Jump Process

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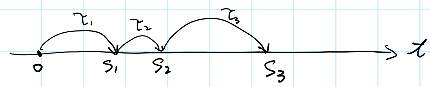
τ 有 density $\begin{cases} \lambda e^{-\lambda t} & t \geq 0 \\ 0 & \text{else}. \end{cases}$

$\mathbb{P}(\tau > t+s \mid \tau > s) = \mathbb{P}(\tau > t)$. Memoriless Property.

① Poisson Process (λ). aka $PP(\lambda)$ is a Counting Process 常用 Integration by Parts!

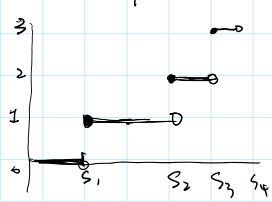
- Arrival times. $S_n \equiv \sum_{i=1}^{n} \tau_i$



$N_t \begin{cases} 0 & t < S_1 \ (=\tau_1). \\ 1 & S_1 \leq t < S_2 \\ \vdots & \\ n & S_n \leq t < S_{n+1} \end{cases}$

有 N_t 为 Stochastic Process

N_t 轨迹



RIGHT-CTS $\left(\lim_{x^+} = x\right)$.

记 $dN_t = N_t - N_{t-} = \begin{cases} 0 & \text{无 jump at } t. \\ 1 & \text{有 jump} \end{cases}$

(N_{t-}: left limit.)

Lemma. 11.2.1

Lemma 11.2.1. *For $n \geq 1$, the random variable S_n defined by (11.2.4) has the gamma density*

S_n 的 PDF
$$g_n(s) = \frac{(\lambda s)^{n-1}}{(n-1)!}\lambda e^{-\lambda s}, \quad s \geq 0. \tag{11.2.5}$$

PROOF: We prove (11.2.5) by induction on n. For $n = 1$, we have that $S_1 = \tau_1$ is exponential, and (11.2.5) becomes the exponential density

$$g_1(s) = \lambda e^{-\lambda s}, \quad s \geq 0.$$

(Recall that $0!$ is defined to be 1.) Having thus established the base case, let us assume that (11.2.5) holds for some value of n and prove it for $n+1$. In other words, we assume S_n has density $g_n(s)$ given in (11.2.5) and we want to compute the density of $S_{n+1} = S_n + \tau_{n+1}$. Since S_n and τ_{n+1} are independent, the density of S_{n+1} can be computed by the convolution

$$\begin{aligned} \int_0^s g_n(v) f(s-v)\,dv &= \int_0^s \frac{(\lambda v)^{n-1}}{(n-1)!}\lambda e^{-\lambda v} \cdot \lambda e^{-\lambda(s-v)}\,dv \\ &= \frac{\lambda^{n+1} e^{-\lambda s}}{(n-1)!}\int_0^s v^{n-1}\,ds = \frac{\lambda^{n+1}e^{-\lambda s}}{n!} v^n \Big|_{v=0}^{v=s} \\ &= \frac{(\lambda s)^n}{n!}\lambda e^{-\lambda s} = g_{n+1}(s). \end{aligned}$$

This completes the induction step and proves the lemma. □

验证，取 $n=1$ $\Rightarrow g_1(s) = \lambda e^{-\lambda s} = f(\tau_1) = f(S_1)$

Pf

$\tau_i \perp \tau_j \quad \forall i \neq j$.

有 $\mathbb{P}(N_t \geq k) = \mathbb{P}(S_k \leq t) = \int_0^t g_k(s)\,ds$

$$= \int_0^t \frac{(\lambda s)^{k-1}}{(k-1)!}\lambda e^{-\lambda s}\,ds$$

同理 $\mathbb{P}(N_t \geq k+1) = \int_0^t \frac{(\lambda s)^k}{k!}\lambda e^{-\lambda s}\,ds$

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let $f' = \lambda e^{-\lambda s} \Rightarrow f = -e^{-\lambda s}$

$g = \frac{(\lambda s)^k}{k!} \Rightarrow g' = \frac{k(\lambda s)^{k-1}\cdot\lambda}{k!} = \frac{(\lambda s)^{k-1}\lambda}{(k-1)!}$

We integrate this last expression by parts to obtain

$$\mathbb{P}\{N(t) \geq k+1\} = -\frac{(\lambda s)^k}{k!} e^{-\lambda s}\Big|_{s=0}^{s=t} + \int_0^t \frac{(\lambda s)^{k-1}}{(k-1)!} \lambda e^{-\lambda s}\, ds$$
$$= -\frac{(\lambda t)^k}{k!} e^{-\lambda t} + \mathbb{P}\{N(t) \geq k\}.$$

This implies that for $k \geq 1$,

$$\mathbb{P}\{N(t) = k\} = \mathbb{P}\{N(t) \geq k\} - \mathbb{P}\{N(t) \geq k+1\} = \frac{(\lambda t)^k}{k!} e^{-\lambda t}.$$

For $k = 0$, we have from (11.2.2)

$$\mathbb{P}\{N(t) = 0\} = \mathbb{P}\{S_1 > t\} = \mathbb{P}\{\tau_1 > t\} = e^{-\lambda t},$$

which is (11.2.6) with $k = 0$. □

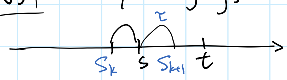
$\Downarrow$ Lemma 证明 $\Downarrow$

Lemma 11.2.2. *The Poisson process $N(t)$ with intensity λ has the distribution*

N_t 的 PDF
$$\mathbb{P}\{N(t) = k\} = \frac{(\lambda t)^k}{k!} e^{-\lambda t}, \quad k = 0, 1, \ldots \tag{11.2.6}$$

下证 N_t 增量独立.

$\boxed{N_t - N_s}$ - independent of $\mathcal{F}_s$ QED.



下计算 $\mathbb{E} N_t = \sum_{k=0}^{+\infty} k \cdot \mathbb{P}(N_t = k)$. $k \in \mathbb{Z}^+$

$= \sum_{k\in\mathbb{Z}^+} k \cdot \frac{(\lambda t)^k}{k!} \cdot e^{-\lambda t}$

$= e^{-\lambda t} \sum_{k\in\mathbb{Z}^+} \frac{(\lambda t)^k}{(k-1)!}$

$= e^{-\lambda t}(\lambda t) \cdot \sum_{k\in\mathbb{Z}^+} \frac{(\lambda t)^{k-1}}{(k-1)!}$

let $n = k-1$

$\mathbb{E}(N_t) = e^{-\lambda t}(\lambda t) \cdot \underbrace{\sum_{n=0}^{\infty} \frac{(\lambda t)^n}{n!}}_{e^{\lambda t}} = \lambda t$

下计算 $Var(N_t) = \mathbb{E}(N_t^2) - [\mathbb{E}(N_t)]^2$

$= \sum_{k\in\mathbb{Z}^+} k^2 \frac{(\lambda t)^k}{k!} e^{-\lambda t} - (\lambda t)^2$

$= \sum_{k\in\mathbb{Z}^+} k \cdot \frac{(\lambda t)^k e^{-\lambda t}}{(k-1)!} - (\lambda t)^2$ $\boxed{拆为\ k-1+1}$

$= \sum_{k\in\mathbb{Z}^+} \frac{(\lambda t)^k e^{-\lambda t}}{(k-2)!} + \sum_{k\in\mathbb{Z}^+} \frac{(\lambda t)^k e^{-\lambda t}}{(k-1)!} - (\lambda t)^2$

$= \sum_{\substack{n=0 \\ n=k-2}}^{+\infty} (\lambda t)^2 \cdot e^{\lambda t} \cdot e^{-\lambda t} + \sum_{\substack{m=0 \\ m=k-1}}^{+\infty} (\lambda t) \cdot e^{\lambda t} \cdot e^{-\lambda t} - (\lambda t)^2$

$= \lambda t$

$$\mathbb{E}[N(t) - N(s)] = \sum_{k=0}^{\infty} k \frac{\lambda^k (t-s)^k}{k!} e^{-\lambda(t-s)}$$
$$= \lambda(t-s) e^{-\lambda(t-s)} \sum_{k=1}^{\infty} \frac{\lambda^{k-1}(t-s)^{k-1}}{(k-1)!}$$
$$= \lambda(t-s) \cdot e^{-\lambda(t-s)} \cdot e^{\lambda(t-s)}$$
$$= \lambda(t-s). \tag{11.2.9}$$

This is consistent with our observation at the end of Subsection 11.2.2 that jumps are arriving at an average rate of λ per unit time. Therefore, the average number of jumps between times s and t is $\mathbb{E}[N(t) - N(s)] = \lambda(t-s)$.

Finally, we compute the second moment of the increment

$$\mathbb{E}\left[(N(t) - N(s))^2\right] = \sum_{k=0}^{\infty} k^2 \frac{\lambda^k (t-s)^k}{k!} e^{-\lambda(t-s)}$$
$$= e^{-\lambda(t-s)} \sum_{k=1}^{\infty} (k-1+1) \frac{\lambda^k (t-s)^k}{(k-1)!}$$
$$= e^{-\lambda(t-s)} \sum_{k=2}^{\infty} \frac{\lambda^k (t-s)^k}{(k-2)!} + e^{-\lambda(t-s)} \sum_{k=1}^{\infty} \frac{\lambda^k (t-s)^k}{(k-1)!}$$
$$= \lambda^2 (t-s)^2 e^{-\lambda(t-s)} \sum_{k=2}^{\infty} \frac{\lambda^{k-2}(t-s)^{k-2}}{(k-2)!}$$
$$+ \lambda(t-s) e^{-\lambda(t-s)} \sum_{k=1}^{\infty} \frac{\lambda^{k-1}(t-s)^{k-1}}{(k-1)!}$$
$$= \lambda^2 (t-s)^2 + \lambda(t-s).$$

This implies

$$Var[N(t) - N(s)] = \mathbb{E}\left[(N(t) - N(s))^2\right] - \left(\mathbb{E}[N(t) - N(s)]\right)^2$$
$$= \lambda^2 (t-s)^2 + \lambda(t-s) - \lambda^2 (t-s)^2$$
$$= \lambda(t-s); \tag{11.2.10}$$

the variance is the same as the mean.

Compensated. PP(λ): $M_t = N_t - \lambda t$

Pf. $\mathbb{E}(M_t | \mathcal{F}_s) = \mathbb{E}(M_t - M_s | \mathcal{F}_s) + \mathbb{E}(M_s | \mathcal{F}_s)$

$= \mathbb{E}(N_t - N_s | \mathcal{F}_s) + \lambda(s-t) + M_s$

$= \mathbb{E}(N_t - N_s) + \lambda(s-t) + M_s$

$= M_s.$