15-6 Triple Integrals

$$\iiint_E \underbrace{f(x,y,z)}\, \underbrace{dV}_{dx\,dy\,dz}$$

belongs in 4-space

Integrate over 3-space

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⑥ $$\int_0^1 \int_0^1 \int_0^{\sqrt{1-z^2}} \frac{z}{y+1}\, dx\, dz\, dy$$

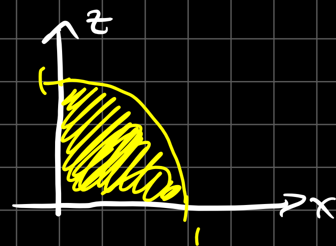
(a) sketch region of integration

$0 \le x \le \sqrt{1-z^2}$

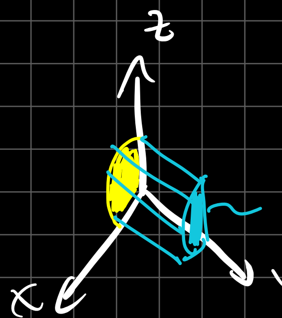
$0 \le z \le 1$

$0 \le y \le 1$

Draw xz plane



$x = \sqrt{1-z^2} \Rightarrow x^2 + z^2 = 1$



quarter of a rod

filled in cylinder

solid cylinder.

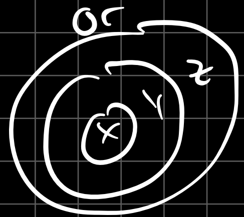
(b) evaluate integral.

$$\int_0^1 \int_0^1 \int_0^{\sqrt{1-z^2}} \frac{z}{y+1}\, dx\, dz\, dy = \int_0^1 \int_0^1 \left(\frac{z}{y+1} x\right)\Bigg|_{x=0}^{x=\sqrt{1-z^2}} dz\, dy$$

x z y

$$= \int_0^1 \int_0^1 \frac{z\sqrt{1-z^2}}{y+1}\, dz\, dy$$

or



$$= \int_0^1 \frac{1}{y+1}\left(\int_0^1 z\sqrt{1-z^2}\,dz\right)dy$$

or

$$= \int_0^1 \frac{1}{y+1}\,dy \int_0^1 z\sqrt{1-z^2}\,dz$$

$u = 1-z^2$
$du = -2z\,dz$

$$= \left[\ln(y+1)\Big|_{y=0}^{y=1}\right]\left(-\frac{1}{2}\right)\int_1^0 \sqrt{u}\,du$$

$$= (\ln 2)\left(\frac{1}{2}\right)\int_0^1 u^{\frac{1}{2}}\,du$$

$$= (\ln 2)\left(\frac{1}{\cancel{2}}\right)\left(\frac{\cancel{2}}{3}\right)\left(u^{3/2}\right)\Big|_{u=0}^{u=1}$$

$$= \frac{\ln 2}{3}$$

(11) $\iiint_E \frac{z}{x^2+z^2}\,dV \qquad E = \{(x,y,z) \mid 1 \le y \le 4,\ y \le z \le 4,\ 0 \le x \le z\}$

$$= \int_1^4 \int_y^4 \int_0^z \frac{z}{x^2+z^2}\,dx\,dz\,dy$$

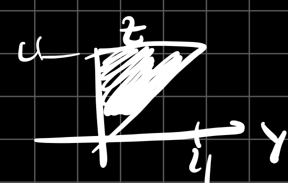
why would this be wrong?

$$\int_1^4 \int_0^z \int_y^4 f(x,y,z)\,dz\,dx\,dy$$

$= g(z)$

to switch
need to draw domain.

$0 \le z \le 4$



back to integral

$0 \le y \le z$

$$= \int_1^4 \int_y^4 \int_0^z \frac{1}{x^2+z^2}\,dx\,dz\,dy$$

$$\frac{d}{dx}\arctan(x) = \frac{1}{x^2+1}$$

$$= \int_1^4 \int_y^4 \frac{z}{z^2} \underbrace{\int_0^z \frac{1}{\left(\frac{x}{z}\right)^2+1}\,dx}\,dz\,dy$$

$u = \frac{x}{z}$

$du = \frac{1}{z}\,dx \Rightarrow dx = z\,du$

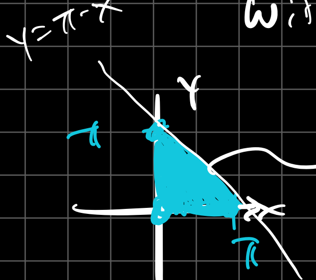
$$= \int_1^4 \int_y^4 \frac{1\,\cancel{z}}{\cancel{z^2}} \int_0^1 \frac{\cancel{z}\,du}{u^2+1}\,dz\,dy$$

$$= \int_1^4 \int_y^4 (1) \underbrace{\int_0^1 \frac{1}{u^2+1}\,du}_{\underbrace{\arctan(1)}_{\frac{\pi}{4}} - \underbrace{\arctan(0)}_{-0}}\,dz\,dy = \int_1^4 \int_y^4 \left(\frac{\pi}{4}\right) dz\,dy = \frac{\pi}{4}\int_1^4 z\Big|_{z=y}^{z=4}\,dy$$

$$= \frac{\pi}{4}\int_1^4 (4-y)\,dy$$

$$= \frac{\pi}{4}\left(4y - \frac{y^2}{2}\right)\Big|_{y=1}^{y=4}$$

(12) $\iiint_E \sin y\,dV$ $\quad E$ below $\boxed{z=x}$ ($z \le x$) and above triangular region with vertices $(0,0,0)$, $(\pi,0,0)$, $(0,\pi,0)$



$0 \le x \le \pi$ $\qquad$ or $\qquad$ $0 \le y \le \pi$

$0 \le y \le -x+\pi$ $\qquad\qquad$ $0 \le x \le \pi - y$

$0 \le z \le x$ $\qquad\qquad$ $0 \le z \le x$

$$\iiint_E \sin y \, dV = \int_0^{\pi}\int_0^{\pi-x}\int_0^{x} \sin y \, dz\,dy\,dx = \int_0^{\pi}\int_0^{\pi-y}\int_0^{x} \sin y \, dz\,dx\,dy$$

Choose one to integrate.

(22) Find volume of solid enclosed by $x^2+z^2=4$ and $y=-1$ lower

$y+z=4$

$y=4-z$ upper limit.

Note:

$$\int_a^b 1\,dx = \ell(a,b)$$

$$\iint_D 1\,dA = A(D)$$

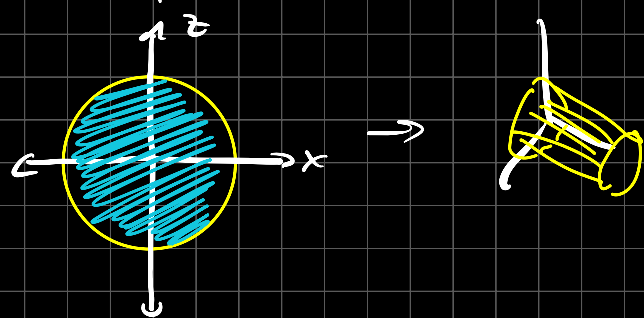
$$\iiint_E 1\,dV = V(E)$$

$$\iint\cdots\int_{\Omega} 1\,dx_1\,dx_2\cdots dx_n = n \text{ dim volume of } \Omega$$

Describe region and set up!

$$\iiint_E 1\,dV$$

xz plane



$-1 \le y \le 4-z$

$-2 \le x \le 2$

$-\sqrt{4-x^2} \le z \le \sqrt{4-x^2}$

$$\text{Volume} = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{-1}^{4-z} 1\,dy\,dz\,dx$$

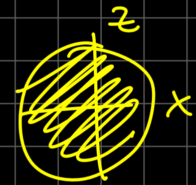
$$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} y\Big|_{y=-1}^{y=4-z} dz\,dx$$

$$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (4-z+1)\,dz\,dx$$

$$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (5-z)\,dz\,dx$$

$x = r\cos\theta$

$z = r\sin\theta$



$$= \int_0^{2\pi}\int_0^{2} (5-r\sin\theta)\,r\,dr\,d\theta$$

x^2+y^2 or x^2+z^2

$0 \le r \le 2$

$0 \le \theta \le 2\pi$

$5-z = 5-r\sin\theta$

$dz\,dx = r\,dr\,d\theta$

$$= \int_0^{2\pi} \left(\frac{5r^2}{2} - \frac{r^3}{3}\sin\theta\right)\Big|_{r=0}^{r=2} d\theta$$

$$= \int_0^{2\pi} \left(10 - \frac{8}{3}\sin\theta\right) d\theta$$

$$= \left[10\theta + \frac{8}{3}\cos\theta\right]\Big|_{\theta=0}^{\theta=2\pi} = 20\pi + \frac{8}{3}\cos 2\pi - 0 - \frac{8}{3}\cos 0$$

$$= 20\pi.$$

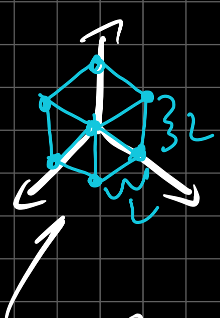
53 $f(x,y,z) = xyz$

Domain solid cube with side $= L$

in first octant

edges parallel to coordinate axes

find avg. value of f.



Volume $= L^3$

$$\frac{\iiint_E f\,dV}{\iiint_E dV} = \frac{\iiint_E f\,dV}{V(E)}$$

$0 \le x \le L$

$0 \le y \le L$

$0 \le z \le L$

$$f_{avg} = \frac{1}{L^3}\int_0^L\int_0^L\int_0^L xyz\,dx\,dy\,dz$$

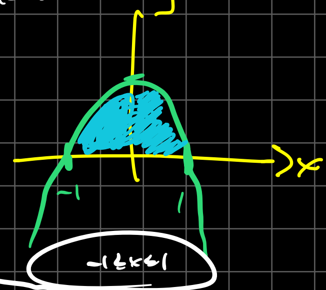
$$= \frac{1}{L^3}\left(\int_0^L x\,dx\right)^3 = \frac{1}{L^3}\left(\frac{x^2}{2}\Big|_{x=0}^{x=L}\right)^3 = \frac{\frac{L^6}{2}}{L^3} = \frac{L^3}{2}$$

55 $\iiint_E 1 - x^2 - 2y^2 - 3z^2\,dU$

find region E for which the integral is a maximum.

Simplify: $\int_I 1 - x^2\,dx$ find I where this is a max.

pick domain that ensures $f(x) \geq 0$



$1 - x^2 \geq 0$

$1 \geq x^2$

$-1 \leq x \leq 1$

repeat: $\iint_D (1 - x^2 - 2y^2)\,dA$

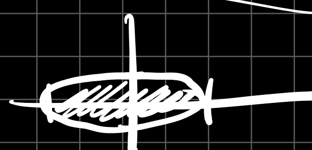
$1 - x^2 - 2y^2 \geq 0$

$x^2 + 2y^2 \leq 1$

elliptic disk.

$-1 \leq x \leq 1$

$-\sqrt{\frac{1-x^2}{2}} \leq y \leq \sqrt{\frac{1-x^2}{2}}$



$\iiint_E 1 - x^2 - 2y^2 - 3z^2\,dU$

$1-x^2-2y^2-3z^2 \geq 0 \Rightarrow x^2+2y^2+3z^2 \leq 1$

elliptic ball.

find relationship b/w x, y, z

$-1 \leq x \leq 1$

$-\sqrt{\frac{1-x^2}{2}} \leq y \leq \sqrt{\frac{1-x^2}{2}}$

$-\sqrt{\frac{1-x^2-2y^2}{3}} \leq z \leq \sqrt{\frac{1-x^2-2y^2}{3}}$