Final Exam Problem Bank Solutions

Examination Date: Monday, December 19th

1. (30 points) In the following, let $S \subset \mathbb{R}$ be non-empty and bounded above. In this problem, we will see an interesting connection between the supremum and infinite sets.

   (a) Show that there exists a sequence $\{x_n\}$ with $x_n \in S$ for all $n \in \mathbb{N}$ such that $\lim_{n\to\infty} x_n = \sup S$.

   (*Hint*: Look at Q8 on HW1 and HW2)

   (b) Suppose $\sup S \notin S$. Show that there exists a strictly monotone increasing sequence $\{y_n\}$ with $y_n \in S$ for all $n \in \mathbb{N}$.

   (*Hint*: You may want to approach this problem inductively. Take some $y_1 \in S$, and note that $\sup S - y_1 > 0$ (why?). Use this with HW1 Q8 to find y_2, etc... When doing induction, you may want to prove that $\sup S - y_p > 0$ for all $p \in \mathbb{N}$.)

   (c) Suppose $\sup S \notin S$. Show that there exists a countably infinite subset $E \subset S$.

   (*Hint*: Consider the set $\{y_n : n \in \mathbb{N}\}$ defined for the sequence $\{y_n\}$ in (b). Can you find some bijection between this set and $\mathbb{N}$?)

   (d) Suppose $\sup S \in S$. Will there always be a countably infinite subset $E \subset S$? Either prove the statement, or give a counterexample.

   (*Note*: finite sets are never countably infinite)

(a) By Q8 on HW1, for all $n \in \mathbb{N}$ there exists some $x_n \in S$ such that $\sup S - 1/n < x_n \leq \sup S < \sup S + 1/n$.

Then for all $\varepsilon > 0$, take $M > 1/\varepsilon$. For all $n \geq M$,

$$|x_n - \sup S| < \frac{1}{n} \leq \frac{1}{M} < \varepsilon$$

Thus, $\{x_n\}$ defined this way converges to $\sup S$, as desired.

(b) Since S is non-empty, there exists some $y_1 \in S$. Note that $\sup S \geq y_1$. Furthermore, $y_1 \neq \sup S$ (otherwise $\sup S \in S$), so we can conclude $\sup S - y_1 > 0$.

Now, let us define the rest of the sequence inductively. Suppose y_p is defined, and furthermore assume that $\sup S - y_p > 0$. Then, by HW1 Q8 there exists some element $y_{p+1} \in S$ satisfying

$$\sup S - (\sup S - y_p) = y_p < y_{p+1} \leq \sup S$$

Again we must have $y_{p+1} \neq \sup S$, so $\sup S - y_{p+1} > 0$. This completes the induction step, showing that there exists a sequence $\{y_n\}$ such that $y_n \in S$ for all $n \in \mathbb{N}$, and $y_{n+1} > y_n$, so the sequence is strictly monotone increasing.

(c) Let $\{y_n\}$ be as defined in (b). Take the set $E := \{y_n : n \in \mathbb{N}\} \subset S$. Note that we can treat $\{y_n\}$ as a function $y : \mathbb{N} \to E$.

To show that this function is injective, note that since $\{y_n\}$ is strictly monotone, $n_1 > n_2$ implies $y(n_1) > y(n_2)$. Taking the contrapositive shows $y(n_2) \geq y(n_1)$ implies $n_1 \geq n_2$. Switching n_1 and n_2 shows that $y(n_2) = y(n_1)$ implies $n_2 = n_1$, so y is injective.

Futhermore, $y(\mathbb{N}) = E$, so y is surjective.

Thus, there exists a bijection between E and $\mathbb{N}$, so E is countably infinite.

(d) False. Take $S = \{1\}$. $\sup S = 1 \in S$, but every subset of S is finite. Hence, S has no countably infinite subsets.

2. (30 points) Let's take a look at limits of functions at infinity.

   Given a function $f : \mathbb{R} \to \mathbb{R}$ and some $L \in \mathbb{R}$, we say $f(x)$ *converges to* L as $x \to \infty$ if for all $\varepsilon > 0$, there exists some $M \in \mathbb{R}$ such that for all $x \geq M$,

   $$|f(x) - L| < \varepsilon$$

   In this case, we write $f(x) \to L$ as $x \to \infty$, or

   $$\lim_{x\to\infty} f(x) := L$$

   If f does not converge to any $L \in \mathbb{R}$ as $x \to \infty$, we say f *diverges* as $x \to \infty$.

   (a) Write down a corresponding definition for $f(x)$ to converge to L as $x \to -\infty$. Then, use the above definition and the definition you wrote to prove that

   $$\lim_{x\to\infty} \frac{1}{1+x^2} = \lim_{x\to-\infty} \frac{1}{1+x^2} = 0$$

   (b) Suppose $f : \mathbb{R} \to \mathbb{R}$ satisfies

   $$\lim_{x\to\infty} f(x) = \lim_{x\to-\infty} f(x) = L$$

   for some $L \in \mathbb{R}$. Define a function $g : \mathbb{R} \to \mathbb{R}$ by

   $$g(y) := \begin{cases} f(1/y) & y \neq 0 \\ L & y = 0 \end{cases}$$

   Show that g is continuous at 0.

   (*Hint*: For the ε-δ definition of continuity, show that you can break $|y| < \delta$ into three cases: $y = 0$, $1/y > 1/\delta$, or $-1/y > 1/\delta$. This might help you find the right value of δ.)

   (c) Continuing from (b), show that if f is continuous at 0, then

   $$\lim_{y\to\infty} g(y) = \lim_{y\to-\infty} g(y) = f(0)$$

(a) We say that $f(x)$ *converges to* L as $x \to -\infty$ if for all $\varepsilon > 0$, there exists some $M \in \mathbb{R}$ such that for all $x \leq M$,

$$|f(x) - L| < \varepsilon$$

Let $\varepsilon > 0$ be arbitrary. Take $M = \sqrt{2/\varepsilon} > 0$. Then for all $x \geq M$,

$$\left|\frac{1}{1+x^2}\right| = \frac{1}{1+x^2} \leq \frac{1}{x^2} \leq \frac{1}{M^2} = \frac{\varepsilon}{2} < \varepsilon$$

Similarly, take $K = -\sqrt{2/\varepsilon} < 0$. Then for all $x \leq K$,

$$\left|\frac{1}{1+x^2}\right| = \frac{1}{1+x^2} \leq \frac{1}{x^2} \leq \frac{1}{K^2} = \frac{\varepsilon}{2} < \varepsilon$$

Thus,

$$\lim_{x\to\infty} \frac{1}{1+x^2} = \lim_{x\to-\infty} \frac{1}{1+x^2} = 0$$

as desired.

(b) Let f, g be as given, and let $\varepsilon > 0$ be arbitrary. Then,

- there exists $M \in \mathbb{R}$ such that for all $x \geq M$, $|f(x) - L| < \varepsilon$
- there exists $K \in \mathbb{R}$ such that for all $x \leq K$, $|f(x) - L| < \varepsilon$

Take $\delta := 1/\max\{|M|, |K|, 1\} > 0$. Then, for all $y \in \mathbb{R}$ with $|y| < \delta$, we have:

- If $y > 0$, then $1/y > 1/\delta \leq |M| \geq M$.
  Thus, $|g(y) - L| = |f(1/y) - L| < \varepsilon$
- If $y < 0$, then $-1/y > 1/\delta \leq |K| \geq -K$, so $1/y \leq K$.
  Thus $|g(y) - L| = |f(1/y) - L| < \varepsilon$
- If $y = 0$, then $|g(y) - L| = |L - L| = 0 < \varepsilon$

Thus, g is continuous at 0.

(c) Suppose f is continuous at 0. Let $\varepsilon > 0$ be arbitrary. Then, there exists $\delta > 0$ such that for all $|x| < \delta$, we have $|f(x) - f(0)| < \varepsilon$

Take $M := 2/\delta$ and $K := -2/\delta$. We have for all $y \geq M > 0$, $0 < 1/y \leq 1/M < \delta$, so

$$|g(y) - f(0)| = |f(1/y) - f(0)| < \varepsilon$$

Similarly for all $y \leq K < 0$, $-\delta < 1/K \leq 1/y < 0$, so

$$|g(y) - f(0)| = |f(1/y) - f(0)| < \varepsilon$$

which shows that

$$\lim_{y\to\infty} g(y) = \lim_{y\to-\infty} g(y) = f(0)$$

3. (40 points) In this problem, we will take a look at different norms on spaces of functions $f : \mathbb{N} \to \mathbb{R}$. Such functions might arise from, e.g. the Fourier sine/cosine series of some periodic function $g : [0, 2\pi] \to \mathbb{R}$.

   Given some $f : \mathbb{N} \to \mathbb{R}$ and $\alpha \geq 0$, we say that $f \in \ell^\infty_\alpha$ if the set $\{k^\alpha |f(k)| : k \in \mathbb{N}\}$ is bounded. This is called the k^α *weighted* ℓ^∞ *space.*

   When $f \in \ell^\infty_\alpha$, we can define the ℓ^∞_α *norm* of f as

$$\|f\|_\alpha := \sup\{k^\alpha |f(k)| : k \in \mathbb{N}\} = \|k^\alpha f\|_u$$

   where $\|\cdot\|_u$ is the uniform norm. Note that $\|f\|_0 = \|f\|_u$, so these norms generalize the uniform norm.

   (a) Prove the triangle inequality for the ℓ^∞_α norm: if $f, g \in \ell^\infty_\alpha$, then

$$\|f + g\|_\alpha \leq \|f\|_\alpha + \|g\|_\alpha$$

   (b) Let $m \in \mathbb{N}$. Show that $\ell^\infty_m \subset \ell^\infty_{m-1}$.

   (c) We say a collection of functions $S \subset \ell^\infty_\alpha$ is *bounded in* ℓ^∞_α *norm* if there exists some $B \in \mathbb{R}$ such that for all $f \in S$, $\|f\|_\alpha \leq B$.

   Show that the set of functions $S = \{g_n : n \in \mathbb{N}\}$ given by

$$g_n(k) := \begin{cases} 1 & k = n \\ 0 & k \neq n \end{cases}$$

   is bounded in ℓ^∞_0 norm, but not bounded in ℓ^∞_m norm for any $m \in \mathbb{N}$.

   (d) We say a collection of functions $K \subset \ell^\infty_0$ is *uniformly small at infinity (USI)* if for all $\varepsilon > 0$, there exists some $C \in \mathbb{N}$ such that for all $k \geq C$ and for all $f \in K$,

$$|f(k)| < \varepsilon$$

   Show that if K is bounded in ℓ^∞_m norm for some $m \in \mathbb{N}$, then K is uniformly small at infinity.

   (*Hint*: If B is the bound of K in ℓ^∞_m norm, try picking M such that $B/k^m < \varepsilon$ for all $k \geq M$ (why can you do this?). Then, what happens to $|f(k)|$?)

   (e) Let $\{f_n\}$ be a sequence of functions in ℓ^∞_0 converging pointwise to some $f \in \ell^\infty_0$. Show that if the collection $\{f_n : n \in \mathbb{N}\}$ is uniformly small at infinity, then f_n converges to f uniformly.

   (*Hint*: Use the USI condition to find some cutoff C such that $|f_n(k) - f(k)| < \varepsilon$ for all $n \in \mathbb{N}$ and all $k \geq C$. Then there are only a finite number of sequences $\{f_n(k)\}^\infty_{n=1}$, $k < C$ to deal with.)

   Remark: The 'weights' used in this problem have something to do with the idea of "decay in Fourier coefficients corresponds to regularity in real space". Parts (d) and (e) have something to do with compact subsets of function spaces – notice that for a sequence $\{f_n\}$ in some bounded USI collection K, each of the sequences $\{f_n(k)\}^\infty_{n=1}$ is a bounded sequence of real numbers. Perhaps we could extract pointwise convergent subequences... (check out the Arzelà-Ascoli theorem if you are interested!)

(a) Let $f, g \in \ell^\infty_\alpha$. Then, using the triangle inequality for the uniform norm,

$$\|f+g\|_\alpha = \|k^\alpha(f+g)\|_u \leq \|k^\alpha f\|_u + \|k^\alpha g\|_u = \|f\|_\alpha + \|g\|_\alpha$$

(b) Let $f \in \ell^\infty_m$. Then, since $k^m = kk^{m-1} \geq k^{m-1}$, we have for all $k \in \mathbb{N}$

$$k^{m-1}|f(k)| \leq k^m|f(k)| \leq \|f\|_\beta$$

Thus, the set $\{k^{m-1}|f(k)| : k \in \mathbb{N}\}$ is bounded, hence $f \in \ell^\infty_{m-1}$. Thus, $\ell^\infty_m \subset \ell^\infty_{m-1}$

(c) Note that we can compute

$$\|g_n\|_\alpha = n^\alpha$$

In particular, $\|g_n\|_0 = 1$ for all $n \in \mathbb{N}$, so the given set S is bounded in ℓ^∞_0 norm. However, $\|g_n\|_m = n^m$ is not bounded (e.g. by the Archimedian property, we can find $n \geq B$ for any possible bound B), so S is not bounded in ℓ^∞_m norm for any $m \in \mathbb{N}$.

(d) Let $\varepsilon > 0$ be arbitrary.

Suppose K is bounded in ℓ^∞_m norm for some $m \in \mathbb{N}$. Then, there exists some B such that for all $f \in K$ and for all $k \in \mathbb{N}$,

$$k^m|f(k)| \leq \|f\|_m \leq B$$

Since $B/k^m \to 0$ as $k \to \infty$, there exists some C such that for all $k \geq C$, $B/k^m < \varepsilon$. Thus, for all $k \geq C$ and for all $f \in K$,

$$|f(k)| \leq B/k^m < \varepsilon$$

Thus, K is uniformly small at infinity.

(e) Suppose $\{f_n\}$ converges pointwise to f, and that the collection $\{f_n : n \in \mathbb{N}\}$ is uniformly small at infinity.

First, let $\varepsilon > 0$ be arbitrary. Then, by USI there exists some $C \in \mathbb{N}$ such that for all $k \geq C$ and for all $n \in \mathbb{N}$,

$$|f_n(k)| < \varepsilon/2$$

Then since $f_n(k) \to f(k)$ as $n \to \infty$, we can take the limit of the above inequality to get that for all $k \geq C$,

$$|f(k)| \leq \varepsilon/2 < \varepsilon$$

Thus, we get that for all $k \geq C$ and all $n \in \mathbb{N}$,

$$|f_n(k) - f(k)| \leq |f_n(k)| + |f(k)| < \varepsilon$$

Now, consider $k < C$. Since $f_n(k) \to f(k)$ as $k \to \infty$, there exists some $M_k \in \mathbb{N}$ such that for all $n \geq M_k$, $|f_n(k) - f(k)| < \varepsilon$.

Take $M := \max\{M_1, M_2, ..., M_{C-1}\}$. Then for all $n \geq M$, we have

$$|f_n(k) - f(k)| < \varepsilon \quad \forall k \in \mathbb{N}$$

or in other words, for all $n \geq M$,

$$\|f_n - f\|_u \leq \varepsilon$$

thus, $f_n \to f$ uniformly as $n \to \infty$.