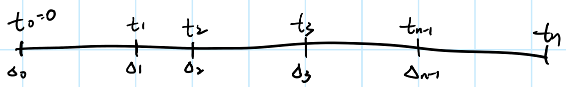
Stochastic Integral / Itô Calculus

2022年6月15日 19:02

X_t: Asset Price at $T=t$



PnL: Profit & Loss

$= \Delta_0 (X_{t_1} - X_{t_0}) + \Delta_1 (X_{t_2} - X_{t_1}) + \cdots + \Delta_{n-1} (X_{t_n} - X_{t_{n-1}})$

$I_t(f) = \int_0^t f(s,\omega)\, dB_s(\omega)$ ~ 考虑 $\int_0^t B_s\, dB_s = r.v. = \frac{1}{2} B_t^2 - \frac{1}{2} t$

限制 1) $f(s,\omega) \in \mathcal{F}_s$

amount of asset at time s, amount based on past ONLY

2) $E\left[\int_0^t f^2(s,\omega)\, ds\right] < \infty$

1) For Simple Function $f(t,\omega)$

↳ 轨迹 piecewise const.

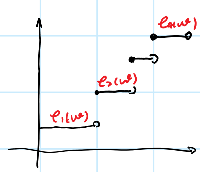
Def^n. ∃ partition $t_0 = 0 < t_1 < t_2 < \cdots < t_n$ s.t.

$f(t) = f(t_0)$, $t \in [t_0, t_1)$

$f(t) = f(t_1)$, $t \in [t_1, t_2)$

⋮

$f(t) = f(t_{n-1})$, $t \in [t_{n-1}, t_n)$



piecewise flat: SIMPLE Function.

Denote $f(t) = e_2(\omega)$, $t \in [t_2, t_3)$ 以此类推

因此 $I_t(f) = e_1 (B_{t_1} - B_{t_0})$

$+ e_2 (B_{t_2} - B_{t_1})$

$+ \cdots + e_n (B_{t_n} - B_{t_{n-1}})$

性质

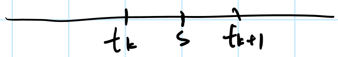
1) stochastic Integral is a Martingale

即 $I_t(f) \in$ Martingale.

Intuitively: trade martingale 则 收获 martingale.

即证 $s \le t$, $\mathbb{E}(I_t(f) \mid \mathcal{F}_s) = I_s(f)$

$$\mathbb{E}\left(\sum_{j=0}^{k} e_j (B_{t_j} - B_{t_{j-1}}) + e_{k+1}(B_{t_{k+1}} - B_{t_k}) + \sum_{j=k+2} e_j (B_{t_j} - B_{t_{j-1}}) \,\Big|\, \mathcal{F}_s\right)$$



考虑 $\mathbb{E}\left(\sum_{j=0}^{k} e_j (B_{t_j} - B_{t_{j-1}}) \mid \mathcal{F}_s\right) = \sum_{j=0}^{k} e_j (B_{t_j} - B_{t_{j-1}})$ ①

且 $e_{k+1} \in \mathcal{F}_{t_k} \Rightarrow \mathbb{E}[e_{k+1}(B_{t_{k+1}} - B_{t_k}) \mid \mathcal{F}_s] = e_{k+1}(B_s - B_{t_k})$ ②

且 $\mathbb{E}\left(\sum_{j=k+2} e_j (B_{t_j} - B_{t_{j-1}}) \mid \mathcal{F}_s\right)$

考虑 Tower Property $\mathbb{E}(\mathbb{E}(X|B)|A)) = \mathbb{E}(X|A)$

let $A = \mathcal{F}_s$, $B = \mathcal{F}_{t_{j-1}}$, 且 $j \ge k+2 \Rightarrow t_{j-1} \ge t_{k+1} \ge s$ $(A \subset B)$

有 $\mathbb{E}(e_j (B_{t_j} - B_{t_{j-1}}) \mid \mathcal{F}_s)$

$= \mathbb{E}(\mathbb{E}(e_j (B_{t_j} - B_{t_{j-1}}) \mid \mathcal{F}_{t_{j-1}}) \mid \mathcal{F}_s)$

考虑 $B_{t_j} - B_{t_{j-1}}$ 与 $\mathcal{F}_{t_{j-1}}$ 独立, 有 $\mathbb{E}(B-B) \sim N \Rightarrow 0$

有 $\mathbb{E} = 0$ ③

①+②+③ $\Rightarrow$ QED

Martingale Property of Itô Integral

Choose $\mathbb{E}(I_t(f) \mid \mathcal{F}_s) = I_s(f)$ 中 $s = 0$.

有 $\mathbb{E}(I_t(f)) = 0$

Itô积分期望为0.

考虑 $Var(I_t(f))$

Itô Isometry

性质2) $Var(I_t(f)) = \mathbb{E}(I_t(f))^2 = \mathbb{E}\int_0^t f^2(s,w)\,ds$

例) $\mathbb{E}\left(\int_0^t \sqrt{s}\, dB_s\right)^2 = \mathbb{E}\int_0^t (\sqrt{s})^2 ds = \int_0^t s\, ds = \frac{1}{2}t^2$

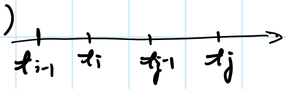
$I_t(f) = \sum_j e_j (B_{t_j} - B_{t_{j-1}})$ 记为 ΔB_j

$= \sum_j (e_j \Delta B_j)$

有 $[I_t(f)]^2 = \sum_{i,j} e_i e_j \Delta B_i \Delta B_j$

WLOG, let $i < j$ i.e. $i+1 \le j \Leftrightarrow i \le j-1$

$\mathbb{E}(e_i e_j \Delta B_i \Delta B_j \mid \mathcal{F}_0)$



$= \mathbb{E}(\mathbb{E}(e_i e_j \Delta B_i \Delta B_j \mid \mathcal{F}_{t_{j-1}}) \mid \mathcal{F}_0)$

$\in \mathcal{F}_{t_{i-1}}$, $\in \mathcal{F}_{t_{j-1}}$, $\in \mathcal{F}_{t_i}$ 均在 $\mathcal{F}_{t_{j-1}}$ 中!

$= \mathbb{E}(e_i e_j \Delta B_i \, \mathbb{E}(\Delta B_j) \mid \mathcal{F}_0)$ $\to 0$

$\mathcal{F}_{t_{j-1}}$ 中!

$$= \mathbb{E}\left(e_i e_j \Delta B_i \, \mathbb{E}(\cancel{\Delta B_j}) \mid \mathcal{F}_j \right) \quad \xrightarrow{} 0$$

$$= \sum \mathbb{E} e_i^2 (\Delta B_i)^2$$

$\mathbb{E}\, e_i^2 (\Delta B_i)^2$ 其中 $e_i \perp \Delta B_i$

$e_i \in \mathcal{F}_{t_{i-1}}$

$$= \left[\mathbb{E}(e_i^2)\right]\left[\cancel{\mathbb{E}(\Delta B_i^2)}\right] \quad \Delta t_i$$

有 $\mathbb{E}(I_t(f))^2 = \sum_i \mathbb{E} e_i^2 (\Delta B_i)^2 = \mathbb{E}\left(\sum_i e_i^2 \Delta t_i\right) = \mathbb{E}\left(\int_0^t f^2(s,\omega)\, ds\right)$ QED!