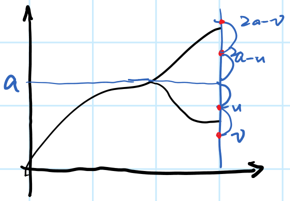
Joint Density of M_t, B_t

2022年6月8日 20:17

$\mathbb{P}(M_t = a, B_t = x)$ Joint Dist.



$a \geq u > v$

$\boxed{对称性}$

落回 u, v 之间　　上涨至 2a-u, 2a-v 之间

$\mathbb{P}(T_a < t, v < B_t < u) = \mathbb{P}(T_a < t, 2a-u < B_t < 2a-v)$

若 $B_t \geq 2a-u$, 则 $T_a \leq t$.

$= \mathbb{P}(2a-u < B_t < 2a-v)$

$=$ 考虑 $B_t \sim N(0,t)$ $= \displaystyle\int_{2a-u}^{2a-v} \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}} dx$

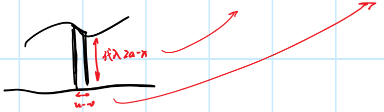
且考虑 $w \in \{T_a < t\} \Leftrightarrow w \in \{M_t > a)$

$\mathbb{P}(M_t > a, v < B_t < u) = \mathbb{P}(M_t > a, 2a-u < B_t < 2a-v)$

当 u, v 收敛至 x 时

$\cancel{(u-v)} \cdot \mathbb{P}(M_t > a, B_t = x) = \mathbb{P}(B_t = 2a-x) \cdot \cancel{(u-v)}$

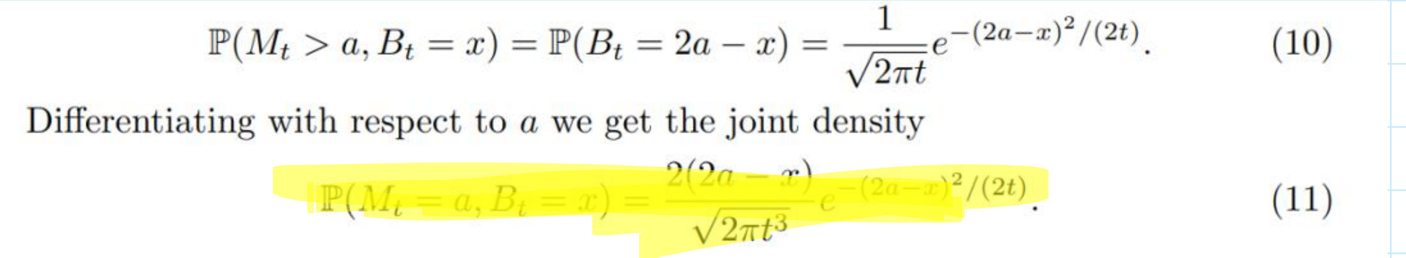
此时 $\displaystyle\int_{2a-u}^{2a-v} \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}} dx \approx \frac{1}{\sqrt{2\pi t}} e^{-\frac{(2a-x)^2}{2t}} \cdot (u-v) = \text{RHS}$



有 $\mathbb{P}(M_t > a, B_t = x) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{(2a-x)^2}{2t}}$

考虑 $\frac{\partial \mathbb{P}(X > x)}{\partial x} = -f(x)$

有 $\mathbb{P}(M_t = a, B_t = x) = -\frac{\partial}{\partial a}\left(\frac{1}{\sqrt{2\pi t}} e^{-\frac{(2a-x)^2}{2t}}\right)$



$$\mathbb{P}(M_t > a, B_t = x) = \mathbb{P}(B_t = 2a - x) = \frac{1}{\sqrt{2\pi t}} e^{-(2a-x)^2/(2t)}. \tag{10}$$

Differentiating with respect to a we get the joint density

$$\mathbb{P}(M_t = a, B_t = x) = \frac{2(2a-x)}{\sqrt{2\pi t^3}} e^{-(2a-x)^2/(2t)}. \tag{11}$$

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knock in / knock out

$$\mathbb{1}_{M_t < a} \underbrace{(B_t - \overset{\text{strike}}{K})}_{\text{call option}}{}_+$$

其中 $X_+ = \max(x, 0)$

有 $\mathbb{E}(\mathbb{1}_{M_t < a} \cdot (B_t - K)_+) = \iint \left[\mathbb{1}_{a < M} (x-K)_+\right] \cdot \overset{\text{joint density}}{f(a,x)}\, da\, dx$

有 $\mathbb{E}(\mathbb{1}_{M_t<a}\cdot(B_t-k)_+) = \iint \left[\mathbb{1}_{a<M}(x-k)_+\right]\cdot f(a,x)\,da\,dx$

In order to calculate the above expectation we use our knowledge of the joint density of M_t and B_t given by (11):

$$\begin{aligned}\mathbb{E}\,\mathbf{1}_{M_t<M}(B_t-K)_+ &= \iint \mathbf{1}_{a<M}(x-K)_+\frac{2(2a-x)}{\sqrt{2\pi t^3}}e^{-(2a-x)^2/(2t)}da\,dx\\ &= \int\limits_K^M\int\limits_x^M (x-K)\frac{2(2a-x)}{\sqrt{2\pi t^3}}e^{-(2a-x)^2/(2t)}da\,dx\\ &= \int\limits_K^M (x-K)\frac{e^{-x^2/2t}}{\sqrt{2\pi t}}dx - \int\limits_K^M (x-K)\frac{e^{-(x-2M)^2/2t}}{\sqrt{2\pi t}}dx. \qquad (13)\end{aligned}$$