Arclength (13.3)

$\vec{r}(t)$ paramet. a curve in $\mathbb{R}^3$

$a \le t \le b$

$$\int_a^b \|\vec{r}\,'(t)\|\,dt. = \text{length of curve.}$$

Arc length function:

$$S(t) = \int_a^t \|\vec{r}\,'(u)\|\,du$$

NOT OK

$$S(t) = \int_a^t \|\vec{r}\,'(t)\|\,dt$$

$$S(1) = \int_a^1 \|\vec{r}\,'(1)\|\,d(1)$$

③ $\vec{r}(t) = \sqrt{2}\,t\,\hat{\imath} + e^{t}\hat{\jmath} + e^{-t}\hat{k}$

find arclength from $t=0$ to $t=1$.

$\vec{r}\,'(t) = \sqrt{2}\,\hat{\imath} + e^{t}\hat{\jmath} - e^{-t}\hat{k}$

$$\|\vec{r}\,'(t)\| = \sqrt{(\sqrt{2})^2 + (e^{t})^2 + (-e^{-t})^2} = \sqrt{2 + e^{2t} + e^{-2t}}$$

$$\cosh t = \frac{e^{t} + e^{-t}}{2} \Longleftrightarrow \sinh(t) = \frac{e^{t} - e^{-t}}{2}$$

$x^2-y^2=1$

$$\sqrt{2+e^{2t}+e^{-2t}} = \underbrace{\sqrt{2+2\cosh(2t)}}_{2\cosh(t)}$$ - nice identity we can use.

OR

$$2\underbrace{e^{t}e^{-t}}_{1}+\underbrace{e^{2t}}_{(e^{t})^2}+\underbrace{e^{-2t}}_{(e^{-t})^2} = (e^{t})^2+2e^{t}e^{-t}+(e^{-t})^2$$

$$=(e^{t}+e^{-t})^2$$

note: $2\cosh t$

then,

$$\|\vec{r}\,'(t)\| = \sqrt{(e^{t}+e^{-t})^2} = e^{t}+e^{-t}$$

$$\int_0^1 \|\vec{r}\,'(t)\|dt = \int_0^1 (e^{t}+e^{-t})dt \quad \text{or} \quad \int_0^1 2\cosh(t)dt$$

$$=(e^{t}-e^{-t})\Big|_{t=0}^{t=1} \qquad = 2\sinh(t)\Big|_0^1$$

$$=e^{1}-e^{-1} \qquad = 2\sinh(1)$$

5 $\vec{r}(t) = \hat{\imath}+t^2\hat{\jmath}+t^3\hat{k} \qquad 0\le t\le 1$

find arc length.

$\vec{r}'(t) = 2t\hat{j} + 3t^2\hat{k}$

$\|\vec{r}'(t)\| = \sqrt{4t^2+9t^4} = t\sqrt{4+9t^2}$

$\int_0^1 t\sqrt{4+9t^2}\,dt = \frac{1}{18}\int_4^{13} \sqrt{u}\,du = \frac{1}{\cancel{18}\,9}\left(\frac{\cancel{2}}{3}\right) u^{3/2}\Big|_{u=4}^{u=13}$

$u = 4+9t^2$

$du = 18t\,dt$

$\frac{1}{18}du = t\,dt$

$= \frac{1}{27}\left(13^{3/2} - 4^{3/2}\right)$

(11) $x^2 = 2y$ & $3z = xy$ have an intersection C

find Length of C from origin to $(6, 18, 36)$.

let $x = t$ $y = \frac{t^2}{2}$ $z = \frac{t^3}{6}$.

$\vec{r}(t) = \langle t, \frac{t^2}{2}, \frac{t^3}{6} \rangle$

$\vec{r}(6) = \langle 6, \frac{36}{2}, \frac{6^3}{6} \rangle = \langle 6, 18, 36 \rangle$

$\vec{r}(0) = \langle 0, 0, 0 \rangle$

$\vec{r}'(t) = \langle 1, t, \frac{t^2}{2} \rangle$, $\|\vec{r}'(t)\| = \sqrt{1+t^2+\frac{t^4}{4}}$

$$\int_0^6 \sqrt{1+t^2+\frac{t^4}{4}}\,dt = \int_0^6 \sqrt{\left(\frac{t^2}{2}+1\right)^2}\,dt = \int_0^6 \frac{t^2}{2}+1\,dt$$

$\vdots$

(14) $\vec{r}(t) = e^t \sin t\,\hat{\imath} + e^t \cos t\,\hat{\jmath} + \sqrt{2}e^t\,\hat{k}$ $\quad P(0, 1, \sqrt{2})$.

(a) find arclength function in direction of increasing t.

need starting t-value $\quad t=0$

$$\vec{r}'(t) = (e^t\cos t + e^t \sin t)\hat{\imath} + (-e^t \sin t + e^t \cos t)\hat{\jmath} + \sqrt{2}e^t\hat{k}$$

$$\|\vec{r}'(t)\| = \sqrt{\left(e^{2t}\cos^2 t + 2e^{2t}\cos t\sin t + e^{2t}\sin^2 t + e^{2t}\cos^2 t - 2e^{2t}\cos t \sin t + e^{2t}\sin^2 t + 2e^{2t}\right)}$$

$$= \sqrt{2e^{2t}\cos^2 t + 2e^{2t}\sin^2 t + 2e^{2t}}$$

$$= \sqrt{4e^{2t}} = 2e^t$$

$$\underline{s(t)} = \int_0^t 2e^u\,du = 2e^u\Big|_{u=0}^{u=t} = \underline{2e^t - 2}$$

(b) reparameterize $\vec{r}(t)$ with respect to arc length.

$$S = 2e^t - 2 \rightarrow \frac{S+2}{2} = \frac{S}{2} + 1 = e^t$$

$$t = \ln\left(\frac{S}{2} + 1\right).$$

$$e^{\ln\left(\frac{S}{2}+1\right)} = \frac{S}{2} + 1$$

Pos vector based on time.

$$\vec{r}(t) = e^t \sin t\,\hat{\imath} + e^t \cos t\,\hat{\jmath} + \sqrt{2}\,e^t\,\hat{k}$$

$$\vec{r}(s) = \left(\frac{s}{2}+1\right)\sin\left(\ln\left(\frac{s}{2}+1\right)\right)\hat{\imath} + \left(\frac{s}{2}+1\right)\cos\left(\ln\left(\frac{s}{2}+1\right)\right)\hat{\jmath} + \sqrt{2}\left(\frac{s}{2}+1\right)\hat{k}$$

position based on distance travelled.

(18) $\vec{r}(t) = \langle t^2, \sin t - t\cos t, \cos t + t\sin t \rangle$, $t > 0$.

(a) find $\vec{T}$, $\vec{N} = \frac{T'}{\|T'\|}$

unit Tangent

unit Normal

$$\vec{r}\,'(t) = \langle 2t, t\sin t, t\cos t \rangle$$

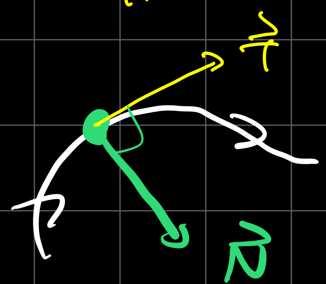
$$\|\vec{r}\,'(t)\| = \sqrt{4t^2 + t^2\sin^2 t + t^2\cos^2 t} = \sqrt{5t^2} = \sqrt{5}\,t$$

$$\vec{T} = \frac{\vec{r}\,'}{\|\vec{r}\,'\|} = \left\langle \frac{2}{\sqrt{5}}, \frac{\sin t}{\sqrt{5}}, \frac{\cos t}{\sqrt{5}} \right\rangle$$

$$\vec{T}' = \left\langle 0, \frac{\cos t}{\sqrt{5}}, \frac{-\sin t}{\sqrt{5}} \right\rangle$$

$$\|\vec{T}'\| = \sqrt{\frac{\cos^2 t}{5} + \frac{\sin^2 t}{5}} = \sqrt{\frac{1}{5}} = \frac{1}{\sqrt{5}}$$

$$\vec{N} = \frac{\vec{T}'}{\|\vec{T}'\|} = \langle 0, \cos t, -\sin t \rangle$$



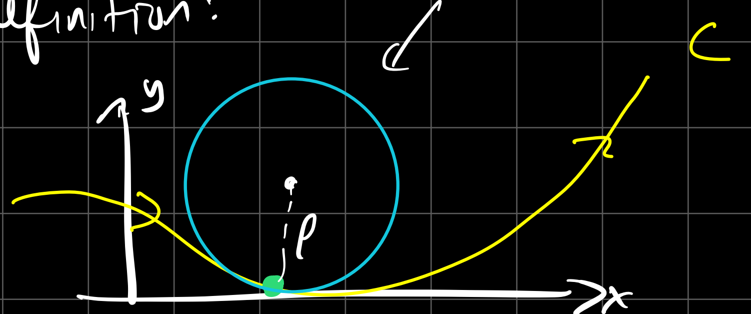
(b) find the curvature, $\kappa(t)$.

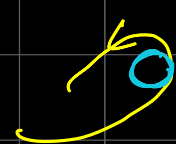
$$\kappa(t) = \left\| \frac{d\vec{T}}{ds} \right\| = \frac{\|\vec{T}'(t)\|}{\|\vec{r}'(t)\|} = \frac{\|\vec{r}'(t) \times r''(t)\|}{\|\vec{r}'(t)\|^3}$$

$$\kappa = \frac{\frac{1}{\sqrt{5}}}{\sqrt{5}\, t} = \frac{1}{5t}$$

Actual definition:

osculating circle.





$$k = \frac{1}{\rho}$$

To find curvature of $f(x)$

$$k(x) = \frac{|f''(x)|}{\left[1 + [f'(x)]^2\right]^{3/2}}$$