Ito (General) Lemma

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$f(t,x) \in C^2$

$$\Rightarrow f(T,B_T) = f(0,B_0) + \int_0^T f_t(t,B_t)\,dt + \int_0^T f_x(t,B_t)\,dB_t + \frac{1}{2}\int_0^T f_{xx}(t,B_t)\,dt$$

Define $dX_t = a(t,\omega)\,dt + \sigma(t,\omega)\,dB_t$

$$\Leftrightarrow X_t = \underbrace{\int_0^t a(t,\omega)\,dt}_{\text{黎曼积分}} + \underbrace{\int_0^t \sigma(t,\omega)\,dB_t}_{\text{伊藤积分}} \quad \text{称为 Itô Process}$$

DRIFT　　VOLATILE (Oscillation)

QV: $\langle X_t, X_t\rangle_{[0,T]} = \lim_{\|\Pi\|\to 0} \sum_i (\Delta X_{t_i})^2 = \int_0^T \sigma^2(s,\omega)\,dt$

因此 $(dX_t)^2 = \sigma^2 dt$.

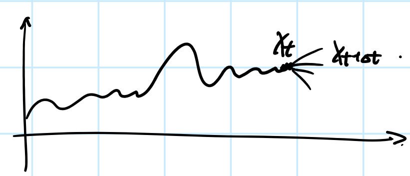
Pf.

考虑 $$(dX_t)^2 = (a\,dt + \sigma\,dB_t)^2 = a^2\underbrace{\cancel{(dt)^2}}_{0} + \underbrace{\cancel{2a\sigma\,dt\,dB_t}}_{0} + \sigma^2\underbrace{\cancel{(dB_t)^2}}_{dt}$$
$$= \sigma^2 dt$$

特别地，当 $a(t,\omega) = a(t,X_t)$

且 $\sigma(t,\omega) = \sigma(t,X_t)$ 时　满足马氏性质 Markov Property

即　Future 只取决于 现在而非未来



∵ $dX_t = a\,dt + \sigma\,dB_t$

∴ $X_{t+\Delta t} - X_t = a\Delta t + \underbrace{\sigma\Delta B_t}_{\text{randomness}}$

考虑 $$f(T,X_T) = f(0,X_0) + \int_0^T f_t(t,X_t)\,dt + \int_0^T f_x(t,X_t)\,dX_t + \frac{1}{2}\int_0^T f_{xx}(t,X_t)\underbrace{\cancel{(dX_t)^2}}_{\sigma^2 dt}$$

$$\boxed{\begin{array}{l}\int g\,dX_t \equiv \int g\cdot a\,dt + \int g\cdot \sigma\,dB_t \\ \text{其中 } dX_t = a\,dt + \sigma\,dB_t\end{array}}$$

$$+\frac{1}{2} f_{xx}(t,X_t)\underbrace{(dX_t)^2}_{\sigma^2 dt}$$

$$= f(0,X_0) + \int_0^T f_t(t,X_t)\,dt + \int_0^T f_x \cdot a\,dt + \int_0^T f_x\,\sigma\,dB_t + \frac{1}{2}\int_0^T f_{xx}\cdot\sigma^2\,dt$$

写作了 $$df(t,X_t) = f_t\,dt + a f_x\,dt + \sigma f_x\,dB_t + \frac{1}{2}\sigma^2 f_{xx}\,dt$$

DRIFT

Volatile.