MA-UY 2314: Discrete Mathematics

Final Exam Solutions

- NYU students are planning to participate in a hackathon. Assume that there are 50 teams from NYU that are participating. Moreover, assume that each team has at least 2 members and no team has more than 13 members. Use the pigeonhole principle to explain why there are at least 5 teams with the same number of teammates.

  Solution: We define the pigeons to be the 50 teams. We define the holes to be the numbers 2-13. A pigeon is placed in a hole based on the number of teammates it has. There are 12 holes and $4 \cdot 12 = 48 < 50$. Therefore, there must be a hole with at least 5 pigeons. That is, at least 5 teams with the same number of teammates. $\square$

- Let $A_1, A_2, \ldots, A_n$ be sets $(n \geq 2)$ such that $A_i \subseteq A_j$ or $A_j \subseteq A_i$. Use the Principle of Mathematical Induction to prove that one of these n sets is a subset of all of them.

  Proof:

  Base Case: For $n = 2$; since $A_1 \subseteq A_2$ or $A_2 \subseteq A_1$ then one of these two sets is a subset of all other sets.

  Induction Step: Assume $k \geq 2$ is any integer such that $P(k)$ is true. Let $X_1, X_2, \ldots X_{k+1}$ be any collection of sets such that $X_i \subseteq X_j$ or $X_j \subseteq X_i$. Remove X_{k+1} from this collection. Then we know from the induction hypothesis that one of the sets $X_1, X_2, \ldots, X_k$, call it X, is a subset of all other sets. That is, $X \subseteq X_i$ for $i = 1, 2, \ldots k$. Now, $X_{k+1} \subseteq X$ or $X \subseteq X_{k+1}$. Therefore, either X or X_{k+1} is a subset of the collection $X_1, X_2, \ldots X_{k+1}$ $\square$

- Determine if the given function is 1-1. If the function is 1-1, simply write "1-1" and move on. If the function is not 1-1, then write "not 1-1" and provide an appropriate counterexample.

  (a) $f = \{(x, y) : x, y \in \mathbb{Z}, \;\; y = x^2\}$
  Solution: Not 1-1. $f(1) = f(-1) = 1$ and $1 \neq -1$. There are infinite number of counterexamples.

  (b) Let $A = \{1, 2, 3\}$ and let f be the function from the power set of A to the integers such that $f(x) = z + 1$ where z is equal to the number of elements in x.
  Solution: Not 1-1. Set $x\{1\}$ and $y = \{2\}$. Then $f(x) = f(y) = 2$ and $x \neq y$. Other counterexamples are possible.

  (c) $f = \{(x, y) : x, y \in \mathbb{Z}, \;\; y = x + 1\}$.
  Solution: 1-1.

- **Prove Directly:**

  Assume x and y are any consecutive integers and let d be the largest positive divisor of both x and y. Then $d = 1$.

  Proof: Assume x and y are any consecutive integers and let d be the largest positive divisor of both x and y. Since x and y are consecutive, it follows that $y = x + 1$. Since $d|x$ and $d|(x + 1)$ we have

  $$x = ds$$
  $$x + 1 = dt$$

  for some integers s and t. Adding 1 to equation $x = ds$ yields $x + 1 = ds + 1$, hence $dt = ds + 1$. Therefore $d(t - s) = 1$ and it follows that $d|1$. By Theorem 4.2.2. (page 191), we must have $d = 1$ or $d = -1$. Since d is positive, it follows that $d = 1$. $\square$

- Fill in the truth table. Use T for true and F for false.

p	q	$p \wedge q$	$p \vee q$	$(p \wedge q) \to (p \vee q)$
T	T	T	T	T
T	F	F	T	T
F	T	F	T	T
F	F	F	F	T

- **Prove by Contradiction:**

  If a is any real number such that

  $$\frac{a^2 - 1}{a + 2} > 0$$

  Then $-2 < a < -1$ or $a > 1$.

  Proof: Assume there exists a real number a such that

  $$\frac{a^2 - 1}{a + 2} > 0 \tag{1}$$

  and ($a \leq -2$ or $a \geq -1$) and $a \leq 1$. Applying DeMorgan's again we have ($a \leq 1$ and $a \leq -2$) or ($a \leq 1$ and $a \leq -1$).

  Case 1: $a \leq 1$ and $a \leq -2$ Therefore $a \leq -2$. Now a cannot be equal to -2 otherwise $\frac{a^2-1}{a+2}$ is undedfined. Moreover, $a^2 - 1 = (a - 1)(a + 1) > 0$ and $a + 2 < 0$ for $a < -2$ which contradicts (1).

Case 2: $a \leq 1$ and $a \leq -1$ Therefore $a \leq -1$. By Case 1, we can restrict a to the case $-2 < a \leq -1$. Now for such values of a, $a^2 - 1 = (a-1)(a+1) \leq 0$ and $a + 2 > 0$ which contradicts (1).

$\square$