1. Find subgroups H and K of D_4 satisfying: H is a normal subgroup of D_4, K is a normal subgroup of H, but K is not a normal subgroup of D_4.

   (Remark. This exercise shows that a normal subgroup of a normal subgroup of a group G may not be a normal subgroup of G)

   Solution: Let $H = \{1, \rho^2, r, \rho^2 r\}$ and $K = \{1, r\}$.

   Since $[D_4 : H] = [H : K] = 2$, H is a normal subgroup of D_4 and K is a normal subgroup of H.

   K is not a normal subgroup of H since $\rho r \rho^{-1} = \rho(\rho r) = \rho^2 r \notin K$.

2. $n \geq 3$. The dihedral group D_n can be regarded as a subgroup of the symmetric group S_n by considering how the n vertices of the regular n-gon (with centre at origin and one vertex at $(1, 0)$) are permuted. Is the element $r \in D_n$ an odd permutation or even permutation? (r denotes the reflection along x-axis)

   Solution: If n is odd, let $n = 2k + 1$. r is a product of $\frac{n-1}{2} = k$ transpositions, so r is odd if k is odd, i.e., $n \equiv 3 \pmod 4$; r is even if k is even, i.e., $n \equiv 1 \pmod 4$

   If n is even, let $n = 2k$. r is a product of $\frac{n-2}{2} = k - 1$ transpositions, so r is odd if k is even, i.e., $n \equiv 0 \pmod 4$; r is even if k is odd, i.e., $n \equiv 2 \pmod 4$

   In summary, r is an odd permutation if $n \equiv 0, 3 \pmod 4$, r is an even permutation if $n \equiv 1, 2 \pmod 4$

3. X is the set of all the basis of $\mathbb{R}^n$. The group $GL_n(\mathbb{R})$ acts on the set X as follows: for any $A \in GL_n(\mathbb{R})$, any $(\vec{v}_1, ..., \vec{v}_n) \in X$,

   $$A.(\vec{v}_1, ..., \vec{v}_n) = (A\vec{v}_1, ..., A\vec{v}_n)$$

   Determine if this action is transitive or not.

   Solution:

   Let $(\vec{e}_1, ..., \vec{e}_n)$ be the standard basis of $\mathbb{R}^n$. For any basis $(\vec{v}_1, ..., \vec{v}_n)$, let A be the matrix whose columns are exactly $\vec{v}_1, ..., \vec{v}_n$, i.e., $A = \begin{bmatrix}\vec{v}_1 & ... & \vec{v}_n\end{bmatrix}$, then we get

   $$A.(\vec{e}_1, ..., \vec{e}_n) = (A\vec{e}_1, ..., A\vec{e}_n) = (\vec{v}_1, ..., \vec{v}_n)$$

   (Recall that $A\vec{e}_j$ is the j-th column vector of A)

We can therefore conclude that $\mathcal{O}((\vec{e}_1, ..., \vec{e}_n)) = X$, hence the action is transitive.

4. G is a group acting on a set X. S is a set. Let $M(X, S)$ to be the set of all functions $X \longrightarrow S$.

   (i). Prove $(g.f)(x) = f(g^{-1}.x)$ defines a group action of G on $M(X, S)$.

   (ii). If S has more than one elements, prove the action defined above is not transitive.

   Solution:

   (i). For any $f \in M(X, S)$, $x \in X$, $(1.f)(x) = f(1^{-1}.x) = f(x)$, so $1.f = f$

   For any $g_1, g_2 \in G$, any $x \in X$:

   $(g_1.(g_2.f))(x) = (g_2.f)(g_1^{-1}.x) = f(g_2^{-1}.(g_1^{-1}.x)) = f((g_2^{-1}g_1^{-1}).x) = f((g_1g_2)^{-1}.x) = ((g_1g_2).f)(x)$

   So $(g_1.(g_2.f)) = (g_1g_2).f$

   (ii). If S has more than one elements, take $s_1 \in S$ and $s_2 \in S$ such that $s_1 \neq s_2$. Let $f_1 : X \longrightarrow S$ be the function with constant value s_1 and $f_2 : X \longrightarrow S$ be the function with constant value s_2. Then for any $g \in G$ and any $x \in X$, $(g.f_1)(x) = f_1(g^{-1}.x) = s_1$, so we see that $g.f_1 \neq f_2$ no matter which $g \in G$ we take, which indicates the action is not transitive.

5. G is a finite group acting on a finite set S. For each $g \in G$, define the set $S^g = \{s \in S | g.s = s\}$.

   (i). Prove $\sum_{s\in S} |G_s| = \sum_{g\in G} |S^g|$.

   (ii). Prove $\sum_{s\in S} |G_s| = |G| \times n$, where n is the number of orbits in S.

   Solution:

   (i). Define a function (We can call it the characteristic function of the group action) $\chi : G \times S \longrightarrow \{0, 1\}$ by:

$$\chi(g, s) = \begin{cases} 1, & \text{if } g.s = s \\ 0, & \text{if } g.s \neq s \end{cases}$$

   Then $|G_s| = \sum_{g\in G} \chi(g, s)$ and $|S^g| = \sum_{s\in S} \chi(g, s)$.

$$\sum_{s\in S} |G_s| = \sum_{s\in S}\sum_{g\in G} \chi(g, s) = \sum_{g\in G}\sum_{s\in S} \chi(g, s) = \sum_{g\in G} |S^g|$$

(ii). Let $\mathcal{O}$ be the set of orbits in this action.

$$\sum_{s\in S}|G_s| = \sum_{s\in S}\frac{|G|}{|O_s|} = |G|\sum_{s\in S}\frac{1}{|O_s|} = |G|\sum_{O\in\mathcal{O}}\sum_{s\in O}\frac{1}{|O|} = |G|\sum_{O\in\mathcal{O}}1 = |G||\mathcal{O}| = |G|\times n$$

6. Given a group action of G on a set X, we define the kernel of the group action to be $K = \{g \in G|\forall x \in X, g.x = x\}$.

(i). Prove that K is a normal subgroup of G.

(ii). We say an action is **faithful** if for any $g \neq g'$ in G, there exists $x \in X$ such that $g.x \neq g'.x$. Prove an action is faithful if and only if its kernel is trivial.

(iii). Prove that there is a well-defined faithful induced group action of G/K on X by $(gK).x = g.x$

Solution:

(i). For any $g_1, g_2 \in K$, any $x \in X$, $(g_1^{-1}g_2).x = (g_1^{-1}).(g_2.x) = g_1^{-1}.x = g_1.(g_1^{-1}.x) = (g_1g_1^{-1}).x = 1.x = x$, so $g_1^{-1}g_2 \in K$, K is a subgroup of G.

For any $g \in K$ and $\gamma \in G$, any $x \in X$,

$$(\gamma g\gamma^{-1}).x = \gamma.(g.(\gamma^{-1}.x)) = \gamma.(\gamma^{-1}.x) = x$$

So $\gamma g\gamma^{-1} \in K$, K is a normal subgroup of G.

(ii). If the action is faithful, for any $g \neq 1$, there exists $x \in X$ such that $g.x \neq 1.x = x$, so $g \notin K$. We get $K = \{1\}$.

If $K = \{1\}$, then for any $g_1 \neq g_2$ in G, $g_1^{-1}g_2 \neq 1$, so $g_1^{-1}g_2 \notin K = \{1\}$, there exists $x \in X$ such that $g_1^{-1}g_2.x \neq x$, i.e. $g_1.x \neq g_2.x$, so the action is faithful.

(iii). For any $g_1K = g_2K$, $g_1^{-1}g_2 \in K$, so for any $x \in X$, $g_1^{-1}g_2.x = x$, i.e. $g_1.x = g_2.x$, $g_1K.x = g_2K.x$, the action is well-defined.

If gK is in the kernel of the induced action, then $gK.x = g.x = x$ for all $x \in X$, we see $g \in K$, $gK = K$, so the action kernel of the induced action is trivial. By (ii), the action is faithful.