14.8 Lagrange Multipliers

$f(x,y)$ and $g(x,y) = c$

$\vec{\nabla} f = \lambda \vec{\nabla} g \rightarrow \begin{matrix} f_x = \lambda g_x \\ f_y = \lambda g_y \end{matrix}$

λ constant

③ $f(x,y) = x^2 - y^2 \qquad x^2 + y^2 = 1$

find max/min.

$$\begin{matrix} \cancel{2}x = \cancel{2}x\lambda \\ -\cancel{2}y = \cancel{2}y\lambda \end{matrix} \rightarrow \begin{matrix} x = x\lambda \\ -y = y\lambda \end{matrix}$$

$$\rightarrow \begin{matrix} x - x\lambda = 0 \\ y + y\lambda = 0 \end{matrix} \rightarrow \begin{matrix} \boxed{x(1-\lambda) = 0} & (I) \\ y(1+\lambda) = 0 & (II) \end{matrix}$$

$x = 0$ or $\lambda = 1$

if $x = 0$ then $x^2 + y^2 = 1 \Rightarrow 0^2 + y^2 = 1 \Rightarrow y = \pm 1$

if $\lambda = 1$ then (II) tells us $y(1+1) = 0 \Rightarrow y = 0$

if $y = 0$ then $x^2 + y^2 = 1 \Rightarrow x = \pm 1$.

Points:	$(0,1)$	$(0,-1)$	$(1,0)$	$(-1,0)$
$f(x,y)=x^2-y^2$	-1	-1	1	1

Max $=1$ at $(\pm 1,0)$

Min $=-1$ at $(0,\pm 1)$

⑤ $f(x,y)=xy$ $\qquad 4x^2+y^2=8$

(i) $y=8x\lambda$

(ii) $x=2y\lambda \Rightarrow x=16x\lambda^2$

$x-16x\lambda^2=0$

$x(1-16\lambda^2)=0$

$x=0$ or $\lambda=\pm\frac{1}{4}$.

if $x=0$ then $4(0)^2+y^2=8 \Rightarrow y=\pm 2\sqrt{2}$

but according to (i) $y=8(0)\lambda=0$

contradiction!

So, $x\neq 0$.

$y=2x,\ \lambda=\frac{1}{4}$

then $\lambda = \pm\frac{1}{4}$ $\quad y = 8x\left(\pm\frac{1}{4}\right) = \begin{cases} \\ y=-2x \quad \lambda=-\frac{1}{4} \end{cases}$ from (1)

$\rightarrow \cancel{4x^2 + (\pm 2x)^2 = 8 \Rightarrow 8x^2 = 8 \Rightarrow x = \pm 1}$

$$\begin{cases} \text{if } \lambda = \frac{1}{4} & y = 2x \quad 4x^2 + (2x)^2 = 8 \quad x = \pm 1 \quad y = \pm 2 \\ \text{if } \lambda = -\frac{1}{4} & y = -2x \quad x = \pm 1 \quad y = \pm 2. \end{cases}$$

Points:	$(1,-2)$	$(1,2)$	$(-1,2)$	$(-1,-2)$
$f(x,y) = xy$	-2	2	-2	2

max $= 2$ @ $(1,2)$, $(-1,-2)$

min $= -2$ @ $(1,-2)$, $(-1,2)$.

(10) $f(x,y,z) = \ln(x^2+1) + \ln(y^2+1) + \ln(z^2+1)$

$x^2+y^2+z^2 = 12.$ find max/min.

$$\frac{\cancel{2}x}{x^2+1} = \lambda\cancel{2}x \qquad x\left(\frac{1}{x^2+1} - \lambda\right) = 0$$

$$\frac{\cancel{2}y}{y^2+1} = \lambda\cancel{2}y \rightarrow y\left(\frac{1}{y^2+1} - \lambda\right) = 0$$

$$\frac{\cancel{2}z}{z^2+1} = \lambda\cancel{2}z \qquad z\left(\frac{1}{z^2+1} - \lambda\right) = 0$$

$x=0$ or $\lambda = \frac{1}{x^2+1}$

$y=0$ or $\lambda = \frac{1}{y^2+1}$

$z=0$ or $\lambda = \frac{1}{z^2+1}$

Can have: (two zero)

$x=y=0$ | $x=z=0$ | $y=z=0$

$z^2=12$ | $y=\pm\sqrt{12}$ | $x=\pm\sqrt{12}$

$z=\pm\sqrt{12}$

$(0,0,\pm\sqrt{12})$ | $(0,\pm\sqrt{12},0)$ | $(\pm\sqrt{12},0,0)$

One zero:

$x=0$, $y,z\neq 0$ | $y=0$, $x,z\neq 0$ | $z=0$ $x,y\neq 0$

$\lambda = \frac{1}{y^2+1}$ | $x^2=z^2$ | $x^2=y^2$

$\lambda = \frac{1}{z^2+1}$ | $x=\pm\sqrt{6}$ | $x=\pm\sqrt{6}$

$y^2=z^2$ | $z=\pm\sqrt{6}$ | $y=\pm\sqrt{6}$

use constraint

$x^2+y^2+z^2=12$ | $(\sqrt{6},0,\pm\sqrt{6})$ | $(\sqrt{6},\pm\sqrt{6},0)$

$2y^2=12$ | $(-\sqrt{6},0,\pm\sqrt{6})$ | $(-\sqrt{6},\pm\sqrt{6},0)$

$y^2=6$

$y=\pm\sqrt{6}$

$z=\pm\sqrt{6}$

$(0, \sqrt{6}, \pm\sqrt{6})$

$(0, -\sqrt{6}, \pm\sqrt{6})$

none are zero: $\frac{1}{x^2+1} = \frac{1}{y^2+1} = \frac{1}{z^2+1}$

$\Rightarrow x^2 = y^2 = z^2$ use $x^2+y^2+z^2 = 12$

$3x^2 = 12$

$x^2 = 4 \to x = \pm 2$

$y = \pm 2$

$z = \pm 2$

$(2, 2, \pm 2)$

$(2, -2, \pm 2)$

$(-2, 2, \pm 2)$

$(-2, -2, \pm 2)$

$\Big\}\ (\pm 2, \pm 2, \pm 2)$

points:	$(\pm 2, \pm 2, \pm 2)$	$(\pm\sqrt{6}, \pm\sqrt{6}, 0)$ $(\pm\sqrt{6}, 0, \pm\sqrt{6})$ $(0, \pm\sqrt{6}, \pm\sqrt{6})$	$(\pm\sqrt{12}, 0, 0)$ $(0, \pm\sqrt{12}, 0)$ $(0, 0, \pm\sqrt{12})$
$f(x,y)$ $= \ln(x^2+1)$ $+ \ln(y^2+1)$ $+ \ln(z^2+1)$	$\ln(5)+\ln(5)+\ln(5)$ $= \ln(5^3)$ $= \ln(125)$	$\ln(7)+\ln(7)$ $= \ln(49)$	$\ln(13)$
	Max		min.

(14) $f(x_1, \ldots, x_n) = x_1 + x_2 + \cdots + x_n$ $\qquad \vec{\nabla} f = \lambda \vec{\nabla} g$

$x_1^2 + x_2^2 + \cdots + x_n^2 = 1$ $\qquad$ find max/min

$$\left.\begin{array}{l} 1 = 2x_1\lambda \\ 1 = 2x_2\lambda \\ \vdots \\ 1 = 2x_n\lambda \end{array}\right\} \quad x_1 = x_2 = \cdots = x_n$$

find x_i $\quad , i = 1, \ldots, n$

$$\underbrace{x_i^2 + x_i^2 + \cdots + x_i^2}_{n} = 1$$

$$n x_i^2 = 1 \Rightarrow x_i^2 = \frac{1}{n} \Rightarrow x_i = \pm \frac{1}{\sqrt{n}}$$

our points: $\left(\pm\frac{1}{\sqrt{n}}, \pm\frac{1}{\sqrt{n}}, \ldots, \pm\frac{1}{\sqrt{n}} \right)$

max occurs at $\left(\frac{1}{\sqrt{n}}, \frac{1}{\sqrt{n}}, \ldots, \frac{1}{\sqrt{n}} \right)$

$$f\left(\frac{1}{\sqrt{n}}, \ldots, \frac{1}{\sqrt{n}} \right) = n\left(\frac{1}{\sqrt{n}} \right) = \sqrt{n}$$

min occurs at $\left(-\frac{1}{\sqrt{n}}, \ldots, -\frac{1}{\sqrt{n}} \right)$

min value $= -\sqrt{n}$.

(17) $f(x,y,z) = x+y+z$ $\quad \underbrace{x^2+z^2}_{g}=2$ and $\underbrace{x+y}_{h}=1$

$$\vec{\nabla} f = \lambda \vec{\nabla} g + \mu \vec{\nabla} h$$

$1 = 2x\lambda + (1)\mu \rightarrow \boxed{2x\lambda = 0}$

$1 = (0)\lambda + (1)\mu \rightarrow \boxed{\mu = 1}$

$\underbrace{1 = 2z\lambda + (0)\mu}$

$\boxed{2z\lambda = 1}$ $\qquad \lambda \neq 0$

$x = 0$

if $x=0$ $\quad x+y=1 \Rightarrow y=1$

$x=0$ $\quad x^2+z^2=2 \Rightarrow z = \pm\sqrt{2}$

points: $(0, 1, \pm\sqrt{2})$

evaluate $f(x,y,z)$

$f(0,1,\sqrt{2}) = 1+\sqrt{2}$ max

$f(0,1,-\sqrt{2}) = 1-\sqrt{2}$ min.

(21) $f(x,y) = x^2+y^2+4x-4y$ $\quad x^2+y^2 \leq 9$

find max/min

$x^2+y^2=9$

use L.m.

$2x+4=\lambda 2x$

$2y-4=\lambda 2y$

$\Rightarrow$

$x+2=\lambda x$

$+\ y-2=\lambda y$

$x+y=\lambda(x+y)$

$(x+y)(1-\lambda)=0$

$y=-x$ or $\lambda=1$

$x^2+(-x)^2=9$

$x^2=\frac{9}{2} \Rightarrow x=\pm\frac{3}{\sqrt{2}}$

$\left(\frac{3}{\sqrt{2}},-\frac{3}{\sqrt{2}}\right),\left(-\frac{3}{\sqrt{2}},\frac{3}{\sqrt{2}}\right)$

$x^2+y^2<9$

set: $\vec{\nabla} f=\vec{0}$

$2x+4=0 \qquad 2y-4=0$

$x=-2 \qquad y=2$

$(-2,2)$

$f(x,y)=x^2+y^2+4x-4y$

$f\left(\frac{3}{\sqrt{2}},-\frac{3}{\sqrt{2}}\right)=\frac{9}{2}+\frac{9}{2}+\frac{12}{\sqrt{2}}+\frac{12}{\sqrt{2}}=9+\frac{24}{\sqrt{2}}=9+12\sqrt{2}$ Max.

$f\left(-\frac{3}{\sqrt{2}},\frac{3}{\sqrt{2}}\right)=\frac{9}{2}+\frac{9}{2}-\frac{12}{\sqrt{2}}-\frac{12}{\sqrt{2}}=9-12\sqrt{2}\approx-7.97$

$f(-2,2)=4+4-8-8=-8$ min.

(24) $f(x,y) = 2x+3y$ $\quad \sqrt{x} + \sqrt{y} = 5$

find <u>max</u>

$$2 = \lambda\left(\frac{1}{2\sqrt{x}}\right) \rightarrow 4\sqrt{x} = \lambda$$

$$\Rightarrow \sqrt{x} = \frac{3}{2}\sqrt{y}$$

$$3 = \lambda\left(\frac{1}{2\sqrt{y}}\right) \rightarrow 6\sqrt{y} = \lambda$$

$$\frac{3}{2}\sqrt{y} + \sqrt{y} = 5 \Rightarrow \frac{5}{2}\sqrt{y} = 5$$

$$\sqrt{y} = 2$$

$$\sqrt{x} = 3$$

$$x = 9 \quad y = 4$$

$$f(9,4) = 18 + 12 = 30.$$

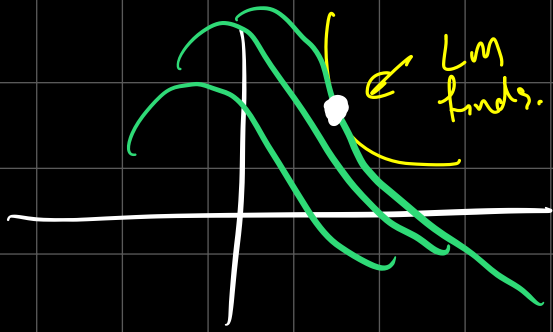
$$f(25,0) = 50$$

$$f(0,25) = 75$$

$\vec{\nabla} g$ fails to exist here.

$\Rightarrow$ L.M. Fails.

use contour curves to check.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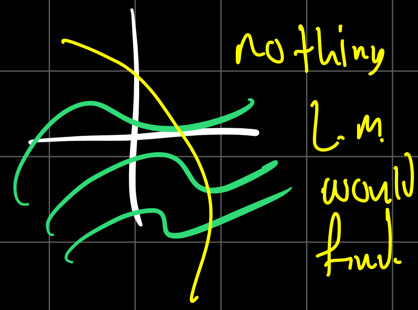
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