Homework 9

Due: Friday Nov. 12, by 11:59pm,
via Gradescope

- Failure to submit homework correctly will result in a zero on the homework.
- Homework must be in LaTeX. Submit the pdf file to Gradescope.
- Problems assigned from the textbook come from the 5th edition.
- No late homework accepted. Lateness due to technical issues will not be excused.

---

1. (6 points) Section 5.3 # 36, 46

   Solution:

   #36. There hint is a rather good hint. In the inductions step, the team we remove (denoted by T') can either win right away, win somewhere in the middle, or at the end. I hope that makes sense. Ok, now on to the proof.

   (a) *Base Case:*

   (b) *Induction Step:* Assume k is any integer ≥ 2 such that $P(k)$ is true. Now consider $k+1$ teams. Let's remove a team. We know we can label the k remaining teams by $T_1, T_2, \ldots T_k$ such that T_i defeats T_{i+1} for $i = 1, 2, 3, \ldots k-1$.

   Case 1: The team we removed, T', defeated T_1. Then we can label the T' by T_1 and the remaining k teams by $T_2, \ldots T_{k+1}$.

   Case 2: The team we removed, T', lost to all teams. Then we can label T' by T_{k+1}.

   Case 3: The team we removed, T', losts to teams $T_1, T_2, \ldots T_m$ and beats T_{m+1} where $1 \leq m \leq k-1$. Then label T' by T_{m+1} and the remaining teams by $T_{m+2}, T_{m+3}, \ldots T_{k+1}$.

   $\square$

   #46. No base case.

2. (6 points) Section 5.4 # 13, 20.

   Solution:

   # 13.

   (a) *Base Case:* We need two base cases. One for $n = 1$ and $n = 2$. I will leave the write up for you.

(b) *Induction Step:* Assume k is any integer ≥ 2 such that i can be written as a product of primes for $i = 2, 3, \ldots k$. NTS that $k+1$ can be written as a product of primes.

Case 1: $k+1$ is prime. Then we're done. Case 2: $k+1$ is composite. Then $k+1 = st$ where $2 \leq s \leq k$ and $2 \leq t \leq t$. We know that s and t can be written as a product of primes by the induction hypothesis. Therefore st can be written as a product of primes. $\square$

20. This problem will force you to understand the definition of the floor function. That is, $\lfloor x \rfloor$ is the greatest integer $\leq x$.

(a) *Base Case:* Two base cases are needed. You will find that the second base case is needed in Case 2.

(b) *Induction Step:* Assume $k \geq 2$ is any integer *geq* such that b_i is divisible by 3 for $i = 1, 2, \ldots k$. NTS b_{k+1} is divisible by 3. We have

$$b_{k+1} = 5 \cdot b_{\lfloor (k+1)/2 \rfloor} + 6$$

Case 1: $k+1$ is even. Then $\lfloor (k+1)/2 \rfloor = \frac{k+1}{2}$. We know that $b_{\lfloor (k+1)/2 \rfloor}$ is divisible by 3 since $1 \leq \frac{k+1}{2} \leq k$. Since $3|6$ it follows that $3|b_{k+1}$

Case 2: $k+1$ is odd. Then $\lfloor (k+1)/2 \rfloor = \frac{k}{2}$ We know that $b_{\lfloor (k+1)/2 \rfloor}$ is divisible by 3 since $1 \leq \frac{k}{2} \leq k$. Since $3|6$ it follows that $3|b_{k+1}$

$\square$

3. (3 points) Consider the Fibonacci sequence $f_0 = f_1 = 1$ and $f_n = f_{n-1} + f_{n-2}$ for all $n \geq 2$. Use the Principle of Strong Mathematical Induction to prove that

$$f_n = \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^{n+1} - \left(\frac{1-\sqrt{5}}{2} \right)^{n+1} \right]$$

4. (3 points) Use the Principle of Strong Mathematical Induction to show that any positive integer can be written as the sum of distinct terms in the Fibonacci sequence. For instance, $4 = f_1 + f_2,\ 6 = f_1 + f_2 + f_3,\ 11 = +f_2 + f_4$.

Solution:

(a) *Base Case* There are two base cases. I will leave them for you to do.

(b) *Induction Step:* Assume k is any integer ≥ 1 such that

$$f_i = \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^{i+1} - \left(\frac{1-\sqrt{5}}{2} \right)^{i+1} \right]$$

for $i = 0, 1, 2, \ldots k$. NTS

$$f_{k+1} = \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^{k+2} - \left(\frac{1-\sqrt{5}}{2} \right)^{k+2} \right]$$

Set

$$a = \frac{1+\sqrt{5}}{2}$$
$$b = \frac{1-\sqrt{5}}{2}$$

We have

$$f_{k+1} = f_k + f_{k-1} = \frac{1}{\sqrt{5}}\left(a^{k+1} - b^{k+1} + a^k - b^k\right)$$

Note that here we've used the induction hypothesis for $i = k-1$ and $i = k$. This is made possible by two base cases. Now we have

$$f_{k+1} = \frac{1}{\sqrt{5}}\left(a^k(a+1) - b^k(b+1)\right)$$

I will leave it for you to show that $a + 1 = a^2$ and $b + 1 = b^2$. Therefore

$$f_{k+1} = \frac{1}{\sqrt{5}}\left(a^{k+2} - b^{k+2}\right)$$

□

5. (6 points) Section 5.4 # 25, 32.

   Solution:

   # 25. With only one base case $k \geq 0$. If $k = 0$, then $k - 1 = -1$.

   # 32. No. $P(5)$ is not necessarily true. There is not integer k such that $3k = 5$.

6. (12 points) Suppose you wish to show that $P(n)$ is true for all integers $n \geq a$. You begin by defining the set

   $$S = \{n \geq a : n \in \mathbb{Z} \wedge P(n) = F\}$$

   Your goal is to show that $S = \emptyset$. You have trouble showing $S = \emptyset$ so you try contradiction.

   Proof: Suppose that $S \neq \emptyset$.

   (a) Explain why S has a smallest element in your contradiction proof.

   Solution: Since S is a non-empty set of integers that is bounded from below (namely by a), it follows from the Well-Ordering Principle.

   (b) If you know that $P(a)$ is T, then explain why the smallest element of S, let's denote it by x, satisfies $x > a$ in your contradiction proof.

   Solution: Every element in $y \in S$ satisfies $a \leq y$. Since $a \notin S$ it follows that the smallest element in S is greater than a.

   (c) Explain why $P(x)$ is F and $P(x-1)$ is T in your contradiction proof.

   Solution: Since $x > a$, then $x - 1 \geq a$. Since $x - 1 \notin S$ (otherwise x is not the least element) we must have $P(x-1)$ is T.

(d) Suppose you don't know that $P(a)$ is T. Explain why you cannot say $P(x-1)$ is T in your contradiction proof.

Solution: That is because x could be a. This means that $x-1$ is $a-1$ and P is defined for $n \geq a$.

7. (6 points) Section 5.4 # 26, 27.

Solution:

26.

Let's define

$$S = \{n > 1 : n \text{ does not have a prime divisor } \}$$

We want to show that $S = \emptyset$. We will do this by contradiction

Proof: Assume that $S \neq \emptyset$. Since S is a non-empty set of integers that is bounded from below, it follows that S has a least element that we will call k. Notice that $2 \notin S$ so that $k > 2$. Moreover, notice that any integer $2 \leq i \leq k-1$ cannot belong to S otherwise we would contradict the fact that k is the lease element. Therefore all integers i, $2 \leq i \leq k-1$ have a prime divisor.

Case 1: k is prime. Then k has a prime divisor, namely itself. This is a contradiction.

Case 2: k is composite. Then $k = st$ where $2 \leq s \leq k-1$ and $2 \leq t \leq k-1$. But s and t have prime divisors, so k must have a prime divisor. This is a contradiction.

Therefore S must be $\emptyset$ $\square$

27. Let's define

$$S = \{n > 1 : n \text{ does not have a prime factorization } \}$$

We want to show that $S = \emptyset$. We will do this by contradiction

Proof: Assume that $S \neq \emptyset$. Since S is a non-empty set of integers that is bounded from below, it follows that S has a least element that we will call k. Notice that $2 \notin S$ so that $k > 2$. Moreover, notice that any integer $2 \leq i \leq k-1$ cannot belong to S otherwise we would contradict the fact that k is the lease element. Therefore all integers i, $2 \leq i \leq k-1$ can be written as a product of primes.

Case 1: k is prime. This is a contradiction.

Case 2: k is composite. Then $k = st$ where $2 \leq s \leq k-1$ and $2 \leq t \leq k-1$. But s and t can be written as products of primes. Hence st can be written as a product of primes. This is a contradiction.

Therefore S must be $\emptyset$ $\square$