2. Sequences and Series

2.1 Sequences and limits & 2.2 Facts about limits of sequences

Given $f : D \to \mathbb{R}$, we say f is bounded if there exists $B \in \mathbb{R}$ such that $|f(x)| \leq B$ for all $x \in D$

If $f : D \to \mathbb{R}$ is bounded, we define $\sup_{x \in D} f(x) := \sup f(D)$, $\inf_{x \in D} f(x) := \inf f(D)$

A sequence (of real numbers) is a function $x : \mathbb{N} \to \mathbb{R}$.

- we write $x_n = x(n)$
- we denote the entire sequence by $\{x_n\}_{n=1}^{\infty}$, or $\{x_n\}$
- $\{x_n\}$ is bounded if there exists $B \in \mathbb{R}$ s.t. $|x_n| \leq B, \forall n \leq N$

  Equivalently, if $\{x_n : n \leq N\}$ is bounded as a set; if $x : N \to R$ is bounded as a function

A sequence $\{x_n\}$ converges to a number $L \in \mathbb{R}$ if

for all $\epsilon > 0$, there exists $M \in \mathbb{N}$ such that $|x_n - L| < \epsilon$ for all $n \geq M$

Symbolically: x_n converges to L means $\forall \epsilon > 0, \exists M \in \mathbb{N}$: $\forall n \geq M, |x_n - L| < \epsilon$

A sequence that converges is convergent, otherwise it is divergent.

We call L the limit of $\{x_n\}$ as $n \to \infty$, and write $\lim_{n \to \infty} = L$

Claim: $\{\frac{1}{n}\}$ is convergent, and $\lim_{n\to\infty} \frac{1}{n} = 0$

Pf. $\underbrace{\text{Given } \varepsilon > 0}_{(1)}$, by the Archimedean prop. $\underbrace{\text{there exists } M \in \mathbb{N} \text{ such that}}_{(2)}$

$M \cdot \varepsilon > 1 \Rightarrow \frac{1}{M} < \varepsilon$. Then, $\underbrace{\text{for all } n \geq M}_{(3)}$, we have

$|x_n - x| = |\frac{1}{n} - 0| = \frac{1}{n} \leq \frac{1}{M} \underbrace{<}_{} \varepsilon$ ✓

Prop. A convergent sequence is bounded.

Cor. An unbounded sequence is divergent.

Note: bounded does not imply convergent e.g. $\{(-1)^n\}$ bounded divergent

Pf. Convergent → for all $n \geq M$, $|x_n| = |x_n - L + L| \leq |x_n - L| + |L| < 1 + |L| = B$

Define $B_2 = \max\{|x_1|, |x_2|, ..., |x_{M-1}|\}$

Take $B = \max\{B_1, B_2\}$

Prop. The limit of a convergent sequence is unique.

Pf. "give yourself an ϵ of room" (pf: $\epsilon' = \frac{\epsilon}{2}$, tri inequality)

Prop. (Continuity of $+$, $-$, $\times$, $\div$ 2.2.5) Let $\{x_n\}$ and $\{y_n\}$ be convergent sequences.

1. $z_n = x_n \pm y_n$, then $\{z_n\}$ converges with limit (pf: $\epsilon' = \frac{\epsilon}{2}$)
   $\lim_{n\to\infty}(x_n \pm y_n) = \lim_{n\to\infty} x_n \pm \lim_{n\to\infty} y_n$
2. $z_n = x_n y_n$, then $\{z_n\}$ converges with limit (pf. $\epsilon' = \min\{\frac{\epsilon}{3|x|}, \frac{\epsilon}{3|y|}, \frac{\epsilon}{3}, 1\}$)
   $\lim_{n\to\infty}(x_n y_n) = (\lim_{n\to\infty} x_n)(\lim_{n\to\infty} y_n)$
3. If $y_n \neq 0$ for all n and $\lim_{n\to\infty} y_n \neq 0$, $z_n = \frac{x_n}{y_n}$, then $\{z_n\}$ converges with limit
   $\lim_{n\to\infty}(\frac{x_n}{y_n}) = \frac{\lim_{n\to\infty} x_n}{\lim_{n\to\infty} y_n}$

Prop. (limits preserve $\leq$, $\geq$ 2.2.3) Let $\{x_n\}$ and $\{y_n\}$ be convergent sequences.
If $x_n \leq y_n$ $\forall n \in \mathbb{N}$, then $\lim_{n\to\infty} x_n \leq \lim_{n\to\infty} y_n$. (pf: $\epsilon' = \frac{\epsilon}{2}$)

A sequence $\{x_n\}$ is monotone increasing if $x_n \leq x_{n+1}$ for all $n \in \mathbb{N}$.
($x_n < x_{n+1}$ — strictly monotone)
A sequence $\{x_n\}$ is monotone decreasing if $x_n \geq x_{n+1}$ for all $n \in \mathbb{N}$.

Monotone Convergence Theorem (MCT). A monotone sequence $\{x_n\}$ is bounded $\iff$ it is convergent.
monotone increasing and bounded: $\lim_{n\to\infty} x_n = \sup\{x_n : n \in \mathbb{N}\}$
monotone decreasing and bounded: $\lim_{n\to\infty} x_n = \inf\{x_n : n \in \mathbb{N}\}$

For a sequence $\{x_n\}$, we call a sequence $\{x_{n_i}\}$ a subsequence if $\{n_i\}$ s a strictly increasing sequence of natural numbers.
Prop. If $\{x_n\}$ is a convergent sequence, then every subsequence $\{x_{n_i}\}$ is also convergent, and $\lim_{n\to\infty} x_n = \lim_{i\to\infty} x_{n_i}$

For a sequence $\{x_n\}$, the K-tail of $\{x_n\}$ for $k \in \mathbb{N}$ is the subsequence
$\{x_{n+k}\}_{n=1}^{\infty}$ or $\{x_n\}_{n=K+1}^{\infty}$.
Prop. (Tail control convergence) Given a sequence $\{x_n\}$, the following are equivalent:
1. $\{x_n\}$ converges.
2. $\{x_{n+k}\}_{n=1}^{\infty}$ converges for all $K \in \mathbb{N}$.
3. $\{x_{n+k}\}_{n=1}^{\infty}$ converges for some $K \in \mathbb{N}$.

If exists, $\lim_{n\to\infty} x_n = \lim_{n\to\infty} x_{n+K}$

Convergence Tests
Let $c > 0$.
1. If $c < 1$, then $\{c^n\}$ converges and $\lim_{n\to\infty} c^n = 0$. (pf. monotone+bounded)
2. If $c > 1$, then $\{c^n\}$ is unbounded (hence divergent). (pf. using $\{\frac{1}{c^n}\}$)

Ratio test. Let $\{x_n\}$ be a sequence such that $x_n \neq 0$ for all n and the limit $L := \lim_{n\to\infty} \frac{|x_{n+1}|}{|x_n|}$ exists.

1. If $L < 1$, then $\{x_n\}$ converges and $\lim x_n = 0$.
2. If $L > 1$, then $\{x_n\}$ is unbounded (hence diverges).

Squeeze Lemma. Suppose $\{a_n\}$, $\{b_n\}$, $\{x_n\}$ satisfy $a_n \leq x_n \leq b_n \; \forall n \in \mathbb{N}$.

If $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$, then $\{x_n\}$ converges and

$\lim_{n\to\infty} x_n = \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$.

2.3 Limit superior, limit inferior, and Bolzano–Weierstrass

Lim Sup/Inf. Let $\{x_n\}$ be a bounded sequence. Define

$a_n = \sup\{x_k, k \geq n\}$

$b_n = \inf\{x_k, k \geq n\}$

Define

$\limsup_{n\to\infty} x_n := \lim_{n\to\infty} a_n$

$\liminf_{n\to\infty} x_n := \lim_{n\to\infty} b_n$

Prop.

1. $\{a_n\}$ is bounded monotone decreasing and $\{b_n\}$ is bounded monotone increasing, so $\limsup_{n\to\infty} x_n$ and $\liminf_{n\to\infty} x_n$ exist. (existence)
2. $\limsup_{n\to\infty} x_n = \inf\{a_n : n \in \mathbb{N}\}$

   $\liminf_{n\to\infty} x_n = \sup\{b_n : n \in \mathbb{N}\}$ (formula)
3. $\liminf_{n\to\infty} x_n \leq \limsup_{n\to\infty} x_n$ (inequality)

Thm 2.3.4 (existence of convergent sequences) or Thm 2.3.8 (Bolzano–Weierstrass theorem) (first part). Suppose $\{x_n\}$ is a bounded sequence (not necessarily convergent). Then there exists a convergent subsequence $\{x_{n_i}\}$ satisfying

$\lim_{k\to\infty} x_{n_k} = \limsup_{n\to\infty} x_n$

Similarly, there exists a (possibly different) subsequence $\{x_{n_i}\}$ satisfying

$\lim_{k\to\infty} x_{m_k} = \liminf_{n\to\infty} x_n$

Pf. Construct a subsequence inductively

Prop. (lim sup/inf convergence test)

Let $\{x_n\}$ be a bounded sequence. Then $\{x_n\}$ converges $\iff$

$\liminf_{n\to\infty} x_n = \limsup_{n\to\infty} x_n$

Furthermore, if so,

$\lim_{n\to\infty} x_n = \liminf_{n\to\infty} x_n = \limsup_{n\to\infty} x_n$

Pf. →: Bolzano–Weierstrass theorem; ←: $a_n \leq x_n \leq b_n$, squeeze lemma

2.4 Cauchy sequences

A sequence $\{x_n\}$ is a Cauchy sequences if for all $\epsilon > 0$ there exists $M \in \mathbb{N}$ such that for all $n, k \geq M$, we have $|x_n - x_k| < \epsilon$.

(Cauchy conpleteness of $\mathbb{R}$) A sequence of real numbers is Cauchy $\iff$ it converges.

Prop. Cauchy → bounded ($\epsilon = 1$, $B = \max\{|x_1|, ..., |x_{M-1}|, 1 + |x_M|\}$)

Pf. Cauchy ← convergent ($|x_n - L| < \frac{\epsilon}{2}$ → $|x_n - x_k| \leq |x_n - L| + |x_k - L| < \epsilon$)

Cauchy → convergent (bounded → by thm 2.3.4, exists subsequences and let $a = \lim_{k\to\infty} x_{n_k} = \lim_{n\to\infty} \sup x_n$, $b = \lim_{k\to\infty} x_{m_k} = \lim_{n\to\infty} \sup x_n$

$|x_n - x_k| < \frac{\epsilon}{3}$, $|x_{n_k} - a| < \frac{\epsilon}{3}$, $|x_{m_k} - b| < \frac{\epsilon}{3}$ → $|a - b| < \epsilon$ → $a = b$ → convergent)

2.5 Series

Given a sequence $\{x_n\}$, we write the “formal object”

$\sum_{n=1}^{\infty} x_n$ or $\sum x_n$

and call it a series.

A series converges if the sequence of partial sums $\{s_k\}$

$s_k = \sum_{n=1}^{k} x_n$

converges. In this case, we write

$\sum_{n=1}^{\infty} x_n = \lim_{k\to\infty} s_k$

If $\{s_k\}$ diverges, we say Σx_n diverges.

Prop. (Convergence of geometric series) Suppose $-1 < r < 1$, then the geometric series $\sum_{n=0}^{\infty} r^n$ converges, and $\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$.

Assigned readings in notes: absolute convergence, comparison series