Homework 6

Due: Friday Oct. 22, by 11:59pm,
via Gradescope

- Failure to submit homework correctly will result in a zero on the homework.
- Homework must be in LaTeX. Submit the pdf file to Gradescope.
- Problems assigned from the textbook come from the 5th edition.
- No late homework accepted. Lateness due to technical issues will not be excused.

---

1. (12 points) Section 4.3 # 32, 33, 36

Solution: # 32. **Proof** Let c be any real root of a polynomial with rational coefficients. That is, $P(c) = 0$, where

$$P(x) = \sum_{k=0}^{n} a_k x^k$$

and $a_k \in \mathbb{Q}$. We know that $a_k = \frac{r_k}{s_k}$ where r_k and s_k are integers and $s_k \neq 0$ for $k = 0, 1, 2, \ldots n$. We multiply the equation $P(c) = 0$ by $s_0 s_2 \cdots s_n$. We obtain

$$\sum_{k=0}^{n} a_k s_1 s_2 \cdots s_{k-1} s_{k+1} \cdots s_n c^k = 0$$

Notice that $Q(c) = 0$ where $Q(x)$ is the polynomial

$$\sum_{k=0}^{n} a_k s_1 s_2 \cdots s_{k-1} s_{k+1} \cdots s_n x^k$$

The real numbers $a_k s_1 s_2 \cdots s_{k-1} s_{k+1} \cdots s_n$ are integers for $k = 0, 1, 2, \ldots n$. Therefore c is the root of a polynomial with integer coefficients $\square$

#33(a) $(x - r)(x - s) = x^2 + (r + s)x + rs$.

If both r and s are odd, then $r + s$ is even (property 2) and rs is odd (property 3).

If both r and s are even, then $r + s$ is even (property 2) and rs is even (property 1).

If one is even and the other is odd, then $r + s$ is odd (property 5) and rs is even (property 4).

(b) Since -1253 is odd, it follows that r and s must have different parity. However, if r and s have different parity, then rs must be even. Hence rs cannot equal 255.

36. Cannot start a universal proof with explicit rationals $r = 1/4$ and $s = 1/2$.

2. (12 points) Section 4.4 # 28-30, 37

   #28 This is false.

   Let $a = 25, b = 5$ and $c = 5$. Then $a|bc$ however $a|b$ and $a|c$ are both false.

   # 29. This is true.

   Proof: Let a and b be any integers that satisfy $a|b$. By the definition of divisibility $b = ka$ for some integer k. Therefore $b^2 = (ka)^2 = k^2a^2$. Set $t = k^2$. Then $b^2 = ta^2$. Since t is an integer, it follows that $a^2|b^2$ $\square$

   #30. This is false. Let $a = -25$ and $n = 5$. Then $a|n^2$ and $a \leq n$. However $a|n$ is false.

   #37. No solution provided.

3. (6 points) Section 4.4 # 45, 48. We should all know the decimal representation of a non-negative integer.

   # 45. **Proof:** Let n be any integer whose decimal representation ends in 5. Then

$$n = \sum_{l=0}^{k} d_l 10^l$$

   where the decimal digits d_l are integers 0-9 and $d_0 = 5$. Notice that $10^l = 2^l 5^l$ for $l = 1, 2, \ldots k$. Therefore

$$n = 5 + \sum_{l=1}^{k} d_l 10^l = 5\left(1 + \sum_{l=1}^{k} d_l 2^l 5^{l-1}\right)$$

   Since

$$t = 1 + \sum_{l=1}^{k} d_l 2^l 5^{l-1}$$

   is an integer, we have $n = 5t$. It follows that 5 divides n $\square$

   #48. Let n be any integer where the sum of its digits are divisible by 3. Then

$$n = \sum_{l=0}^{k} d_l 10^l$$

   where 3 divides $d_0 + d_1 + \cdots d_k$. Note that 3 divides $10^l - 1$ for $l = 1, 2, \ldots k$. Therefore

$$\begin{aligned} n = \sum_{l=0}^{k} d_l 10^l &= d_0 + \sum_{l=1}^{k} d_l(10^l - 1 + 1) \\ &= d_0 + \sum_{l=1}^{k} d_l(10^l - 1) + \sum_{l=1}^{k} d_l \\ &= \sum_{l=1}^{k} d_l(10^l - 1) + \sum_{l=0}^{k} d_l \end{aligned}$$

Set

$$t = \sum_{l=1}^{k} d_l(10^l - 1)$$
$$s = \sum_{l=0}^{k} d_l$$

Notice that 3 divides t since 3 divides $10^l - 1$ for $l = 1, 2, \ldots k$. 3 also divides s, since s is the sum of the digits in n. Therefore 3 divides $t + s$, that is 3 divides n $\square$

4. (9 points) Section 4.5 # 12, 17, 21

#12. Not sure if I am prove this the best way. But here ya go!!

Proof: Let N be the number of days between DayT and DayN. By the Quotient-Remainder Theorem, we have

$$N = 7q + r \tag{1}$$

where q and r are unique integers and $0 \leq r < 7$.

Case 1: $r = 0$, therefore DayT = DayN. Add DayT to (1). Then DayT + N = 7q + DayT = 7q + DayN. Therefore (DayT + N) mod 7 = DayN.

Case 2: $r = 1$, therefore DayN = DayT + 1 . Add DayT to (1). Then DayT + N = 7q + 1 + DayT = 7q + DayN. Therefore (DayT + N) mod 7 = DayN.

Case 3: $r = 2$, therefore DayN = DayT + 2 . Add DayT to (1). Then DayT + N = 7q + 2 + DayT = 7q + DayN. Therefore (DayT + N) mod 7 = DayN.

Case 4: $r = 3$, therefore DayN = DayT + 3 . Add DayT to (1). Then DayT + N = 7q + 3 + DayT = 7q + DayN. Therefore (DayT + N) mod 7 = DayN.

Case 5: $r = 4$, therefore DayN = DayT + 4 . Add DayT to (1). Then DayT + N = 7q + 4 + DayT = 7q + DayN. Therefore (DayT + N) mod 7 = DayN.

Case 6: $r = 1$, therefore DayN = DayT + 5 . Add DayT to (1). Then DayT + N = 7q + 5 + DayT = 7q + DayN. Therefore (DayT + N) mod 7 = DayN.

Case 7: $r = 1$, therefore DayN = DayT + 6 . Add DayT to (1). Then DayT + N = 7q + 6 + DayT = 7q + DayN. Therefore (DayT + N) mod 7 = DayN.

$\square$

17. **Proof:** Let n be any integer.

Case 1: Assume that n is an even integer. Then $n = 2k$ for some integer k. Therefore $n^2 - n + 3 = 4k^2 - 2k + 3 = 4k^2 - 2k + 2 + 1 = 2(2k^2 - k + 1) + 1$. Set $t = (2k^2 - k + 1$. Since t is an integer and $n^2 - n + 3 = 2t + 1$, it follows $n^2 - n + 3$ is odd.

Case 2: Assume that n is an even integer. Then $n = 2m + 1$ for some integer m. Therefore $n^2 - n + 3 = 4m^2 + 4m + 1 - 2m - 1 + 3 = 4m^2 + 2m + 3 = 4m^2 + 2m + 2 + 1 = 2(2m^2 + m + 1) + 1$. Set $t = (2m^2 + m + 1$. Since t is an integer and $n^2 - n + 3 = 2t + 1$, it follows $n^2 - n + 3$ is odd.

$\square$

21. **Proof:** We know that $b = 12q + 5$ for some integer q. Therefore, $8b = 12 \cdot 8 \cdot q + 40 = 12 \cdot 8 \cdot q + 36 + 4 = 12(8q + 3) + 4$. Set $t = 8q + 3$. Then $8b = 12t + 4$ and by the uniqueness in the Quotient-Remainder Theorem, it follows that $8b$ mod 12 $= 4$.

5. (9 points) Section 4.5 # 25, 31(a), 33

   # 25. **Proof:** Let a and b be any integers that satisfy a mod $7 = 5$ and b mod $7 = 6$. Then $a = 7q + 5$ and $b = 7k + 6$ where q and k are integers. We multiple ab and obtain

$$\begin{aligned} ab &= 49qk + 42q + 35k + 30 \\ &= 49qk + 42q + 35k + 28 + 2 \\ &= 7(7qk + 6q + 5k + 4) + 2 \end{aligned}$$

   Set $t = 7qk + 6q + 5k + 4$. Then $ab = 7t + 2$ and by the uniqueness in the Quotient-Remainder Theorem it follows that ab mod $7 = 2$ $\square$.

   #31(a). **Proof:** Let m and n be any integers.

   Case 1: m and n are both odd. Then by property 2 it follows that $m + n$ and $m - n$ are even.

   Case 2: m and n are both even. Then by property 1 it follows that $m + n$ and $m - n$ are both even.

   Case 3: m and n have different parity. Then properties 5 and 6 it follows that $m + n$ and $m - n$ are odd.

   Therefore $m + n$ and $m - n$ are either both even or both odd $\square$

   #33. **Proof** : Let a, b and c be any integers such that $a - b$ is even and $b - c$ is even. Since $a - c = (a - b) + (b - c)$ it follows that $a - c$ is the sum of two even integers. Then by property 1 it follows that $a - c$ is even $\square$

6. (9 points) Section 4.5 # 38, 42, 47.

   # 38. **Proof:** Let m be any integer. Then by the Quotient-Remainder Theorem we have $m = 5q + r$ where $0 \leq r < 5$..

   Case 1: Assume that $r = 0$. Then $m^2 = 25q^2 = 5t$ where $t = 5q$.

   Case 2: Assume that $r = 1$. Then $m^2 = 25q^2 + 10q + 1 = 5(5q^2 + 2q) + 1 = 5t + 1$ where $t = 5q^2 + 2q$.

   Case 3: Assume that $r = 2$. Then $m^2 = 25q^2 + 20q + 4 = 5(5q^2 + 4q) + 4 = 5t + 1$ where $t = 5q^2 + 4q$.

   Case 4: Assume that $r = 3$. Then $m^2 = 25q^2 + 30q + 9 = 25q^2 + 30q + 5 + 4 = 5(5q^2 + 6q + 1) + 4 = 5t + 1$ where $t = 5q^2 + 6q + 1$.

   Case 5: Assume that $r = 3$. Then $m^2 = 25q^2 + 40q + 16 = 25q^2 + 40q + 15 + 1 = 5(5q^2 + 8q + 3) + 1 = 5t + 1$ where $t = 5q^2 + 8q + 3$.

Notice that in all of the cases above, $m^2 = 5t + r$ for some integer t and r is 0, 1, or 4
$\square$

42. This is an if and only if. This will require us to prove the following two if-then statements

1. For all real numbers r and $c \geq 0$, if $-c \leq r \leq c$, then $|r| \leq c$.

2. For all real numbers r and $c \geq 0$, if $|r| \leq c$, then $-c \leq r \leq c$.

Proof of 1.: Let r be any real number and c be any non-negative number which satisfy $-c \leq r \leq c$.

Case 1: Assume that r is non-negative. Then $r = |r|$. Therefore $-c \leq |r| \leq c$. Therefore $|r| \leq c$.

Case 2: Assume that $r < 0$. Then $-r = |r|$. Since $-c \leq r \leq c$ we multiply by -1 and obtain $-c \leq -r \leq c$. Therefore $-c \leq |r| \leq c$, that is $|r| \leq c$ $\square$

Proof of 2: Let r be any real number and c be any non-negative number which satisfy $|r| \leq c$. By Lemma 4.5.4. we know that $-|r| \leq r \leq |r|$. Therefore $-|r| \leq r \leq c$. Notice that $|r| \leq c$ implies $-c \leq -|r|$. Therefore $-c \leq r \leq c$ $\square$

$\square$

47. Set $r = m \bmod d$ and $s = n \bmod d$.

Theorem. *d— r-s*

Proof: By the Quotient Remainder Theorem we know that $m = dq+r$ and $n = dq'+s$. Therefore $m - n = d(q = q') + r - s$. We also know that $m - n = ds$ for some integer s. Therefore $ds = d(q - q') + r - s \rightarrow d(s - q + q') = r - s$. Set $t = s - q - q'$, then t is an integer (since integers are closed under addition). Therefore d divides r-s $\square$