Quadratic Variation of BM (QV)

2022年6月8日 18:12

partition of $[0,T]$: $\{t_0=0<t_1<t_2<\cdots<t_n=T\}$

记 Norm of π $\|\pi\| = \max(t_{j+1}-t_j)$, where $j=0,1,\dots,n-1$

Function f on $[0,T]$

有 QV of f on $[0,T]$ 记为 $\langle f,f\rangle_{[0,T]}$

Then quadratic variation is

$$\langle f,f\rangle_T = \lim_{\max_j |t_{j+1}-t_j|\to 0} \sum_{j=0}^{n-1} (f(t_{j+1})-f(t_j))^2 = \lim_{\|\Pi\|\to 0} \sum_{j=0}^{n-1} (f(t_{j+1})-f(t_j))^2. \quad (8)$$

为何QV重要? ~ 用以研究 处处不 cts diff'ble 的函数.

Statement.

If f cts diff'ble on $[0,T]$, 有 $\langle f,f\rangle_{[0,T]} = 0$ (无穷个0相加)

考虑 $\sum_{j=0}^{n-1} (f_{t_{j+1}} - f_{t_j})^2$

recall Mean Value Thm. f $\begin{cases} \text{diff'ble } (a,b) \\ \text{cts } [a,b] \end{cases}$ $\Rightarrow \exists\, c\in[a,b]$ s.t. $f'(c) = \frac{f(b)-f(a)}{b-a}$

存在 t_j^*

$= \sum_{j=0}^{n-1} \left[f'(t_j^*)(t_{j+1}-t_j)\right]^2$

Therefore

$$\begin{aligned}\sum_{j=0}^{n-1}(f(t_{j+1})-f(t_j))^2 &= \sum_{j=0}^{n-1}|f'(t_j^*)|^2(t_{j+1}-t_j)^2 \\ &\le \max_j |t_{j+1}-t_j| \sum_{j=0}^{n-1}|f'(t_j^*)|^2(t_{j+1}-t_j) \\ &= \|\Pi\| \sum_{j=0}^{n-1}|f'(t_j^*)|^2(t_{j+1}-t_j). \quad (9)\end{aligned}$$

因为 $(t_{j+1}-t_j)^2 < \|\pi\|\cdot(t_{j+1}-t_j)$

Sum $\sum_{j=0}^{n-1}|f'(t_j^*)|^2(t_{j+1}-t_j)$ is a Riemann sum for the integral $\int_0^T |f'(t)|^2 dt$. $<\infty$

$$\begin{aligned}\langle f,f\rangle_T &\le \lim_{\|\Pi\|\to 0} \|\Pi\| \sum_{j=0}^{n-1}|f'(t_j^*)|^2(t_{j+1}-t_j) \\ &= \lim_{\|\Pi\|\to 0}\|\Pi\| \lim_{\|\Pi\|\to 0}\sum_{j=0}^{n-1}|f'(t_j^*)|^2(t_{j+1}-t_j). \quad (10)\end{aligned}$$

结论

重要

很重要

Theorem 4. *Let $B(t)$ be a Brownian motion. Then $\langle B,B\rangle_T = T$.*

$\langle B_t, B_t\rangle_{[0,T]} = T$

$(dB_t)^2 = dt$

The fact that $\langle B,B\rangle_t = t$ is informally written as

$$dB(t)dB(t) = dt.$$

Pf. By def^n QV $= \langle B_t,B_t\rangle_{[0,T]} = \lim_{\|\pi\|\to 0}\sum_{j=0}^{n-1}(B_{t_{j+1}} - B_{t_j})^2$

记 $\Delta B_j = B_{t_{j+1}} - B_{t_j}$

有 QV $= \lim \sum(\Delta B_j)^2$ → 记为 Z

有 $E Z = E \sum_{j=0}^{n-1}(\Delta B_j)^2$

$= \sum^{n-1} E(\Delta B_j)^2$ → $t_{j+1}-t_j = \Delta t_j$

有 $\mathbb{E} Z = \mathbb{E} \sum_{j=0} (\Delta B_j)^2$ —

$$= \sum_{j=0}^{m-1} \mathbb{E}(\Delta B_j)^2 \quad t_{j+1} - t_j = \Delta t_j$$

$$= \sum_j \Delta t_j = T$$

考虑 $\mathrm{Var}\, Z = \mathbb{E}(Z - \mathbb{E}Z)^2 = \mathbb{E}\left(\sum (\Delta B_j)^2 - T\right)^2$

$$= \mathbb{E}\left(\sum (\Delta B_j)^2 - \sum \Delta t_j\right)^2$$

$$= \mathbb{E}\left(\sum_j \left[(\Delta B_j)^2 - \Delta t_j\right]\right)^2$$

$$= \mathbb{E} \sum_{i,j} \left[(\Delta B_i)^2 - \Delta t_i\right]\cdot\left[(\Delta B_j)^2 - \Delta t_j\right]$$

$$= \sum_{i,j} \mathbb{E}[\ ]\cdot[\ ]$$

trivial: $\left(\sum_i a_i\right)^2 = \sum_{i,j} a_i a_j$

分类讨论: ① $i \neq j \Rightarrow$ independent $X \perp Y \Rightarrow f_1(X) \perp f_2(Y)$

$$\mathrm{Var}\, Z = \mathbb{E}\left((\Delta B_j)^2 - \Delta t_j\right)\left((\Delta B_i)^2 - \Delta t_i\right)$$

$$= \mathbb{E}\left[(\Delta B_j)^2 - \Delta t_j\right]\cdot \mathbb{E}\left[(\Delta B_i)^2 - \Delta t_i\right]$$

$$= 0 \cdot 0 = 0$$

② $i = j$

$$\mathrm{Var}\, Z = \left[\mathbb{E}\left((\Delta B_j)^2 - \Delta t_j\right)\right]^2$$

$$= \mathbb{E}\left[(\Delta B_j)^4 - 2(\Delta B_j)^2 \Delta t_j + (\Delta t_j)^2\right]$$

考虑 $\mathbb{E}X^4 = \int_{\mathbb{R}} x^4 \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$

$$= \int_{\mathbb{R}} x^3 \left(\frac{x}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\right) dx$$

是 $\frac{-1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$ 的导数

$$= x^3\left(-\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}\right)\Bigg|_{-\infty}^{+\infty} + \int 3x^2 \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$

exp 比 x^3 decay 更快 所以是0。

从而 $\mathrm{Var}\, Z = \mathbb{E}\left(3(\Delta t_i)^2 - 2(\Delta t_i)^2 + (\Delta t_i)^2\right) = 2(\Delta t_i)^2$

综上 $\mathrm{Var}\, Z = 2\sum_i (\Delta t_i)^2$ 收敛了。

从而得证!

Black-Scholes & Time-Series

Volatility of the Brownian Motion. Let α and σ be constants and define the geometric Brownian motion

$$S(t) = s(0)e^{\sigma B(t) + (\alpha - \sigma^2/2)t}.$$

$S(t)$: Asset Price 在时间 t; α: r - risk free overnight rate, σ: Black vol

This is the asset-price model used in the Black-Scholes formula. We show how to use the quadratic variation of BM to identify the volatility σ from a path of the process.

⇝ estimate σ from $\{t_1, t_2, \dots, t_n\}$ & $\{S_1, S_2, \dots, S_n\}$

通过过去预测未来价格

Let $0 \le T_1 < T_2$ be give and suppose we observe geometric BM $S(t)$ on time interval $T_1 \le tT_2$. We may then choose a partition of this interval $T_1 = t_0 < t_1 < \cdots < t_m = T_2$ and observe "log-returns"

$\frac{S(t_{j+1})}{S(t_j)}$ 再取对

$$\log \frac{S(t_{j+1})}{S(t_j)} = \sigma(B(t_{j+1}) - B(t_j)) + (\alpha - \sigma^2/2)(t_{j+1} - t_j)$$

over each of the subintervals $[t_j, t_{j+1}]$. The sum of the squares of the log-returns, sometimes called the realized volatility is

完全平方公式展开

$$\sum_{j=0}^{m-1}\left(\log\frac{S(t_{j+1})}{S(t_j)}\right)^2 = \sigma^2 \sum_{j=0}^{m-1}(B(t_{j+1}) - B(t_j))^2 + (\alpha - \sigma^2/2)\sum_{j=0}^{m-1}(t_{j+1} - t_j)^2$$
$$+ 2\sigma(\alpha - \sigma^2/2)\sum_{j=0}^{m-1}(B(t_{j+1}) - B(t_j))(t_{j+1} - t_j)$$
$$\to \sigma^2(T_2 - T_1). \qquad (11)$$

$$\rightsquigarrow \sigma \approx \left(\frac{1}{T}\sum_j \left(\log \frac{S_{t_{j+1}}}{S_{t_j}}\right)^2\right)^{\frac{1}{2}}$$

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下证 $\lim\limits_{\|\pi\|\to 0}\left|\sum_j \Delta B_j \Delta t_j\right| = 0$

$$\le \sum_j |\Delta B_j| \Delta t_j$$

$$\le \max_j |\Delta B_j| \cdot T \quad \rightsquigarrow \text{因为每个 } |\Delta B_j| \le \max$$

$\to 0$

$\rightsquigarrow$ 也写为 $dB_t \cdot dt = 0 \iff \langle B_t, t\rangle_{[0,T]} = 0$

$$\langle f, g\rangle_{[0,T]} = \lim_{\|\pi\|\to 0} \sum \left[f(t_{j+1}) - f(t_j)\right]\left[g(t_{j+1}) - g(t_j)\right]$$

let $f = B_t$, $g = t$