First order

homogeneous

$\frac{dy}{dt} + a(t)y = 0$

sol:$y(t) = c\exp\Big(-\int a(t)dt\Big)$

init-val: $\ln|y(t)| - \ln|y(t_0)| = -\int_{t_0}^{t} a(s)ds \Rightarrow y(t) = y_0\exp\Big(-\int_{t_0}^{t} a(s)ds\Big)$

non-homo

$\frac{dy}{dt} + a(t)y = b(t)$

select $\mu(t) = \exp\Big(\int a(t)dt\Big)$

$y = \frac{1}{\mu(t)}\Big(\int \mu(t)b(t)dt + c\Big)$

init-val: $\mu(t)y - \mu(t_0)y_0 = \int_{t_0}^{t} \mu(s)b(s)ds$

seprable

$\frac{dy}{dt} = \frac{g(t)}{f(y)}$

$\int f(y)dy = \int g(t)dt + C$

init-value: $\int_{y_0}^{y} f(r)dr = \int_{t_0}^{t} g(s)ds$

if $\frac{dy}{dt} = f(y)g(t), \text{ and } f(y_0) = 0$, then $y(t) = y_0$ is the only solution.

$M(y,t) + N(y,t)\frac{dy}{dt} = 0$

$M_y = N_t \to$ exact

$\phi = \int M$

$\frac{M_y - N_t}{N}$

exact

$M(y,t) + N(y,t)\frac{dy}{dx} = 0$

test:$M_y = N_t$?

if yes, find $\phi(y,t) s.t. \phi_t = M, \phi_y = N$ (by $\int M$), $\phi = C$ is the implicit solution. $C = \phi(t_0, y_0)$ if init-val given

if not, exist $\mu(t,y)$ to make equation exact if

- $p(t) = \frac{M_y - N_t}{N}$ is a function of t
- $p(y) = \frac{N_t - M_y}{M}$ is a function of y

then $\mu(t) = \exp\Big(\int pdt\Big)$ or $\mu(y) = \exp\Big(\int pdy\Big)$

picard iter: $y_{n+1}(t) = y_0 + \int_{t_0}^{t} f(s, y_n(s))ds$

existence-uniqueness: $M = \max_{(t,y) \text{ in } R} |f(t,y)|$, $\alpha = \min\Big(a, \frac{b}{M}\Big)$ $\Rightarrow$unique solution $y(t)$ on $[t_0, t_0 + \alpha]$

Example 4. Show that the solution $y(t)$ of the initial-value problem

$$\frac{dy}{dt} = e^{-t^2} + y^3, \qquad y(0) = 1$$

exists for $0 \leqslant t \leqslant 1/9$, and in this interval, $0 \leqslant y \leqslant 2$.

Solution. Let R be the rectangle $0 \leqslant t \leqslant \frac{1}{9}$, $0 \leqslant y \leqslant 2$. Computing

$$M = \max_{(t,y) \text{ in } R} e^{-t^2} + y^3 = 1 + 2^3 = 9,$$

we see that $y(t)$ exists for

$$0 \leqslant t \leqslant \min\left(\tfrac{1}{9}, \tfrac{1}{9}\right)$$

and in this interval, $0 \leqslant y \leqslant 2$.

Second-order linear differential equations

$\frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(y)y = 0$
Existence-uniqueness Theorem: let $p(t), q(t)$ continuious for $t \in (\alpha, \beta)$, then there is a unique $y(t)$ satisfying the equation in the interval and IV.
linear combination of solution is the general solution

linear equations with constant coefficient**

$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = 0$
characteristic equation: $ar^2 + br + c = 0$

Case1: distinct root:
$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$
Case 2: complex roots $r = \alpha \pm \beta i$
$y(t) = e^{\alpha t}(c_1 \cos(\beta t) + c_2 \sin(\beta t))$
Case 3: equale root
$y(t) = (c_1 + c_2 t)e^{rt}$

non-homo
general solution: $y(t) = c_1 y_1(t) + c_2 y_2(t) + \psi(t)$ where ψ is a particular solution
difference of non homo equaiotn is a solution of homo equation

method of reduction of order
for $y'' + p(t)y' + q(t)y = 0$ given $y_1(t)$, want to find $y_2(t)$
calculate $u(t) = \frac{\exp(-\int p(t)dt)}{y_1^2(t)}$
then $y_2(t) = y_1(t)\int u(t)dt$

method of variation of parm
know $y_1(t), y_2(t)$, want solve $L[y] = g(t)$
$\psi(t) = u_1(t)y_1(t) + u_2(t)y_2(t)$
where:
$u_1(t) = \int -\frac{g(t)y_2(t)}{W[y_1, y_2]}dt,$
$u_2(t) = \int \frac{g(t)y_1(t)}{W[y_1, y_2]}dt$

Series solution

$p(t)\frac{d^2y}{dt^2} + q(t)\frac{dy}{dt} + r(t)y = 0$
solve when $p, q. r$ are polynomial

1. let $y(t) = \sum_{n=0}^{\infty} c_n(t-a)^n$, find y', y''
2. insert to DE
3. change index of $\sum$ to collect similar power term
4. set coeff of like power term to 0 to get recurrence relation
5. use initial value to solve recurrence

Example 4. Solve the initial-value problem

$$L[y]=(t^2-2t)\frac{d^2y}{dt^2}+5(t-1)\frac{dy}{dt}+3y=0; \qquad y(1)=7, \quad y'(1)=3. \quad (15)$$

Hence, the differential equation (15) can be written in the form

$$L[y]=\left[(t-1)^2-1\right]\frac{d^2y}{dt^2}+5(t-1)\frac{dy}{dt}+3y=0.$$

Setting $y(t)=\sum_{n=0}^{\infty}a_n(t-1)^n$, we compute

$$\begin{aligned}
L[y](t)&=\left[(t-1)^2-1\right]\sum_{n=0}^{\infty}n(n-1)a_n(t-1)^{n-2}\\
&\quad+5(t-1)\sum_{n=0}^{\infty}na_n(t-1)^{n-1}+3\sum_{n=0}^{\infty}a_n(t-1)^n\\
&=-\sum_{n=0}^{\infty}n(n-1)a_n(t-1)^{n-2}\\
&\quad+\sum_{n=0}^{\infty}n(n-1)a_n(t-1)^n+\sum_{n=0}^{\infty}(5n+3)a_n(t-1)^n\\
&=-\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}(t-1)^n+\sum_{n=0}^{\infty}(n^2+4n+3)a_n(t-1)^n.
\end{aligned}$$

Setting the sums of the coefficients of like powers of t equal to zero gives $-(n+2)(n+1)a_{n+2}+(n^2+4n+3)a_n=0$, so that

$$a_{n+2}=\frac{n^2+4n+3}{(n+2)(n+1)}a_n=\frac{n+3}{n+2}a_n, \qquad n\geqslant 0. \qquad (16)$$

To satisfy the initial conditions, we set $a_0=7$ and $a_1=3$. Then, from (16),

$$\begin{aligned}
&a_2=\frac{3}{2}a_0=\frac{3}{2}\cdot 7, \qquad a_4=\frac{5}{4}a_2=\frac{5\cdot 3}{4\cdot 2}\cdot 7, \qquad a_6=\frac{7}{6}a_4=\frac{7\cdot5\cdot3}{6\cdot4\cdot2}\cdot 7,\ldots\\
&a_3=\frac{4}{3}a_1=\frac{4}{3}\cdot 3, \qquad a_5=\frac{6}{5}a_3=\frac{6\cdot 4}{5\cdot 3}\cdot 3, \qquad a_7=\frac{8}{7}a_5=\frac{8\cdot6\cdot4}{7\cdot5\cdot3}\cdot 3,\ldots
\end{aligned}$$

and so on. Proceeding inductively, we find that

$$a_{2n}=\frac{3\cdot5\cdots(2n+1)}{2\cdot4\cdots(2n)}\cdot 7 \quad\text{and}\quad a_{2n+1}=\frac{4\cdot6\cdots(2n+2)}{3\cdot5\cdots(2n+1)}\cdot 3 \qquad (\text{for } n\geqslant 1).$$

Hence,

$$\begin{aligned}
y(t)&=7+3(t-1)+\frac{3}{2}\cdot7(t-1)^2+\frac{4}{3}\cdot3(t-1)^3+\ldots\\
&=7+7\sum_{n=1}^{\infty}\frac{3\cdot5\cdots(2n+1)(t-1)^{2n}}{2^n n!}+3(t-1)+3\sum_{n=1}^{\infty}\frac{2^n(n+1)!(t-1)^{2n+1}}{3\cdot5\cdots(2n+1)}.
\end{aligned}$$

Singular points

$L[y]=P(t)y''+Q(t)y'+R(t)y=0$ is singular at $t=t_0$ if $P(t_0)=0$

Euler equation $t^2y''+\alpha ty'+\beta y=0$

solution of Euler equation:

solve for r:$r^2+(\alpha-1)r+\beta=0$

Case1: distinct root of r

$y(t)=c_1t^{r_1}+c_2t^{r_2}$

Case2: equal root:

$y(t)=(c_1+c_2\ln t)t^r$

Case3: complex root:

$$r=\underbrace{\frac{1-\alpha}{2}}_{\lambda}\pm i\underbrace{\frac{\left(4\beta-(\alpha-1)^2\right)^{\frac{1}{2}}}{2}}_{\mu}$$

$y=t^{\lambda}(c_1\cos(\mu\ln t)+c_2\sin(\mu\ln t))$

Regular **Singular points**

$$L[y]=\frac{d^2y}{dt^2}+p(t)\frac{dy}{dt}+q(t)y=0$$

$(t-t_0)p(t),(t-t_0)^2q(t)\quad \text{analytic at}\quad t=t_0\Rightarrow$ regular singular points at $t=t_0$

Frobenius method

solve $L[y] = \dfrac{d^2y}{dt^2} + p(t)\dfrac{dy}{dt} + q(t)y = 0$ where $p \text{ and } q$ are rational function

let $y(t) = (t - t_0)^r \sum\limits_{k=0}^{\infty} a_k (t - t_0)^k$

collect similar term, solve recurrence of a to get 2 linear independent solution

3.1 convert to system

3.4

Theorem 4 (Existence–uniqueness theorem). *There exists one, and only one, solution of the initial-value problem*

$$\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x}, \qquad \mathbf{x}(t_0) = \mathbf{x}^0 = \begin{bmatrix} x_1^0 \\ x_2^0 \\ \vdots \\ x_n^0 \end{bmatrix}. \tag{2}$$

Moreover, this solution exists for $-\infty < t < \infty$.

Theorem 6 (Test for linear independence). *Let* $\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^k$ *be* k *solutions of* $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$. *Select a convenient* t_0. *Then,* $\mathbf{x}^1, \ldots, \mathbf{x}^k$ *are linear independent solutions if, and only if,* $\mathbf{x}^1(t_0), \mathbf{x}^2(t_0), \ldots, \mathbf{x}^k(t_0)$ *are linearly independent vectors in* $\mathbf{R}^n$.

test linear independence -> basis

3.8-3.10 eigenvalue eigenvector method

try $x(t) = e^{\lambda t} v$, $x(t) = e^{\lambda t} v$ iff $\lambda, v \ \text{ s.t.} Av = \lambda v$

$M(y,t) + N(y,t)\frac{dy}{dt} = 0$

solve $\det(A - \lambda x) = 0$

$M_y = N_t$?

- case distinct root λ_i
  find corresponding v_i
  $\overrightarrow{x}(t) = c_1 e^{\lambda_1 t} \overrightarrow{v}_1 + c_2 e^{\lambda_2 t} \overrightarrow{v}_2 + \ldots$

$p = \frac{M_y - N_t}{N} = \frac{N_t - M_y}{M}$

$\mu = e^{\int p}$

- case complex root
  lemma: if $x(t) = y(t) + iz(t)$, then y and z are real valued solution of $\dot{x} = Ax$
  $x(t) = e^{(\alpha + i\beta)t}\left(v^1 + iv^2\right) = e^{\alpha t}\left[\left(v^1 \cos \beta t - v^2 \sin \beta t\right) + i\left(v^1 \sin \beta t + v^2 \cos \beta t\right)\right]$
  so:
  $x_1(t) = e^{\alpha t}\left(v^1 \cos \beta t - v^2 \sin \beta t\right)$
  $x_2(t) = e^{\alpha t}\left(v^1 \sin \beta t + v^2 \cos \beta t\right)$

- case equal root

$$(A-\lambda I)\overrightarrow{v}_1=\overrightarrow{0} \quad \rightarrow \quad \overrightarrow{v}_1$$
$$(A-\lambda I)^2\overrightarrow{v}_2=\overrightarrow{0}, (A-\lambda I)\overrightarrow{v}_2 \neq \overrightarrow{0} \quad \rightarrow \quad \overrightarrow{v}_2$$
$$\overrightarrow{x}(t)=c_1e^{\lambda t}\overrightarrow{v}_1+c_2e^{\lambda t}[I+t(A-\lambda I)]\overrightarrow{v}_2+...$$

$\frac{-4 I \int}{8}$ $-t$

3.11

$e^{At}=X(t)X^{-1}(0)$
to calculate e^{At}:
calculate $A=P\Lambda P^{-1}$
$e^{At}=Pe^{\Lambda t}P^{-1}$

3.12

To solve $x'=Ax+f(t),\ \ x(t_0)=x^0$:

$$x(t)=e^{At}e^{-At_0}x^0+e^{At}\int_{t_0}^{t}e^{-As}f(s)ds=e^{A(t-t_0)}x^0+\int_{t_0}^{t}e^{A(t-s)}f(s)ds$$

4.1 eq point

Equilibuirm point -> set derivative to 0

4.2 stability of linear system

def stability: $\phi(t)$ is stable if
$\forall\varepsilon>0,\ \exists\delta=\delta(\varepsilon) s.t |\varphi(0)-\phi(0)|<\delta\rightarrow|\varphi(t)-\phi(t)|<\varepsilon$ for every φ and for all $t>0$

for $x'=Ax$, every solution:

$$\forall j, Re(\lambda_j)<0\rightarrow \quad \text{asymptotically stable}$$
$$\exists j, Re(\lambda_j)>0\rightarrow \quad \text{unstable}$$
$$Re(\lambda_j)=0$$

have k linear independent e-vector $\rightarrow$stable, otherwise not stable.

4.3 stability of of equilibrium solution $x(t)=x^0$ of $x'=f(x)$

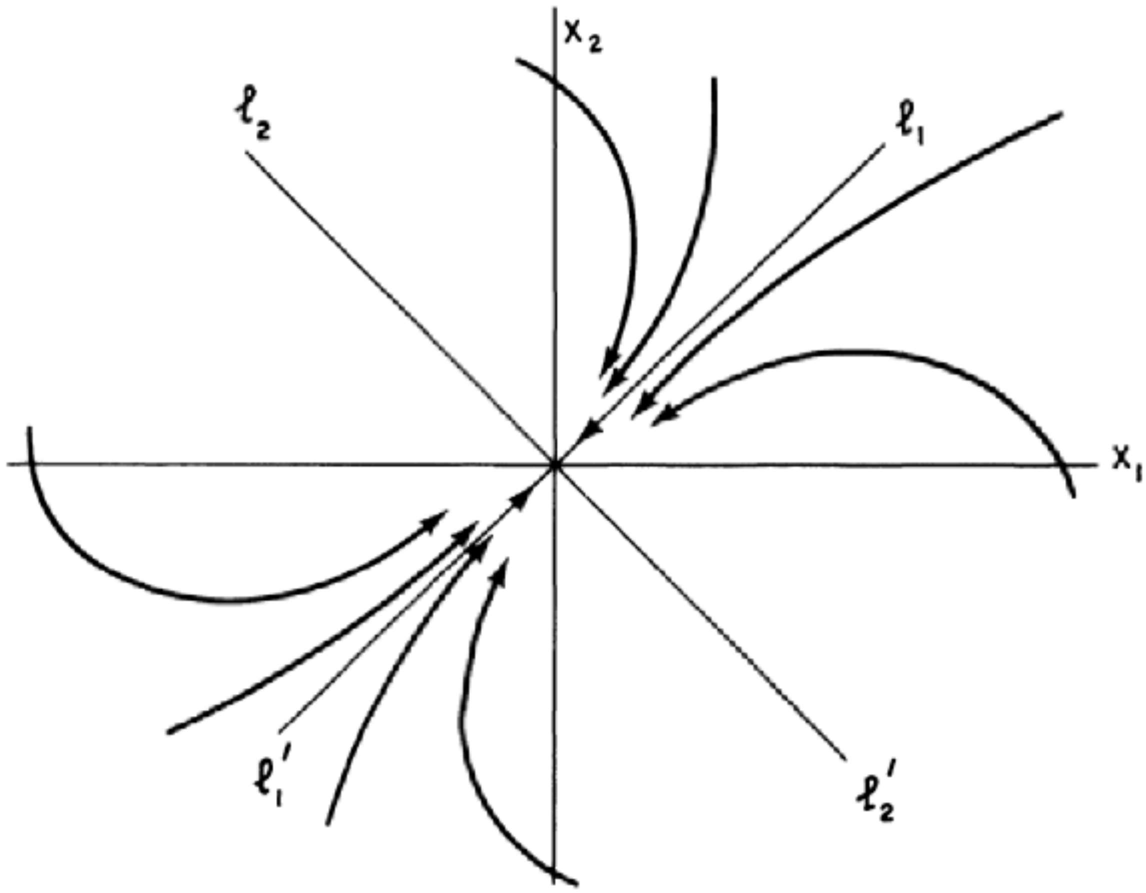
1. set $z = x - x^0$
2. write $f\left(x^0 + z\right)$ in form $Az + g(z)$, where g is at least order 2
3. compute the e-value of A, if all negative, then asymptotically stable. if positive real, unstable.
   Note some time need tyler expansion to get Az

4.4 orbit

try solve y(x)

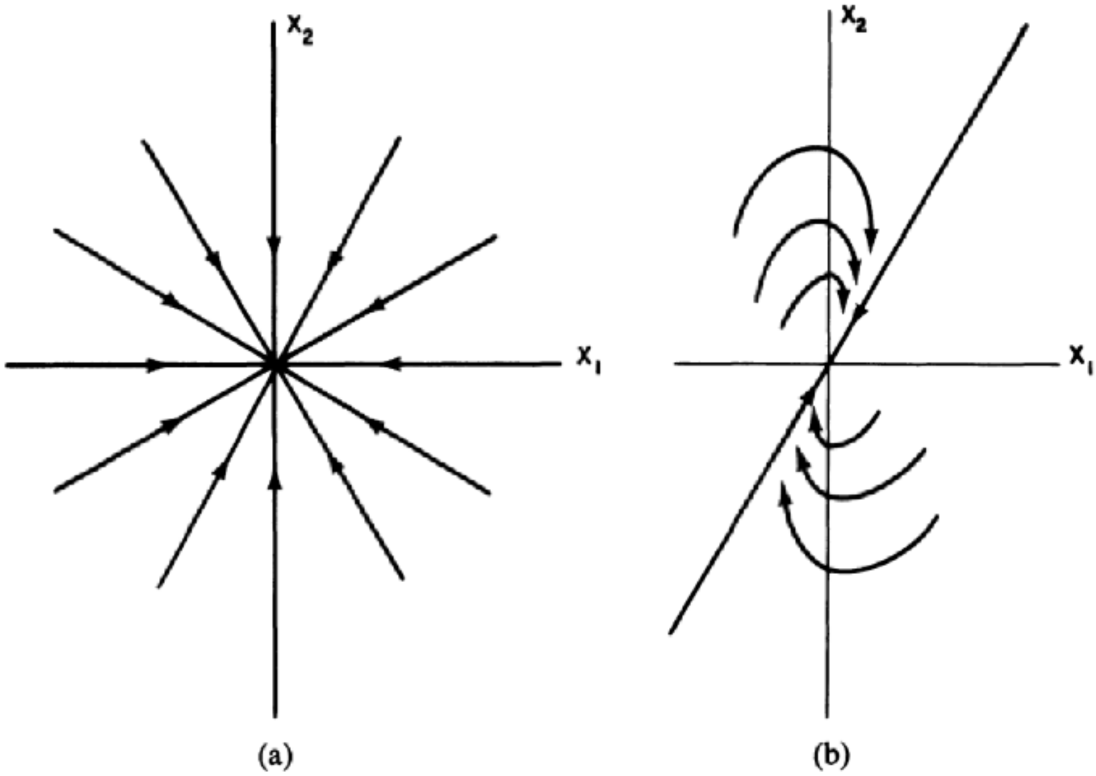
4.7 phase portraits of linear system

$\lambda_1 < \lambda_2 < 0$



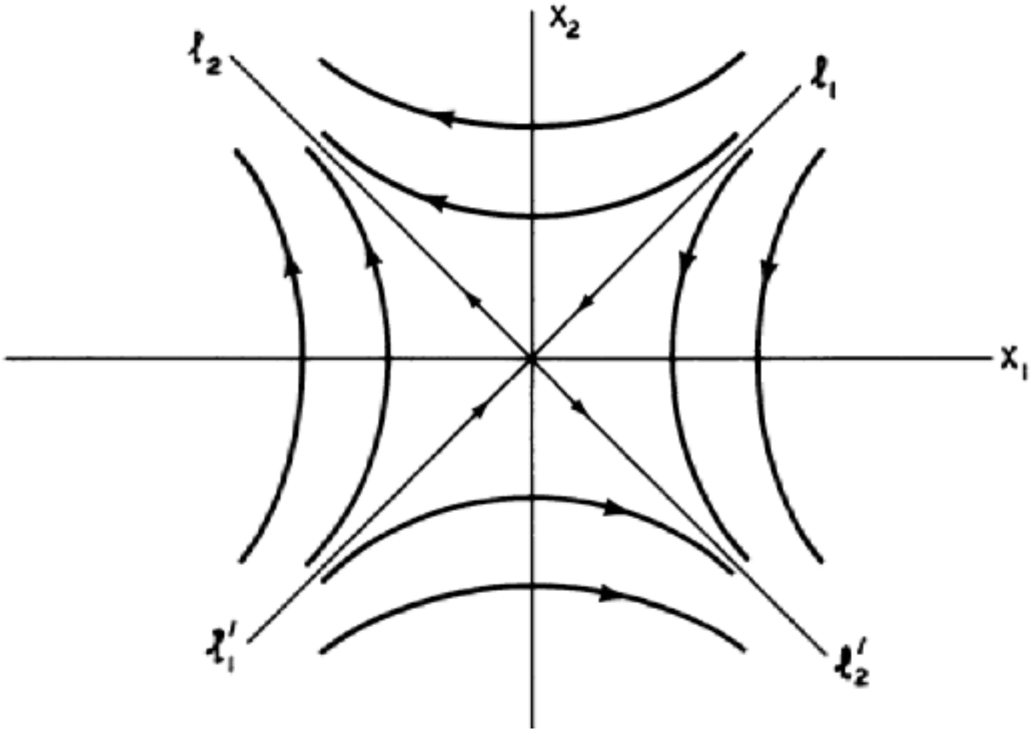
$0 < \lambda_1 < \lambda_2$
reversed arrow

same negative e-value: multiplicity 2 vs 1



same positive e-value: reversed arrow

$\lambda_1 < 0 < \lambda_2$



$\lambda_1 = \alpha + i\beta, \lambda_2 = \alpha - i\beta$

$\alpha = 0$:circle

$\alpha < 0$:stable focus

$\alpha < 0$:unstable focus

directions to be determined by checking sign of $x_2{}'(+,0)$

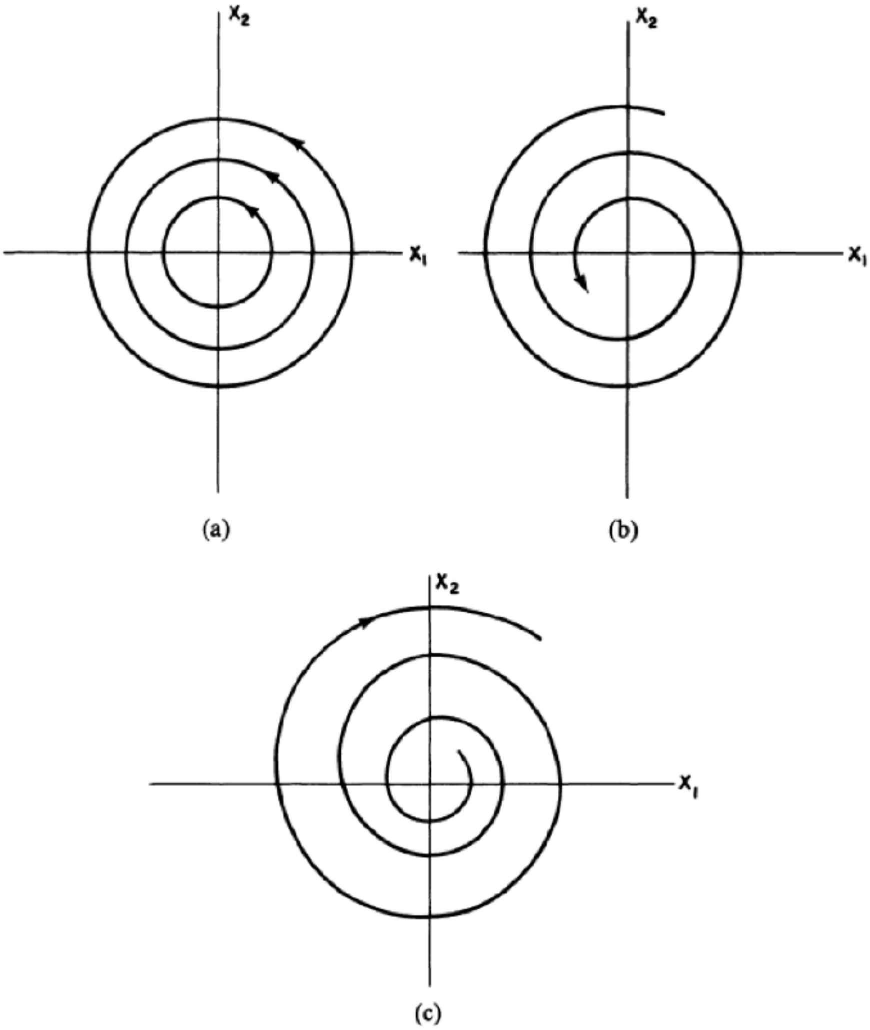


Figure 6. (a) $\alpha = 0$; (b) $\alpha < 0$; (c) $\alpha > 0$

f	L
1	$\frac{1}{s}$
e^{at}	$\frac{1}{s-a}$
t^n	$\frac{n!}{s^{n+1}}$
$\cos at$	$\frac{s}{s^2+a^2}$
$\sin at$	$\frac{a}{s^2+a^2}$
$\sinh kt$	$\frac{k}{s^2-k^2}$
$\cosh kt$	$\frac{s}{s^2-k^2}$
$H_a(t)\, f(t-a)$	$e^{-as} F(s)$
f'	$s L(f) - f(0)$
f''	$s^2 L(f) - s f(0) - f'(0)$
$-tf$	$\frac{d}{ds} L(f)$
$e^{at} f(s)$	$F(s-a)$

$$L(f) = \int_0^\infty e^{-st} f(t) \cdot dt$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\cdot dx$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\cdot dx$$

$$f = \frac{a_0}{2} + \sum_{1} a_n\cos\frac{n\pi x}{l} + b_n\sin\frac{n\pi x}{l}$$

f $\quad X'' + \lambda X = 0, \quad X(0) = 0, \quad X(l) = 0$

imply that $X(0)=0$ and $X(l)=0$ (otherwise, u must be identically zero). Thus, $u(x,t)=X(x)T(t)$ is a solution of (5) if

from (8)
$$X''+\lambda X=0; \qquad X(0)=0, \quad X(l)=0 \tag{9}$$

and

$$T'+\lambda\alpha^2 T=0. \tag{10}$$

At this point, the constant λ is arbitrary. However, we know from Example 1 of Section 5.1 that the boundary-value problem (9) has a nontrivial solution $X(x)$ only if $\lambda=\lambda_n=n^2\pi^2/l^2$, $n=1,2,\ldots$; and in this case,

$$X(x)=X_n(x)=\sin\frac{n\pi x}{l}.$$

Equation (10), in turn, implies that

$$T(t)=T_n(t)=e^{-\alpha^2 n^2\pi^2 t/l^2}.$$

(Actually, we should multiply both $X_n(x)$ and $T_n(t)$ by constants; however, we omit these constants here since we will soon be taking linear combinations of the functions $X_n(x)T_n(t)$.) Hence,

$$u_n(x,t)=\sin\frac{n\pi x}{l}e^{-\alpha^2 n^2\pi^2 t/l^2}$$

is a nontrivial solution of (5) for every positive integer n.

(b) Suppose that $f(x)$ is a finite linear combination of the functions $\sin n\pi x/l$; that is,

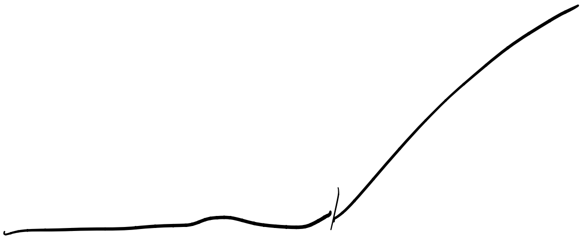
$$f(x)=\sum_{n=1}^{N} c_n\sin\frac{n\pi x}{l}.$$

Then,

$$u(x,t)=\sum_{n=1}^{N} c_n\sin\frac{n\pi x}{l}e^{-\alpha^2 n^2\pi^2 t/l^2}$$

is the desired solution of (1), since it is a linear combination of solutions of (5), and it satisfies the initial condition

$$u(x,0)=\sum_{n=1}^{N} c_n\sin\frac{n\pi x}{l}=f(x), \qquad 0<x<l.$$



3

CH_3C) H