Lecture 6

Planar Graphs:

General Graph!

def G is planar If G can be drawn w/o edge crossings

def A face of planar G is a cctd subset of $\mathbb{R}^2 \setminus G$.

Lem $\sum_{f \in F} |\partial f| = 2e$ $\qquad \partial f = \{\text{edges that walk around } f\}$.

"Handshake for dual".

Prop For G planar: $V - E + F = 2$.

Pf: Take a Spanning Tree: $V - E = 1$, $F = 1$

$V - E + F = 2$ Then each new edge adds: 1-edge & 1 new face.

$\Rightarrow V - E + F = 2$. ∎

Thm a cctd planar graph has a vertex w/ $\deg \leq 5$. ← Both verts have $d=1$.

Pf: No loops or bigons $\Rightarrow \forall f \in F_G \,|\, |\partial f| \geq 3$. (Except •—•)

$$\Rightarrow \sum_{f \in F} 3 \leq \sum_{f \in F} |\partial f| = 2|E| \Rightarrow 3|F| \leq 2|E|$$

$$\Rightarrow |E| - |V| + 2 = |F| \leq \frac{2}{3}|E| \Rightarrow \frac{1}{3}|E| + \frac{6}{3} \leq |V|$$

$$\Rightarrow |E| + 6 \leq 3|V| \Rightarrow \boxed{|E| \leq 3|V| - 6}$$ Test if planar.

Now compute avg degree:

$$\frac{1}{|V|}\sum d(v) \leq \frac{6|V| - 12}{|V|} \leq 6 - \frac{12}{|V|} < 6 \Rightarrow \left[\exists\, v \in V \,|\, d(v) \leq 5\right]$$

∎

<u>Thm</u> Every planar graph has a 6-coloring.

Pf: Bwoc. Let G be a minimal graph w/o 6-coloring.

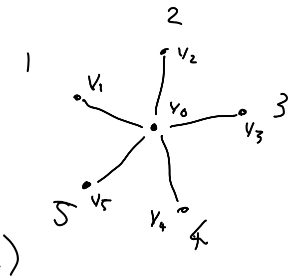
G planar $\Rightarrow \left(\exists v_0 \in V_G \,\middle|\, d(v_0) \le 5\right) \Rightarrow G \setminus v_0$ has a 6 coloring

$\Rightarrow$ greedy can color v_0 ∎

<u>Thm</u> Every planar graph has a 5-coloring:

<u>Pf</u> Let $P \ni v_0$ be given, $\deg(v_0) \le 5$.

If $d(v_0) \le 3 \rightsquigarrow$ done by greedy.



If $d(v_0) = 5$ and all are diff color, then:

<u>Idea</u> <u>def</u>: $P_{i,j} \subseteq P$ The subgraph w/ $V_{i,j} = \sigma^{-1}(\{i,j\})$

"All verts colored i, j" Consider $P_{1,3}$.

Either v_1, v_3 are in the same comp of $(P \setminus \{v_0\})_{1,3}$, or they're Not. If so there's $\gamma: v_1 \to v_3 \subseteq P_{1,3} \setminus \{v_0\}$.

But in a planar diagram of P, γ cuts v_2 from v_4 so they are in diff comps. of $P_{2,4}$.

<u>Now</u>: Either v_1, v_3 are discctd in $P_{1,3}$ <u>OR</u> v_2, v_4 discctd in $P_{2,4}$.

↳ Change $v_3 \rightsquigarrow 1$ ↳ Change $v_4 \rightsquigarrow 2$.

which free's a color for v_0! ∎