Paul-Levy Thm

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X_t CTS martingale → all trajectories 无 jump

Then 3者等价

考虑 $\log \|B_t\|$ 是 local martingale 但非 CTS martingale.

$$\begin{cases} X_t \quad BM \\ \langle X_t, X_t \rangle = t \\ X_t^2 - t \quad \text{martingale} \end{cases}$$

只需证 有独立增量 $X_{s+t} - X_s \mid \mathcal{F}_s \sim N(0,t)$

即证 $\mathbb{E}\left(e^{iu(X_{s+t} - X_t)} \mid \mathcal{F}_s\right) = e^{-\frac{u^2 t}{2}}$ ← $N(0,1)$ 分布

考虑 $\mathbb{E}(X \mid \mathcal{F}) = y$

若 1) $y \in \mathcal{F}$

2) $\forall A \in \mathcal{F}$ 有 $\mathbb{E}(X \cdot \mathbb{1}_A) = \mathbb{E}(y \cdot \mathbb{1}_A)$

只需证 $\forall A \in \mathcal{F}_s$ 有 $\mathbb{E}\left(e^{iu(X_{s+t} - X_s)} \cdot \mathbb{1}_A\right) = \mathbb{P}(A) \cdot e^{-\frac{u^2 t}{2}}$

对 $f(x) = e^{iux}$ $f_t = 0$, $f_x = iue^{iux}$, $f_{xx} = -u^2 e^{iux}$

有 $e^{iu(X_{s+t} - s)} = 1 + \int_0^t iue^{iu(X_{s+r} - X_s)} dX_{s+r} - \frac{1}{2}u^2 \int_0^t e^{iu(X_{r+s} - X_s)} dr$ — 考虑 2) 性质 $\langle X_t, X_t \rangle = t$

有 $\mathbb{1}_A e^{iu(X_{s+t} - X_s)} = \mathbb{1}_A + \mathbb{1}_A \int_0^t iue^{iu(X_{s+r} - X_s)} dX_{s+r} - \mathbb{1}_A \cdot \frac{1}{2}u^2 \int_0^t e^{iu(X_{r+s} - X_s)} dr$

$\therefore \mathbb{E}(LHS) = \mathbb{P}(A) + \boxed{\mathbb{E}\left(iu \int_0^t e^{iu(X_{s+r} - X_s)} dX_{s+r} \, \mathbb{1}_A\right)} \boxed{- \frac{u^2}{2}\mathbb{E}\left(\int_0^t \mathbb{1}_A e^{iu(X_{s+r} - X_s)} dr\right)}$

考虑 $= \frac{u^2}{2}\int_0^t \mathbb{E}\left(\mathbb{1}_A e^{iu(X_{s+r} - X_s)}\right) dr$

考虑 $\varphi_t = \mathbb{P}(A) - \frac{u^2}{2}\int_0^t \varphi_r \, dr$

有 $\varphi_t' = -\frac{u^2}{2}\varphi_t$. 解得 $\varphi_t = Ce^{-\frac{u^2 t}{2}}$

求解 c: $t = 0$ 时 有 $\varphi_0 = \mathbb{P}(A) \Rightarrow c = \mathbb{P}(A)$.

考虑 $= iu \cdot \mathbb{E}\left(\mathbb{E}\left(\int_0^t e^{iu(X_{s+r} - X_s)} dX_{s+r} \cdot \mathbb{1}_A \mid \mathcal{F}_s\right) \mid \mathcal{F}_0\right)$

$= iu\, \mathbb{E}\left(\mathbb{1}_A \underbrace{\mathbb{E}\left(\int_0^t \sigma \, dX_{s+r} \mid d\hat{\mathcal{F}}_s\right)}_{\text{martingale}} \mid \mathcal{F}_0\right)$

$= iu\left(\mathbb{E}(\mathbb{1}_A \cdot 0) \mid \mathcal{F}_0\right) = 0$

从而有 $\mathbb{E}(LHS) = 0$