5. The Riemann Integral

5.1 The Riemann integral

A partition P of an interval $[a, b]$ is a finite set of real numbers $\{x_0, x_1, ..., x_n\}$ such that $a = x_0 < x_1 < ... < x_n = b$, we write $\Delta x_1 = x_i - x_{i-1}$.

Let $f : [a, b] \to \mathbb{R}$ be a bounded function, and P be a partition of $[a, b]$. Define

$m_i = \inf\{f(x) : x \in [f_{i-1}, f_i]\}$

$M_i = \sup\{f(x) : x \in [f_{i-1}, f_i]\}$

lower Darboux sum $L(P, f) = \sum_{i=1}^{n} m_i \Delta x_i$

upper Darboux sum $U(P, f) = \sum_{i=1}^{n} M_i \Delta x_i$

Prop. (Darboux sums are bounded) Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Let $m, M \in \mathbb{R}$ be such that $m \le f(x) \le M$ for all $x \in [a, b]$. Then for every partition P of $[a, b]$, we have $m(b - a) \le L(P, f) \le U(P, f) \le M(b - a)$.

Pf. $m(b - a) = m(\sum_{i=1}^{n} \Delta x_i) = \sum_{i=1}^{n} m\Delta x_i \le \sum_{i=1}^{n} m_i \Delta x_i \le \sum_{i=1}^{n} M_i \Delta x_i$

As the sets of lower and upper Darboux sums are bounded, we define

$\underline{\int_a^b} f(x)dx = \sup\{L(P, f) : P \text{ is a partition of } [a, b]\}$

$\overline{\int_a^b} f(x)dx = \inf\{U(P, f) : P \text{ is a partition of } [a, b]\}$

Let $P = \{x_0, x_1, ..., x_n\}$, $\tilde{P} = \{\tilde{x_0}, \tilde{x_1}, ..., \tilde{x_n}\}$ be partition of $[a, b]$. We set $\tilde{P}$ is a refinement of P if as sets $P \subset \tilde{P}$.

Example: $P = \{0, \frac{1}{2}, 1\}$, $\tilde{P} = \{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\}$

Then $L(\tilde{P}, f) \ge L(P, f)$ and $U(\tilde{P}, f) \le U(P, f)$.

Pf. $x_0 = \tilde{x}_0$, $x_n = \tilde{x}_l$, $x_j = \tilde{x_{k_j}}$, $j = 0, 1, ..., n$

$$L(P, f) = \sum_{j=1}^{n} m_j \Delta x_j \le \sum_{j=1}^{n} \sum_{p=k_{j-1}+1}^{k_j} \tilde{m}_p \Delta \tilde{x}_j = \sum_{j=1}^{l} \tilde{m}_j \Delta \tilde{x}_j = L(\tilde{P}, f)$$

Let $f : [a, b] \to \mathbb{R}$ be a bounded function. If $\underline{\int_a^b} f(x)dx = \overline{\int_a^b} f(x)dx$, we say f is Riemann integrable. We denote the set of Riemann integrable functions on $[a, b]$ as $\mathcal{R}(a, b)$.

If $f \in \mathcal{R}$, then $\int_a^b f(x)dx := \underline{\int_a^b} f(x)dx = \overline{\int_a^b} f(x)dx$. We call this the Riemann integral of f.

Prop. Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Then f is Riemann integrable if for every $\epsilon > 0$, there exists a partition P of $[a, b]$ such that $U(P, f) - L(P, f) < \epsilon$.

Pf. $0 \le \overline{\int_a^b} f(x)dx - \underline{\int_a^b} f(x)dx \le U(P, f) - L(P, f) < \epsilon \to \underline{\int_a^b} f(x)dx = \overline{\int_a^b} f(x)dx$

5.2 Properties of the integral

Additivity.

Lemma. (Additivity of Darboux sum) Suppose $a < b < c$ and $f : [a, b] \to \mathbb{R}$ is a bounded function. Then $\underline{\int_a^c} f(x)dx = \underline{\int_a^b} f(x)dx + \underline{\int_b^c} f(x)dx$ and $\overline{\int_a^c} f(x)dx = \overline{\int_a^b} f(x)dx + \overline{\int_b^c} f(x)dx$.

Pf. $\underline{\int_a^c} f(x)dx = \sup\{L(P, f) : P \text{ is a partition of } [\text{a,c}]\}$
$= \sup\{L(P, f) : P \text{ is a partition of } [\text{a,c}], b \in P\}$
$= \sup\{L(P_1, f) + L(P_2, F) : P_1 \text{ is a partition of } [\text{a,b}], P_2 \text{ is a partition of } [\text{b,c}]\}$
$= \sup\{L(P_1, f) : P_1 \text{ is a partition of } [\text{a,b}]\} + \sup\{L(P_1, f) : P_2 \text{ is a partition of } [\text{b,c}]\}$
$= \underline{\int_a^b} f(x)dx + \underline{\int_b^c} f(x)dx$

Prop. Let $a < b < c$. A function $f : [a, b] \to \mathbb{R}$ is Riemann integrable $\iff$ f is Riemann integrable on $[a, b]$ and $[b, c]$. If f Riemann integrable, then $\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx$.

Cor. If $f \in \mathcal{R}[a, b]$ and $[c, d] \subset [a, b]$, then the restriction $f|_{[c,d]}$ is in $\mathcal{R}[c, d]$.

Linearity.

Prop. Let f and g in $\mathcal{R}[a, b]$ and $\alpha \in \mathbb{R}$. Then

1. αf is in $\mathcal{R}[a, b]$ and $\int_a^b \alpha f(x)dx = \alpha \int_a^b f(x)dx$
2. $f + g$ is in $\mathcal{R}[a, b]$ and $\int_a^b (f(x) + g(x))dx = \int_a^b f(x)dx + \int_a^b g(x)dx$

Monotonicity.

Prop. Let $f, g : [a, b] \to \mathbb{R}$ be bounded, and $f(x) \le g(x)$ for all $x \in [a, b]$. Then

$\underline{\int_a^b} f(x)dx \le \underline{\int_a^b} g(x)dx$ and $\overline{\int_a^b} f(x)dx \le \overline{\int_a^b} g(x)dx$

Furthermore, if $f, g \in \mathcal{R}[a, b]$, then $\int_a^b f = \int_a^b g$.

Refined forms of continuity

Def. Let $S \subset \mathbb{R}$. $f : S \to \mathbb{R}$.

We say f is uniformly continuous if for all $\epsilon > 0$, there exists $\delta > 0$ such that for all $x, y \in S$ with $|x - y| < \delta$,

$|f(x) - f(y)| < \epsilon$

f is Lipschitz continous if there exists $K \in \mathbb{R}$ such that for all $x, y \in S$

$|f(x) - f(y)| \le K|x - y|$

We call K a Lipschitz constant.

Hierarchy of continuity.

For $c \in S$,

f differentiable at c → f continuous at c

For an interval $I \subset \mathbb{R}$. $f : I \to \mathbb{R}$.

differentiable + bounded derivative → Lipschitz continuous → uniformly continuous → continuous

For a closed and bounded interval $f : [a, b] \to \mathbb{R}$

continuous derivative → bounded derivative uniformly continuous $\iff$ continuous

Prop. uniformly continuous → continuous

Pf. Let $c \in S$, $\epsilon > 0$ be arbitrary

uniformly continuous → $\exists \delta > 0$: $\forall x, y \in S$ with $|x - y| < \delta$, $|f(x) - f(y)| < \delta$

Take $y = c$. $\forall x \in S$ with $|x - c| < \delta$, $|f(x) - f(c)| < \delta$

Claim: $f : (0, 1) \to \mathbb{R}$, $f(x) = \frac{1}{x}$ is continuous but not uniformly continuous

Prop. $f : [a, b] \to \mathbb{R}$, continous → uniformly continuous

Prop. f differentiable + f' bounded → Lipschitz continuous

Claim: $f : [-1, 1] \to \mathbb{R}$, $f(x) = |x|$ is Lipschitz continuous but not differentiable

Prop. Lipschitz continuous → uniformly continuous

Pf. Lipschitz continuous → $\exists K \in \mathbb{R}$: $\forall x, y \in S$, $|f(x) - f(y)| \leq K|x - y|$

Let $\epsilon > 0$ be arbitrary. Take $\delta = \frac{\epsilon}{K}$. $\forall x, y \in S$ with $|x - y| < \delta$, $|f(x) - f(y)| \leq K|x - y| < K \cdot \delta = K \cdot \frac{\epsilon}{K} = \epsilon$

Claim: $f : [0, 1] \to \mathbb{R}$, $f(x) = \sqrt{x}$ is uniformly continuous but not Lipschitz continuous.

Lemma. If $f : [a, b] \to \mathbb{R}$ is continuous, then it is Riemann integrable.

Pf. $f : [a, b] \to \mathbb{R}$, continous → uniformly continuous

Let $\epsilon > 0$ be arbitrary. $\exists \delta > 0$: $\forall x, y \in [a, b]$ with $|x - y| < \delta$, $|f(x) - f(y)| < \frac{\epsilon}{b-a}$

$$\begin{aligned}
&\overline{\int_a^b} f - \underline{\int_a^b} f \leq U(P, f) - L(P, f) \\
&= \left(\sum_{i=1}^n M_i \Delta x_i\right) - \left(\sum_{i=1}^n m_i \Delta x_i\right) \\
&= \sum_{i=1}^n (M_i - m_i) \Delta x_i \\
&< \frac{\epsilon}{b-a} \sum_{i=1}^n \Delta x_i \\
&= \frac{\epsilon}{b-a}(b - a) = \epsilon
\end{aligned}$$

$\overline{\int_a^b} f = \underline{\int_a^b} f$ → Riemann integrable

5.3 Fundamental theorem of calculus

First form of the fundamental theorem of calculus. Let $F : [a, b] \to \mathbb{R}$ be a continuous function, differentiable on (a, b). Let $f \in \mathcal{R}[a, b]$ be such that $f(x) = F'(x)$, $\forall x \in (a, b)$. Then $\int_a^b f = F(b) - F(a)$.

Pf. Let $P = \{x_0, ..., x_n\}$ be an arbitrary partition of $[a, b]$. For each inerval $[x_{i-1}, x_i]$, by MVT,

$\exists c_i$ s.t. $F(x_i) - F(x_{i-1}) = F'(c_i)(x_i - x_{i-1})$.

$m_i \leq f(c_i) \leq M_i \to m_i \Delta x_i \leq F(x_i) - F(x_{i-1}) \leq M_i \Delta x_i \to L(P, f) \leq F(b) - F(a) \leq U(P, f) \to \underline{\int_a^b} f \leq F(b) - F(a) \leq \overline{\int_a^b} f \to \int_a^b f = F(b) - F(a)$

Second form of the fundamental theorem of calculus. Let $f \in \mathcal{R}[a,b]$. Define

$F(x) = \int_a^x f(x)dx$.

Then

1. F is Lipschitz continuous on $[a,b]$.
2. If f is continous at $c \in [a,b]$, then F is differentiable at c, $F'(c) = f(c)$.

Pf. Since $f \in \mathcal{R}[a,b]$, it is bounded, then $\exists M > 0$: $\forall x \in [a,b]$, $|f(x)| \leq M$

Suppose $x, y \in [a,b]$ with $x > y$, then $|F(x) - F(y)| = |\int_a^x f(x)dx - \int_a^y f(x)dx| = |\int_x^y f(x)dx| \leq M|x-y| \to$ Lipschitz continuous

Suppose f is continous at c,

$\forall \epsilon > 0$, $\exists \delta > 0$: $\forall x \in [a,b]$ with $|x-c| < \delta$, $|f(x) - f(c)| < \epsilon$

$\to f(c) - \epsilon < f(x) < f(c) + \epsilon$

If $x > c$, $(f(c) - \epsilon)(x-c) < \int_c^x f(x) < (f(c)+\epsilon)(x-c)$; If $x < c$, the inequalities are reversed. Therefore, if $x \neq c$, $f(c) - \epsilon \leq \frac{\int_c^x f(x)}{x-c} \leq f(c) + \epsilon$

since $\frac{F(x)-F(c)}{x-c} = \frac{\int_a^x f(x) - \int_a^c f(x)}{x-c} = \frac{\int_c^x f(x)}{x-c} \to |\frac{F(x)-F(c)}{x-c} - f(c)| \leq \epsilon \to F'(c) = f(c)$