1. Let $f : S_3 \longrightarrow Aut(S_3)$ be defined by $f(\sigma) = \phi_\sigma$, where $\phi_\sigma(\tau) = \sigma\tau\sigma^{-1}$ for any $\tau \in S_3$.

   (i). Prove $|Aut(S_3)| \leq 6$ [Hint: Note that the 3-cycles are products of 2-cycles]

   (ii). Prove f is an isomorphism.

   Solution:

   (i). If f is an automorphism, then $|f((1\ 2))| = |f((1\ 3))| = |f((2\ 3))| = 2$, it follows $\{(1\ 2), (1\ 3), (2\ 3)\} = \{f((1\ 2)), f((1\ 3)), f((2\ 3))\}$. There are $3! = 6$ ways to arrange for the images of $f((1\ 2)), f((1\ 3)), f((2\ 3))$.

   Note $(1\ 2\ 3) = (1\ 3)(1\ 2)$ and $(1\ 3\ 2) = (1\ 2)(1\ 3)$, which means $f((1\ 2\ 3))$ and $f((1\ 3\ 2))$ will be determined by $f((1\ 2)), f((1\ 3)), f((2\ 3))$. We conclude there are at most 6 possible f.

   (ii). $Z(S_3)$ is trivial, so f is injective, $6 = |S_3| \leq |\operatorname{Im}(f)| \leq |Aut(S_3)| \leq 6$, we conclude $\operatorname{Im}(f) = Aut(S_3)$, so f is bijective. And we know f is a homomorphism, we conclude f is an automorphism.

2. Prove that every subgroup of index two is a normal subgroup.

   Solution: When $g \in H$, it is obvious that $gHg^{-1} = H$.

   When $g \notin H$, since the index of H is two, there are two left cosets H and gH, two right cosets H and Hg. Since cosets make a partition of G,

   $$H \sqcup gH = G = H \sqcup Hg$$

   This implies $gH = Hg$, i.e. $gHg^{-1} = H$.

   We conclude $gHg^{-1} = H$ for all $g \in G$, so H is a normal subgroup of G.

3. G is a group. H and K are subgroups of G.

   (i). For any $x, y \in G$, prove either $xH \cap yK = \emptyset$ or $xH \cap yK = g(H \cap K)$ for some $g \in G$.

   (ii). If $[G : H]$ and $[G : K]$ are finite, prove $[G : H \cap K]$ is finite.

   Solution:

   (i). If $xH \cap yK \neq \emptyset$, let $g \in xH \cap yK$, then $g \in xH$ implies $gH = xH$, and $g \in yK$ implies $gK = yK$. So $xH \cap yK = gH \cap gK = g(H \cap K)$.

(ii). $g(H \cap K) = gH \cap gK$, so each left coset of $H \cap K$ in G is the intersection of some left coset H in G with some left coset of K in G, and $[G : H] < \infty, [G : K] < \infty$ implies there are finitely many left cosets of H and K in G respectively, so the number of their intersection is finite, and it follows the number of left cosets of $H \cap K$ in G is finite.

4. Let H and K be subgroups of G. Let $g \in G$, the set

$$HgK = \{hgk \in G | h \in H, k \in K\}$$

is called a double coset. The set of double cosets of the above form is denoted by $H\backslash G/K$

(i). Prove the double cosets in $H\backslash G/K$ form a partition of G.

(ii). Let $G = S_3$, $H = \{id, (1\ 2)\}$, $K = \{id, (1\ 3)\}$. How many elements are there in $H\backslash G/K$?

(iii). If N is a normal subgroup of G. prove $HN = \{hn \in G | h \in H, n \in N\}$ is a subgroup of G.

(iv). If N is a normal subgroup of G, prove $H\backslash G/N$ has $[G : HN]$ elements.

Solution:

(i). Define a relation on G by $x \sim y$ if $x \in HyK$. This is an equivalence relation.

- For any $x \in G$, $x = 1x1 \in HxK$
- If $x \sim y$, then $x \in HyK$, there exists $h \in H$ and $k \in K$ such that $x = hyk$, so $y = h^{-1}xk^{-1} \in HxK$, $y \sim x$.
- If $x \sim y$ and $y \sim z$, then $x \in HyK$ and $y \in HzK$, there exists $h_1 \in H, k_1 \in K$ such that $x = h_1yk_1$, and there exists $h_2 \in H, k_2 \in K$ such that $y = h_2zk_2$, so $x = h_1yk_1 = h_1(h_2zk_2)k_1 = (h_1h_2)z(k_2k_1) \in HzK$, $x \sim z$.

By the definition of this equivalence relation, the double cosets are exactly the equivalence classes, so they form a partition of G.

(ii). $H(id)K = \{id, (1\ 2), (1\ 3), (1\ 3\ 2)\}$

$H(2\ 3)K = \{(2\ 3), (1\ 2\ 3)\}$

The disjoint union of the above two double cosets s already S_3, so there are 2 elements in $H\backslash G/K$.

(iii). For any $h_1n_1 \in HN$ and $h_2n_2 \in HN$ ($h_1, h_2 \in H, n_1, n_2 \in N$), we see

$$(h_1n_1)^{-1}(h_2n_2) = n_1^{-1}h_1^{-1}h_2n_2 = h_1^{-1}h_2(h_1^{-1}h_2)^{-1}n_1^{-1}(h_1^{-1}h_2)n_2$$

Since N is a normal subgroup of G, $n_1^{-1} \in N$, so $(h_1^{-1}h_2)^{-1}n_1^{-1}(h_1^{-1}h_2) \in N$, and $(h_1^{-1}h_2)^{-1}n_1^{-1}(h_1^{-1}h_2)n_2 \in N$. And $h_1^{-1}h_2 \in H$, thus

$$(h_1n_1)^{-1}(h_2n_2) = h_1^{-1}h_2(h_1^{-1}h_2)^{-1}n_1^{-1}(h_1^{-1}h_2)n_2 \in HN$$

We conclude HN is a subgroup of G.

(iv). N is a normal subgroup of G, for any $g \in G$, $gN = Ng$. It follows $HgN = HNg$, we find there is a one-to-one correspondence between the double cosets in $H\backslash G/K$ and right cosets of HN in G, so the number of double cosets is $[G : HN]$.

5. $\mathbb{R}$ is the group of real numbers with addition. Prove that $r + \mathbb{Z}$ is an element of finite order in $\mathbb{R}/\mathbb{Z}$ if and only if $r \in \mathbb{Q}$.

Solution:

If $r + \mathbb{Z}$ is of order $k < \infty$, then $k(r + \mathbb{Z}) = 0 + \mathbb{Z}$, i.e., $rk + \mathbb{Z} = 0 + \mathbb{Z}$, we get $rk \in \mathbb{Z}$, so $r \in \mathbb{Q}$.

Conversely, for any rational number $\frac{a}{b}$ ($a, b \in \mathbb{Z}$, $b > 0$), we see

$$b(\frac{a}{b} + \mathbb{Z}) = a + \mathbb{Z} = 0 + \mathbb{Z}$$

which implies the order of $\frac{a}{b} + \mathbb{Z}$ is finite.