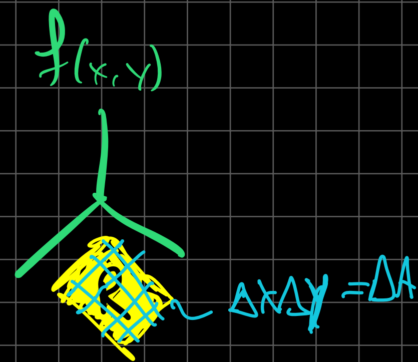
# Integrals of $f(x,y)$ 15.1 & 15.2

(10) $\iint_R (2x+1)\,dA$ $\quad dA = dx\,dy$ $\qquad R = \{(x,y) \mid 0 \le x \le 2,\ 0 \le y \le 4\}$

Rectangle

bounded by constants.

$$= \int_0^4 \int_0^2 (2x+1)\,dx\,dy$$




$$= \int_0^2 \underbrace{\int_0^4 (2x+1)\,dy}\,dx$$

treat 2x+1 constant.

$$= \int_0^2 (2x+1)y \Big|_{y=0}^{y=4} dx$$

$$= \int_0^2 (2x+1)4\,dx = 4(x^2+x)\Big|_{x=0}^{x=2} = 4(6) = 24.$$

Note:

$$\int_0^4 \int_0^2 (2x+1)\,dx\,dy = \int_0^4 (x^2+x)\Big|_{x=0}^{x=2} dy = \int_0^4 6\,dy = 24.$$

(13) $f(x,y) = x + 3x^2y^2$

find $\int_0^2 f(x,y)\,dx = \int_0^2 x + 3x^2y^2\,dx = \left(\frac{x^2}{2} + x^3y^2\right)\Big|_{x=0}^{x=2}$

$$= \frac{2^2}{2} + 2^3y^2 = 2 + 8y^2$$

(17) $\int_0^1 \int_1^2 (x+e^{-y})\,dx\,dy$

$$= \int_0^1 \left(\frac{x^2}{2} + xe^{-y}\right)\Big|_{x=1}^{x=2} dy$$

$$= \int_0^1 \left[2+2e^{-y} - \left(\frac{1}{2} + e^{-y}\right)\right] dy$$

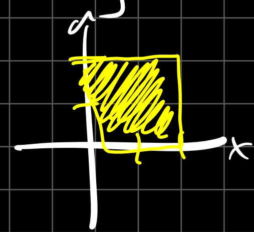
$$= \int_0^1 \left(\frac{3}{2} + e^{-y}\right) dy$$

$$= \left(\frac{3}{2}y - e^{-y}\right)\Big|_{y=0}^{y=1}$$

$$= \left(\frac{3}{2} - e^{-1}\right) - \left(0 - e^{0}\right)$$

$$= \frac{3}{2} - e^{-1} + 1 = \frac{5}{2} - e^{-1}$$

(33) $\iint_R ye^{-xy}\,dA \qquad R = [0,2]\times[0,2]$




$$= \int_0^2 \int_0^2 \underline{y}\, e^{-xy} \boxed{dx}\, dy = \int_0^2 ?\ dy$$

what is a function that satisfies $\frac{\partial}{\partial x}\left(\ ?\ \right) = ye^{-xy}$

$? = -e^{-xy}$

alternative way of thinking:

y = constant. $\qquad \int_0^2 ye^{-xy}\,dx = y\int_0^2 e^{-xy}\,dx = \cancel{y}\left(-\frac{e^{-xy}}{\cancel{y}}\right)\Big|_{x=0}^{x=2}$

$$= \int_0^2 \int_0^2 \underbrace{y e^{-xy}}\, dx\, dy = \int_0^2 \left(-e^{-xy}\right)\Big|_{x=0}^{x=2} dy$$

$$= \int_0^2 \left(-e^{-2y} + 1\right) dy$$

$$= \left(\frac{e^{-2y}}{2} + y\right)\Big|_{y=0}^{y=2}$$

$$= \frac{e^{-4}}{2} + 2 - \left(\frac{1}{2}\right)$$

what if we tried:

$$\int_0^2 \int_0^2 y e^{-xy}\, dy\, dx \quad \text{need IBP.}$$

(47) $f(x,y) = x^2 y$

Calc II avg $\dfrac{\int_a^b f(x)dx}{b-a} = \dfrac{\int_a^b f(x)dx}{\int_a^b 1\, dx}$

$R$ = Rectangle with vertices $(-1,0), (-1,5), (1,5), (1,0)$

find avg value of $f$ over $R$.

$$f_{avg} := \frac{\iint_R f(x,y)\, dA}{\iint_R 1\, dA = A(R)}$$

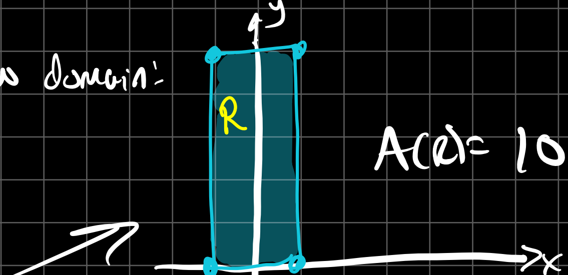
Calc I


$\underbrace{\quad}_{b-a}$

$\int_a^b (1) = (b-a)(1) = b-a$

$=$ size of domain

Draw domain:




$A(R) = 10$

$-1 \le x \le 1 \quad 0 \le y \le 5$

$$f_{avg}=\frac{\iint_R x^2y\,dA}{10}=\frac{\int_0^5\int_{-1}^1 x^2y\,dx\,dy}{10}=\frac{\int_0^5 \frac{x^3y}{3}\Big|_{x=-1}^{x=1}dy}{10}$$

$$=\frac{\int_0^5\left(\overbrace{\frac{y}{3}+\frac{y}{3}}^{\frac{2y}{3}}\right)dy}{10}=\frac{\left(\frac{y^2}{3}\right)\Big|_{y=0}^{y=5}}{10}=\frac{25}{30}=\frac{5}{6}.$$

**Special functions** $f(x,y)=p(x)q(y)$ = separable. over $R$

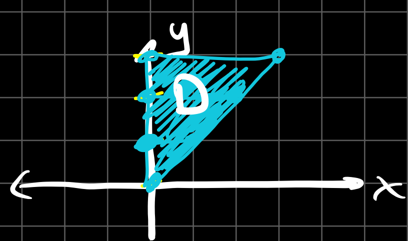
Ex) $f(x,y)=x^2y=p(x)q(y)$ where $p(x)=x^2$ $q(y)=y$

then $\int_a^b\int_c^d f(x,y)\,dx\,dy=\int_a^b q(y)\,dy\int_c^d p(x)\,dx$

Ex) $\int_0^5\int_{-1}^1 x^2y\,dx\,dy=\int_{-1}^1 x^2\,dx\int_0^5 y\,dy=\left(\frac{x^3}{3}\Big|_{x=-1}^{x=1}\right)\left(\frac{y^2}{2}\Big|_{y=0}^{y=5}\right)$

15.2

9) $\iint_D e^{-y^2}\,dA \quad D=\{(x,y)\mid 0\le y\le 3,\ 0\le x\le y\}.$



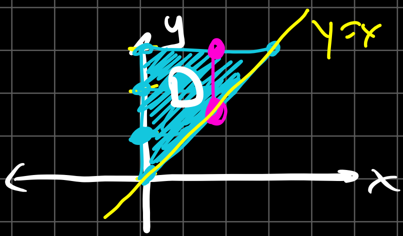

$$\iint_D e^{-y^2}\,dA=\underbrace{\int_0^3 \underbrace{\int_0^y e^{-y^2}\,dx}\,dy}=\int_0^3\left(xe^{-y^2}\Big|_{x=0}^{x=y}\right)dy$$

$$=\int_0^3 ye^{-y^2}\,dy$$

Could also switch

$u=-y^2 \Rightarrow du=-2y\,dy$

but how?




let $x$ have constant bounds.

$0 \le x \le 3$

find $y$ in terms of $x$

$x \le y \le 3$

$$\iint_D e^{-y^2}\,dA = \int_0^3 \int_x^3 e^{-y^2}\,dy\,dx$$ too difficult!!

Sometimes one order is easier than the other!

$$= -\frac{1}{2}\int_0^{-9} e^u\,du$$

$$= -\frac{1}{2}(e^{-9} - e^0)$$

$$= \frac{1}{2} - \frac{1}{2}e^{-9}$$

(15) $\iint_D y\,dA$    $D$ is region bounded by $y = x-2$, $x = y^2$

find points of intersection.

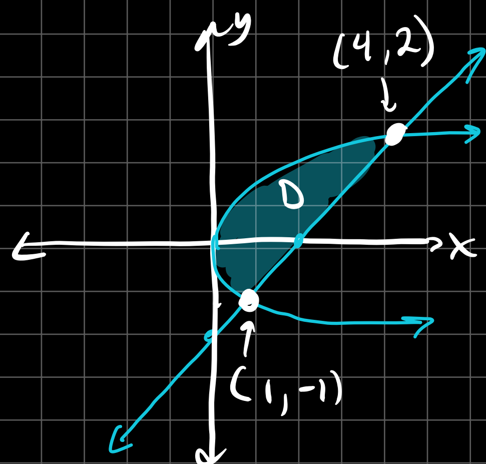
$x = (x-2)^2$

$x = x^2 - 4x + 4$

$x^2 - 5x + 4 = 0$

$(x-1)(x-4) = 0$

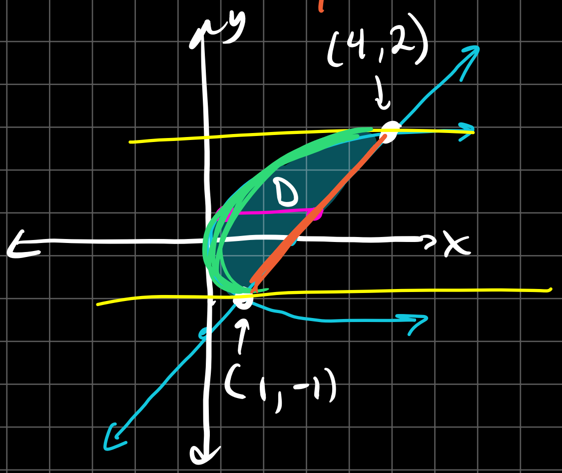
$x = 1$, $x = 4$



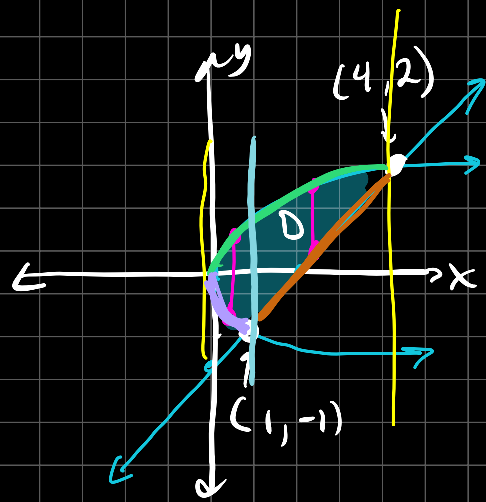

Describe domain using inequalities

$y = x-2$, $x = y^2$

$y+2 = x$




vs




$y^2 = x \Rightarrow y = \pm\sqrt{x}$

$-1 \le y \le 2$

$y^2 \le x \le y+2$

$0 \le x \le 4$

$0 \le x \le 1$ | $1 \le x \le 4$

$-\sqrt{x} \le y \le \sqrt{x}$ | $x-2 \le y \le \sqrt{x}$

$$\iint_D y\,dA = \int_{-1}^{2}\int_{y^2}^{y+2} y \; dx\, dy$$

$$= \int_{-1}^{2} (xy)\Big|_{x=y^2}^{x=y+2} dy$$

$$= \int_{-1}^{2} \Big[\underbrace{(y+2)y - y^2(y)}_{y^2+2y-y^3}\Big] dy$$

$$= \int_{-1}^{2} y^2+2y-y^3\,dy = \left(\frac{y^3}{3} + y^2 - \frac{y^4}{4}\right)\Bigg|_{y=-1}^{y=4}$$

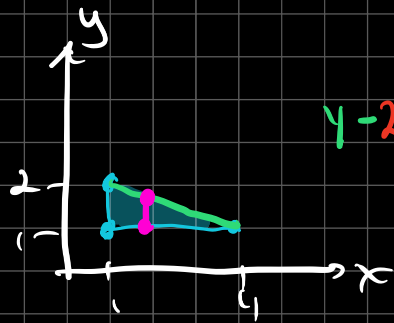
**OR**

$$\iint_D y\,dA = \int_0^1\int_{-\sqrt{x}}^{\sqrt{x}} y\,dy\,dx + \int_1^4\int_{x-2}^{\sqrt{x}} y\,dy\,dx$$

What does $\iint_D f(x,y)\,dA$ mean? $\int_I f(x)\,dx \Rightarrow$ area.

Volume under graph of $f(x,y)$ above domain $D$.

(25) Find volume under $z = xy$ above triangle with vertices $(1,1)$, $(4,1)$, $(1,2)$




$y - 2 = -\frac{1}{3}(x-1) \Rightarrow y = -\frac{1}{3}x + \frac{1}{3} + 2 = -\frac{1}{3}x + \frac{7}{3}$

$1 \le x \le 4$

$1 \le y \le -\frac{1}{3}x + \frac{7}{3}$

$$\text{Volume} = \int_1^4 \int_1^{-\frac{1}{3}x+\frac{7}{3}} xy \, dy\, dx$$

$$= \int_1^4 \frac{xy^2}{2} \Big|_{y=1}^{y=-\frac{1}{3}x+\frac{7}{3}} dx$$

$$= \int_1^4 \left[\frac{x}{2}\left(-\frac{1}{3}x + \frac{7}{3}\right)^2 - \frac{x}{2}\left(\frac{1}{2}\right)\right] dx$$

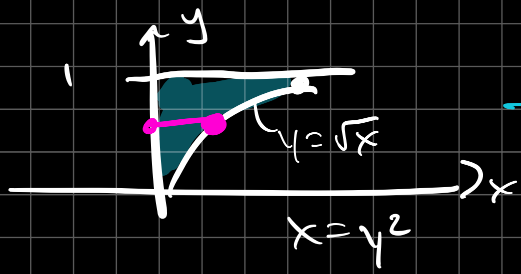
Integrate a polynomial.

(53) $\int_0^1 \underbrace{\int_{\sqrt{x}}^1 \sqrt{y^3+1}\, dy}_{\text{can't do!}}\, dx$

Sketch domain

$\sqrt{x} \le y \le 1$

$0 \le x \le 1$




$0 \le y \le 1$

$0 \le x \le y^2$

$$\int_0^1 \int_{\sqrt{x}}^1 \sqrt{y^3+1}\, dy\, dx$$

$$= \int_0^1 \int_0^{y^2} \sqrt{y^3+1}\, dx\, dy$$

easy!!!

53–56 switching