3. Continuous Functions

3.1 Limits of functions

Motivation: In order to say $f(x) \to L$ as $x \to c$, we first need to figure out conditions to "allow" x to approach c.

Let $S \subset \mathbb{R}$ be a set. A number $c \in \mathbb{R}$ is called a cluster point of S if for all $\delta > 0$, there exists $x \in S \setminus \{c\}$ such that $|x - c| < \delta$.

Example: $S = \{0\} \cup [1, 2) \to c \in [1, 2]$

Prop. (Limit Characterization of Cluster Points) Let $S \subset \mathbb{R}$. Then $c \in \mathbb{R}$ is called a cluster point of S $\iff$ there exists a convergent sequence $\{x_n\}$ such that $x_n \in S \setminus \{c\}$ $\forall n \in \mathbb{N}$ and $\lim_{n\to\infty} x_n = c$.

Pf. $\to$: c is a cluster point, pick $x_n \in (c - \frac{1}{n}, c + \frac{1}{n}) \cap S \setminus \{c\}$

$\leftarrow$: $\{x_n\}$ converges to c, let $\delta > 0$, $\exists M \in \mathbb{N}$: $\forall n \geq M, |x_n - c| < \delta$, so $x_M \in (c - \frac{1}{n}, c + \frac{1}{n}) \cap S \setminus \{c\}$

Limit of a function. Let $f : S \to \mathbb{R}$ be a function, and let $c \in \mathbb{R}$ be a cluster point of $S \subset \mathbb{R}$. Suppose there exists $L \in \mathbb{R}$ such that for all $\epsilon > 0$, there exists $\delta > 0$ such that whenever $x \in S \setminus \{c\}$ and $|x - c| < \delta$, we have

$|f(x) - L| < \epsilon$.

Then we say $f(x)$ converges to L as x goes to c.

We write $\lim_{x\to c} f(x) = L$

or $f(x) \to L$ as $x \to c$.

If no such L exists, we say $f(x)$ diverges at c.

Symbolically: $\exists L \in \mathbb{R}$: $\forall \epsilon > 0$, $\exists \delta > 0$: $\forall x \in (c - \delta, c + \delta) \cap S \setminus \{c\}$, $|f(x) - L| < \epsilon$.

Sequential limits lemma. — function limits $\iff$ sequence limits

Let $S \subset R$, and c be a cluster point of S. Then $f(x) \to L$ as $x \to c$ $\iff$ for every sequence $\{x_n\}$ satisfying $x_n \in S \setminus \{c\}$ $\forall n$ and $\lim x_n = c$, we have that the sequence $\{f(x_n)\}$ converges to L.

Symbolically: $\lim_{x\to c} f(x) = L \iff \forall \{x_n\}$ s.t. $x_n \in S \setminus \{c\}$ & $\lim x_n = c$, $\lim_{n\to\infty} f(x_n) = L$

3.2 Continuous functions

Let $S \subset R$, $c \in S$. We say that f is continous at c if for every $\epsilon > 0$ there is a $\delta > 0$ such that whenver $x \in S$ and $|x - c| < \delta$, then $|f(x) - f(c)| < \epsilon$.

When f is continous at all $c \in S$, then we say f is a continous function.

Prop. (Characterization of Continuity) Let $S \subset R$, $c \in S$, $f : S \to R$. Then:

1. If c is not a cluster point of S, then f is continous at c.
2. If c is a cluster point of S, then f is continous at c $\iff$ the limit of $f(x)$ as $x \to c$ exists and $\lim_{x\to c} f(x) = f(c)$.

3. (Sequential Characterization of Continuity) f is continous at $c \iff$ for every sequence $\{x_n\}$ where $x_n \in S$ and $\lim x_n = c$, the sequence $\{f(x_n)\}$ converges to $f(c)$.

Prop. (Continuity of algebraic operations) Let $f, g : S \to \mathbb{R}$ be functions continuous at $c \in S$,

1. $h : S \to \mathbb{R}$ defined by $h(x) := f(x) + g(x)$ is continous at c.
2. $-, \times, \div$

Prop. (Compositions preserve continuity) Let $A, B \subset \mathbb{R}$ and $f : B \to \mathbb{R}$ and $g : A \to B$ be functions. If g is continuous at $c \in A$ and f is continuous at $g(c)$, then $f \circ g : A \to \mathbb{R}$ is continous at c.

Prop. (Negation of Sequential Characterization of Continuity) Let $S \subset R$, $c \in S$, $f : S \to R$. If there exists $\{x_n\}$ with $x_n \in S$ and $\lim x_n = c$ s.t. $\{f(x_n)\}$ does not converge to $f(c)$, then $f(n)$ is discontinous at c.

3.3 Min-max and intermediate value theorems

Lemma. A continuous function $f : [a, b] \to \mathbb{R}$ is bounded.

We say a function $f : S \to \mathbb{R}$ achieves an absolute maximum if there exists $c \in S$ such that $f(x) \leq f(c)\ \forall x \in S$.

Similarly for absolute minimum: $f(x) \geq f(c)\ \forall x \in S$.

Min-Max Theorem: A continous function $f : [a, b] \to \mathbb{R}$ achieves both an abs min and abs max on the closed and bounded interval $[a, b]$.

Pf. bounded → has an inf → $\exists\{f(x_n)\}$ approaches the inf

→ by Bolzano-Weierstrass, $\exists$ convergent subsequences $\{x_{n_k}\}$, let $x = \lim_{k\to\infty} x_{n_k}$

→ by Characterization of Continuity, $\inf f([a, b]) = \lim_{n\to\infty} x_n = \lim_{k\to\infty} x_{n_k} = f(\lim_{k\to\infty} x_{n_k}) = f(x)$

Lemma. (Bisection Method for Finding Roots) Let $f : [a, b] \to \mathbb{R}$ be continous. Suppose $f(a) < 0$ and $f(b) > 0$, then there exists $c \in (a, b)$ such that $f(c) = 0$.

Pf. Let $a_1 = a$, $b_1 = b$.

If $f(\frac{a_n+b_n}{2}) \geq 0$, let $a_{n+1} = a_n$, $b_{n+1} = \frac{a_n+b_n}{2}$. → $\lim f(a_n) \leq 0$ → squeeze $f(c) = 0$

If $f(\frac{a_n+b_n}{2}) < 0$, let $a_{n+1} = \frac{a_n+b_n}{2}$, $b_{n+1} = b_n$. → $\lim f(b_n) < 0$

Bolzano's Intermediate Value Theorems. Let $f : [a, b] \to \mathbb{R}$ be continous. Suppose $y \in \mathbb{R}$ satisfies $f(a) < y < f(b)$ and $f(b) < y < f(a)$. Then there exists $c \in (a, b)$ such that $f(c) = y$.

Pf: Let $g(x) = f(x) - y$.

Prop. Every polynomial of odd degree has a real root.

Pf. $f(x) = a_d x^d + ... + a_1 x_1 + a_0$, $g(x) = x^d + ... + b_1 x_1 + b_0$

$\lim_{n\to\infty} |\frac{b_{d-1}n^{d-1}+...+b_0}{n^d}| = \lim_{n\to\infty} b = 0$ → $\exists M$ s.t. $|\frac{b_{d-1}M^{d-1}+...+b_0}{M^d}| < 1$ → $g(M) > 0$, since d is odd, $\exists K$ s.t. $g(K) < 0$ → squeeze $g(c) = 0$