Backward and Forward Kolmogorov's Eqn

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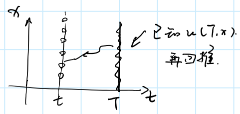
https://www.math.nyu.edu/~kohn/pde.finance/2015/section1.pdf pp. 10 - 16

1. Backward, Type 1 PDE / "Expected Payoff"

$u(t,x) = \mathbb{E}_{X_t=x} V(X_T)$ $\quad \begin{cases} u_t + \mathcal{L}u = 0 \\ u(T,x) = V(x) \end{cases}$



2. Forward Kolmogorov's Eqn [Conditional Density] ~ density 固定. fwd 只需算一次. backward. 给 $V(x_1)\ V(x_2) \cdots V(x_{10,000})$ 向前推 多次!

$p(z,s;x,t)$ = probability of being at z at time s, given that it started at x at time t. $\mathbb{P}(X_s = z \mid X_t = x)$

类似马氏链 finite MC ~ stationary probability.

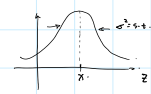
More precisely: $p(\cdot,s;x,t)$ is the probability density of the state at time s, given that it started at x at time t. Of course p is only defined for $s > t$. To describe a Markov process, p must satisfy the Chapman-Kolmogorov equation

$$p(z,s;x,t) = \int_{R^n} p(z_1,s_1;x,t)p(z,s;z_1,s_1)\,dz_1$$

for any s_1 satisfying $t < s_1 < s$. Intuitively: the state can get from (x,t) to (z,s) by way of being at various intermediate states z_1 at a chosen intermediate time s_1. The Chapman-Kolmogorov equation calculates $p(z,s;x,t)$ by adding up (integrating) the probabilities of getting from (x,t) to (z,s) via (z_1,s_1), for all possible intermediate positions z_1.

例: BM: $p(s,z;t,x) = \frac{1}{\sqrt{2\pi(s-t)}} e^{-\frac{(z-s)^2}{2(s-t)}}$ $\xrightarrow{p(s,z;t,x)}$

考虑 $X_s - X_t \sim N(0, s-t)$.



As $s \to t$, 有 $f_x(s,z;t,x) \to \delta_x(z) = \begin{cases} 0 & z \neq x \\ +\infty & z = x \end{cases}$ 且 $\int_{\mathbb{R}} \delta_x(z)\,dz = 1$

考虑 $dX_t^1 = a^1 dt + \sigma^{11} dB_t^1 + \cdots \sigma^{1n} dB_t^n$

$dX_t^2 = a^2 dt + \sigma^{21} dB_t^2 + \cdots + \sigma^{2n} dB_t^n$

$\vdots$

$dX_t^m = a^m dt + \cdots + \sigma^{mn} dB_t^n$

The initial position of a Markov process need not be deterministic. Even if it is (e.g. if $y(0) = x$ is fixed), we may wish to consider a later time as the "initial time." The transition probability determines the evolution of the spatial distribution, no matter what its initial value: if $\rho_0(x)$ is the probability density of the state at time t then

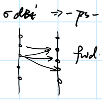
$$\rho(z,s) = \int_{R^n} p(z,s;x,t)\rho_0(x)\,dx \tag{12}$$

gives the probability density (as a function of z) at any time $s > t$.

The crucial fact about the transition probability is this: it solves the *forward Kolmogorov equation* in s and z:

$$-p_s - \sum_i \frac{\partial}{\partial z_i}(f_i(z,s)p) + \frac{1}{2}\sum_{i,j,k} \frac{\partial^2}{\partial z_i \partial z_j}(g_{ik}(z,s)g_{jk}(z,s)p) = 0 \text{ for } s > t, \tag{13}$$

1-D w/ $dX_t = a\,dt + \sigma\,dB_t^1 \Rightarrow -p_s - (a(s,z)p)_z + \frac{1}{2}(\sigma^2(s,z)p)_{zz} = 0$



with initial condition

$$p = \delta_x(z) \text{ at } s = t.$$

已知 initial position. 向未来推. ⇒ forward.

We can write the forward Kolmogorov equation as

$$-p_s + \mathcal{L}^* p = 0 \tag{14}$$

w/ $p(t,z;t,x) = \delta_x(z)$

with

$$\mathcal{L}^* p = -\sum_i \frac{\partial}{\partial z_i}(f_i p) + \sum_{i,j} \frac{\partial^2}{\partial z_i \partial z_j}(a_{ij}p). \tag{15}$$

Here $a_{ij} = \frac{1}{2}(gg^T)_{ij}$ just as before. The initial condition $p = \delta_x(z)$ encapsulates the fact, already noted, that the graph of $p(\cdot,s;x,t)$ becomes infinitely tall and thin at x as s decreases to t. The technical meaning is that

考虑 $u(t,x) = \mathbb{E}_{X_t=x} V(X_T)$.

$= \int_{-\infty}^{+\infty} V(z) \cdot p(T,z;t,x)\,dz$

Note: 内积 $\langle \mathcal{L}x, y \rangle = \langle x, \mathcal{L}^* y \rangle$.

例: 矩阵中 $\mathcal{L}^* = \mathcal{L}^T$

在 function space 中 $\langle f, g \rangle = \int_{-\infty}^{+\infty} f g\,dx$

Pf:

1) $u(t,x) = \mathbb{E}_{X_t=x} V(X_T)$ – backward.

$= \int_{-\infty}^{+\infty} V(z)\, p(T,z;t,x)\,dz$

$u_t + \mathcal{L}u = 0$. $\quad \mathbb{E}_{X_t=x}(\mathbb{E}_{X_s=z}(V(X_T))) = \mathbb{E}_{X_t=x} V(X_T)$.

$\int_{-\infty}^{+\infty} u(s,z) \cdot p(s,z;t,x)\,dz = u(t,x)$.

2) $\langle \mathcal{L}f, g \rangle = \langle f, \mathcal{L}^* g \rangle$

思路 $\mathcal{L}f = a f_x + \frac{1}{2}\sigma^2 f_{xx}$

$\mathcal{L}^* g = (ag)_x + \frac{1}{2}(\sigma^2 g)_{xx}$

$\Rightarrow \int \mathcal{L}f g\,dx = \int a f_x g\,dx + \int \frac{1}{2}\sigma^2 f_{xx} g\,dx$

$= \underbrace{f(ag)\Big|_{-\infty}^{+\infty}}_{0} - \int f(ag)_x\,dx + \frac{1}{2}\Big(\underbrace{f_x \cdot (\sigma^2 g)\Big|_{-\infty}^{+\infty}}_{0} - \int_{-\infty}^{+\infty} f_x (\sigma^2 g)_x\,dx\Big)$.

$= -\int f(ag)_x\,dx - \frac{1}{2}\Big[f \cdot (\sigma^2 g)_x\Big|_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} f(\sigma^2 g)_{xx}\,dx\Big]$

$= -\int_{\mathbb{R}} f \cdot (ag)_x\,dx + \frac{1}{2}\int_{\mathbb{R}} f \cdot (\sigma^2 g)_{xx}\,dx = \int f \mathcal{L}^* g\,dx$ $\quad \square$.

3) $u(T,x) - u(t,x) \overset{\text{Itô}}{=} \int_t^T u_t\,ds + \int_t^T u_x\,dX_s + \frac{1}{2}\int_t^T u_{xx}(dX_s)^2$

$= \int_{\mathbb{R}} \cdots dx + \frac{1}{2}\int_{\mathbb{R}} \cdots \sigma^2 f_{xx}\, dx \cdots \iint \mathcal{L} f \, dx$ □.

3) $u(T,x) - u(t,x) \overset{\text{Itô}}{=} \int_t^T u_t \, ds + \int_t^T u_x \, dX_s + \frac{1}{2}\int_t^T u_{xx} (dX_s)^2$

$= \int_t^T u_t \, ds + \int_t^T u_x (a\,ds + \sigma\, dB_s) + \frac{1}{2}\int_t^T \sigma^2 u_{xx} \, ds$

$= \int_t^T (\overset{u_t + \mathcal{L}u}{\cancel{u_t + a u_x + \frac{1}{2}\sigma^2 u_{xx}}}) \, ds$ + Itô Integral.

有 $\mathbb{E}_{X_t = x}\left[u(T, X_T) - u(t,x) \right] = \mathbb{E}_{X_t=x}\left[\int_t^T (u_t + \mathcal{L}u) \, ds + \overset{0}{\cancel{\int \text{Itô}\, dB_s}} \right]$

考虑 $\mathbb{E}_{X_t=x}\left[u(T,X_T)\right] - u(t,x) = \mathbb{E}_{X_t=x}\int_t^T (u_t + \mathcal{L}u)\, ds$

利用1)结论 $\Rightarrow \int_{-\infty}^{+\infty} u(T,z)\, p(T,z;t,x)\, dz - u(t,x) = \int_t^T \mathbb{E}_{X_t=x}\left(u_t + \mathcal{L}u\right) ds = \int_t^T\int_{-\infty}^{+\infty} (u_t + \mathcal{L}u)\, p(s,z;t,x)\, dz\, ds$

考虑 RHS $= \int_t^T \int_{\mathbb{R}} u_t \, p \, dz\, ds + \int_t^T\int_{\mathbb{R}} \mathcal{L}u \cdot p \, dz\, ds$

其中 $\int_{\mathbb{R}} \mathcal{L}u\, p\, dz = \int_{\mathbb{R}} u \cdot \mathcal{L}^* p \, dz$ by 2).

则有 RHS $= \int_{-\infty}^{+\infty}\int_t^T u_t\, p\, ds\, dz + \int_t^T \int_{\mathbb{R}} u \cdot \mathcal{L}^* p \, dz\, ds.$

其中 $\int_t^T u_s \cdot p \, ds = u\cdot p \Big|_{s=t}^{T} - \int_t^T u \cdot p_s \, ds$

$= u(T,z)\cdot p(T,z) - \int_t^T u \cdot p_s\, ds. - u(t,z)\cdot p(t,z).$

则 RHS $= \int_{-\infty}^{+\infty} \left(u(T,z)\cdot p(T,z) - u(t,z)\cdot p(t,z)\right) dz - \int_{-\infty}^{+\infty}\int_t^T u\cdot p_s \, ds\, dz + \int_t^T\int_{-\infty}^{+\infty} u \cdot \mathcal{L}^* p \, dz\, ds$

又 LHS $= \int_{-\infty}^{+\infty} u(T,z)\cdot p(T,z)\, dz - u(t,x).$

则 $- u(t,x) = \int_{-\infty}^{+\infty} u(t,z)\cdot p(t,z)\, dz - \underbrace{\int_t^T \int_{-\infty}^{+\infty} u(-p_s + \mathcal{L}^* p)\, dz\, ds}_{M.}$

若 $M \neq 0$ $\exists\ p^t(z,s)$ s.t. $M \neq 0$ $\Rightarrow$ $\exists\ u$ s.t. u 不动. M changes

$\Rightarrow M = 0$.

$\Rightarrow \quad -u(t,x) = \int_{\mathbb{R}} u(t,z)\cdot p(t,z)\, dz$ QED!