WS 4 Due 12/8 - 3 questions. // Final 12/17 3^{30}pm - 5^{30}pm.

16.7 Surface Integrals

$$\iint_S f(x,y,z)\,dS = \iint_D f(\vec{r}(u,v))\,\|\vec{r}_u \times \vec{r}_v\|\,dA$$

S is param by $\vec{r}(u,v)$

S: surface

D: Domain of u, v that give the param surface.

Looks like $\displaystyle\int_C f(x,y)\,ds = \int_a^b f(\vec{r}(t))\,\|\vec{r}\,'(t)\|\,dt$

$$\iint_S f(x,y,z)\,dS = \iint_D f(x,y,g(x,y))\sqrt{g_x^2+g_y^2+1}\,dA$$

S is param by a function $g(x,y)$.

S be an oriented surface:

$$\iint_S \vec{F}\cdot d\vec{S} = \iint_S \vec{F}\cdot\vec{n}\,dS$$

Flux of $\vec{F}$ through S.

$\vec{n}$ ↳ normal vector to surface.

$$= \iint_D \vec{F}\cdot(\vec{r}_u\times\vec{r}_v)\,dA$$

$$\iint_S \vec{F}\cdot d\vec{S} = \iint_D (-Pg_x - Qg_y + R)\,dA$$

surface param by $z=g(x,y)$

$\vec{F} = \langle P, Q, R\rangle$

upwards: $\vec{n} = -g_x\hat{i} - g_y\hat{j} + \hat{k}$

downwards: $\vec{n} = g_x\hat{i} + g_y\hat{j} - \hat{k}$

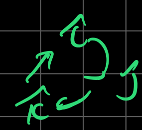
⑤ $\iint_S (x+y+z)\,dS$ S is the parallelogram with

$x = u+v$
$y = u-v$ $\quad 0 \le u \le 2$
$z = 1+2u+v$ $\quad 0 \le v \le 1$

$$\iint_S f(x,y,z)\,dS = \iint_D f(\vec r(u,v))\,\|\vec r_u \times \vec r_v\|\,dA$$

$\vec r(u,v) = \langle u+v,\ u-v,\ 1+2u+v \rangle$

$\vec r_u(u,v) = \langle 1, 1, 2 \rangle \qquad \vec r_v(u,v) = \langle 1, -1, 1 \rangle$



$$\vec r_u \times \vec r_v = (\hat i + \hat j + 2\hat k) \times (\hat i - \hat j + \hat k)$$
$$= -\hat k + (-\hat j) - \hat k + \hat i + 2\hat j - 2(-\hat i)$$
$$= 3\hat i + \hat j - 2\hat k$$

$$\|\vec r_u \times \vec r_v\| = \sqrt{3^2 + 1^2 + (-2)^2} = \sqrt{14}$$

$$f(\vec r(u,v)) = u + \cancel{v} + u - \cancel{v} + 1 + 2u + v = 4u + v + 1$$

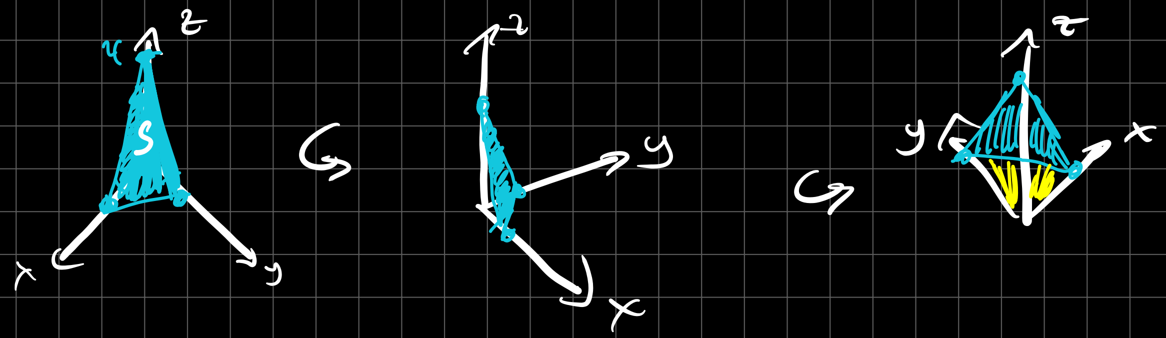
$$\iint_S (x+y+z)\,dS = \int_0^2 \int_0^1 (4u+v+1)\sqrt{14}\,dv\,du$$

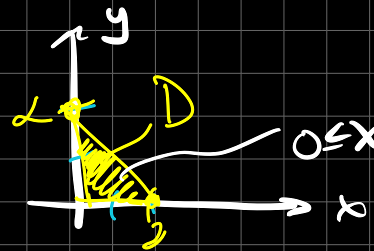
⑩ $\iint_S \underline{xz}\,dS$ S is part of plane that lies in first octant.

$x=x,\ y=y,$ $\quad 2x+2y+z=4 \quad$ $z = 4 - \underbrace{2x - 2y}_{g(x,y)}$ $\quad x, y, z \ge 0.$

$$\iint_S f(x,y,z)\,dS = \iint_D f(x,y,g(x,y))\sqrt{g_x^2 + g_y^2 + 1}\,dA$$

$g_x = -2 \qquad g_y = -2 \qquad f(x,y,g(x,y)) = x(4-2x-2y) = 4x - 2x^2 - 2xy$





$0 \le x \le 2 \qquad 0 \le y \le 2 - x.$

$$\iint_S xz\,dS = \int_0^2 \int_0^{2-x} (4x - 2x^2 - 2xy)\underbrace{\sqrt{(-2)^2 + (-2)^2 + 1}}_{3}\,dy\,dx$$

$$= 3\int_0^2 \int_0^{2-x} (4x - 2x^2 - 2xy)\,dy\,dx$$

(14) $\iint_S y^2 z^2\,dS$ $\quad S$ is part of cone $y = \underbrace{\sqrt{x^2+z^2}}_{g(x,z)}$ given by $0 \le y \le 5$

two options

$y = g(x,z) = \sqrt{x^2+z^2}$ $\qquad y^2 z^2 = \sqrt{x^2+z^2}^{\,2} z^2 = (x^2+z^2)z^2$

$$\left.\begin{aligned} g_x &= \frac{x}{\sqrt{x^2+z^2}} \\ g_z &= \frac{z}{\sqrt{x^2+z^2}} \end{aligned}\right\} \sqrt{g_x^2 + g_z^2 + 1} = \sqrt{\frac{x^2}{x^2+z^2} + \frac{z^2}{x^2+z^2} + 1} = \sqrt{2}$$

Domain:

$-5 \le x \le 5$

$-\sqrt{25-x^2} \le z \le \sqrt{25-x^2}$

$$\iint_S y^2 z^2\,dS = \int_{-5}^{5} \int_{-\sqrt{25-x^2}}^{\sqrt{25-x^2}} (x^2+z^2)z^2 \sqrt{2}\,dz\,dx \rightarrow \text{Convert to polar!!}$$

$\underline{2^{nd}\text{ option}}$: $y = \sqrt{x^2+z^2}$ let $x = r\cos\theta$ $z = r\sin\theta$.

$\boxed{y = r}$ $0 \le r \le 5$ $0 \le \theta \le 2\pi$.

$y = g(r,\theta)$

$g_r = 1$ $\sqrt{g_r^2+g_\theta^2+1} = \sqrt{2}$

$g_\theta = 0$

proceed in this param.

⋮

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(23) $\vec{F}(x,y,z) = \underbrace{xy}_{P}\hat{i} + \underbrace{yz}_{Q}\hat{j} + \underbrace{zx}_{R}\hat{k}$

S is given by: $z = 4 - x^2 - y^2$ above

$0 \le x \le 1$

$0 \le y \le 1$

and has a $\underline{\text{positive orientation}}$

$\underline{\text{upwards}}$

Find flux of $\vec{F}$ through S.

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_D (-P g_x - Q g_y + R)\, dA$$

$z = g(x,y) = \underline{4 - x^2 - y^2}$

$g_x = -2x$

$g_y = -2y$

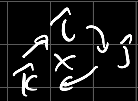
$$= \int_0^1 \int_0^1 \left(-xy(-2x) - y(4-x^2-y^2)(-2y) + (4-x^2-y^2)x\right) dx\,dy$$

(25) $\vec{F}(x,y,z) = x\hat{i} + y\hat{j} + z^2\hat{k}$ S = sphere of radius = 1 center at origin positively oriented.

find flux of $\vec{F}$ through S.

$\vec{r}(u,v) = \langle \underbrace{\sin u\cos v}_{x}, \underbrace{\sin u\sin v}_{y}, \underbrace{\cos u}_{z} \rangle$ $0 \le u \le \pi$ $0 \le v \le 2\pi$

$\vec{r}_u = \langle \cos u\cos v, \cos u\sin v, -\sin u \rangle$



$\vec{r}_v = \langle -\sin u \sin v, \ \sin u \cos v, \ 0 \rangle$

$\hat{k} \times \hat{j}$

$\hat{k} \times \hat{i}$

$\vec{r}_u \times \vec{r}_v = \langle \cos u \cos v, \ \cos u \sin v, \ -\sin u \rangle \times \langle -\sin u \sin v, \ \sin u \cos v, \ 0 \rangle$

$= \underbrace{\cos u \cos v \sin u \cos v \,\hat{k}}_{\cos u \sin u \cos^2 v} - \underbrace{\cos u \sin v \sin u \sin v (-\hat{k})}_{\cos u \sin u \sin^2 v} + \sin^2 u \sin v \,\hat{j} - \sin^2 u \cos v (-\hat{i})$

$= \cos u \sin u \,\hat{k} + \sin^2 u \sin v \,\hat{j} + \sin^2 u \cos v \,\hat{i}$

$= \sin^2 u \cos v \,\hat{i} + \sin^2 u \sin v \,\hat{j} + \cos u \sin u \,\hat{k}$

$$\text{Flux} = \iint_D \underset{\vec{F}(\vec{r}(u,v))}{\vec{F}} \cdot (\vec{r}_u \times \vec{r}_v)\, dA =$$

$$\int_0^{\pi} \int_0^{2\pi} \left[(\sin u \cos v \,\hat{i} + \sin u \sin v \,\hat{j} + \cos^2 u \,\hat{k}) \cdot (\sin^2 u \cos v \,\hat{i} + \sin^2 u \sin v \,\hat{j} + \cos u \sin u \,\hat{k}) \right] dv\, du$$

$$= \int_0^{\pi} \int_0^{2\pi} \left(\underbrace{\sin^3 u \cos^2 v + \sin^3 u \sin^2 v}_{\sin^3 u} + \cos^3 u \sin u \right) dv\, du$$

$$= \int_0^{\pi} \int_0^{2\pi} \sin^3 u + \cos^3 u \sin u \, dv\, du = 2\pi \int_0^{\pi} \sin^3 u + \cos^3 u \sin u \, du$$

$$= 2\pi \int_0^{\pi} (\sin^2 u + \cos^3 u) \sin u \, du$$

$$= 2\pi \int_0^{\pi} (1 - \cos^2 u + \cos^3 u) \sin u \, du$$

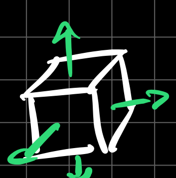
$w = \cos u$

$dw = -\sin u \, du$

$$= -2\pi \int_1^{-1} (1 - w^2 + w^3)\, dw$$

Comment

Outwards



6 diff. normal vectors.

Flux through cube $=$ 6 surface integrals.