Stochastic Calculus
Problem Sets w/ Solutions

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Disclaimer:

These are the problem sets for the course *Stochastic Calculus (MATH.GA 2902)*, given by professor Alexey Kuptsov at New York University in Summer 2022. The solutions are mostly given by Rex Liu, with help from Xiang Fang (UCSB PSTAT), Theodore Plotkin (NYU Courant), and Efe Dikmen (NYU Stern).

If you see any mistakes or think that the presentation is unclear and could be improved, please send an email to: cl5682@nyu.edu. All comments and suggestions are appreciated.

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Chapter 1

Questions

1.1 Symmetric Random Walk

1. Let $M(n)$ be the symmetric random walk defined in class. Calculate
   (a) $\mathbb{E}\left[e^{M(n)}\right]$ using the fact that $X_1, X_2, \ldots, X_n$ are independent;
   (b) $\mathbb{P}\left[M(n) = k\right]$ for a given integer $k \in [-n, n]$.
2. Find the characteristic function of a random variable $X \sim N(\mu, \sigma^2)$.
3. Let $M(t)$ be a symmetric random walk. Check if $M(t)^2 - t$ is a martingale with respect to filtration $\mathcal{F}_t = (X_1, X_2, \ldots, X_t)$.

1.2 Multivariate Normal Distribution

1. Let $\mathbf{X} = (X_1, X_2, \ldots, X_n)$ be an n-dimensional random vector that has density function given by

$$f(x_1, \ldots, x_n) := \frac{1}{(2\pi)^{n/2}\sqrt{\det(\Sigma)}} \exp\left\{-\frac{1}{2}(\mathbf{x}-\mathbf{a})^\top \Sigma^{-1}(\mathbf{x}-\mathbf{a})\right\}$$

   where $\mathbf{a} = (a_1, \ldots, a_n)\,, \mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$, and $\Sigma \in \mathbb{R}^{n \times n}$ is a symmetric positive definite matrix.

   (a) Show that the density function f is well-defined.

   Hint: you might find useful a fact that positive definite matrix could be represented as a product of a lower triangular matrix and its transpose (Cholesky decomposition) + linear change of variables in the integral.

(b) Show that $\mathbf{a} = (a_1, \dots, a_n)$ is indeed a vector of expectations, i.e.

$$a_i = \mathbb{E}\left[X_i\right] \qquad \forall i = 1, 2, \dots, n$$

(c) Show that Σ is indeed a covariance matrix.

2. (*Wald's*) Let B_t be a standard Brownian motion starting from 0 and θ be an arbitrary constant. Prove that

$$X_t = e^{\theta B_t - \frac{1}{2}\theta^2 t}$$

is a martingale with respect to filtration generated by Brownian motion.

1.3 Running Maximum

1. The first variation of function $f(t)$ on interval $[0, T]$ is defined as

$$FV(f) = \lim_{\|\Pi\|\to 0} \sum_{i=0}^{n-1} |f(t_{i+1}) - f(t_i)|$$

Estimate the first variation of Brownian Motion.

2. Let B_t be a standard Brownian Motion starting from 0 and M_t be its running maximum. Calculate

(a) for a fixed level a, the expectation of the first passage time T_a;

(b) the quadratic variation of the running maximum M_t on interval $[0, T]$;

(c) the probability density function of M_t.

1.4 Stochastic Integral

1. Let B_t be a standard Brownian motion starting from 0, ε be a number in $[0, 1]$, and $\Pi = \{t_0, t_1, \dots, t_n\}$ be a partition of $[0, t]$ with $0 = t_0 < t_1 < \dots < t_n = t$. Consider the approximating sum

$$S_\varepsilon^\Pi = \sum_{i=0}^{n-1} \left((1-\varepsilon)B(t_i) + \varepsilon B(t_{i+1})\right)\left(B(t_{i+1}) - B(t_i)\right)$$

for the stochastic integral $\int_0^t B_s \mathrm{d}B_s$.

(a) Show that

$$\lim_{\|\Pi\|\to 0} S_\varepsilon^\Pi = \frac{1}{2}B_t^2 + \left(\varepsilon - \frac{1}{2}\right)t$$

where the limit is in L^2.

(b) Show that the right-hand side of the above identity is a martingale if and only if $\varepsilon = 0$.

2. Let $\sigma(t)$ be deterministic function of time, β be constant and define

$$X(T) = \int_0^T \sigma(t) e^{-\beta t} \mathrm{d}B_t$$

Find the expectation and variance of $X(T)$. What is the distribution of $X(T)$?

3. Let B_t be a standard Brownian Motion starting from 0 and M_t be its running maximum. Calculate $\mathbb{P}[M_t > 2B_t]$.

1.5 Itô's Formula

1. Use Itô's formula to show that the following stochastic processes are martingales.
   (a) $X_t = e^{t/2} \cos B_t$;
   (b) $X_t = e^{t/2} \sin B_t$;
   (c) $X_t = (B_t + t)e^{-B_t - t/2}$.

2. Assume that $f(x)$ is twice continuously differentiable. Find all functions f such that $f(B_t)$ is a martingale.

   Hint: apply Itô's lemma to $f(B_t)$.

3. Let B_t be a standard Brownian motion and let $P(x)$ be a polynomial of degree $n, n \geq 1$.
   (a) Find values of the constants a, b, and a differential equation satisfied by $P(x)$, so that the process

   $$X_t = t^a P\left(\frac{B_t}{t^b}\right)$$

   is a martingale.

   Hint: apply Itô's lemma to $t^a P\left(x/t^b\right)$.

   (b) List the polynomials that have this property for $n = 2, 3, 4$.

1.6 Stochastic Differential Equations

1. Solve the following SDE:

$$\mathrm{d}X_t = (\beta - \log X_t)X_t \mathrm{d}t + \sigma X_t \mathrm{d}B_t, \quad X(0) = X_0$$

where β and σ are given constants.

2. (*Bessel Process*) Let $B_t = \left(B_t^1, B_t^2, \dots, B_t^d\right)$ be a d-dimensional Brownian motion and

$$R_t = \|B_t\| = \sqrt{\left(B_t^1\right)^2 + \ldots + \left(B_t^d\right)^2}$$

   (a) Find the distribution of the process

$$X_t = \sum_{i=1}^{d} \int_0^t \frac{B_s^i}{R_s} \mathrm{d}B_s^i$$

   Hint: use Lévy's Characterization of Brownian Motion.

   (b) Find a stochastic differential equation that R_t satisfies.

3. Using mutlivariate Ito's formula prove that for Ito processes X_t, Y_t holds the product rule formula

$$\mathrm{d}(X_t Y_t) = X_t \mathrm{d}Y_t + Y_t \mathrm{d}X_t + \mathrm{d}X_t \mathrm{d}Y_t$$

1.7 Midterm Practices

1. Consider the following model for stock price

$$S_t = S_0 e^{\sigma B_t}$$

   where $\sigma > 0$ is a constant and B_t is a standard Brownian motion starting from 0.

   (a) Find expectation and variance of S_t.

   (b) Compute the price of ATM vanilla call option in this model, i.e. $\mathbb{E}(S_T - S_0)_+$ for given maturity T.

   (c) Write the formula for the price of knock-out option with barrier $K > S_0$, i.e. the value of option that pays $\$1$ if stock price doesn't reach level K till maturity T and 0 otherwise.

   (d) Use Itô's formula to check if S_t is a martingale.

   (e) Compute quadratic variation of S_t on interval $[0, T]$.

2. Consider the following model for the stock price

$$X_t = e^{\int_0^t s \mathrm{d}B_s}, t \geq 0$$

   (a) For fixed time t, what is the distribution of the random variable $\int_0^t s \mathrm{d}B_s$? What is then the distribution of X_t?

   (b) From (a), calculate the price of the call option with maturity T and strike K, i.e.

$$\mathbb{E}(X_T - K)_+$$

3. Let B_t be a standard Brownian Motion starting from 0 and M_t be its running maximum, i.e. $M_t = \max_{0\le s\le t} B_s$. For fixed time t, compute the density of the random variable $M_t - B_t$.

4. Using Itô's lemma check if $X_t = e^{B_t/2-t}$ is a martingale. What is the quadratic variation of process X_t on interval $[0,T]$?

5. Let $B_t, t \geq 0$ be a standard Brownian motion starting from 0. Fix $t < T$ and a constant α. Compute $\mathbb{P}(B_t > B_T + \alpha)$.

1.8 Midterm

1. For given σ, b and S_0, consider the stochastic process S_t

$$S_t = S_0 + \frac{\sigma}{2b}\left(e^{2bB_t - 2b^2 t} - 1\right)$$

where B_t is a standard Brownian motion.

(a) Find $\mathbb{E}[S_t]$ and $\operatorname{Var}(S_t)$ for fixed t.

(b) Is S_t a martingale?

(c) Compute the quadratic variation of S_t on $[0,T]$.

2. Let B_t^1 and B_t^2 be two independent standard Brownian motions. Show that $\left\langle B_t^1, B_t^2\right\rangle_{[0,T]} = 0$ for any $T \geq 0$.

3. Let σ be a fixed constant and $X_t = e^{\sigma B_t}$. What is the probability that the maximum of X_t on the interval $[0,T]$ exceeds a, given $a > 0$?

1.9 Girsanov Theorem

1. Let B_t be a Brownian motion defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $\gamma(t,w)$ be an adapted to Brownian filtration $(\mathcal{F}_t)_{t\geq 0}$ process. Let

$$Z_t = \exp\left\{\int_0^t \gamma(t,w)\mathrm{d}B_s - \frac{1}{2}\int_0^t \gamma^2(s,w)\mathrm{d}s\right\}$$

and for any fixed $T < \infty$ define a new measure $\widetilde{\mathbb{P}}$ on F_T as

$$\widetilde{\mathbb{P}}(\mathrm{d}w) = Z_T(w)\mathbb{P}(\mathrm{d}w)$$

We show in the following that $\widetilde{B}_t = B_t - \int_0^t \gamma(s,w)\mathrm{d}s$ is a Brownian motion under $\widetilde{\mathbb{P}}$.

(a) Apply Itô's Lemma to obtain an SDE for Z_t.

(b) Apply Itô's Lemma to $Z_t\widetilde{B}_t$ and show that it is a martingale under $\mathbb{P}$.

(c) Assume you know that for any random variable Y and $s \leq t \leq T$, one has

$$\widetilde{\mathbb{E}}\left(Y|\mathcal{F}_s\right) = \frac{1}{Z_s(w)}\mathbb{E}\left(YZ_t(w)|\mathcal{F}_s\right)$$

Use this to show that $\widetilde{B}_t$ is a martingale under $\widetilde{\mathbb{P}}$.

(d) Similarly, apply Itô's Lemma to $Z_t\left(\widetilde{B}_t^2 - t\right)$ and show that it is a martingale under $\mathbb{P}$.

(e) Show that $\widetilde{B}_t^2 - t$ is a martingale under $\widetilde{\mathbb{P}}$.

(f) Apply Lévy's Theorem to conclude that $\widetilde{B}_t$ is a Brownian motion under $\widetilde{\mathbb{P}}$.

1.10 Connection with PDE

1. Consider the Ito process $Y_t, t \geq 0$ given by

$$\mathrm{d}Y_t = e^t\mathrm{d}B_t, \quad Y_0 = 0$$

and let $\tau = \min\{t \geq 0 : |Y_t| = 1\}$.

(a) Find a PDE for $f(t,x)$ satisfied whenever $f(t, Y_t)$ is a martingale.

(b) Verify that

$$f(t,x) = x^2 - \frac{e^{2t}}{2}$$

satisfies the PDE and that $f(t, Y_t)$ is a martingale.

(c) Show that $\mathbb{E}\left[e^{2\tau}\right] = 3$.

(d) Show that $\mathbb{E}[\tau] \leq \frac{\log 3}{2}$.

2. Let B_t be a standard Brownian motion and T a fixed maturity. Compute

$$\mathbb{E}_{B_t=x}\left[\int_t^T B_s^2\mathrm{d}s\right]$$

Check your answer by calculating the above expectation using the properties of the Brownian motion.

1.11 Boundary Value Problems

1. Let B_t be a standard Brownian motion. Let γ be a constant and $T > 0$ be a fixed maturity. Use the Feynman-Kac theorem to compute

$$\mathbb{E}_{B_0=x}\left[\exp\left\{-\gamma\int_0^T B_s\mathrm{d}s\right\}\right]$$

2. Let X_t satisfy the following SDE,

$$\mathrm{d}X_t = X_t\mathrm{d}t + \sigma\mathrm{d}B_t,$$

where $X_0 = x \in [a,b]$ and σ is some fixed constant. Let $\tau = \inf\{t \geq 0 : X_t \notin [a,b]\}$.

   (a) Find $\mathbb{E}_{X_0=x}X_\tau$.

   (b) Is $\mathbb{E}_{X_0=x}\tau < \infty$?

1.12 Forward Kolmogorov Equation

1. (*Smoluchowski's equation*) We consider the diffusion $(X_t, t \geq 0)$ given by the SDE

$$\mathrm{d}X_t = \mathrm{d}B_t - V'(X_t)\mathrm{d}t$$

where $V : \mathbb{R} \to \mathbb{R}$ is some smooth function such that

$$\int_{\mathbb{R}} e^{-2V(x)}\mathrm{d}x < \infty$$

   (a) Verify that the invariant distribution $f(x) = Ce^{-2V(x)}$ is a solution to $\mathcal{L}^* f = 0$.

   (b) Consider the specific example of $V(x) = |x|$. What is the SDE in this case? What is the exact invariant distribution?

2. (*Invariant probability of the Ornstein-Uhlenbeck process*) Consider the Ornstein-Uhlenbeck process with SDE

$$\mathrm{d}X_t = -X_t\mathrm{d}t + \mathrm{d}B_t, \quad X_0 = x$$

   (a) Find the transition probability density $p(t, z; 0, x)$.

   (b) Find $\lim_{t\to\infty} p(t, z; 0, x)$.

   (c) Let $f(x) = e^{-x^2}$. Prove that $\mathcal{L}^* f = 0$.

   (d) When considering the more general SDE

$$\mathrm{d}X_t = -\alpha X_t\mathrm{d}t + \sigma\mathrm{d}B_t, \quad X_0 = x,$$

   how are the above results affected?

1.13 Processes with Jumps

1. Let $N_t, t \geq 0$ be a Poisson process with intensity $\lambda \geq 0$ defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with respect to a filtration $\mathcal{F}_t$. Let $M_t = N_t - \lambda t$ be a compensated Poisson process. Prove that $M_t^2 - \lambda t$ is a martingale.

2. Let N_t and B_t be Poisson process and Brownian motion, respectively, relative to the same filtration $\mathcal{F}_t$. Consider the process

$$X_t = e^{\alpha N_t + \beta B_t}$$

where $\alpha, \beta \in \mathbb{R}$. Use Itô's formula to find an SDE for X_t. Show that $m_t = \mathbb{E}[X_t]$ solves

$$\frac{\mathrm{d}m_t}{\mathrm{d}t} - \left[(e^\alpha - 1)\lambda + \frac{\beta^2}{2} \right] m_t = 0$$

Solve the above ODE for m_t and deduce that N_t is independent of B_t.

1.14 Final Practices

1. Let B_t be a standard Brownian motion. Compute

$$\mathbb{E}\left(\int_0^T \sqrt{s}\mathrm{d}B_s \right)^3$$

2. Using Itô's lemma check if $X_t = e^{B_t^2 - t}$ is a martingale. What is the quadratic variation of process X_t on interval $[0, T]$?

3. Let B_t be a standard Brownian motion starting from 0.

   (a) Using PDE approach, find $u(t,x) = \mathbb{E}_{B_t = x} e^{-\gamma B_T^2}$ for given $T > t$ and $\gamma > 0$.

   (b) By the distribution of B_T, calculate directly (without solving PDE) $\mathbb{E}_{B_t = x} e^{-\gamma B_T^2}$.

4. Consider the SDE:

$$\mathrm{d}X_t = -\frac{X_t}{2-t}\mathrm{d}t + \sqrt{t(2-t)}\mathrm{d}B_t, \quad X(0) = 0, 0 \le t < 1$$

   Suppose the solution is of the form

$$X_t = a(t)Y_t, \quad Y_t = \int_0^t b(s)\mathrm{d}B_s$$

   for some smooth functions a, b. Apply Itô's formula to find an ODE satisfied by $a(t)$ and $b(t)$ and solve it.

5. Let X_t be a one-dimensional stochastic process with drift

$$\mathrm{d}X_t = f(X_t)\mathrm{d}t + \mathrm{d}B_t$$

   for some smooth function f with $X_0 = x \in [a, b]$. Let $\tau = \inf\{t \ge 0 : X_t \notin [a, b]\}$. Find $\mathbb{E}_{X_0 = x} X_\tau$.

6. Let X_t be stochastic process satisfying the SDE

$$\mathrm{d}X_t = \sigma(t)X_t\mathrm{d}B_t$$

with $X_0 = x$ and $\sigma(t)$ some deterministic function.

   (a) Use PDE approach to find $\mathbb{E}_{X_t=x}X_T^2$.

   (b) Solve the given SDE.

   (c) Calculate $\mathbb{E}_{X_t=x}X_T^2$ without sovling the PDE.

7. Let $X_t = (T-t)\int_0^t \frac{\mathrm{d}B_s}{T-s}$. Find the quadratic variation of X_t on $[0,T]$.

8. Let $\gamma(t)$ be a given deterministic function. Compute

$$\mathbb{E}_{B_0=0}\left[B_t e^{\int_0^t \gamma(s)\mathrm{d}B_s}\right]$$

1.15 Final

1. Let X_t be a stochastic process following the SDE

$$\mathrm{d}X_t = -X_t\mathrm{d}t + \sigma\mathrm{d}B_t,$$

where σ is a constant and X_0 is the start point at time $t = 0$. Let X_0 be within the interval $[a,b]$ and $\tau := \min\{t : X_t \notin [a,b]\}$ is the exit time. Find $\mathbb{P}_{X_0}(X_\tau = a)$.

2. Let T be fixed and X_t be a stochastic process defined in Q1. Also, let

$$u(t,x) = \mathbb{E}_{X_t=x}X_T^2$$

   (a) Using PDE approach, calculate $u(t,x)$.

   (b) Alternatively, solve the SDE given in Q1 and then calculate $\mathbb{E}_{X_t=x}X_T^2$.

3. Let $N_t^1, N_t^2, \ldots, N_t^k$ be independent Poisson processes with intensities $\lambda_1, \lambda_2, \ldots, \lambda_k$. Show that

$$N_t := \sum_{i=1}^k N_t^i$$

is also a Poisson process and find its intensity.

4. Calculate $\mathbb{E}_{B_0=x}\left[B_t^2 e^{\alpha B_t - \alpha^2/2}\right]$ where B_t is a brownian motion and α is a fixed constant.

5. Find the stable distribution of X_t as the stochastic process defined in Q1.

Chapter 2

Solutions

2.1 Symmetric Random Walk

1. Let $M(n)$ be the symmetric random walk defined in class. Calculate

   (a) $\mathbb{E}\left[e^{M(n)}\right]$ using the fact that $X_1, X_2, \ldots, X_n$ are independent;

   (b) $\mathbb{P}\left[M(n) = k\right]$ for a given integer $k \in [-n, n]$.

Solution. (a) Suppose $n \geq 1$. (Otherwise, $\mathbb{E}\left[e^{M(0)}\right] = \mathbb{E}[1] = 1$.) By definition, we have

$$\begin{aligned}
\mathbb{E}\left[e^{M(n)}\right] &= \mathbb{E}\left[\exp\left\{\sum_{j=1}^{n} X_j\right\}\right] \\
&= \mathbb{E}\left[\prod_{j=1}^{n} e^{X_j}\right] \\
&= \prod_{j=1}^{n} \mathbb{E}\left[e^{X_j}\right] && \text{since } f(x) = e^x \text{ is measurable and } X_j \text{ are indep} \\
&= \prod_{j=1}^{n} \left(\frac{e^1 + e^{-1}}{2}\right) \\
&= \boxed{\left(\frac{e + e^{-1}}{2}\right)^n}
\end{aligned}$$

(b) Note that when $M(n) = k$, we have exactly $(n+k)/2$ heads and $(n-k)/2$ tails. Following the binomial distribution, one has

$$\boxed{\mathbb{P}\left[M(n)=k\right]=\binom{n}{(n+k)/2}\frac{1}{2^n}, \qquad k=-n,-n+2,\ldots,n-2,n}$$

□

2. Find the characteristic function of a random variable $X \sim N(\mu,\sigma^2)$.

Solution. The characteristic function of $X \sim N(\mu,\sigma^2)$ is defined as

$$\varphi(k)=\mathbb{E}\left[e^{ikX}\right]=\frac{1}{\sqrt{2\pi\sigma^2}}\int_{-\infty}^{\infty}e^{-\frac{(x-\mu)^2}{2\sigma^2}}e^{ikx}\mathrm{d}x$$

We integrate it by completing the square:

$$\begin{aligned}-\frac{(x-\mu)^2}{2\sigma^2}+ikx&=-\frac{1}{2\sigma^2}\left((x-\mu)^2-2ik\sigma^2x\right)\\&=-\frac{1}{2\sigma^2}\left(\left(x-\left(\mu+ik\sigma^2\right)\right)^2-\left(\mu+ik\sigma^2\right)^2+\mu^2\right)\\&=-\frac{1}{2\sigma^2}\left(x-\left(\mu+ik\sigma^2\right)\right)^2-\frac{k^2\sigma^2}{2}+ik\mu\end{aligned}$$

Hence, we have

$$\begin{aligned}\varphi(k)&=\frac{1}{\sqrt{2\pi\sigma^2}}\int_{-\infty}^{\infty}\exp\left\{-\frac{1}{2\sigma^2}\left(x-\left(\mu+ik\sigma^2\right)\right)^2-\frac{k^2\sigma^2}{2}+ik\mu\right\}\mathrm{d}x\\&=\exp\left\{-\frac{k^2\sigma^2}{2}+ik\mu\right\}\frac{1}{\sqrt{2\pi\sigma^2}}\int_{-\infty}^{\infty}\exp\left\{-\frac{1}{2\sigma^2}\left(x-\left(\mu+ik\sigma^2\right)\right)^2\right\}\mathrm{d}x\\&=\boxed{\exp\left\{-\frac{k^2\sigma^2}{2}+ik\mu\right\}}\qquad\text{(shifted Gaussian integral}=1)\end{aligned}$$

i.e. $\varphi_X(k)=e^{ik\mu-k^2\sigma^2/2}$ shown in lecture notes. □

3. Let $M(t)$ be a symmetric random walk. Check if $M(t)^2-t$ is a martingale with respect to filtration $\mathcal{F}_t=(X_1,X_2,\ldots,X_t)$.

Solution. We check by definition.

Note that $M(t)$ is proved to be $\mathcal{F}_t$-measurable, and the composition of measurable functions remains measurable[1], so that $M(t)^2-t$ is $\mathcal{F}_t$-measurable.

[1] Let (X,Σ) be a measurable space. If $f:X\to\mathbb{R}$ is measurable and $g:\mathbb{R}\to\mathbb{R}$ is Borel measurable, then $g\circ f:X\to\mathbb{R}$ is measurable. Also, every continuous function is Borel measurable.

Also note that $|M(t)| \leq t \implies M(t)^2 - t \in [-t, t^2 - t]$, so $\mathbb{E}\left[|M(t)^2 - t|\right] < \infty$ obviously.

Finally, by *independence of increments*, one has

$$\begin{aligned}\mathbb{E}\left[M(t+1)^2 - (t+1)|\mathcal{F}_t\right] &= \mathbb{E}\left[(M(t) + X_{t+1})^2 - (t+1)\Big|\mathcal{F}_t\right] \\ &= \mathbb{E}\left[M(t)^2 + 2M(t)X_{t+1} + X_{t+1}^2 - (t+1)|\mathcal{F}_t\right] \\ &= M(t)^2 - (t+1) + 2M(t)\mathbb{E}\left[X_{t+1}|\mathcal{F}_t\right] + \mathbb{E}\left[X_{t+1}^2|\mathcal{F}_t\right] \\ &= M(t)^2 - (t+1) + 1 = M(t)^2 - t\end{aligned}$$

Hence, $M(t)^2 - t$ *is* a martingale with respect to filtration $\mathcal{F}_t = (X_1, X_2, \ldots, X_t)$. □

2.2 Multivariate Normal Distribution

1. Let $\mathbf{X} = (X_1, X_2, \ldots, X_n)$ be an n-dimensional random vector that has density function given by

$$f(x_1, \ldots, x_n) := \frac{1}{(2\pi)^{n/2}\sqrt{\det(\Sigma)}} \exp\left\{-\frac{1}{2}(\mathbf{x} - \mathbf{a})^\top \Sigma^{-1}(\mathbf{x} - \mathbf{a})\right\}$$

where $\mathbf{a} = (a_1, \ldots, a_n), \mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$, and $\Sigma \in \mathbb{R}^{n\times n}$ is a symmetric positive definite matrix.

(a) Show that the density function f is well-defined.

Hint: you might find useful a fact that positive definite matrix could be represented as a product of a lower triangular matrix and its transpose (Cholesky decomposition) + linear change of variables in the integral.

Proof. By the hint, we first claim two lemmas.

Lemma 1 (Cholesky Factorization Theorem[2]). *Given a SPD matrix A, there exists a lower triangular matrix L such that $A = LL^\top$.*

Lemma 2 (Multivariate Change of Variables Theorem[3]). *Let V and W be bounded open sets in $\mathbb{R}^n$. Let $h : V \to W$ be a bijective map, given by*

$$\mathbf{h}(\mathbf{u}) := (h_1(u_1, \ldots, u_n), \ldots, h_n(u_1, \ldots, u_n))$$

Let $f : W \to \mathbb{R}$ be a continuous, bounded function. Then

$$\int_W f(\mathbf{x})\mathrm{d}\mathbf{x} = \int_V f(\mathbf{h}(\mathbf{u}))J\mathrm{d}\mathbf{u}$$

where $J = \det(\partial\mathbf{h}/\partial\mathbf{u})$ is the Jacobian.

[2] See the proof here.

[3] See the proof here.

Now, suppose that $\mathbf{X}$ is a random variable taking values in $S \subset \mathbb{R}^n$ and that $\mathbf{X}$ has a continuous distribution on S with probability density function f. Let $\mathbf{Y} = \mathbf{a} + B\mathbf{X}$ where $\mathbf{a} \in \mathbb{R}^n$ and B is an invertible $n \times n$ matrix. Note that $\mathbf{Y}$ takes values in $T = \{\mathbf{a} + B\mathbf{x} : \mathbf{x} \in S\} \subset \mathbb{R}^n$.

Also note that the transformation $\mathbf{y} = \mathbf{a} + B\mathbf{x}$ maps $\mathbb{R}^n$ bijectively to $\mathbb{R}^n$. The inverse transformation is $\mathbf{x} = B^{-1}(\mathbf{y} - \mathbf{a})$. The Jacobian of the inverse transformation is the constant function $\det(B^{-1}) = 1/\det(B)$. Hence, by Lemma 2, we have the following proposition:

Proposition 3 (Linear Transformation of Random Variables)**.** $\mathbf{Y}$ *has probability density function* g *given by*

$$g(\mathbf{y}) = \frac{1}{|\det(B)|} f\left(B^{-1}(\mathbf{y} - \mathbf{a})\right), \quad \mathbf{y} \in T$$

Now we approach the problem. By Lemma 1, let $\Sigma^{-1} = H^\top H$ be its Cholesky decomposition. Consider $\mathbf{y} = H(\mathbf{x} - \mathbf{a})$. One has

$$\mathbf{y}^\top \mathbf{y} = (\mathbf{x} - \mathbf{a})^\top H^\top H(\mathbf{x} - \mathbf{a}) = (\mathbf{x} - \mathbf{a})^\top \Sigma^{-1}(\mathbf{x} - \mathbf{a})$$

and $\det(H) = 1/\sqrt{\det(\Sigma)}$. Then, by Proposition 3, one has

$$\begin{aligned} f_{\mathbf{Y}}(\mathbf{y}) &= \frac{1}{\det(H)} f_{\mathbf{X}}(\mathbf{x}) \\ &= \frac{\sqrt{\det(\Sigma)}}{(2\pi)^{n/2}\sqrt{\det(\Sigma)}} \exp\left\{-\frac{1}{2}(\mathbf{x} - \mathbf{a})^\top \Sigma^{-1}(\mathbf{x} - \mathbf{a})\right\} \\ &= \frac{1}{(2\pi)^{n/2}} \exp\left\{-\frac{1}{2}\mathbf{y}^\top \mathbf{y}\right\} \end{aligned}$$

Hence, it is enough to show that the density $f_{\mathbf{Y}}(\mathbf{y})$ for $\mathbf{Y} = H(\mathbf{X} - \mathbf{a})$ is well-defined. In fact, one has

$$\begin{aligned} &\int_{\mathbb{R}^n} f_{\mathbf{Y}}(\mathbf{y}) \mathrm{d}\mathbf{y} \\ &= \frac{1}{(2\pi)^{n/2}} \int\int \cdots \int \exp\left\{-\frac{1}{2}\left(y_1^2 + y_2^2 + \ldots + y_n^2\right)\right\} \mathrm{d}y_1 \mathrm{d}y_2 \ldots \mathrm{d}y_n \\ &= \frac{\left(\sqrt{2\pi}\right)^n}{(2\pi)^{n/2}} = 1 \end{aligned}$$

Therefore, $f_{\mathbf{Y}}(\mathbf{y})$, and thus $f_{\mathbf{X}}(\mathbf{x})$, are indeed densities for $\mathbf{Y}$ and $\mathbf{X}$, respectively. $\square$

(b) Show that $\mathbf{a} = (a_1, \ldots, a_n)$ is indeed a vector of expectations, i.e.

$$a_i = \mathbb{E}[X_i] \qquad \forall i = 1, 2, \ldots, n$$

Proof. Note that

$$\begin{aligned} f_{\mathbf{Y}}(\mathbf{y}) &= \frac{1}{(2\pi)^{n/2}} \exp\left\{-\frac{1}{2}\mathbf{y}^\top \mathbf{y}\right\} \\ &= \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}} e^{-y_i^2/2} \end{aligned}$$

implying that $Y_1, Y_2, \ldots, Y_n$ are independent standard normal random variables, i.e. $Y_i \sim N(0,1), \forall i$. From the fact that $\mathbb{E}[A\mathbf{X} + \mathbf{b}] = A\mathbb{E}[\mathbf{X}] + \mathbf{b}$, one has $\mathbf{0} = \mathbb{E}[\mathbf{Y}] = \mathbb{E}[H(\mathbf{X} - \mathbf{a})] = H\mathbb{E}[\mathbf{X}] - H\mathbf{a}$. Hence, $\mathbb{E}[\mathbf{X}] = \mathbf{a}$, i.e. $a_i = \mathbb{E}[X_i], \forall i$. □

(c) Show that Σ is indeed a covariance matrix.

Proof. By (a) and (b), as $Y_1, Y_2, \ldots, Y_n$ are i.i.d, one has $\mathrm{Cov}(\mathbf{Y}) = I$. From the fact that $\mathrm{Cov}(A\mathbf{X} + \mathbf{b}) = A\mathrm{Cov}(\mathbf{X})A^\top$, one has $I = \mathrm{Cov}(\mathbf{Y}) = \mathrm{Cov}(H(\mathbf{X} - \mathbf{a})) = H\mathrm{Cov}(\mathbf{X})H^\top$. Hence, $\mathrm{Cov}(\mathbf{X}) = H^{-1}\left(H^\top\right)^{-1} = \left(H^\top H\right)^{-1} = \Sigma$. □

2. (*Wald's*) Let B_t be a standard Brownian motion starting from 0 and θ be an arbitrary constant. Prove that

$$X_t = e^{\theta B_t - \frac{1}{2}\theta^2 t}$$

is a martingale with respect to filtration generated by Brownian motion.

Proof. We follow the definition. Denote the filtration generated by Brownian motion as $\mathcal{F}$.

Similar to Problem 1.3, one can see that X_t is $\mathcal{F}$-measurable for all t.

Given that $B_t \sim N(0,t)$ with $\mathbb{E}[B_t] = 0$, one has

$$\begin{aligned} \mathbb{E}[X_t] &= \mathbb{E}\left[e^{\theta B_t - \frac{1}{2}\theta^2 t}\right] \\ &= e^{-\frac{1}{2}\theta^2 t} \int_{\mathbb{R}} e^{\theta x} f_{B_t}(x) \mathrm{d}x \\ &= e^{-\frac{1}{2}\theta^2 t} \int_{\mathbb{R}} e^{\theta x} \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}} \mathrm{d}x \\ &= e^{-\frac{1}{2}\theta^2 t} e^{\frac{1}{2}\theta^2 t} = 1 < \infty \end{aligned}$$

Finally, by *independence of increments*, for all $s \leq t$, one has

$$
\begin{aligned}
\mathbb{E}\left[X_t|\mathcal{F}_s\right] &= \mathbb{E}\left[e^{\theta B_t - \frac{1}{2}\theta^2 t}\middle|\mathcal{F}_s\right] \\
&= \mathbb{E}\left[e^{\theta(B_t - B_s) + \theta B_s - \frac{1}{2}\theta^2 t}\middle|\mathcal{F}_s\right] \\
&= e^{\theta B_s - \frac{1}{2}\theta^2 t}\mathbb{E}\left[e^{\theta(B_t - B_s)}\middle|\mathcal{F}_s\right]
\end{aligned}
$$

Note that $B_t - B_s \sim N(0, t-s)$, which is independent of $\mathcal{F}_s$, and similarly from above,

$$
\begin{aligned}
&= e^{\theta B_s - \frac{1}{2}\theta^2 t} e^{\theta^2(t-s)/2} \\
&= e^{\theta B_s - \theta^2 s/2} = X_s
\end{aligned}
$$

Hence, X_t is a martingale. □

2.3 Running Maximum

1. The first variation of function $f(t)$ on interval $[0, T]$ is defined as

$$
FV(f) = \lim_{\|\Pi\| \to 0} \sum_{i=0}^{n-1} |f(t_{i+1}) - f(t_i)|
$$

Estimate the first variation of Brownian Motion.

Solution. Note that[4] the quadratic variation of Brownian motion is

$$
QV(B) = \lim_{\|\Pi\| \to 0} \sum_{i=0}^{n-1} (B(t_{i+1}) - B(t_i))^2 = T
$$

but

$$
\sum_{i=0}^{n-1} (B(t_{i+1}) - B(t_i))^2 \leq \left(\sum_{i=0}^{n-1} |B(t_{i+1}) - B(t_i)|\right) \max_{0 \leq i \leq n-1} |B(t_{i+1}) - B(t_i)|
$$

Take limit on both sides, when $\|\Pi\| \to 0$, $n \to \infty$, and since B is continuous, one has $\max_{0 \leq i \leq n-1} |B(t_{i+1}) - B(t_i)| \to 0$. However, the left side converges to $T > 0$, so it is only possible that

$$
\boxed{FV(B) = \lim_{\|\Pi\| \to 0} \sum_{i=0}^{n-1} |B(t_{i+1}) - B(t_i)| = \infty}
$$

□

[4] Lecture Notes 2, Remark 5

2. Let B_t be a standard Brownian Motion starting from 0 and M_t be its running maximum. Calculate

   (a) for a fixed level a, the expectation of the first passage time T_a;

   (b) the quadratic variation of the running maximum M_t on interval $[0, T]$;

   (c) the probability density function of M_t.

Solution. (a) Note that [5] the density function of T_a is

$$f_{T_a}(t) = \frac{a}{\sqrt{2\pi t^3}} \exp\left\{\frac{-a^2}{2t}\right\}, \quad t > 0,$$

so that

$$\begin{aligned} \mathbb{E}\left[T_a\right] &= \int_0^\infty t f_{T_a}(t)\mathrm{d}t \\ &= \int_0^\infty \frac{a}{\sqrt{2\pi t}} \exp\left\{\frac{-a^2}{2t}\right\} \mathrm{d}t \end{aligned}$$

Note that

$$\lim_{t\to 0} \exp\left\{\frac{-a^2}{2t}\right\} = 0, \quad \lim_{t\to\infty} \exp\left\{\frac{-a^2}{2t}\right\} = 1,$$

and the function $f(x) = e^x$ is continuous and monotonically increasing, so that there exists some $t = s > 0$ s.t. $e^{-a^2/2s} = 1/2$, then

$$\begin{aligned} \mathbb{E}\left[T_a\right] &= \int_0^\infty \frac{a}{\sqrt{2\pi t}} \exp\left\{\frac{-a^2}{2t}\right\} \mathrm{d}t \\ &\geq \int_s^\infty \frac{a}{\sqrt{2\pi t}} \exp\left\{\frac{-a^2}{2t}\right\} \mathrm{d}t && \text{non-negativity} \\ &> \int_s^\infty \frac{a}{2\sqrt{2\pi t}} \mathrm{d}t \\ &= \left[\frac{a\sqrt{t}}{\sqrt{2\pi}}\right]_s^\infty = \infty \end{aligned}$$

implying that the integral is divergent, i.e. $\boxed{\mathbb{E}\left[T_a\right] = \infty.}$

[5] Lecture Notes 3, Remark 2

(b) By definition, one has

$$\begin{aligned}\langle M_t, M_t\rangle &= \lim_{\|\Pi\|\to 0}\sum_{i=0}^{n-1}\left(M(t_{i+1})-M(t_i)\right)^2\\ &\le \lim_{\|\Pi\|\to 0}\sum_{i=0}^{n-1}\left(M(t_{i+1})-M(t_i)\right)\max_{0\le i\le n-1}|M(t_{i+1})-M(t_i)|\\ &= \lim_{\|\Pi\|\to 0} M(T)\max_{0\le i\le n-1}|M(t_{i+1})-M(t_i)|\end{aligned}$$

Similar to Problem 1, when $\|\Pi\| \to 0$, $n \to \infty$, and since M is continuous, one has $\max_{0\le i\le n-1}|M(t_{i+1})-M(t_i)| \to 0$, and as $M(T)$ is finite, $\boxed{\langle M_t, M_t\rangle = 0.}$

(c) Note that[6] the joint density of M_t and B_t is

$$f_{M_t,B_t}(a,x) = \frac{2(2a-x)}{t\sqrt{2\pi t}}\exp\left\{-\frac{(2a-x)^2}{2t}\right\}$$

Hence, by definition, the density of M_t is

$$\begin{aligned}f_{M_t}(a) &= \int_{-\infty}^{a}\frac{2(2a-x)}{t\sqrt{2\pi t}}\exp\left\{-\frac{(2a-x)^2}{2t}\right\}\mathrm{d}x \qquad \text{since } M_t \ge B_t\\ &= \sqrt{\frac{2}{\pi t}}\int_{-\infty}^{-\frac{a^2}{2t}} e^u \mathrm{d}u \qquad \text{subst. } u = -\frac{(2a-x)^2}{2t} \implies \mathrm{d}x = \frac{t\mathrm{d}u}{2a-x}\\ &= \boxed{\sqrt{\frac{2}{\pi t}}\exp\left\{-\frac{a^2}{2t}\right\}}\end{aligned}$$

□

2.4 Stochastic Integral

1. Let B_t be a standard Brownian motion starting from 0, ε be a number in $[0,1]$, and $\Pi = \{t_0, t_1, \ldots, t_n\}$ be a partition of $[0,t]$ with $0 = t_0 < t_1 < \ldots < t_n = t$. Consider the approximating sum

$$S_\varepsilon^\Pi = \sum_{i=0}^{n-1}\left((1-\varepsilon)B(t_i)+\varepsilon B(t_{i+1})\right)\left(B(t_{i+1})-B(t_i)\right)$$

for the stochastic integral $\int_0^t B_s \mathrm{d}B_s$.

[6] Lecture Notes 3, Equation 11

(a) Show that

$$\lim_{\|\Pi\|\to 0} S_\varepsilon^\Pi = \frac{1}{2}B_t^2 + \left(\varepsilon - \frac{1}{2}\right)t$$

where the limit is in L^2.

Proof. Note that the quadratic variation for Brownian motion is

$$\sum_{i=0}^{n-1} (B(t_{i+1}) - B(t_i))^2 = t, \quad t_n = t$$

and that

$$\sum_{i=0}^{n-1} B(t_{i+1})^2 - B(t_i)^2 = B(t_n)^2 - B(t_0)^2 = B_t^2$$

Hence, we have

$$\begin{aligned}
&\lim_{\|\Pi\|\to 0} S_\varepsilon^\Pi \\
&= \lim \sum_{i=0}^{n-1} \left((1-\varepsilon)B(t_i) + \varepsilon B(t_{i+1})\right)\left(B(t_{i+1}) - B(t_i)\right) \\
&= \lim \sum_{i=0}^{n-1} (1-\varepsilon)B(t_i)\left(B(t_{i+1}) - B(t_i)\right) + \varepsilon B(t_{i+1})\left(B(t_{i+1}) - B(t_i)\right) \\
&= (1-\varepsilon)\lim \sum_{i=0}^{n-1}\left[\left(B(t_i)B(t_{i+1}) - \frac{B(t_{i+1})^2}{2} - \frac{B(t_i)^2}{2}\right) + \frac{B(t_{i+1})^2 - B(t_i)^2}{2}\right] \\
&\quad + \varepsilon \lim \sum_{i=0}^{n-1}\left[\left(\frac{B(t_{i+1})^2}{2} + \frac{B(t_i)^2}{2} - B(t_i)B(t_{i+1})\right) + \frac{B(t_{i+1})^2 - B(t_i)^2}{2}\right] \\
&= (1-\varepsilon)\left[-\frac{1}{2}\lim \sum_{i=0}^{n-1}(B(t_{i+1}) - B(t_i))^2 + \frac{1}{2}\lim \sum_{i=0}^{n-1}\left(B(t_{i+1})^2 - B(t_i)^2\right)\right] \\
&\quad + \varepsilon\left[\frac{1}{2}\lim \sum_{i=0}^{n-1}(B(t_{i+1}) - B(t_i))^2 + \frac{1}{2}\lim \sum_{i=0}^{n-1}\left(B(t_{i+1})^2 - B(t_i)^2\right)\right] \\
&= \frac{1-\varepsilon}{2}\left(-t + B_t^2\right) + \frac{\varepsilon}{2}\left(t + B_t^2\right) \\
&= \frac{1}{2}B_t^2 + \left(\varepsilon - \frac{1}{2}\right)t
\end{aligned}$$

□

(b) Show that the right-hand side of the above identity is a martingale if and only if $\varepsilon = 0$.

Proof. ($\implies$) When $\varepsilon = 0$, one has RHS $= B_t^2/2 - t/2$. It is enough to show that $B_t^2 - t$ is a martingale. Similar to Problem 1.3, $B_t^2 - t$ is $\mathcal{F}_t$-measurable. Then, as $\mathbb{E}\left[B_t^2\right] = t$, one has $\mathbb{E}\left[B_t^2 - t\right] < \infty$. Lastly, by *independence of increments*, for all $s \leq t$, one has

$$\begin{aligned}
\mathbb{E}\left[B_t^2 - t \middle| \mathcal{F}_s\right] &= \mathbb{E}\left[B_t^2 \middle| \mathcal{F}_s\right] - t \\
&= \mathbb{E}\left[(B_t - B_s + B_s)^2 \middle| \mathcal{F}_s\right] - t \\
&= \mathbb{E}\left[(B_t - B_s)^2 \middle| \mathcal{F}_s\right] + 2B_s\mathbb{E}\left[(B_t - B_s)|\mathcal{F}_s\right] + B_s^2 - t \\
&= (t-s) + 0 + B_s^2 - t \\
&= B_s^2 - s
\end{aligned}$$

Hence, $B_t^2 - t$ is a martingale.

($\impliedby$) When $\frac{1}{2}B_t^2 + \left(\varepsilon - \frac{1}{2}\right)t$ is a martingale, one has

$$\mathbb{E}\left[\frac{1}{2}B_t^2 + \left(\varepsilon - \frac{1}{2}\right)t\right] < \infty$$

implying $|\varepsilon| < \infty$ and

$$\mathbb{E}\left[\frac{1}{2}B_t^2 - \left(\varepsilon - \frac{1}{2}\right)t \middle| \mathcal{F}_s\right] = \frac{1}{2}B_s^2 - \left(\varepsilon - \frac{1}{2}\right)s, \qquad \forall s \leq t$$

$$\begin{aligned}
&\iff \quad \frac{1}{2}\mathbb{E}\left[B_t^2 \middle| \mathcal{F}_s\right] - \left(\varepsilon - \frac{1}{2}\right)t = \frac{1}{2}B_s^2 - \left(\varepsilon - \frac{1}{2}\right)s \\
&\iff \quad \mathbb{E}\left[B_t^2 \middle| \mathcal{F}_s\right] = B_s^2 - (2\varepsilon - 1)(t-s) \\
&\iff \quad t - s + B_s^2 = B_s^2 - (2\varepsilon - 1)(t-s) \quad (\text{see} \implies \text{part}) \\
&\iff \quad 2\varepsilon(t-s) = 0
\end{aligned}$$

The equation holds for every $s \leq t$ only when $\varepsilon = 0$. □

2. Let $\sigma(t)$ be deterministic function of time, β be constant and define

$$X(T) = \int_0^T \sigma(t)e^{-\beta t}\mathrm{d}B_t$$

Find the expectation and variance of $X(T)$. What is the distribution of $X(T)$?

Solution. Since Itô integral is a martingale[7], one has $\mathbb{E}\left[X(T)\right] = \boxed{0}$.

[7] Lecture Notes 4, Theorem 2

By the isometry property[8], one has

$$\begin{aligned}
\operatorname{Var}[X(T)] &= \mathbb{E}\left[X(T)^2\right] - \mathbb{E}[X(T)]^2 \\
&= \mathbb{E}\left[X(T)^2\right] \\
&= \mathbb{E}\left[\int_0^T \sigma(t)^2 e^{-2\beta t} \mathrm{d}t\right] \\
&= \int_0^T \mathbb{E}\left[\sigma(t)^2 e^{-2\beta t}\right] \mathrm{d}t \\
&= \boxed{\int_0^T \sigma(t)^2 e^{-2\beta t} \mathrm{d}t}
\end{aligned}$$

Note that an Itô integral is a linear combination of normal variables[9], where ΔB_i is by definition Gaussian, and by the theorem[10] that linear combination of normally distributed random variables is normal, one has

$$\boxed{X(T) \sim N\left(0, \int_0^T \sigma(t)^2 e^{-2\beta t} \mathrm{d}t\right)}$$

$\square$

3. Let B_t be a standard Brownian Motion starting from 0 and M_t be its running maximum. Calculate $\mathbb{P}[M_t > 2B_t]$.

Solution. Note that the joint density for M_t, B_t is[11]

$$f_{(M_t,B_t)}(a,x) = \sqrt{\frac{2}{\pi}} \frac{2a-x}{t^{3/2}} \exp\left(-\frac{(2a-x)^2}{2t}\right) \qquad a \geq 0, x \leq a$$

[8] Lecture Notes 4, Theorem 3
[9] Lecture Notes 4, Equation 16
[10] See proof here.
[11] Lecture Notes 3, Equation 11

Hence,

$$\begin{aligned}
\mathbb{P}[M_t > 2B_t] &= \iint_{a>2x} f_{(M_t,B_t)}(a,x)\mathrm{d}x\mathrm{d}a \\
&= \sqrt{\frac{2}{\pi}}t^{-3/2}\int_0^\infty \int_{-\infty}^{a/2} (2a-x)\exp\left(-\frac{(2a-x)^2}{2t}\right)\mathrm{d}x\mathrm{d}a \\
&= \sqrt{\frac{2}{\pi}}t^{-3/2}\int_0^\infty \int_{-\infty}^{-3a/2} -b\exp\left(-\frac{b^2}{2t}\right)\mathrm{d}b\mathrm{d}a \quad \text{Subst. } b = x - 2a^{[12]} \\
&= \sqrt{\frac{2}{\pi}}t^{-3/2}\int_0^\infty te^{-\frac{9a^2}{8t}}\mathrm{d}a \\
&= \sqrt{\frac{2}{\pi}}t^{-1/2}\int_0^\infty e^{-\frac{9a^2}{8t}}\mathrm{d}a \\
&= \sqrt{\frac{2}{\pi}}t^{-1/2}\frac{\sqrt{2\pi t}}{3} \\
&= \boxed{\frac{2}{3}}
\end{aligned}$$

□

2.5 Itô's Formula

1. Use Itô's formula[13] to show that the following stochastic processes are martingales.
   (a) $X_t = e^{t/2}\cos B_t$;
   (b) $X_t = e^{t/2}\sin B_t$;
   (c) $X_t = (B_t + t)e^{-B_t - t/2}$.

Solution. (a) Let $f(t,x) = e^{t/2}\cos x$. One has

$$\begin{cases} \frac{\partial f}{\partial t}(t,x) = f(t,x)/2, \\ \frac{\partial f}{\partial x}(t,x) = -e^{t/2}\sin x, \\ \frac{\partial^2 f}{\partial x^2}(t,x) = -f(t,x). \end{cases}$$

[12] It is important to note that $|J| = 1$, so that by the *Change of Variables for Double Integral*, one has

$$\iint_R f(x,a)\mathrm{d}x\mathrm{d}a = \iint_S f(b+2a,a)\mathrm{d}b\mathrm{d}a$$

[13] Lecture Notes 5, Theorem 1

Hence, by Itô's formula, one has

$$\begin{aligned}X_t &= f(0,B_0) + \int_0^T \left[f_t(t,B_t) + \frac{1}{2}f_{xx}(t,B_t)\right]\mathrm{d}t + \int_0^T f_x(t,B_t)\mathrm{d}B_t \\ &= 1 + 0 + \int_0^T f_x(t,B_t)\mathrm{d}B_t \\ &= 1 - \int_0^T e^{t/2}\sin B_t \mathrm{d}B_t\end{aligned}$$

Note that (1) Itô integral is a martingale, and (2) any linear transformation of Brownian motion is a martingale, X_t is thus a martingale. One should also note that it is in fact an Itô integral[14], i.e. $e^{t/2}\sin B_t$ is $\mathcal{F}_t$-adapted and $\mathbb{E}\left[\int_0^T e^t(\sin B_t)^2\mathrm{d}t\right] \le \int_0^T e^t\mathrm{d}t < \infty, \forall t \ge 0$.

(b) Let $f(t,x) = e^{t/2}\sin x$. One has

$$\begin{cases} \frac{\partial f}{\partial t}(t,x) = f(t,x)/2, \\ \frac{\partial f}{\partial x}(t,x) = e^{t/2}\cos x, \\ \frac{\partial^2 f}{\partial x^2}(t,x) = -f(t,x). \end{cases}$$

Hence, by Itô's formula, one has

$$\begin{aligned}X_t &= f(0,B_0) + \int_0^T \left[f_t(t,B_t) + \frac{1}{2}f_{xx}(t,B_t)\right]\mathrm{d}t + \int_0^T f_x(t,B_t)\mathrm{d}B_t \\ &= 0 + 0 + \int_0^T f_x(t,B_t)\mathrm{d}B_t \\ &= \int_0^T e^{t/2}\cos B_t \mathrm{d}B_t\end{aligned}$$

Similar to (a), X_t is a martingale. (Check that $\mathbb{E}\left[\int_0^T e^t(\cos B_t)^2\mathrm{d}t\right] \le \int_0^T e^t\mathrm{d}t < \infty$.)

(c) Let $f(t,x) = (x+t)e^{-x-t/2}$. One has

$$\begin{cases} \frac{\partial f}{\partial t}(t,x) = -\frac{1}{2}\left(t+x-2\right)e^{-\frac{t}{2}-x}, \\ \frac{\partial f}{\partial x}(t,x) = -\left(x+t-1\right)e^{-x-\frac{t}{2}}, \\ \frac{\partial^2 f}{\partial x^2}(t,x) = \left(x+t-2\right)e^{-x-\frac{t}{2}}. \end{cases}$$

[14] Lecture Notes 4, Definition 5

Hence, by Itô's formula, one has

$$
\begin{aligned}
X_t &= f(0, B_0) + \int_0^T \left[f_t(t, B_t) + \frac{1}{2} f_{xx}(t, B_t) \right] \mathrm{d}t + \int_0^T f_x(t, B_t) \mathrm{d}B_t \\
&= 0 + 0 + \int_0^T f_x(t, B_t) \mathrm{d}B_t \\
&= - \int_0^T (B_t + t - 1)\, e^{-B_t - \frac{t}{2}} \mathrm{d}B_t
\end{aligned}
$$

Similar to (a), X_t is a martingale. (Check that $\mathbb{E}\left[(B_t + t - 1)^2 e^{-2B_t - t} \mathrm{d}t\right] < \infty$.)

Note: alternatively, they can all be easily shown by the fact that

Theorem 4. *If $u(t, x)$ is a polynomial in t and x with*

$$
\frac{\partial u}{\partial t} + \frac{1}{2} \frac{\partial^2 u}{\partial x^2} = 0,
$$

then $u(t, B_t)$ is a martingale.

□

2. Assume that $f(x)$ is twice continuously differentiable. Find all functions f such that $f(B_t)$ is a martingale. *Hint*: apply Itô's lemma to $f(B_t)$.

   Solution. f can only be linear functions, i.e. $f(x) = ax + b$ for some constants a, b, such that $f(B_t)$ is a martingale.

   It is enough to show that

   Proposition 5. *$f(B_t)$ is a martingale if and only if $f(x) = ax + b$ for some constants a, b.*

   ($\Longleftarrow$) Any linear transformation of Brownian motion is a martingale.

   ($\Longrightarrow$) By Itô's lemma[15] (for Brownian motion), one has

$$
\begin{aligned}
\mathrm{d}f(B_t) &= \left(\frac{\partial f}{\partial t} + \frac{1}{2} \frac{\partial^2 f}{\partial x^2} \right) \mathrm{d}t + \frac{\partial f}{\partial x} \mathrm{d}B_t \\
&= \frac{1}{2} f_{xx}(B_t) \mathrm{d}t + f_x(B_t) \mathrm{d}B_t \qquad (f \text{ does not depend on } t)
\end{aligned}
$$

   Note that a martingale is driftless[16], i.e. $\frac{1}{2} f_{xx}(B_t) = 0$, so that $f''(x) \equiv 0$, thus leading us to $f(x) = ax + b$ for some a, b. □

[15] Lecture Notes 5, Theorem 4

[16] This property is deduced from the *Martingale Representation Theorem*.

3. Let B_t be a standard Brownian motion and let $P(x)$ be a polynomial of degree $n, n \geq 1$.

   (a) Find values of the constants a, b, and a differential equation satisfied by $P(x)$, so that the process
   $$X_t = t^a P\left(\frac{B_t}{t^b}\right)$$
   is a martingale.

   Hint: apply Itô's lemma to $t^a P\left(x/t^b\right)$.

   (b) List the polynomials that have this property for $n = 2, 3, 4$.

Solution. (a) Let $X_t = f(B_t, t)$ where $f(x,t) = t^a P\left(x/t^b\right)$. Note that

$$\begin{cases} f_t(x,t) = at^{a-1}P(t^{-b}x) - bt^{a-b-1}xP'(t^{-b}x) \\ f_x(x,t) = t^{a-b}P'\left(x/t^b\right) \\ f_{xx}(x,t) = t^{a-2b}P''(\frac{x}{t^b}) \end{cases}$$

By Itô's lemma (for Brownian motion), one has

$$\begin{aligned} &\mathrm{d}f(B_t,t) \\ =& \left(\frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\right)\mathrm{d}t + \frac{\partial f}{\partial x}\mathrm{d}B_t \\ =& \left(at^{a-1}P\left(\frac{B_t}{t^b}\right) - bt^{a-b-1}B_tP'\left(\frac{B_t}{t^b}\right) + \frac{t^{a-2b}}{2}P''\left(\frac{B_t}{t^b}\right)\right)\mathrm{d}t + t^{a-b}P'\left(\frac{B_t}{t^b}\right)\mathrm{d}B_t \end{aligned}$$

Similar to Problem 2, by the driftlesssness of martingale, one has

$$\begin{aligned} & at^{a-1}P\left(\frac{B_t}{t^b}\right) - bt^{a-b-1}B_tP'\left(\frac{B_t}{t^b}\right) + \frac{t^{a-2b}}{2}P''\left(\frac{B_t}{t^b}\right) \equiv 0 \\ \Longleftrightarrow \quad & aP\left(\frac{B_t}{t^b}\right) - bt^{-b}B_tP'\left(\frac{B_t}{t^b}\right) + \frac{t^{1-2b}}{2}P''\left(\frac{B_t}{t^b}\right) \equiv 0 \\ \Longleftrightarrow \quad & aP(W_t) - bW_tP'(W_t) + \frac{t^{1-2b}}{2}P''(W_t) \equiv 0 \end{aligned}$$

substituting $W_t := t^{-b}B_t$. Note that the first two terms do not depend (directly) on t, so that the last term should also be independent of t, i.e. (1) $b = 1/2$ or (2) $P''(W_t) = 0$.

Case (1): $b = 1/2$. Then, one has

$$2aP(W_t) - W_tP'(W_t) + P''(W_t) \equiv 0$$

Suppose that $P(x) := c_nx^n + c_{n-1}x^{n-1} + \ldots c_1x + c_0$ for some constants $c_i, i = 0, 1, \ldots, n$. Then coefficient of the highest degree has $2ac_n - nc_n \equiv 0$. Hence, (as

$c_n \neq 0$,) $a = n/2$. Then, all other c_i are uniquely determined by a, b, c_n. (Specifically, note that $c_{n-1} = c_{n-3} = c_{n-5} = \ldots = 0$.)

Case (2): $P''(W_t) = 0$. Then, similar to Problem 2, one has $P(x) = ux + v$ for some constants u and v, so that $(a-b)ux + av \equiv 0$, leading us to (i) $a = b$ (as $u \neq 0$), $v = 0$, or (ii) $a = b = 0, v \in \mathbb{R}$.

(b) For $n = 2, 3, 4$, we consider Case (1) in (a).

When $n = 2$, one has $a = 1, b = 1/2, c_1 = 0$, and $c_0 = -c_2$. WLOG, suppose $P(x)$ is monic, i.e. $c_2 = 1$, then one has $\boxed{P(x) = x^2 - 1, a = 1, b = 1/2}$, so that $X_t = B_t^2 - t$.

When $n = 3$, one has $a = 3/2, b = 1/2, c_2 = c_0 = 0$, and $c_1 = -3c_3$. Suppose $c_3 = 1$, then $\boxed{P(x) = x^3 - 3x, a = 3/2, b = 1/2}$, so that $X_t = B_t^3 - 3tB_t$.

When $n = 4$, one has $a = 2, b = 1/2, c_3 = c_1 = 0$, and

$$\begin{cases} c_2 + 6c_4 = 0 \\ 2c_0 + c_2 = 0 \end{cases}$$

Suppose $c_4 = 1$, then $c_2 = -6, c_0 = 3$. Hence, $\boxed{P(x) = x^4 - 6x^2 + 3, a = 2, b = 1/2}$, and $X_t = B_t^4 - 6tB_t^2 + 3t^2$.

□

2.6 Stochastic Differential Equations

1. Solve the following SDE:

$$\mathrm{d}X_t = (\beta - \log X_t)X_t \mathrm{d}t + \sigma X_t \mathrm{d}B_t, \quad X(0) = X_0$$

where β and σ are given constants.

Solution. We let $Y_t = \log X_t$, and that $\mu_t = (\beta - \log X_t)X_t = (\beta - Y_t)X_t, \sigma_t = \sigma X_t$. By Itô's formula, one has

$$\begin{aligned} \mathrm{d}Y_t &= \left(\frac{\partial Y_t}{\partial t} + \mu_t \frac{\partial Y_t}{\partial x} + \frac{\sigma_t^2}{2} \frac{\partial^2 Y_t}{\partial x^2} \right) \mathrm{d}t + \sigma_t \frac{\partial Y_t}{\partial x} \mathrm{d}B_t \\ &= \left(0 + \mu_t \frac{1}{X_t} + \frac{\sigma_t^2}{2} \frac{-1}{X_t^2} \right) \mathrm{d}t + \sigma_t \frac{1}{X_t} \mathrm{d}B_t \\ &= \left(\beta - Y_t - \frac{\sigma^2}{2} \right) \mathrm{d}t + \sigma \mathrm{d}B_t \end{aligned}$$

Now we solve Y_t by

$$\mathrm{d}Y_t + Y_t\mathrm{d}t = \left(\beta - \frac{\sigma^2}{2}\right)\mathrm{d}t + \sigma\mathrm{d}B_t$$

Note that $A(t) := \int_0^t -1\mathrm{d}s = -t$. Multiplying $e^{-A(t)}$ on both sides, one has

$$\mathrm{d}\left(e^t Y_t\right) = e^t\left(\beta - \frac{\sigma^2}{2}\right)\mathrm{d}t + e^t\sigma\mathrm{d}B_t$$

Integrating on both sides from 0 to t, one has

$$\begin{aligned}
e^t Y_t - Y_0 &= \left(\beta - \frac{\sigma^2}{2}\right)\left(e^t - 1\right) + \sigma\int_0^t e^s\mathrm{d}B_s \\
&= \left(\beta - \frac{\sigma^2}{2}\right)\left(e^t - 1\right) + \sigma\left(e^t B_t - \int_0^t e^s B_s\mathrm{d}s\right) \\
\iff \quad Y_t &= Y_0 e^{-t} + \left(\beta - \frac{\sigma^2}{2}\right)\left(1 - e^{-t}\right) + \sigma B_t - \sigma e^{-t}\int_0^t e^s B_s\mathrm{d}s
\end{aligned}$$

Hence,

$$\boxed{X_t = e^{Y_t} = \exp\left\{e^{-t}\log X_0 + \left(\beta - \frac{\sigma^2}{2}\right)\left(1 - e^{-t}\right) + \sigma B_t - \sigma e^{-t}\int_0^t e^s B_s\mathrm{d}s\right\}}$$

□

2. (*Bessel Process*) Let $B_t = \left(B_t^1, B_t^2, \ldots, B_t^d\right)$ be a d-dimensional Brownian motion and

$$R_t = \|B_t\| = \sqrt{\left(B_t^1\right)^2 + \ldots + \left(B_t^d\right)^2}$$

(a) Find the distribution of the process

$$X_t = \sum_{i=1}^d \int_0^t \frac{B_s^i}{R_s}\mathrm{d}B_s^i$$

Hint: use Lévy's Characterization of Brownian Motion.

Solution. We compute $\langle X_t, X_t\rangle$. Note that $\left\langle B_t^i, B_t^j\right\rangle = 0, \forall i \neq j$, so that one has[17]

$$\langle X_t, X_t\rangle_T = \sum_{i=1}^d \int_0^T \frac{\left(B_s^i\right)^2}{R_s^2}\mathrm{d}s = T$$

Hence, by *Lévy's Characterization* (note that X_t is a local martingale with $X_0 = 0$ since Itô integral is martingale), X_t is a standard Brownian motion, i.e. $\boxed{X_t \sim N(0, t)}$. □

[17] By the generalization of quadratic variation of Itô processes

(b) Find a stochastic differential equation that R_t satisfies.

Solution. Consider $R_t^2 = \left(B_t^1\right)^2 + \ldots + \left(B_t^d\right)^2$[18]. Integrating by parts[19], one has

$$\mathrm{d}R_t^2 = 2\sum_{i=1}^{d} B_t^i \,\mathrm{d}B_t^i + \sum_{i=1}^{d} \mathrm{d}\left\langle B_t^i, B_t^i\right\rangle$$

Note that $\left\langle B_t^i, B_t^i\right\rangle = t$, and substituting with X_t in (a), one has

$$\boxed{\mathrm{d}R_t^2 = 2R_t\mathrm{d}X_t + d\mathrm{d}t}$$

□

3. Using mutlivariate Itô's formula prove that for Ito processes X_t, Y_t holds the product rule formula

$$\mathrm{d}(X_tY_t) = X_t\mathrm{d}Y_t + Y_t\mathrm{d}X_t + \mathrm{d}X_t\mathrm{d}Y_t$$

Proof. By the multivariate Itô's formula that

$$\begin{aligned}\mathrm{d}f(X_t,Y_t) = &\frac{\partial f(X_t,Y_t)}{\partial X_t}\mathrm{d}X_t + \frac{\partial f(X_t,Y_t)}{\partial Y_t}\mathrm{d}Y_t \\ &+ \frac{1}{2}\left(\frac{\partial^2 f(X_t,Y_t)}{\partial X_t^2}(\mathrm{d}X_t)^2 + 2\frac{\partial^2 f(X_t,Y_t)}{\partial X_t \partial Y_t}\mathrm{d}X_t\,\mathrm{d}Y_t + \frac{\partial^2 f(X_t,Y_t)}{\partial Y_t^2}(\mathrm{d}Y_t)^2\right)\end{aligned}$$

and given that $f(X_t,Y_t) = X_tY_t$, one has

$$\mathrm{d}X_tY_t = Y_t\mathrm{d}X_t + X_t\mathrm{d}Y_t + \frac{1}{2}\left(0 + 2\mathrm{d}X_t\mathrm{d}Y_t + 0\right) = X_t\mathrm{d}Y_t + Y_t\mathrm{d}X_t + \mathrm{d}X_t\mathrm{d}Y_t$$

□

2.7 Midterm Practices

1. Consider the following model for stock price

$$S_t = S_0e^{\sigma B_t}$$

where $\sigma > 0$ is a constant and B_t is a standard Brownian motion starting from 0.

[18] This is the 'n-dimensional' squared Bessel process, $\mathrm{BES}_\mathrm{d}^2$.

[19] One has the following theorem (integration by parts): if X, Y are semimartingales, then

$$XY = X_0Y_0 + \int X_-\,\mathrm{d}Y + \int Y_-\,\mathrm{d}X + \langle X, Y\rangle$$

(a) Find expectation and variance of S_t.

(b) Compute the price of ATM vanilla call option in this model, i.e. $\mathbb{E}(S_T - S_0)_+$ for given maturity T.

(c) Write the formula for the price of knock-out option with barrier $K > S_0$, i.e. the value of option that pays $1 if stock price doesn't reach level K till maturity T and 0 otherwise.

(d) Use Itô's formula to check if S_t is a martingale.

(e) Compute quadratic variation of S_t on interval $[0, T]$.

Solution. (a) Note that $B_t \sim N(0,t)$, i.e. $B_t = \sqrt{t}Z$ where $Z \sim N(0,1)$, so that

$$\begin{aligned}
\mathbb{E}\left[S_0 e^{\sigma B_t}\right] &= \mathbb{E}\left[S_0 e^{\sigma\sqrt{t}Z}\right] \\
&= S_0 \int_{-\infty}^{\infty} e^{\sigma\sqrt{t}z} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \mathrm{d}z \\
&= S_0 \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{\sigma\sqrt{t}z - \frac{z^2}{2}} \mathrm{d}z \\
&= S_0 e^{\sigma^2 t/2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-(z-\sigma\sqrt{t})^2/2} \mathrm{d}z \\
&= S_0 e^{\sigma^2 t/2}
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{E}\left[\left(S_0 e^{\sigma B_t}\right)^2\right] &= S_0^2 \mathbb{E}\left[e^{2\sigma\sqrt{t}Z}\right] \\
&= S_0^2 \int_{-\infty}^{\infty} e^{2\sigma\sqrt{t}Z} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \mathrm{d}z \\
&= S_0^2 e^{2\sigma^2 t}
\end{aligned}$$

Hence, $\mathrm{Var}(S_t) = \mathbb{E}\left[\left(S_0 e^{\sigma B_t}\right)^2\right] - \mathbb{E}^2\left[S_0 e^{\sigma B_t}\right] = S_0^2\left(e^{2\sigma^2 t} - e^{\sigma^2 t}\right)$.

(b) Following the general solution of Black-Scholes, i.e.

$$S_t = S_0 \exp\left(\left(r - \frac{\sigma^2}{2}\right)t + \sigma B_t\right)$$

we notice that $r - \sigma^2/2 = 0$. Hence,

$$d_1 = \frac{0 + \sigma^2 t/2 + \sigma^2 t/2}{\sigma\sqrt{t}} = \sigma\sqrt{t}, \quad d_2 = d_1 - \sigma\sqrt{t} = 0,$$

so we have

$$\begin{aligned}
e^{-\sigma^2 t/2}\mathbb{E}(S_T - S_0)_+ &= S_0\left(N\left(\sigma\sqrt{t}\right) - e^{-\sigma^2 t/2}N(0)\right) \\
&= S_0\left(N\left(\sigma\sqrt{t}\right) - e^{-\sigma^2 t/2}/2\right) \\
\iff \quad \mathbb{E}(S_T - S_0)_+ &= e^{\sigma^2 t/2}S_0 N\left(\sigma\sqrt{t}\right) - \frac{S_0}{2} \\
&= \frac{e^{\sigma^2 t/2}S_0}{\sqrt{2\pi}}\int_{-\infty}^{\sigma\sqrt{t}} e^{-y^2/2}\mathrm{d}y - \frac{S_0}{2}
\end{aligned}$$

(c) By assumption,

$$\mathbb{E}\mathbb{1}_{M_t<K}(S_T - S_0)_+ = \int_{S_0}^{K}\int_{x}^{K}(x - S_0)\frac{2(2a-x)}{\sqrt{2\pi t^3}}e^{-(2a-x)^2/(2t)}\mathrm{d}a\mathrm{d}x$$

(d) Ignoring the constant coefficients S_0 and σ, we show that $X_t = e^{B_t}$ is not a martingale.

$$\begin{aligned}
\mathbb{E}\left[X_t|\mathcal{F}_s\right] &= \mathbb{E}\left[e^{W_t}|\mathcal{F}_s\right] \\
&= \mathbb{E}\left[e^{W_t - W_s + W_s}|\mathcal{F}_s\right] \\
&= e^{W_s}\mathbb{E}\left[e^{W_t - W_s}|\mathcal{F}_s\right] \\
&= e^{W_s}e^{\frac{t-s}{2}} \neq X_s
\end{aligned}$$

Hence, X_t is not a martingale.

(e) By Itô's formula, one has

$$S_t = S_0 + \int_0^T \sigma S_0 e^{\sigma B_t}\mathrm{d}B_t + \frac{1}{2}\int_0^T \sigma^2 S_0 e^{\sigma B_t}\mathrm{d}t,$$

so that

$$\langle S_t, S_t\rangle = \int_0^T \left(\sigma S_0 e^{\sigma B_t}\right)^2\mathrm{d}t$$

□

2. Consider the following model for the stock price

$$X_t = e^{\int_0^t s\mathrm{d}B_s}, t \geq 0$$

(a) For fixed time t, what is the distribution of the random variable $\int_0^t s\mathrm{d}B_s$? What is then the distribution of X_t?

(b) From (a), calculate the price of the call option with maturity T and strike K, i.e.

$$\mathbb{E}(X_T - K)_+$$

Solution. (a) Using the fact that for a deterministic square integrable function $f(t)$, one has

$$\int_0^t f(\tau)dW_\tau \sim N\left(0, \int_0^t |f(\tau)|^2 d\tau\right),$$

we see that $\int_0^t s\mathrm{d}B_s \sim N(0, t^3/3)$. X_t is then a log-normal distribution that $X_t = e^{\sigma Z}$ where $\sigma = \sqrt{t^3/3}$ and $Z \sim N(0,1)$.

(b) By the log-normal distribution density function for $X = e^{\mu+\sigma Z}, Z \sim N(0,1)$ that

$$f_X(x) = \frac{1}{x\sigma\sqrt{2\pi}} \exp\left(-\frac{(\ln x - \mu)^2}{2\sigma^2}\right),$$

one has

$$f_{X_t}(x) = \frac{1}{x\sqrt{\frac{t^3}{3}}\sqrt{2\pi}} \exp\left(-\frac{3\ln^2 x}{2t^3}\right)$$

Therefore,

$$\begin{aligned}\mathbb{E}(X_T - K)_+ &= \int_K^\infty (x-K) f_{X_t}(x)\mathrm{d}x \\ &= \mathrm{e}^{\frac{t^3}{6}} \Phi\left(\frac{t^3 - 3\log K}{\sqrt{3}t^{3/2}}\right) + K\Phi\left(\frac{\sqrt{3}\log K}{t^{3/2}}\right) - K\end{aligned}$$

□

3. Let B_t be a standard Brownian Motion starting from 0 and M_t be its running maximum, i.e. $M_t = \max_{0\le s\le t} B_s$. For fixed time t, compute the density of the random variable $M_t - B_t$.

Solution. Note that[20] the joint density of M_t and B_t is

$$f_{M_t,B_t}(a,x) = \frac{2(2a-x)}{t\sqrt{2\pi t}} \exp\left\{-\frac{(2a-x)^2}{2t}\right\}$$

[20] Lecture Notes 3, Equation 11

Hence, the density of M_t is

$$\begin{aligned}
f_{M_t - B_t}(a) &= \int_{-a}^{\infty} f_{M_t, B_t}(a + x, x)\mathrm{d}x \\
&= \int_{-a}^{\infty} \frac{2(2a + x)}{t\sqrt{2\pi t}} \exp\left\{-\frac{(2a + x)^2}{2t}\right\} \mathrm{d}x \\
&= \sqrt{\frac{2}{\pi t}} \int_{-\infty}^{-\frac{a^2}{2t}} e^u \mathrm{d}u \qquad \text{subst. } u = -\frac{(2a + x)^2}{2t} \\
&= \boxed{\sqrt{\frac{2}{\pi t}} \exp\left\{-\frac{a^2}{2t}\right\}}
\end{aligned}$$

Note: $|B_t|$, M_t, $M_t - B_t$ all have the same distribution, $\forall t > 0$. □

4. Using Itô's lemma check if $X_t = e^{B_t/2 - t}$ is a martingale. What is the quadratic variation of process X_t on interval $[0, T]$?

   Solution. X_t is not a martingale as it does not satisfy

$$\frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial^2 u}{\partial x^2} = 0$$

   By Itô's formula,

$$X_t = f(0, B_0) + \int_0^T \left[f_t(t, B_t) + \frac{1}{2} f_{xx}(t, B_t)\right] \mathrm{d}t + \int_0^T f_x(t, B_t)\mathrm{d}B_t$$

   so that

$$[X_t] = \int_0^T f_x^2(t, B_t)\mathrm{d}t = \int_0^T \frac{1}{4} e^{B_t - 2t}\mathrm{d}t$$

   □

5. Let $B_t, t \geq 0$ be a standard Brownian motion starting from 0. Fix $t < T$ and a constant α. Compute $\mathbb{P}(B_t > B_T + \alpha)$.

   Solution. Note that

$$\mathbb{P}(B_t > B_T + \alpha) = \mathbb{P}(B_T - B_t < -\alpha),$$

   and by definition, $B_T - B_t \sim N(0, T - t)$, so that

$$\mathbb{P}(B_T - B_t < -\alpha) = \Phi\left(\frac{-\alpha}{\sqrt{T - t}}\right)$$

   □

2.8 Midterm

1. For given σ, b and S_0, consider the stochastic process S_t
$$S_t = S_0 + \frac{\sigma}{2b}\left(e^{2bB_t - 2b^2 t} - 1\right)$$
where B_t is a standard Brownian motion.

   (a) Find $\mathbb{E}[S_t]$ and $\mathrm{Var}(S_t)$ for fixed t.

   (b) Is S_t a martingale?

   (c) Compute the quadratic variation of S_t on $[0, T]$.

Solution. (a) Note that $e^{2bB_t - 2b^2 t}$ follows a log-normal distribution with $\sqrt{t}Z = B_t$ where $Z \sim N(0,1)$, so that $\mu = -2b^2 t$ and $\sigma = 2b\sqrt{t}$. Hence, $\mathbb{E}\left[e^{2bB_t - 2b^2 t}\right] = e^{\mu + \sigma^2/2} = 1$, so that
$$\mathbb{E}[S_t] = S_0 + \frac{\sigma}{2b}\left(\mathbb{E}\left[e^{2bB_t - 2b^2 t}\right] - 1\right) = \boxed{S_0}$$

Also,
$$\begin{aligned}
\mathbb{E}\left[S_t^2\right] &= \mathbb{E}\left[\left(S_0 + \frac{\sigma}{2b}\left(e^{2bB_t - 2b^2 t} - 1\right)\right)^2\right] \\
&= \mathbb{E}\left[S_0^2 + 2S_0\frac{\sigma}{2b}\left(e^{2bB_t - 2b^2 t} - 1\right) + \frac{\sigma^2}{4b^2}\left(e^{2bB_t - 2b^2 t} - 1\right)^2\right] \\
&= S_0^2 + \frac{\sigma^2}{4b^2}\left(\mathbb{E}\left[e^{4bB_t - 4b^2 t}\right] - 2\mathbb{E}\left[e^{2bB_t - 2b^2 t}\right] + 1\right) \\
&= S_0^2 + \frac{\sigma^2}{4b^2}\left(\mathbb{E}\left[e^{4bB_t - 4b^2 t}\right] - 1\right) \\
&= S_0^2 + \frac{\sigma^2}{4b^2}\left(e^{4b^2 t} - 1\right) \qquad \text{log-normal with } \mu = -4b^2 t, \sigma = 4b\sqrt{t}
\end{aligned}$$

Hence, the variance is
$$\mathrm{Var}(S_t) = \mathbb{E}\left[S_t^2\right] - \mathbb{E}[S_t]^2 = \boxed{\frac{\sigma^2}{4b^2}\left(e^{4b^2 t} - 1\right)}$$

(b) Consider $u(t,x) = S_0 + \frac{\sigma}{2b}\left(e^{2bx - 2b^2 t} - 1\right)$. One has
$$\frac{\partial u}{\partial t} = -b\sigma e^{2bx - 2b^2 t}, \qquad \frac{\partial^2 u}{\partial x^2} = 2b\sigma e^{2bx - 2b^2 t},$$
hence satisfying the condition
$$\frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial^2 u}{\partial x^2} = 0,$$
so that S_t is a martingale.

(c) By Itô's formula and the generalization of quadratic variation of Itô processes, one has

$$\langle S_t, S_t\rangle_T = \int_0^T u_x^2(t, B_t)\mathrm{d}t = \boxed{\int_0^T \sigma^2 e^{4bB_t - 4b^2 t}\mathrm{d}t}$$

□

2. Let X and Y be two independent standard Brownian motions. Show that $\langle X, Y\rangle_{[0,T]} = 0$ for any $T \geq 0$.

Proof. By definition,

$$\langle X, Y\rangle_t = \lim_{||\Delta||\to 0} \sum_i (X_{t_{i+1}} - X_{t_i})(Y_{t_{i+1}} - Y_{t_i})$$

Let $U_i^\Delta = \left(X_{t_{i+1}} - X_{t_i}\right)\left(Y_{t_{i+1}} - Y_{t_i}\right)$, then, since the processes X and Y are independent, for every i,

$$\mathbb{E}\left(\left(U_i^\Delta\right)^2\right) = \mathbb{E}\left(\left(X_{t_{i+1}} - X_{t_i}\right)^2\right)\mathbb{E}\left(\left(Y_{t_{i+1}} - Y_{t_i}\right)^2\right) = (t_{i+1} - t_i)(t_{i+1} - t_i),$$

and, since the increments of X are independent and the increments of Y are independent, for every $i \neq j$,

$$\mathbb{E}\left(U_i^\Delta U_j^\Delta\right) = \mathbb{E}\left(X_{t_{i+1}} - X_{t_i}\right)\mathbb{E}\left(Y_{t_{i+1}} - Y_{t_i}\right)\mathbb{E}\left(X_{t_{j+1}} - X_{t_j}\right)\mathbb{E}\left(Y_{t_{j+1}} - Y_{t_j}\right) = 0^4 = 0$$

Hence the square of the L^2 norm of the RHS for subdivision Δ is

$$\sum_i (t_{i+1} - t_i)^2 \leq \|\Delta\| \cdot t$$

Taking the limit, one has $\mathbb{E}\left[\langle X, Y\rangle_t^2\right] = 0$, i.e. $\mathbb{E}\left[\langle X, Y\rangle_t^2\right] = \mathrm{Var}(\langle X, Y\rangle) + \mathbb{E}(\langle X, Y\rangle)^2 = 0$, implying that $\langle X, Y\rangle_{[0,T]} = 0$ for any $T \geq 0$. □

3. Let σ be a fixed constant and $X_t = e^{\sigma B_t}$. What is the probability that the maximum of X_t on the interval $[0, T]$ exceeds a, given $a > 0$?

Solution. WLOG, suppose $\sigma > 0$[21], so that $f(x) = e^{\sigma x}$ is monotonically increasing. Note that

[21] If $\sigma < 0$, $f(x) = e^{\sigma x}$ is decreasing, so that $\max_{[0,T]} X_t = e^{\sigma \underline{M_t}}$, where $\underline{M_t}$ is the running minimum satisfying $f_{\underline{M_t}, B_t} = -f_{M_t, B_t}$.

$\max_{[0,T]} X_t = e^{\sigma M_t}$ where M_t is the running maximum of B_t. Therefore, one has

$$\begin{aligned}
\mathbb{P}\left[\max_{[0,T]} X_t > a\right] &= \mathbb{P}\left[e^{\sigma M_t} > a\right] \\
&= \mathbb{P}\left[\sigma M_t > \log a\right] \\
&= \mathbb{P}\left[M_t > \frac{\log a}{\sigma}\right] \quad \text{where } f_{M_t}(a) = \sqrt{\frac{2}{\pi t}} \exp\left\{-\frac{a^2}{2t}\right\} \\
&= \int_{\frac{\log a}{\sigma}}^{\infty} \sqrt{\frac{2}{\pi t}} \exp\left\{-\frac{x^2}{2t}\right\} \mathrm{d}x \\
&= 2\int_{\frac{\log a}{\sigma\sqrt{t}}}^{\infty} \frac{e^{-v^2/2}}{\sqrt{2\pi}} \mathrm{d}v \quad \text{Subst. } v = \frac{x}{\sqrt{t}} \implies \mathrm{d}x = \mathrm{d}v\sqrt{t} \\
&= \boxed{2 - 2\Phi\left(\frac{\log a}{\sigma\sqrt{t}}\right)}
\end{aligned}$$

□

2.9 Girsanov Theorem

1. Let B_t be a Brownian motion defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $\gamma(t, w)$ be an adapted to Brownian filtration $(\mathcal{F}_t)_{t\geq 0}$ process. Let

$$Z_t = \exp\left\{\int_0^t \gamma(s, w)\mathrm{d}B_s - \frac{1}{2}\int_0^t \gamma^2(s, w)\mathrm{d}s\right\}$$

and for any fixed $T < \infty$ define a new measure $\widetilde{\mathbb{P}}$ on $\mathcal{F}_T$ as

$$\widetilde{\mathbb{P}}(\mathrm{d}w) = Z_T(w)\mathbb{P}(\mathrm{d}w)$$

We show in the following that $\widetilde{B}_t = B_t - \int_0^t \gamma(s, w)\mathrm{d}s$ is a Brownian motion under $\widetilde{\mathbb{P}}$.

   (a) Apply Itô's Lemma to obtain an SDE for Z_t.

   (b) Apply Itô's Lemma to $Z_t\widetilde{B}_t$ and show that it is a martingale under $\mathbb{P}$.

   (c) Assume you know that[22] for any random variable Y and $s \leq t \leq T$, one has

$$\widetilde{\mathbb{E}}\left(Y|\mathcal{F}_s\right) = \frac{1}{Z_s(w)}\mathbb{E}\left(YZ_t(w)|\mathcal{F}_s\right)$$

   Use this to show that $\widetilde{B}_t$ is a martingale under $\widetilde{\mathbb{P}}$.

[22] This can be proved by the tower law for conditional expectation.

(d) Similarly, apply Itô's Lemma to $Z_t\left(\widetilde{B}_t^2 - t\right)$ and show that it is a martingale under $\mathbb{P}$.

(e) Show that $\widetilde{B}_t^2 - t$ is a martingale under $\widetilde{\mathbb{P}}$.

(f) Apply Lévy's Theorem to conclude that $\widetilde{B}_t$ is a Brownian motion under $\widetilde{\mathbb{P}}$.

Proof. [23] We first show that $\widetilde{B}_t$ is a martingale under $\widetilde{\mathbb{P}}$.

Note that $Z_t = f(Y_t, t)$ where $f(x,t) := \exp\left\{x - \frac{1}{2}\int_0^t \gamma_s^2 \mathrm{d}s\right\}$ and $Y_t := \int_0^t \gamma_s \mathrm{d}B_s$. By Itô's formula, one has

$$\begin{aligned}
\mathrm{d}Z_t &= f_t(Y_t,t)\mathrm{d}t + \frac{1}{2}f_{xx}(Y_t,t)\mathrm{d}\langle Y,Y\rangle_t + f_x(Y_t,t)\mathrm{d}Y_t \\
&= -\frac{1}{2}\gamma_t^2 Z_t \mathrm{d}t + \frac{1}{2}Z_t\gamma_t^2\mathrm{d}t + Z_t\gamma_t\mathrm{d}B_t \\
&= Z_t\gamma_t\mathrm{d}B_t
\end{aligned}$$

Also, note that $\mathrm{d}\widetilde{B}_t = \mathrm{d}B_t - \gamma_t\mathrm{d}t$. Hence, by Itô's product rule, one has

$$\begin{aligned}
\mathrm{d}\left(Z_t\widetilde{B}_t\right) &= \widetilde{B}_t\mathrm{d}Z_t + Z_t\mathrm{d}\widetilde{B}_t + \mathrm{d}Z_t\mathrm{d}\widetilde{B}_t \\
&= \widetilde{B}_t Z_t\gamma_t\mathrm{d}B_t + Z_t\left(\mathrm{d}B_t - \gamma_t\mathrm{d}t\right) + Z_t\gamma_t\mathrm{d}B_t\left(\mathrm{d}B_t - \gamma_t\mathrm{d}t\right) \\
&= \widetilde{B}_t Z_t\gamma_t\mathrm{d}B_t + Z_t\mathrm{d}B_t - Z_t\gamma_t\mathrm{d}t + Z_t\gamma_t\mathrm{d}t - 0 \\
&= \left(\widetilde{B}_t Z_t\gamma_t + Z_t\right)\mathrm{d}B_t
\end{aligned}$$

which is driftless (no $\mathrm{d}t$ term), so that $Z_t\widetilde{B}_t$ is a martingale (under $\mathbb{P}$). Now, applying the theorem given in (c), for $T \geq t \geq s$, one has

$$\begin{aligned}
\widetilde{\mathbb{E}}\left(\widetilde{B}_t\middle|\mathcal{F}_s\right) &= \frac{1}{Z_s(w)}\mathbb{E}\left(\widetilde{B}_t Z_t(w)\middle|\mathcal{F}_s\right) \\
\implies \widetilde{\mathbb{E}}\left(\widetilde{B}_t\middle|\mathcal{F}_s\right) &= \widetilde{B}_s \quad \text{since } Z_t\widetilde{B}_t \text{ is a martingale}
\end{aligned}$$

Also, notice that $\widetilde{B}_t$ is $\mathcal{F}_T$ measurable ($T \geq t$) and $\mathbb{E}\left[\left|\widetilde{B}_t\right|\right] < \infty$ ($\int_0^t \gamma_s \mathrm{d}s$ is bounded), implying that $\widetilde{B}_t$ is a martingale under $\widetilde{\mathbb{P}}$.

Then we show that $\widetilde{B}_t^2 - t$ is a martingale under $\widetilde{\mathbb{P}}$.

Similarly, by Itô's product rule, one has

$$\mathrm{d}\left(\widetilde{B}_t^2\right) = 2\widetilde{B}_t\mathrm{d}\widetilde{B}_t + \mathrm{d}t = 2\widetilde{B}_t\mathrm{d}\widetilde{B}_t + \mathrm{d}t = 2\widetilde{B}_t\left(\mathrm{d}B_t - \gamma_t\mathrm{d}t\right) + \mathrm{d}t$$

[23] Reference.

and

$$\begin{aligned}
\mathrm{d}\left(Z_t\widetilde{B}_t^2\right) &= \widetilde{B}_t^2\mathrm{d}Z_t + Z_t\mathrm{d}\widetilde{B}_t^2 + \mathrm{d}Z_t\mathrm{d}\widetilde{B}_t^2 \\
&= \widetilde{B}_t^2 Z_t\gamma_t\mathrm{d}B_t + Z_t\left(2\widetilde{B}_t\left(\mathrm{d}B_t - \gamma_t\mathrm{d}t\right) + \mathrm{d}t\right) + Z_t\gamma_t\mathrm{d}B_t\left(2\widetilde{B}_t\left(\mathrm{d}B_t - \gamma_t\mathrm{d}t\right) + \mathrm{d}t\right) \\
&= \widetilde{B}_t^2 Z_t\gamma_t\mathrm{d}B_t + 2Z_t\widetilde{B}_t\mathrm{d}B_t + Z_t\mathrm{d}t
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{d}\left(Z_t t\right) &= Z_t\mathrm{d}t + t\mathrm{d}Z_t + \mathrm{d}Z_t\mathrm{d}t \\
&= Z_t\mathrm{d}t + tZ_t\gamma_t\mathrm{d}B_t
\end{aligned}$$

so that

$$\begin{aligned}
\mathrm{d}\left(Z_t\left(\widetilde{B}_t^2 - t\right)\right) &= \mathrm{d}\left(Z_t\widetilde{B}_t^2\right) - \mathrm{d}\left(Z_t t\right) \\
&= \widetilde{B}_t^2 Z_t\gamma_t\mathrm{d}B_t + 2Z_t\widetilde{B}_t\mathrm{d}B_t - tZ_t\gamma_t\mathrm{d}B_t
\end{aligned}$$

which is driftless (no $\mathrm{d}t$ term), so that $Z_t\left(\widetilde{B}_t^2 - t\right)$ is a martingale (under $\mathbb{P}$). And similarly, applying the theorem given in (c), for $T \geq t \geq s$, one has

$$\widetilde{\mathbb{E}}\left(\widetilde{B}_t^2 - t\middle|\mathcal{F}_s\right) = \widetilde{B}_s^2 - s \quad \text{since } Z_t\left(\widetilde{B}_t^2 - t\right) \text{ is a martingale}$$

Also, notice that $\widetilde{B}_t^2 - t$ is $\mathcal{F}_T$ measurable ($T \geq t$) and $\mathbb{E}\left[\left|\widetilde{B}_t^2 - t\right|\right] < \infty$ ($\int_0^t \gamma_s\mathrm{d}s$ is bounded), implying that $\widetilde{B}_t^2 - t$ is a martingale under $\widetilde{\mathbb{P}}$.

Hence, both $\widetilde{B}_t$ and $\widetilde{B}_t^2 - t$ are martingales under $\widetilde{\mathbb{P}}$. By the theorem[24] that

Theorem 6 (Lévy's)**.** *Suppose that both M and $(M_t^2 - t)_{t\geq 0}$ are local martingales. Then M is a Brownian motion with respect to $(\mathcal{F}_t)$.*

one has $\widetilde{B}_t$ is a Brownian motion under $\widetilde{\mathbb{P}}$. □

2.10 Connection with PDE

1. Consider the Ito process $Y_t, t \geq 0$ given by

$$\mathrm{d}Y_t = e^t\mathrm{d}B_t, \quad Y_0 = 0$$

and let $\tau = \min\{t \geq 0 : |Y_t| = 1\}$.

[24] See the proof here.

(a) Find a PDE for $f(t,x)$ satisfied whenever $f(t,Y_t)$ is a martingale.

(b) Verify that

$$f(t,x) = x^2 - \frac{e^{2t}}{2}$$

satisfies the PDE and that $f(t,Y_t)$ is a martingale.

(c) Show that $\mathbb{E}\left[e^{2\tau}\right] = 3$.

(d) Show that $\mathbb{E}[\tau] \leq \frac{\log 3}{2}$.

Solution. (a) By Itô's formula, one has

$$\mathrm{d}f(t,Y_t) = \left(f_t + \frac{e^{2t}}{2} f_{xx}\right)\mathrm{d}t + e^t f_x \mathrm{d}B_t$$

To let f be a martingale, one has

$$\boxed{\frac{\partial f}{\partial t} + \frac{e^{2t}}{2}\frac{\partial^2 f}{\partial x^2} = 0}$$

(b) Note that $f_t = -e^{2t}$ and $f_{xx} = 2$, hence satisfying $-e^{2t} + 2e^{2t}/2 = 0$.

Since $f(t,Y_t)$ is driftless, i.e. the $\mathrm{d}t$ term is zero, $f(t,Y_t)$ is a martingale.

(c) Note that $f(t,Y_t)$ is a martingale, and $\tau = \min\{t \geq 0 : |Y_t| = 1\}$ is almost surely bounded. Given that $Y_0 = 0$, by the *optional stopping theorem*, one has

$$\mathbb{E}\left[f(\tau, Y_\tau)\right] = \mathbb{E}\left[f(0, Y_0)\right]$$

where

$$\mathbb{E}\left[f(\tau, Y_\tau)\right] = \mathbb{E}\left[Y_\tau^2 - \frac{e^{2\tau}}{2}\right] = \mathbb{E}\left[Y_\tau^2\right] - \frac{1}{2}\mathbb{E}\left[e^{2\tau}\right] = 1 - \frac{1}{2}\mathbb{E}\left[e^{2\tau}\right]$$

and

$$\mathbb{E}\left[f(0, Y_0)\right] = \mathbb{E}\left[f(0,0)\right] = -\frac{1}{2}$$

Hence,

$$1 - \frac{1}{2}\mathbb{E}\left[e^{2\tau}\right] = -\frac{1}{2},$$

i.e. $\mathbb{E}\left[e^{2\tau}\right] = 3$.

(d) Note that $g(x) = e^{2x}$ is a convex function and monotonically increasing. By *Jensen's inequality in probabilistic form*, one has

$$\begin{aligned} & \mathbb{E}\left[g(\tau)\right] \geq g\left(\mathbb{E}[\tau]\right), \\ \text{i.e.} \qquad & e^{2\mathbb{E}[\tau]} \leq 3, \\ \iff \qquad & \mathbb{E}[\tau] \leq \frac{\log 3}{2} \end{aligned}$$

□

2. Let B_t be a standard Brownian motion and T a fixed maturity. Compute

$$\mathbb{E}_{B_t=x}\left[\int_t^T B_s^2 \mathrm{d}s\right]$$

Check your answer by calculating the above expectation using the properties of the Brownian motion.

Solution. Consider

$$u(t,x) = \mathbb{E}_{B_t=x}\left[\int_t^T b(s,B_s)\mathrm{d}s\right]$$

where $b(s, B_s) = B_s^2$. By the *running payoff* formula, u solves

$$u_t + \frac{1}{2}u_{xx} + x^2 = 0, \quad u(T,x) = 0$$

We calculate $u(t,x)$ by the properties of the Brownian motion.

Note that $\forall s > t$, one has $B_s = B_t + B_{s-t}$ with independence of increments. Also note that $\mathbb{E}[B_u] = 0$ and $\mathbb{E}[B_u^2] = u, \forall u$. Therefore, one has

$$\begin{aligned} u(t,x) &= \mathbb{E}_{B_t=x}\left[\int_t^T (B_t + B_{s-t})^2 \mathrm{d}s\right] \\ &= \mathbb{E}_{B_t=x}\left[\int_t^T \left(B_t^2 + 2B_tB_{s-t} + B_{s-t}^2\right)\mathrm{d}s\right] \\ &= \mathbb{E}_{B_t=x}\left[\int_0^{T-t} \left(B_t^2 + 2B_tB_u + B_u^2\right)\mathrm{d}u\right] \\ &= \int_0^{T-t} \left(\mathbb{E}_{B_t=x}\left[B_t^2 + 2B_tB_u + B_u^2\right]\right)\mathrm{d}u \\ &= \int_0^{T-t} \left(x^2 + 0 + u\right)\mathrm{d}u \\ &= \boxed{(T-t)x^2 + \frac{1}{2}(T-t)^2} \end{aligned}$$

Check answer: note that $u_t = t - x^2 - T$ and $u_{xx} = 2(T-t)$, so that it satisfies the PDE conditions given above. □

2.11 Boundary Value Problems

1. Let B_t be a Brownian motion. Let γ be a constant and $T > 0$ be a fixed maturity. Use the Feynman-Kac theorem to compute

$$\mathbb{E}_{B_0=x}\left[\exp\left\{-\gamma\int_0^T B_s \mathrm{d}s\right\}\right]$$

Solution. Applying the Feynman-Kac theorem on

$$u(t,x) := \mathbb{E}_{B_t=x}\left[\exp\left\{-\int_t^T \gamma B_s \mathrm{d}s\right\}\right],$$

one has

$$u_t + \frac{u_{xx}}{2} - \gamma x u = 0$$

with boundary condition $u(T,x) = 1$. One may guess that the solution is of the form

$$u(t,x) = e^{-a(t)x - b(t)},$$

so that

$$\left(-a'x - b'\right)u + a^2 u/2 - \gamma x u = 0$$

Regrouping the terms, one has

$$-xu\left(a' + \gamma\right) + u\left(a^2/2 - b'\right) = 0$$

In order for the equation and the boundary condition $e^{-a(T)x - b(T)} = 1$ to hold with all x, one has

$$\begin{cases} a' + \gamma = 0, \\ a^2/2 - b' = 0, \\ a(T) = b(T) = 0, \end{cases}$$

so that $a(t) = \gamma(T-t)$ (satisfying the first and third equation), and thus $b' = \gamma^2(T-t)^2/2$. Hence,

$$b(t) = \int_t^T \frac{\gamma^2(T-s)^2}{2}\mathrm{d}s = -\frac{\gamma^2(T-t)^3}{6}$$

Check that $b(T) = 0$ as desired. Therefore, the solution is given as

$$u(t,x) = e^{-\gamma(T-t)x + \gamma^2(T-t)^3/6},$$

showing that

$$\mathbb{E}_{B_0=x}\left[\exp\left\{-\gamma\int_0^T B_s \mathrm{d}s\right\}\right] = u(0,x) = \boxed{e^{-\gamma Tx+\gamma^2T^3/6}}$$

Alternatively, one may check the answer by calculating it through probabilistic methods.

Note that

$$\int_0^T B_s \mathrm{d}s \sim N\left(0, \frac{T^3}{3}\right),$$

which implies that given $B_0 = x$,

$$-\gamma\int_0^T B_s \mathrm{d}s \sim N\left(-\gamma xT, \gamma^2\frac{T^3}{3}\right)$$

Applying LOTUS on $Y = e^X$ where $X \sim N(\mu, \sigma^2)$[25], one has

$$\mathbb{E}[Y] = e^{\mu+\sigma^2/2}$$

Hence,

$$\mathbb{E}_{B_0=x}\left[\exp\left\{-\gamma\int_0^T B_s \mathrm{d}s\right\}\right] = \exp\left\{-\gamma xT + \gamma^2\frac{T^3}{3}\cdot\frac{1}{2}\right\} = \boxed{e^{-\gamma xT+\gamma^2T^3/6}}$$

□

2. Let X_t satisfy the following SDE,

$$\mathrm{d}X_t = X_t\mathrm{d}t + \sigma\mathrm{d}B_t,$$

where $X_0 = x \in [a,b]$ and σ is some fixed constant. Let $\tau = \inf\{t \geq 0 : X_t \notin [a,b]\}$.

(a) Find $\mathbb{E}_{X_0=x}X_\tau$.

(b) Is $\mathbb{E}_{X_0=x}\tau < \infty$?

Solution. One shall assume that $\sigma \neq 0$.

(a) Let $u(t,x) = \mathbb{E}_{X_t=x}X_\tau$. By Itô's lemma, one has

$$\mathrm{d}u(t,x) = u_t\mathrm{d}t + u_x\mathrm{d}X_t + \frac{1}{2}u_{xx}(\mathrm{d}X_t)^2$$

where

$$(\mathrm{d}X_t)^2 = (X_t\mathrm{d}t + \sigma\mathrm{d}B_t)^2 = \sigma^2\mathrm{d}t,$$

[25] See the proof here.

so that

$$\begin{aligned} \mathrm{d}u(t,x) &= u_t \mathrm{d}t + u_x(x\mathrm{d}t + \sigma \mathrm{d}B_t) + \frac{1}{2}u_{xx}\sigma^2 \mathrm{d}t \\ &= \left(u_t + u_x x + u_{xx}\sigma^2/2\right)\mathrm{d}t + u_x \sigma \mathrm{d}B_t \end{aligned}$$

For u to be a martingale, the drift term should be 0, i.e.

$$u_t + u_x x + u_{xx}\sigma^2/2 = 0$$

Also note that, as the expectation u does not depend on the initial condition at t, u is time homogeneous, implying that

$$u(t,x) = u(0,x), \quad \forall t > 0$$

Hence, $u_t = 0$, remaining to solve the ODE

$$u_x x + u_{xx}\sigma^2/2 = 0$$

with initial conditions $u(a) = a, u(b) = b$. One may guess u_x is of the form $u_x = ke^{\ell x^2}$ where k, ℓ are some constants, so that one has

$$kxe^{\ell x^2} + kl\sigma^2 xe^{\ell x^2} = 0,$$

implying that $1 + \ell\sigma^2 = 0$, i.e. $\ell = -1/\sigma^2$. Hence, u is of the form

$$u(x) = k\int_0^x e^{-s^2/\sigma^2}\mathrm{d}s + c = \frac{k\sqrt{\pi}\,\sigma \mathrm{erf}\left(\frac{x}{\sigma}\right)}{2} + c = d \cdot \mathrm{erf}\left(\frac{x}{\sigma}\right) + c$$

for some constants c, d. Solving the system with initial conditions,

$$\begin{cases} d \cdot \mathrm{erf}\left(\frac{a}{\sigma}\right) + c = a, \\ d \cdot \mathrm{erf}\left(\frac{b}{\sigma}\right) + c = b, \end{cases}$$

one gets

$$d = \frac{a-b}{\mathrm{erf}\left(\frac{a}{\sigma}\right) - \mathrm{erf}\left(\frac{b}{\sigma}\right)}, \quad c = \frac{b \cdot \mathrm{erf}\left(\frac{a}{\sigma}\right) - a \cdot \mathrm{erf}\left(\frac{b}{\sigma}\right)}{\mathrm{erf}\left(\frac{a}{\sigma}\right) - \mathrm{erf}\left(\frac{b}{\sigma}\right)}$$

Hence,

$$\mathbb{E}_{X_0=x}X_\tau = u(0,x) = u(x) = \boxed{\frac{(a-b)\cdot \mathrm{erf}\left(\frac{x}{\sigma}\right) + b\cdot \mathrm{erf}\left(\frac{a}{\sigma}\right) - a \cdot \mathrm{erf}\left(\frac{b}{\sigma}\right)}{\mathrm{erf}\left(\frac{a}{\sigma}\right) - \mathrm{erf}\left(\frac{b}{\sigma}\right)}}$$

(b) Yes, $\mathbb{E}_{X_0=x}\tau < \infty$. Note that it is equivalent to show that X_t is expected to break the boundary ($[a,b]$) in finite time. Recall that an *Ornstein-Uhlenbeck process* is defined as

Definition 2.11.1 (Ornstein-Uhlenbeck process)**.** is a stochastic process that satisfies the following stochastic differential equation

$$\mathrm{d}X_t = \kappa(\theta - X_t)\mathrm{d}t + \sigma \mathrm{d}B_t$$

where B_t is a standard Brownian motion on $t > 0$; $\kappa > 0$ is the rate of mean reversion, θ is the long-term mean of the process, and $\sigma > 0$ is the volatility or average magnitude, per square-root time, of the random fluctuations that are modelled as Brownian motions.

Also, recall that the variance[26] of an Ornstein-Uhlenbeck process X_t is given as

$$\mathrm{Var}(X_t) = \frac{\sigma^2}{2\kappa}\left(1 - e^{-2\kappa t}\right)$$

Back to the question, here X_t is a similar process with $\kappa = -1 < 0$[27] and $\theta = 0$. Therefore, one has

$$\mathrm{Var}(X_t) = \frac{\sigma^2}{2}\left(e^{2t} - 1\right) \xrightarrow{t\to\infty} \infty$$

implying that the variance of X_t is unbounded, but the boundary $[a, b]$ has a, b as finite values. Hence, X_t is expected to break the boundary in finite time, so $\mathbb{E}_{X_0=x}\tau < \infty$.

$\square$

2.12 Forward Kolmogorov Equation

1. (*Smoluchowski's equation*) We consider the diffusion $(X_t, t \geq 0)$ given by the SDE

$$\mathrm{d}X_t = \mathrm{d}B_t - V'(X_t)\mathrm{d}t$$

where $V : \mathbb{R} \to \mathbb{R}$ is some smooth function such that

$$\int_{\mathbb{R}} e^{-2V(x)}\mathrm{d}x < \infty$$

   (a) Verify that the invariant distribution $f(x) = Ce^{-2V(x)}$ is a solution to $\mathcal{L}^* f = 0$.

   (b) Consider the specific example of $V(x) = |x|$. What is the SDE in this case? What is the exact invariant distribution?

[26] See the proof here.

[27] The definition of the *Ornstein-Uhlenbeck process* requires κ to be strictly larger than 0, but it does not affect the calculation of the variance, and similar with σ.

Solution. (a) By definition, $\mathcal{L}^*$, as the dual of $\mathcal{L}$, is defined as

$$\begin{aligned}\mathcal{L}^* f &= -\left(-V'(x)f\right)_x + \frac{1}{2}(1\cdot f)_{xx} \\ &= V_{xx}f + V_x f_x + \frac{f_{xx}}{2}\end{aligned}$$

where the assumed solution $f(x) = Ce^{-2V(x)}$ has

$$f_x(x) = -2V_x f \implies f_{xx} = -2V_{xx}f - 2V_x f_x,$$

so that

$$\begin{aligned}\mathcal{L}^* f &= V_{xx}f + V_x f_x + \frac{f_{xx}}{2} \\ &= V_{xx}f + V_x f_x - V_{xx}f - V_x f_x = 0,\end{aligned}$$

hence satisfying the equation.

(b) Given $V(x) = |x|$, one has

$$V'(x) = \operatorname{sgn}(x) := \begin{cases} -1 & \text{if } x < 0, \\ 0 & \text{if } x = 0, \\ 1 & \text{if } x > 0 \end{cases}$$

Hence, in this case, the SDE is

$$\boxed{\mathrm{d}X_t = \mathrm{d}B_t - \operatorname{sgn}(X_t)\mathrm{d}t^{28}}$$

Since f is a distribution function, one has

$$\int_{\mathbb{R}} f(x)\mathrm{d}x = C\left(\int_{-\infty}^{0} e^{2x}\mathrm{d}x + \int_0^{\infty} e^{-2x}\mathrm{d}x\right) = 1 \implies \boxed{C = 1}$$

□

2. (*Invariant probability of the Ornstein-Uhlenbeck process*) Consider the Ornstein-Uhlenbeck process with SDE

$$\mathrm{d}X_t = -X_t\mathrm{d}t + \mathrm{d}B_t, \quad X_0 = x$$

   (a) Find the transition probability density $p(t, z; 0, x)$.

   (b) Find $\lim_{t\to\infty} p(t, z; 0, x)$.

[28] Note that $\forall t \geq 0$, one has $\mathbb{P}(X_t \neq 0) = 1$ almost surely. Hence, $V(X_t) = |X_t|$ is almost surely twice continuously differentiable.

(c) Let $f(x) = e^{-x^2}$. Prove that $\mathcal{L}^* f = 0$.

(d) When considering the more general SDE

$$\mathrm{d}X_t = -\alpha X_t \mathrm{d}t + \sigma \mathrm{d}B_t, \quad X_0 = x,$$

how are the above results affected?

Solution. (a) Recall (*will prove it in (d)*) that for a general Ornstein-Uhlenbeck process

$$\mathrm{d}X_t = -\alpha X_t \mathrm{d}t + \sigma \mathrm{d}B_t,$$

conditional on $s (s \leq t)$, one has

$$X_t \sim N\left(X_s e^{-\alpha(t-s)}, \frac{\sigma^2}{2\alpha}\left(1 - e^{-2\alpha(t-s)}\right)\right)$$

Given $s = 0, X_0 = x, \alpha = 1, \sigma = 1$, one has

$$X_t \sim N\left(xe^{-t}, \frac{1 - e^{-2t}}{2}\right)$$

Hence, by definition,

$$\boxed{p(t, z; 0, x) = \frac{1}{\sqrt{\pi\left(1 - e^{-2t}\right)}} \exp\left\{\frac{(z - xe^{-t})^2}{e^{-2t} - 1}\right\}}$$

(b) When $t \to \infty$, $e^{-t} \to 0$. Hence,

$$\boxed{\lim_{t \to \infty} p(t, z; 0, x) = \frac{e^{-z^2}}{\sqrt{\pi}}}$$

(c) By definition,

$$\begin{aligned}
\mathcal{L}^* f &= -(-xf)_x + \frac{1}{2}(1 \cdot f)_{xx} \\
&= f + xf_x + \frac{f_{xx}}{2} \\
&= e^{-x^2} - 2x^2 e^{-x^2} + 2x^2 e^{-x^2} - e^{-x^2} \\
&= \boxed{0}
\end{aligned}$$

(d) We shall now prove the proposition given in (a) that

$$X_t \sim N\left(X_s e^{-\alpha(t-s)}, \frac{\sigma^2}{2\alpha}\left(1 - e^{-2\alpha(t-s)}\right)\right)$$

Proof. [29] Note that in the given SDE, X_t has a drift towards the value zero, at an exponential rate α. This motivates the change of variables

$$X_t = e^{-\alpha t} Z_t \iff Z_t = e^{\alpha t} X_t,$$

so that

$$\begin{aligned} \mathrm{d}Z_t &= \alpha e^{\alpha t} X_t \mathrm{d}t + e^{\alpha t} \mathrm{d}X_t \\ &= \alpha e^{\alpha t} X_t \mathrm{d}t + e^{\alpha t} \left(-\alpha X_t \mathrm{d}t + \sigma \mathrm{d}B_t\right) \\ &= 0\mathrm{d}t + \sigma e^{\alpha t} \mathrm{d}B_t \end{aligned}$$

Integrating on both sides, one has

$$Z_t = Z_s + \sigma \int_s^t e^{\alpha u} \mathrm{d}B_u,$$

implying that

$$X_t = e^{-\alpha(t-s)} X_s + \sigma e^{-\alpha t} \int_s^t e^{\alpha u} \mathrm{d}B_u$$

Note that this is a sum of deterministic terms and an integral of a deterministic function with respect to a Wiener process with normally distributed increments. The distribution is thus normal with mean $X_s e^{-\alpha(t-s)}$ and variance $\frac{\sigma^2}{2\alpha}\left(1 - e^{-2\alpha(t-s)}\right)$[30]. □

In this specific question, we still have $s = 0$, $X_0 = x$, so that

$$\boxed{p(t, z; 0, x) = \frac{1}{\sqrt{\pi \sigma^2 \left(1 - e^{-2\alpha t}\right)/\alpha}} \exp\left\{\frac{\alpha (z - x e^{-\alpha t})^2}{\sigma^2 \left(e^{-2\alpha t} - 1\right)}\right\}}$$

and

$$\boxed{\lim_{t\to\infty} p(t, z; 0, x) = \frac{e^{-\alpha z^2/\sigma^2}}{\sqrt{\pi \sigma^2/\alpha}}}$$

[29] Reference.

[30] It can be calculated through Itô isometry. Specifically,

$$\mathbb{E}\left(\int_s^t e^{\alpha u} \mathrm{d}B_u\right)^2 = \mathbb{E}\left[\int_s^t e^{2\alpha u} \mathrm{d}u\right] = \frac{1}{2\alpha}\left(e^{2\alpha t} - e^{2\alpha s}\right)$$

so that

$$\mathrm{Var}\left(\sigma e^{-\alpha t} \int_s^t e^{\alpha u} \mathrm{d}B_u\right) = \sigma^2 e^{-2\alpha t} \frac{1}{2\alpha}\left(e^{2\alpha t} - e^{2\alpha s}\right) = \frac{\sigma^2}{2\alpha}\left(1 - e^{-2\alpha(t-s)}\right)$$

but

$$
\begin{aligned}
\mathcal{L}^* f &= -(-\alpha x f)_x + \frac{1}{2}(\sigma^2 f)_{xx} \\
&= \alpha f + \alpha x f_x + \frac{\sigma^2}{2} f_{xx} \\
&= \alpha\left(1-2x^2\right)e^{-x^2} + \sigma^2\left(2x^2-1\right)\mathrm{e}^{-x^2} \\
&= \boxed{\left(\alpha-\sigma^2\right)\left(1-2x^2\right)e^{-x^2}}
\end{aligned}
$$

is not always zero.

□

2.13 Processes with Jumps

1. Let $N_t, t \geq 0$ be a Poisson process with intensity $\lambda \geq 0$ defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with respect to a filtration $\mathcal{F}_t$. Let $M_t = N_t - \lambda t$ be a compensated Poisson process. Prove that $M_t^2 - \lambda t$ is a martingale.

   Proof. We verify that $\mathbb{E}\left[M_t^2 - \lambda t \middle| \mathcal{F}_s\right] = M_s^2 - \lambda s, \forall s < t$.

   Given $s < t$, note that

$$
\begin{aligned}
\mathbb{E}\left[M_t^2 \middle| \mathcal{F}_s\right] &= \mathrm{Var}(M_t|\mathcal{F}_s) + (\mathbb{E}[M_t|\mathcal{F}_s])^2 \\
&= \mathrm{Var}(N_t - \lambda t|\mathcal{F}_s) + M_s^2 && M_t \text{ is a martingale} \\
&= \mathrm{Var}(N_t|\mathcal{F}_s) + M_s^2 \\
&= \mathrm{Var}(N_t - N_s + N_s|\mathcal{F}_s) + M_s^2 \\
&= \mathrm{Var}(N_t - N_s|\mathcal{F}_s) + \mathrm{Var}(N_s|\mathcal{F}_s) + M_s^2 && \text{indep. increment} \\
&= \mathrm{Var}(N_t - N_s) + M_s^2 && \mathrm{Var}(N_s|\mathcal{F}_s) = 0 \\
&= \mathrm{Var}(N_{t-s}) + M_s^2 = \lambda(t-s) + M_s^2 && N_{t-s} \sim \mathrm{Pois}\left(\lambda(t-s)\right)
\end{aligned}
$$

   Hence,

$$
\begin{aligned}
\mathbb{E}\left[M_t^2 - \lambda t \middle| \mathcal{F}_s\right] &= \mathbb{E}\left[M_t^2 \middle| \mathcal{F}_s\right] - \lambda t \\
&= \lambda(t-s) + M_s^2 - \lambda t \\
&= \boxed{M_s^2 - \lambda s}
\end{aligned}
$$

□

2. Let N_t and B_t be Poisson process and Brownian motion, respectively, relative to the same filtration $\mathcal{F}_t$. Consider the process

$$X_t = e^{\alpha N_t + \beta B_t}$$

where $\alpha, \beta \in \mathbb{R}$. Use Itô's formula to find an SDE for X_t. Show that $m_t = \mathbb{E}[X_t]$ solves

$$\frac{\mathrm{d}m_t}{\mathrm{d}t} - \left[(e^\alpha - 1)\lambda + \frac{\beta^2}{2}\right] m_t = 0$$

Solve the above ODE for m_t and deduce that N_t is independent of B_t.

Solution. Let $Y_t = \alpha N_t + \beta B_t$. Applying *Itô's formula for jump-processes* on $f(x) = e^x$, one has

$$e^{Y_t} = e^{Y_0} + \int_0^t e^{Y_s} \mathrm{d}Y_s^c + \frac{1}{2}\int_0^t e^{Y_s} (\mathrm{d}Y_s^c)^2 + \sum_{0 \le s \le t} \left(e^{Y_s} - e^{Y_{s-}}\right)$$

where $Y_t^c = \beta B_t$, so that $(\mathrm{d}Y_s^c)^2 = \beta^2 \mathrm{d}s$. Also, since $\Delta N_s = N_s - N_{s-} = 1$, one has

$$\begin{aligned} e^{Y_s} - e^{Y_{s-}} &= e^{\alpha(1+N_{s-}) + \beta B_s} - e^{Y_{s-}} \\ &= (e^\alpha - 1) e^{Y_{s-}} && B_{s-} = B_s \\ &= (e^\alpha - 1) e^{Y_{s-}} \Delta N_s \end{aligned}$$

implying that

$$\begin{aligned} \sum_{0 \le s \le t} \left(e^{Y_s} - e^{Y_{s-}}\right) &= \sum_{0 \le s \le t} \left((e^\alpha - 1) e^{Y_{s-}} \Delta N_s\right) \\ &= (e^\alpha - 1) \int_0^t e^{Y_{s-}} \mathrm{d}N_s \end{aligned}$$

Hence, the SDE for X_t is given as

$$\boxed{\mathrm{d}X_t = \beta X_t \mathrm{d}B_t + \frac{1}{2}\beta^2 X_t \mathrm{d}s + (e^\alpha - 1) X_{t-} \mathrm{d}N_t}$$

Note that

$$\begin{aligned} X_t &= e^0 + \beta \int_0^t X_s \mathrm{d}B_s + \frac{\beta^2}{2} \int_0^t X_s \mathrm{d}s + (e^\alpha - 1) \int_0^t X_{s-} \mathrm{d}N_s \\ &= 1 + \beta \int_0^t X_s \mathrm{d}B_s + \frac{\beta^2}{2} \int_0^t X_s \mathrm{d}s + (e^\alpha - 1) \int_0^t X_{s-} (\mathrm{d}N_s - \lambda \mathrm{d}s + \lambda \mathrm{d}s) \\ &= 1 + \beta \int_0^t X_s \mathrm{d}B_s + \frac{\beta^2}{2} \int_0^t X_s \mathrm{d}s + (e^\alpha - 1) \int_0^t X_{s-} \mathrm{d}M_s + \lambda(e^\alpha - 1) \int_0^t X_{s-} \mathrm{d}s \end{aligned}$$

where $M_s = N_s - \lambda s$ is a compensated Poisson process and thus a martingale. Following the theorem that

Proposition 7. *If the integrator M_t is an arbitrary martingale, and the integrand f is bounded, then the integral $\int f \mathrm{d}M_t$ is a martingale*[31]*, and hence*

$$\mathbb{E}\left[\int f \mathrm{d}M\right] = 0,$$

one can take the expectation on both sides:

$$\begin{aligned}
\mathbb{E}\left[X_t\right] &= 1 + \frac{\beta^2}{2}\mathbb{E}\left[\int_0^t X_s \mathrm{d}s\right] + \lambda(e^\alpha - 1)\mathbb{E}\left[\int_0^t X_{s-} \mathrm{d}s\right] \\
&= 1 + \frac{\beta^2}{2}\mathbb{E}\left[\int_0^t X_s \mathrm{d}s\right] + \lambda(e^\alpha - 1)\mathbb{E}\left[\int_0^t X_s \mathrm{d}s\right] \\
&= 1 + \left(\frac{\beta^2}{2} + \lambda(e^\alpha - 1)\right)\mathbb{E}\left[\int_0^t X_s \mathrm{d}s\right] \\
&= 1 + \left(\frac{\beta^2}{2} + \lambda(e^\alpha - 1)\right)\int_0^t \mathbb{E}[X_s] \mathrm{d}s
\end{aligned}$$

Given $m_t = \mathbb{E}[X_t]$, taking derivatives on both sides, one has

$$\frac{\mathrm{d}m_t}{\mathrm{d}t} - \left[\left(e^\alpha - 1\right)\lambda + \frac{\beta^2}{2}\right] m_t = 0$$

Solving this ODE, one has

$$m_t = \exp\left\{\left(\frac{\beta^2}{2} + \lambda(e^\alpha - 1)\right) t\right\}$$

On the other hand, noting that $B_t \sim N(0, t)$ and $N_t \sim \text{Pois}\left(\lambda t\right)$, one has[32]

$$\mathbb{E}\left[e^{\alpha N_t}\right] = e^{\lambda t(e^\alpha - 1)}, \quad \mathbb{E}\left[e^{\beta B_t}\right] = e^{\beta^2 t/2}$$

Hence,

$$m_t = \mathbb{E}\left[e^{\alpha N_t + \beta B_t}\right] = \mathbb{E}\left[e^{\alpha N_t}\right]\mathbb{E}\left[e^{\beta B_t}\right],$$

implying that N_t is independent of B_t[33]. □

[31] See the proof here (theorem 5).
[32] By the moment generating function.
[33] Necessary and sufficient condition on joint MGF for independence.

2.14 Final Practices

1. Let B_t be a standard Brownian motion. Compute

$$\mathbb{E}\left(\int_0^T \sqrt{s}\mathrm{d}B_s\right)^3$$

Solution. The Itô integral for a deterministic square integrable function $f(t)$ follows a normal distribution with mean 0, and if $X \sim N(0, \sigma^2)$, one has $\mathbb{E}\left[X^{2k+1}\right] = 0, k \in \mathbb{N}$. □

2. Using Itô's lemma check if $X_t = e^{B_t^2 - t}$ is a martingale. What is the quadratic variation of process X_t on interval $[0, T]$?

Solution. Consider $u(t, x) = e^{x^2 - t}$. Since

$$\frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial^2 u}{\partial x^2} \neq 0,$$

X_t is not a martingale.

By Itô's formula,

$$X_t = f(0, B_0) + \int_0^T \left[f_t(t, B_t) + \frac{1}{2} f_{xx}(t, B_t)\right] \mathrm{d}t + \int_0^T f_x(t, B_t)\mathrm{d}B_t$$

so that

$$[X_t] = \int_0^T f_x^2(t, B_t)\mathrm{d}t = \boxed{\int_0^T 4B_t^2 e^{2B_t^2 - 2t}\mathrm{d}t}$$

□

3. Let B_t be a standard Brownian motion starting from 0.
   (a) Using PDE approach, find $u(t, x) = \mathbb{E}_{B_t = x} e^{-\gamma B_T^2}$ for given $T > t$ and $\gamma > 0$.
   (b) By the distribution of B_T, calculate directly (without solving PDE) $\mathbb{E}_{B_t = x} e^{-\gamma B_T^2}$.

Solution. (a) Following the *expected value of payoff* formula, we let $u(t, x) = \mathbb{E}_{B_t = x} e^{-\gamma B_T^2}$ which solves

$$u_t + a(t, x)u_x + \frac{1}{2}\sigma^2(t, x)u_{xx} = 0, \forall t < T, \quad u(T, x) = e^{-\gamma x^2}$$

where $a(t, x) = 0, \sigma^2(t, x) = 1$. Hence, we need to solve

$$u_t + \frac{u_{xx}}{2} = 0, \forall t < T, \quad u(T, x) = e^{-\gamma x^2}$$

Guess that the solution is of the form

$$u(t,x) = e^{a(t)x^2+c(t)},$$

so that $u_t = (a'x^2 + c')u$, $u_x = 2axu$, $u_{xx} = 2au + 4a^2x^2u$. Plugging into the previous PDE, one has

$$(2a^2 + a')x^2 + c' + a = 0$$

with the condition $u(T,x) = e^{-\gamma x^2}$, which gives us the solution

$$a(t) = \frac{\gamma}{2(t-T)\gamma - 1}, \quad c(t) = \frac{-\ln\left(2\gamma\left(T-t\right)+1\right)}{2}$$

Hence,

$$u(t,x) = \mathbb{E}_{B_t=x} e^{-\gamma B_T^2} = \boxed{\exp\left\{\frac{\gamma}{2(t-T)\gamma - 1}x^2 - \frac{\ln\left(2\gamma\left(T-t\right)+1\right)}{2}\right\}}$$

(b) Given $B_t = x$, one has

$$\begin{aligned}
&\mathbb{E}_{B_t=x}\left[e^{-\gamma B_T^2}\right] \\
=&\mathbb{E}_{B_t=x}\left[e^{-\gamma\left[(B_T-B_t)^2+2B_t(B_T-B_t)+B_t^2\right]}\right] \\
=&\mathbb{E}_{B_t=x}\left[e^{-\gamma\left[B_{T-t}^2+2B_tB_{T-t}+B_t^2\right]}\right] \\
=&\mathbb{E}\left[e^{-\gamma\left[(T-t)Z^2+2x\sqrt{T-t}Z+x^2\right]}\right] \qquad Z \sim N(0,1) \\
=&e^{-\gamma x^2}\mathbb{E}\left[e^{-\gamma\left[(T-t)Z^2+2x\sqrt{T-t}Z\right]}\right] \\
=&\frac{e^{-\gamma x^2}}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp\left\{-\frac{z^2}{2} - \gamma(T-t)z^2 - 2\gamma x\sqrt{T-t}z\right\}\mathrm{d}z
\end{aligned}$$

We integrate by completing the square. Let $K = \frac{1}{2} + \gamma(T-t)$, then

$$\begin{aligned}
&\int_{-\infty}^{\infty}\exp\left\{-\frac{z^2}{2} - \gamma(T-t)z^2 - 2\gamma x\sqrt{T-t}z\right\}\mathrm{d}z \\
=&\int_{-\infty}^{\infty}\exp\left\{-K\left(z+\frac{\gamma x\sqrt{T-t}}{K}\right)^2 + \frac{\lambda^2x^2(T-t)}{K}\right\}\mathrm{d}z \\
=&e^{\frac{2\lambda^2x^2(T-t)}{1+2\gamma(T-t)}}\int_{-\infty}^{\infty}\exp\left\{-K\left(z+\frac{\gamma x\sqrt{T-t}}{K}\right)^2\right\}\mathrm{d}z \\
=&e^{\frac{2\lambda^2x^2(T-t)}{1+2\gamma(T-t)}}\sqrt{\frac{2\pi}{1+2\gamma(T-t)}}
\end{aligned}$$

Hence,

$$
\begin{aligned}
&\mathbb{E}_{B_t=x}\left[e^{-\gamma B_T^2}\right] \\
&= \frac{e^{-\gamma x^2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left\{-\frac{z^2}{2} - \gamma(T-t)z^2 - 2\gamma x\sqrt{T-t}z\right\} \mathrm{d}z \\
&= \frac{e^{-\gamma x^2}}{\sqrt{2\pi}} e^{\frac{2\lambda^2 x^2 (T-t)}{1+2\gamma(T-t)}} \sqrt{\frac{2\pi}{1+2\gamma(T-t)}} \\
&= e^{-\gamma x^2 + \frac{2\lambda^2 x^2(T-t)}{1+2\gamma(T-t)}} \left(1+2\gamma(T-t)\right)^{-1/2} \\
&= \exp\left\{-\gamma x^2 + \frac{2\lambda^2 x^2 (T-t)}{1+2\gamma(T-t)} - \frac{\ln\left(1+2\gamma(T-t)\right)}{2}\right\} \\
&= \boxed{\exp\left\{\frac{\gamma}{2(t-T)\gamma - 1}x^2 - \frac{\ln\left(2\gamma\left(T-t\right)+1\right)}{2}\right\}}
\end{aligned}
$$

□

4. Consider the SDE:

$$
\mathrm{d}X_t = -\frac{X_t}{2-t}\mathrm{d}t + \sqrt{t(2-t)}\mathrm{d}B_t, \quad X(0) = 0, 0 \le t < 1
$$

Suppose the solution is of the form

$$
X_t = a(t)Y_t, \quad Y_t = \int_0^t b(s)\mathrm{d}B_s
$$

for some smooth functions a, b. Apply Itô's formula to find an ODE satisfied by $a(t)$ and $b(t)$ and solve it.

Solution. Let $f(x) = a(t)x$, so that $X_t = f(Y_t)$. Applying Itô's lemma on X_t, one has

$$
\begin{aligned}
\mathrm{d}X_t &= a'(t)Y_t\mathrm{d}t + a(t)\mathrm{d}Y_t + \frac{1}{2}\cdot 0 \cdot (\mathrm{d}Y_t)^2 \\
&= a'(t)Y_t\mathrm{d}t + a(t)b(t)\mathrm{d}B_t
\end{aligned}
$$

Therefore, one has

$$
\begin{cases}
-\frac{a(t)Y_t}{2-t} = -\frac{X_t}{2-t} = a'(t)Y_t \implies a'(t) = \frac{a(t)}{t-2} \\
a(t)b(t) = \sqrt{t(2-t)},
\end{cases}
$$

which gives us $\boxed{a(t) = t-2, b(t) = -\sqrt{\frac{t}{2-t}}}$ (noting that $0 \le t < 1$). □

5. Let X_t be a one-dimensional stochastic process with drift

$$\mathrm{d}X_t = f(X_t)\mathrm{d}t + \mathrm{d}B_t$$

for some smooth function f with $X_0 = x \in [a,b]$. Let $\tau = \inf\{t \geq 0 : X_t \notin [a,b]\}$. Find $\mathbb{E}_{X_0=x}X_\tau$.

Solution. Note that the maturity T is not given, so that it can be considered as ∞. Also, as $a = f(x), \sigma = 1$ do not depend on t, $u(t,x) = \mathbb{E}_{X_0=x}X_\tau$ is stationary in time. Applying the *boundary value problem and exit times* formula, one has

$$\begin{cases} u_t + \mathcal{L}u = 0 & \implies f(x)u_x + \frac{1}{2}u_{xx} = 0, \\ u(x) = V(x) = x, x \in \partial D & \implies u(a) = a, u(b) = b, \end{cases}$$

which gives us (flexible for the lower bound of the integral, but would be easier to find the constant coefficients later by choosing a)

$$\begin{aligned} u_x &= e^{-2\int_a^x f(y)\mathrm{d}y + C_0}, \\ \implies u(x) &= \int_a^x e^{-2\int_a^y f(z)\mathrm{d}z + C_0}\mathrm{d}y + C_1 \end{aligned}$$

Now, plugging in $u(a) = a, u(b) = b$, one has

$$\begin{cases} u(a) = 0 + C_1 = a & \implies C_1 = a \\ e^{C_0}\int_a^b e^{-2\int_a^y f(z)\mathrm{d}z}\mathrm{d}y + C_1 = b & \implies e^{C_0} = \frac{b-a}{\int_a^b e^{-2\int_a^y f(z)\mathrm{d}z}\mathrm{d}y} \end{cases}$$

Hence,

$$\mathbb{E}_{X_0=x}X_\tau = u(t,x) = u(x) = \boxed{\frac{b-a}{\int_a^b e^{-2\int_a^y f(z)\mathrm{d}z}\mathrm{d}y}\int_a^x e^{-2\int_a^y f(z)\mathrm{d}z}\mathrm{d}y + a}$$

□

6. Let X_t be stochastic process satisfying the SDE

$$\mathrm{d}X_t = \sigma(t)X_t\mathrm{d}B_t$$

with $X_0 = x$ and $\sigma(t)$ some deterministic function.

(a) Use PDE approach to find $\mathbb{E}_{X_t=x}X_T^2$.

(b) Solve the given SDE.

(c) Calculate $\mathbb{E}_{X_t=x}X_T^2$ without sovling the PDE.

Solution. (a) Following the *expected value of payoff* formula, we let $u(t,x) = \mathbb{E}_{X_t=x} X_T^2$ which solves

$$u_t + a(t,x)u_x + \frac{1}{2}\sigma^2(t,x)u_{xx} = 0, \forall t < T, \quad u(T,x) = x^2$$

where $a(t,x) = a(t) = 0, \sigma^2(t,x) = \sigma^2(t)x^2$. Hence, we need to solve

$$u_t + \frac{\sigma^2(t)x^2}{2} u_{xx} = 0, \forall t < T, \quad u(T,x) = x^2$$

Guess that the solution is of the form $u(t,x) = a(t)x^2 + b(t)x + c(t)$, so that

$$\begin{cases} a'(t) + \sigma^2(t)a = 0, \\ b'(t) = 0, \\ c'(t) = 0, \end{cases} \qquad \begin{cases} a(T) = 1, \\ b(T) = c(T) = 0, \end{cases}$$

which gives us the solution $a(t) = e^{\int_t^T \sigma^2(s)\mathrm{d}s}, b(t) = 0, c(t) = 0$. Hence,

$$\mathbb{E}_{X_t=x} X_T^2 = u(t,x) = \boxed{e^{\int_t^T \sigma^2(s)\mathrm{d}s} x^2}$$

(b) Applying Itô's lemma on $\log(X_t)$, one has

$$\begin{aligned} \mathrm{d}\log X_t &= \frac{\mathrm{d}X_t}{X_t} - \frac{(\mathrm{d}X_t)^2}{2X_t^2} \\ &= \frac{\sigma(t)X_t\mathrm{d}B_t}{X_t} - \frac{\sigma^2(t)X_t^2\mathrm{d}t}{2X_t^2} \\ &= \sigma(t)\mathrm{d}B_t - \frac{\sigma^2(t)}{2}\mathrm{d}t \\ \implies \quad \log\frac{X_t}{X_0} &= \int_0^t \sigma(s)\mathrm{d}B_s - \frac{1}{2}\int_0^t \sigma^2(s)\mathrm{d}s \end{aligned}$$

With $X_0 = x$, one has

$$\boxed{X_t = x\exp\left\{\int_0^t \sigma(s)\mathrm{d}B_s - \frac{1}{2}\int_0^t \sigma^2(s)\mathrm{d}s\right\}}$$

(c) Given $X_t = x$, one has

$$\begin{aligned} \mathbb{E}_{X_t=x} X_T^2 &= \mathbb{E}_{X_t=x}\left[X_t^2 \exp\left\{2\int_t^T \sigma(s)\mathrm{d}B_s - \int_t^T \sigma^2(s)\mathrm{d}s\right\}\right] \\ &= x^2 e^{-\int_t^T \sigma^2(s)\mathrm{d}s}\mathbb{E}_{X_t=x}\left[e^{2\int_t^T \sigma(s)\mathrm{d}B_s}\right] \end{aligned}$$

where $2\int_t^T \sigma(s)\mathrm{d}B_s \sim N\left(0, 4\int_t^T \sigma^2(s)\mathrm{d}s\right)$. Following the log-normal distribution, one has

$$\mathbb{E}_{X_t=x}\left[e^{2\int_t^T \sigma(s)\mathrm{d}B_s}\right] = e^{2\int_t^T \sigma^2(s)\mathrm{d}s},$$

implying that

$$\mathbb{E}_{X_t=x}X_T^2 = \boxed{e^{\int_t^T \sigma^2(s)\mathrm{d}s}x^2}$$

□

7. Let $X_t = (T-t)\int_0^t \frac{\mathrm{d}B_s}{T-s}$. Find the quadratic variation of X_t on $[0, T]$.

Solution. Let $Y_t = \int_0^t \frac{\mathrm{d}B_s}{T-s}$, then for $f(t,x) = (T-t)x$, $X_t = f(t, Y_t)$. Applying Itô's lemma on X_t, one has

$$\begin{aligned}
\mathrm{d}X_t &= -Y_t\mathrm{d}t + (T-t)\mathrm{d}Y_t + 0 \\
&= -Y_t\mathrm{d}t + (T-t)\frac{\mathrm{d}B_t}{T-t} \\
&= -Y_t\mathrm{d}t + \mathrm{d}B_t
\end{aligned}$$

Then,

$$[X_t]_T = \int_0^T 1^2\mathrm{d}s = \boxed{T}$$

□

8. Let B_t be a standard Brownian motion and $\gamma(t)$ be a given deterministic function. Compute

$$\mathbb{E}\left[B_t e^{\int_0^t \gamma(s)\mathrm{d}B_s}\right]$$

Solution. We apply the *Girsanov theorem.*

$$\mathbb{E}\left[B_t e^{\int_0^t \gamma(s)\mathrm{d}B_s}\right] = \mathbb{E}\left[B_t e^{\int_0^t \gamma(s)\mathrm{d}B_s - \frac{1}{2}\int_0^t \gamma^2(s)\mathrm{d}s}\right]e^{\frac{1}{2}\int_0^t \gamma^2(s)\mathrm{d}s} = \widetilde{\mathbb{E}}[B_t]e^{\frac{1}{2}\int_0^t \gamma^2(s)\mathrm{d}s}$$

where $\widetilde{B}_t = B_t - \int_0^t \gamma(s)\mathrm{d}s$ is a Brownian motion under the new measure, so that

$$\begin{aligned}
\widetilde{\mathbb{E}}[B_t]e^{\frac{1}{2}\int_0^t \gamma^2(s)\mathrm{d}s} &= \widetilde{\mathbb{E}}\left[\widetilde{B}_t + \int_0^t \gamma(s)\mathrm{d}s\right]e^{\frac{1}{2}\int_0^t \gamma^2(s)\mathrm{d}s} \\
&= \left(\widetilde{\mathbb{E}}\left[\widetilde{B}_t\right] + \int_0^t \gamma(s)\mathrm{d}s\right)e^{\frac{1}{2}\int_0^t \gamma^2(s)\mathrm{d}s} \\
&= \boxed{\int_0^t \gamma(s)\mathrm{d}s \cdot e^{\frac{1}{2}\int_0^t \gamma^2(s)\mathrm{d}s}}
\end{aligned}$$

□

2.15 Final

1. Let X_t be a stochastic process following the SDE

$$\mathrm{d}X_t = -X_t \mathrm{d}t + \sigma \mathrm{d}B_t,$$

where σ is a constant and X_0 is the start point at time $t = 0$. Let X_0 be within the interval $[a, b]$ and $\tau := \min\{t : X_t \notin [a, b]\}$ is the exit time. Find $\mathbb{P}_{X_0}(X_\tau = a)$.

Solution. Let $X_0 = x, \mathbb{E}_{X_0=x}[X_\tau] = u$ and $\mathbb{P}_{X_0}(X_\tau = a) = p$. Noting that X_τ can either be a or b, one has

$$\mathbb{E}_{X_0=x}[X_\tau] = pa + (1-p)b \implies p = \frac{u-b}{a-b}$$

Now, we calculate u following the boundary value problem formula. First note that it is stationary in time, so $u_t = 0$, and one has

$$\begin{cases} -xu_x + \frac{\sigma^2}{2} u_{xx} = 0 \\ u(x) = x, x \in \partial D \end{cases} \implies u(a) = a, u(b) = b$$

Solving the equation, one has

$$u(x) = \sqrt{\pi} c_0 \sigma \mathrm{erfi}\left(\frac{x}{\sigma}\right) + c_2 = c_1 \sigma \mathrm{erfi}\left(\frac{x}{\sigma}\right) + c_2$$

for some constants c_0, c_1, c_2. Plugging in the boundary condition, one has

$$\begin{cases} c_1 \sigma \mathrm{erfi}\left(\frac{a}{\sigma}\right) + c_2 = a \\ c_1 \sigma \mathrm{erfi}\left(\frac{b}{\sigma}\right) + c_2 = b \end{cases}$$

which gives us the solution

$$\boxed{c_1 = \frac{b-a}{\sigma\left(\mathrm{erfi}\left(\frac{b}{\sigma}\right) - \mathrm{erfi}\left(\frac{a}{\sigma}\right)\right)}, \quad c_2 = \frac{a \cdot \mathrm{erfi}\left(\frac{b}{\sigma}\right) - b \cdot \mathrm{erfi}\left(\frac{a}{\sigma}\right)}{\mathrm{erfi}\left(\frac{b}{\sigma}\right) - \mathrm{erfi}\left(\frac{a}{\sigma}\right)}}$$

Hence,

$$\begin{aligned} \mathbb{P}_{X_0}(X_\tau = a) &= \frac{u-b}{a-b} \\ &= \boxed{\frac{\left(c_1 \sigma \mathrm{erfi}\left(\frac{x}{\sigma}\right) + c_2\right) - b}{a-b}} \end{aligned}$$

where c_1, c_2 are calculated above. $\square$

2. Let T be fixed and X_t be a stochastic process defined in Q1. Also, let

$$u(t,x) = \mathbb{E}_{X_t=x} X_T^2$$

(a) Using PDE approach, calculate $u(t,x)$.

(b) Alternatively, solve the SDE given in Q1 and then calculate $\mathbb{E}_{X_t=x} X_T^2$.

Solution. (a) Following the *expected value of payoff* formula, one has

$$u_t + a(t,x)u_x + \frac{1}{2}\sigma^2(t,x)u_{xx} = 0, \forall t < T, \quad u(T,x) = x^2,$$

where $a(t,x) = -x$ and $\sigma(t,x) = \sigma$. Guess that the solution is of the form $u(t,x) = f(t)x^2 + g(t)$. It follows that

$$\begin{cases} f'(t)x^2 + g'(t) - 2x^2 f(t) + \sigma^2 f(t) = 0 & \implies f'(t) = 2f(t), g'(t) = -\sigma^2 f(t), \\ f(T)x^2 + g(T) = x^2 & \implies f(T) = 1, g(T) = 0, \end{cases}$$

which gives us the solution $f(t) = e^{2(t-T)}, g(t) = \sigma^2 \left(1 - e^{2(t-T)}\right)/2$. Hence,

$$\boxed{u(t,x) = e^{2(t-T)}x^2 + \frac{\sigma^2\left(1 - e^{2(t-T)}\right)}{2}}$$

(b) Similar to Q2 from section 2.12, this is a special case of the *Ornstein-Uhlenbeck process* with $\alpha = 1$. Hence, the solution is of the form

$$X_T = e^{t-T}X_t + \sigma e^{-T}\int_t^T e^u \mathrm{d}B_u$$

Given $X_t = x$, one has

$$X_T \sim N\left(xe^{t-T}, \frac{\sigma^2}{2}\left(1 - e^{-2(T-t)}\right)\right),$$

implying that

$$\mathbb{E}_{X_t=x}X_T^2 = \mathrm{Var}_{X_t=x}X_T + \left(\mathbb{E}_{X_t=x}X_T\right)^2 = \boxed{e^{2(t-T)}x^2 + \frac{\sigma^2\left(1 - e^{2(t-T)}\right)}{2}}$$

□

3. Let $N_t^1, N_t^2, \ldots, N_t^k$ be independent Poisson processes with intensities $\lambda_1, \lambda_2, \ldots, \lambda_k$. Show that

$$N_t := \sum_{i=1}^k N_t^i$$

is also a Poisson process and find its intensity.

Proof. WLOG, we consider the case $k = 2$. We show by definition that $N_t = N_t^1 + N_t^2$ is a Poisson process.

(i) $N(0) = N_0^1 + N_0^2 = 0$. (ii) N_t has independent increments (the numbers of occurrences counted in disjoint intervals are independent of each other) since N_t^1 and N_t^2 are independent. (iii) $N(t)$ has a Poisson distribution. This follows from the fact that both N_t^1 and N_t^2 have Poisson distribution with rates λ_1 and λ_2. Then, $N_t^1 + N_t^2 = N_t \sim \text{Pois}\left(\lambda_1 + \lambda_2\right)$[34]. Hence, N_t has intensity $\lambda_1 + \lambda_2$.

Thus, for the more general case, one has N_t is a Poisson process with intensity $\sum_i \lambda_i$. ◻

4. Calculate $\mathbb{E}_{B_0=x}\left[B_t^2 e^{\alpha B_t - \alpha^2 t/2}\right]$ where B_t is a brownian motion and α is a fixed constant.

Solution. Consider $\Theta(u) = \alpha$.

$$\begin{aligned} Z_t &= \exp\left\{\int_0^t \Theta(u)\mathrm{d}B_u - \frac{1}{2}\int_0^t \Theta^2(u)\mathrm{d}u\right\} \\ &= e^{\alpha B_t - \alpha^2 t/2} \end{aligned}$$

Then, by Girsanov theorem, one has

$$\begin{aligned} \mathbb{E}_{B_0=x}\left[B_t^2 e^{\alpha B_t - \alpha^2 t/2}\right] &= \mathbb{E}_{B_0=x}\left[B_t^2 Z_t\right] \\ &= \widetilde{\mathbb{E}}\left[B_t^2\right] && \text{change of measure} \\ &= \widetilde{\mathbb{E}}\left[\left(\widetilde{B}_t + \int_0^t \alpha \mathrm{d}s\right)^2\right] && \text{Girsanov} \\ &= \widetilde{\mathbb{E}}\left[\widetilde{B}_t^2 + 2\alpha t\widetilde{B}_t + \alpha^2 t^2\right] \\ &= \widetilde{\mathbb{E}}\left[\widetilde{B}_t^2\right] + 2\alpha t\widetilde{\mathbb{E}}\left[\widetilde{B}_t\right] + \alpha^2 t^2 \\ &= \boxed{t + x^2 + 2\alpha t x + \alpha^2 t^2} && \widetilde{B}_0 = x \end{aligned}$$

◻

5. Find the stable distribution of X_t as the stochastic process defined in Q1.

Solution. Similar to Q1 from section 2.12, one has

$$\mathcal{L}^* f = 0$$

where $\mathcal{L}^*$ is the dual of $\mathcal{L}$ and f is the density function of the stable distribution of X_t. By

[34] See the proof here.

definition, one has

$$\begin{aligned}\mathcal{L}^* f &= -\frac{\partial(-xf)}{\partial x} + \frac{\sigma^2}{2}\frac{\partial f}{\partial x^2} \\ &= f + x f_x + \frac{\sigma^2}{2} f_{xx} = 0\end{aligned}$$

Guess the solution is of the form $f(x) = c \cdot e^{ax^2+bx}$. Plugging it in, one has

$$f + x(2ax+b)x + \frac{\sigma^2}{2}\left((2ax+b)^2 f + 2af\right) = 0, \quad \int_{\mathbb{R}} f(x)\mathrm{d}x = 1,$$

which gives us the solution $a = -1/\sigma^2, b = 0, c = \frac{1}{\sqrt{\pi\sigma^2}}$. Hence, the stable distribution (density function) is

$$\boxed{f(x) = \frac{1}{\sqrt{\pi\sigma^2}} e^{-\frac{x^2}{\sigma^2}}}$$

$\square$

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