Lagrange Multipliers

- max/min with constraint.

$f(x,y)$ with $g(x,y) = C$

find where $\vec{\nabla} f = \lambda \vec{\nabla} g$

constant.

(3) $f(x,y) = x^2 - y^2$ $\quad \underbrace{x^2+y^2}_{g(x,y).} = 1$ find max & min.

$\vec{\nabla} f = \langle 2x, -2y \rangle$ $\quad \vec{\nabla} g = \langle 2x, 2y \rangle$

$\vec{\nabla} f = \lambda \vec{\nabla} g \Rightarrow$ $\quad \cancel{2}x = \lambda \cancel{2}x$

$-\cancel{2}y = \lambda \cancel{2}y$

now solve for x, y.

$\lambda x - x = 0 \qquad x(\lambda - 1) = 0 \quad$ (I)

$\lambda y + y = 0 \Rightarrow y(\lambda + 1) = 0 \quad$ (II)

according to (I), $x = 0$ OR $\lambda = 1$

use constraint and other equation.

if $x = 0$ $\quad 0^2 + y^2 = 1 \Rightarrow y = \pm 1$

if $\lambda = 1$ use (II) $y(1+1) = 0 \Rightarrow y = 0$

constraint: $x^2 + (0)^2 = 1 \Rightarrow x = \pm 1$

collect points:	$(0,1)$	$(0,-1)$	$(1,0)$	$(-1,0)$
evaluate $f(x,y)$	-1	-1	1	1

min $= -1$ at $(0, \pm 1)$

max $= 1$ at $(\pm 1, 0)$.

(5) $f(x,y) = xy \qquad 4x^2 + y^2 = 8$

$f_x = \lambda g_x \qquad \Rightarrow \qquad \boxed{y = \lambda 8x} \quad (1)$

$f_y = \lambda g_y \qquad\qquad x = \lambda 2y \quad$ sub (1)

$x = \lambda 2(\lambda 8x)$

$x = 16\lambda^2 x$

$x - 16\lambda^2 x = 0$

$x(1 - 16\lambda^2) = 0$

$x = 0$ or $\lambda = \pm \frac{1}{4}$

if $x = 0$ we have $4(0)^2 + y^2 = 8 \Rightarrow y = \pm 2\sqrt{2}$.

if $\lambda = \frac{1}{4}$, use (1) to obtain: $y = 2x$

if $\lambda = -\frac{1}{4}$ use (1) to obtain $y = -2x$

put into constraint

$4x^2 + (\pm 2x)^2 = 8$

$8x^2 = 8$

$x = \pm 1$

$x = 1 \Rightarrow y = -2$

$x = -1 \Rightarrow y = 2$

Collect points:	$(0, 2\sqrt{2})$	$(0, -2\sqrt{2})$	$(1, -2)$	$(-1, 2)$
Evaluate $f(x,y)$	0	0	-2	-2

max: $(0, 2\sqrt{2})$, $(0, -2\sqrt{2})$ — min: $(1,-2)$, $(-1,2)$

(10) $f(x,y,z) = \ln(x^2+1) + \ln(y^2+1) + \ln(z^2+1)$

$x^2 + y^2 + z^2 = 12$

find max/min.

$$f_x = \lambda g_x \qquad \frac{\cancel{2}x}{x^2+1} = \lambda \cancel{2}x \Rightarrow \frac{x}{x^2+1} - \lambda x = 0 \Rightarrow x\left(\frac{1}{x^2+1} - \lambda\right) = 0$$

$$f_y = \lambda g_y \quad \Rightarrow \qquad x = 0 \text{ or } \lambda = \frac{1}{x^2+1}$$

$$\frac{\cancel{2}y}{y^2+1} = \lambda \cancel{2}y \Rightarrow y = 0 \text{ or } \lambda = \frac{1}{y^2+1}$$

$$f_z = \lambda g_z \qquad \frac{\cancel{2}z}{z^2+1} = \lambda \cancel{2}z \Rightarrow z = 0 \text{ or } \lambda = \frac{1}{z^2+1}$$

if $x, y, z \neq 0$ $\quad \frac{1}{x^2+1} = \frac{1}{y^2+1} = \frac{1}{z^2+1}$

$$\Rightarrow \quad x^2+1 = y^2+1 = z^2+1$$

$x^2 = y^2 = z^2$ use $x^2+y^2+z^2 = 12$

get: $3x^2 = 12$ $\quad 3y^2 = 12$ $\quad 3z^2 = 12$

$x^2 = 4$

$x = \pm 2$ $\quad y = \pm 2$ $\quad z = \pm 2$

now how can we find our points?

$x = y = 0$ $\quad z = \pm\sqrt{12}$

$x = z = 0$ $\quad y = \pm\sqrt{12}$

$y = z = 0$ $\quad x = \pm\sqrt{12}$

also have

$$\boxed{\begin{array}{l} (0, 0, \pm\sqrt{12}) \\ (0, \pm\sqrt{12}, 0) \\ (\pm\sqrt{12}, 0, 0) \\ (\pm 2, \pm 2, \pm 2) \end{array}}$$

if $x = 0$ $\quad y, z$ nonzero $\quad y^2+z^2 = 12$ and $y^2 = z^2$

$y^2 = 6 \quad z^2 = 6$

$y = \pm\sqrt{6}$, $z = \pm\sqrt{6}$

we obtain:

$$\boxed{\begin{array}{l} (0, \pm\sqrt{6}, \pm\sqrt{6}) \\ (\pm\sqrt{6}, 0, \pm\sqrt{6}) \\ (\pm\sqrt{6}, \pm\sqrt{6}, 0) \end{array}}$$

$$f(x,y,z) = \ln(x^2+1) + \ln(y^2+1) + \ln(z^2+1)$$

$$\left.\begin{aligned}&(0,0,\pm\sqrt{12})\\&(0,\pm\sqrt{12},0)\\&(\pm\sqrt{12},0,0)\end{aligned}\right\} \rightarrow \ln(13) \leftarrow \text{min}$$

$$\left.\begin{aligned}&(0,\pm\sqrt{6},\pm\sqrt{6})\\&(\pm\sqrt{6},0,\pm\sqrt{6}\\&(\pm\sqrt{6},\pm\sqrt{6},0)\end{aligned}\right\} \rightarrow \ln(7)+\ln 7 = 2\ln 7 = \ln(49)$$

$$(\pm2,\pm2,\pm2) \rightarrow \ln(5)+\ln(5)+\ln(5) = 3\ln 5 = \ln(125)$$

max.

(14) $f(x_1,\dots,x_n) = x_1+x_2+\dots+x_n$

$x_1^2+\dots+x_n^2=1$

find max/min.

$$\begin{aligned}1&=\lambda 2x_1\\1&=\lambda 2x_2\\&\vdots\\1&=\lambda 2x_n\end{aligned} \Rightarrow \lambda 2x_1=\lambda 2x_2=\dots=\lambda 2x_n$$

$$x_1=x_2=\dots x_n$$

for some x_i $\quad \underbrace{x_i^2+x_i^2+\dots+x_i^2}_{n \text{ terms.}}=1$

$$n x_i^2=1 \rightarrow x_i^2=\frac{1}{n}$$

$$x_i=\pm\frac{1}{\sqrt{n}}$$

max $= \frac{n}{\sqrt{n}} = \sqrt{n}$ $\qquad$ min $= -\sqrt{n}$.

(17) $f(x,y,z) = x+y+z$ $\qquad x^2+z^2=2$ and $x+y=1$

$$\vec{\nabla} f = \lambda \vec{\nabla} g + \mu \vec{\nabla} h$$

$1 = 2x\lambda + \mu(1)$ $\qquad 1 = 2x\lambda + 1 \Rightarrow 2x\lambda = 0$

$x=0$ or $\cancel{\lambda=0}$

$\boxed{1 = \mu}$ $\qquad \to \qquad 1 = 2z\lambda$

$1 = 2z\lambda$

$x=0$ and $x+y=1 \Rightarrow 0+y=1 \Rightarrow y=1$

$0^2+z^2=2 \Rightarrow z = \pm\sqrt{2}$

	$(0,1,\sqrt{2})$	$(0,1,-\sqrt{2})$
$f(x,y,z)$	$1+\sqrt{2}$	$1-\sqrt{2}$
	Max	min.

(21) $f(x,y) = x^2+y^2+4x-4y$ $\quad , \quad x^2+y^2 \le 9$

$x^2+y^2<9$

use $\vec{\nabla} f = \vec{0}$

find c.p.

$f_x = 2x+4=0$

$f_y = 2y-4=0$

$x=-2 \quad y=2$

$(-2,2)$

$x^2+y^2=9$

use L.M.

$2x+4 = \lambda 2x$

$2y-4 = \lambda 2y$

$x, y \neq 0$

$\left.\begin{array}{l} \dfrac{x+2}{x} = \lambda \\ \dfrac{y-2}{y} = \lambda \end{array}\right\} \quad \dfrac{x+2}{x} = \dfrac{y-2}{y}$

$yx + 2y = xy - 2x$

$y = -x$

use $x^2+y^2=9$

$x^2+(-x)^2=9$

$x^2=\frac{9}{2} \Rightarrow x=\pm\frac{3}{\sqrt{2}}$

$y=\mp\frac{3}{\sqrt{2}}$

$\left(\frac{3}{\sqrt{2}},-\frac{3}{\sqrt{2}}\right),\left(-\frac{3}{\sqrt{2}},\frac{3}{\sqrt{2}}\right)$.

collect points:	$(-2,2)$	$\left(\frac{3}{\sqrt{2}},-\frac{3}{\sqrt{2}}\right)$	$\left(-\frac{3}{\sqrt{2}},\frac{3}{\sqrt{2}}\right)$
$f(x,y)=x^2+y^2+4x-4y$	$4+4-8-8$ $=-8$	$\frac{9}{2}+\frac{9}{2}+\frac{12}{\sqrt{2}}+\frac{12}{\sqrt{2}}$ $=\frac{18}{2}+\frac{24}{\sqrt{2}}$	$\frac{9}{2}+\frac{9}{2}-\frac{12}{\sqrt{2}}-\frac{12}{\sqrt{2}}$ $=\frac{18}{2}-\frac{24}{\sqrt{2}}=9-12\sqrt{2}$ ≈ -7.9
	min	max	

(24) $f(x,y)=2x+3y$ $\qquad \sqrt{x}+\sqrt{y}=5$

$2=\lambda\left(\frac{1}{2\sqrt{x}}\right) \rightarrow 4\sqrt{x}=\lambda$

$3=\lambda\left(\frac{1}{2\sqrt{y}}\right) \qquad 6\sqrt{y}=\lambda$

$\rightarrow 4\sqrt{x}=6\sqrt{y}$

$\sqrt{x}=\frac{3}{2}\sqrt{y}$

$\frac{3}{2}\sqrt{y}+\sqrt{y}=5$

$\frac{5}{2}\sqrt{y}=5$

$\sqrt{y}=2$

$y=4$

$x=9$

$f(9,4)=2(9)+3(4)=30.$

Q: $f(25,0)=50$

$(25,0)$ satisfies constraint.

$f(0,25) = 75$

∇g undefined at these points.

more importantly

∇g needs to be defined

at max/min.

verify using level curves of $f(x,y)$ and overlay $\sqrt{x}+\sqrt{y}=5$