

We close this section with an important definition.

Definition 7.18. Let $A \subseteq \mathbb{C}$ be open, $f : A \to \mathbb{C}$ holomorphic and z_0 an isolated singularity of f. Let $R > 0$ with $\dot{D}(z_0, R) \subseteq A$. The number b_1 in (7.36) is called the *residue* of f in z_0. We write

$$b_1 = \operatorname{Res}(f; z_0). \tag{7.46}$$

8. The Residue theorem

(Reference: Marsden-Hoffman, Basic complex analysis, 3rd Ed., Sections 4.1-4.4, 6.2)

Recall that if $A \subseteq \mathbb{C}$ is a simply connected domain and $f : A \to \mathbb{C}$, Cauchy's integral theorem 4.31 yields that

$$\int_\gamma f(z)\mathrm{d}z = 0 \tag{8.1}$$

for every closed, piecewise C^1 curve $\gamma : [a,b] \to \mathbb{C}$, $a < b$ with $\gamma([a,b]) \subseteq A$. The *residue theorem* allows us calculate integrals a wider class of integrals: Namely, if $f : A \setminus \{z_1, \ldots, z_k\} \to \mathbb{C}$ is holomorphic with $z_1, \ldots, z_k \in A$ pairwise distinct, we will show that for any closed, piecewise C^1 curve $\gamma : [a,b] \to \mathbb{C}$, $a < b$ with $\gamma([a,b]) \subseteq A \setminus \{z_1, \ldots, z_k\}$, one has

$$\int_\gamma f(z)\mathrm{d}z = 2\pi i \sum_{j=1}^{k} I(\gamma; z_j)\mathrm{Res}(f; z_j). \tag{8.2}$$

Here $I(\gamma; z_j)$ is the *index* or *winding number* of γ with respect to z_j (which we will introduce below) and $\mathrm{Res}(f; z_j)$ is the *residue* of f in z_j.

8.1. Calculation of residues

Throughout this section, let $A \subseteq \mathbb{C}$ open, $f : A \to \mathbb{C}$ holomorphic and z_0 a singularity of f. Recall the Definition 7.18 of $\mathrm{Res}(f; z_j)$ in the previous section. We give some rules to effectively calculate the residue.

In some cases, the Laurent expansion of a function is explicit, and we can read off the residue explicitly.

Example 8.1.
(i) $\mathrm{Res}\left(\frac{\cos(z)}{z}; 0\right) = 1$, since $\frac{\cos(z)}{z} = \frac{1}{z} - \frac{z}{2!} \pm \ldots$ for $z \in \mathbb{C} \setminus \{0\}$.

(ii) $\mathrm{Res}\left(\exp\left(\frac{1}{z}\right); 0\right) = 1$, since $\exp\left(\frac{1}{z}\right) = \sum_{n=0}^{\infty} (n! z^n)^{-1}$ for $z \in \mathbb{C} \setminus \{0\}$.

(iii) Analogously, one has $\mathrm{Res}\left(\exp\left(\frac{1}{z^k}\right); 0\right) = 0$ for $k \in \mathbb{N} \setminus \{1\}$.

End of Lecture 11

Lemma 8.2. *Let $R > 0$ be such that $\dot{D}(z_0, R) \subseteq A$.*

(i) For $\rho \in (0, R)$, one has

$$b_1 = \frac{1}{2\pi i} \int_{\partial D(z_0, \rho)} f(z)\mathrm{d}z. \tag{8.3}$$

(ii) If z_0 is removable, then $\mathrm{Res}(f; z_0) = 0$.

Proof. Part (i) is just (7.37) for $n = 1$. Part (ii) follows immediately from Theorem 7.17, (i). □

The next lemma gives a direct method to calculate the residue of a function in a singularity z_0 that can be written as the quotient of a holomorphic function and a polynomial with a zero in z_0.

Lemma 8.3. *Suppose we can write $f : \dot{D}(z_0, R) \to \mathbb{C}$ as*

$$f(z) = \frac{g(z)}{(z - z_0)^m}, \qquad z \in \dot{D}(z_0, R), \tag{8.4}$$

where $g : D(z_0, R) \to \mathbb{C}$ is holomorphic and fulfills $g(z_0) \neq 0$, and $m \in \mathbb{N}$. Then

$$\operatorname{Res}(f; z_0) = \frac{g^{(m-1)}(z_0)}{(m-1)!}. \tag{8.5}$$

In particular, for a simple pole (where $m = 1$), one has

$$\operatorname{Res}(f; z_0) = \lim_{z \to z_0} (z - z_0) f(z). \tag{8.6}$$

Proof. Since g is holomorphic on $D(z_0, R)$, we can apply Taylor's Theorem 6.13 and obtain

$$g(z) = \sum_{n=0}^{\infty} \frac{g^{(n)}(z_0)}{n!} (z - z_0)^n, \qquad z \in D(z_0, R), \tag{8.7}$$

so we obtain for $z \neq z_0$ the Laurent series for f

$$f(z) = \frac{g(z)}{(z - z_0)^m} = \sum_{n=0}^{\infty} \frac{g^{(n)}(z_0)}{n!} (z - z_0)^{n-m}. \tag{8.8}$$

We can read off the residue as the term corresponding to $n = m - 1$:

$$\operatorname{Res}(f; z_0) = \frac{g^{(m-1)}(z_0)}{(m-1)!}. \tag{8.9}$$

For $m = 1$ we have (using the continuity of g):

$$\operatorname{Res}(f; z_0) = g(z_0) = \lim_{z \to z_0} g(z) = \lim_{z \to z_0} (z - z_0) f(z). \tag{8.10}$$

□

Example 8.4. Consider $f : \mathbb{C} \setminus \{\pm i\} \to \mathbb{C}$, $f(z) = \frac{\exp(iz)}{z^2+1}$, which has simple poles at $z = \pm i$. We see that

$$\operatorname{Res}(f; i) = \lim_{z \to i} (z - i) f(z) = \lim_{z \to i} \frac{\exp(iz)}{z + i} = -\frac{i}{2e}, \tag{8.11}$$

$$\operatorname{Res}(f; -i) = \lim_{z \to -i} (z + i) f(z) = \lim_{z \to -i} \frac{\exp(iz)}{z - i} = \frac{ei}{2}. \tag{8.12}$$

We immediately obtain the following corollary, which gives the residue for a more general class of functions.

Corollary 8.5. *Let g and h be holomorphic functions in $D(z_0, R)$ with $g(z_0) \neq 0$, $h(z_0) = 0$ and $h'(z_0) \neq 0$. Then $f = \frac{g}{h}$ has a residue in z_0 given by*

$$\operatorname{Res}(f; z_0) = \frac{g(z_0)}{h'(z_0)}. \tag{8.13}$$

Proof. Note that f has a simple pole in z_0. Indeed,

$$\lim_{z \to z_0} \frac{h(z) - h(z_0)}{z - z_0} = \lim_{z \to z_0} \frac{h(z)}{z - z_0} = h'(z_0) \neq 0, \tag{8.14}$$

so the function $z \mapsto \begin{cases} \frac{z - z_0}{h(z)}, & z \neq z_0 \\ \frac{1}{h'(z_0)}, & z = z_0, \end{cases}$ is holomorphic in $D(z_0, r)$ for some $r \in (0, R)$ and we can apply the previous proposition to obtain

$$\operatorname{Res}\left(\frac{g}{h}; z_0\right) = \lim_{z \to z_0} (z - z_0) \frac{g(z)}{h(z)} = \frac{g(z_0)}{h'(z_0)}. \tag{8.15}$$

□

The situation is more intricate if h has a pole of higher order. Here we state

Proposition 8.6. *Let g and h be holomorphic in $D(z_0, R)$ with $g(z_0) \neq 0$, $h(z_0) = h'(z_0) = 0$ and $h''(z_0) \neq 0$. Then $f = \frac{g}{h}$ has a residue in z_0 given by*

$$\operatorname{Res}(f; z_0) = 2\frac{g'(z_0)}{h''(z_0)} - \frac{2}{3} \frac{g(z_0) h'''(z_0)}{(h''(z_0))^2}. \tag{8.16}$$

We give the proof as an ⇝ *Exercise*. More general formulas for obtaining the residue can be found in *Marsden-Hoffman, Basic complex analysis, 3rd Ed., Section 4.1.*

8.2. Winding number

Definition 8.7. Let $\gamma : [a, b] \to \mathbb{C}$ (with $a < b$) be a closed, piecewise C^1 curve and $z \in \mathbb{C} \setminus \gamma([a, b])$. The number

$$I(\gamma; z) = \frac{1}{2\pi i} \int_\gamma \frac{\mathrm{d}w}{w - z} \tag{8.17}$$

is called the *index* or *winding number* of γ with respect to z. We furthermore define

$$\begin{aligned} \operatorname{Int}(\gamma) &= \{z \in \mathbb{C} \setminus \gamma([a, b])\,;\, I(\gamma; z) \neq 0\}, \text{ the } \textit{interior} \text{ of } \gamma, \\ \operatorname{Ext}(\gamma) &= \{z \in \mathbb{C} \setminus \gamma([a, b])\,;\, I(\gamma; z) = 0\}, \text{ the } \textit{exterior} \text{ of } \gamma. \end{aligned} \tag{8.18}$$

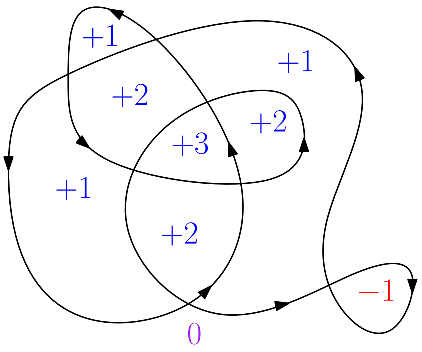

Figure 8.1.: A curve γ with respective winding numbers.

With Cauchy's integral formula, one can see that for $k \in \mathbb{Z} \setminus \{0\}$, and $z_0 \in \mathbb{C}$ for

$$\gamma(t) = z_0 + r\exp(2\pi ikt), \qquad t \in [0,1], \tag{8.19}$$

the winding number of any z with $|z - z_0| \neq r$ is

$$I(\gamma; z) = \begin{cases} 0, & |z - z_0| > r, \\ k, & |z - z_0| < r. \end{cases} \tag{8.20}$$

So at least in the case of a circle, the winding number describes how often the curve γ travels around z in a counterclockwise fashion. This intuition is correct in a more general sense. We will show that the index is an integer that is constant on the connected components of $\mathbb{C} \setminus \gamma([a,b])$.

Proposition 8.8. *Let $\gamma : [a,b] \to \mathbb{C}$, $a < b$, be a closed, piecewise C^1 curve and $z_0 \in \mathbb{C} \setminus \gamma([a,b])$. Then $I(\gamma; z_0) \in \mathbb{Z}$.*

Proof. We define

$$g(t) = \int_a^t \frac{\gamma'(s)}{\gamma(s) - z_0} \mathrm{d}s. \tag{8.21}$$

At all points t where the integrand is continuous, we have by the fundamental theorem of calculus that

$$g'(t) = \frac{\gamma'(t)}{\gamma(t) - z_0} \qquad \Rightarrow \qquad \frac{\mathrm{d}}{\mathrm{d}t}\big\{\exp(-g(t))(\gamma(t) - z_0)\big\} = 0. \tag{8.22}$$

Therefore the function $t \mapsto \exp(-g(t))(\gamma(t) - z_0)$ must be piecewise constant and by continuity, it is constant. This means that

$$\exp(-g(b))(\gamma(b) - z_0) = \exp(-g(a))(\gamma(a) - z_0), \tag{8.23}$$

which in turn implies that $\exp(-g(b)) = \exp(-g(a))$ (since $\gamma(b) = \gamma(a)$), and since $g(a) = 0$, we see that

$$\exp(-g(b)) = 1 \qquad \Rightarrow \qquad g(b) \in 2\pi i\mathbb{Z}. \tag{8.24}$$

□

Proposition 8.9. *Let $\gamma : [a,b] \to \mathbb{C}$, $a < b$, be a closed, piecewise C^1 curve and assume $z_0, z_1 \in \mathbb{C} \setminus \gamma([a,b])$ can be connected by a continuous path, i.e. there exists a $\varphi : [0,1] \to \mathbb{C} \setminus \gamma([a,b])$ with $\varphi(0) = z_0$ and $\varphi(1) = z_1$. Then*

$$I(\gamma; z_0) = I(\gamma; z_1). \tag{8.25}$$

Proof. The set $\mathbb{C} \setminus \gamma([a,b])$ is open. Consider the set[1]

$$C(z_0) = \{z \in \mathbb{C} \setminus \gamma([a,b]) \,;\, z \text{ can be connected to } z_0 \text{ by a continuous path in } \mathbb{C} \setminus \gamma([a,b])\}. \tag{8.26}$$

By construction, $C(z_0)$ is path-connected, and therefore also connected. The map

$$I(\gamma; \cdot) : \begin{cases} \mathbb{C} \setminus \gamma([a,b]) \to \mathbb{C}, \\ z \mapsto I(\gamma; z) \end{cases} \tag{8.27}$$

is holomorphic, and thus also continuous. Therefore also $I(\gamma; C(z_0))$ is connected, but $I(\gamma; C(z_0)) \subseteq \mathbb{Z}$ consists of isolated points. Therefore, $I(\gamma; C(z_0))$ can only consist of a single point, and therefore $I(\gamma; z_0) = I(\gamma; z_1)$. ☐

8.3. The residue theorem and its proof

Theorem 8.10. *Let $A \subseteq \mathbb{C}$ be a simply connected domain and $z_1, z_2, \ldots, z_k$ finitely many, pairwise distinct points in A. Let $f : A \setminus \{z_1, \ldots, z_k\} \to \mathbb{C}$ be holomorphic and $\gamma : [a,b] \to \mathbb{C}$, $a < b$ with $\gamma([a,b]) \subseteq A \setminus \{z_1, \ldots, z_k\}$ a closed, piecewise C^1 curve. Then one has*

$$\int_\gamma f(z)\mathrm{d}z = 2\pi i \sum_{j=1}^{k} I(\gamma; z_j)\mathrm{Res}(f; z_j). \tag{8.28}$$

Proof. By definition, all z_j, $j = 1, \ldots, k$ are (isolated) singularities of f. For some $R_j > 0$, we therefore have a Laurent expansion

$$f(z) = \sum_{n=1}^{\infty} \frac{b_{n,j}}{(z - z_j)^n} + \sum_{n=0}^{\infty} a_{n,j}(z - z_j)^n, \qquad z \in \dot{D}(z_j, R_j). \tag{8.29}$$

Note that the function

$$\mathbb{C} \setminus \{z_j\} \to \mathbb{C}, \qquad z \mapsto h_j\left(\frac{1}{z - z_j}\right) = \sum_{n=1}^{\infty} \frac{b_{n,j}}{(z - z_j)^n} \tag{8.30}$$

is holomorphic by Theorem 7.24. We can therefore set

$$g : A \setminus \{z_1, \ldots, z_k\} \to \mathbb{C},\ z \mapsto f(z) - \sum_{j=1}^{k} h_j\left(\frac{1}{z - z_j}\right) = f(z) - \sum_{j=1}^{k}\sum_{n=1}^{\infty} \frac{b_{n,j}}{(z - z_j)^n}, \tag{8.31}$$

[1]This is the *connected component* of z_0 in $\mathbb{C} \setminus \gamma([a,b])$.

which is holomorphic and has (isolated) singularities in $z_1, \ldots, z_k$. For every $j = 1, \ldots, k$, the Laurent expansion of g in $\dot{D}(z_j, R_j)$ has a vanishing principal part[2], so by Theorem 7.17, (i), all singularities are removable. By removing the singularities one by one, we obtain a holomorphic function $\widetilde{g} : A \to \mathbb{C}$ with $\widetilde{g}|_{A\setminus\{z_1,\ldots,z_k\}} = g$. Since A is simply connected, we use Cauchy's integral theorem 4.31 for simply connected domains and see that

$$\begin{aligned} 0 &= \int_\gamma g(z)\mathrm{d}z \\ &= \int_\gamma \left(f(z) - \sum_{j=1}^k h_j\left(\frac{1}{z-z_j}\right)\right)\mathrm{d}z \\ &= \int_\gamma f(z)\mathrm{d}z - \sum_{j=1}^k \int_\gamma h_j\left(\frac{1}{z-z_j}\right)\mathrm{d}z \\ &= \int_\gamma f(z)\mathrm{d}z - \sum_{j=1}^k \int_\gamma \left(\sum_{n=1}^\infty \frac{b_{n,j}}{(z-z_j)^n}\right)\mathrm{d}z. \end{aligned} \tag{8.32}$$

The Laurent series in the above expression converge uniformly on every compact set contained in $A \setminus \{z_1, \ldots, z_k\}$ in view of Theorem 7.15. We can exchange summation and integration by Proposition 6.6. This gives

$$\begin{aligned} 0 &= \int_\gamma f(z)\mathrm{d}z - \sum_{j=1}^k \sum_{n=1}^\infty b_{n,j} \int_\gamma \frac{1}{(z-z_j)^n}\mathrm{d}z \\ &= \int_\gamma f(z)\mathrm{d}z - \sum_{j=1}^k b_{1,j} \int_\gamma \frac{1}{z-z_j}\mathrm{d}z \\ &= \int_\gamma f(z)\mathrm{d}z - \sum_{j=1}^k \mathrm{Res}(f; z_j)\cdot(2\pi i I(\gamma; z_j)), \end{aligned} \tag{8.33}$$

and we used that the functions $z \mapsto \frac{1}{(z-z_j)^n}$ have a primitive $-\frac{1}{n-1}\frac{1}{(z-z_j)^{n-1}}$ on $\mathbb{C}\setminus\{z_j\}$ for $n \geq 2$ for the second equality. □

8.4. Applications of the residue theorem

Definition 8.11. Let $A \subseteq \mathbb{C}$ be open, $f : A \to \mathbb{C}$ holomorphic and let z_0 be a pole or removable singularity such that f is not constant to zero on some $\dot{D}(z_0, R)$, $R > 0$. Consider the Laurent expansion

$$f(z) = \sum_{n=0}^\infty a_n(z-z_0)^n + \sum_{n=1}^\infty \frac{b_n}{(z-z_0)^n}, \qquad z \in \dot{D}(z_0, R). \tag{8.34}$$

[2] We use that the Laurent coefficients are additive, which follows for instance from (7.37).

We define the *zero order*

$$0\text{-ord}(f;z_0)=\begin{cases}\min\{n\in\mathbb{N}_0\,;\,a_n\neq 0\}, & \text{if there is no } m\geq 1 \text{ with } b_m\neq 0,\\ 0, & \text{if there exists an } m\geq 1 \text{ with } b_m\neq 0\end{cases}\tag{8.35}$$

and the *pole order*

$$\infty\text{-ord}(f;z_0)=\begin{cases}0, & \text{if there is no } m\geq 1 \text{ with } b_m\neq 0,\\ \max\{n\in\mathbb{N}\,;\,b_n\neq 0\}, & \text{if there exists an } m\geq 1 \text{ with } b_m\neq 0.\end{cases}\tag{8.36}$$

Remark 8.12. The notion of the pole order above generalizes the one given in Definition 7.5, see also Theorem 7.17, (ii).

We now prove the *root-pole counting theorem.*

Theorem 8.13. *Let $A\subseteq\mathbb{C}$ a simply connected domain and $f:A\setminus\{z_1,\dots,z_k\}\to\mathbb{C}$ holomorphic and non-zero on $A\setminus\{z_1,\dots,z_k\}$, with pairwise distinct zeros*[3] *or poles $z_1,\dots,z_k\in A$. Suppose that $\gamma:[a,b]\to\mathbb{C}$, $a<b$ is a closed piecewise C^1 curve with $\gamma([a,b])\subseteq A\setminus\{z_1,\dots,z_k\}$. Then one has the* root-pole counting formula

$$\frac{1}{2\pi i}\int_\gamma\frac{f'(z)}{f(z)}\mathrm{d}z=\sum_{j=1}^k I(\gamma;z_j)\big(0\text{-ord}(f;z_j)-\infty\text{-ord}(f;z_j)\big).\tag{8.37}$$

Proof. The function $z\mapsto\frac{f'(z)}{f(z)}$ is holomorphic on $A\setminus\{z_1,\dots,z_k\}$, and using the residue theorem 8.10 yields

$$\frac{1}{2\pi i}\int_\gamma\frac{f'(z)}{f(z)}\mathrm{d}z=\sum_{j=1}^k I(\gamma;z_j)\mathrm{Res}\left(\frac{f'}{f};z_j\right).\tag{8.38}$$

We show: If z_0 is a zero or pole of f, then

$$\mathrm{Res}\left(\frac{f'}{f};z_0\right)=0\text{-ord}(f;z_0)-\infty\text{-ord}(f;z_0)=:m\in\mathbb{Z}.\tag{8.39}$$

In both cases, there exists $R>0$ and $g:D(z_0,R)\to\mathbb{C}$ holomorphic with $g(z_0)\neq 0$ such that

$$f(z)=(z-z_0)^m g(z),\qquad z\in\dot{D}(z_0,R).\tag{8.40}$$

Indeed, if z_0 is a pole this follows by definition, and if z_0 is a zero, we use Taylor's theorem for the continuation of f to obtain such a g. For $z\in\dot{D}(z_0,R)$ we find that

$$f'(z)=m(z-z_0)^{m-1}g(z)+(z-z_0)^m g'(z),\tag{8.41}$$

by the product rule, and therefore

$$\frac{f'(z)}{f(z)}=\frac{m}{z-z_0}+\frac{g'(z)}{g(z)},\qquad z\in\dot{D}(z_0,R).\tag{8.42}$$

[3] Understood as removable singularities in which the holomorphic continuation attains the value zero.

Since g is holomorphic on $D(z_0, R)$ and $g(z_0) \neq 0$, $\frac{g'}{g}$ is holomorphic on some disk $D(z_0, \delta)$ ($\delta \in (0, R)$). It follows that

$$\operatorname{Res}\left(\frac{f'}{f}; z_0\right) = \lim_{z \to z_0} (z - z_0) \frac{f'(z)}{f(z)} = m. \tag{8.43}$$

□

The following result, called the *argument principle* is immediate.

Corollary 8.14. *Under the same hypotheses as Theorem 8.13 one has*

$$\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)} \mathrm{d}z = N(0) - N(\infty), \tag{8.44}$$

where

$$N(0) = \sum_{j=1}^{k} 0\text{-ord}(f; z_j), \qquad N(\infty) = \sum_{j=1}^{k} \infty\text{-ord}(f; z_j), \tag{8.45}$$

if every zero or pole has winding number 1 *for* γ.

Remark 8.15. The name "argument principle" for the previous corollary comes with the following observation: Let $\gamma : [a, b] \to \mathbb{C}$, $a < b$ a closed, piecewise C^1 curve with $\gamma([a, b]) \subseteq A \setminus \{z_1, ..., z_k\}$, $f : A \setminus \{z_1, ..., z_k\} \to \mathbb{C}$, and all $z_1, ..., z_k$ poles or zeros, and f has no zeros on $A \setminus \{z_1, ..., z_k\}$. We consider $w_0 = f(\gamma(a))$ and ask how much the argument of $w_t = f(\gamma(t))$ is changed when moving continuously along the curve $[a, b] \to \mathbb{C}$, $t \mapsto f(\gamma(t))$, that is we are interested in

$$\Delta_\gamma \arg(f) = 2\pi I(f \circ \gamma; 0). \tag{8.46}$$

Note that the latter is well-defined since f does not have zeros on $\gamma([a, b])$. Without loss of generality, assume that γ is C^1. We note that

$$\begin{aligned} i\Delta_\gamma \arg(f) = 2\pi i I(f \circ \gamma; 0) &= \int_{f \circ \gamma} \frac{\mathrm{d}z}{z} \\ &= \int_a^b \frac{f'(\gamma(t))}{f(\gamma(t))} \gamma'(t) \mathrm{d}t \\ &= \int_\gamma \frac{f'(z)}{f(z)} \mathrm{d}z. \end{aligned} \tag{8.47}$$

Combined with the argument principle (8.44), we see that if every zero or pole of f has winding number 1 for γ, one can compute

$$\Delta_\gamma \arg(f) = 2\pi (N(0) - N(\infty)). \tag{8.48}$$

We will now prove *Rouché's Theorem.*

Theorem 8.16. *Let $A \subseteq \mathbb{C}$ be a simply connected domain, $f, g : A \to \mathbb{C}$ holomorphic and let $\gamma : [a, b] \to \mathbb{C}$, $a < b$ a closed, piecewise C^1 curve with $\gamma([a, b]) \subseteq A$ which surrounds every point in its interior exactly once[4]. Assume that f and g have finitely many zeros[5] in A and*

$$|f(z) - g(z)| < |f(z)|, \qquad z \in \gamma([a, b]). \tag{8.49}$$

Then f and g have no zeros on $\gamma([a, b])$ and have the same number of zeros in $\mathrm{Int}(\gamma)$*, counting multiplicities.*

Proof. Let $\lambda \in [0, 1]$. We consider the interpolation $h_\lambda = f + \lambda(g - f) : A \to \mathbb{C}$ between f and g. Note that for $z \in \gamma([a, b])$, we have

$$\begin{aligned} &|h_\lambda(z)| \geq |f(z)| - \lambda|g(z) - f(z)| > |f(z)|(1 - \lambda) \geq 0, \ \lambda \neq 0, \\ &|h_0(z)| = |f(z)| > |f(z) - g(z)| \geq 0. \end{aligned} \tag{8.50}$$

So h_λ has no zeros on $\gamma([a, b])$. By the root-pole counting formula, we see that

$$\frac{1}{2\pi i} \int_\gamma \frac{h_\lambda'(z)}{h_\lambda(z)} \mathrm{d}z = \frac{1}{2\pi i} \int_\gamma \frac{f'(z) - \lambda(g'(z) - f'(z))}{f(z) - \lambda(g(z) - f(z))} \mathrm{d}z = N_\lambda(0), \tag{8.51}$$

where $N_\lambda(0)$ counts the number of zeros of h_λ in $\mathrm{Int}(\gamma)$. By considering the integral expression (8.51), we see that $\lambda \mapsto N_\lambda(0)$ must be continuous on $[0, 1]$, but since $N_\lambda(0) \in \mathbb{Z}$, it is necessarily constant and we must have $N_0(0) = N_1(0)$. □

Rouché's Theorem is useful for locating the zeros of functions by comparison with functions that are simpler to analyze:

Example 8.17. Consider the polynomial $P(z) = z^4 - 6z + 3$. How many zeros does it have in $\mathscr{A}_1^2(0)$?

Take $\gamma_{\partial D(0,2)} : [0, 2\pi] \to \mathbb{C}, t \mapsto 2\exp(it)$, and take $f(z) = z^4$, $g(z) = P(z) = z^4 - 6z + 3$.

$$\text{For } z \in \partial D(0, 2) \ , \ |f(z) - g(z)| = |6z - 3| \leq 15 < 16 = |f(z)|.$$

Thus both f and $g = P$ have 4 zeros inside $D(0, 2)$.

Now consider $\gamma_{\partial D(0,1)} : [0, 2\pi] \to \mathbb{C}, t \mapsto \exp(it)$, and define $\widetilde{f}(z) = -6z$, $g(z) = P(z) = z^4 - 6z + 3$.

$$\text{For } z \in \partial D(0, 1) \ , \ |\widetilde{f}(z) - g(z)| = |z^4 + 3| \leq 4 < 6 = |\widetilde{f}(z)|$$

So both $\widetilde{f}$ and $g = P$ have 1 zero inside $D(0, 1)$. We conclude that P has 3 zeros in the annulus.

End of Lecture 12

We give another application of Rouché's Theorem, which is known as *Hurwitz' Theorem.*

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[4] This means that for all $z \in A \setminus \gamma([a, b])$, $I(\gamma; z) \in \{0, 1\}$.

[5] One can show that this condition is in fact not necessary.

Theorem 8.18. *Let $A \subseteq \mathbb{C}$ be a domain and $(f_n)_{n\in\mathbb{N}}$ a sequence of holomorphic functions $f_n : A \to \mathbb{C}$ such that $f_n \rightrightarrows f$ on every compact $K \subseteq A$ for some $f : A \to \mathbb{C}$. Assume that f is not identically zero, and let $z_0 \in A$. Then $f(z_0) = 0$ if and only if there is a sequence $(z_n)_{n\in\mathbb{N}} \subseteq A$ with $z_n \to z_0$ and $N \in \mathbb{N}$ such that $f_n(z_n) = 0$ for all $n \geq N$.*

Proof. First we note that f is holomorphic on A by the Weierstrass approximation theorem 6.7. Assume that f is not identically equal to zero. We first show that if $\gamma : [a, b] \to \mathbb{C}$, $a < b$ with $\gamma([a, b]) \subseteq B \setminus \{z \in A\,;\, f(z) = 0\}$ (with $B \subseteq A$ is a simply connected subdomain) is a closed curve avoiding all zeros of f, then there exists $N(\gamma) \in \mathbb{N}$ such that every f_n with $n \geq N(\gamma)$ has the same numbers of zeros in $\operatorname{Int}(\gamma)$ as f.

Indeed, since $|f|$ is continuous and non-zero on the compact set $\gamma([a, b])$, we have

$$|f(z)| \geq \min_{w\in\gamma([a,b])} |f(w)| =: m > 0, \qquad \text{for all } z \in \gamma([a, b]). \tag{8.52}$$

Clearly $f_n \rightrightarrows f$ on $\gamma([a, b])$, so there exists an $N(\gamma)$ with

$$|f_n(z) - f(z)| < m \leq |f(z)|, \qquad \text{for all } z \in \gamma([a, b]), n \geq N(\gamma). \tag{8.53}$$

We apply Rouché's Theorem 8.16 to conclude that f_n and f have the same numbers of zeros in $\operatorname{Int}(\gamma)$ for $n \geq N(\gamma)$.

Now suppose that $f(z_0) = 0$. The zeros of f are isolated by Proposition 6.18, meaning that there is a number $\delta > 0$ such that $f(z) \neq 0$ for $z \in \dot{D}(z_0, \delta)$. For each $k \in \mathbb{N}$, consider

$$\gamma_{\partial D(z_0,\frac{\delta}{k})} : [0, 2\pi] \to \mathbb{C}, \qquad t \mapsto z_0 + \frac{\delta}{k}\exp(it). \tag{8.54}$$

Consider $N_k = N(\gamma_{\partial D(z_0,\frac{\delta}{k})})$, then for $n \geq N_k$, f_n must have a zero z_n in $\operatorname{Int}(\gamma_{\partial D(z_0,\frac{\delta}{k})}) = D(z_0, \frac{\delta}{k})$, so $f_n(z_n) = 0$. For $n \geq N_k$, we have $|z_n - z_0| < \frac{\delta}{k}$, which proves the theorem by setting $N = N_1$ and choosing $z_n \in D(z_0, \frac{\delta}{k})$ for $n \geq N_k$.

Conversely, assume that $f_n(z_n) = 0$ for a sequence $(z_n)_{n\in\mathbb{N}} \subseteq A$ with $z_n \to z_0$ as $n \to \infty$. Let $\varepsilon > 0$, then by uniform continuity (note that $(z_n)_{n\in\mathbb{N}} \subseteq \overline{D(z_0, \delta)}$ for some $\delta > 0$) there exists an $N \in \mathbb{N}$ such that

$$|f(z_n) - f_n(z_n)| \leq \frac{\varepsilon}{2}, \qquad n \geq N. \tag{8.55}$$

By continuity, there exists $N' \in \mathbb{N}$ such that

$$|f(z_0) - f(z_n)| \leq \frac{\varepsilon}{2}, \qquad n \geq N', \tag{8.56}$$

therefore

$$|f(z_0) - f_n(z_n)| \leq |f(z_0) - f(z_n)| + |f(z_n) - f_(z_n)| \leq \varepsilon, \qquad n \geq \max\{N, N'\}. \tag{8.57}$$

This shows that $0 = \lim_{n\to\infty} f_n(z_n) = f(z_0)$. □

A special case of Hurwitz' Theorem (sometimes stated under the same name) is the following: Assume that $f_n \rightrightarrows f$ on every compact $K \subseteq A$ as $n \to \infty$, and f_n has no zeros on A. Then either $f \equiv 0$ or f has no zeros on A. The following Corollary is an important application of Hurwitz' Theorem.

Corollary 8.19. *Let $A \subseteq \mathbb{C}$ a domain and $(f_n)_{n\in\mathbb{N}}$ a sequence of injective holomorphic functions $f_n : A \to \mathbb{C}$ such that $f_n \rightrightarrows f$ on every compact $K \subseteq A$ for some $f : A \to \mathbb{C}$. Then f is either constant or injective.*

Proof. Assume that f is not constant and let $z_0 \in A$ be fixed. The functions $z \mapsto f_n(z) - f_n(z_0)$ have no zeros in $A \setminus \{z_0\}$ by injectivity. By Hurwitz' Theorem 8.18, $z \mapsto f(z) - f(z_0)$ is either identically zero on $A \setminus \{z_0\}$ or it has no zeros there. The first case is excluded by assumption, so we see that

$$f(z) \neq f(z_0), \qquad z \in A \setminus \{z_0\}. \tag{8.58}$$

Since $z_0 \in A$ was arbitrary, we obtain the injectivity of f on A. ☐

8.5. Evaluation of real integrals using the residue theorem

We give some examples on how to calculate certain real integrals using the residue theorem. Some of these techniques are also used in the ⇝ *Exercises*.

Rational functions of sine and cosine

An important class of functions that can be treated using residue calculus are rational functions involving sin and cos. The main tool is the observation that for $z = \exp(i\theta)$, $\theta \in [0, 2\pi]$, we can write

$$\begin{aligned} \sin(\theta) &= \frac{\exp(i\theta) - \exp(-i\theta)}{2i} = \frac{z - \frac{1}{z}}{2i}, \\ \cos(\theta) &= \frac{\exp(i\theta) + \exp(-i\theta)}{2} = \frac{z + \frac{1}{z}}{2}. \end{aligned} \tag{8.59}$$

This observation leads to the following result:

Proposition 8.20. *Let $P(X,Y), Q(X,Y) \in \mathbb{C}[X,Y]$ be two polynomials in two variables and consider*

$$R(x,y) = \frac{P(x,y)}{Q(x,y)} \tag{8.60}$$

as the quotient of P and Q. Assume that $Q(x,y) \neq 0$ for all $(x,y) \in \mathbb{R}$ with $x^2 + y^2 = 1$. Then

$$\int_0^{2\pi} R(\cos(\theta), \sin(\theta))\mathrm{d}\theta = 2\pi \sum_{z_0 \in D(0,1)} \operatorname{Res}(f; z_0), \tag{8.61}$$

where f is the rational function

$$f(z) = \frac{1}{z} R\left(\frac{1}{2}\left(z + \frac{1}{z}\right), \frac{1}{2i}\left(z - \frac{1}{z}\right)\right). \tag{8.62}$$

Proof. For $|z| = 1$, we have that $\frac{1}{z} = \overline{z}$ and $\frac{1}{2}(z + \frac{1}{z}) = \mathrm{Re}(z) = x$, $\frac{1}{2i}(z - \frac{1}{z}) = \mathrm{Im}(z) = y$ are real. Since $1 = |z|^2 + x^2 + y^2$, we see that for $|z| = 1$ the function $Q(x, y)$ has no zeros. We can rewrite the integral in (8.61) as

$$\begin{aligned}
&\int_0^{2\pi} R(\cos(\theta), \sin(\theta))\mathrm{d}\theta \\
&\quad = \frac{1}{i}\int_0^{2\pi} R\left(\frac{1}{2}(\exp(i\theta) + \exp(-i\theta)), \frac{1}{2i}(\exp(i\theta) - \exp(-i\theta))\right) i\exp(-i\theta)\exp(i\theta)\mathrm{d}\theta \\
&\quad = \frac{1}{i}\int_{\partial D(0,1)} f(z)\mathrm{d}z.
\end{aligned} \tag{8.63}$$

The function f has only finitely many poles in $\overline{D(0,1)}$, so we can use the residue theorem 8.10 to conclude that

$$\frac{1}{i}\int_{\partial D(0,1)} f(z)\mathrm{d}z = 2\pi \sum_{z_0 \in D(0,1)} \mathrm{Res}(f; z_0), \tag{8.64}$$

where we used that for $z_0 \in D(0, 1)$, we always have $I(\gamma_{\partial D(0,1)}; z_0) = 1$. □

Example 8.21. We show that for $a \in D(0, 1)$, one has

$$\int_0^{2\pi} \frac{1}{1 - 2a\cos(\theta) + a^2}\mathrm{d}\theta = \frac{2\pi}{1 - a^2}. \tag{8.65}$$

This is trivial for $a = 0$. Otherwise, we associate the rational function

$$f(z) = \frac{1}{z(1 + a^2 - az - a/z)} = -\frac{1/a}{(z - a)(z - 1/a)}. \tag{8.66}$$

There is exactly one pole of f in $D(0, 1)$, which is simple and we have

$$\mathrm{Res}(f; a) = \lim_{z\to a}(z - a)f(z) = -\frac{1}{a^2 - 1}, \tag{8.67}$$

and the claim follows by (8.61).

Integrals over the whole real line

We recall the definition of integrability on $\mathbb{R}$, restricting our attention to continuous functions.

Definition 8.22. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. Then f is called *integrable over* $\mathbb{R}$ if the limits

$$\lim_{A\to\infty}\int_0^A f(x)\mathrm{d}x, \qquad \text{and} \qquad \lim_{B\to\infty}\int_{-B}^0 f(x)\mathrm{d}x \tag{8.68}$$

both exist. We then define

$$\int_{-\infty}^{\infty} f(x)\mathrm{d}x = \lim_{A\to\infty}\int_0^A f(x)\mathrm{d}x + \lim_{B\to\infty}\int_{-B}^0 f(x)\mathrm{d}x. \tag{8.69}$$

If $|f|$ is integrable over $\mathbb{R}$, we say that f is *absolutely integrable over* $\mathbb{R}$. It is easy to see that absolute integrability implies integrability.

Proposition 8.23. *Let $P(X) = a_n X^n + \ldots + a_0$, $Q(X) = b_m X^m + \ldots + b_0 \in \mathbb{R}[X]$ be two polynomials with real coefficients and $a_n, b_m \neq 0$. Assume that*

$$m \geq n+2, \qquad Q(x) \neq 0 \textit{ for all } x \in \mathbb{R}. \tag{8.70}$$

We define the quotient function on $\mathbb{C}$ (except for a finite number of singularities) as

$$R(z) = \frac{P(z)}{Q(z)}. \tag{8.71}$$

Then $R(x)$ is absolutely integrable over $\mathbb{R}$ and

$$\int_{-\infty}^{\infty} R(x)\mathrm{d}x = 2\pi i \sum_{j=1}^{k} \mathrm{Res}(R; z_j), \tag{8.72}$$

where $z_1, \ldots, z_k$ are the poles of R in the upper half plane $\mathscr{H} = \{z \in \mathbb{C}\,;\, Im(z) > 0\}$.

Proof. We see that for $z \in \mathbb{C} \setminus \{0\}$, one has

$$R(z) = \frac{P(z)}{Q(z)} = z^{n-m} \cdot \frac{\frac{a_0}{z^n} + \ldots + a_n}{\frac{b_0}{z^m} + \ldots + b_m}. \tag{8.73}$$

For $|z| \to \infty$, the second factor tends to $\frac{a_n}{b_m}$, and is in particular bounded. In other words, there are $M, c > 0$ such that

$$|R(z)| \leq M|z|^{n-m} \leq \frac{M}{|z|^2}, \qquad \text{for all } |z| > c. \tag{8.74}$$

The existence of the limit $\lim_{A\to\infty} \int_0^A |R(x)|\mathrm{d}x$ follows easily, by considering separately the (compact) interval $[0, c]$ on which the continuous function R is bounded, and $[c, A]$, for $A \to \infty$, where we can use the bound (8.74). The existence of the limit $\lim_{B\to\infty} \int_{-B}^0 |R(x)|\mathrm{d}x$ follows similarly. We now show (8.72). Let $r > \max\{|z_1|, \ldots, |z_k|\}$ and consider the closed, piecewise curve $\gamma_r : [-r, r+\pi]$ given by

$$\gamma_r(t) = \begin{cases} t, & t \in [-r, r], \\ r\exp(i(t-r)), & t \in [r, r+\pi]. \end{cases} \tag{8.75}$$

By our requirement on r, γ_r avoids the singularities of R. We also write as β_r the parametrization of the half-circle $\beta_r = \gamma_r|_{[r,r+\pi]}$. With the residue theorem 8.10, we see that

$$\int_{\beta_r} R(z)\mathrm{d}z + \int_{-r}^{r} R(x)\mathrm{d}x = \int_{\gamma_r} R(z)\mathrm{d}z = 2\pi i \sum_{j=1}^{k} \mathrm{Res}(R; z_j), \tag{8.76}$$

since we have $I(\gamma; z_j) = 1$ for all $j = 1, \ldots, k$. Furthermore, we have

$$\begin{aligned} \left|\int_{\beta_r} R(z)\mathrm{d}z\right| &= \left|\int_0^{\pi} R(r\exp(it)) ri\exp(it)\mathrm{d}t\right| \\ &\leq r\int_0^{\pi} |R(r\exp(it))|\mathrm{d}t \leq r\frac{M}{r^2}\pi \to 0, \qquad \text{as } r \to \infty, \end{aligned} \tag{8.77}$$

which shows the claim upon combination with (8.76). □

Example 8.24. We want to calculate

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}x}{1+x^2} \tag{8.78}$$

using the residue theorem. Note that $z^2+1=(z-i)(z+i)$. Thus $f: z \mapsto \frac{1}{z^2+1}$ has exactly one pole in $\mathscr{H}$, namely i, and

$$\operatorname{Res}(f;i) = \lim_{z\to i} \frac{1}{z+i} = \frac{1}{2i}. \tag{8.79}$$

Using Theorem 8.23, we obtain

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}x}{1+x^2} = 2\pi i \frac{1}{2i} = \pi. \tag{8.80}$$

Of course the approach presented above is not limited to rational functions. We can prove with the same approach as in Proposition 8.23 the following result.

Proposition 8.25. *(i) Suppose f is holomorphic on an open set containing $\overline{\mathscr{H}}$ with $\mathscr{H} = \{z \in \mathbb{C}\,;\, Im(z) > 0\}$, except for a finite number of isolated singularities, which are not on the real axis. Furthermore, suppose that for $M, c > 0$ and $\alpha > 1$, one has*

$$|f(z)| \leq \frac{M}{|z|^\alpha}, \qquad |z| \geq c. \tag{8.81}$$

Then $f(x)$ is absolutely integrable over $\mathbb{R}$ and

$$\int_{-\infty}^{\infty} f(x)\mathrm{d}x = 2\pi i \sum_{j=1}^{k} \operatorname{Res}(f;z_j), \tag{8.82}$$

where $z_1, \ldots, z_k$ are the singularities of f in $\mathscr{H}$.

(ii) If the conditions of (i) hold with $\mathscr{H}$ replaced by the lower half-plane $\mathscr{L} = \{z \in \mathbb{C}\,;\, Im(z) < 0\}$, then

$$\int_{-\infty}^{\infty} f(x)\mathrm{d}x = -2\pi i \sum_{j=1}^{k} \operatorname{Res}(f;z_j), \tag{8.83}$$

where $z_1, \ldots, z_k$ are the singularities of f in $\mathscr{L}$.

Integrals involving branch cuts

We have so far studied integrals over functions that are holomorphic on $\mathbb{C}$ except for a finite number of isolated singularities. Functions involving non-integer powers involve logarithms, and their treatment using the residue theorem requires more sophisticated contours. We explain here the use of *keyhole-shaped contours*.

Proposition 8.26. *Let $P(X), Q(X) \in \mathbb{R}[X]$ be two polynomials with real coefficients and consider the function $z \mapsto \frac{P(z)}{Q(z)}$. Assume that the denominator Q has no roots on the positive real axis $\mathbb{R}_+$ and assume that $R(0) \neq 0$ and*

$$\lim_{x\to\infty} x^{\lambda}|R(x)| = 0, \tag{8.84}$$

for $\lambda \in \mathbb{R} \setminus \mathbb{Z}$, $\lambda > 0$. The function $x \mapsto x^{\lambda-1}R(x)$ is absolutely integrable on $\mathbb{R}_+$ and we have that

$$\int_0^{\infty} x^{\lambda-1}R(x)\mathrm{d}x = \frac{\pi}{\sin(\pi\lambda)} \sum_{z_0\in\mathbb{C}_+} \operatorname{Res}(f; z_0), \tag{8.85}$$

where

$$f(z) = (-z)^{\lambda-1}R(z) = \exp\big((\lambda-1)\operatorname{Log}(-z)\big)R(z), \tag{8.86}$$

and $\mathbb{C}_+ = \mathbb{C} \setminus [0, \infty)$.

Proof. We remark that the function f is holomorphic on $\mathbb{C}_+$ except for a finite number of poles since Log is holomorphic on $\mathbb{C}_-$. For $r > 0$, we consider the contour $\gamma_r = \gamma_{1,r} * \gamma_{2,r} * \gamma_{3,r} * \gamma_{4,r}$, where the curves $\gamma_{j,r}$, $1 \leq j \leq 4$ are given up to translation of the parameter intervals by

$$\begin{cases} \gamma_{1,r}(t) = \exp(i\varphi)t, & t \in [\frac{1}{r}, r], \\ \gamma_{2,r}(t) = r\exp(it), & t \in [\varphi, 2\pi - \varphi], \\ \gamma_{3,r}(t) = -\exp(-i\varphi)t, & t \in [-r, -\frac{1}{r}], \\ \gamma_{4,r}(t) = \frac{1}{r}\exp(i(2\pi - t)), & t \in [\varphi, 2\pi - \varphi], \end{cases} \tag{8.87}$$

where $\varphi \in (0, \pi)$ and $r > 1$.

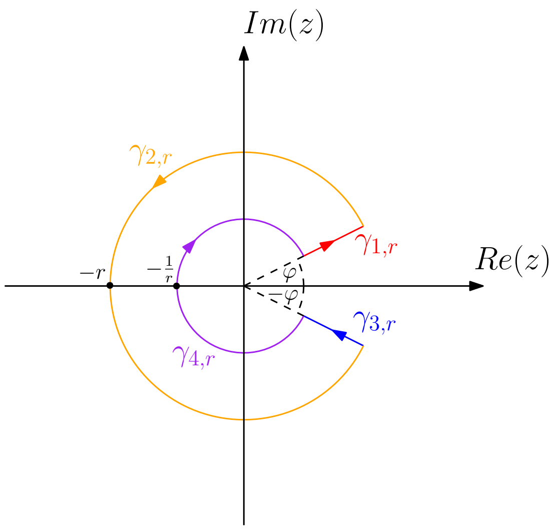

Figure 8.2.: The curves $\gamma_{1,r}$, $\gamma_{2,r}$, $\gamma_{3,r}$ and $\gamma_{4,r}$ constituting the keyhole shape.

Since $\mathbb{C}_+$ is a simply connected domain, for $r > 1$ large enough we can use the residue theorem 8.10 to obtain (for φ small enough)

$$\begin{aligned}\int_{\gamma_r} f(z)\mathrm{d}z &= \sum_{j=1}^{4} \int_{\gamma_{j,r}} f(z)\mathrm{d}z \\ &= 2\pi i \sum_{z_0 \in \mathbb{C}_+} \operatorname{Res}(f; z_0).\end{aligned} \tag{8.88}$$

For this fixed r, we take the limit $\varphi \to 0$ and by the definition of $(-z)^{\lambda-1}$, the integrals over $\gamma_{1,r}$ and $\gamma_{3,r}$ converge to

$$\begin{aligned}&\exp\big(-(\lambda-1)\pi i\big) \int_{\frac{1}{r}}^{r} x^{\lambda-1} R(x)\mathrm{d}x, \text{ and} \\ &-\exp\big((\lambda-1)\pi i\big) \int_{\frac{1}{r}}^{r} x^{\lambda-1} R(x)\mathrm{d}x,\end{aligned} \tag{8.89}$$

respectively. The other two integrals do not contribute to the result since one can easily show that

$$\lim_{r\to\infty} \int_{\gamma_{2,r}} f(z)\mathrm{d}z = \lim_{r\to\infty} \int_{\gamma_{4,r}} f(z)\mathrm{d}z = 0, \tag{8.90}$$

uniformly in φ. □

Example 8.27. For $\lambda \in (0,1)$, we have

$$\int_0^\infty \frac{x^{\lambda-1}}{1+x}\mathrm{d}x = \frac{\pi}{\sin(\pi\lambda)}. \tag{8.91}$$

9. Conformal mappings

(Reference: Marsden-Hoffman, Basic complex analysis, 3rd Ed., Sections 5.1–5.3)

9.1. Motivation

Let $A \subseteq \mathbb{C}$ be open and consider $f : A \to \mathbb{C}$ holomorphic. We are interested in the local mapping behavior of f. Intuitively we have for $z_0 \in A$:

$$\begin{aligned} f(z) - f(z_0) &= f'(z_0)(z - z_0) + O((z - z_0)^2) \\ &= |f'(z_0)| \exp(i\mathrm{Arg}(f'(z_0))(z - z_0) + O((z - z_0)^2). \end{aligned} \tag{9.1}$$

So, the map f involves a local rotation around the point z_0 of size $\mathrm{Arg}(f'(z_0))$ and a stretching with a factor $|f'(z_0)|$. We make this precise:

Definition 9.1. Let $A \subseteq \mathbb{C}$ be open and consider a map $f : A \to \mathbb{C}$.

(i) We say that f *preserves angles at* $z_0 \in A$ if there exists $\theta \in [0, 2\pi)$ and $r > 0$ such that for every differentiable curve $\gamma : [-\varepsilon, \varepsilon] \to \mathbb{C}$, $\varepsilon > 0$, $\gamma([-\varepsilon, \varepsilon]) \subseteq A$ with

$$\gamma(0) = z_0, \qquad \gamma'(0) \neq 0, \tag{9.2}$$

the curve $\sigma : [-\varepsilon, \varepsilon] \to \mathbb{C}$, given by $\sigma(t) = f(\gamma(t))$ is differentiable in $t = 0$ and setting $u = \sigma'(0)$ and $v = \gamma'(0)$, we have

$$|u| = r|v|, \qquad \arg(u) = \arg(v) + \theta. \tag{9.3}$$

(ii) f is *conformal at* $z_0 \in A$ if it is holomorphic and fulfills $f'(z_0) \neq 0$. f is conformal if it is conformal at every $z_0 \in A$.

The following is immediate

Lemma 9.2. *If $f : A \to \mathbb{C}$ is conformal at $z_0 \in A$, it preserves angles at z_0.*

Proof. From the chain rule, we have $u = \sigma'(0) = f'(z_0)\gamma'(0) = f'(z_0)v$, so $\arg(u) = \arg(f'(z_0)) + \arg(v)$ and $|u| = |f'(z_0)||v|$. □

Definition 9.3. Let $A, B \subseteq \mathbb{C}$ be open. A map $f : A \to B$ is called *biholomorphic*[1] if

(i) f is bijective.

[1] In some texts, "conformal" is used synonymously with "biholomorphic".

(ii) f is holomorphic.

(iii) f^{-1} is holomorphic.

Two domains $A_1, A_2 \subseteq \mathbb{C}$ are *conformally equivalent* if there is a biholomorphic map $f : A_1 \to A_2$. It is straightforward to show that conformal equivalence is indeed an equivalence relation on the set of domains contained in $\mathbb{C}$.

Example 9.4. (i) The upper half-plane $\mathscr{H}$ and $D(0,1)$ are conformally equivalent, using the biholomorphic map

$$f : \mathscr{H} \to D(0,1), \qquad z \mapsto \frac{z-i}{z+i}. \tag{9.4}$$

Indeed, one can see easily that $\left|\frac{z-i}{z+i}\right| < 1$ if and only if $\mathrm{Im}(z) > 0$.

(ii) $\mathbb{C}$ and $D(0,1)$ are not conformally equivalent, since any holomorphic $f : \mathbb{C} \to D(0,1)$ is bounded, and thus would have to be constant by Liouville's theorem 4.36.

(iii) Many further important examples for conformal maps can be found in *Marsden-Hoffman, Basic complex analysis, 3rd Ed., Section 5.2* in particular pp. 340–341.

We establish right away the relation between conformal and biholomorphic maps.

Lemma 9.5. *Let $A, B \subseteq \mathbb{C}$ be domains and $f : A \to B$ a map. The following are equivalent:*

(i) f is biholomorphic.

(ii) f is bijective and conformal.

(iii) f is bijective and holomorphic.

Proof. We show that (i) implies (ii). Assume that f is biholomorphic. Then $f \circ f^{-1} = \mathrm{id}$. By the chain rule,

$$f'(f^{-1}(w))(f^{-1})'(w) = 1, \qquad w \in B. \tag{9.5}$$

In particular,

$$f'(f^{-1}(w)) \neq 0, \qquad w \in B, \tag{9.6}$$

and since f^{-1} is bijective, we also have that

$$f'(z) \neq 0, \qquad z \in A. \tag{9.7}$$

Clearly, (ii) implies (iii).
Now assume that (iii) holds. So let f be holomorphic and bijective. We show that f^{-1} is continuous first. Indeed, let $U \subseteq A$ be open. Since f is injective, it is not constant on any connected component of U, so we apply the open mapping theorem **??** to see that $(f^{-1})^{-1}(U) = f(U)$ is open. Note again by injectivity that f' is not identically equal to zero on a subdomain of A, so the zero set $N(f')$ of f' only has isolated points and is closed in A by Proposition 6.18. Since f is open, also $\widetilde{N} = f(N(f'))$ has only isolated points and is closed in B.

Let $w_0 \in B \setminus \widetilde{N}$ with inverse image $z_0 = f^{-1}(w_0)$. Then

$$f(z) = f(z_0) + (z - z_0)\psi(z) \tag{9.8}$$

with a function $\psi : A \to \mathbb{C}$ continuous in z_0 and $\psi(z_0) = f'(z_0) \neq 0$. Set $z = f^{-1}(w)$ for $w \in B$, then

$$w = w_0 + (f^{-1}(w) - f^{-1}(w_0))\psi(f^{-1}(w)). \tag{9.9}$$

The function $q = \psi \circ f^{-1}$ is continuous in w_0 and fulfills $q(w_0) = \psi(z_0) \neq 0$, so there exists $r > 0$ with

$$f^{-1}(w) = f^{-1}(w_0) + \frac{w - w_0}{q(w)}, \qquad \text{for all } w \in D(w_0, r) \cap B. \tag{9.10}$$

We see that f^{-1} is differentiable in w_0 with the derivative

$$(f^{-1})'(w_0) = \frac{1}{q(w_0)} = \frac{1}{\psi(z_0)} = \frac{1}{f'(z_0)} = \frac{1}{f'(f^{-1}(w_0))}. \tag{9.11}$$

We have therefore seen that f^{-1} is continuous on B and holomorphic on $B \setminus \widetilde{N}$, so by the Riemann removability condition (Theorem 7.4), f^{-1} is holomorphic on B. □

9.2. The Riemann mapping theorem

Definition 9.6. A domain $\emptyset \neq A \subseteq \mathbb{C}$ is an *elementary domain*, if every holomorphic function $f : A \to \mathbb{C}$ has a primitive on A.

Proposition 9.7. *Let $A, B \subseteq \mathbb{C}$ be domains and $f : A \to B$ biholomorphic. If A is an elementary domain, then also B.*

Proof. Let $g : B \to \mathbb{C}$ be holomorphic. We show that g has a primitive $G : B \to \mathbb{C}$. To this end, consider the holomorphic map

$$g \circ f : A \to \mathbb{C}.$$

Since A is an elementary domain and since the function

$$(g \circ f) \cdot f' : A \to \mathbb{C}$$

is holomorphic as well, we obtain a primitive $F : A \to \mathbb{C}$ with $F' = (g \circ f) \cdot f'$. We set $G = F \circ f^{-1}$ (which is holomorphic, since f is biholomorphic). Then

$$\begin{aligned} G'(w) &= (F \circ f^{-1})'(w) \\ &= F'(f^{-1}(w)) \cdot (f^{-1})'(w) \\ &= (g \circ f)(f^{-1}(w)) \cdot f'(f^{-1}(w))) \cdot (f^{-1})'(w) \\ &= g(w) \cdot (f \circ f^{-1})'(w) \\ &= g(w). \end{aligned}$$

So G is a primitive of g and thus B is an elementary domain. □

We are now able to state the following version of the *Riemann mapping theorem.*

Theorem 9.8. *Let $A \subseteq \mathbb{C}$ be an elementary domain with $A \neq \mathbb{C}$. Then A is conformally equivalent to the unit disk $D(0,1)$.*

Before we prove the Riemann mapping theorem, let us shortly discuss its interpretation.

Corollary 9.9. *(i) Let $\emptyset \neq A \subseteq \mathbb{C}$ a domain. Then A is an elementary domain if and only if it is simply connected.*

(ii) Let $A \subseteq \mathbb{C}$ be a simply connected domain with $\emptyset \neq A \neq \mathbb{C}$. Then A is conformally equivalent to the unit disk $D(0,1)$.

Proof. We first show (i). If A is a simply connected domain, every holomorphic function $f : A \to \mathbb{C}$ has a primitive by Cauchy's integral theorem 4.31. On the other hand, let us assume that A is an elementary domain. By the Riemann mapping theorem 9.8, either $A = \mathbb{C}$ or A is conformally equivalent to $D(0,1)$. Both are convex and therefore simply connected. Therefore also A is simply connected, since homotopies are stable under biholomorphic maps.
The claim (ii) follows from (i) and the Riemann mapping theorem 9.8. □

Remark 9.10. The notion of elementary domains is not standard. Typically the Riemann mapping theorem is stated in the form of Corollary 9.9, (ii). In our set-up, we also see that simply connected domains are exactly the domains on which every holomorphic function has a primitive, see also the comment below the proof of Theorem 4.31.

End of Lecture 13

Proof of Theorem 9.8. The proof is performed in a number of steps.

Step 1: Logarithms and roots of holomorphic functions

Lemma 9.11. *Let $B \subseteq \mathbb{C}$ an elementary domain and $f : B \to \mathbb{C}$ holomorphic. Assume that f does not have a zero on B. Then*

(i) There is a holomorphic function $h : B \to \mathbb{C}$ with $f(z) = \exp(h(z))$ for all $z \in B$.

(ii) For every $n \in \mathbb{N}$, there is a holomorphic function $H : B \to \mathbb{C}$ with $H^n(z) = f(z)$ for all $z \in B$.

Proof. Since f does not have zeros on B, the function $\frac{f'}{f}$ is holomorphic on B, and since B is an elementary domain, $\frac{f'}{f}$ has a primitive $F : B \to \mathbb{C}$. We set

$$G(z) = \frac{\exp(F(z))}{f(z)}, \qquad z \in B. \tag{9.12}$$

Then

$$\begin{aligned} G'(z) &= \frac{F'(z)f(z)\exp(F(z)) - \exp(F(z))f'(z)}{f^2(z)} \\ &= \frac{f'(z)\exp(F(z)) - f'(z)\exp(F(z))}{f^2(z)} = 0. \end{aligned} \tag{9.13}$$

Since B is a domain, $G(z) = C$ is constant on B for some $C \in \mathbb{C}\setminus\{0\}$. Since $\exp : \mathbb{C} \to \mathbb{C}\setminus\{0\}$ is surjective, there exists $c \in \mathbb{C}$ with $C = \exp(c)$. Thus,

$$\exp(c) = \frac{\exp(F(z))}{f(z)} \qquad \Rightarrow \qquad f(z) = \exp(F(z) - c). \tag{9.14}$$

We can choose $h = F - c$, to show (i). Then (ii) follows by setting

$$H(z) = \exp\left(\frac{1}{n}h(z)\right) \qquad \Rightarrow \qquad H^n(z) = \exp(h(z)) = f(z), \tag{9.15}$$

showing also (ii). □

Step 2: Biholomorphic map into a subset of $D(0,1)$

Lemma 9.12. *Let $\emptyset \neq B_1 \neq \mathbb{C}$ be an elementary domain. Then B_1 is conformally equivalent to an elementary domain B_2 with $0 \in B_2 \subseteq D(0,1)$.*

Proof. By assumption, there is $c \in \mathbb{C} \setminus B_1$. The holomorphic function $f(z) = z - c$ does not have a zero on B_1. Since B_1 is an elementary domain, by Lemma 9.11, there exists a function $g : B_2 \to \mathbb{C}$ with $g^2(z) = f(z)$ on B_1. For $z_1, z_2 \in B_1$, we have

$$g(z_1) = \pm g(z_2) \qquad \Rightarrow \qquad f(z_1) = f(z_2) \qquad \Rightarrow z_1 = z_2. \tag{9.16}$$

It follows that

(i) g is injective, so $g : B_1 \to g(B_1)$ is biholomorphic and $g(B_1)$ is an elementary domain (see Lemma 9.6).

(ii) If $w \in g(B_1) \setminus \{0\}$, then $-w \notin g(B_1)$. Indeed, if $w = g(z_1) = -g(z_2)$ with $z_1, z_2 \in B_1$, then $z_1 = z_2$. Inserting in the previous equation, we see that $w = -w$, so $w = 0$, contradiction.

Since $g(B_1)$ is open and non-empty, there is $z_0 \in \mathbb{C} \setminus \{0\}$ and $r >$ with $0 \notin D(z_0, r) \subseteq g(B_1)$. Then

$$|w + z_0| \geq r \qquad \text{for all } w \in g(B_1). \tag{9.17}$$

Indeed, if $w \in g(B_1)$ with $|w + z_0| < r$, then $w \in D(-z_0, r) = -D(z_0, r)$, which is impossible due to (ii), since $-D(z_0, r) \cap g(B_1)$ would have to be empty.

Now consider the map

$$h : w \mapsto \frac{1}{w + z_0}, \tag{9.18}$$

which is holomorphic for $w \neq -z_0$ and injective. Thus, h maps $g(B_1)$ into a conformally equivalent elementary domain $h(g(B_1))$, which is bounded, since for $w \in g(B_1)$ we have $|w + z_0| \geq r$ and thus $\left|\frac{1}{w+z_0}\right| \leq \frac{1}{r}$. After applying an appropriate translation $t : w \mapsto w + a$, we obtain a conformally equivalent elementary domain $t(h(g(B_1)))$, which contains 0. After an appropriate dilation $s : w \mapsto \rho \cdot w$, $\rho \in (0, \infty)$, we obtain a conformally equivalent elementary domain

$$B_2 = s(t(h(g(B_1)))), \qquad 0 \in B_2 \subseteq D(0,1). \tag{9.19}$$

□

Step 3: Reduction to an extremal problem

In view of the previous step, we now assume that $0 \in A \subsetneq D(0,1)$ (there is nothing to show for $A = D(0,1)$).

Lemma 9.13. *Let A be an elementary domain with $0 \in A \subsetneq D(0,1)$. Then there exists an injective holomorphic map $f : A \to D(0,1)$ with $f(0) = 0$ and $|f'(0)| > 1$.*

Proof. Choose $a \in D(0,1)$ with $a \notin A$. We consider the map

$$h(z) = \frac{z-a}{\bar{a}z - 1}, \tag{9.20}$$

which is a biholomorphic map from $D(0,1)$ to itself (see ⇝ *Exercise*) that fulfills $h(a) = 0$. The function h does not have a zero in A, so again by Lemma 9.11, there exists a holomorphic function $H : A \to \mathbb{C}$ with $H^2(z) = h(z)$. The function H still maps A injectively onto a subset of $D(0,1)$. The function

$$f(z) = \frac{H(z) - H(0)}{\overline{H(0)}H(z) - 1} \tag{9.21}$$

maps A injectively into $D(0,1)$ and fulfills $f(0) = 0$. We compute the derivative:

$$f'(0) = \frac{H'(0)}{|H(0)|^2 - 1}. \tag{9.22}$$

We have

$$H^2(z) = \frac{z-a}{\bar{a}z - 1} \qquad \Rightarrow \qquad 2H(0) \cdot H'(0) = |a|^2 - 1. \tag{9.23}$$

Furthermore

$$|H(0)|^2 = |a| \qquad \Rightarrow \qquad |H(0)| = \sqrt{|a|}. \tag{9.24}$$

Finally,

$$\begin{aligned} |f'(0)| &= \frac{|H'(0)|}{||H(0)|^2 - 1|} = \frac{\big||a|^2 - 1\big|}{2\sqrt{|a|}} \frac{1}{||a| - 1|} = \frac{|a| + 1}{2\sqrt{|a|}} \\ &= 1 + \frac{\left(\sqrt{|a|} - 1\right)^2}{2\sqrt{|a|}} > 1. \end{aligned} \tag{9.25}$$

□

From the previous result, we obtain

Corollary 9.14. *Let A be an elementary domain with $0 \in A \subsetneq D(0,1)$ with*

$$\mathcal{K}(A) = \{f : A \to D(0,1)\,;\ f \textit{ injective, holomorphic, and } f(0) = 0\}. \tag{9.26}$$

If there is $g \in \mathcal{K}(A)$ with

$$|g'(0)| \geq |f'(0)|, \qquad \textit{for all } f \in \mathcal{K}(A), \tag{9.27}$$

then this g is a biholomorphic map from A to $D(0,1)$.

Proof. If g was not surjective, we would have $g(A) \subsetneq D(0,1)$. By Proposition 9.7 since A is an elementary domain, so is $g(A)$, so we can apply Lemma 9.13 to $g(A)$. Then, we obtain $f \in \mathscr{K}(g(A))$ with $|f'(0)| > 1$. Therefore, $f \circ g \in \mathscr{K}(A)$ and

$$|(f \circ g)'(0)| = |f'(g(0))g'(0)| = |f'(0)| \cdot |g'(0)| > |g'(0)|. \tag{9.28}$$

This is a contradiction to the maximality of $|g'(0)|$. So g is surjective and therefore is a biholomorphic map. □

Therefore, the proof of the Riemann mapping theorem is reduced to showing that $g \in \mathscr{K}(A)$ exists that fulfills (9.27).

Step 4: Reduction to Montel's theorem

Define

$$K = \sup_{f \in \mathscr{K}(A)} |f'(0)| \in [0, \infty) \cup \{\infty\}, \tag{9.29}$$

which exists since $\mathrm{id}_A \in \mathscr{K}(A)$ and therefore $\mathscr{K}(A) \neq \emptyset$. By the definition of the supremum, there exists a sequence $(f_n)_{n\in\mathbb{N}}$ of functions $f_n \in \mathscr{K}(A)$, such that $|f_n'(0)| \to K$ as $n \to \infty$.

We now state *Montel's theorem.*

Theorem 9.15. *Let $B \subseteq \mathbb{C}$ be open and $(u_n)_{n\in\mathbb{N}}$ a sequence of holomorphic functions $u_n : B \to \mathbb{C}$, that are uniformly bounded on B, meaning that there exists $C > 0$ such that*

$$|u_n(z)| \leq C, \qquad n \in \mathbb{N}, z \in B. \tag{9.30}$$

Then $(u_n)_{n\in\mathbb{N}}$ has a convergent subsequence, which converges uniformly on every compact $K \subseteq A$.

The proof of this theorem is postponed to the next subsection. We finish the proof of the Riemann mapping theorem. Since $f_n(A) \subseteq D(0,1)$, the sequence $(f_n)_{n\in\mathbb{N}}$ is bounded from above, and we can apply Montel's Theorem 9.15 to it. There is a subsequence $(f_{n_\nu})_{\nu\in\mathbb{N}}$, which converges uniformly on every compact $K \subseteq A$ to some $g : A \to \mathbb{C}$. Since $f_{n_\nu}(0) = 0$, we clearly have

$$g(0) = \lim_{\nu\to\infty} f_{n_\nu}(0) = 0. \tag{9.31}$$

Since $f_{n_\nu} \rightrightarrows g$ on every compact $K \subseteq A$, as $\nu \to \infty$, we see that g is holomorphic on A, and

$$g'(z) = \lim_{\nu\to\infty} f_{n_\nu}'(z), \qquad z \in A, \tag{9.32}$$

by Weierstrass' approximation theorem 6.7. in particular, setting $z = 0$ in (9.32), we find that

$$|g'(0)| = \lim_{\nu\to\infty} |f_{n_\nu}'(0)| = K(< \infty), \tag{9.33}$$

and so we obtain that $|f'(0)| \leq |g'(0)|$ for every $f \in \mathscr{K}(A)$. Note that since $\mathrm{id}_A \in \mathscr{K}(A)$ and the maximality of g, we have $g'(0) \neq 0$, so g cannot be constant on A. Since all f_{n_ν} are injective, g is injective as well by the Corollary 8.19 to Hurwitz' theorem.

Since $f_{n_\nu}(z)$ converges to $g(z)$ and $|f_{n_\nu}(z)| < 1$ for all $z \in A$, we obtain $|g(z)| \leq 1$. If there was a $z_0 \in A$ with $|g(z_0)| = 1$, g would have to be constant 1 by the maximum modulus principle (Theorem 5.2), which we already excluded. Therefore $g(A) \subseteq D(0,1)$. This all shows that $g \in \mathscr{K}(A)$ fulfills (9.27), and by the previous step we see that g is the required biholomorphic map. □

Step 5: Proof of Montel's theorem

We will prove Montel's theorem by relying on a classical theorem in functional analysis, namely the *Arzela-Ascoli theorem.* Before we can state it, we need a definition.

Definition 9.16. Let $K \subseteq \mathbb{C}$ be compact and $\mathscr{F}$ a family of functions $f : K \to \mathbb{C}$. We say that $\mathscr{F}$ is *equicontinuous* if for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\text{for all } f \in \mathscr{F}, x, y \in K \,:\, |x-y| < \delta \qquad \Rightarrow \qquad |f(x) - f(y)| < \varepsilon. \tag{9.34}$$

The next result is the aforementioned *Arzela-Ascoli theorem.*

Theorem 9.17. *Let $K \subseteq \mathbb{C}$ be compact and $(f_n)_{n\in\mathbb{N}}$ a sequence of functions that is uniformly bounded, i.e.*

$$|f_n(z)| \le C, \qquad n \in \mathbb{N}, z \in K, \tag{9.35}$$

for some $C > 0$ and equicontinuous. Then there exists a subsequence $(f_{n_\nu})_{\nu\in\mathbb{N}}$ and a continuous $f : K \to \mathbb{C}$ with $f_{n_\nu} \rightrightarrows f$ on K as $\nu \to \infty$.

Proof of Montel's theorem 9.15. We consider an exhaustion of $B \subseteq \mathbb{C}$ open with compact sets $K_n \subseteq B$, $n \in \mathbb{N}$, where

$$B = \bigcup_{n=1}^{\infty} K_n, \qquad K_n \subseteq \mathring{K}_{n+1}, n \in \mathbb{N}, \tag{9.36}$$

for instance $K_n = \{z \in B\,;\, |z| \le n, d(z, \partial B) \ge \frac{1}{n}\}$ will be sufficient. For any compact $K \subseteq B$, there exists an $n \in \mathbb{N}$ with $K \subseteq \mathring{K}_n$. We show that the sequence $(u_n)_{n\in\mathbb{N}}$ in the statement of Montel's theorem is equicontinuous on $K = K_n$. Since $K_n \subseteq \mathring{K}_{n+1}$, we find for every $z \in K_n$ some $r = r(z) > 0$ with $D(z, 3r) \subseteq \mathring{K}_{n+1}$, and finitely many disks $D(z_j, r_j)$, $z_j \in K_n$, $r_j = r(z_j)$, $1 \le j \le J$ cover K_n. Now by the Cauchy inequality (Proposition 4.35), we see that

$$|u_m'(z_0)| \le \frac{\max_{z\in D(z_0,r_j)} |u_m(z)|}{r_j} \le \frac{\max_{z\in K_{n+1}} |u_m(z)|}{\min\{r_j\,;\, 1 \le j \le J\}} \le \frac{C}{\min\{r_j\,;\, 1 \le j \le J\}} = C_1, \tag{9.37}$$

for $z_0 \in D(z_j, 2r_j)$, since $D(z_0, r_j) \subseteq \overline{D(z_j, 3r_j)} \subseteq K_{n+1} \subseteq B$, uniformly for all u_m in the sequence. Let $\varepsilon > 0$ be given. For $z, w \in K_n$ with $|z-w| < \delta < \min(\{r_j\,;\, 1 \le j \le J\} \cup \{\frac{\varepsilon}{C_1}\})$ we can choose z_j with $z \in D(z_j, r_j)$. Then $w \in D(z_j, 2r_j)$ and we see that

$$\begin{aligned} |u_m(z) - u_m(w)| &= \left| \int_0^1 u_m'(w + t(z-w))(z-w)\mathrm{d}t \right| \\ &\le \max_{t\in[0,1]} |u_m'(w + t(z-w))||z-w| < C_1\delta < \varepsilon, \end{aligned} \tag{9.38}$$

for every $m \in \mathbb{N}$. This establishes the equicontinuity of $(u_m)_{m\in\mathbb{N}}$ on K_n. By assumption, we also have that the sequence is uniformly bounded on B, in particular on K_n. By the Arzela-Ascoli theorem 9.17, we see that $(u_m)_{m\in\mathbb{N}}$ has a uniformly convergent subsequence on K_n. Since K_n was an exhaustion of B, we find a convergent subsequence which converges uniformly on every compact $K \subseteq B$ using a diagonal argument. ☐

A. Appendix

A.1. Differentiability in $\mathbb{R}^n$

We briefly recall the notions of partial derivatives and total differentiability of functions mapping open subsets $U \subseteq \mathbb{R}^n$ into $\mathbb{R}^m$, where $n, m \geq 1$.

Definition A.1. (i) Let $r > 0$ and $x_0 \in \mathbb{R}^n$. We call $D(x_0, r) = \{x \in \mathbb{C}; \|x - x_0\| < r\}$ the *(open) r ball around* x_0. We call $\dot{D}(x_0, r) = D(x_0, r) \setminus \{x_0\}$ the *deleted (open) r ball around* x_0. Here $\|y\| = \|(y_1, ..., y_n)\| = \sqrt{y_1^2 + ... + y_n^2}$ is the *Euclidean norm* on $\mathbb{R}^n$.

(ii) Let $U \subseteq \mathbb{R}^n$. A point $x_0 \in U$ is called an *interior point* of U, if there is $r > 0$ such that $D(x_0, r) \subseteq U$.

(iii) A set $U \subseteq \mathbb{R}^n$ is called *open* if every point $x_0 \in U$ is an interior point of U.

(iv) A set $U \subseteq \mathbb{C}$ is called *closed* if $U^c = \mathbb{R}^n \setminus U$ is open.

The results of Section 2.2 about interior, closure, boundary, compactness, (path-)connectedness, continuity etc. are valid for subsets of $\mathbb{R}^n$ and functions $f : U \to \mathbb{R}$ with the necessary replacements of $|\cdot|$ by $\|\cdot\|$.

Definition A.2. Let $U \subseteq \mathbb{R}^n$ be open and $f : U \to \mathbb{R}$ a real function. We say that f has a *partial derivative* $\frac{\partial}{\partial x_i} f(x_0)$ in $x_0 \in U$ in coordinate direction i, if the limit

$$\frac{\partial f}{\partial x_i}(x_0) = \lim_{h \to 0} \frac{f(x_0 + he_i) - f(x_0)}{h} \tag{A.1}$$

exists in $\mathbb{R}$, where $e_i \in \mathbb{R}^n$ is the unit vector in direction i, meaning that

$$(e_i)_j = \begin{cases} 1, & j = i, \\ 0, & j \neq i. \end{cases} \tag{A.2}$$

Of course this is nothing else but the ordinary derivative in $x_{0,i}$ of the function

$$\xi \mapsto f_i(\xi) = f(x_{0,1}, ..., x_{0,i-1}, \xi, x_{0,i+1}, ..., x_{0,n}), \tag{A.3}$$

meaning $\frac{\partial f}{\partial x_i}(x_0) = f_i(x_{0,i})$. To calculate the partial derivative in direction i of a function $f : U \to \mathbb{R}$ in $x = (x_1, ..., x_n)$, we treat every $(x_j)_{j \neq i}$ as constant, and take the one-dimensional derivative with respect to x_i.

Definition A.3. Let $U \subseteq \mathbb{R}^n$ be open and $f : U \to \mathbb{R}$. Assume that f has partial derivatives in all directions $i = 1, ..., n$ in x_0. We call the vector

$$\text{grad } f(x_0) = \left(\frac{\partial f}{\partial x_1}(x_0), ..., \frac{\partial f}{\partial x_n}(x_0) \right) \tag{A.4}$$

the *gradient* of f in $x_0 \in U$.

We now show that partial derivatives commute, if the resulting expressions are continuous. This is the important *Schwarz' Theorem.*

Theorem A.4. *Let $U \subseteq \mathbb{R}^n$ open and $f : U \to \mathbb{R}$ a function with continuous second partial derivatives at $x_0 \in U$, then for every $1 \leq i, j \leq n$, one has*

$$\frac{\partial^2 f}{\partial x_i \partial x_j}(x_0) = \frac{\partial}{\partial x_i}\left(\frac{\partial f}{\partial x_j}\right)(x_0) = \frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial x_i}\right)(x_0) = \frac{\partial^2 f}{\partial x_j \partial x_i}(x_0). \tag{A.5}$$

Proof. Without loss of generality, assume $n = 2$, $i = 1$, $j = 2$ and $x_0 = (0, 0)$. Write (x, y) instead of (x_1, x_2). There exists a $\delta > 0$ such that

$$\{(x, y) \in \mathbb{R}^2 ; |x| < \delta, |y| < \delta\} \subseteq U. \tag{A.6}$$

For fixed $|y| < \delta$, consider the function $h_y : (-\delta, \delta) \to \mathbb{R}$ defined by

$$h_y(x) = f(x, y) - f(x, 0). \tag{A.7}$$

By the mean value theorem for functions depending on one real variable, there exists $\xi \in \mathbb{R}$, $|\xi| \leq |x|$ with

$$h_y(x) - h_y(0) = h_y'(\xi)x = (\tfrac{\partial f}{\partial x}(\xi, y) - \tfrac{\partial f}{\partial x}(\xi, 0))x. \tag{A.8}$$

We apply the mean value theorem again to the function $y \mapsto \frac{\partial f}{\partial x}(\xi, y)$ (with fixed ξ as above) which gives the existence of $\eta \in \mathbb{R}$, with $|\eta| \leq |y|$, such that

$$\tfrac{\partial f}{\partial x} f(\xi, y) - \tfrac{\partial f}{\partial x} f(\xi, 0) = \tfrac{\partial}{\partial y}\left(\tfrac{\partial f}{\partial x}\right)(\xi, \eta)y. \tag{A.9}$$

We combine (A.7) and (A.8) with the expression above and obtain

$$f(x, y) - f(x, 0) - f(0, y) + f(0, 0) = \tfrac{\partial}{\partial y}\left(\tfrac{\partial f}{\partial x}\right)(\xi, \eta)xy. \tag{A.10}$$

We can interchange the roles of x and y and repeat this argument for the function $\widetilde{h}_x : (-\delta, \delta) \to \mathbb{R}$, defined by

$$\widetilde{h}_x(y) = f(x, y) - f(0, y). \tag{A.11}$$

We obtain $\widetilde{\eta}, \widetilde{\xi} \in \mathbb{R}$ with $|\widetilde{\eta}| \leq |y|$ and $|\widetilde{\xi}| \leq |x|$ such that

$$f(x, y) - f(x, 0) - f(0, y) + f(0, 0) = \tfrac{\partial}{\partial x}\left(\tfrac{\partial f}{\partial y}\right)(\widetilde{\xi}, \widetilde{\eta})xy. \tag{A.12}$$

By comparing (A.10) and (A.12), we see that for $xy \neq 0$, we obtain

$$\tfrac{\partial}{\partial y}\left(\tfrac{\partial f}{\partial x}\right)(\xi, \eta) = \tfrac{\partial}{\partial x}\left(\tfrac{\partial f}{\partial y}\right)(\widetilde{\xi}, \widetilde{\eta}). \tag{A.13}$$

The claim follows by letting $(x, y) \to (0, 0)$, since then also $(\xi, \eta) \to (0, 0)$ and $(\widetilde{\xi}, \widetilde{\eta}) \to (0, 0)$, and the second partial derivatives are assumed to be continuous. $\square$

We now turn to the notion of total differentiability.

Definition A.5. Let $U \subseteq \mathbb{R}^n$ open and $f : U \to \mathbb{R}^m$ a function. We say that f is *totally differentiable* in $x_0 \in U$ if there exists a matrix $M \in \mathbb{R}^{m \times n}$ and a continuous function $\varphi : D(0,r) \to \mathbb{R}^m$ (for some $r > 0$) with

$$f(x) = f(x_0) + M \cdot (x - x_0) + \varphi(x - x_0) \tag{A.14}$$

such that

$$\lim_{\xi \to 0} \frac{\varphi(\xi)}{\|\xi\|} = 0. \tag{A.15}$$

The latter condition means that for every $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$\left\| \frac{\varphi(x - x_0)}{\|x - x_0\|} \right\| = \frac{\|f(x) - f(x_0) - M \cdot (x - x_0)\|}{\|x - x_0\|} < \varepsilon, \qquad \text{for } x \in \dot{D}(x_0, \delta). \tag{A.16}$$

We write a function $f : U \to \mathbb{R}^m$ in components $f = (f_1, ..., f_m)^\top$ with $f_i : U \to \mathbb{R}$, $i = 1, ..., m$. The next two theorems explain the relation between the partial derivatives of the f_i and the total differentiability of f.

Theorem A.6. *Let $U \subseteq \mathbb{R}^n$ be open and $f : U \to \mathbb{R}^m$ totally differentiable in $x_0 \in U$ with a matrix $M \in \mathbb{R}^{m \times n}$ such that* (A.14) *holds. One has*

(i) f is continuous in x_0.

(ii) All component functions $f_i : U \to \mathbb{R}$ have all partial derivatives in x_0 and it holds that

$$\frac{\partial f_i}{\partial x_j}(x_0) = M_{ij}, \qquad 1 \leq i \leq m, 1 \leq j \leq n. \tag{A.17}$$

Thus, M is necessarily the Jacobian *of f in x_0, defined by*

$$Df(x_0) = \left(\frac{\partial f_i}{\partial x_j}(x_0) \right)_{1 \leq i \leq m, 1 \leq j \leq n}. \tag{A.18}$$

Proof. For (i), use that for $x \neq x_0$

$$\begin{aligned} \|f(x) - f(x_0)\| &\leq \|M \cdot (x - x_0)\| + \frac{\|\varphi(x - x_0)\|}{\|x - x_0\|} \|x - x_0\| \\ &\leq \left(\|M\|_{\max} + \frac{\|\varphi(x - x_0)\|}{\|x - x_0\|} \right) \|x - x_0\|, \end{aligned} \tag{A.19}$$

where $\|M\|_{\max} = \max_{1 \leq i \leq m, 1 \leq j \leq n} |M_{ij}|$. The latter expression converges to zero for $x \to x_0$.

For (ii), we write for $1 \leq i \leq m$

$$f_i(x) = f_i(x_0) + \sum_{k=1}^{n} M_{ik}(x - x_0)_k + \varphi_i(x - x_0). \tag{A.20}$$

We then specify $x = x_0 + he_j$, so that this equation yields

$$f_i(x_0 + he_j) - f_i(x_0) = h \underbrace{\sum_{k=1}^{n} M_{ik} \cdot (e_j)_k}_{=M_{ij}} + \varphi_i(he_j). \tag{A.21}$$

We infer that

$$\frac{\partial f_i}{\partial x_j}(x_0) = \lim_{h \to 0} \frac{f_i(x_0 + he_j) - f_i(x_0)}{h} = M_{ij} + \lim_{h \to 0} \frac{\varphi_i(he_j)}{h} = M_{ij}. \tag{A.22}$$

□

Theorem A.7. *Let $U \subseteq \mathbb{R}^n$ open and $f : U \to \mathbb{R}^m$ a function such that all partial derivatives $\frac{\partial f_i}{\partial x_j}$ exist in U and are continuous in $x_0 \in U$. Then f is totally differentiable in x_0.*

Proof. We first show the case where $m = 1$. Let $\delta > 0$ be such that $D(x_0, \delta) \subseteq U$. For $x \in D(x_0, \delta)$, define $\xi = (\xi_1, ..., \xi_n) = x - x_0 \in \mathbb{R}^n$. We set

$$x^{(k)} = x_0 + \sum_{j=1}^{k} \xi_j e_j, \qquad k = 0, 1, ..., n, \tag{A.23}$$

so $x^{(0)} = x_0$, $x^{(n)} = x_0 + \xi = x$. By the mean value theorem for differentiable functions in one variable, there are $\theta_k \in [0, 1]$ for $k = 1, ..., n$ with

$$f(x^{(k)}) - f(x^{(k-1)}) = \frac{\partial f}{\partial x_k}(x^{(k-1)} + \theta_k \xi_k e_k)\xi_k. \tag{A.24}$$

Summing up these terms, we therefore obtain that

$$\begin{aligned} f(x) - f(x_0) &= \sum_{k=1}^{n} \frac{\partial f}{\partial x_k}(x^{(k-1)} + \theta_k \xi_k e_k)\xi_k \\ &= \sum_{k=1}^{n} \frac{\partial f}{\partial x_k}(x_0)\xi_k + \varphi(\xi), \end{aligned} \tag{A.25}$$

where we defined

$$\varphi(\xi) = \sum_{k=1}^{n} \left(\frac{\partial f}{\partial x_k}(x^{(k-1)} + \theta_k \xi_k e_k) - \frac{\partial f}{\partial x_k}(x_0) \right) \xi_k. \tag{A.26}$$

Now since the partial derivatives are all continuous by assumption, one has that as $\xi \to 0$, also $\lim_{\xi \to 0} \frac{\partial f}{\partial x_k}(x^{(k-1)} + \theta_k \xi_k e_k) - \frac{\partial f}{\partial x_k}(x_0) = 0$ for $k = 1, ..., n$. So by the Cauchy-Schwarz inequality, it follows that $\lim_{\xi \to 0} \frac{\varphi(\xi)}{\|\xi\|} = 0$. We now turn to the case of general $m \geq 1$. Note that for $i = 1, ..., m$, we have

$$f_i(x) - f_i(x_0) = \sum_{k=1}^{n} \frac{\partial f_i}{\partial x_k}(x_0)(x - x_0)_k + \varphi_i(x - x_0). \tag{A.27}$$

We therefore obtain

$$f(x) - f(x_0) - Df(x_0) \cdot (x - x_0) = \varphi(x - x_0), \tag{A.28}$$

where $\varphi = (\varphi_1, ..., \varphi_m)^\top$, and we can bound the above expression for $x \neq x_0$ (using $\|y\| \leq \sum_{i=1}^m |y_j|$ for $y \in \mathbb{R}^m$):

$$\frac{\|f(x) - f(x_0) - Df(x_0) \cdot (x - x_0)\|}{\|x - x_0\|} \leq \frac{\sum_{i=1}^m |\varphi_i(x - x_0)|}{\|x - x_0\|}, \tag{A.29}$$

which goes to zero as $x \to x_0$. $\square$

We will also use the *Chain rule*, which we now state and prove.

Theorem A.8. *Let $U \subseteq \mathbb{R}^n$, $V \subseteq \mathbb{R}^m$ open and $f : U \to \mathbb{R}^m$, $f(U) \subseteq V$ and $g : V \to \mathbb{R}^p$. Assume that f is totally differentiable in $x_0 \in U$ and g is totally differentiable in $f(x_0) \in V$. Then also $g \circ f : U \to \mathbb{R}^p$ is totally differentiable in x_0 and it holds that*

$$D(g \circ f)(x_0) = \underbrace{Dg(f(x_0))}_{\in \mathbb{R}^{p \times m}} \cdot \underbrace{Df(x_0)}_{\in \mathbb{R}^{m \times n}}. \tag{A.30}$$

Proof. We write for simplicity $y_0 = f(x_0)$. One has

$$\begin{aligned} f(x_0 + \xi) - f(x_0) &= Df(x_0) \cdot \xi + \varphi(\xi), \\ g(y_0 + \eta) - g(y_0) &= Dg(y_0) \cdot \eta + \psi(\eta), \end{aligned} \tag{A.31}$$

where $\lim_{\xi \to 0} \frac{\varphi(\xi)}{\|\xi\|} = 0$ and $\lim_{\eta \to 0} \frac{\psi(\eta)}{\|\eta\|} = 0$. We insert $\eta = f(x_0 + \xi) - f(x_0) = Df(x_0)\xi + \varphi(\xi)$ (which is small for ξ sufficiently small, by continuity of f in x_0) and we see that

$$\begin{aligned} (g \circ f)(x_0 + \xi) &= g(f(x_0 + \xi)) = g(f(x_0) + \eta) \\ &= g(f(x_0)) + Dg(f(x_0)) \cdot Df(x_0)\xi \\ &+ \underbrace{Dg(f(x_0)) \cdot \varphi(\xi) + \psi(Df(x_0) \cdot \xi + \varphi(\xi))}_{=:\chi(\xi)}. \end{aligned} \tag{A.32}$$

Note that there exists a $\delta > 0$ such that

$$\|\varphi(\xi)\| \leq \|\xi\| \qquad \text{for } \|\xi\| < \delta. \tag{A.33}$$

Now write $\psi(\eta) = \|\eta\|\psi_1(\eta)$ with $\lim_{\eta \to 0} \psi_1(\eta) = 0$ (ψ_1 is defined in a deleted r ball around 0), then for $\|\xi\| < \delta$,

$$\|\psi(Df(x_0) \cdot \xi + \varphi(\xi))\| \leq (\|Df(x_0)\|_{\max} + 1)\|\xi\| \cdot \|\psi_1(Df(x_0) \cdot \xi + \varphi(\xi))\|. \tag{A.34}$$

Now as $\xi \to 0$, also $\frac{\varphi(\xi)}{\|\xi\|} \to 0$, $Df(x_0) \cdot \xi \to 0$ and $\varphi(\xi) \to 0$. We obtain that

$$\lim_{\xi \to 0} \frac{\chi(\xi)}{\|\xi\|} = 0. \tag{A.35}$$

By (A.32), we obtain the claim. $\square$

A.2. C^k path-connectedness

In this section we discuss some aspects of (path-)connectedness and the existence of C^k paths as well as C^k homotopies. An important technical tool is the *mollification* of continuous functions. We start with the following definition.

Definition A.9. A smooth function $\eta : \mathbb{R}^n \to [0, \infty)$ with $\operatorname{supp}(\eta) = \overline{\{x \in \mathbb{R}^n \,;\, \eta(x) \neq 0\}} \subseteq D(0,1)$ and

$$\int_{\mathbb{R}^n} \eta(x) \mathrm{d}^n x = 1 \tag{A.36}$$

is called a *(Friedrichs) mollifier*. For $\varepsilon > 0$ the function $\eta_\varepsilon : \mathbb{R}^n \to \mathbb{R}$ is defined as

$$\eta_\varepsilon(x) = \frac{1}{\varepsilon^n} \eta\left(\frac{x}{\varepsilon}\right). \tag{A.37}$$

The set of functions $(\eta_\varepsilon)_{\varepsilon>0}$ is called the *Dirac sequence* associated with η.

One immediately verifies that η_ε is smooth with $\operatorname{supp}(\eta_\varepsilon) \subseteq D(0, \varepsilon)$ and

$$\int_{\mathbb{R}^n} \eta_\varepsilon(x) \mathrm{d}^n x = \int_{D(0,\varepsilon)} \eta_\varepsilon(x) \mathrm{d}^n x = 1. \tag{A.38}$$

An explicit example for a mollifier η is can be constructed by definining

$$\varphi(x) = \begin{cases} \exp\left(-\frac{1}{1-\|x\|^n}\right), & \|x\| < 1 \\ 0, & \|x\| \geq 1. \end{cases} \tag{A.39}$$

and setting $\eta = \frac{1}{\int_{\mathbb{R}^n} \varphi(x) \mathrm{d}^n x} \varphi$. The relevance of mollifiers is that they can be used to “smoothen” a given continuous[1] function f. Namely let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuous function. We set

$$(f * \eta_\varepsilon)(x) = \int_{\mathbb{R}^n} f(x - y) \eta_\varepsilon(y) \mathrm{d}^n y = \int_{\mathbb{R}^n} f(z) \eta_\varepsilon(x - z) \mathrm{d}^n z. \tag{A.40}$$

The relevance of this definition is that the function $f_\varepsilon := f * \eta_\varepsilon$ (the *convolution* of f with η_ϵ) has features described in the following proposition.

Proposition A.10. *Let $f : \mathbb{R}^n \to \mathbb{R}$ continuous and $f_\varepsilon = f * \eta_\varepsilon$ defined as in* (A.40).

(i) *f_ε is smooth.*

(ii) $\operatorname{supp}(f_\varepsilon) \subseteq \operatorname{supp}(f) + \varepsilon = \bigcup_{x \in \operatorname{supp}(f)} \overline{D(x, \varepsilon)}$.

(iii) *If $|f|$ is bounded by $M \geq 0$, then $|f_\varepsilon|$ is also bounded by $M \geq 0$.*

(iv) *If $K \subseteq \mathbb{R}^n$ is compact then*

$$\sup_{x \in K} |f_\varepsilon(x) - f(x)| \to 0, \qquad \textit{as } \varepsilon \downarrow 0. \tag{A.41}$$

[1] or more generally: locally integrable

Proof. For (i), we write (for $i = 1, ..., n$, e_i the unit vector in direction i, and $h \neq 0$)

$$\frac{f_\varepsilon(x+he_i) - f_\varepsilon(x)}{h} = \int_{\mathbb{R}^n} f(z) \frac{\eta_\varepsilon(x+he_i-z) - \eta_\varepsilon(x-z)}{h} \mathrm{d}z. \tag{A.42}$$

Since η_ε is smooth, we find by the mean value theorem from real analysis a $\xi \in [0,h]$ such that

$$\frac{\eta_\varepsilon(x+he_i-z) - \eta_\varepsilon(x-z)}{h} = \frac{\partial \eta_\varepsilon}{\partial x_i}(x_0 + \xi e_i - z). \tag{A.43}$$

We find that for given $\epsilon > 0$, we find h small enough such that

$$\left| \frac{f_\varepsilon(x+he_i) - f_\varepsilon(x)}{h} - \int_{\mathbb{R}^n} f(z) \frac{\partial \eta_\varepsilon}{\partial x_i}(x_0 - z)\mathrm{d}z \right| < \epsilon c(f,\varepsilon), \tag{A.44}$$

where $c(f,\varepsilon)$ depends only on f and ε (note that the integrals are in fact only over a compact set, on which $z \mapsto f(z)\frac{\partial \eta_\varepsilon}{\partial x_i}(x_0 - z)$ is uniformly continuous). This procedure can be iterated since η_ε is smooth.

For (ii), simply note that for $x \notin \mathrm{supp}(f) + \varepsilon$, by definition one has $|x - z| > \varepsilon$ for every $z \in \mathrm{supp}(f)$. But by (A.40), this means that $f(z)\eta_\varepsilon(x-z) = 0$ for every $z \in \mathbb{R}^n$, and so $f_\varepsilon(x) = 0$.

The claim (iii) follows by the inequality for $x \in \mathbb{R}^n$

$$|f_\varepsilon(x)| = \left| \int_{\mathbb{R}^n} f(x-y)\eta_\varepsilon(y)\mathrm{d}^n y \right| \leq \int_{\mathbb{R}^n} \underbrace{|f(x-y)|}_{\leq M} \eta_\varepsilon(y)\mathrm{d}^n y \leq M, \tag{A.45}$$

using (A.38) in the last step.

Finally, (iv) follows from

$$\begin{aligned} \sup_{x\in K} |f_\varepsilon(x) - f(x)| &= \sup_{x \in K} \left| \int_{\mathbb{R}^n} \left(f(x-y) - f(y)\right) \eta_\varepsilon(y)\mathrm{d}^n y \right| \\ &\leq \underbrace{\sup_{x\in K, y \in \mathbb{R}^n, |y|<\varepsilon} |f(x-y) - f(x)|}_{\to 0, \text{ as } \varepsilon \downarrow 0} \underbrace{\int_{\mathbb{R}^n} \eta_\varepsilon(y)\mathrm{d}^n y}_{=1, \text{by (A.38)}} \end{aligned} \tag{A.46}$$

using that f is uniformly continuous on the compact set K. □

Mollifications can be used to smoothen curves, which is our main application.

Lemma A.11. *Let $\gamma : [0,1] \to \mathbb{C}$ be continuous curve with $\gamma = \gamma_1 + i\gamma_2$, $\gamma_1, \gamma_2 \in \mathbb{R}$. For every $\delta > 0$ there exists a curve $\widehat{\gamma} : [0,1] \to \mathbb{C}$ of class C^k for every $k \in \mathbb{N}$ with $\widehat{\gamma}(0) = \gamma(0)$, $\widehat{\gamma}(1) = \gamma(1)$ and $\sup_{t\in[0,1]} |\gamma(t) - \widehat{\gamma}(t)| < \delta$.*

Proof. By reparametrizing γ, we can assume without loss of generality that $\gamma(t) = \gamma(0)$ for every $t \in [0, \varepsilon_0]$ and $\gamma(t) = \gamma(1)$ for every $t \in [1-\varepsilon_0, 1]$ for some $\varepsilon_0 > 0$. We extend γ to a continuous function on $\mathbb{R}$ by setting $\gamma(t) = \gamma(0)$ for $t < 0$ and $\gamma(t) = \gamma(1)$ for $t > 1$. Now take $\varepsilon \in (0, \varepsilon_0/2)$ and consider

$$\gamma_{j,\varepsilon} = \gamma_j * \eta_\varepsilon, \qquad j = 1, 2. \tag{A.47}$$

We see that

$$\gamma_{1,\varepsilon}(0) = \int_{\mathbb{R}} \gamma_1(0-y)\eta_\varepsilon(y)\mathrm{d}y = \int_{(-\varepsilon,\varepsilon)} \underbrace{\gamma_1(0-y)}_{=\gamma_1(0)} \eta_\varepsilon(y)\mathrm{d}y = \gamma_1(0), \tag{A.48}$$

and similarly $\gamma_{2,\varepsilon}(0) = \gamma_2(0)$, $\gamma_{1,\varepsilon}(1) = \gamma_1(1)$, $\gamma_{2,\varepsilon}(1) = \gamma_2(1)$. Now we have that for $j = 1, 2$ by Proposition A.10, (iv),

$$\sup_{t\in[0,1]} |\gamma_{j,\varepsilon}(t) - \gamma_j(t)| \to 0, \tag{A.49}$$

as $\varepsilon \downarrow 0$. Thus choosing $\varepsilon > 0$ small enough we obtain that for $\widehat{\gamma} = \gamma_{1,\varepsilon} + i\gamma_{2,\varepsilon}$, it holds that

$$\sup_{t\in[0,1]} |\widehat{\gamma}(t) - \gamma(t)| \leq \sup_{t\in[0,1]} |\gamma_{1,\varepsilon}(t) - \gamma_1(t)| + \sup_{t\in[0,1]} |\gamma_{2,\varepsilon}(t) - \gamma_2(t)| < \delta. \tag{A.50}$$

$\square$

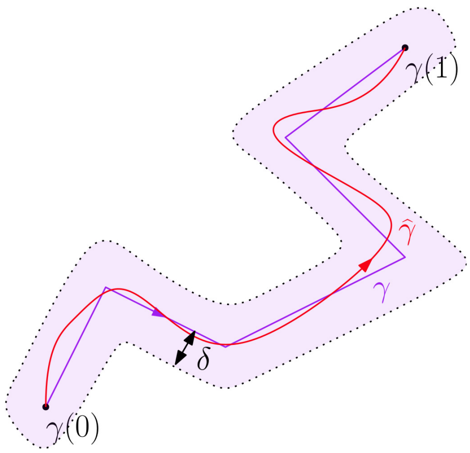

Figure A.1.: The continuous curve γ and the "smoothened" replacement $\widehat{\gamma}$, which stays closer to γ than δ.

We are now able to show the following claim.

Proposition A.12. *Let $A \subseteq \mathbb{C}$.*

(i) If A is C^k path-connected (for $k \in \mathbb{N}_0$), meaning that for every $z, w \in A$ exists a path $\gamma : [0,1] \to A$ of class C^k with $\gamma(0) = z$ and $\gamma(1) = w$, then A is connected.

(ii) If A is open and connected[2]*, then it is also C^k path-connected for every $k \in \mathbb{N}_0$.*

Proof. In both cases, there is nothing to show if $A = \emptyset$, so suppose A is non-empty. Let us first assume that A is C^k path-connected. If A was not connected, then there are non-empty relatively open sets $A_1, A_2 \subseteq A$ with $A_1 \cup A_2 = A$ and $A_1 \cap A_2 = \emptyset$. Define the function

$$f : A \to \mathbb{C}, \qquad z \mapsto \begin{cases} 1, & z \in A_1, \\ 0, & z \in A_2. \end{cases} \tag{A.51}$$

We note that this function is continuous. Indeed, if $(z_n)_{n\in\mathbb{N}} \subseteq A$ with $z_n \to z \in A_1$ then by the relative openness, $z_n \in A_1$ for large enough n, so $f(z_n) = 1 \to f(z) = 1$ as $n \to \infty$, and similarly for $z \in A_2$. Now let $z \in A_1$ and $w \in A_2$ (which exist since these sets are non-empty), and let $\gamma : [0,1] \to A$ be a C^k path (in particular continuous) with $\gamma(0) = z$ and $\gamma(1) = w$. The map $f \circ \gamma : [0,1] \to \mathbb{R}$ is then a continuous map with $(f \circ \gamma)([0,1]) = \{0,1\}$, with $f(\gamma(0)) = 1$ and $f(\gamma(1)) = 0$, which violates the intermediate value theorem. This contradiction shows that A must be connected.

Conversely suppose that A is open and connected and let $z \in A$. We consider the set

$$A' = \{w \in A\,;\ \text{there is a } C^k \text{ path } \gamma : [0,1] \to A \text{ with } \gamma(0) = z \text{ and } \gamma(1) = w\}. \tag{A.52}$$

We will show that A' is non-empty, open and closed (in the relative topology of A).

- Clearly $z \in A'$ since the constant map $\gamma_{\{z\}} : [0,1] \to A$, $\gamma_{\{z\}}(t) = z$ for all $t \in [0,1]$ of class C^k, so A' is non-empty.
- We now show that A' is closed: Assume $(w_n)_{n\in\mathbb{N}} \subseteq A'$ is a sequence with $w_n \to w$. Then for some $\varepsilon > 0$, $D(w,\varepsilon) \subseteq A$, and for some $N = N(\varepsilon) \in \mathbb{N}$, we know that $w_n \in D(w,\varepsilon)$ for every $n \geq N$. There exists a continuous path γ from z to w_N, and by adding a straight line segment from w_N to w (which is contained in $D(w,\varepsilon)$ by convexity), we obtain a continuous path $\widetilde{\gamma} : [0,1] \to \mathbb{C}$ connecting z to w. Note that $\widetilde{\gamma}([0,1])$ is compact (since $\widetilde{\gamma}$ is continuous), and so by Lemma 4.30, $d(\widetilde{\gamma}([0,1]), \partial A) > r$ for some $r > 0$. Choose $\varepsilon \in (0,r)$. By Lemma A.11, we can obtain a C^k curve $\widehat{\gamma}$ such that $\sup_{t\in[0,1]} |\widetilde{\gamma}(t) - \widehat{\gamma}(t)| < \varepsilon$ with $\widehat{\gamma}(0) = z$ and $\widehat{\gamma}(1) = w$ and fulfills $\widehat{\gamma}([0,1]) \subseteq A$. Therefore $w \in A'$, and this shows that A' is closed.
- Finally we argue that A' is open: Let $w \in A'$. Let $\varepsilon > 0$ such that $D(w,\varepsilon) \subseteq A$. There exists a continuous path γ from z to w, and similar as in the previous step, for every $u \in D(w,\varepsilon)$ we can extend γ to a continuous path $\gamma_u : [0,1] \to \mathbb{C}$ from z to u with $\gamma_u([0,1]) \subseteq A$, fulfilling $d(\gamma_u([0,1]), \partial A) > r_u$ for some $r_u > 0$. We can again replace γ_u by a C^k curve $\widehat{\gamma}_u$ with $\widehat{\gamma}_u(0) = z$ and $\widehat{\gamma}_u(1) = u$ with $\widehat{\gamma}_u([0,1]) \subseteq A$. This shows that $D(w,\varepsilon) \subseteq A'$ and so A' is open.

Since A is connected, we obtain that $A = A'$, so A is indeed C^k path-connected. □

[2] i.e. a domain

Mollification is also used to show that a homotopy between C^1 curves γ_0 and γ_1 can be turned into a C^1 homotopy. We present a sketch of the

Proof of Lemma 4.29. After reparametrization, we assume

$$\begin{aligned} \gamma_j(t) = \gamma_j(0) =: z_0, \qquad & \text{for } t \in [0, \varepsilon_0], j = 1, 2 \\ \gamma_j(t) = \gamma_j(1) =: z_1, \qquad & \text{for } t \in [1-\varepsilon_0, 1], j = 1, 2 \\ H(s, \cdot) = \gamma_0, \qquad & \text{for } s \in [0, \varepsilon_0] \\ H(s, \cdot) = \gamma_1, \qquad & \text{for } s \in [1-\varepsilon_0, 1]. \end{aligned} \tag{A.53}$$

Let now $\eta : \mathbb{R}^2 \to [0, \infty)$ be a two-dimensional Friedrichs mollifier. For $0 < \varepsilon \leq \frac{\varepsilon_0}{2}$, consider

$$H^{(\varepsilon)}(s,t) = (H * \eta_\varepsilon)(s,t), \qquad \varepsilon \leq s, t \leq 1 - \varepsilon. \tag{A.54}$$

Moreover we smoothen $\gamma_0 = \gamma_s = H(s, \cdot)$ $(0 \leq s \leq \varepsilon_0)$ by setting

$$H^{(\varepsilon)}(s,t) = (H * \eta_{r(s)})(s,t), \qquad 0 \leq s \leq \varepsilon \leq t \leq 1 - \varepsilon, \tag{A.55}$$

where $r : [0,1] \to \mathbb{R}$ is a C^1 function such that $r(s) = s$ for $0 \leq s \leq \varepsilon/2$, $r(\varepsilon) = \varepsilon$. We see that $\gamma_s^{(\varepsilon)} = H^{(\varepsilon)}(s, \cdot) \in C^1$ for $0 \leq s \leq \varepsilon$ and $\gamma_s^{(\varepsilon)} \to \gamma_0$ in C^1 as $s \downarrow 0$. We also smoothen $\gamma_1 = \gamma_s = H(s, \cdot)$ $(1 - \varepsilon_0 \leq s \leq 1)$ by setting

$$H^{(\varepsilon)}(s,t) = (H * \eta_{1-r(s)})(s,t), \qquad \varepsilon \leq t \leq 1 - \varepsilon \leq s \leq 1. \tag{A.56}$$

Finally, we set $H^{(\varepsilon)}(s,t) = z_0$ for $s \in [0,1]$ and $0 \leq t \leq \varepsilon$, $H^{(\varepsilon)}(s,t) = z_1$ for $s \in [0,1]$ and $1 - \varepsilon \leq t \leq 1$. Putting all parts of the definition together, we obtain the required C^1 homotopy. $\square$