1. $\phi : X \longrightarrow Y$ is a function from a set X to a set Y. Z is a set. Define $M(Z, X)$ to be the set of all functions from Z to X, and define $M(Z, Y)$ to be the set of all functions from Z to Y. Define

$$\Phi : M(Z, X) \longrightarrow M(Z, Y)$$

$$f \mapsto \phi \circ f$$

If ϕ is injective, prove Φ is injective.

Solution:

For any $f_1 \neq f_2$ in $M(Z, X)$, there exists $z \in Z$ such that $f_1(z) \neq f_2(z)$. ϕ is injective, so $\phi(f_1(z)) \neq \phi(f_2(z))$. We get

$$\Phi(f_1)(z) = \phi \circ f_1(z) = \phi(f_1(z)) \neq \phi(f_2(z)) = \phi \circ f_2(z) = \Phi(f_2)(z)$$

So $\Phi(f_1) \neq \Phi(f_2)$. we conclude Φ is injective.

2. Define a relation on $\mathbb{Q}$ by $a \sim b$ if $a - b = \frac{k}{3^n}$ for some $k, n \in \mathbb{Z}$ and $n > 0$.

(i). Prove that it is an equivalence relation.

(ii). Prove $\mathbb{Z} \subseteq [0]$.

(iii). Prove that there are infinitely many distinct equivalence classes for this equivalence relation.

Solution:

(i). Reflexive: $\forall r \in \mathbb{Q}$, $r - r = 0 = \frac{0}{3^1}$, so $r \sim r$

Symmetric: If $a \sim b$, then $a - b = \frac{k}{3^n}$ for some $k, n \in \mathbb{Z}$ and $n > 0$, $b - a = \frac{-k}{3^n}$, so $b \sim a$

Transitive: If $a \sim b$ and $b \sim c$, then $a - b = \frac{k_1}{3^{n_1}}$ for some $k_1, n_1 \in \mathbb{Z}$ and $n_1 > 0$, and $b - c = \frac{k_2}{3^{n_2}}$ for some $k_2, n_2 \in \mathbb{Z}$ and $n_2 > 0$. It follows

$$a - c = (a - b) + (b - c) = \frac{k_1}{3^{n_1}} + \frac{k_2}{3^{n_2}} = \frac{3^{n_2}k_1 + 3^{n_1}k_2}{3^{n_1+n_2}}$$

i.e., $a \sim c$

We conclude it is an equivalence relation.

(ii). $\forall k \in \mathbb{Z}$, $k - 0 = k = \frac{3k}{3^1}$, $k \sim 0$, so $k \in [0]$, $\mathbb{Z} \subseteq [0]$

(iii). Take positive prime numbers p, q that are not equal to 3. If $\frac{1}{p} \sim \frac{1}{q}$, then

$$\frac{1}{p} - \frac{1}{q} = \frac{q-p}{pq} = \frac{k}{3^n}$$

for some $k, n \in \mathbb{Z}$ and $n > 0$. This means

$$3^n(q-p) = kpq$$

Note that $p|kpq$, so $p|3^n(q-p)$, and together with $p \neq 3$ we get

$$p|q-p$$

This implies $p|q$, it has to be $p = q$ since both p and q are primes.

We thus conclude $p \neq q \implies [\frac{1}{p}] \neq [\frac{1}{q}]$. There are infinitely many primes, so we already obtain infinitely many distinct equivalence classes.

3. How many different equivalence relations can we define on a set of four elements?

   Solution: We know equivalence relations are in one-to-one correspondence with partition of a set, so we only need to find all the partitions of a set of four elements.

   Denote this set by $\{a, b, c, d\}$, we see the possible partitions are as follows:

   $\{a\} \sqcup \{b\} \sqcup \{c\} \sqcup \{d\}$,

   $\{a\} \sqcup \{b\} \sqcup \{c, d\}$, $\{a\} \sqcup \{c\} \sqcup \{b, d\}$, $\{a\} \sqcup \{d\} \sqcup \{b, c\}$, $\{b\} \sqcup \{c\} \sqcup \{a, d\}$, $\{b\} \sqcup \{d\} \sqcup \{a, c\}$, $\{c\} \sqcup \{d\} \sqcup \{a, b\}$,

   $\{a, b\} \sqcup \{c, d\}$, $\{a, c\} \sqcup \{b, d\}$, $\{a, d\} \sqcup \{b, c\}$

   $\{a\} \sqcup \{b, c, d\}$, $\{b\} \sqcup \{a, c, d\}$, $\{c\} \sqcup \{a, b, d\}$, $\{d\} \sqcup \{a, b, c\}$

   $\{a, b, c, d\}$

   So there are in total 15 of them.

   Remark: In general, the number of partitions of a set of n elements is called the Bell Number. You may read this Wikipedia Page for more story on that: https://en.wikipedia.org/wiki/Bell_number

4. A relation on a set is reflexive, and it also satisfies

$$a \sim b \text{ and } a \sim c \Longrightarrow b \sim c$$

Prove it is an equivalence relation.

Solution:

Symmetry: If $a \sim b$, together with $a \sim a$, we get $b \sim a$

Transitivity: If $a \sim b$ and $b \sim c$, by the symmetry we just proved, $b \sim a$. Together with $b \sim c$, we get $a \sim c$.

5. Let G be the set of all functions $\mathbb{R} \longrightarrow \mathbb{R}$. Given f_1 and f_2 in G, define $f_1 + f_2$ to be the function $(f_1 + f_2)(x) = f_1(x) + f_2(x)$ for any $x \in \mathbb{R}$. Show that G is an abelian group with the above law of composition.

Solution: (1). The law of composition is associative: for any $f_1, f_2, f_3 \in G$, $((f_1 + f_2) + f_3)(x) = (f_1 + f_2)(x) + f_3(x) = (f_1(x) + f_2(x)) + f_3(x) = f_1(x) + (f_2(x) + f_3(x)) = f_1(x) + (f_2 + f_3)(x) = (f_1 + (f_2 + f_3))(x)$ for any $x \in \mathbb{R}$, so $(f_1 + f_2) + f_3 = f_1 + (f_2 + f_3)$.

(2). The identity element is the zero function $f_0(x) \equiv 0$: for any $f \in G$, $(f + f_0)(x) = f(x) + f_0(x) = f(x) = f_0(x) + f(x) = (f_0 + f)(x)$, so $f + f_0 = f_0 + f = f$.

(3). The inverse of $f \in G$ is the function $-f \in G$ defined by $(-f)(x) = -f(x)$ for any $x \in \mathbb{R}$: $(f + (-f))(x) = f(x) + (-f)(x) = f(x) + (-f(x)) = 0$, and similarly $((-f) + f)(x) = 0$, so $f + (-f) = (-f) + f = f_0$.

So G with composition of functions is a group. It is an abelian group because for any $f_1, f_2 \in G$, $(f_1 + f_2)(x) = f_1(x) + f_2(x) = f_2(x) + f_1(x) = (f_2 + f_1)(x)$, so $f_1 + f_2 = f_2 + f_1$

6. G is a group.

(i). If $g \in G$, prove the inverse element of g is unique in G.

(ii). If $a, b \in G$, prove $(ab)^{-1} = b^{-1}a^{-1}$

(iii). If $x, y, z \in G$ and $xyz = 1$, prove $yzx = 1$

Solution:

(i). If h, k are both inverse of g, then

$$h = h.1 = h(gk) = (hg)k = 1.k = k$$

(Alternatively, you can also apply the Cancellation Law to $hg = 1 = kg$ to conclude $h = k$)

(ii). $(b^{-1}a^{-1})(ab) = b^{-1}(a^{-1}a)b = b^{-1}b = 1$, $(ab)(b^{-1}a^{-1}) = a(bb^{-1})a^{-1} = aa^{-1} = 1$, so $(ab)^{-1} = b^{-1}a^{-1}$

(iii). If $xyz = 1$, then $yzx = (x^{-1}x)yzx = x^{-1}(xyz)x = x^{-1}1x = 1$