Homework Solutions

Name: Notes and Solutions

May 10, 2022

1. Homework 9

Exercise 1.1 (3.10.1). *Find the general solution of*

$$\frac{d\vec{x}}{dt} = \begin{pmatrix} 0 & -1 & 1 \\ 2 & -3 & 1 \\ 1 & -1 & -1 \end{pmatrix} \vec{x}.$$

Sol. We make use of the eigenmethod, first searching for eigenvalues of

$$\mathbf{A} = \begin{pmatrix} 0 & -1 & 1 \\ 2 & -3 & 1 \\ 1 & -1 & -1 \end{pmatrix}$$

solving the characteristic equation

$$\begin{aligned} 0 = \det(\mathbf{A} - \lambda\mathbf{I}) &= \begin{vmatrix} -\lambda & -1 & 1 \\ 2 & -3-\lambda & 1 \\ 1 & -1 & -1-\lambda \end{vmatrix} \\ &= -\lambda((-3-\lambda)(-1-\lambda)+1) + (2(-1-\lambda)-1) + -2 - (-3-\lambda) \\ &= -\lambda(4 + 4\lambda + \lambda^2) + -3 - 2\lambda + -2 + 3 + \lambda \\ &= -\lambda^3 - 4\lambda^2 - 5\lambda - 2 \\ &= -(\lambda+2)(\lambda+1)^2. \end{aligned}$$

We see that the eigenvalues of $\mathbf{A}$ are $\lambda = -2$ and $\lambda = -1$ (with multiplicity 2).

First we consider $\lambda = -2$, and search for an eigenvector $\vec{v}_1$ satisfying

$$\vec{0} = (\mathbf{A} + 2\mathbf{I})\vec{v}_1 = \begin{pmatrix} 2 & -1 & 1 \\ 2 & -1 & 1 \\ 1 & -1 & 1 \end{pmatrix} \vec{v}_1.$$

Proceeding via Gaussian elimination, we find

$$\begin{pmatrix} 2 & -1 & 1 \\ 2 & -1 & 1 \\ 1 & -1 & 1 \end{pmatrix} \to \begin{pmatrix} 2 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} \end{pmatrix} \to \begin{pmatrix} 2 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Thus, the above equation is equivalent to solving

$$\vec{0} = (\mathbf{A} + 2\mathbf{I})\vec{v}_1 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \vec{v}_1.$$

We conclude that the first component of $\vec{v}_1$ must be zero and the second and third components must be equal, so any eigenvector is a constant multiple of $\vec{v}_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$.

Next, we consider $\lambda = -1$, and first search for an eigenvector $\vec{v}_2$ satisfying

$$\vec{0} = (\mathbf{A} + \mathbf{I})\vec{v}_2 = \begin{pmatrix} 1 & -1 & 1 \\ 2 & -2 & 1 \\ 1 & -1 & 0 \end{pmatrix} \vec{v}_2.$$

Proceeding via Gaussian elimination, we find

$$\begin{pmatrix} 1 & -1 & 1 \\ 2 & -2 & 1 \\ 1 & -1 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Thus, the above equation is equivalent to solving

$$\vec{0} = (\mathbf{A} + 2\mathbf{I})\vec{v}_2 = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \vec{v}_2.$$

We conclude that the third component of $\vec{v}_2$ must be zero, and the first and second components must be equal. Hence, any eigenvector is a constant multiple of $\vec{v}_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$.

This yields two linearly independent solutions, but we need a third. To find this, we seek a generalized eigenvector $\vec{v}_3$ for $\lambda = -1$ satisfying $(\mathbf{A} + \mathbf{I})^2\vec{v}_3 = 0$, as then we have the simple form

$$\begin{aligned} e^{\mathbf{A}t}\vec{v}_3 = e^{-t}e^{(\mathbf{A}+\mathbf{I})t}\vec{v}_3 &= e^{-t}\left(\mathbf{I} + t(\mathbf{A} + \mathbf{I}) + \frac{t^2}{2!}(\mathbf{A} + \mathbf{I})^2 + \cdots\right)\vec{v}_3 \\ &= e^{-t}\left(\vec{v}_3 + t(\mathbf{A} + \mathbf{I})\vec{v}_3 + \frac{t^2}{2!}(\mathbf{A} + \mathbf{I})^2\vec{v}_3 + \cdots\right) \\ &= e^{-t}\left(\vec{v}_3 + t(\mathbf{A} + \mathbf{I})\vec{v}_3 + \mathbf{0}\right) \\ &= e^{-t}\left(\vec{v}_3 + t(\mathbf{A} + \mathbf{I})\vec{v}_3\right). \end{aligned}$$

Computing, we need

$$\vec{0} = (\mathbf{A} + \mathbf{I})^2\vec{v}_3 = \begin{pmatrix} 1 & -1 & 1 \\ 2 & -2 & 1 \\ 1 & -1 & 0 \end{pmatrix}^2 \vec{v}_3 = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 1 & 0 \end{pmatrix} \vec{v}_3.$$

Proceeding via Gaussian elimination, we find

$$\begin{pmatrix} 0 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 1 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and so the above equation is equivalent to solving

$$\vec{0} = (\mathbf{A} + \mathbf{I})\vec{v}_3 = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \vec{v}_3.$$

We conclude only that the first two components must be equal. Since we seek a vector that is linearly independent from $\vec{v}_2$, we set the third component equal to 1 and choose $\vec{v}_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ as our generalized eigenvector. Observe that

$$(\mathbf{A}+\mathbf{I})\vec{v}_3 = \begin{pmatrix} 1 & -1 & 1 \\ 2 & -2 & 1 \\ 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

Hence, the general solution is given by

$$\begin{aligned} \vec{x}(t) &= c_1 e^{-2t}\vec{v}_1 + c_2 e^{-t}\vec{v}_2 + c_3 e^{-t}\left(\vec{v}_3 + t(\mathbf{A}+\mathbf{I})\vec{v}_3\right) \\ &= c_1 e^{-2t}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + c_2 e^{-t}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + c_3 e^{-t}\left(\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + t\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}\right) \\ &= c_1 e^{-2t}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + e^{-t}\left(c_2\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + c_3\begin{pmatrix} t+1 \\ t+1 \\ 1 \end{pmatrix}\right). \end{aligned}$$

Up to redefining constants, we can rewrite this solution as

$$\vec{x}(t) = c_1 e^{-2t}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + e^{-t}\left(c_2\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + c_3\begin{pmatrix} t \\ t \\ 1 \end{pmatrix}\right).$$

Exercise 1.2 (3.10.15)**.** *Suppose that* $\mathbf{A}^2 = \alpha\mathbf{A}$. *Find* $e^{\mathbf{A}t}$.

Sol. We determine the powers of $\mathbf{A}$ recursively. Notice that

$$\begin{aligned} \mathbf{A}^2 &= \alpha\mathbf{A} \\ \mathbf{A}^3 &= \mathbf{A}^2\mathbf{A} = \alpha^2\mathbf{A} \\ \mathbf{A}^4 &= \mathbf{A}^3\mathbf{A} = \alpha^2\mathbf{A}^2 = \alpha^3\mathbf{A} \\ \mathbf{A}^5 &= \mathbf{A}^4\mathbf{A} = \alpha^3\mathbf{A}^2 = \alpha^4\mathbf{A} \\ &\vdots \\ \mathbf{A}^n &= \mathbf{A}^{n-1}\mathbf{A} = \alpha^{n-2}\mathbf{A}^2 = \alpha^{n-1}\mathbf{A} \\ &\vdots \end{aligned}$$

Hence, we see that

$$\begin{aligned} e^{\mathbf{A}t} &= \mathbf{I} + t\mathbf{A} + \frac{t^2}{2!}\mathbf{A}^2 + \cdots + \frac{t^n}{n!}\mathbf{A}^n + \cdots \\ &= \mathbf{I} + t\mathbf{A} + \frac{\alpha t^2}{2!}\mathbf{A} + \cdots + \frac{\alpha^{n-1}t^n}{n!}\mathbf{A} + \cdots \\ &= \mathbf{I} + \frac{1}{\alpha}\left(\alpha t\mathbf{A} + \frac{\alpha^2t^2}{2!}\mathbf{A} + \cdots + \frac{\alpha^n t^n}{n!}\mathbf{A} + \cdots\right) \\ &= \mathbf{I} + \frac{1}{\alpha}\left(-1 + 1 + \alpha t + \frac{\alpha^2t^2}{2!} + \cdots + \frac{\alpha^n t^n}{n!} + \cdots\right)\mathbf{A} \\ &= \mathbf{I} + \frac{e^{\alpha t}-1}{\alpha}A. \end{aligned}$$

Exercise 1.3 (3.10.17)**.** *Let*

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

1. *Show that* $\mathbf{A}^2 = -\mathbf{I}$.

2. *Show that*

$$e^{\mathbf{A}t} = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}.$$

Proof. The first item is a straightforward computation, and we see

$$\mathbf{A}^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0-1 & \end{pmatrix} = -\mathbf{I}.$$

For the second item we could solve the equation $\frac{d\vec{x}}{dt} = \mathbf{A}\vec{x}$, but instead we exploit the symmetry from the first point. Since $\mathbf{A}^2 = -\mathbf{I}$, we also have $\mathbf{A}^3 = -\mathbf{A}$ and $\mathbf{A}^4 = -\mathbf{A}^2 = \mathbf{I}$. Continuing in this way, for any $k \geq 0$, we have

$$\mathbf{A}^{4k} = \mathbf{I}, \quad \mathbf{A}^{4k+1} = \mathbf{A}, \quad \mathbf{A}^{4k+2} = -\mathbf{I}, \quad \mathbf{A}^{4k+3} = -\mathbf{A}.$$

Hence, we can write

$$\begin{aligned} e^{\mathbf{A}t} = \sum_{n=0}^{\infty} \frac{t^n}{n!}\mathbf{A}^n &= \sum_{\substack{n=4k \\ k\geq 0}} \frac{t^n}{n!}\mathbf{I} + \sum_{\substack{n=4k+1 \\ k\geq 0}} \frac{t^n}{n!}\mathbf{A} - \sum_{\substack{n=4k+2 \\ k\geq 0}} \frac{t^n}{n!}\mathbf{I} - \sum_{\substack{n=4k+3 \\ k\geq 0}} \frac{t^n}{n!}\mathbf{A} \\ &= \left(\sum_{k=0}^{\infty} \frac{(-1)^k t^{2k}}{(2k)!} \right) \mathbf{I} + \left(\sum_{k=0}^{\infty} \frac{(-1)^k t^{2k+1}}{(2k+1)!} \right) \mathbf{A} \\ &= \cos(t)\mathbf{I} + \sin(t)\mathbf{A}, \end{aligned}$$

where we have identified the Taylor series expansions of $\cos t$ and $\sin t$. Hence

$$e^{\mathbf{A}t} = \cos(t)\mathbf{I} + \sin(t)\mathbf{A} = \begin{pmatrix} \cos t & 0 \\ 0 & \cos t \end{pmatrix} + \begin{pmatrix} 0 & \sin t \\ -\sin t & 0 \end{pmatrix} = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix},$$

as desired. □

Exercise 1.4 (3.11.1)**.** *Compute* $e^{\mathbf{A}t}$ *for*

$$\mathbf{A} = \begin{pmatrix} 1 & -1 & -1 \\ 1 & 3 & 1 \\ -3 & 1 & -1 \end{pmatrix}.$$

Sol. We determine a fundamental matrix solution $\mathbf{X}$ for the equation

$$\frac{d\vec{x}}{dt} = \mathbf{A}\vec{x}$$

and compute $\mathbf{X}(t)\mathbf{X}(0)^{-1}$. To do so, we make use of the eigenmethod, finding eigenvalues of $\mathbf{A}$ which satisfy

$$\begin{aligned} 0 = \det(\mathbf{A} - \lambda\mathbf{I}) &= \begin{vmatrix} 1-\lambda & -1 & -1 \\ 1 & 3-\lambda & 1 \\ -3 & 1 & -1-\lambda \end{vmatrix} \\ &= (1-\lambda)((3-\lambda)(-1-\lambda) - 1) + (-1-\lambda+3) - (1+3(3-\lambda)) \\ &= (1-\lambda)(\lambda^2 - 2\lambda - 4) + 2 - \lambda - 10 + 3\lambda \\ &= -\lambda^3 + 3\lambda^2 + 4\lambda - 12 \\ &= -(\lambda-3)(\lambda-2)(\lambda+2). \end{aligned}$$

Thus, the eigenvalues of $\mathbf{A}$ are $\lambda = 3$, $\lambda = 2$ and $\lambda = -2$. We seek associated eigenvectors.

First, we consider $\lambda = 3$, and search for an eigenvector $\vec{v}_1$ satisfying

$$\vec{0} = (\mathbf{A} - 3\mathbf{I})\vec{v}_1 = \begin{pmatrix} -2 & -1 & -1 \\ 1 & 0 & 1 \\ -3 & 1 & -4 \end{pmatrix} \vec{v}_1.$$

Proceeding via Gaussian elimination, we find

$$\begin{pmatrix} -2 & -1 & -1 \\ 1 & 0 & 1 \\ -3 & 1 & -4 \end{pmatrix} \to \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{1}{2} \\ 0 & \frac{5}{2} & -\frac{5}{2} \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Thus, the above equation is equivalent to

$$\vec{0} = (\mathbf{A} - 3\mathbf{I})\vec{v}_1 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \vec{v}_1.$$

We conclude that the second and third components of $\vec{v}_1$ must be equal, and the first and third components must be opposites. Hence, any eigenvector for $\lambda = 3$ is a constant multiple of $\vec{v}_1 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$, and $\vec{x}_1(t) = \begin{pmatrix} -e^{3t} \\ e^{3t} \\ e^{3t} \end{pmatrix}$.

Next, we consider $\lambda = 2$ and search for an eigenvector $\vec{v}_2$ satisfying

$$\vec{0} = (\mathbf{A} - 2\mathbf{I})\vec{v}_2 = \begin{pmatrix} -1 & -1 & -1 \\ 1 & 1 & 1 \\ -3 & 1 & -3 \end{pmatrix} \vec{v}_2.$$

Proceeding via Gaussian elimination, we find

$$\begin{pmatrix} -1 & -1 & -1 \\ 1 & 1 & 1 \\ -3 & 1 & -3 \end{pmatrix} \to= \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 4 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Thus, the above equation is equivalent to

$$\vec{0} = (\mathbf{A} - 2\mathbf{I})\vec{v}_1 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \vec{v}_2.$$

We conclude that the second component of $\vec{v}_2$ must vanish, and the first and third components must be opposites. Hence, any eigenvector for $\lambda = 2$ must be a constant multiple of $\vec{v}_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$, and $\vec{x}_2(t) = \begin{pmatrix} -e^{2t} \\ 0 \\ e^{2t} \end{pmatrix}$.

Finally, we consider $\lambda = -2$ and search for an eigenvector $\vec{v}_3$ satisfying

$$\vec{0} = (\mathbf{A} + 2\mathbf{I})\vec{v}_3 = \begin{pmatrix} 3 & -1 & -1 \\ 1 & 5 & 1 \\ -3 & 1 & 1 \end{pmatrix} \vec{v}_3.$$

Proceeding via Gaussian elimination, we find

$$\begin{pmatrix} 3 & -1 & -1 \\ 1 & 5 & 1 \\ -3 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & -\frac{1}{3} & -\frac{1}{3} \\ 0 & \frac{16}{3} & \frac{4}{3} \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -\frac{1}{4} \\ 0 & 1 & \frac{1}{4} \\ 0 & 0 & 0 \end{pmatrix}.$$

We conclude that the third component of $\vec{v}_3$ is 4 times the first, and -4 times the second. Thus, any eigenvector for $\lambda = -2$ is a constant multiple of $\vec{v}_3 = \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix}$, and $\vec{x}_3(t) = \begin{pmatrix} e^{-2t} \\ -e^{-2t} \\ 4e^{-2t} \end{pmatrix}$. Thus,

$$\mathbf{X}(t) = \begin{pmatrix} -e^{3t} & -e^{2t} & e^{-2t} \\ e^{3t} & 0 & -e^{-2t} \\ e^{3t} & e^{2t} & 4e^{-2t} \end{pmatrix}.$$

Hence

$$\mathbf{X}(0) = \begin{pmatrix} -1 & -1 & 1 \\ 1 & 0 & -1 \\ 1 & 1 & 4 \end{pmatrix},$$

and we can find the inverse by Gaussian elimination on the augmented matrix

$$\begin{aligned}
\begin{pmatrix} -1 & -1 & 1 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 \\ 1 & 1 & 4 & 0 & 0 & 1 \end{pmatrix} &\to \begin{pmatrix} 1 & 1 & -1 & -1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 5 & 1 & 0 & 1 \end{pmatrix} \\
&\to \begin{pmatrix} 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & \frac{1}{5} & 0 & \frac{1}{5} \end{pmatrix} \\
&\to \begin{pmatrix} 1 & 0 & 0 & \frac{1}{5} & 1 & \frac{1}{5} \\ 0 & 1 & 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & \frac{1}{5} & 0 & \frac{1}{5} \end{pmatrix}.
\end{aligned}$$

Thus,

$$\mathbf{X}^{-1}(0) = \frac{1}{5}\begin{pmatrix} 1 & 5 & 1 \\ -5 & -5 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

and so

$$\begin{aligned}
e^{\mathbf{A}t} = \mathbf{X}(t)\mathbf{X}^{-1}(0) &= \frac{1}{5}\begin{pmatrix} -e^{3t} & -e^{2t} & e^{-2t} \\ e^{3t} & 0 & -e^{-2t} \\ e^{3t} & e^{2t} & 4e^{-2t} \end{pmatrix}\begin{pmatrix} 1 & 5 & 1 \\ -5 & -5 & 0 \\ 1 & 0 & 1 \end{pmatrix} \\
&= \frac{1}{5}\begin{pmatrix} -e^{3t} + 5e^{2t} + e^{-2t} & -5e^{3t} + 5e^{2t} & -e^{3t} + e^{-2t} \\ e^{3t} - e^{-2t} & 5e^{3t} & e^{3t} - e^{-2t} \\ e^{3t} - 5e^{2t} + 4e^{-2t} & 5e^{3t} - 5e^{2t} & e^{3t} + 4e^{-2t} \end{pmatrix}.
\end{aligned}$$

Exercise 1.5 (3.11.15)**.** *Let* $\mathbf{X}(t)$ *be a fundamental matrix solution of*

$$\frac{d\vec{x}}{dt} = \mathbf{A}\vec{x}.$$

Prove that the solution $\vec{x}(t)$ *of the initial-value problem*

$$\begin{cases} \frac{d\vec{x}}{dt} = \mathbf{A}\vec{x} \\ \vec{x}(t_0) = \vec{x}_0 \end{cases}$$

is $\vec{x}(t) = \mathbf{X}(t)\mathbf{X}(t_0)^{-1}$.

Proof. Given a fundamental matrix solution $\mathbf{X}(t)$, we need then only find the constant vector $\vec{c}$ such that $\vec{x}(t) = \mathbf{X}(t)\vec{c}$ satisfies the initial conditions. Setting $t = t_0$, we see that $\vec{x}_0 = \vec{x}(t_0) = \mathbf{X}(t_0)\vec{c}$, and so $\vec{c} = \mathbf{X}(t_0)^{-1}\vec{x}_0$. Hence, our particular solution is

$$\vec{x}(t) = \mathbf{X}(t)\vec{c} = \mathbf{X}(t)\mathbf{X}(t_0)^{-1}.$$

□

Another way to see this would simply be by differentiating the formula for $\vec{x}(t)$ to see that it satisfies the requisite ODE, and plugging in $t = t_0$ directly to see that the initial conditions match.

2. Homework 10

Exercise 2.1 (4.1.5). *Find the equilibrium values of the system*

$$\begin{cases} \frac{dx}{dt} &= xy^2 - x \\ \frac{dy}{dt} &= x\sin(\pi y). \end{cases}$$

Sol. In order for $\frac{dx}{dt} = 0$, we require $xy^2 - x = 0$. Hence, we must have $x = 0$ or $y^2 - 1 = 0$, which implies $y = \pm 1$. In order for $\frac{dy}{dt} = 0$, we require $x\sin(\pi y) = 0$. Hence, we again require that either $x = 0$ or $\sin(\pi y) = 0$, which just tells us that $y \in \mathbb{Z}$. Combining these pieces of information, we see that both $\frac{dx}{dt}$ and $\frac{dy}{dt}$ vanish at the points $(x, y) = (0, y)$ with y arbitrary, and $(x, y) = (x, 1)$ with x arbitrary and $(x, y) = (x, -1)$ with x arbitrary; these are the equilibrium points of the system.

Exercise 2.2 (4.1.9). *Consider the system of differential equations*

$$\begin{cases} \frac{dx}{dt} &= ax + by \\ \frac{dy}{dt} &= cx + dy. \end{cases}$$

1. *Show that* $(x, y) = (0, 0)$ *is the only equilibrium point of the above system if* $ad - bc \neq 0$.
2. *Show that the above system has a line of equilibrium points if* $ad - bc = 0$.

Proof. Finding an equilibrium point amounts to solving the system of equations

$$\begin{cases} ax + by &= 0 \\ cx + dy &= 0, \end{cases}$$

which in matrix form can be written as

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

If $ad - bc = \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} \neq 0$, then $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is invertible and so the above equation only has solution $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Hence, in this situation, $(x, y) = (0, 0)$ is the only equilbrium point.

Otherwise, $ad - bc = 0$ and the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is not invertible. In that case, the rank of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is no more than 1 and so has at least a one dimensional nullspace. This corresponds to the above system of equations having at least a line (and possibly all of $\mathbb{R}^2$) as its solution, and hence at least a line of equilibrium points. □

Exercise 2.3 (4.2.3). *Determine the stability or instability of all solutions of the system of ODEs*

$$\frac{d\vec{x}}{dt} = \begin{pmatrix} -5 & 3 \\ -1 & 1 \end{pmatrix} \vec{x}.$$

Sol. As implied by Theorem 1 in §4.2, it suffices to consider the stability of the trivial solution $\vec{x} = \vec{0}$, which is determined by the eigenvalues of $\begin{pmatrix} -5 & 3 \\ -1 & 1 \end{pmatrix}$. These solve

$$0 = \det \left(\begin{pmatrix} -5 & 3 \\ -1 & 1 \end{pmatrix} - \lambda \mathbf{I} \right) = \det \begin{pmatrix} -5 - \lambda & 3 \\ -1 & 1 - \lambda \end{pmatrix} = (-5 - \lambda)(1 - \lambda) + 3 = \lambda^2 + 4\lambda - 2,$$

and so are given by

$$\lambda = \frac{-4 \pm \sqrt{16+8}}{2} = \frac{-4 \pm 2\sqrt{6}}{2} = -2 \pm \sqrt{6}.$$

Since $\sqrt{6} > 2$, one of the values of λ is positive and we conclude that the solutions of the given system are unstable.

Exercise 2.4 (4.2.11)**.** *Determine whether the solutions $x(t) \equiv 0$ and $x(t) \equiv 1$ of the single scalar equation $\frac{dx}{dt} = x(1-x)$ are stable or unstable.*

Sol. We provide a heuristic solution and then give a very explicit argument as well.

Let's first consider the solution $x_1(t) \equiv 0$. If we perturb this solution slightly in the positive direction (say by setting $x(0) = \epsilon > 0$), then notice that $\frac{dx}{dt} > 0$ and the solution moves away from zero. Similarly, if we perturb this solution slightly in the negative direction (say by setting $x(0) = \epsilon < 0$), then notice that $\frac{dx}{dt} < 0$ and the solution again moves further away from zero. Hence, the solution is unstable.

Similarly, let us look at $x_2(t) \equiv 1$. If we perturb this solution slightly in the positive direction (say by setting $x(0) = 1 + \epsilon > 1$), then notice that $\frac{dx}{dt} < 0$ and the solution moves back towards 1. Similarly, if we perturb this solution slightly in the negative direction (say by setting $x(0) = 1 - \epsilon < 1$), then notice that $\frac{dx}{dt} > 0$ and the solution again moves back towards 1. Hence, the solution is stable.

The above argument suffices, but if we want to directly use the definition of stability then notice that we can solve this equation analytically since it is separable. Notice that dividing through by $x(1-x)$ away from $x = 0$ and $x = 1$ we have

$$\begin{aligned}
\frac{dx}{dt} &= x(1-x) \\
\frac{1}{x(1-x)}\frac{dx}{dt} &= 1 \\
\left(\frac{1}{x} + \frac{1}{1-x}\right)\frac{dx}{dt} &= 1 \\
\frac{d}{dt}\left(\ln|x| - \ln|1-x|\right) &= 1 \\
\ln\left|\frac{x}{1-x}\right| &= t + c \\
\frac{x}{1-x} &= ke^t \quad k \text{ possibly } < 0 \\
x &= ke^t - xke^t \\
x(1+ke^t) &= ke^t \\
x(t) &= \frac{ke^t}{1+ke^t}.
\end{aligned}$$

Suppose $x(0) = x_0 \notin \{0, 1\}$. Then, $x_0 + x_0 k = k$ and $k = \frac{x_0}{1-x_0}$. This tells us what $x(t)$ looks like away from $x = 0$ or $x = 1$. With this solution form in mind, let us consider initial conditions that are not zero or one.

Let's look at $x_1(t) \equiv 0$ first. If x_0 is such that $k > 0$, then $x(t) \to 1$ as $t \to \infty$, and in particular no matter how small we make x_0 (and as such, k), $x(t)$ does not stay close to $x_1(t)$; since we only need one solution that starts close to x_1 and does not return to call the equilibrium solution unstable, we conclude that this solution is unstable.

Next, we consider $x_2(t) \equiv 1$. Let $\epsilon > 0$ be arbitrary and choose $\delta = \frac{2}{\epsilon}$. First, suppose $x_0 > 1 - \delta$. Then, $k > \frac{1-\delta}{\delta} > 0$ and we find

$$|x(t) - x_2(t)| = |x(t) - 1| = \left|\frac{ke^t}{1+ke^t} - 1\right| = \left|\frac{ke^t - 1 - ke^t}{1+ke^t}\right| = \frac{1}{1+ke^t} \le \frac{1}{1+k} < \frac{1}{1+\frac{1-\delta}{\delta}} = \frac{1}{\delta} = \frac{\epsilon}{2} < \epsilon.$$

Next, suppose $x_0 < 1 + \delta$. Then, $k < 0$ and $|k| < \frac{1+\delta}{\delta}$. Furthermore, $|k| > 1$ because for some $\alpha > 0$, $k = \frac{1+\alpha}{-\alpha}$ and $|k| = \frac{1+\alpha}{\alpha} > 1$. In particular, the denominator of $x(t)$ never approaches zero, and our solution never blows up. We find then

$$|x(t) - x_2(t)| = |x(t) - 1| = \left|\frac{ke^t}{1+ke^t} - 1\right| = \left|\frac{ke^t - 1 - ke^t}{1+ke^t}\right| = \frac{1}{|k|e^t - 1} \leq \frac{1}{|k| - 1} < \frac{1}{\frac{1+\delta}{\delta} - 1} = \frac{1}{\delta} = \frac{\epsilon}{2} < \epsilon.$$

Thus, if $|x_0 - 1| < \delta$, $|x(t) - x_2(t)| < \epsilon$ for all time and the equilibrium solution x_2 is stable.

Exercise 2.5 (4.2.13)**.** *Consider the differential equation $\frac{dx}{dt} = x^2$. Show that all solutions $x(t)$ with $x(0) \geq 0$ are unstable while all solutions $x(t)$ with $x(0) < 0$ are asymptotically stable.*

Proof. First consider a solution $x(0) \geq 0$. A heuristic argument for instability is that for small positive perturbations away from zero (i.e. $y(0) = x(0) + \epsilon > 0$), $\frac{dx}{dt} > 0$ and so solutions grow further away from zero. We can prove this directly by looking at analytic solutions for the equation away from zero. Notice that for x away from zero, we can write

$$\begin{aligned} \frac{1}{x^2}\frac{dx}{dt} &= 1 \\ \frac{d}{dt}\left(-\frac{1}{x}\right) &= 1 \\ -\frac{1}{x} &= t + c \\ x(t) &= \frac{-1}{t+c}. \end{aligned}$$

If $x(0) = 0$ then $x(t) \equiv 0$ for all $t > 0$ and if $x(0) = x_0 \neq 0$, we have

$$x_0 = \frac{-1}{c} \implies x(t) = \frac{-1}{t - \frac{1}{x_0}} = \frac{1}{\frac{1}{x_0} - t}.$$

Suppose that $x_0 \geq 0$. Let $y(0) = x_0 + \delta > 0$ for $\delta > 0$ arbitrarily small. Then, as $t \to \frac{1}{x_0+\delta}$, $y(t) \uparrow \infty$. However, $x\left(\frac{1}{x_0+\delta}\right) < \infty$ since $x(t)$ does not blow up until $t = \frac{1}{x_0}$ (or never, in the case $x_0 = 0$). In particular, as $t \to \frac{1}{x_0+\delta}$, $|x(y) - y(t)| \to \infty$ and so solutions will not stay close to $x(t)$ no matter how small the perturbation is. It follows that solutions $x(t)$ with $x(0) \geq 0$ are unstable.

Let us now consider solution $x(t)$ with $x(0) = x_0 < 0$. Then, by the above we can write

$$x(t) = \frac{1}{\frac{1}{x_0} - t} = \frac{1}{t + \frac{1}{|x_0|}}.$$

Suppose $|y(0) - x_0| < \frac{|x_0|}{2}$, so that in particular $y(0) < 0$ and $|y(0)| > \frac{|x_0|}{2}$. Then

$$y(t) = \frac{1}{t + \frac{1}{|y(0)|}}$$

and

$$\begin{aligned} |x(t) - y(t)| &= \left|\frac{1}{t + \frac{1}{|y(0)|}} - \frac{1}{t + \frac{1}{|x_0|}}\right| \\ &= \left|\frac{t + \frac{1}{|x_0|} - t - \frac{1}{|y(0)|}}{\left(t + \frac{1}{|y(0)|}\right)\left(t + \frac{1}{|x_0|}\right)}\right| \\ &\leq \frac{|y(0) - x_0|}{|y(0)||x_0|}\frac{1}{t^2} \end{aligned}$$

$$\leq \frac{|x_0|}{2\frac{|x_0|}{2}|x_0|}\frac{1}{t^2}$$
$$= \frac{1}{|x_0|}\frac{1}{t^2} \to 0.$$

so that $|x(t) - y(t)| \to 0$ as $t \to \infty$ and $y(t)$ ultimately returns to $x(t)$. In particular, the solution $x(t)$ is asymptotically stable. ☐

Exercise 2.6 (4.3.3)**.** *Find all equilibrium solutions of*

$$\begin{cases} \frac{dx}{dt} &= x^2 + y^2 - 1 \\ \frac{dy}{dt} &= 2xy \end{cases}$$

and determine, if possible, whether they are stable or unstable.

Sol. First, observe that we can write the system as $\frac{d}{dt}\vec{x} = \mathbf{F}(\vec{x})$, with $\vec{x} = (x, y)$ and

$$\mathbf{F}(\vec{x}) = \begin{pmatrix} f_1(x,y) \\ f_2(x,y) \end{pmatrix} = \begin{pmatrix} x^2 + y^2 - 1 \\ 2xy \end{pmatrix}.$$

Notice that $\frac{dy}{dt} = 0$ if and only if at least one of x or y is zero, and that $\frac{dx}{dt} = 0$ if and only if $x^2 + y^2 = 1$. Thus, the four points $(0, 1)$, $(0, -1)$, $(1, 0)$ and $(-1, 0)$ are the four equilibrium points of this system.

Let us first expand the system about $(0, 1)$, and let $\vec{x} = (x, y)$, $\vec{z} = \vec{x} - (0, 1)$. Using a Taylor series expansion, we have that

$$\frac{d}{dt}\vec{z} = \frac{d}{dt}\vec{x} = \mathbf{F}\begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix}\Big|_{(0,1)}\vec{z} + \mathbf{g}(\vec{z})$$
$$= \begin{pmatrix} 2x & 2y \\ 2y & 2x \end{pmatrix}\Big|_{(0,1)}\vec{z} + \mathbf{g}(\vec{z})$$
$$= \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}\vec{z} + \mathbf{g}(\vec{z})$$

with $\frac{|\mathbf{g}(z)|}{|z|} \to 0$ as $|z| \to 0$. Notice that the eigenvalues of $\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$ are solutions of

$$0 = \det\left(\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} - \lambda\mathbf{I}\right) = \det\begin{pmatrix} -\lambda & 2 \\ 2 & -\lambda \end{pmatrix} = \lambda^2 - 4 = (\lambda - 2)(\lambda + 2),$$

or $\lambda = \pm 2$. Since one of the eigenvalues of A is positive, we conclude by Theorem 2 in §4.3 that this equilibrium solution is unstable.

Similarly, we consider $(0, -1)$. We let $\vec{x} = (x, y)$, $\vec{z} = \vec{x} - (0, -1)$. Using a Taylor series expansion, we have that

$$\frac{d}{dt}\vec{z} = \frac{d}{dt}\vec{x} = \mathbf{F}\begin{pmatrix} 0 \\ -1 \end{pmatrix} + \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix}\Big|_{(0,-1)}\vec{z} + \mathbf{g}(\vec{z})$$
$$= \begin{pmatrix} 2x & 2y \\ 2y & 2x \end{pmatrix}\Big|_{(0,-1)}\vec{z} + \mathbf{g}(\vec{z})$$
$$= \begin{pmatrix} 0 & -2 \\ -2 & 0 \end{pmatrix}\vec{z} + \mathbf{g}(\vec{z}).$$

with $\frac{|\mathbf{g}(z)|}{|z|} \to 0$ as $|z| \to 0$. Notice that the eigenvalues of $\begin{pmatrix} 0 & -2 \\ -2 & 0 \end{pmatrix}$ are solutions of

$$0 = \det\left(\begin{pmatrix} 0 & -2 \\ -2 & 0 \end{pmatrix} - \lambda\mathbf{I}\right) = \det\begin{pmatrix} -\lambda & -2 \\ -2 & -\lambda \end{pmatrix} = \lambda^2 - 4 = (\lambda - 2)(\lambda + 2),$$

or $\lambda = \pm 2$. Since one of the eigenvalues of A is again positive, we conclude by Theorem 2 in §4.3 that this equilibrium solution is unstable.

Next, we consider $(1,0)$. We let $\vec{x} = (x,y)$, $\vec{z} = \vec{x} - (1,0)$. Using a Taylor series expansion, we have that

$$\begin{aligned}\frac{d}{dt}\vec{z} = \frac{d}{dt}\vec{x} &= \mathbf{F}\begin{pmatrix}1\\0\end{pmatrix} + \begin{pmatrix}\frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y}\\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y}\end{pmatrix}\Big|_{(1,0)}\vec{z} + \mathbf{g}(\vec{z})\\ &= \begin{pmatrix}2x & 2y\\ 2y & 2x\end{pmatrix}\Big|_{(1,0)}\vec{z} + \mathbf{g}(\vec{z})\\ &= \begin{pmatrix}2 & 0\\ 0 & 2\end{pmatrix}\vec{z} + \mathbf{g}(\vec{z}).\end{aligned}$$

with $\frac{|\mathbf{g}(z)|}{|z|} \to 0$ as $|z| \to 0$. Notice that the eigenvalues of $\begin{pmatrix}2 & 0\\ 0 & 2\end{pmatrix}$ are solutions of

$$0 = \det\left(\begin{pmatrix}2 & 0\\ 0 & 2\end{pmatrix} - \lambda\mathbf{I}\right) = \det\begin{pmatrix}2-\lambda & 0\\ 0 & 2-\lambda\end{pmatrix} = (2-\lambda)^2,$$

or $\lambda = 2$. Since one of the eigenvalues of A is again positive, we conclude by Theorem 2 in §4.3 that this equilibrium solution is unstable.

Finally, we consider $(-1,0)$. We let $\vec{x} = (x,y)$, $\vec{z} = \vec{x} - (-1,0)$. Using a Taylor series expansion, we have that

$$\begin{aligned}\frac{d}{dt}\vec{z} = \frac{d}{dt}\vec{x} &= \mathbf{F}\begin{pmatrix}-1\\0\end{pmatrix} + \begin{pmatrix}\frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y}\\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y}\end{pmatrix}\Big|_{(-1,0)}\vec{z} + \mathbf{g}(\vec{z})\\ &= \begin{pmatrix}2x & 2y\\ 2y & 2x\end{pmatrix}\Big|_{(-1,0)}\vec{z} + \mathbf{g}(\vec{z})\\ &= \begin{pmatrix}-2 & 0\\ 0 & -2\end{pmatrix}\vec{z} + \mathbf{g}(\vec{z}).\end{aligned}$$

with $\frac{|\mathbf{g}(z)|}{|z|} \to 0$ as $|z| \to 0$. Notice that the eigenvalues of $\begin{pmatrix}-2 & 0\\ 0 & -2\end{pmatrix}$ are solutions of

$$0 = \det\left(\begin{pmatrix}-2 & 0\\ 0 & -2\end{pmatrix} - \lambda\mathbf{I}\right) = \det\begin{pmatrix}-2-\lambda & 0\\ 0 & -2-\lambda\end{pmatrix} = (2+\lambda)^2,$$

or $\lambda = -2$. Since both of the eigenvalues of A are negative, we conclude by Theorem 2 in §4.3 that this equilibrium solution is in fact stable.

Exercise 2.7 (4.3.9)**.** *Verify that the origin is an equilibrium point of*

$$\begin{cases}\frac{dx}{dt} &= e^{x+y} - 1\\ \frac{dy}{dt} &= \sin(x+y)\end{cases}$$

and determine, if possible, if it is stable or unstable.

Sol. We first observe that at $(0,0)$, we have $\frac{dx}{dt} = e^0 - 1 = 1 - 1 = 0$ and $\frac{dy}{dt} = \sin(0+0) = 0$ so that the origin is indeed an equilibrium point. To determine whether or not it is stable or unstable, we notice that this equation is of the form $\frac{d}{dt}\vec{x} = \mathbf{F}(\vec{x})$ with $\vec{x} = (x,y)$ and

$$\mathbf{F}(\vec{x}) = \begin{pmatrix}f_1(x,y)\\ f_2(x,y)\end{pmatrix} = \begin{pmatrix}e^{x+y} - 1\\ \sin(x+y)\end{pmatrix}.$$

Taylor expanding the solution about $(0,0)$, we have with $\vec{z} = \vec{x} - (0,0)$ that

$$\begin{aligned}
\frac{d}{dt}\vec{z} = \frac{d}{dt}\vec{x} &= \mathbf{F}\begin{pmatrix}0\\0\end{pmatrix} + \begin{pmatrix}\frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y}\\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y}\end{pmatrix}\Big|_{(0,0)}\vec{z} + \mathbf{g}(\vec{z})\\
&= \begin{pmatrix}e^{x+y} & e^{x+y}\\ \cos(x+y) & \cos(x+y)\end{pmatrix}\Big|_{(0,0)}\vec{z} + \mathbf{g}(\vec{z})\\
&= \begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}\vec{z} + \mathbf{g}(\vec{z}).
\end{aligned}$$

with $\frac{|\mathbf{g}(z)|}{|z|} \to 0$ as $|z| \to 0$. Notice that the eigenvalues of $\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}$ are solutions of

$$0 = \det\left(\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix} - \lambda\mathbf{I}\right) = \det\begin{pmatrix}1-\lambda & 1\\ 1 & 1-\lambda\end{pmatrix} = (1-\lambda)^2 - 1 = -2\lambda + \lambda^2 = \lambda(\lambda - 2),$$

i.e. $\lambda = 0$ or $\lambda = 2$. Since one of the eigenvalues is positive, we conclude by Theorem 2 in §4.3 that this equilibrium is unstable.

Exercise 2.8 (4.3.11)**.** *Verify that the origin is an equilibrium point of*

$$\begin{cases}\frac{dx}{dt} &= \cos y - \sin x - 1\\ \frac{dy}{dt} &= x - y - y^2\end{cases}$$

and determine, if possible, if it is stable or unstable.

Sol. We first observe that at $(0,0)$, we have $\frac{dx}{dt} = \cos(0) - \sin(0) - 1 = 1 - 0 - 1 = 0$ and $\frac{dy}{dt} = 0 - 0 - 0^2 = 0$ so that the origin is indeed an equilibrium point. To determine whether or not it is stable or unstable, we notice that this equation is of the form $\frac{d}{dt}\vec{x} = \mathbf{F}(\vec{x})$ with $\vec{x} = (x,y)$ and

$$\mathbf{F}(\vec{x}) = \begin{pmatrix}f_1(x,y)\\ f_2(x,y)\end{pmatrix} = \begin{pmatrix}\cos y - \sin x - 1\\ x - y - y^2\end{pmatrix}.$$

Taylor expanding the solution about $(0,0)$, we have with $\vec{z} = \vec{x} - (0,0)$ that

$$\begin{aligned}
\frac{d}{dt}\vec{z} = \frac{d}{dt}\vec{x} &= \mathbf{F}\begin{pmatrix}0\\0\end{pmatrix} + \begin{pmatrix}\frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y}\\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y}\end{pmatrix}\Big|_{(0,0)}\vec{z} + \mathbf{g}(\vec{z})\\
&= \begin{pmatrix}-\cos x & -\sin y\\ 1 & -1-2y\end{pmatrix}\Big|_{(0,0)}\vec{z} + \mathbf{g}(\vec{z})\\
&= \begin{pmatrix}-1 & 0\\ 1 & -1\end{pmatrix}\vec{z} + \mathbf{g}(\vec{z}).
\end{aligned}$$

with $\frac{|\mathbf{g}(z)|}{|z|} \to 0$ as $|z| \to 0$. Notice that the eigenvalues of $\begin{pmatrix}-1 & 0\\ 1 & -1\end{pmatrix}$ are solutions of

$$0 = \det\left(\begin{pmatrix}-1 & 0\\ 1 & -1\end{pmatrix} - \lambda\mathbf{I}\right) = \det\begin{pmatrix}-1-\lambda & 0\\ 1 & -1-\lambda\end{pmatrix} = (1+\lambda)^2,$$

with solution $\lambda = -1$. Since all eigenvalues are all negative, we conclude by Theorem 2 in §4.3 that the equilibrium point is stable.

3. Homework 11

Exercise 3.1 (4.4.1)**.** *Verify that*

$$\begin{cases} x(t) = 1 + t \\ y(t) = \cos(t^2) \end{cases}$$

is a solution of

$$\begin{cases} \dot{x} = 1 \\ \dot{y} = 2(1-x)\sin(1-x)^2 \end{cases}$$

and find its orbit.

Sol. We first observe that

$$\frac{dx}{dt} = 1$$

and

$$\frac{dy}{dt} = -2t\sin(t^2) = -2(x-1)\sin(x-1)^{)} = 2(1-x)\sin(1-x)^2$$

with $t = x - 1$. It's orbit is simply the trajectory traced out by the corresponding curve $\Phi(x, y) = 0$ parametrized by $x(t)$ and $y(t)$, which can be found by substituting $t = x - 1$:

$$y(x) = \cos(x-1)^2.$$

Exercise 3.2 (4.4.5)**.** *Find the orbits of*

$$\begin{cases} \dot{x} = y \\ \dot{y} = -x. \end{cases}$$

Sol. Notice first that the only equilibrium point $\dot{x} = \dot{y} = 0$ is given by $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Away from this equilibrium, trajectories satisfy the first order differential equation

$$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{-x}{y}.$$

This equation is separable, and we can solve

$$\begin{aligned} y\frac{dy}{dx} &= -x \\ \frac{d}{dx}\left(\frac{1}{2}y^2\right) &= -x \\ \frac{1}{2}y^2 &= -\frac{1}{2}x^2 + c \\ x^2 + y^2 &= c. \end{aligned}$$

So, we see that the trajectories are the equilibrium point $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and the circles $x^2 + y^2 = c$.

Exercise 3.3 (4.4.13)**.** *Find the orbits of*

$$\begin{cases} \dot{x} = 2xy \\ \dot{y} = x^2 - y^2. \end{cases}$$

Sol. We first find the equilibrium points, which satisfy $\dot{x} = \dot{y} = 0$. So, $2xy = 0$ which implies that x or y is zero, which coupled with $x^2 - y^2 = 0$ forces both x and y to be zero. Outside of the equilibrium point $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, trajectories satisfy the first order differential equation

$$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{x^2 - y^2}{2xy}.$$

We can rewrite this equation as

$$(y^2 - x^2) + 2xy\frac{dy}{dx} = 0,$$

which is exact since $\frac{\partial}{\partial y}\left(y^2 - x^2\right) = 2y = \frac{\partial}{\partial x}(2xy)$. Integrating $2xy$ with respect to y, we find that the solution $\Phi(x, y) = c$ has to satisfy

$$\Phi(x, y) = \int 2xy\ dy = xy^2 + k(x)$$

for some $k(x)$. Differentiating with respect to x,

$$y^2 - x^2 = \frac{\partial \Phi}{\partial x} = y^2 + k'(x),$$

so $k'(x) = -x^2$ and $k(x) = -\frac{1}{3}x^3 + c$. Thus, the trajectories are given by the equilibrium point $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and the curves $xy^2 - \frac{1}{3}x^3 = c$.

Exercise 3.4 (4.7.1)**.** *Draw the phase portrait of*

$$\frac{d\vec{x}}{dt} = \begin{pmatrix} -5 & 1 \\ 1 & -5 \end{pmatrix} \vec{x}.$$

Sol. We use the eigenmethod, and find the eigenvalues and eigenvectors of the matrix $\mathbf{A} = \begin{pmatrix} -5 & 1 \\ 1 & -5 \end{pmatrix}$. The eigenvalues λ must satisfy

$$0 = \det(\mathbf{A} - \lambda \mathbf{I}) = \begin{vmatrix} -5-\lambda & 1 \\ 1 & -5-\lambda \end{vmatrix} = (5+\lambda)^2 - 1 = \lambda^2 + 10\lambda + 24 = (\lambda + 4)(\lambda + 6),$$

and so the eigenvalues are $\lambda_1 = -4$ and $\lambda_2 = -6$.

Now, for $\lambda_1 = -4$ our eigenvector $\vec{v}_1$ must satisfy $(\mathbf{A} + 4\mathbf{I})\vec{v}_1 = \vec{0}$. Row reducing, we have

$$\begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

so any eigenvector is a multiple of $\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

For $\lambda_2 = -6$, our eigenvector $\vec{v}_2$ must satisfy $(\mathbf{A} + 6\mathbf{I})\vec{v}_2 = \vec{0}$. Row reducing, we have

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

so any eigenvector is a multiple of $\vec{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

It follows that the general solution is $\vec{x}(t) = c_1 e^{-4t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-6t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$. Solutions with $c_1 = 0$ converge asymptotically to the equilibrium point $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ along the vector $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$, but otherwise solutions converge asymptotically to equilibrium approaching the vector $\begin{pmatrix}1\\1\end{pmatrix}$ since $-4 > -6$. The phase diagram is given in Figure 1 in the handwritten notes at the end of this document.

Exercise 3.5 (4.7.5)**.** *Draw the phase portrait of*

$$\frac{d\vec{x}}{dt} = \begin{pmatrix}1 & -4\\ -8 & 4\end{pmatrix}\vec{x}.$$

Sol. We use the eigenmethod, and find the eigenvalues and eigenvectors of the matrix $\mathbf{A} = \begin{pmatrix}1 & -4\\ -8 & 4\end{pmatrix}$. The eigenvalues λ must satisfy

$$0 = \det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix}1-\lambda & -4\\ -8 & 4-\lambda\end{vmatrix} = (1-\lambda)(4-\lambda) - 32 = \lambda^2 - 5\lambda - 28,$$

and so the eigenvalues are $\lambda_1 = \frac{5+\sqrt{137}}{2}$ and $\lambda_2 = \frac{5-\sqrt{137}}{2}$.

Now, for $\lambda_1 = \frac{5+\sqrt{137}}{2}$ our eigenvector $\vec{v}_1$ must satisfy $(\mathbf{A} - \lambda_1\mathbf{I})\vec{v}_1 = \vec{0}$. Row reducing, we have

$$\begin{pmatrix}\frac{-3-\sqrt{137}}{2} & -4 & 0\\ -8 & \frac{3-\sqrt{137}}{2} & 0\end{pmatrix} \to \begin{pmatrix}\frac{3+\sqrt{137}}{8} & 1 & 0\\ 0 & 0 & 0\end{pmatrix},$$

so any eigenvector is a multiple of $\vec{v}_1 = \begin{pmatrix}1\\ \frac{-3-\sqrt{137}}{8}\end{pmatrix}$.

For $\lambda_2 = \frac{5-\sqrt{137}}{2}$, our eigenvector $\vec{v}_2$ must satisfy $(\mathbf{A} - \lambda_2\mathbf{I})\vec{v}_2 = \vec{0}$. Row reducing, we have

$$\begin{pmatrix}\frac{-3+\sqrt{137}}{2} & -4 & 0\\ -8 & \frac{3+\sqrt{137}}{2} & 0\end{pmatrix} \to \begin{pmatrix}\frac{3-\sqrt{137}}{8} & 1 & 0\\ 0 & 0 & 0\end{pmatrix},$$

so any eigenvector is a multiple of $\vec{v}_2 = \begin{pmatrix}1\\ \frac{-3+\sqrt{137}}{8}\end{pmatrix}$.

It follows that the general solution is $\vec{x}(t) = c_1e^{\lambda_1 t}\vec{v}_1 + c_2e^{\lambda_2 t}\vec{v}_2$. Observe that $\lambda_1 > 0 > \lambda_2$. Solutions with $c_1 = 0$ converge asymptotically to the equilibrium point $\begin{pmatrix}0\\0\end{pmatrix}$ along the vector $\vec{v}_2$, but otherwise solutions diverge from equilibrium approaching the vector $\vec{v}_1$ since $\lambda_1 > 0$. The phase diagram is given in Figure 2 in the handwritten notes at the end of this document.

Exercise 3.6 (4.7.11)**.** *The equation of motion of a spring-mass system with damping (see Section 2.6) is $m\ddot{z} + c\dot{z} + kz = 0$, where m, c, and k are positive numbers. Convert this equation to a system of first-order equations for $x = z$, $y = \dot{z}$, and draw the phase portrait of this system. Distinguish the overdamped, critically damped, and underdamped cases.*

Sol. Observe that with $x = z$ and $y = \dot{z}$, we have the system of equations

$$\frac{d\vec{x}}{dt} = \begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = \begin{pmatrix}\dot{z}\\ \ddot{z}\end{pmatrix} = \begin{pmatrix}y\\ -\frac{k}{m}x - \frac{c}{m}y\end{pmatrix} = \begin{pmatrix}0 & 1\\ -\frac{k}{m} & -\frac{c}{m}\end{pmatrix}\vec{x}.$$

To determine the phase portrait of this system, we need to characterize the eigenvalues which govern the solutions. These solve the characteristic equation

$$0 = \det\left(\begin{pmatrix}0 & 1\\ -\frac{k}{m} & -\frac{c}{m}\end{pmatrix} - \lambda\mathbf{I}\right) = \begin{vmatrix}-\lambda & 1\\ -\frac{k}{m} & -\frac{c}{m} - \lambda\end{vmatrix} = \lambda\left(\lambda + \frac{c}{m}\right) + \frac{k}{m} = \lambda^2 + \frac{c}{m}\lambda + \frac{k}{m},$$

whose solutions $m\lambda^2 + c\lambda + k = 0$ satisfy $\lambda = \frac{-c \pm \sqrt{c^2 - 4km}}{2m}$. We distinguish three cases.

Case 1: $c^2 - 4km > 0$, or **overdamped** motion.

In this case, observe that both eigenvalues λ are real. Furthermore, $c^2 - 4km < c^2$ since $k, m, c > 0$, so $\sqrt{c^2 - 4km} < c$ and both eigenvalues are negative. Hence, we have asymptotic convergence to equilibrium $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ along the corresponding eigenvectors, as in Figure 3a of the handwritten notes at the end of this document.

Case 2: $c^2 - 4km = 0$, or **critically damped** motion.

Notice that we have a double root $\lambda = \frac{-c}{2m}$. We are only able to find a single linearly independent eigenvector though, since row reduction yields solutions to $\left(\begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{pmatrix} - \lambda \mathbf{I}\right)\vec{v} = \vec{0}$ satisfying

$$\begin{pmatrix} \frac{c}{2m} & 1 & 0 \\ -\frac{k}{m} & -\frac{c}{2m} & 0 \end{pmatrix} \to \begin{pmatrix} 1 & \frac{2m}{c} & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The components a and b of $\vec{v}$ must solve $a + \frac{2m}{c}b = 0$, which means that we only have one free parameter to determine the eigenvalue and hence a one dimensional eigenspace. Let's call the eigenvector $\vec{v}$; then, we must look for a linearly independent generalized eigenvector $\vec{u}$ satisfying $\left(\begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{pmatrix} - \lambda \mathbf{I}\right)^2 \vec{u} = \vec{0}$, and our general solution is of the form $\vec{x}(t) = c_1 e^{\lambda t}\vec{v} + c_2 e^{\lambda t}\left(\vec{u} + \left(\begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{pmatrix} - \lambda \mathbf{I}\right) t\vec{u}\right)$. Since we are only in two dimensions and $\left(\begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{pmatrix} - \lambda \mathbf{I}\right)\vec{u}$ is linearly independent from $\vec{u}$, it must be some constant k times $\vec{v}$, i.e. $k\vec{v}$. Thus, we can write

$$\vec{x}(t) = c_1 e^{\lambda t}\vec{v} + c_2 e^{\lambda t}(\vec{u} + kt\vec{v}) = (c_1 + c_2 kt)e^{\lambda t}\vec{v} + c_2 e^{\lambda t}\vec{u}.$$

Notice that since $c_2 kt$ asymptotically dominates c_2, even in the $c_1 = 0$ case we find asymptotic convergence to the equilibrium point $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ along $\vec{v}$ as in Figure 3b of the handwritten notes at the end of this document.

Case 3: $c^2 - 4km < 0$, or **underdamped motion**.

Notice that we have two complex eigenvalues with negative imaginary part, so trajectories spiral towards the equilibrium point $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Furthermore, notice that at $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ y \end{pmatrix}$ for positive y, $\frac{dx}{dt} = y > 0$, and so the spirals move clockwise as in Figure 3c of the handwritten notes at the end of this document.

Exercise 3.7 (4.7.13)**.** *In this problem, we consider the system*

$$\begin{cases} \dot{x} = y \\ \dot{y} = x + 2x^3. \end{cases}$$

1. *Show that the equilibrium solution $x = 0$, $y = 0$ of the linearized system is a saddle, and draw the phase portrait of the linearized system.*
2. *Find the orbits of the given system, and draw its phase portrait.*
3. *Show that there are exactly two orbits of the the nonlinear system (one for $x > 0$ and one for $x < 0$) on which $x \to 0$, $y \to 0$ as $t \to \infty$. Similarly, there are exactly two orbits of the nonlinear system on which $x \to 0$, $y \to 0$ as $t \to -\infty$. Thus, observe that the phase portraits for the nonlinear and linearized systems look the same near the origin.*

Sol. We first determine the linearized system about $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, which is given by

$$\frac{d\vec{z}}{dt} = \begin{pmatrix} \frac{\partial}{\partial x}(y) & \frac{\partial}{\partial y}(y) \\ \frac{\partial}{\partial x}(x+2x^3) & \frac{\partial}{\partial y}(x+2x^3) \end{pmatrix} \Big|_{(0,0)} \vec{z} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \vec{z}.$$

To determine the phase portrait of this system, we consider the eigenvalues and eigenvectors of the matrix $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. The eigenvalues solve

$$0 = \det(\mathbf{A} - \lambda \mathbf{I}) = \begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - 1,$$

or $\lambda = \pm 1$. Since $\lambda_1 = 1 > 0 > -1 = \lambda_2$, the equilibrium solution is a saddle. To draw the phase portrait, we determine the associated eigenvectors. For $\lambda_1 = 1$, we need $(\mathbf{A} - \mathbf{I})\vec{v}_1 = \vec{0}$ and so by row reduction we see that

$$\begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

so that any eigenvector is a multiple of $\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. For $\lambda_2 = -1$, we need $(\mathbf{A} + \mathbf{I})\vec{v}_2 = \vec{0}$ and so by row reduction we see that

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

so that any eigenvector is a multiple of $\vec{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

It follows that the general solution is given by $\vec{z}(t) = c_1 e^t \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. Solutions with $c_1 = 0$ converge asymptotically to the equilibrium solution $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ along the vector $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$, and otherwise diverge from equilibrium approaching the vector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$. The associated phase portrait is given in Figure 4a of the handwritten notes at the end of this document.

For the nonlinear system, away from the equilibrium point $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ trajectories satisfy the first order ODE

$$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{x + 2x^3}{y}.$$

This equation is separable, and has solutions

$$\begin{aligned} y\frac{dy}{dx} &= x + 2x^3 \\ \frac{d}{dx}\left(\frac{1}{2}y^2\right) &= x + 2x^3 \\ \frac{1}{2}y^2 &= \frac{1}{2}x^2 + \frac{1}{2}x^4 + c, \end{aligned}$$

so trajectories satisfy $y^2 = x^2 + x^4 + c$ or $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. The phase portrait is given in Figure 4b of the handwritten notes at the end of this document, as is the identification of the requested orbits that converge to equilibrium as $t \to \infty$ and $t \to -\infty$.

4. Homework 12

All of the problems in this section begin with the same diffential equation,

$$y'' + \lambda y = 0,$$

differing only in the boundary conditions. For each problem, you must consider three cases.

1. If $\lambda = 0$, then $y'' = 0$ and the solution is
$$y(x) = c_1 + c_2 t$$
2. If $\lambda < 0$, guess $y(x) = e^{rx}$ and the characteristic equation is $r^2 = \lambda$ and the solution is then
$$y(x) = c_1 e^{\sqrt{-\lambda}x} + c_2 e^{-\sqrt{-\lambda}x}.$$
It is sometimes more convenient to consider a different set of basis functions for this case,
$$\cosh\sqrt{-\lambda}x = \frac{e^{\sqrt{-\lambda}x} + e^{-\sqrt{-\lambda}x}}{2}$$
and
$$\sinh\sqrt{-\lambda}x = \frac{e^{\sqrt{-\lambda}x} - e^{-\sqrt{-\lambda}x}}{2}$$
which are formed by making linear combinations of the our first solutions, $c_1 = c_2 = 1/2$ for cosh and $c_1 = -c_2 = 1/2$ for sinh. (You can check to see that sinh and cosh both solve the ODE!) In this case, the solution can be written
$$y(x) = c_1 \cosh\sqrt{-\lambda}x + c_2 \sinh\sqrt{-\lambda}x,$$
which has the convenience that the cosh part of the solution is even and the sinh part is purely odd. They work a bit like cosine and sine, as in the next case.
3. If $\lambda > 0$, the the characteristic equations is $r^2 = -\lambda$, or $r = \pm i\sqrt{\lambda}$, and the solutions are
$$y(x) = c_1 \cos\sqrt{\lambda}x + c_2 \sin\sqrt{\lambda}x,$$

The three questions then ask us to consider when we can match the boundary conditions with these various solutions and get non-trivial solutions, where c_1 or c_2 is a free parameter.

5.1.1 Here, the boundary conditions are $y(0) = 0$ and $y'(l) = 0$.

Sol. We consider the three cases:

1. $y(0) = c_1$, so $c_1 = 0$. $y'(l) = c_2$, so $c_2 = 0$. No non-trivial solutions are possible.
2. Here the cosh and sinh version of the solutions are convenient, as $\cosh(0) = 1$ and $\sinh(0) = 0$. Thus, $y(0) = c1$, implying that $c_1 = 0$.

   We are left with $y(x) = c_2 \sinh(x)$, so $y'(x) = \sqrt{-\lambda}\cosh\sqrt{-\lambda}x$, such that $y'(l) = \sqrt{-\lambda}\cosh\sqrt{\lambda}l$. But $\cosh x > 0 \ \forall x \in R$; $y'(l) = 0$ can only be satisfied if $\lambda = 0$, violating our assumption that $\lambda < 0$. No non-trivial solutions possible.
3. $y(0) = c_1$, so again, $c_1 = 0$. $y'(l) = c_2\sqrt{\lambda}\cos\sqrt{\lambda}l$. c_2 is thus a free parameter if $\cos\sqrt{\lambda}l = 0$, which requires that $\sqrt{\lambda}l = (2n+1)\pi/2$ for any integer n, or equivalently
$$\lambda = \left(\frac{(n+1/2)\pi}{l}\right)^2$$
These eigenvalues are associated with the solution $y(x) = c\sin\sqrt{\lambda}x$

5.1.5 Here, the boundary conditions are $y(0) = 0$ and $y(\pi) - y'(\pi) = 0$.

Sol. Again, consider the three cases:

1. $y(0) = c_1$, so $c_1 = 0$. $y'(x) = c2$, Thus, $y(\pi) - y'(\pi) = -c_2$, so c_2 must also be zero. No non-trivial solutions.

2. As in the problem above, $y(0) = 0$ implies that $c_1 = 0$. Thus we are left with

$$y(\pi) - y'(\pi) = 0$$

or,

$$\sinh\sqrt{-\lambda}\pi - \sqrt{-\lambda}\cosh\sqrt{-\lambda\pi} = 0$$

This is an unpleasant algebra problem to solve, but we could find solutions by graphing the functions.

3. The beauty of cosh and sinh vs. cosine and sine is that they behave almost identically. The first condition $y(0) = 0$ implies that c_1. The second condition yields a nearly identical condition:

$$\sin\sqrt{\lambda}\pi - \sqrt{\lambda}\cos\sqrt{\lambda\pi} = 0$$

or equivalently,

$$\tan\sqrt{\lambda}\pi = \sqrt{\lambda}$$

Again, this isn't a trivial problem to solve, but again, if your life depended on it, you could graph the tangent function and see where it lines up with $\sqrt{\lambda}$; there will be infinitely many solutions!

Ugh, it was the same ODE, but with a slight twist on the boundary conditions, the eigenvalues became very different, allowing some new eigenfunctions as well (the sinh solutions associated with case 2!)

5.1.9 Here, the boundary conditions are $y(0) = y(2\pi)$ and $y'(0) = y'(2\pi)$.

Sol. This is asking us to find functions that are periodic on the interval $[0, pi)$, which we know to be the trigonometric functions. But lets show this by turning the crank. Again, consider the cases. I know it's becoming tedious, but new things will happen.

1. Case 1 is now interesting. $y(0) = y(2\pi)$ implies $c_1 = c_1 + 2\pi c_2$, or that $c_2 = 0$. Hence we are left with $y(x) = c_1$, which automatically solves our second boundary condition. Then, any constant function on the interval is an eigenfunction associated with eigenvalue $\lambda = 0$

3. I think it's more intuitive to skip the case 2 first, and start with case 3. Here, $y(0) = y(2\pi)$ implies

$$\begin{aligned} c_1 &= c_1\cos 2\pi\sqrt{\lambda} + c_2\sin 2\pi\sqrt{\lambda} \\ 0 &= c_1(\cos 2\pi\sqrt{\lambda} - 1) + c_2\sin 2\pi\sqrt{\lambda} \end{aligned}$$

and $y'(0) = y'(2\pi)$ implies

$$\begin{aligned} \sqrt{\lambda}c_2 &= -c_1\sqrt{\lambda}\sin 2\pi + c_2\sqrt{\lambda}\cos 2\pi\sqrt{\lambda} \\ c_2 &= -c_1\sin 2\pi\sqrt{\lambda} + c_2\cos 2\pi\sqrt{\lambda} \\ 0 &= -c_1\sin 2\pi\sqrt{\lambda} + c_2(\cos 2\pi\sqrt{\lambda} - 1) \end{aligned}$$

I can tell by inspection that both of these will be satisfied if $\sqrt{\lambda} = n$ for any integer n. I think we could prove this more carefully by setting this up as a linear system:

$$\begin{pmatrix} \cos 2\pi\sqrt{\lambda} - 1 & \sin 2\pi\sqrt{\lambda} \\ -\sin 2\pi\sqrt{\lambda} & \cos 2\pi\sqrt{\lambda} - 1 \end{pmatrix} \vec{c} = \vec{0}$$

where $\vec{c}$ is a vector with components c_1 and c_2. This linear system will have nontrivial solutions for $\vec{c}$ if the determinant

$$\det \begin{pmatrix} \cos 2\pi\sqrt{\lambda} - 1 & \sin 2\pi\sqrt{\lambda} \\ -\sin 2\pi\sqrt{\lambda} & \cos 2\pi\sqrt{\lambda} - 1 \end{pmatrix} = 0$$

or

$$\begin{aligned} \cos^2 2\pi\sqrt{\lambda} - 2\cos 2\pi\sqrt{\lambda} + 1 + \sin^2 2\pi\sqrt{\lambda} &= 0 \\ -2\cos 2\pi\sqrt{\lambda} + 2 &= 0 \\ \cos 2\pi\sqrt{\lambda} &= 1 \end{aligned}$$

and there we have that $\sqrt{\lambda} = n$. (In the second line we recalled that $\sin^2 x + \cos^2 x = 1$.

So our eigenvalues are $\lambda = n^2$ and eigenfunctions are

$$y(x) = c_1 \cos nx + c_2 \sin nx$$

Now consider case 2. It turns out that everything we did with the cosines and sines will be the same for hyperbolic sines and cosines. Just repeating the steps will get us to the result with the determinant:

$$\det \begin{pmatrix} \cosh 2\pi\sqrt{-\lambda} - 1 & \sinh 2\pi\sqrt{-\lambda} \\ -\sinh 2\pi\sqrt{-\lambda} & \cosh 2\pi\sqrt{-\lambda} - 1 \end{pmatrix} = 0$$

or

$$\cosh^2 2\pi\sqrt{-\lambda} - 2\cosh 2\pi\sqrt{-\lambda} + 1 + \sin^2 2\pi\sqrt{-\lambda} = 0 \tag{1}$$

Ugh, we'll need some new identities.

$$\cosh^2 x + \sinh^2 x = \cosh 2x$$

and

$$\cosh 2x = 2\cosh^2 x - 1$$

Using these, we can reduce (1) to

$$2\cosh^2 2\pi\sqrt{-\lambda} - 2\cosh 2\pi\sqrt{-\lambda} = 0$$

or

$$\cosh^2 2\pi\sqrt{-\lambda} = \cosh 2\pi\sqrt{-\lambda}$$

This is equivalent to the equation $x^2 = x$, which has only solutions $x = 0$ and $x = 1$. cosh can never be equal to 0 (it's strictly positive) and is equal to 1 only at $\cosh 0$, implying λ would have to be 0, violating our assumption that it is negative. There are no non-trivial solutions associated with $\lambda < 0$.