1. G is a group and H is a subgroup of G. G acts on the set of left cosets G/H by $g.xH = (gx)H$.

   (i). Prove this is a well-defined group action.

   (ii). Is this action transitive? Why or Why not?

   (iii). Give a proof of the Lagrange Theorem using Counting Formula.

   Solution:

   (i). If $xH = yH$, then $x^{-1}y \in H$. For any $g \in G$, $(gx)^{-1}(gy) = x^{-1}g^{-1}gy = x^{-1}y \in H$, so $(gx)H = (gy)H$, i.e. $g.(xH) = g.(yH)$.

   (ii). It is transitive: for any $xH \in G/H$, $x.H = xH$, so $xH \in O_H$, we see there is a single orbit, so the action is transitive.

   (iii). $O_H = G/H$, and the stabiliser $G_H = H$, by the Counting Formula:

$$|G/H| = |O_H| = \frac{|G|}{|G_H|} = \frac{|G|}{|H|}$$

2. G is a group with $|G| = p^2$ for some prime p. Prove either $G \cong \mathbb{Z}/p^2\mathbb{Z}$ or $G \cong \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$

   Solution: $|G| = p^2$ and p is a prime, so G is an abelian group.

   If G is a cyclic group of order p^2, then $G \cong \mathbb{Z}/p^2\mathbb{Z}$.

   If G is not a cyclic group, then all the non-identity elements of G have order p. Choose $x \in G \setminus \{1\}$, and choose $y \in G \setminus < x >$. In particular, $|x| = |y| = p$. G is an abelian group, so $< x >$ and $< y >$ are normal subgroups of G. $y \notin < x >$ implies $< x > \cap < y >= \{1\}$ since p is a prime. $| < x >< y > | = \frac{|<x>| \times |<y>|}{|<x> \cap <y>|} = p^2$, so $< x >< y >= G$. We conclude

$$G =< x > \times < y >\cong \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$$

3. Prove any group of order 77 is cyclic.

   Solution: If G is a group of order $77 = 7 \times 11$, it will have Sylow 7-subgroups and Sylow 11-subgroups, i.e. subgroups of order 7 and order 11. The number of Sylow 7-subgroups divides 11 and is congruent to 1 modulo 7, so it has to be

1, which then implies this unique Sylow 7-subgroup is a normal subgroup of G, and call it H. Similarly, we can show that there is a unique Sylow 5-subgroup of G that is a normal subgroup, and call it K. Since 7 and 11 are primes, we know $H \cong \mathbb{Z}/7\mathbb{Z}$ and $K \cong \mathbb{Z}/11\mathbb{Z}$.

$|H| = 7$ and $|K| = 11$ implies $|H \cap K| = 1$, so $H \cap K = \{1\}$.

$|HK| = \frac{|H|\times|K|}{|H\cap K|} = \frac{7\times 11}{1} = 77 = |G|$, so $HK = G$

and together with the fact H, K are normal subgroups of G, we conclude

$$G \cong H \times K \cong \mathbb{Z}/7\mathbb{Z} \times \mathbb{Z}/11\mathbb{Z} \cong \mathbb{Z}/77\mathbb{Z}$$

4. Prove a group of order 90 is not simple.

Solution:

$|G| = 90 = 2\times 3^2 \times 5$. The number of Sylow 5-subgroups divides $2\times 3^2 = 18$ and is congruent to 1 modulo 5, so it is 1 or 6. The number of Sylow 3-subgroups divides $2 \times 5 = 10$ and is congruent to 1 modulo 3, so it is 1 or 10.

Suppose G is a simple group, then the number of Sylow 5-groups has to be 6, and the number of Sylow 3-subgroups has to be 10.

If any pair of the 10 Sylow 3-subgroups intersect only at 1, then the union of all Sylow 5-subgroups and Sylow 3-subgroups have $(5-1)\times 6 + (9-1)\times 10 = 104 > 90 = |G|$ elements, so there exists Sylow 3-subgroups H and K such that $H \cap K \neq \{1\}$, and $|H| = |K| = 9$, H, K are distinct, so $|H \cap K| = 3$.

Consider the normalizer of $H\cap K$, $N(H\cap K) = \{g \in G | g(H\cap K)g^{-1} = H\cap K\}$. Since $|H| = |K| = 9 = 3^2$, H and K are abelian, so $H \cup K \subseteq N(H \cap K)$

$$|N(H \cap K)| \geq |H \cup K| = 9 + 9 - 3 = 15$$

and $|N(H \cap K)|$ divides $|G| = 90$, $|H| = 9$ divides $N(H \cap K)$, so $|N(H \cap K)|$ may be $18, 45, 90$.

If $|N(H \cap K)| = 90$, then $N(H \cap K)$ is a normal subgroup, contradict to the assumption G is simple.

If $|N(H \cap K)| = 45$, then $[G : N(H \cap K)] = 2$, again $N(H \cap K)$ is normal, contradiction.

If $|N(H \cap K)| = 18$, let S be the set of subgroups of G of form $g(H \cap K)g^{-1}$, $g \in G$, then G acts on S by conjugation and this is a transitive action, with the stabilizer for $H \cap K$ to be $N(H \cap K)$. By the Counting Formula, $|S| = \frac{|G|}{|N(H\cap K)|} = \frac{90}{18} = 5$, so this action corresponds to a homomorphism

$$\phi : G \longrightarrow S_5$$

$\ker(\phi) \neq G$ since it is not the trivial action, and $\ker(\phi) \neq \{1\}$ since $|G| = 90$ does not divide $|S_5| = 120$. So $\ker(\phi)$ is a proper normal subgroup of G, contradiction.

We conclude G cannot be a simple group.

5. $\phi : \mathbb{Z}/2\mathbb{Z} \longrightarrow Aut(\mathbb{Z})$ is defined by

$$\phi(\bar{m}) : \mathbb{Z} \longrightarrow \mathbb{Z}$$

$$k \mapsto (-1)^m k$$

Let $G = \mathbb{Z} \rtimes_\phi (\mathbb{Z}/2\mathbb{Z})$. Find all the elements of finite order in G.

Solution:

The elements of finite order are $(0, \bar{0})$ and $(k, \bar{1})$ for any $k \in \mathbb{Z}$.

Case 1. The identity element $(0, \bar{0})$ is of finite order.

Case 2. For any $n \in \mathbb{Z}$:

$$(n, \bar{1})(n, \bar{1}) = (n + \phi(\bar{1})n, \bar{1} + \bar{1}) = (n + (-1)^{-1}n, \bar{0}) = (0, \bar{0})$$

So $|(n, \bar{1})| = 2$.

Case 3: For any $n \in \mathbb{Z} \setminus \{0\}$, we will show by mathematical induction that $(n, \bar{0})^k = (kn, \bar{0}) \neq (0, \bar{0})$ for any positive integer k, which will imply $(n, \bar{0})$ is of infinite order:

(1). when $k = 1$, $(n, \bar{0})^1 = (n, \bar{0}) \neq (0, \bar{0})$.

(2). Assume $(n, \bar{0})^k = (kn, \bar{0})$, then

$$(n, \bar{0})^{k+1} = (n, \bar{0})^k(n, \bar{0}) = (kn, \bar{0})(n, \bar{0}) = (kn+(-1)^0 n, \bar{0}+\bar{0}) = ((k+1)n, \bar{0}) \neq (0, \bar{0})$$

6. Prove $O_2(\mathbb{R}) = SO_2(\mathbb{R}) \rtimes < r >$, where $r = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

Solution: $SO_2(\mathbb{R})$ is a normal subgroup of $O_2(\mathbb{R})$ and $< r >$ is a subgroup of $O_2(\mathbb{R})$.

Denote the identity 2×2 matrix by I_2. In $< r >= \{I_2, r\}$, $\det(r) = -1$, so $SO_2(\mathbb{R}) \cap < r >= \{I_2\}$.

$[O_2(\mathbb{R}) : SO_2(\mathbb{R})] = 2$ and $r \in O_2(\mathbb{R}) \setminus SO_2(\mathbb{R})$, so $O_2(\mathbb{R}) = SO_2(\mathbb{R}) \sqcup SO_2(\mathbb{R})r$, $O_2(\mathbb{R}) = SO_2(\mathbb{R}) < r >$

We conclude $O_2(\mathbb{R}) = SO_2(\mathbb{R}) \rtimes < r >$.

7. If H and K are subgroups of G such that $G = H \rtimes K$, prove

$$G/H \cong K$$

Solution:

Define $f : G = H \rtimes K \longrightarrow K$ by $f(h, k) = k$. We first show it is a homomorphism:

$f((h_1, k_1)(h_2, k_2)) = f(h_1 k_1 h_2 k_1^{-1}, k_1 k_2) = k_1 k_2 = f(h_1, k_1) f(h_2, k_2)$

Next, $\ker(f) = \{(h, k) \in G = H \rtimes K | f(h, k) = 1\} = \{(h, 1) \in G = H \rtimes K | h \in H\} = H$, and f is surjective since for any $k \in K$, $k = f(1, k)$.

By First Isomorphism Theorem, $G/H = H \rtimes K / \ker(f) \cong K$