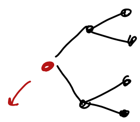
Lecture 10

Trees and searching: Recall: Tree $\iff$ cctd & ① All bridges
② $\exists!$ paths
③ No circuits
④ $V - E = 1$

def a root is a distinguished vertex:

Eg: 

Root.

$\rightsquigarrow$ Outward flow assigns direction

$d_{in} = 1$, $d_{out} = n-1$ $\boxed{d_{in} = 0 \iff v = r}$

leaf: $d_{out} = 0$, internal: $d_{out} \neq 0$.

def If $d_{out} \equiv m$ then m-ary tree   binary for case $m = 2$.

$n = \ell + i$

Prop for m-ary Tree, $n = mi + 1$, $i = |\text{internal verts}|$.

Pf Each vertex is the child of unique parent so $n = mi + \underline{\text{root}}$. ∎

Eg 56 people sign up for a bracket: How many matches?

Ans: $n = ?$, $m = 2$, $\ell = 56$, $i = ?$ $\Rightarrow$ $\ell + i = 2i + 1$

$\ell = i + 1 \Rightarrow \ell - 1 = \boxed{i = 55.}$

def height of tree: length of longest path from root.

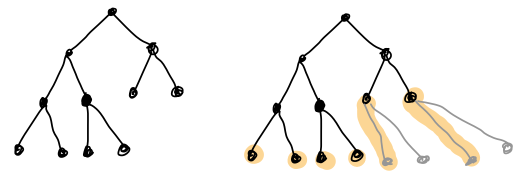
def balanced iff all leaves at height h or $h-1$. } accomodate m-ary.

Thm T an m-ary tree w/ height h and ℓ leaves.

(a) $\ell \leq m^h$ (if all leaves at height $h \Rightarrow \ell = m^h$)

(b) $h \geq \log_m(\ell)$ If balanced then $h = \lceil \log_m(\ell) \rceil$

Pf: At each height there are $\le m^h$ verts. and if the tree is complete then $l = m^h$. Otoh If not then complete it and total leaves $\le m^h$ by flowing down.



$l \le m^h$. (a) ✓

Take log both sides, then balanced $\Rightarrow$ $m^{h-1} \le l \le m^h \Rightarrow$
$$h-1 \le \log_m l \le h$$
$$\Rightarrow h = \lceil \log_m l \rceil.$$

Eg. How many binary tests to find a number? $\in \{1, \dots, N\}$.

Soln Searching gives binary tree w/ N verts (could stop at internal vertex!)

$$\Rightarrow \begin{matrix} 2i+1 = n \\ i + l = n \end{matrix} \Rightarrow \frac{n-1}{2} = n - l \Rightarrow n-1 = 2n - 2l$$

$$\Rightarrow 2l = n+1 \Rightarrow l = \frac{n+1}{2} \Rightarrow h = \lceil \log_2(n+1) - 1 \rceil = \lfloor \log_2(n+1) \rfloor.$$

Qn How many trees are there on n labeled vertices?

(Meaning $i \mapsto i$ does not give an isom)

Thm Cayley: n^{n-2} trees on n-labels.

Pf n-lab trees $\longleftrightarrow$ sequences of length $n-1$.

Idea: Append to the sequence the parent of lowest labeled leaf then delete the leaf. Stop at 2 leaves.

Note length = #labels − 2.

Reconstruction: Note Leaves do not appear.

Smallest missing # is adj to 1^{st} term. Do $(6,2,2,3,3,3) \rightsquigarrow n=8$

$\cancel{1}, \cancel{2}, 3, \cancel{4}, \cancel{5}, \cancel{6}, \cancel{7}, 8$