Black-Scholes Equations

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Itô Process: $dX_t = a\,dt + \sigma\,dB_t$

Special Case: $a(t,\omega) = a(t,X_t)$

$\sigma(t,\omega) = \sigma(t,X_t)$

则有 $dX_t = a(t,X_t)dt + \sigma(t,X_t)dB_t$ $\boxed{SDE}$ Stochastic "Diff eq"

B-S. $a(t,x) = rx$ $r \in \mathbb{R}$

$\sigma(t,x) = \sigma_0 x$ $\sigma \in \mathbb{R}$.

此时 SDE 有 $dX_t = rX_t\,dt + \sigma_0 X_t\,dB_t$

从而: $\dfrac{dX_t}{X_t} = r\,dt + \sigma_0\,dB_t$

$\dfrac{dX_t}{X_t} \neq d\log X_t$!

令 $f(t,x) = \ln x$

$\Rightarrow f_t = 0$

$f_x = \dfrac{1}{x}$

$f_{xx} = -\dfrac{1}{x^2}$

$\Rightarrow d\log X_t = \dfrac{dX_t}{X_t} - \dfrac{1}{2X_t^2}(dX_t)^2$

Itô

考虑 $(dX_t)^2 = \sigma_0^2 X_t^2\,dt$.

则 $d\log X_t = \dfrac{dX_t}{dt} - \frac{1}{2}\sigma_0^2\,dt$

则 $d\log X_t + \frac{1}{2}\sigma_0^2\,dt = \dfrac{dX_t}{dt} = r\,dt + \sigma_0\,dB_t$

PDE

$dX_t = rX_t\,dt$

有解 $X_T = X_0 e^{rT}$

积分 有 $\log\dfrac{X_T}{X_0} = \left(r - \dfrac{\sigma_0^2}{2}\right)T + \sigma_0 B_T$

从而 $X_T = X_0 e^{\left(r-\frac{\sigma_0^2}{2}\right)T + \sigma B_T}$ $\boxed{\text{log normal Dist}}$

Call Option: $(X_T - K)_+$

$$\text{Price} = e^{-rT}\mathbb{E}(X_T - K)_+$$
$$= X_0 N(d_1) - Ke^{-rT}N(d_2).$$

其中 $d_1 = \dfrac{\log\left(\frac{X_0}{K}\right) + \sigma^2 T/2}{\sigma\sqrt{T}}$

$d_2 = d_1 - \sigma\sqrt{T}$

N: cumulative normal function.