Homework 11

Due: Friday Dec. 3, by 11:59pm,
via Gradescope

- Failure to submit homework correctly will result in a zero on the homework.
- Homework must be in LaTeX. Submit the pdf file to Gradescope.
- Problems assigned from the textbook come from the 5th edition.
- No late homework accepted. Lateness due to technical issues will not be excused.

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1. (30 points) Section 6.3 # 2, 22, 28.

   Solution:

   # 2. Let $A = \{1, 2\}$, $B = 2, 3$ and let $U = 1, 2, 3, 4$ be the universe. Then $A^c \cup B^c = \{1, 3, 4\}$ while $(A \cup B)^c = \{4\}$.

   # 22(a) $\exists$ set S, $\forall$ sets T, $S \cap T \neq \emptyset$.

   # 22(b) $\forall$ sets S, $\exists$ set T, $S \cup T \neq \emptyset$.

   # 28. (a) Set difference laws. (b) Set difference laws. (c) Commutative law. (d) DeMorgan's law. (e) Double complement law. (f) Distribution law. (g) Set difference laws.

2. (9 points) Section 6.3 # 33, 38, 43. Annotate!!!

   Solution:

   # 33. **Proof:**

$$
\begin{aligned}
(A - B) \cap (A \cap B) &= (A \cap B^c) \cap (A \cap B) \ (set\ difference\ laws) \\
&= (A \cap B^c) \cap Z \quad \text{I've set } Z = A \cap B \\
&= A \cap (B^c \cap Z) \quad (Associative) \\
&= A \cap (Z \cap B^c) \quad (Commutative) \\
&= A \cap ((A \cap B) \cap B^c) \quad (substitution) \\
&= A \cap (A \cap (B \cap B^c)) \quad (Associative) \\
&= A \cap (A \cap \emptyset) \quad (Complement\ law) \\
&= A \cap \emptyset \quad (Identity\ law) \\
&= \emptyset \quad (Identity\ law)
\end{aligned}
$$

$\square$

\# 38. **Proof:**

$$\begin{aligned}(A \cap B)^c \cap A &= (A^c \cup B^c) \cap A \ (DeMorgan's \quad law) \\ &= A \cap (A^c \cup B^c) \quad (Commutative\ law) \\ &= (A \cap A^c) \cup (A \cap B^c) \quad (Distributive\ law) \\ &= \emptyset \cup (A \cap B^c \quad (Complement\ law) \\ &= (A \cap B^c) \cup \emptyset \quad (Commutative\ law) \\ &= A \cap B^c \quad (Identity\ law) \\ &= A - B \quad (Set\ difference\ law)\end{aligned}$$

□

\# 43.

$$\begin{aligned}&(A \cap (B \cup C)) \cap (A - B)) \cap (B \cup C^c) \\ &= (A \cap (B \cup C)) \cap (A \cap B^c)) \cap (B \cup C^c) \quad (Set\ difference\ laws) \\ &= (A \cap A \cap (B \cup C) \cap B^c) \cap (B \cup C^c) \quad (Commutative\ law) \\ &= (A \cap (B \cup C) \cap B^c) \cap (B \cup C^c) \quad (Idempotent\ law) \\ &= (A \cap [(B^c \cap B) \cup (B^c \cap C)]) \cap (B \cup C^c) \quad (Distribution\ law) \\ &= (A \cap [\emptyset \cup (B^c \cap C)]) \cap (B \cup C^c) \quad (Complement\ law) \\ &= (A \cap (B^c \cap C)) \cap (B \cup C^c) \quad (Identity\ law) \\ &= A \cap B^c \cap (C \cap (B \cup C^c)) \quad (Associative\ law) \\ &= A \cap B^c \cap ((C \cap B) \cup (C \cap C^c)) \quad (Distributive\ law) \\ &= A \cap B^c \cap ((C \cap B) \cup \emptyset) \quad (Complement\ law) \\ &= A \cap B^c \cap (C \cap B) \quad (Identity\ law) \\ &= A \cap B^c \cap (B \cap C) \quad (Commutative\ law) \\ &= A \cap (B^c \cap B) \cap C \quad (Associative\ law) \\ &= A \cap \emptyset \cap C \quad (Complement\ law) \\ &= \emptyset \quad (Identity\ law)\end{aligned}$$

3. (15 points) Section 6.3 # 46, 52. Use Theorem 6.2.2. when doing problem 52. Annotate as well.

Solution:

\# 46. (a) $\{1, 2, 5, 6\}$. (b) $\{3, 4, 7, 8\}$. (c) $\{1, 2, 3, 4, 5, 6, 7, 8\}$. (d) $\{1, 2, 7, 8\}$.

\# 52. **Proof:** I owe you a proof using Theorem 6.2.2. I was getting a mess. Even the element method is not very pretty. However, the element method is what I'll give. Lots of applications of DeMorgan's. I will prove just one half of the proof. Namely, I'll prove $(A \triangle B) \triangle C \subseteq A \triangle (B \triangle C)$.

Remark. *I may have gone about this a round about way.*

Proof: Assume that $x \in (A \triangle B) \triangle C$. Then $x \in (A \triangle B) - C$ or $x \in C - (A \triangle B)$.

Case 1: Let's assume that $x \in (A \triangle B) - C$. Then ($x \in A - B$ or $x \in B - A$) and $x \in C^c$. By DeMorgan's, this implies that ($x \in A - B$ and $x \in C^c$) or ($x \in B - A$ and $x \in C^c$).

Case 1': $x \in A - B$ and $x \notin C$. Then $x \in A$ and $x \notin B$ and $x \notin C$. Since $x \notin B$ then $x \notin (B - C)$. Since $x \notin C$ then $x \notin (C - B)$. Therefore $x \notin B \triangle C$. Therefore $x \in A - (B \triangle C)$. It follows that $x \in A \triangle (B \triangle C)$.

Case 1": $x \in (B - A)$ and $x \notin C$. Then $x \in B$ and $x \notin A$ and $x \notin C$. Since $x \in B$ and $x \notin C$ we have $x \in B - C$. Therefore $x \in B \triangle C$. Since $x \notin A$ we conclude that $x \in (B \triangle C) - A$. That is, $x \in A \triangle (B \triangle C)$.

Case 2: Let's assume that $x \in C - (A \triangle B)$. Then $x \in C$ and $x \notin (A \triangle B)$. Since $x \notin (A \triangle B)$ it follows that $x \notin (A - B)$ and $x \notin (B - A)$ by DeMorgan's. Since $x \notin (A - B)$ it follows that $x \notin A$ or $x \in B$. Similarly, $x \notin (B - A)$ implies that $x \notin B$ or $x \in A$. Therefore $x \in C$ and ($x \notin A$ or $x \in B$) and ($x \notin B$ or $x \in A$).

Suppose that $x \notin A$. Then $x \notin A$ or $x \in B$ implies $x \in B$. However, $x \notin B$ or $x \in A$ implies that $x \in A$. Contradiction. Therefore $x \in A$.

Since $x \in A$, then $x \notin A$ or $x \in B$ implies that $x \in B$.

Therefore $x \in A$ and $x \in B$ and $x \in C$. Since $x \in B$ and $x \in C$ it follows that $x \notin (B - C)$ or $x \notin (C - B)$, i.e. $x \notin B \triangle C$. Therefore $x \in A - (B \triangle C)$. That is, $x \in A \triangle (B \triangle C)$ $\square$

4. (12 points) Section 7.1 # 4, 14.

Solution:

4(a) .

$$
\begin{aligned}
f_1 &= \{(a,u),(b,u)\} \\
f_2 &= \{(a,u),(b,v)\} \\
f_3 &= \{(a,v),(b,u)\} \\
f_4 &= \{(a,v),(b,v)\}
\end{aligned}
$$

4(b). $f = \{(a,u),(b,u),(c,u)\}$.

4(c).

$$
\begin{aligned}
f_1 &= \{(a,u),(b,u),(c,u)\} \\
f_2 &= \{(a,v),(b,v),(c,v)\} \\
f_3 &= \{(a,u),(b,u),(c,v)\} \\
f_4 &= \{(a,u),(b,v),(c,u)\} \\
f_5 &= \{(a,v),(b,u),(c,u)\} \\
f_6 &= \{(a,v),(b,v),(c,u)\} \\
f_7 &= \{(a,v),(b,u),(c,v)\} \\
f_8 &= \{(a,u),(b,v),(c,v)\}
\end{aligned}
$$

14. No. $H(0) = 1$ while $K(0) = 0$.

5. (15 points) Section 7.1 # 22, 25, 27

   Solution:

   # 22. **Proof by Contradiction:** Assume that $\log_3 7$ is a rational number. Then there exists integers a and b, $b \neq 0$ such that $\log_3 7 = \frac{a}{b}$. Now $a/b > 0$ so let's assume that both a and b are positive. Let's also assume that a/b is in simplest form. Then $3^{a/b} = 7$. It follows that $3^a = 7^b$. It follows that the prime factorization of 3^a is 7^a. However, the prime factorization of 3^a is 3^a. This contradicts the fact that the prime factorization is unique.

   Now what if the integers a and b are both negative. Just set $x/y = -a/-b$. Then $3^x = 7^y$ and again the contradiction comes from the uniqueness of the prime factorization. $\square$

   Therefore $3^{a/b} = 7 \to 3^a = 7^b$.

   # 25(a). $p_1(2, y) = 2$ and $p_2(5, x) = 5$. The range of p_1 is A.

   # 25 (b). $p_1(2, y) = y$ and $p_2(5, x) = x$. The range of p_2 is B.

   # 27(a). $f(aba) = 0,\ f(bbab) = 2,\ f(b) = 0.$

   # 27(b). $g(aba) = aba,\ g(bbab) = babb,\ g(b) = b.$

6. (9 points) Section 7.1 # 32(b), 42, 44, 45.

   Solution:

   # 42. This is true. Let's prove it.

   Proof: Let $y \in F(A)$. Then there exists an $x \in A$ such that $f(x) = y$. Since $A \subseteq B$ we have that $x \in B$. Therefore $f(x) = y$ for $x \in B$. That is, $y \in B$ $\square$

   # 44. This is false. Let $A = \{1\}$ and $B = \{2\}$ and $f = \{(1, a), (2, a)\}$. Now $f(A - B) = f(A) = \{a\}$. Note that $f(B) = \{a\}$ so that $f(A) - f(B) = \emptyset$.

   # 45. This is true. Let's prove it.

   Proof: Let $x \in f^{-1}(C)$. Then $f(x) \in C$. Since $C \subseteq D$ it follows that $f(x) \in D$. Therefore $x \in f^{-1}(D)$ $\square$

7. (6 points) Section 7.2 # 25.

   # 25. Yes, C is 1-1. No, C is not onto. Let's first prove 1-1.

   Proof: Assume that $C(s) = C(s')$. Then $as = as'$. Note that strings are ordered collection of objects. A string of size n is denoted as an n-tuple $x_1, x_2, \ldots, x_n)$ where x_i is 0 or 1. Two n-tuples $(x_1, x_2, \ldots, x_n)$, $(y_1, y_2, \ldots, y_n)$ are equal to one another if and only if $x_i = y_i$ for $i = 1, 2, \ldots n$. It follows that $s = s'$ $\square$

   About onto. Note that $s = a$ does not belong to the image of C. This is because the strings in the image of C are at least of length 2.

Remark. *A string of a's and b's of length k is simply an element of* $\{a, b\}^k$. *If S is the set of all strings of a's and b's, then*

$$S = \bigcup_{k=1}^{\infty} \{a, b\}^k$$

Regarding # 25 in section 7.2. The string as *where* $s \in S$ *is simply appending the string* s *by* a *on the left (see page 20). Therefore, if* $s = (a, b)$. *Then* $as = (a, a, b)$. *If* $s = (b)$, *then* $as = (a, b)$.