Girsanov's Theorem

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Bt is a BM on (Ω, F, P)

γ is a constant (可推广至 time-dependent & random. i.e. stochastic 过程)

$$Z_t = e^{\int_0^t \gamma dB_s - \frac{1}{2}\int_0^t \gamma^2 ds}$$

$$\tilde{\mathbb{P}}(dz) = e^{\mu z - \frac{1}{2}\mu^2}\mathbb{P}(dz) \sim N(\mu, 1)$$

定义 $\tilde{\mathbb{P}}$ on $\mathcal{F}_T$: $\mathbb{E}(\mathbb{1}_A Z_T) = \tilde{\mathbb{E}}\,\mathbb{1}_A = \tilde{\mathbb{P}}(A)$

$\tilde{B}_t = B_t - \gamma t$ 为 BM$(\Omega, \mathcal{F}, \tilde{\mathbb{P}})$

So assume that $B_t, t \ge 0$ defined on $(\Omega, \mathcal{F}, \mathbb{P})$ is a Brownian motion starting from zero. Now define for a given constant γ

$$Z_t = e^{\int_0^t \gamma dB_s - \frac{1}{2}\int_0^t \gamma^2 ds} = e^{\gamma B_t - \frac{1}{2}\gamma^2 t} \tag{3}$$

and for each $T > 0$ define a new probability measure $\tilde{\mathbb{P}}_T$ on $\mathcal{F}_T$ (i.e. on the σ-algebra of the events known at time T) by

$$\tilde{\mathbb{P}}_T = \mathbb{E}(\mathbf{1}_A Z_T) \quad \text{for} \quad A \in \mathcal{F}_T. \tag{4}$$

Theorem 1 (Girsanov, Cameron-Martin). *Define a process $\tilde{B}_t = B_t - \gamma t$. Then for each fixed $T < \infty$ the process $\tilde{B}_t$ is a Brownian motion on $(\Omega, \mathcal{F}_T, \tilde{\mathbb{P}}_T)$.*

tivial case. $\gamma = 0 \Rightarrow Z_t = 1 \Rightarrow \tilde{\mathbb{P}}(A) = \mathbb{P}(A)$

Pf: ① Z_t 在P下 Martingale. (证明: Itô's Lemma)

② $\tilde{\mathbb{P}}_T$ 测度 are consistent. i.e., if $t < T$, $A \in \mathcal{F}_t$, $\tilde{\mathbb{P}}_T(A) = \tilde{\mathbb{P}}_t(A)$

Lemma 3. *Measures $\tilde{\mathbb{P}}_T$ are consistent, that is for $t < T$ and $A \in \mathcal{F}_t$*

$$\tilde{\mathbb{P}}_T(A) = \tilde{\mathbb{P}}_t(A). \tag{5}$$

Proof of Lemma 3. Indeed, using the tower property of conditional expectation

$$\begin{aligned}\tilde{\mathbb{P}}_T(A) &= \tilde{\mathbb{E}}_T(\mathbf{1}_A) = \mathbb{E}(Z_T\mathbf{1}_A) = \mathbb{E}(\mathbb{E}(Z_T\mathbf{1}_A|\mathcal{F}_t)) \\ &= \mathbb{E}(\mathbf{1}_A\mathbb{E}(Z_T|\mathcal{F}_t)) = \mathbb{E}(\mathbf{1}_A Z_t) = \tilde{\mathbb{P}}_t(A).\end{aligned} \tag{6}$$

□

Fix a sequence $t_1 \le t_2 \le \cdots \le t_n$ and consider Brownian increments $B(t_1), B(t_2) - B(t_1), \ldots, B(t_n) - B(t_{n-1})$. By the definition of $\tilde{\mathbb{P}}_T$

$$\begin{aligned}&\tilde{\mathbb{P}}_T[B(t_1) \in dz_1, B(t_2) - B(t_1) \in dz_2, \ldots, B(t_n) - B(t_{n-1}) \in dz_n] \\ &= e^{\gamma B(t_n) - \gamma^2 t_n/2}\mathbb{P}_T[B(t_1) \in dz_1, B(t_2) - B(t_1) \in dz_2, \ldots, B(t_n) - B(t_{n-1}) \in dz_n] \\ &= e^{\gamma(z_1 + \cdots + z_n) - \gamma^2 t_n/2}\frac{1}{(\sqrt{2\pi})^n}e^{-z_1^2/(2t_1) - z_2^2/(2(t_2 - t_1)) - \cdots - z_n^2/(2(t_n - t_{n-1}))} \\ &= \frac{1}{(\sqrt{2\pi})^n}e^{(\gamma z_1 - \gamma^2 t_1/2 - z_1^2/(2t_1)) + \cdots + (\gamma z_n - \gamma^2(t_n - t_{n-1})/2 - z_n^2/(2(t_n - t_{n-1})))} \\ &= \frac{1}{(\sqrt{2\pi})^n}e^{-(z_1 - \gamma t_1)^2/(2t_1) - \cdots - (z_n - \gamma(t_n - t_{n-1}))^2/(2(t_n - t_{n-1}))}.\end{aligned} \tag{7}$$

Thus under $\tilde{\mathbb{P}}$ random variables $B(t_1) - \gamma t_1, B(t_2) - B(t_1) - \gamma(t_2 - t_1), \ldots, B(t_n) - B(t_{n-1}) - \gamma(t_n - t_{n-1})$ are independent normal random variables with expectation zero and variances $t_1, t_2 - t_1, \ldots, t_n - t_{n-1}$. So we proved that the finite dimensional distributions of $\tilde{B}_t = B_t - \gamma t$ under $\tilde{\mathbb{P}}$ coincide with the finite dimensional distributions of Brownian motion. □

Define $A = \begin{cases} B_{t_1} \in [z_1, z_1 + dz_1] \\ B_{t_2} - B_{t_1} \in [z_2, z_2 + dz_2] \\ \vdots \\ B_{t_n} - B_{t_{n-1}} \in [z_n, z_n + dz_n] \end{cases}$

有 $\mathbb{P}(A) = \mathbb{P}(B_{t_1} \in dz_1)\cdot\mathbb{P}(B_{t_2} - B_{t_1} \in dz_2)\cdots\mathbb{P}(B_{t_n} - B_{t_{n-1}} \in dz_n)$

考虑 $\int_{z_1}^{z_1 + dz_1} f(y)dy \doteq f(z_1)\cdot dz_1$ 近似

$\Rightarrow \mathbb{P}(A) = \frac{1}{\sqrt{2\pi t_1}}\exp\left(-\frac{z_1^2}{2t_1}\right)dz_1 \frac{1}{\sqrt{2\pi(t_2 - t_1)}}\exp\left(-\frac{z_2^2}{2(t_2 - t_1)}\right)dz_2 \cdots$

By Lemma 3 $\Rightarrow \tilde{\mathbb{P}}_T(A) = \tilde{\mathbb{P}}_{t_n}(A)$

By def $\Rightarrow \tilde{\mathbb{P}}_T = \mathbb{E}(Z_{t_n}\cdot\mathbb{1}_A)$

考虑 $Z_{t_n} = \exp(\gamma B_{t_n} - \frac{1}{2}\gamma^2 t_n)$, $B_{t_n} = z_1 + z_2 + \cdots + z_n$

$\therefore Z_{t_n} = \exp(\gamma(z_1 + \cdots + z_n) - \frac{1}{2}\gamma^2 t_n) = \mathbb{E}(Z_{t_n}\mathbb{1}_A) = e^{\gamma(z_1 + \cdots + z_n) - \frac{\gamma^2}{2}(t_1 + t_2 - t_1 + \cdots + t_n - t_{n-1})}$

考虑 $\exp^{\gamma(z_1 + \cdots + z_n)} = \frac{1}{\sqrt{2\pi t_1}}e^{\frac{z_1^2}{-2t_1}}\cdot e^{\gamma z_1 - \frac{1}{2}\gamma^2 t}$

考虑 $\gamma z_1 - \frac{1}{2}\gamma^2 t + \frac{z_1^2}{2t_1}$ 为完全平方

$Z_{t_n} = \frac{1}{\sqrt{2\pi t_1}}e^{-\frac{(z_1 - \gamma t_1)^2}{2t_1}}dz_1 \times \frac{1}{\sqrt{2\pi(t_2 - t_1)}}\exp^{\frac{-(z_2 - \gamma(t_2 - t_1))^2}{2(t_2 - t_1)}} \times \cdots \times \frac{1}{\sqrt{2\pi(t_n - t_{n-1})}}\exp - \left(\frac{(z_n - \gamma(t_n - t_{n-1}))^2}{2(t_n - t_{n-1})}\right)$

从而得 $B_t - \gamma t$.

$\mathbb{P}\{z > b\} = \tilde{\mathbb{P}} = N[a, b]$