homogeneous

$\dfrac{dy}{dt}+a(t)y=0$

sol:$y(t)=c\exp\left(-\int a(t)dt\right)$

init-val: $\ln|y(t)|-\ln|y(t_0)|=-\int_{t_0}^{t}a(s)ds\Rightarrow y(t)=y_0\exp\left(-\int_{t_0}^{t}a(s)ds\right)$

non-homo

$\dfrac{dy}{dt}+a(t)y=b(t)$

select $\mu(t)=\exp\left(\int a(t)dt\right)$

$y=\dfrac{1}{\mu(t)}\left(\int\mu(t)b(t)dt+c\right)$

init-val: $\mu(t)y-\mu(t_0)y_0=\int_{t_0}^{t}\mu(s)b(s)ds$

seprable

$\dfrac{dy}{dt}=\dfrac{g(t)}{f(y)}$

$\int f(y)dy=\int g(t)dt+C$

init-value:$\int_{y_0}^{y}f(r)dr=\int_{t_0}^{t}g(s)ds$

if $\dfrac{dy}{dt}=f(y)g(t),\ \text{and}\ f(y_0)=0$, then $y(t)=y_0$ is the only solution.

exact

$M(y,t)+N(y,t)\dfrac{dy}{dx}=0$

test:$M_y=N_t$?

if yes, find $\phi(y,t) s.t. \phi_t=M,\phi_y=N$ (by $\int M$), $\phi=C$ is the implicit solution. $C=\phi(t_0,y_0)$ if init-val given

if not, exist $\mu(t,y)$ to make equation exact if

- $p(t)=\dfrac{M_y-N_t}{N}$ is a function of t
- $p(y)=\dfrac{N_t-M_y}{M}$ is a function of y

then $\mu(t)=\exp\left(\int pdt\right)$ or $\mu(y)=\exp\left(\int pdy\right)$

picard iter: $y_{n+1}(t)=y_0+\int_{t_0}^{t}f(s,y_n(s))ds$

existence-uniqueness: $M=\max\limits_{(t,y)\ \text{in}\ R}|f(t,y)|$, $\alpha=\min\left(a,\dfrac{b}{M}\right)$ ⇒unique solution $y(t)$ on $[t_0,t_0+\alpha]$

Example 4. Show that the solution $y(t)$ of the initial-value problem

$$\frac{dy}{dt}=e^{-t^2}+y^3,\qquad y(0)=1$$

exists for $0\leqslant t\leqslant 1/9$, and in this interval, $0\leqslant y\leqslant 2$.

Solution. Let R be the rectangle $0\leqslant t\leqslant\frac{1}{9}$, $0\leqslant y\leqslant 2$. Computing

$$M=\max_{(t,y)\text{ in }R}e^{-t^2}+y^3=1+2^3=9,$$

we see that $y(t)$ exists for

$$0\leqslant t\leqslant\min\left(\tfrac{1}{9},\tfrac{1}{9}\right)$$

and in this interval, $0\leqslant y\leqslant 2$.