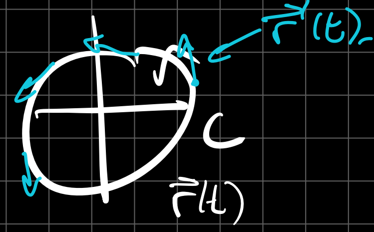
Worksheet : Vector is a flow line or not.



check $\boxed{\vec{F}(\vec{r}(t)) = \vec{r}\,'(t)}$

(3) $x = ar\cos\theta$
$y = br\sin\theta$

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = r^2$$

Jacobian $\begin{vmatrix} x_r & x_\theta \\ y_r & y_\theta \end{vmatrix}$

$$dx\,dy = |x_r y_\theta - x_\theta y_r|\,dr\,d\theta$$

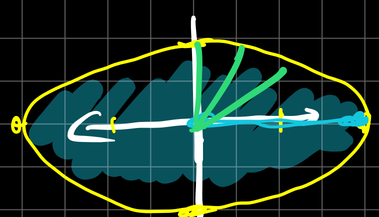
previously if $x = r\cos\theta$
$y = r\sin\theta$ then $x_r y_\theta - x_\theta y_r = \cos\theta\, r\cos\theta - (-r\sin\theta)\sin\theta$
$= r$

$$dx\,dy = |r|\,dr\,d\theta \quad \text{but } r > 0$$
$$= r\,dr\,d\theta.$$

question is what does r do?

$x^2 + y^2 = 1$ $\quad x = r\cos\theta$, $y = r\sin\theta$ $\quad 0 \le r \le 1$

$\frac{x^2}{4} + y^2 = 1$ $\quad x = 2r\cos\theta$, $y = r\sin\theta$



$0 \le r \le 1$ b/c $x = 2r\cos\theta$

if we had 3 variables:

$x(\rho, \theta, \phi) = \rho\sin\phi\cos\theta$
$y(\rho, \theta, \phi) = \rho\sin\phi\sin\theta$
$z(\rho, \theta, \phi) = \rho\cos\phi$

$$J = \begin{vmatrix} x_\rho & x_\theta & x_\phi \\ y_\rho & y_\theta & y_\phi \\ z_\rho & z_\theta & z_\phi \end{vmatrix} = \rho^2\sin\phi$$

Fundamental Thms

$$\int_a^b F'(x)\,dx = F(b)-F(a)$$

endpoints

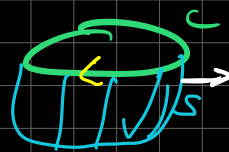
$$\int_C \vec{\nabla} f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$$

$$\iint_D (Q_x - P_y)\,dA = \int_C P\,dx + Q\,dy$$

boundary

$$\iint_S \vec{\nabla}\times\vec{F}\cdot d\vec{S} = \int_C \vec{F}\cdot d\vec{r}$$

Boundary



$$\iiint_E \vec{\nabla}\cdot\vec{F}\,dV = \iint_S \vec{F}\cdot d\vec{S}$$

boundary

Special Scenarios

$\text{curl}\,\vec{F} = \vec{0}$

if $\vec{F} = \vec{\nabla} f$ then $\vec{F}$ is conservative.
and $\text{curl}\,\vec{F} = \vec{0}$

but, if $\text{curl}\,\vec{F} = \vec{0}$, is $\vec{F}$ conservative?

Curl test - if $\text{curl}\,\vec{F} = \vec{0}$

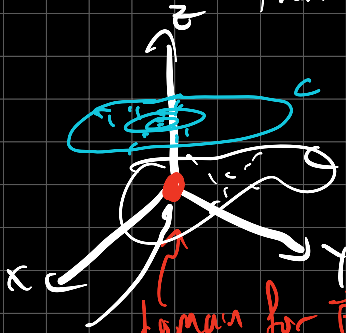
- $\vec{F}$ smooth on 3space
- domain of $\vec{F}$ has property that every

Closed curve can be smoothly contracted to a point while staying in the domain

then $\vec{F}$ is conservative.

$\vec{F} = \vec{\nabla} f$

$\vec{F}$ is cons.

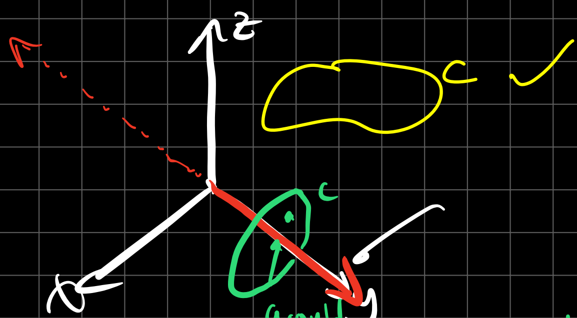


domain for $\vec{F}$ can't have origin.

all closed curves can be contracted to a point in the domain w/o touching the bad point

now let domain of $\vec{F}$ be everything except y-axis and curl $\vec{F} = \vec{0}$

not conservative.



would touch the bad line.

Comment about worksheet

$\vec{F} = \dfrac{\vec{r}}{\|\vec{r}\|^3}$

undefined at origin.

undefined at $\|\vec{r}\| = 0$

$\vec{r} = \langle x, y, z \rangle$

$\Rightarrow \vec{r} = \vec{0}$

curl $\vec{F} \stackrel{?}{=} \vec{0}$ if it is, then ~~it~~ $\vec{F}$ is a conservative vector field

$\vec{F} = \vec{\nabla} f$

why? b/c Every closed curve in the domain of $\vec{F}$ can be contracted to a point while avoiding the origin.

Divergence test:

$\vec{F}$ is smooth in 3 space.

Domain of $\vec{F}$ satisfies:

Every closed surface is the boundary of a solid that is in the domain of $\vec{F}$

$\operatorname{div}\vec{F} = 0$

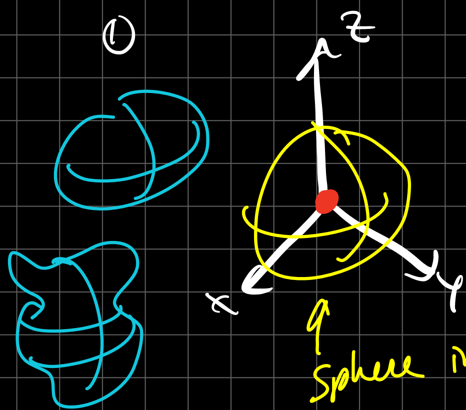
then $\vec{F}$ is a curl field $\Rightarrow \vec{F} = \vec{\nabla} \times \vec{G}$

Recall: if $\vec{F} = \vec{\nabla} \times \vec{G}$ then $\operatorname{div}\vec{F} = 0$

but converse is not true. we have to add

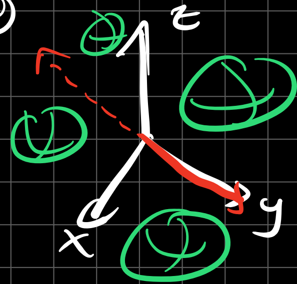
assume for these ex $\operatorname{div}\vec{F} = 0$ and $\vec{F}$ is smooth.

①



sphere in domain
but ball inside
contains the origin
Not in domain!

②



all avoid y-axis.

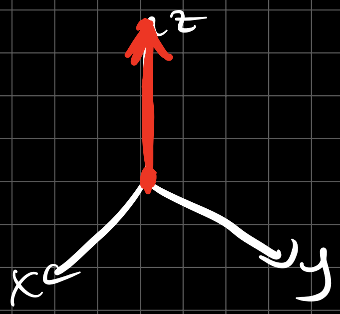
$\vec{F}$ is a curl field.

20.3 in Hughes book:
Do given domains satisfy Curl or div Test?

⑬ all points (x, y, z) such that $x > 0$

curl test ✓
div test ✓

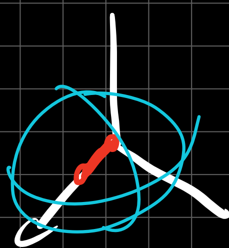
(15) all points (x,y,z) not on positive z-axis.



curl test ✓

div test. ✓

(16) All points (x,y,z) except x-axis with $0 \le x \le 1$



curl test ✓

div test ✗

Point: if $\vec{F}$ is a curl field, $\vec{F} = \vec{\nabla} \times \vec{G}$ it has a vector potential.

if $\vec{F}$ is conservative $\Rightarrow$ gradient field $\Rightarrow \vec{F} = \vec{\nabla} f$

scaler potential.

(26) $\vec{E} = \dfrac{\vec{r}}{\|\vec{r}\|^p} = \dfrac{\langle x,y,z\rangle}{(x^2+y^2+z^2)^{p/2}} = \left\langle \dfrac{x}{(x^2+y^2+z^2)^{p/2}}, \dfrac{y}{(x^2+y^2+z^2)^{p/2}}, \dfrac{z}{(\quad)^{p/2}} \right\rangle$

find curl $\vec{E} = \vec{\nabla} \times \vec{E}$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{x}{(x^2+y^2+z^2)^{p/2}} & \frac{y}{(x^2+y^2+z^2)^{p/2}} & \frac{z}{(x^2+y^2+z^2)^{p/2}} \end{vmatrix}$$

$$= \left[\frac{\partial}{\partial y}\left(\frac{z}{(x^2+y^2+z^2)^{p/2}}\right) - \frac{\partial}{\partial z}\left(\frac{y}{(x^2+y^2+z^2)^{p/2}}\right)\right]\hat{i}$$

$$- \left[\frac{\partial}{\partial x} \frac{z}{(x^2+y^2+z^2)^{p/2}} - \frac{\partial}{\partial z} \frac{x}{(x^2+y^2+z^2)^{p/2}}\right]\hat{j}$$

$\frac{\partial}{\partial x}(x^2+y^2+z^2)^{p/2}$

$= \frac{p}{2}(x^2+y^2+z^2)^{\frac{p}{2}-1} 2x$

$= xp(x^2+y^2+z^2)^{p/2-1}$

$= xp(\|\vec{r}\|^2)^{\frac{p}{2}-1}$

$= xp\|\vec{r}\|^{p-2}$

Error in class These should be $-p-2$

$$+\left[\frac{\partial}{\partial x}\frac{y}{(x^2+y^2+z^2)^{p/2}}-\frac{\partial}{\partial y}\frac{x}{(x^2+y^2+z^2)^{p/2}}\right]\hat{k}$$

$$=\left(zyp\|\vec{r}\|^{p-2}-yzp\|\vec{r}\|^{p-2}\right)\hat{\imath}-\left(zxp\|\vec{r}\|^{p-2}-xzp\|\vec{r}\|^{p-2}\right)\hat{\jmath}$$

$$+\left(yxp\|\vec{r}\|^{p-2}-xyp\|\vec{r}\|^{p-2}\right)\hat{k}$$

$$=\vec{0}.$$

What is domain of $\vec{E}$? $\dfrac{\vec{r}}{\|\vec{r}\|^p}$

$p=0$: $\vec{E}=\vec{r}$ domain $\mathbb{R}^3$

$p<0$ E domain $\mathbb{R}^3$

curl test good.

(b) due to exponent being $-p-2$ we have $-2p<0$ as a range where curl test does not apply.

$p>0$ domain everything except origin.

curl test satisfied here.

$\vec{E}$ is conservative!!

(29) let $\vec{A}$ be a vector potential for $\vec{B}$

show $\vec{A}+\text{grad}\,\psi$ is also a vector potential for $\vec{B}$.

$$\underline{\vec{B}=\vec{\nabla}\times\vec{A}}$$

think:

if $\frac{d}{dx}f=e^x$

then $f=e^x+\underline{\underline{C}}$

b/c $\frac{d}{dx}e^x=e^x$

$\frac{d}{dx}(C)=0.$

show

$$\vec{B}=\vec{\nabla}\times(\vec{A}+\text{grad}\,\psi)$$

$$=\vec{\nabla}\times\vec{A}+\underbrace{\vec{\nabla}\times(\text{grad}\,\psi)}_{\text{curl}(\nabla\psi)=\vec{0}}$$

$$=\vec{\nabla}\times\vec{A}+\vec{0}=\vec{\nabla}\times\vec{A}.$$

What is $\text{div}(A+\text{grad}\,\psi)$? $=\text{div}A+\underbrace{\text{div}(\text{grad}\,\psi)}_{\vec{\nabla}\cdot(\vec{\nabla}\psi)}$

$$\left(\frac{\partial}{\partial x}\hat{i}+\frac{\partial}{\partial y}\hat{j}+\frac{\partial}{\partial z}\hat{k}\right)\cdot\left(\psi_x\hat{i}+\psi_y\hat{j}+\psi_z\hat{k}\right)$$

$$=\underbrace{\psi_{xx}+\psi_{yy}+\psi_{zz}}_{\text{Laplacian of }\psi}$$

$\Delta\psi$

$$\operatorname{div}(\vec{A}+\operatorname{grad}\psi)=\operatorname{div}\vec{A}+\Delta\psi$$

how can we choose ψ so that

$\vec{A}+\operatorname{grad}\psi$ has zero div.

$$\operatorname{div}\vec{A}+\Delta\psi=0 \Rightarrow \Delta\psi=-\operatorname{div}A$$

then $\vec{A}$ is in Coulomb gauge.