1. (5 points) G is a group. H is a subgroup of G and N is a normal subgroup of G. H acts on G/N by left multiplication, i.e., for any $h \in H$, $gN \in G/N$,

$$h.gN = (hg)N$$

Prove that the above action is transitive if and only if $G = HN$.

Solution:

If the action is transitive, for any $g \in G$, $gN \in O(N)$, so there exists $h \in H$ such that $hN = h.N = gN$, in particular $g \in hN \subseteq HN$, we conclude $G = HN$.

Conversely, if $G = HN$, for any $gN \in G/N$, there exists $h \in H, n \in N$ such that $g = hn \in hN$, so $gN = hN = h.N$, which means $gN \in O(N)$, we conclude $O(N) = G/N$, the action is transitive.

2. M_2 is the group of isometries on $\mathbb{R}^2$. Let H be the subset of M_2 consisting of all translations and all rotations (around any point).

(i). (5 points) Prove that H is a subgroup of M_2

(ii). (5 points) If T is the group of translations on $\mathbb{R}^2$ and R is the group of rotations around origin on $\mathbb{R}^2$, prove that $H = T \rtimes R$.

(iii). (5 points) What is $[M_2 : H]$, the index of H in M_2? Prove your answer.

Solution:

Method I:

(i). H consists of all isometries of the form $t_{\vec{a}}\rho_\theta$. For any $t_{\vec{a}}\rho_\alpha, t_{\vec{b}}\rho_\beta \in H$,

$$(t_{\vec{a}}\rho_\alpha)^{-1}(t_{\vec{b}}\rho_\beta) = \rho_{-\alpha}t_{\vec{b}-\vec{a}}\rho_\beta = t_{\rho_{-\alpha}(\vec{b}-\vec{a})}\rho_{-\alpha+\beta} \in H$$

(ii). T is a normal subgroup in M_2, so it is also a subgroup in H. Also we have $T \cap R = \{1\}$ and $H = TR$, so $H = T \rtimes R$.

(iii). Every element of M_2 is of the form $t_{\vec{a}}\rho_\theta$ or $t_{\vec{a}}\rho_\theta r$, and $r \notin H$, so there are two distinct right cosets H and Hr, $[M_2 : H] = 2$.

Method II for (i),(iii):

In Homework, we have proved

$$\Psi : M_2 \longrightarrow \{\pm 1\}$$

$$t_{\vec{a}}\rho_\theta r^k \mapsto (-1)^k$$

is a well-defined homomorphism.

$\ker(\Psi) = H$, so H is a subgroup of M_2, and $[M_2 : H] = |\operatorname{Im}(\Psi)| = 2$

3. (5 points) G is a group. $Z(G) = \{g \in G | \forall x \in G, gx = xg\}$. If $[G : Z(G)] = k$, prove that each conjugacy class of G has at most k elements.

   Solution: Let G act on itself by conjugation. For any $g \in G$, its conjugacy class C_x is its orbit, and its normalizer $N(x)$ is its stablizer. Note that $Z(G) \subseteq N(x)$, so

$$[G : N(x)] \leq [G : Z(G)] = k$$

   Apply the counting formula,

$$|C_x| = [G : N(x)] \leq k$$

4. (5 points) G is a finite group, p is a prime and p divides $|G|$. N is a normal subgroup of G and P is a Sylow p-subgroup of G. Prove that PN/N is a Sylow p-subgroup of G/N.

   Solution: Let $|G| = p^e m$, where m is relatively prime to p. $|P| = p^e$.

   First,
$$|PN/N| = \frac{|PN|}{|N|} = \frac{|P|}{|P \cap N|} = \frac{p^e}{|P \cap N|}$$
   so $|PN/N|$ is a p-subgroup of G/N.

   Next, $\frac{|G/N|}{|PN/N|} = \frac{|G|}{|PN|}$, which divides $\frac{|G|}{|P|} = m$, and m is relatively prime to p, so $|PN/N|$ has the same number of p-factor as $|G/N|$, which implies PN/N is a Sylow p-subgroup of G/N.

5. (5 points) Let $\mathbb{Z}$ be the group of integers with addition as composition.
   $G = \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$. $\phi : A_3 \longrightarrow \mathrm{Aut}(G)$ is the homomorphism defined by

$$\phi_\sigma : G \longrightarrow G$$

$$(a_1, a_2, a_3) \mapsto (a_{\sigma(1)}, a_{\sigma(2)}, a_{\sigma(3)})$$

   Find all the elements of finite order in $G \rtimes_\phi A_3$.

   Solution:

   $(a_1, a_2, a_3, id)^n = (na_1, na_2, na_3, id)$, which cannot be $(0, 0, 0, id)$ for any $n \neq 0$ unless $a_1 = a_2 = a_3 = 0$.

   Let $\sigma = (1\ 2\ 3)$, $|\sigma| = 3$, so $(a_1, a_2, a_3, \sigma)^n$ cannot be $(0, 0, 0, id)$ when n is not a multiple of 3.

   When $n = 3k$:

$$\begin{aligned}
(a_1, a_2, a_3, \sigma)^3 &= (a_1, a_2, a_3, \sigma)^2 (a_1, a_2, a_3, \sigma) \\
&= ((a_1, a_2, a_3) + \phi_\sigma(a_1, a_2, a_3), \sigma^1)(a_1, a_2, a_3, \sigma) \\
&= ((a_1, a_2, a_3) + (a_2, a_3, a_1), \sigma^2)(a_1, a_2, a_3, \sigma) \\
&= (a_1 + a_2, a_2 + a_3, a_3 + a_1, \sigma^2)(a_1, a_2, a_3, \sigma) \\
&= ((a_1 + a_2, a_2 + a_3, a_3 + a_1) + \phi_{\sigma^2}(a_1, a_2, a_3), \sigma^3) \\
&= (a_1 + a_2 + a_3, a_1 + a_2 + a_3, a_1 + a_2 + a_3, id)
\end{aligned}$$

   So $(a_1, a_2, a_3, \sigma)^{3k} = (k(a_1 + a_2 + a_3), k(a_1 + a_2 + a_3), k(a_1 + a_2 + a_3), id)$, this cannot be the identity for nonzero k unless $a_1 + a_2 + a_3 = 0$.

   We can have similar argument for $\sigma^{-1} = (1\ 3\ 2)$.

   In summary, the elements of finite order are $(0, 0, 0, id)$ and all elements of the form $(a_1, a_2, a_3, (1\ 2\ 3))$ or $(a_1, a_2, a_3, (1\ 3\ 2))$ for $a_1 + a_2 + a_3 = 0$.

6. (5 points) Let $I = (x^2 + 2) \subseteq \mathbb{R}[x]$. Find the multiplicative inverse of

$$2x + 1 + I \in \mathbb{R}[x]/I$$

Solution: Let the inverse be $ax + b + I \in \mathbb{R}[x]/I$.

$1+I = (2x+1+I)(ax+b+I) = 2ax^2+(a+2b)x+b+I = 2a(-2)+(a+2b)x+b+I$

so $\begin{cases} a + 2b = 0 \\ -4a + b = 1 \end{cases} \implies \begin{cases} a = -\frac{2}{9} \\ b = \frac{1}{9} \end{cases} \implies ax + b + I = -\frac{2}{9}x + \frac{1}{9} + I$

7. (5 points) R is a ring. Prove that $I = (x)$ is a maximal ideal of $R[x]$ if and only if R is a field.

   Solution:

   Method I: Let $\phi : R[x] \longrightarrow R$ be the evaluation map $\phi(p(x)) = p(0)$. It is a surjective homomrophism, with $\ker(\phi) = \{p(x) \in R[x] | p(0) = 0\} = (x)$

   First Isomorphism Theorem implies

   $$R[x]/(x) \cong R$$

   (x) is a maximal ideal $\iff$ $R[x]/(x)$ is a field $\iff$ R is a field.

   Method II:

   If R is a field, then $(p(x))$ is a maximal ideal of $R[x]$ if and only if $p(x)$ is irreducible. x is irreducible, so (x) is a maximal ideal.

   Conversely, If $I = (x)$ is a maximal ideal, suppose R is not a field, then there exists $a \neq 0$ in R that is not a unit. Let $J = \{f(x) \in R[x] | f(0) \in aR\}$. Since $x \in J$, $(x) \subseteq J$. $a \in J$ and $a \notin (x)$, so $(x) \neq J$. a is not a unit, so $1 \notin aR$, $J \neq R[x]$. We get

   $$(x) \underset{\neq}{\subsetneq} J \underset{\neq}{\subsetneq} R[x]$$

   contradict to (x) maximal ideal. We conclude R is a field.

8. (5 points) Classify groups of order 45 up to isomorphism.

   Solution:

   If G is a group of order $45 = 3^2 \times 5$, it has Sylow 3-group and Sylow 5-group.

   Let n_p be the number of Sylow p-subgroups of G, then

$$\begin{cases} n_3 | 5 \\ n_3 \equiv 1 (\mod 3) \end{cases} \implies n_3 = 1$$

$$\begin{cases} n_5 | 3^2 \\ n_5 \equiv 1 (\mod 5) \end{cases} \implies n_5 = 1$$

   This implies the unique Sylow 3-subgroup H and the unique Sylow 5-subgroup K are normal.

   $|H| = 9$ and $|K| = 5$ are relatively prime, so $H \cap K = \{1\}$.

   $|HK| = \frac{|H| \times |K|}{|H \cap K|} = 45 = |G|$, so $G = HK$.

   We get $G = H \times K$. $|H| = 3^2$, so $H \cong \mathbb{Z}/9\mathbb{Z}$ or $H \cong \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$. $|K| = 5$, so $K \cong \mathbb{Z}/5\mathbb{Z}$.

   We conclude $G \cong \mathbb{Z}/9\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z}$ or $G \cong \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z}$.