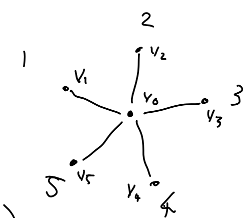
Lecture 7

Thm Every planar graph has a 5-coloring:

Pf Let $P \ni v_0$ be given, $\deg(v_0) \le 5$.

If $d(v_0) \le 4 \rightsquigarrow$ done by greedy.



If $d(v_0) = 5$ and all are diff color, then:

Idea **def**: $P_{i,j} \subseteq P$ The subgraph w/ $V_{i,j} = \mathcal{C}^{-1}(\{i,j\})$

"All verts colored i, j" Consider $P_{1,3}$.

Either v_1, v_3 are in the same comp of $(P \setminus \{v_0\})_{1,3}$, or they're Not. If so there's $\gamma : v_1 \to v_3 \subseteq P_{1,3} \setminus \{v_0\}$.

But in a planar diagram of P, γ cuts v_2 from v_4 so they are in diff comps. of $P_{2,4}$.

Now: Either v_1, v_3 are disccted in $P_{1,3}$ **or** v_2, v_4 disccted in $P_{2,4}$.

$\hookrightarrow$ Change $v_3 \rightsquigarrow 1$ $\qquad$ $\hookrightarrow$ Change $v_4 \rightsquigarrow 2$.

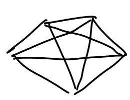
which free's a color for v_0 !

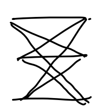
$\blacksquare$

Thm [Kuratowski]: A graph is non-planar $\iff$ It contains a subdivision of K_5 or $K_{3,3}$.

Idea: $3v - 6 \ge e$ for planar.

$15 - 6 \ge 10$ $\cancel{\times}$ $\Rightarrow K_5$ non-planar





$3 \cdot 6 - 6 \ge 9$

$12 \ge 9$

Show $K_{3,3}$ non-planar:

Assume it is, then bipartite $\Rightarrow |\partial f| \ge 4$. $\Rightarrow \sum_{f \in F} |\partial f| = 2|E|$

$\Rightarrow 4|F| \le 2|E| \Rightarrow F \le \frac{1}{2} E$

Now $E - V + 2 = F \leq \frac{1}{2} E \Rightarrow 2E - 2V + 4 \leq E \Rightarrow$

$-2V + 4 \leq -E \Rightarrow 2V - 4 \geq E$ "Bipartite planar Graphs"

Prop $\boxed{\begin{array}{l} G \text{ planar, cctd, } |E| \geq 2 \\ \text{I.F.T. } \dfrac{L(V-2)}{(L-2)} \geq E \end{array}}$ Then let L = length of shortest circuit or 3 if no circuits.

Pf: For $f \in F$, $\partial f = E_{ff'} \sqcup E_{ff}$ but

$E_{ff'}$ is a circuit so $|\partial f| \geq L$

(If no circuits then there's 1 face w/ $|\partial f| = 2E \geq 4$).

$$E - V + 2 \leq \frac{2}{L} E \Rightarrow LE - LV + 2L \leq 2E$$

$$\Rightarrow -LV + 2L \leq (2-L)E \Rightarrow \boxed{\frac{L}{L-2}(V-2) \geq E}$$

$K_{3,3}$: $\frac{4}{4-2}(6-2) \geq 9$?

$2(4) \geq 9$?

$\not\geq$.