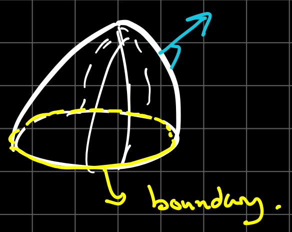
Stoke's Thm



$\vec{F}$ has continuous p.d. on an open region containing S.

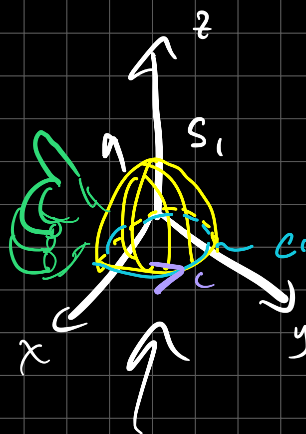
$$\int_C \vec{F}\cdot d\vec{r} = \iint_S \underbrace{\text{curl}\,\vec{F}}_{\vec{\nabla}\times\vec{F}}\cdot d\vec{S}$$

Circulation.

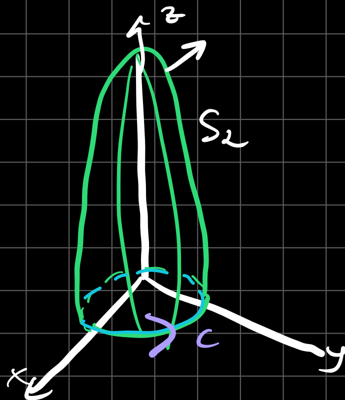
Basic Surface integral.

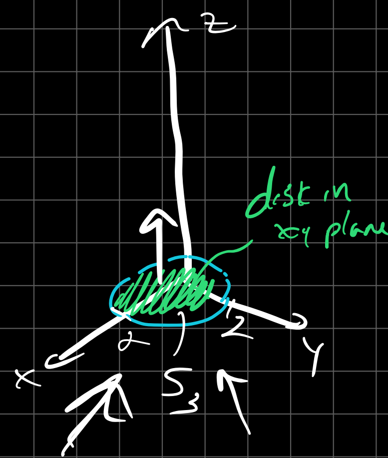
16.8

①

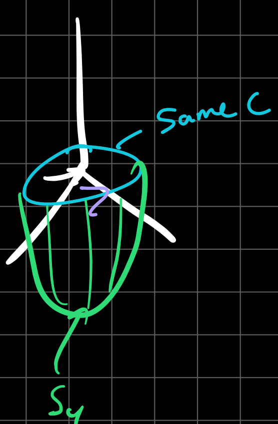


circle of radius 2 in xy plane.





what can we say about $\iint_{S_1} \text{curl}\,\vec{F}\cdot d\vec{S}$ compared to



$$= \iint_{S_2} \text{curl}\,\vec{F}\cdot d\vec{S} = \iint_{S_3} \text{curl}\,\vec{F}\cdot d\vec{S}$$

both equal to : $\boxed{\int_C \vec{F}\cdot d\vec{r}}$

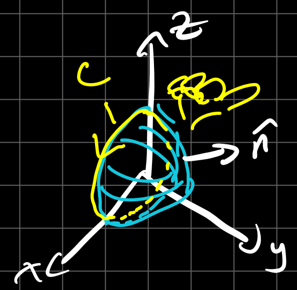
In order for $\iint_{S_4} \text{curl}\,\vec{F}\cdot d\vec{S}$ to equal $\iint_{S_1}, \iint_{S_2}, \iint_{S_3}$

S_4 must be oriented inward. /upward.

③ $\vec{F}(x,y,z) = ze^{y}\hat{\imath} + x\cos y\,\hat{\jmath} + xz\sin y\,\hat{k}$

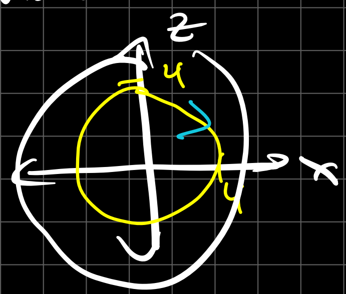
S is hemisphere $x^2+y^2+z^2=16$, $y \geq 0$ oriented in direction of positive y-axis.

Evaluate $\iint_S \text{curl}\,\vec{F}\cdot d\vec{S}$



$= \int_C \vec{F}\cdot d\vec{r}$

need to param. C.



$y = 0$

$x = 4\cos(t)$

$z = -4\sin(t)$

$0 \leq t \leq 2\pi$

$\vec{r}(t) = \langle 4\cos t, 0, -4\sin t\rangle$

$$\int_C \vec{F}\cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t)\,dt$$

$\vec{F}(x,y,z) = ze^{y}\hat{\imath} + x\cos y\,\hat{\jmath} + xz\sin y\,\hat{k}$

$\vec{F}(\vec{r}(t)) = -4\sin t\,\hat{\imath} + 4\cos t\,\hat{\jmath}$ $\qquad$ $\vec{r}\,'(t) = \langle -4\sin t, 0, -4\cos t\rangle$

$\vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t) = 16\sin^2 t$

$\cos(2t) = \cos^2 t - \sin^2 t$

$= 1 - 2\sin^2 t$

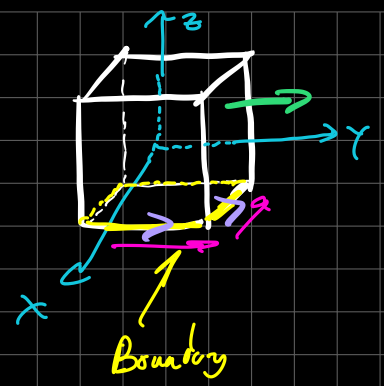
$\sin^2 t = \frac{1-\cos 2t}{2}$

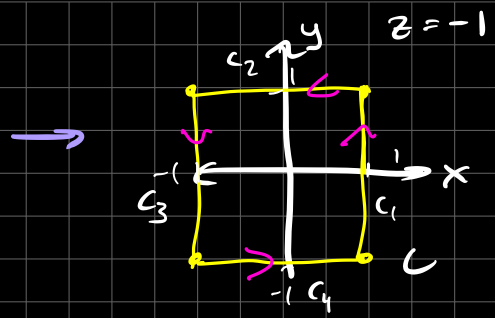
$$\iint_S \text{curl}\,\vec{F}\cdot d\vec{S} = \int_0^{2\pi} 16\sin^2 t\,dt$$

$$= \frac{16}{2}\int_0^{2\pi}(1-\cos 2t)\,dt \qquad \underline{\text{easy}}$$

⑤ $\vec{F}(x,y,z) = xyz\,\hat{\imath} + xy\,\hat{\jmath} + x^2yz\,\hat{k}$

S is top & 4 sides of cube with vertices $(\pm 1, \pm 1, \pm 1)$ oriented out.





find $\iint_S \text{curl}\,\vec{F}\cdot d\vec{S} = \int_C \vec{F}\cdot d\vec{r}$

Need 5 surface integrals + curl calculation.

4 integrals.

C_1: $x=1$, $y=t$, $z=-1$: $\vec{r}(t)=\langle 1,t,-1\rangle$
$-1\le t\le 1$ $\vec{r}\,'(t)=\langle 0,1,0\rangle$

C_2: $x=-t$ $y=1$, $z=-1$: $\vec{r}(t)=\langle -t,1,-1\rangle$
$-1\le t\le 1$ $\vec{r}\,'(t)=\langle -1,0,0\rangle$

C_3: $x=-1$, $y=-t$, $z=-1$: $\vec{r}(t)=\langle -1,-t,-1\rangle$
$-1\le t\le 1$ $\vec{r}\,'(t)=\langle 0,-1,0\rangle$

C_4: $x=t$, $y=-1$, $z=-1$: $\vec{r}(t)=\langle t,-1,-1\rangle$
$-1\le t\le 1$ $\vec{r}\,'(t)=\langle 1,0,0\rangle$

$\vec{F}(x,y,z) = xyz\hat{i} + xy\hat{j} + x^2yz\hat{k}$

$$\int_C \vec{F}\cdot d\vec{r} = \int_{-1}^{1}(-t\hat{i}+t\hat{j}-t\hat{k})\cdot\hat{j}\,dt$$

$$+\int_{-1}^{1}(t\hat{i}+-t\hat{j}-t^2\hat{k})\cdot(-\hat{i})\,dt$$

$$+\int_{-1}^{1}(\_\_+t\hat{j}+\_\_)\cdot(-\hat{j})\,dt$$

$$+\int_{-1}^{1}(t\hat{i}+\_\_\_\_)\cdot(\hat{i})\,dt = \int_{-1}^{1}t\,dt+\int_{-1}^{1}-t\,dt+\int_{-1}^{1}-t\,dt+\int_{-1}^{1}t\,dt$$

$$=0.$$

Comment: $\oint_C \vec{F}\cdot d\vec{r}$ if $\vec{F} = \nabla f$ then $\oint_C \vec{F}\cdot d\vec{r} = 0$.

$\vec{F}(x,y,z) = xyz\hat{\imath} + xy\hat{\jmath} + x^2yz\hat{k}$ if $\vec{F}$ has a scalar potential then $\vec{F}$ is conservative.

$f_x = xyz \longrightarrow f(x,y,z) = \dfrac{x^2yz}{2} + g(y,z)$

$\downarrow f_y$

$f_y = xy = \dfrac{x^2z}{2} + \dfrac{\partial}{\partial y} g(y,z)$ $\qquad \dfrac{x^2z}{2} + \dfrac{\partial}{\partial y} g(y,z) = xy$

$f_z = x^2yz$ $\qquad \dfrac{\partial}{\partial y} g(y,z) = xy - \dfrac{x^2z}{2}$

Contradiction.

$\vec{F}$ not conservative.

Note: if $\oint_C \vec{F}\cdot d\vec{r} = 0$ ($\vec{F}$ cons.) and components of $\vec{F}$ has cont p.d.

for every closed curve C.

then $\iint_S \text{curl}\vec{F}\cdot d\vec{S} = 0 \iff \text{curl}\vec{F} = \text{curl}(\nabla f) = \vec{0}$

S has C as its boundary.

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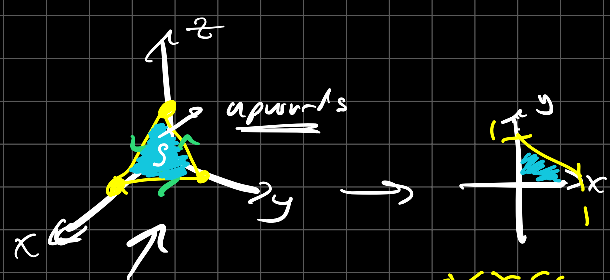
(7) $\vec{F}(x,y,z) = (x+y^2)\hat{\imath} + (y+z^2)\hat{\jmath} + (z+x^2)\hat{k}$

C is the triangle with vertices $(1,0,0)$, $(0,1,0)$, $(0,0,1)$

oriented counterclockwise when viewed from above.

evaluate $\int_C \vec{F}\cdot d\vec{r}$ Use Stokes

find S s.t. $\int_C \vec{F}\cdot d\vec{r} = \iint_S \text{curl}\vec{F}\cdot d\vec{S}$



S is: $x+y+z=1 \rightsquigarrow z = \underbrace{1-x-y}_{g(x,y)}$

$0 \le x \le 1$

$0 \le y \le 1-x$

$$\iint_S \text{curl}\,\vec{F}\cdot d\vec{S} = \int_0^1\int_0^{1-x}\left(-Pg_x - Qg_y + R\right)\,dy\,dx$$

$\hat{n} = -g_x\hat{i} - g_y\hat{j} + \hat{k}$ ~ upwards

P, Q, R come from $\text{curl}\,\vec{F}$

$$\text{curl}\,\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+y^2 & y+z^2 & z+x^2 \end{vmatrix}$$

$$= (0-2z)\hat{i} - (2x-0)\hat{j} + (0-2y)\hat{k} = \underbrace{-2z}_{P}\hat{i}\ \underbrace{-2x}_{Q}\hat{j}\ \underbrace{-2y}_{R}\hat{k}$$

Make sure z in terms of x, y.

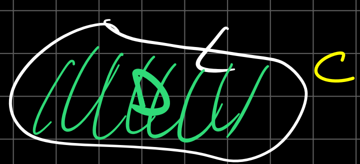
$z = 1-x-y$.

$$\iint_S \text{curl}\,\vec{F}\cdot d\vec{S} = \int_0^1\int_0^{1-x}\left(2(1-x-y)(-1) + (2x)(-1) - 2y\right)dy\,dx$$

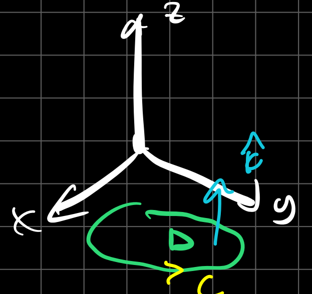
$$-2 \cancel{+2x} \cancel{+2y} \cancel{-2x} \cancel{-2y}$$

$$= \int_0^1\int_0^{1-x}(-2)\,dy\,dx = -2A(\text{Triangle}) = -1.$$

Compare G.T to Stokes Thm.



$$\iint_D (Q_x - P_y)\,dA = \int_C \vec{F}\cdot d\vec{r}$$



$$\text{curl}\,\vec{F}\cdot\hat{k} = Q_x - P_y$$

GT = Stokes Thm where surface lies in xy plane.