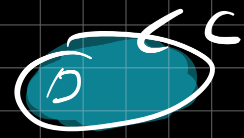
Fundamental Theorems

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$ FTC

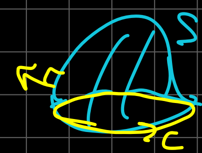
$$\int_C \vec{\nabla} f \cdot d\vec{r} = \underline{f(\vec{r}(b)) - f(\vec{r}(a))}$$ FT Line integrals

$$\iint_D Q_x - P_y \, dA = \int_C P\,dx + Q\,dy$$



G.T.

$$\iint_S (\vec{\nabla} \times \vec{F}) \cdot d\vec{S} = \int_C \vec{F} \cdot d\vec{r}$$



Stoke's Thm.

$$\iiint_E \vec{\nabla} \cdot \vec{F} \, dV = \iint_S \vec{F} \cdot d\vec{S}$$ div Thm.

Integral of a deriv over Region = Evaluating Function over Boundary of no deriv.

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look at §20.3 in Hughes

Note: if $\vec{F} = \vec{\nabla} f$ then $\text{curl}\,\vec{F} = \vec{0}$

Converse is Not necessarily true.

Curl test

- $\vec{F}$ smooth on $\mathbb{R}^3$
- $\text{curl}\,\vec{F} = \vec{0}$
- Domain of $\vec{F}$ to satisfy: if every closed curve in the domain of $\vec{F}$ can be smoothly contracted

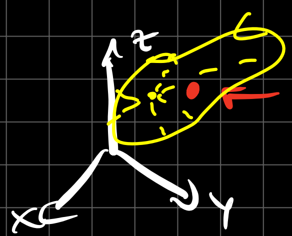
to a point while staying inside the domain of $\vec{F}$

then $\vec{F}$ is conservative.

$\Rightarrow \vec{F} = \vec{\nabla} f$

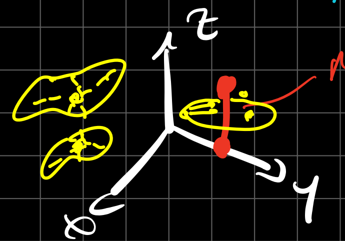
$\quad\quad\quad$ ↳ Scalar potential.

Ex)



Satisfies domain condition of curl test.

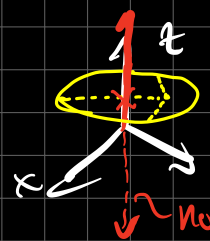
Ex)



we can slide above or under segment.

Satisfies domain condition of curl test.

Ex)



Can not contract w/o hitting z-axis for any curve that wraps around z-axis.

domain does not satisfy curl test.

Divergence test

Recall: if $\vec{F} = \vec{\nabla} \times \vec{G}$ then $\operatorname{div} \vec{F} = 0$

but converse not necess. true.

need condition on domain.

div Test: if $\vec{F}$ smooth

$\operatorname{div} \vec{F} = 0$

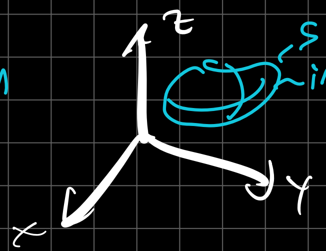
Domain of $\vec{F}$ satisfies:

every closed surface in domain of $\vec{F}$ only contains points in the domain of $\vec{F}$

in the interior.

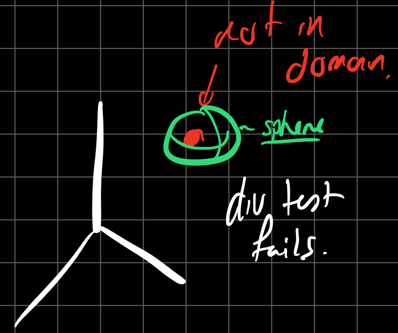
then $\vec{F} = \vec{\nabla} \times \vec{G}$ ← vector Potential.

Domain $\mathbb{R}^3$



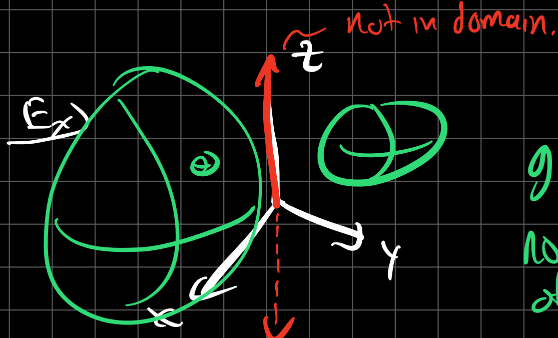
interior contains only points in domain.

Ex)



div test fails.

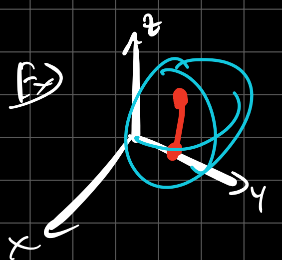
Ex)



good!!

no "bad" points are inside of any closed surface.

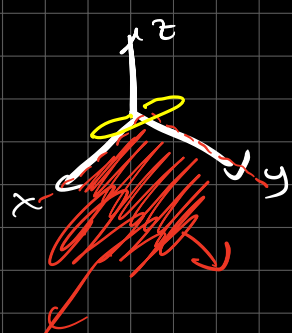
Ex)



Surface contains "bad" points on the inside.

$\vec{F}$ not curl field.

Ex)

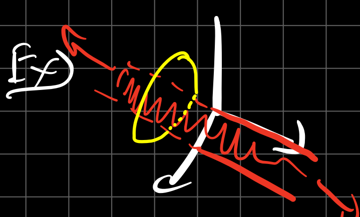


Domain everything except part of xy plane when $x, y > 0$

curl test ✓

div test ✓

Ex)



curl test : ✗

div test : ✓

⑥ $\vec{E} = \frac{\vec{r}}{\|\vec{r}\|^p}$ domain: $p \le 0$ domain $\mathbb{R}^3$

$p > 0$ domain $\mathbb{R}^3 \setminus$ origin.

$\mathbb{R}^3 \setminus \{(0,0,0)\}$.

$\vec{r} = \langle x, y, z \rangle$

$$\text{Curl } \vec{E} = \text{curl } \frac{\langle x, y, z\rangle}{(x^2+y^2+z^2)^{p/2}}$$

$$= \Big\langle \underbrace{\frac{x}{(x^2+y^2+z^2)^{p/2}}}_{P}, \underbrace{\frac{y}{(x^2+y^2+z^2)^{p/2}}}_{Q}, \underbrace{\frac{z}{(x^2+y^2+z^2)^{p/2}}}_{R} \Big\rangle.$$

when computing curl, we will need to find:

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = (R_y - Q_z)\hat{i} - (R_x - P_z)\hat{j} + (Q_x - P_y)\hat{k}$$

$$R_x = \frac{\partial}{\partial x} \frac{z}{(x^2+y^2+z^2)^{p/2}} = z \frac{\partial}{\partial x} (x^2+y^2+z^2)^{-p/2}$$

$$= z\left(-\frac{p}{2}\right)(x^2+y^2+z^2)^{-\frac{p}{2}-1}(2x)$$

$$= -xzp(x^2+y^2+z^2)^{-\frac{p}{2}-1}$$

$$= -xzp(\|\vec{r}\|^2)^{-\frac{p}{2}-1}$$

$$= -zxp\|\vec{r}\|^{-p-2}$$

$$\text{curl } \vec{F} = \left(-zyp\|\vec{r}\|^{-p-2} - (-yz)p\|\vec{r}\|^{-p-2}\right)\hat{i}$$

$$- (\underbrace{-zx - (-zx)}_{0})p\|\vec{r}\|^{-p-2}\hat{j} + (\underbrace{-xy + xy}_{0})p\|\vec{r}\|^{-p-2}\hat{k}$$

$$\|\vec{r}\|^{-p-2} = \frac{1}{\|\vec{r}\|^{p+2}}$$

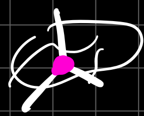
if $-p-2 < 0$

$p > -2$

then we have an issue.

E is defined at origin but curl is not.

Does $\vec{E}$ satisfy curl test?

$p \le -2$	$-2 < p \le 0$	$p > 0$
Domain $\mathbb{R}^3$	Domain $\mathbb{R}^3$	$\text{curl}\,\vec{F} = \vec{0}$
$\text{curl}\,\vec{E} = \vec{0}$	$\text{curl}\,\vec{E}$ undefined at origin.	
curl test ✓	Fails	curl test ✓

$\vec{E}$ has potential

when $p=3$ ≈ gravitational force b/w 2 objects

we can find f s.t. $\boxed{\vec{E} = \vec{\nabla} f}$

worksheet: f yields $\vec{F} = -GMm \dfrac{\vec{r}}{\|\vec{r}\|^3}$

$f = \dfrac{GMm}{r}$ $\quad r = \|\vec{r}\|$

gravitational force b/w two objects.

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29. $\vec{A}$ is a vector potential for $\vec{B}$

$$\vec{B} = \text{curl}\,\vec{A} = \vec{\nabla} \times \vec{A}$$

Show that $\vec{A} + \text{grad}\,\psi$ is also a vector potential for $\vec{B}$.

$$\text{curl}\left(\vec{A} + \text{grad}\,\psi\right)$$

$$= \vec{\nabla} \times \left(\vec{A} + \text{grad}\,\psi\right)$$

$$= \underbrace{\vec{\nabla} \times \vec{A}} + \vec{\nabla} \times \text{grad}\,\psi$$

$$= \vec{B} + \underbrace{\text{curl}\left(\vec{\nabla}\psi\right)}_{0} = \vec{B}.$$

$$\frac{d}{dx}\left(f(x) + C\right) = \frac{d}{dx}\left(f(x)\right)$$

What is divergence of $\vec{A} + \text{grad}\,\psi$?

$$\vec{\nabla} \cdot \left(\vec{A} + \text{grad}\,\psi\right) = \underbrace{\vec{\nabla} \cdot \vec{A}}_{\text{div}\vec{A}} + \vec{\nabla} \cdot \text{grad}\,\psi$$

$$\text{div}\vec{A} +$$

$$\vec{\nabla} \cdot (\vec{\nabla}\psi) = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) \cdot \left(\psi_x\hat{i} + \psi_y\hat{j} + \psi_z\hat{k}\right)$$

$$= \underbrace{\psi_{xx} + \psi_{yy} + \psi_{zz}}_{\text{Laplacian of } \psi} = \Delta\psi = \nabla^2\psi$$

$$\text{div}\left(\vec{A} + \text{grad}\,\psi\right) = \text{div}A + \Delta\psi$$

What can pick ψ to be so $\vec{A} + \text{grad}\,\psi$ has 0 divergence?

$\text{div}A + \Delta\psi = 0$ choose ψ s.t.

$$\underbrace{\Delta\psi = -\text{div}A}_{\text{Poisson's equation.}}$$

if $\operatorname{div} \vec{A} = 0$ then we want $\Delta \psi = 0$

ψ to be harmonic.

vector potential $\vec{A}$ is said to be in Coulomb gauge.