12-3 Dot Product

$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$ – scalar

$= \|\vec{u}\| \, \|\vec{v}\| \cos\theta$

$\vec{u} \cdot \vec{v} = 0 \iff \vec{u} \perp \vec{v}$

① $(\vec{a} \cdot \vec{b}) \cdot \vec{c}$ ✗ Are any of these valid.

✓ $\|\vec{a}\| (\vec{b} \cdot \vec{c})$ – Scalar

✓ $(\vec{a} + \vec{b}) \cdot \vec{c}$ – Scalar

$\vec{a} \cdot \vec{b} + \vec{c}$ ✗

⑥ $\vec{a} = \langle p, -p, 2p \rangle$ $\vec{b} = \langle 2q, q, -q \rangle$

$\vec{a} \cdot \vec{b} = 2pq + (-pq) + (-2pq) = -pq.$

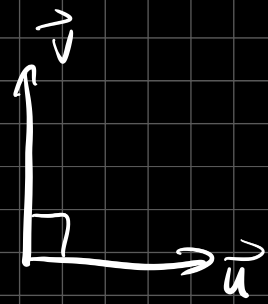
⑧ $\vec{a} = 3\hat{i} + 2\hat{j} - \hat{k}$ $\vec{b} = 4\hat{i} + 5\hat{k}$

$= 4\hat{i} + 0\hat{j} + 5\hat{k}$

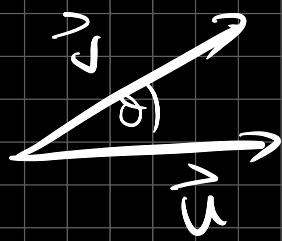
$\vec{a} \cdot \vec{b}$

$= (3)(4) + (2)(0) + (-1)(5) = 7.$

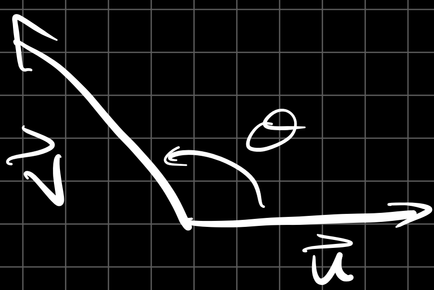
Comment: $\vec{u} \cdot \vec{v} = \|\vec{u}\| \, \|\vec{v}\| \cos\theta.$



$\vec{u}\cdot\vec{v} = 0$



$\vec{u}\cdot\vec{v} > 0$



$\vec{u}\cdot\vec{v} < 0$

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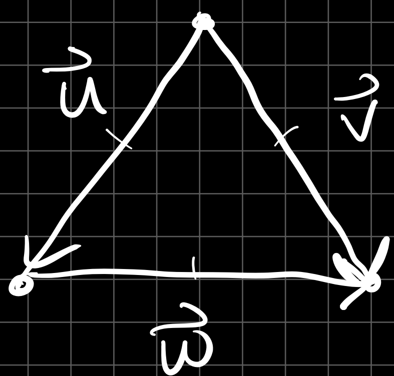
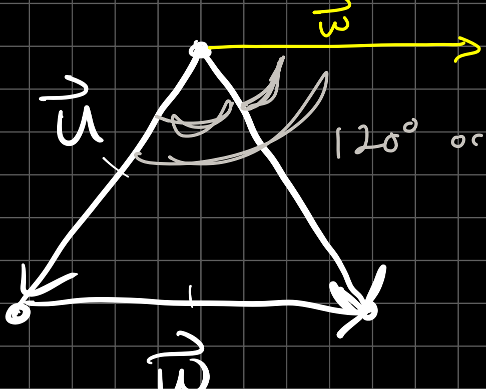


$\|\vec{u}\| = 1$

find $\vec{u}\cdot\vec{v}$

$\vec{u}\cdot\vec{w}$

$\vec{u}\cdot\vec{v} = \underbrace{\|\vec{u}\|}_{1}\,\underbrace{\|\vec{v}\|}_{1} \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}.$



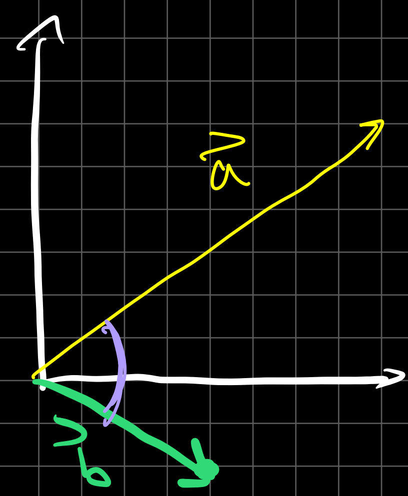
120° or $\frac{2\pi}{3}$ $\quad \cos\left(\frac{2\pi}{3}\right) = -\frac{1}{2}$

$\vec{u}\cdot\vec{w} = \|\vec{u}\|\,\|\vec{w}\|\left(-\frac{1}{2}\right) = -\frac{1}{2}.$

(15) Find the angle between

$\vec{a} = \langle 4, 3 \rangle$

$\vec{b} = \langle 2, -1 \rangle$



$$\vec{a} \cdot \vec{b} = \|\vec{a}\| \|\vec{b}\| \cos\theta$$

$$4(2) + (3)(-1) = \sqrt{(4)^2 + (3)^2} \sqrt{2^2 + (-1)^2} \cos\theta$$

$$5 = 5\sqrt{5} \cos\theta \Rightarrow \cos\theta = \frac{1}{\sqrt{5}}$$

$$\theta = \arccos\left(\frac{1}{\sqrt{5}}\right) = \cos^{-1}\left(\frac{1}{\sqrt{5}}\right)$$

(24)

$\vec{u} = \langle -5, 4, -2 \rangle \quad \vec{v} = \langle 3, 4, -1 \rangle$

determine if $\vec{u} \perp \vec{v}$, $\vec{u} \parallel \vec{v}$, or neither.

$\vec{u} = \alpha \vec{v}$

$\vec{u} \cdot \vec{v} \neq 0$ not $\perp$.

Can we find α s.t. $\vec{u} = \alpha \vec{v}$

$\langle -5, 4, -2 \rangle = \langle 3\alpha, 4\alpha, -\alpha \rangle$

need: $-5 = 3\alpha \rightarrow \alpha = -\frac{5}{3}$

$4 = 4\alpha \Rightarrow \alpha = 1$ X

$-2 = -\alpha \rightarrow \alpha = 2$

contradiction

not $\parallel$

(*) Could also have tried to find θ or $\cos\theta$

θ = 0, π then $\parallel$

$\cos\theta$ = -1, 1 then $\parallel$

(26) find x s.t. angle b/w $\langle 2, 1, -1 \rangle$, $\langle 1, x, 0 \rangle$ is 45°. $\frac{\pi}{4}$. $\Big\}$ $\cos 45^\circ = \frac{1}{\sqrt{2}}$

$\langle 2, 1, -1 \rangle \cdot \langle 1, x, 0 \rangle = 2 + x$

$$\|\langle 2,1,-1\rangle\| = \sqrt{6}$$

$$\|\langle 1,x,0\rangle\| = \sqrt{1+x^2}$$

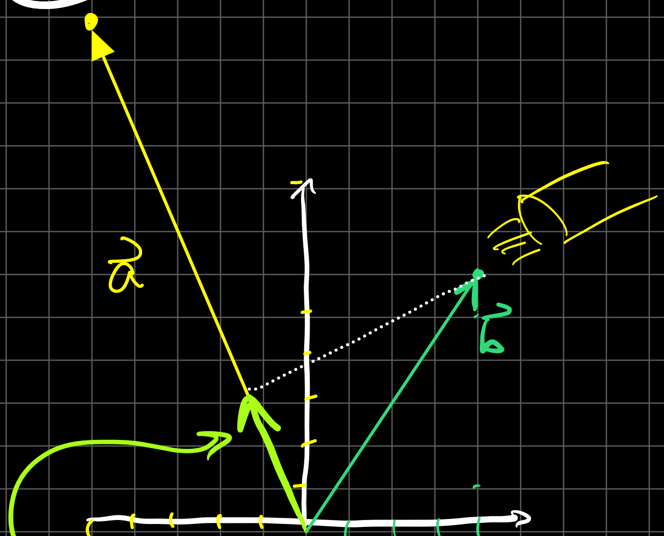
$$(2+x)^2 = \left(\sqrt{6}\sqrt{1+x^2}\left(\frac{1}{\sqrt{2}}\right)\right)^2$$

$$4+4x+x^2 = 3+3x^2 \Rightarrow 2x^2-4x-1=0$$

$$x = \frac{4 \pm \sqrt{16-4(2)(-1)}}{4}$$

$$= \frac{4 \pm \sqrt{24}}{4} = 1 \pm \frac{1}{2}\sqrt{6}$$

39 $\vec{a} = \langle -5, 12\rangle$ $\vec{b} = \langle 4, 6\rangle$



find scalar proj. of $\vec{b}$ onto $\vec{a}$

$$\text{Comp}_{\vec{a}}\vec{b} = \frac{\vec{a}\cdot\vec{b}}{\|\vec{a}\|}$$

$$= \frac{-20+72}{13} = \frac{52}{13} = 4$$

$$\text{proj}_{\vec{a}}\vec{b} = \text{Comp}_{\vec{a}}\vec{b}\left(\frac{\vec{a}}{\|\vec{a}\|}\right)$$

$$= 4\left(\frac{\langle -5,12\rangle}{13}\right) = \left\langle \frac{-20}{13}, \frac{48}{13}\right\rangle$$

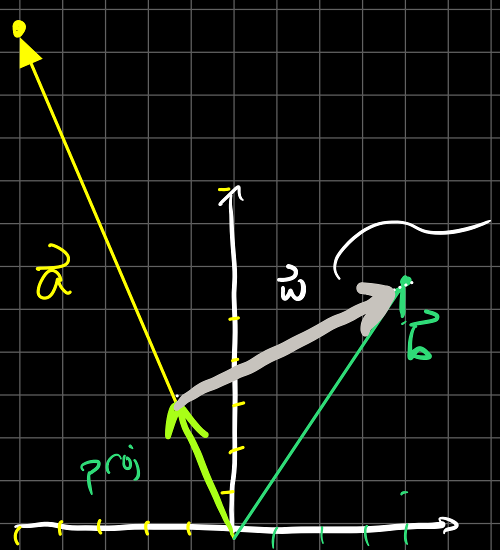
What would happen if $\vec{a}$ changed length?

$\alpha > 0$ $\quad \alpha\vec{a} = \langle -5\alpha, 12\alpha \rangle$ $\quad$ leave $\vec{b}$

$$\text{comp}_{\alpha\vec{a}}\vec{b} = \frac{\alpha\vec{a}\cdot\vec{b}}{\|\alpha\vec{a}\|} = \frac{52\alpha}{13\alpha} = 4$$

$$\text{proj}_{\alpha\vec{a}}\vec{b} = 4\frac{\alpha\vec{a}}{\|\alpha\vec{a}\|} = \text{Same as before.}$$

Further Investigation



$\vec{w} \perp \vec{a}$, $\vec{w} \perp \text{proj}_{\vec{a}}\vec{b}$

$$\text{proj}_{\vec{a}}\vec{b} + \vec{w} = \vec{b}$$

$$\text{orth}_{\vec{a}}\vec{b} = \vec{b} - \text{proj}_{\vec{a}}\vec{b}$$

64 if $\vec{u}+\vec{v}$ and $\vec{u}-\vec{v}$ are orthogonal

then $\|\vec{u}\| = \|\vec{v}\|$.

Prove this statement.

Let $(\vec{u}+\vec{v})\cdot(\vec{u}-\vec{v}) = 0$

$$(\vec{u}+\vec{v})\cdot\vec{u} + (\vec{u}+\vec{v})\cdot(-\vec{v}) = 0$$

$$\underbrace{\vec{u}\cdot\vec{u}}_{\|\vec{u}\|^2} + \vec{v}\cdot\vec{u} + \vec{u}\cdot(-\vec{v}) + \underbrace{\vec{v}\cdot(-\vec{v})}_{-\|\vec{v}\|^2} = 0$$

$\vec{v}\cdot\vec{u}$ $\quad -\vec{u}\cdot\vec{v}$

0

$\vec{u}\cdot\vec{v} = \vec{v}\cdot\vec{u}$

$u_1v_1 + u_2v_2 + \cdots$

$v_1u_1 + v_2u_2$

$$\|\vec{u}\|^2 - \|\vec{v}\|^2 = 0$$

$$\left(\|\vec{u}\| + \|\vec{v}\|\right)\left(\|\vec{u}\| - \|\vec{v}\|\right) = 0$$

$\|\vec{u}\| = \|\vec{v}\|$

$\|\vec{u}\| = -\|\vec{v}\|$

only if

$\vec{u} = \vec{v} = \vec{0}$

$\|\vec{u}\| = \|\vec{v}\|$

$\therefore \|\vec{u}\| = \|\vec{v}\|.$

$\vec{u}\cdot(-\vec{v})$

$= u_1(-v_1) + \cdots$

$= -u_1v_1 + \cdots$

$= -\vec{u}\cdot\vec{v}$