Lecture 18

Recurrance Relations: a formula which counts solutions for size n in terms of sizes of previous problems:

Such as:

① $a_n = \sum_{k=1}^{r} c_k a_{n-k}$ $\rightsquigarrow$ fibonacci

② $a_n = c\, a_{n-1} + f(n)$

③ $a_n = \sum_{k=0}^{n-1} a_k a_{n-1-k}$, $a_0 a_{n-1} + \dots + a_{n-1} a_0$

④ $a_{n,m} = a_{n-1,m} + a_{n-1,m-1}$ Two sizes!

Eg Arrangements of n objects:

$$a_n = n\, a_{n-1} \rightsquigarrow a_0 = 1 \rightsquigarrow n!$$

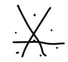
Eg: Climbing stairs one or two at a time:

$a_1 = 1$, $a_2 = 2$, Idea You know how many ways to do $< n$ steps so how to reduce the problem?

Ans Take 1 step! there are now either $n-1$ steps to go or $n-2$ steps dep. on your first size.

Thus: $a_n = a_{n-1} + a_{n-2}$

Eg Regions in the plane: (formed by n generic lines).

Low cases $a_0 = 1$, $a_1 = 2$, $a_2 = 4$, $a_3 =$  $= 7$

In general you get $n-1$ pts of intersection so n faces get cut in two

so $a_n = a_{n-1} + n$.

Eg: How many binary sequences w/ no 111's:

$a_0 = 1$, $a_1 = 2$, $a_2 = 4$, $a_3 = 7$ $(8-1)$

$$\underset{\underline{0}\ldots}{a_n} = \underset{\underline{1}\,\underline{0}\ldots}{a_{n-1}} + \underset{\underline{1}\,\underline{1}\,\underline{0}\ldots}{a_{n-2}} + a_{n-3} \ldots \Rightarrow \begin{array}{l} a_4 = 13 \\ a_5 = 24 \\ a_6 = 44 \\ a_7 = 81 \checkmark \end{array}$$

(0,0) case!

<u>Solving linear Recurrance:</u>

$a_n = \sum_{k=1}^{r} c_k a_{n-k}$ find clsd form for a_n.

Ideas: If $f(n)$, $g(n)$ satisfy linear recurrence reln then $p f(n) + q g(n)$ does too!

Pf: $p\, f(n) + q\, g(n) \overset{?}{=} \sum c_k (p f(n-k) + q g(n-k)) = p \sum c_k f(n-k) + q \sum c_k g(n-k)$

$= p\, f(n) + q\, g(n)$ ✓.

Ideas: $a_n = x\, a_{n-1}$, $a_0 = 1 \Rightarrow a_n = x^n$!

Now subst: $a_n = x^n \rightsquigarrow x^n = c_1 x^{n-1} + \ldots + c_r x^{n-r}$

$\Rightarrow \quad x^r - c_1 x^{r-1} - \ldots - c_r = 0$ If x solves this then $a_n = x^n$ solves Reln!

$a_n = 6a_{n-1} - 9a_{n-2}$

Idea: define characteristic eqn

$\alpha^n = c_1 \alpha^{n-1} + \ldots + c_r \alpha^{n-r} \rightsquigarrow$ Has roots (maybe $\mathbb{C}$ but assume mult $= 1$).

Let $\alpha_1, \ldots, \alpha_r$ be roots (divide by α^{n-r}) then

$a_n = Z_1 \alpha_1^n + Z_2 \alpha_2^n + \ldots + Z_r \alpha_r^n$

Z_i sets initial conditions.

If there is a root α of mult. 3 then $\underset{1}{\alpha^n} + \underset{2}{n\alpha^n} + \underset{3}{n^2\alpha^n}$ goes.

mult $= m \rightsquigarrow \underbrace{\alpha^n + n\alpha^n + n^2\alpha^n + \ldots + n^{m-1}\alpha^n}_{m \text{ terms}}$.

To see where multiple roots come from:

Note that being a multi root $\Rightarrow$ root of derivative:.

(1) $a^n = c_1 a^{n-1} + \dots + c_k a^{n-k}$ assume r is a double root.

$\frac{d}{da} \rightsquigarrow na^{n-1} = c_1(n-1)a^{n-2} + \dots c_k(n-k)a^{n-k-1}$

$\cdot a \rightsquigarrow na^n = c_1 \underbrace{(n-1)a^{n-1}}_{b_{n-1}} + \dots + c_k \underbrace{(n-k)a^{n-k}}_{b_{n-k}}$

let $b_n = na^n = c_1 b_{n-1} + \dots + c_k b_{n-k}$

so b_n solves (1)!

Eg's

(1) $a_n = 2a_{n-1} + 3a_{n-2}$ $a_0 = a_1 = 1$

Guess $a_n = x^n \Rightarrow x^n = 2x^{n-1} + 3x^{n-2} \Leftrightarrow x^n - 2x^{n-1} - 3x^{n-2} = 0$

$\Leftrightarrow x^2 - 2x - 3 = 0 \Leftrightarrow (x-3)(x+1) = 0$

$a_n = z_0 3^n + z_1(-1)^n$

$a_0 = 1 = z_0 + z_1 \Rightarrow 2 = 4z_0 = z_0 = 1/2$
$a_1 = 1 = 3z_0 - z_1$
$\Rightarrow 1 = \frac{3}{2} - z_1 \Rightarrow z_1 = \frac{3}{2} - 1 = 1/2$

Thus $a_n = 1/2 \cdot 3^n + 1/2(-1)^n = \dfrac{3^n + (-1)^n}{2}$

(2) $a_n = -2a_{n-2} - a_{n-4}$; $a_0 = 0, a_1 = 1, a_2 = 2, a_3 = 3$

Guess $a_n = x^n \Rightarrow x^n = -2x^{n-2} - x^{n-4} \Leftrightarrow x^n + 2x^{n-2} + x^{n-4} = 0$

$\Leftrightarrow x^4 + 2x^2 + 1 = 0 \Leftrightarrow (x^2+1)^2 = 0 \Leftrightarrow (x-i)^2(x+i)^2 = 0$
i $-i$ mult = 2.

$a_n = z_0(i)^n + z_1 n(i)^n + z_2(-i)^n + z_3 n(-i)^n$

$a_0 = 0 = z_0 + z_2$, etc. $\rightsquigarrow$ four eqns & four unknowns, solve for z_0, z_1, z_2, z_3 ...