Today:

$\boxed{5.4}$ Strong Mathematical Induction

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Last time:

$\boxed{5.3}$ Mathematical Induction II

$\boxed{5.4}$ Strong Mathematical Induction

5.4 Strong Mathematical Induction and the Well-Ordering Principle for the Integers

Principle of Strong Mathematical Induction

Let $P(n)$ be a property that is defined for $n \in \mathbb{Z}$ and let $a, b \in \mathbb{Z}$ be fixed integers such that $a \leq b$. Suppose the two statements are true:

① $P(a), P(a+1), \ldots, P(b)$ are all true

(base case, basis step)

② For every integer $k \geq b$, if $P(i)$ is true for all $i \in \{a, \ldots, k\}$ then $P(k+1)$ is true

(induction hypothesis, inductive step)

Then the statement

for every integer $n \geq a$, $P(n)$

is true.

Another way to state the inductive hypothesis is to say that $P(a), P(a+1), \ldots, P(k)$ are all true.

Proving a Property of a Sequence with Strong Induction

#2

Suppose $\{b_n\}_{n=1}^{\infty}$ is a sequence such that $b_1 = 4$, $b_2 = 12$, and $b_k = b_{k-1} + b_{k-2}$ for all $k \geq 3$.

Prove that b_n is divisible by 4 for all $n \geq 1$.

Proof:

Let $P(n)$ be the claim $4 \mid b_n$.

① Base cases

$b_1 = 4$ and $4 \mid 4$ because $4(1) = 4$

$b_2 = 12$ and $4 \mid 12$ because $4(3) = 12$

② Suppose $P(i)$, i.e. $4 \mid b_i$, for all $i \in \{2, \dots, k\}$ for $k \geq 2$. Goal: Show $P(k+1)$ is true.

$$b_{k+1} = b_k + b_{k-1}$$

by definition of the sequence.

By (strong) induction hypothesis

$4 \mid b_k$ meaning there exists $l_1 \in \mathbb{Z}$ such that $b_k = 4l_1$

$4 \mid b_{k-1}$ meaning there exists $l_2 \in \mathbb{Z}$ such that $b_{k-1} = 4l_2$

so
$$
\begin{aligned}
b_{k+1} &= b_k + b_{k-1} \\
&= 4l_1 + 4l_2 \\
&= 4(l_1 + l_2)
\end{aligned}
$$

where $l_1 + l_2 \in \mathbb{Z}$ and $4 \mid b_{k+1}$.

Theorem 5.4.1

Existence and Uniqueness of Binary Integer Representations

For any $n \in \mathbb{Z}^+$, n has a unique representation in the form

$$n = \sum_{i=0}^{r} c_i 2^i$$

$$= c_r 2^r + c_{r-1} 2^{r-1} + \cdots + c_2 2^2 + c_1 2 + c_0$$

where $r \in \mathbb{Z}$ such that $r \geq 0$, $c_r = 1$ and either $c_i = 0$ or $c_i = 1$ for all $i \in \{0, \ldots, r-1\}$.

Existence proof? either $k+1$ is even or odd

Uniqueness proof?

Proof:

Let $P(n)$ be the statement that,

$$n = \sum_{i=0}^{r} c_i 2^i \quad \text{where } r \geq 0, \quad c_r = 1$$

and either $c_i = 0$ or $c_i = 1$ for all $i \in \{0, \dots, r-1\}$.

① Base case:

Demonstrating $P(1)$

$$n = 1 = 1(2^0) = \sum_{i=0}^{0} c_i 2^i,$$

where $c_0 = 1$.

② Suppose $P(i)$ for all $i \in \{1, \dots, k\}$ where $k \geq 1$, $k \in \mathbb{Z}$. Since $k+1 \in \mathbb{Z}$, either $k+1$ even or $k+1$ is odd.

Suppose $k+1$ is even.
Then $\frac{k+1}{2} \in \mathbb{Z}$ and $\frac{k+1}{2} \leq k$
so, by induction hypothesis,

$$\frac{k+1}{2} = \sum_{i=0}^{r_0} c_i 2^i$$

$$= c_{r_0} 2^{r_0} + c_{r_0 - 1} 2^{r_0 - 1} + \dots + c_1 2 + c_0 2^0.$$

Then, multiplying both sides by 2,

$$K+1 = 2\sum_{i=0}^{r_0} c_i 2^i = \sum_{i=0}^{r_0} c_i 2^{i+1}$$

$$= c_{r_0} 2^{r_0+1} + c_{r_0-1} 2^{r_0} + \cdots + c_1 2^2 + c_0 2^1$$

so $K+1$ has a binary representation.

Suppose $K+1$ is odd. Then, by parity, K is even and $K/2 \leq K$ so,

by induction hypothesis,

$$\frac{K}{2} = \sum_{i=0}^{r_1} c_i 2^i = c_{r_1} 2^{r_1} + \cdots + c_1 2^1 + c_0 2^0$$

but, multiplying by 2,

$$K = 2\sum_{i=0}^{r_1} c_i 2^i = \sum_{i=0}^{r_1} c_i 2^{i+1}$$

$$= c_{r_1} 2^{r_1+1} + c_{r_1-1} 2^{r_1} + \cdots + c_1 2^2 + c_0 2^1$$

and, adding 1,

$$K+1 = c_{r_1}2^{r_1+1} + c_{r_1-1}2^{r_1} + \cdots + c_1 2^2 + c_0 2^1 + 1$$

$$= c_{r_1}2^{r_1+1} + c_{r_1-1}2^{r_1} + \cdots + c_1 2^2 + c_0 2^1 + 1(2^0).$$

So $K+1$ has a binary representation.
Thus we established any $K \in \mathbb{Z}^+$ has this representation. (existence)

Suppose K has at least two distinct representations, i.e. there are $r, s \in \mathbb{Z}$ such that

$$K = \sum_{i=0}^{r} c_i 2^i = c_r 2^r + c_{r-1}2^{r-1} + \cdots + c_1 2 + c_0 2^0$$

$$= \sum_{i=0}^{s} b_i 2^i = b_s 2^s + b_{s-1}2^{s-1} + \cdots + b_1 2 + b_0 2^0$$

where $c_r, b_s = 1$ and c_i, b_t are all either 1 or 0 for all $i \in \{0, \ldots, r-1\}$ or $t \in \{0, \ldots, s-1\}$. Without loss of generality, suppose $r<s$ such that

$$K = c_r 2^r + c_{r-1} 2^{r-1} + \cdots + c_1 2^1 + c_0 2^0$$

$$K \leq 2^r + 2^{r-1} + \cdots + 2 + 1 = \frac{1-2^{r+1}}{1-2} = \frac{1-2^{r+1}}{-1} = 2^{r+1} - 1$$

$K \leq 2^{r+1} - 1 < 2^s$ because $r<s$ so the smallest s that satisfies $r<s$ is $s=r+1>r$

$$K < 2^s \leq 2^s + b_{s-1} 2^{s-1} + b_{s-2} 2^{s-2} + \cdots + b_1 2^1 + b_0 2^0 = K$$

so $K<K$, a contradiction. So K only has one representation. (uniqueness)

$\square$

$$13 = (1)2^3 + (1)2^2 + (0)2^1 + (1)(2^0)$$