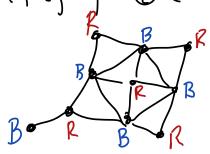
Lecture 4 and 5

Graph Colorings:

"Each # is a color"

def a coloring of a graph is a fn $\sigma : V \to \{1, \dots, n\}$ s.t.

$$\forall v_i, v_j \mid \sigma(v_i) = \sigma(v_j) \Rightarrow \mathrm{Adj}(v_i, v_j) = 0 .$$

Eg 

$\iff$ Partition V into $V_1 \cup V_2 \cup \dots \cup V_n$ all indep sets.

2 coloring $\iff$ Bipartite $\iff$ 2 indep sets partition V.

def $\chi(G)$ the chromatic number of G is the min k s.t. one can color G with k colors.

Eg. $\chi(K_n) = n$ $\quad \left(\chi(G) = |V_G| \iff G \cong K_{|V_G|} \right)$.

Lem: $H \leq G$ then $\chi(H) \leq \chi(G)$ "coloring induces coloring" delete edges.

Lem Let $q = \mathrm{Max}\{ |I| \mid I \subseteq V \text{ indep} \}$, $d_{max} = \mathrm{Max}\{ d(v) \mid v \in V \}$

Then $$\left\lceil \frac{n}{q} \right\rceil \leq \chi(G) \leq d_{max} + \cancel{1}$$ 0 by Brooks.

Pf: Each color set is indep so its size is $\leq q$. thus

$$n \leq \chi(G) \cdot q \Rightarrow \frac{n}{q} \leq \chi(G) \Rightarrow \left\lceil \frac{n}{q} \right\rceil \leq \chi(G) \quad ✓$$

(*) Enumerate the verts $\{v_1, \dots, v_n\}$, color each vertex in order with the lowest color not already among its neighbors (blank neighbors are OK).

Since each vertex has at most d_{max} neighbors, the lowest number not on the list of adj. numbers must be $\leq d_{max} + 1$. $\square$

It follows that $\frac{|V|}{q} \leq d_{max} + 1$ for ALL graphs.

A Polyn Invt

Let G be given: Pick $e \in E_G$.

def $G_{lose} = G \setminus e$

G_{fuse} = The graph obtained by fusing ∂e and removing duplicate edges.

def $P_G(k) \in \mathbb{Z}[k]$

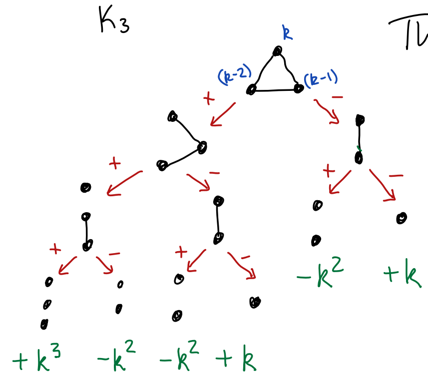
$$\left.\begin{array}{l} P_{Null(n)}(k) = k^n \\ \text{For any } e \in E \quad P_G(k) = P_{G_{lose}}(k) - P_{G_{fuse}}(k) \end{array}\right\} \text{Counts Colors.}$$

Works b/c $|E_{G_{fuse}}| \le |E_{G_{lose}}| \le |E_G| - 1$

Thus after $|E_G|$ steps $p_g(k) = \sum_n a_n k^n$.

K_3

Eg.



$\chi_{K_3}(k) = \boxed{k^3 - 3k^2 + 2k}$

Otoh: $\chi_{K_3}(k) = k(k-1)(k-2)$ OMG!

$= k(k^2 - 3k + 2) = \boxed{k^3 - 3k^2 + 2k}$

Eg: $G_1 \sqcup G_2 \rightsquigarrow P_{G_1} \cdot P_{G_2}$ $\quad (P_{G \cup \bullet} = k P_G)$

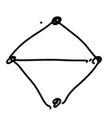
$P_{K_n}(k) = k(k-1)(k-2)\dots(k-n+1)$

$P_{Tree}(k) = k(k-1)^{|Tree|}$ (Idea: Pick a root then flow out)

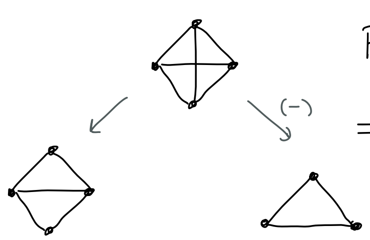
↳ Includes Paths.

$P_{C_n}(k) = (k-1)^n + (-1)^n(k-1)$ (Do C_3, then Lose-Fuse for strong induction)

Eg: G_4



Note This is K_4 Lose !!



$P_{K_4} = P_G - P_{C_3} \Rightarrow P_G = P_{K_4} + P_{C_3}$

$= k(k-1)(k-2)(k-3) + \left[(k-1)^3 - (k-1)\right]$

$= \boxed{k^4 - 5k^3 + 8k^2 - 4k}$

Lemma: The 2^{nd} leading coeff of $\chi_G(k) = -|E|$.

Pf: The null-graphs of order $n-1$ are those obtained by fusing one edge. This means they're all $(-)$ & there's $|E|$ of them ∎

Tutte Polynomial (of Multi Graph)

def: $T_G(x,y)$ def'd by:

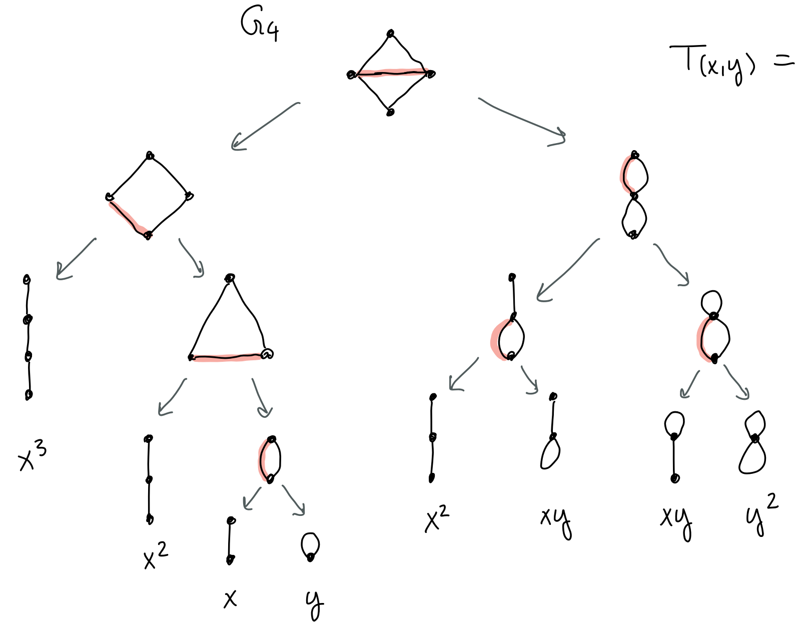
General Graph Fuse

$$T_G = T_{G;\,\text{Lose}} + T_{G;\,\text{Fuse}}$$

Let G be a tree with loops, then $T_G(x,y) = x^{|\text{Bridges}|}\, y^{|\text{Loops}|}$

G_4

$$T(x,y) = x^3 + 2x^2 + y^2 + 2xy + x + y.$$



Specializations: Let $c(G)$ be the # of cctd comps of G.

$$\chi_G(k) = (-1)^{|V_G| - c(G)}\, k^{c(G)}\, T_G(1-k, 0)$$

Eg: $T_{G_4}(1-k,0) = \left[(1-k)^3 + 2(1-k)^2 + (1-k)\right]$

$$\chi_{G_4}(k) = (-1)^{4-1}\, k\left[(1-k)^3 + 2(1-k)^2 + (1-k)\right]$$

$$= -k\left[1 - 3k + 3k^2 - k^3 + 2 - 4k + 2k^2 + 1 - k\right]$$

$$= -k\left[-k^3 + 5k^2 - 8k + 4\right] = \boxed{k^4 - 5k^3 + 8k^2 - 4k}$$

Liste:

$\chi_G(k) = (-1)^{|V| - |G|}\, k^{|G|}\, T_G(1-k, 0)$

$T(2,1) = $ # of Forests

$V_{\text{Kalt}}(q) = T_{\text{Black}(K)}\left(q, \frac{1}{q}\right)$

$T(1,1) = $ # Spanning forests

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