Divergence Thm

E - simple solid region with S = piecewise smooth boundary (surface), oriented outwards (positive).

$\vec{F}$ has continuous p.d. on an open region containing E.

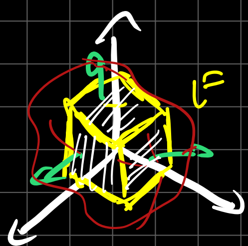
$$\underbrace{\iint_S \vec{F}\cdot d\vec{S}}_{\text{Flux of } \vec{F} \text{ through } S.} = \iiint_E \operatorname{div}\vec{F}\, dV$$

$\operatorname{div}\vec{F} \downarrow \nabla\cdot\vec{F}$

① $\vec{F}(x,y,z) = 3x\hat{i} + xy\hat{j} + 2xz\hat{k}$

E is cube bounded by $x=0,\ x=1,\ y=0,\ y=1,\ z=0,\ z=1$

oriented out.



6 sides

Can you apply div Thm? S ✓

Yes!

$\vec{F} = P\hat{i} + Q\hat{j} + R\hat{k}$

$\left.\begin{matrix} P_x \\ Q_y \\ R_z \end{matrix}\right\}$ are these "ok"? ✓

let's set it up!

$$\operatorname{div}\vec{F} = \frac{\partial}{\partial x}(3x) + \frac{\partial}{\partial y}(xy) + \frac{\partial}{\partial z}(2xz)$$

$$= 3 + x + 2x = 3 + 3x$$

$E:\ 0 \le x \le 1,\quad 0 \le y \le 1,\quad 0 \le z \le 1$

$$\iint_S \vec{F}\cdot d\vec{S} = \int_0^1\int_0^1\int_0^1 (3+3x)\,dx\,dy\,dz = \int_0^1\int_0^1 \left(3x+\tfrac{3}{2}x^2\right)\Big|_{x=0}^{x=1} dy\,dz$$

$$= \int_0^1\int_0^1 \left(3+\tfrac{3}{2}\right) dy\,dz = \boxed{\tfrac{9}{2}}$$

Flux of $\vec{F}$ through S.

(*) This is **WAY** better than doing 6 surface integrals

But if $\vec{F}$ was not "nice" inside of S

then you would NEED to use 6 surface integrals.

③ $\vec{F} = \langle z, y, x\rangle$

E is solid ball oriented out. $x^2+y^2+z^2 \le 16$

Can we use div Thm to find Flux of $\vec{F}$ through S. Yes!

Surface of ball = Sphere of radius 4

E = inside or the solid.

now Find Flux of $\vec{F}$ through S.

$$\operatorname{div}\vec{F} = \frac{\partial}{\partial x}(z) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(x) = 1$$

$$\iint_S \vec{F}\cdot d\vec{S} = \iiint_E 1\,dV = V(E) = \frac{4}{3}\pi(4)^3$$

⑤ $\vec{F} = xye^z\hat{\imath} + xy^2z^3\hat{\jmath} - ye^z\hat{k}$

S = surface of box bounded by coordinate planes and $x=3$, $y=2$, $z=1$

oriented out.

Find Flux of $\vec{F}$ across S.

$$\operatorname{div}\vec{F} = \frac{\partial}{\partial x}(xye^z) + \frac{\partial}{\partial y}(xy^2z^3) + \frac{\partial}{\partial z}(-ye^z)$$

$$= ye^{z} + 2xyz^3 - ye^{z} = 2xyz^3$$

$E: \quad 0 \le x \le 3, \quad 0 \le y \le 2, \quad 0 \le z \le 1$

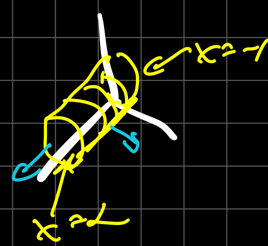
$$\iint_S \vec{F} \cdot d\vec{S} = \int_0^3 \int_0^2 \int_0^1 2xyz^3 \, dz\,dy\,dx$$

$$= 2 \int_0^3 x\,dx \int_0^2 y\,dy \int_0^1 z^3\,dz \quad \text{- simple!!}$$

7. $\vec{F}(x,y,z) = 3xy^2\,\hat{i} + xe^{z}\,\hat{j} + z^3\,\hat{k}$

S surface of solid bounded by $y^2+z^2=1$ and $x=-1$, $x=2$ oriented out.

Find Flux of $\vec{F}$ through (across) S.



$$\operatorname{div}\vec{F} = 3y^2 + 0 + 3z^2 = 3(y^2+z^2)$$

$E: \quad -1 \le x \le 2 \qquad y^2+z^2 \le 1$ cylindrical

$-1 \le y \le 1$

$-\sqrt{1-y^2} \le z \le \sqrt{1-y^2}$

$$\int_{-1}^{2} \int_{-1}^{1} \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} 3(y^2+z^2)\,dz\,dy\,dx = 3\int_{-1}^{1} \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} 3(y^2+z^2)\,dy\,dx$$

polar.

$z = r\cos\theta \qquad 0 \le r \le 1$

$y = r\sin\theta \qquad 0 \le \theta \le 2\pi$

$y^2+z^2 = r^2$

$$= 9 \int_0^{2\pi} \int_0^1 r^2 \, r\,dr\,d\theta = 9(2\pi)\left(\frac{r^4}{4}\right)\Big|_{r=0}^{r=1} = \frac{18\pi}{4} = \frac{9\pi}{2}.$$

(7^*) Suppose we have same $\vec{F}$ and same E but S is missing "top" $x=2$ part.

Can not use div thm.

unless.... we pretend the top is there then

Subtract Flux of $\vec{F}$ through top:

$$\text{Flux} = \frac{9\pi}{2} - \underbrace{\iint_{S_1} \vec{F}\cdot d\vec{S}}$$

$x=2$ part

$\hat{n} = \hat{\imath}$

$$\vec{F}(x,y,z) = 3xy^2\hat{\imath} + xe^z\hat{\jmath} + z^3\hat{k}$$

on S_1 $\quad \vec{F} = 6y^2\hat{\imath} + 2e^z\hat{\jmath} + z^3\hat{k} \qquad y^2+z^2 \le 1$

$$\vec{F}\cdot\hat{n} = 6y^2$$

then $\displaystyle\iint_{S_1} \vec{F}\cdot d\vec{S} = \iint_{S_1} 6y^2\, dS = \int_0^{2\pi}\int_0^1 6r^2\sin^2\theta\, r\, dr\, d\theta$

$z = r\cos\theta$

$y = r\sin\theta$

$0 \le r \le 1$

$0 \le \theta \le 2\pi$

and $\quad \text{Flux} = \dfrac{9\pi}{2} - $

(13) $\vec{F} = \|\vec{r}\|\vec{r} \qquad \vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$

$S =$ hemisphere $\quad z = \sqrt{1-x^2-y^2}$

and disk $\quad x^2+y^2 \le 1$ in xy plane.

Find Flux of $\vec{F}$ through S if S is oriented out.

$$\vec{F}(x,y,z) = \sqrt{x^2+y^2+z^2}\,(x\hat{\imath} + y\hat{\jmath} + z\hat{k})$$

$$= x\sqrt{x^2+y^2+z^2}\,\hat{\imath} + y\sqrt{x^2+y^2+z^2}\,\hat{\jmath} + z\sqrt{x^2+y^2+z^2}\,\hat{k}$$

$$\operatorname{div}\vec{F} = \vec{\nabla}\cdot\vec{F} = \sqrt{x^2+y^2+z^2} + x\frac{\cancel{2}x}{\cancel{2}\sqrt{x^2+y^2+z^2}} + \sqrt{x^2+y^2+z^2} + y\frac{\cancel{2}y}{\cancel{2}\sqrt{x^2+y^2+z^2}}$$

$$+ \sqrt{x^2+y^2+z^2} + z\frac{\cancel{2}z}{\cancel{2}\sqrt{x^2+y^2+z^2}}$$

$$= 3\sqrt{x^2+y^2+z^2} + \frac{x^2+y^2+z^2}{\sqrt{x^2+y^2+z^2}}$$

$$= 4\sqrt{x^2+y^2+z^2}$$

div looks good!

but P_x, Q_y, R_z have an issue at $(0,0,0)$.

Options: find flux via surface integrals.

OR

~~think back to~~ FTC.

~~$\int_{-1}^{1} \frac{1}{x}\,dx = \int_{-1}^{1} (\ln(x))'\,dx = \ln(1) - \ln(-1)$ ¡¡BAD!! $= 0$~~

~~thinking $\frac{1}{x}$ as $(\ln x)'$~~

Surface

Solid

S_1 for E_1

$0 \le \theta \le 2\pi$

$0 \le \phi \le \frac{\pi}{2}$

$a \le \rho \le 1$

hemisphere of radius a

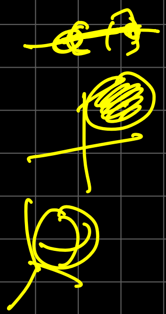
$x^2+y^2+z^2=a^2$

let S be the surface with this hemisphere

$$\iint_{S_1} \vec{F}\cdot d\vec{S} = \iiint_{E_1} 4\sqrt{x^2+y^2+z^2}\,dV$$

$$= \int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_a^1 4\rho\,\rho^2 \sin\phi\,d\rho\,d\phi\,d\theta$$

$$= 8\pi \int^{\pi/2} \sin\phi\,d\phi \int_a^1 \rho^3\,d\rho$$



$$= 8\pi \underbrace{(-\cos\phi)\Big|_{\phi=0}^{\phi=\frac{\pi}{2}}}_{1} \left(\frac{\rho^4}{4}\right)\Big|_{\rho=a}^{\rho=1}$$

$$= 2\pi(\rho^4)\Big|_{\rho=a}^{\rho=1} = 2\pi - 2\pi a^4$$

Flux of $\vec{F}$ through S $= \lim_{a\to 0^+} \iint_{S_1} \vec{F}\cdot d\vec{S} = \lim_{a\to 0^+}(2\pi - 2\pi a^4) = 2\pi.$