Homework 5 Solutions

Due: Friday Oct. 15, by 11:59pm,
via Gradescope

- Failure to submit homework correctly will result in a zero on the homework.
- Homework must be in LaTeX. Submit the pdf file to Gradescope.
- Problems assigned from the textbook come from the 5th edition.
- No late homework accepted. Lateness due to technical issues will not be excused.

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1. (9 points) Section 4.1 # 7, 11, 16

   Solution:

   # 7. **Proof:** $a = b = 0$. Notice $\sqrt{0+0} = \sqrt{0} + \sqrt{0}$ $\square$.

   # 11. **Proof:** $n = 0$. Notice $2n^2 - 5n + 2 = 2$ when $n = 0$ and 2 is prime. $\square$.

   # 16. **Disproof:** Let $n = 0$. Then $n^2 + 1 = 1$ and 1 is not prime.

2. (15 points) Section 4.1 # 22, 29

   Solution:

   # 22. **Proof:** Set $p(n) = n^2 - n + 11$.

   $p(1) = 11$ and 11 is prime.

   $p(2) = 13$ and 13 is prime.

   $p(3) = 17$ and 17 is prime.

   $p(4) = 23$ and 23 is prime.

   $p(5) = 31$ and 31 is prime.

   $p(6) = 41$ and 41 is prime.

   $p(7) = 53$ and 53 is prime.

   $p(8) = 67$ and 67 is prime.

   $p(9) = 83$ and 83 is prime.

   $p(10) = 101$ and 101 is prime.

   $\square$

   # 29.

   (a) By definition of an even integer.

   (b) Substitution.

   (c) The integers are closed under addition and multiplication.

   (d) by the definition of an even integer.

3. (9 points) Section 4.2 # 8, 9, 14

   Solution:

   # 8.

   Proof: Let m be any even integer and let n be any odd integer. By the definition of even, $m = 2k$ for some integer k. By the definition of odd, $n = 2l + 1$ for some integer l.

$$\begin{aligned} 5m + 3n &= 5(2k) + 3(2l + 1) \text{ (substition)} \\ &= 10k + 6l + 3 \text{ (distribution)} \\ &= 10k + 6l + 2 + 1 \\ &= 2(5k + 3l + 1) + 1 \text{ (factor out)} \end{aligned}$$

   Set $t = 5k + 3l + 1$. Then t is an integer since integers are closed under addition and multiplication. Hence $5m + 2n = 2t + 1$, and so it is odd by the definition of an odd integer □

   Remark. *Our proofs will not be identical. But we should all have something roughly the same. I get it, the annotation is a hard to figure out. For now, do the best you can and try to follow the spirit of section 4.1.*

   #9 **Proof:** Let m be any integer greater than 4 that is a perfect square. By the definition of a perfect square, $m = k^2$ for some integer k. Set $t = m^2 - 1$. Note that t is the integer immediately preceding m. We factor and obtain $t = (m - 1)(m + 1)$. Since $m > 4$, we can subtract 1 from both sides and see that $m - 1 > 3$. Therefore t cannot be prime □

   #14 **Proof:** Let $k = 10$. Then $2k^2 - 5k + 2 = 157$ and 157 is prime □

   Remark. *It took me a while to find $k = 10$.*

4. (6 points) Section 4.2 # 18, 19

   Solution:

   # 18. There is an incorrect jump to conclusion in the line $mn = 2p(2q+) = 2r$.

   # 19. Cannot use the same integer k for both m and n.

5. (9 points) Section 4.2 # 26, 30, 31

   Solution:

   # 26. This is false. $a = 0, b = 1, c = 2$. Notice that $a + b + c = 3$ and 3 is odd.

   # 30. This is false. Let $m = 3$. Then $m^2 - 4 = 5$ and 5 is not composite.

   # 31. This is false. Let $n = 11$. Then $n^2 - n + 11 = 112$ and 112 is not prime.

   Remark. *Based on # 22. from section 4.1, I simply started looking for counterexamples larger than 10.*

6. (9 points) Section 4.3 # 14, 18.

Solution:

14.

(a) $\forall x\mathbb{R}$, If $x \in \mathbb{Q}$, then $x^3 \in \mathbb{Q}$.

(b) This is a true statement. Let's prove it.

Proof Let x be any rational number. Then by the defintion of a rational number $x = \frac{a}{b}$ where a and b are integers and $b \neq 0$.

$$x^3 = \left(\frac{a}{b}\right)^3 = \frac{a^3}{b^3}$$

Let $t = a^3$ and $s = b^3$. Then t and s are integers since the integers are closed under multiplication. Now $b^2 = b \cdot b$ and $b \neq 0$, hence b^2 is not zero by the **Zero Property**. Since $b^3 = b \cdot b^2$, we again conclude that $b^3 \neq 0$ by the **Zero Property**. By the definition of a rational number $x^3 = \frac{t}{s}$ is rational □

18. This is a true statement. Let's prove it.

Proof: Let r and s be any two rational numbers. By the definition of a rational number $r = \frac{a}{b}$ and $s = \frac{a'}{b'}$ where a, a', b, b' are integers and $b \neq 0$ and $b' \neq 0$. Now

$$\frac{r+s}{2} = \frac{\frac{a}{b} + \frac{a'}{b'}}{2} = \frac{\frac{ab'a'b}{bb'}}{2} = \frac{ab' + a'b}{2bb'}$$

Set $t = ab' + a'b$ and $s = 2bb'$. Then t and s are integers since the integers are closed under addition and multiplication. $bb' \neq 0$ by the **Zero Property**. Therefore $s = 2 \cdot bb' \neq 0$ by the **Zero Property**. Hence $r = \frac{t}{s}$ is a rational number by the definition of a rational number □