Dot Product (12.3)

- Scalar mult of vectors.

$$\vec{u} \cdot \vec{v} = c \in \mathbb{R}. \quad \text{Scalar.}$$

what does $\vec{u} \cdot \vec{v}$ tell us?

$\vec{u} \cdot \vec{v} = 0 \Rightarrow \vec{u} \perp \vec{v}$

$\vec{u} \cdot \vec{v} = \|\vec{u}\| \, \|\vec{v}\| \cos\theta$

how much $\vec{u}, \vec{v}$ point in same direction

Tends to work in!

$\vec{u} \cdot \vec{v} > 0 \rightarrow$ same direction

$\vec{u} \cdot \vec{v} < 0 \Rightarrow$ opposite direction

12.3

① which is/are meaningful?

$(\vec{a} \cdot \vec{b}) \cdot \vec{c}$ ✗

Scalar $\|\vec{a}\| \cdot (\vec{b} + \vec{c})$ ✗

$\vec{a} \cdot \vec{b} + \vec{c}$

$\underbrace{\vec{a} \cdot (\vec{b} + \vec{c})}_{\text{vector} \cdot \text{vector}}$ ✓

add scalar to vector
can't do this!

$\underbrace{(\vec{a} \cdot \vec{b})}_{\text{Scalar}} \cdot \underbrace{\vec{c}}_{\text{Vector}}$ ✗

⑥ $\vec{a} = \langle p, -p, 2p \rangle$

$\vec{b} = \langle 2q, q, -q \rangle$

$\vec{a} \cdot \vec{b} = 2pq - pq - 2pq$

$= -pq.$

$\vec{u} = \langle u_1, u_2, u_3 \rangle$

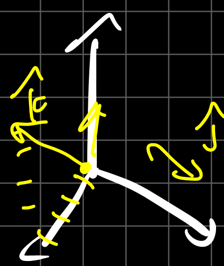
$\vec{v} = \langle v_1, v_2, v_3 \rangle$

$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + u_3 v_3$

8. $\vec{a} = 3\hat{i} + 2\hat{j} - \hat{k} = \langle 3, 2, -1 \rangle$

$\vec{b} = 4\hat{i} + 5\hat{k} = 4\hat{i} + 0\hat{j} + 5\hat{k} = \langle 4, 0, 5 \rangle$

$\vec{a} \cdot \vec{b} = 12 + 0 - 5 = 7.$

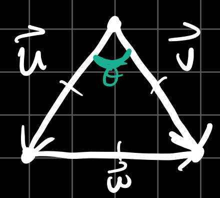


follow up: if $\vec{a} = 3\hat{i} + 2\hat{j} - \hat{k}$ ($\hat{i} = \langle 1, 0, 0 \rangle$)

$\vec{b} = 4\hat{i} + 5\hat{j}$

can we do $\vec{a} \cdot \vec{b}$? Yes

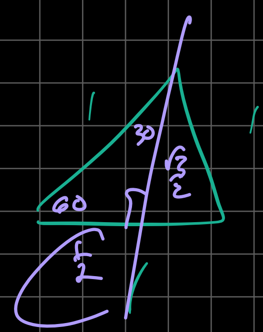
11.



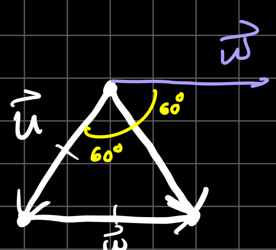
$\|\vec{u}\| = 1$

find $\vec{u} \cdot \vec{v} = \|\vec{u}\| \|\vec{v}\| \cos\theta$ $\quad \theta = \frac{\pi}{3}$

$\vec{u} \cdot \vec{w}$



$\vec{u} \cdot \vec{v} = (1)(1)\left(\frac{1}{2}\right) = \frac{1}{2}$



$\vec{u} \cdot \vec{w} = \underbrace{\|\vec{u}\| \|\vec{w}\|}_{1} \underbrace{\cos\theta}_{-\frac{1}{2}} = -\frac{1}{2}$

Comment:

Since $\vec{u}\cdot\vec{v} = \|\vec{u}\|\,\|\vec{v}\|\cos\theta$

$$|\vec{u}\cdot\vec{v}| = \big|\underbrace{\|\vec{u}\|\,\|\vec{v}\|}_{\text{positive}}\cos\theta\big|$$

$$= \|\vec{u}\|\,\|\vec{v}\|\,\underbrace{|\cos\theta|}_{\leq 1}$$

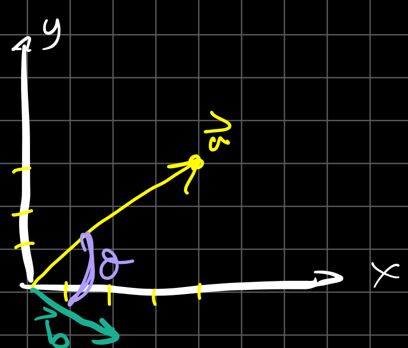
$$\leq \|\vec{u}\|\,\|\vec{v}\|$$

also

$$-\|\vec{u}\|\,\|\vec{v}\| \leq \vec{u}\cdot\vec{v} \leq \|\vec{u}\|\,\|\vec{v}\|$$

(15) $\vec{a} = \langle 4, 3\rangle \qquad \vec{b} = \langle 2, -1\rangle$

find angle b/w $\vec{a}$ & $\vec{b}$



$$\vec{a}\cdot\vec{b} = \|\vec{a}\|\,\|\vec{b}\|\cos\theta$$

$$\frac{\vec{a}\cdot\vec{b}}{\|\vec{a}\|\,\|\vec{b}\|} = \cos\theta$$

$\vec{a}\cdot\vec{b} = 8 - 3 = 5$

$\|\vec{a}\| = \sqrt{16+9} = 5$

$\|\vec{b}\| = \sqrt{4+1} = \sqrt{5}$

$$\theta = \cos^{-1}\left(\frac{\vec{a}\cdot\vec{b}}{\|\vec{a}\|\,\|\vec{b}\|}\right)$$

$$= \cos^{-1}\left(\frac{5}{5\sqrt{5}}\right) = \boxed{\cos^{-1}\left(\frac{1}{\sqrt{5}}\right)}$$

Comment: $\vec{u}\cdot\vec{u} = \|\vec{u}\|^2 \iff \|\vec{u}\| = \sqrt{\vec{u}\cdot\vec{u}}$

(18) $\vec{a} = \langle -1, 3, 4\rangle$ $\qquad \vec{b} = \langle 5, 2, 1\rangle$

find angle b/w $\vec{a}, \vec{b}$

$$\|\vec{a}\| = \sqrt{(-1)^2 + (3)^2 + (4)^2} = \sqrt{26}$$

$$\|\vec{b}\| = \sqrt{30}$$

$$\vec{a}\cdot\vec{b} = (-1)(5) + (3)(2) + (4)(1) = -5 + 6 + 4 = 5$$

$$\theta = \cos^{-1}\left(\frac{5}{\sqrt{30}\sqrt{26}}\right)$$

(24) $\vec{u} = \langle -5, 4, -2\rangle$ $\qquad \vec{v} = \langle 3, 4, -1\rangle$

$\perp$, $\parallel$, neither?

$\vec{u}\cdot\vec{v} = 0$

$\vec{u} = \alpha\vec{v}$

$$\vec{u}\cdot\vec{v} = |\alpha|\,\|\vec{v}\|\,\|\vec{v}\|\cos\theta$$

θ: 0, π

$|\alpha|\,\|\vec{v}\|\,\|\vec{v}\| = \underline{|\alpha|\,\|\vec{v}\|^2}$

$-|\alpha|\,\|\vec{v}\|\,\|\vec{v}\| = \underline{-|\alpha|\,\|\vec{v}\|^2}$

Check $\perp$: $\vec{u}\cdot\vec{v} = (-5)(3) + (4)(4) + (-2)(-1)$

$= -15 + 16 + 2 \neq 0$ $\qquad$ not $\perp$

Check $\parallel$: find α s.t. $\vec{u} = \alpha\vec{v}$

$\alpha\vec{v} = \langle 3\alpha, 4\alpha, -\alpha\rangle = \langle -5, 4, -2\rangle = \vec{u}$

$3\alpha = -5$

$4\alpha = 4 \Rightarrow \alpha = 1$

$-\alpha = -2 \Rightarrow \alpha = 2$

$\Big\}$ contradiction. NO solution.

Not parallel

(26) find x s.t. angle b/w $\langle 2,1,-1\rangle$, $\langle 1,x,0\rangle$ is $45°$?

$\cos 45° = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$

$\|\langle 2,1,-1\rangle\| = \sqrt{(2)^2+(1)^2+(-1)^2} = \sqrt{6}$

$\|\langle 1,x,0\rangle\| = \sqrt{1+x^2}$

$\langle 2,1,-1\rangle \cdot \langle 1,x,0\rangle = 2+x$

$$(2+x)^2 = \left(\sqrt{6}\sqrt{1+x^2}\left(\frac{1}{\sqrt{2}}\right)\right)^2$$

$4+4x+x^2 = 3+3x^2 \Rightarrow 2x^2-4x-1=0$

$$x = \frac{4 \pm \sqrt{16-4(2)(-1)}}{2(2)}$$

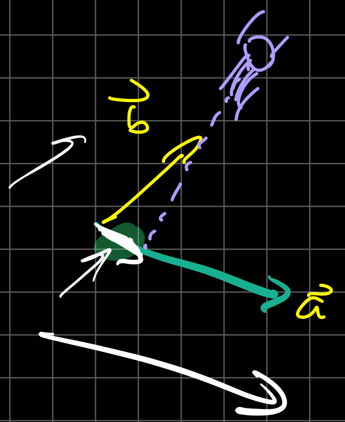
$$= \frac{4 \pm \sqrt{24}}{4}$$

$\sqrt{4}\sqrt{6} = 2\sqrt{6}$

$$= 1 \pm \frac{1}{2}\sqrt{6}$$

(43) $\vec{a} = 3\hat{i} - 3\hat{j} + \hat{k}$ $\qquad \vec{b} = 2\hat{i} + 4\hat{j} - \hat{k}$

find scalar & vector projections of $\vec{b}$ onto $\vec{a}$



$$\text{comp}_{\vec{a}} \vec{b} = \frac{\vec{a} \cdot \vec{b}}{\|\vec{a}\|}$$

$$\text{proj}_{\vec{a}} \vec{b} = \frac{\vec{a} \cdot \vec{b}}{\|\vec{a}\|^2} \vec{a} = \left(\text{comp}_{\vec{a}} \vec{b}\right) \frac{\vec{a}}{\|\vec{a}\|}$$

unit vector in direction of $\vec{a}$

$$\text{comp}_{\vec{a}} \vec{b} = \frac{3(2) + (-3)(4) + (1)(-1)}{\sqrt{3^2 + (-3)^2 + 1}} = \frac{6 - 12 - 1}{\sqrt{19}} = \frac{-7}{\sqrt{19}}$$

$$\text{proj}_{\vec{a}} \vec{b} = \frac{-7}{\sqrt{19}} \left(\frac{3\hat{i} - 3\hat{j} + \hat{k}}{\sqrt{19}} \right)$$

$$= \frac{-7}{19} \left(3\hat{i} - 3\hat{j} + \hat{k} \right)$$

what if $\vec{a} \parallel \vec{b}$ and we wanted to find $\text{proj}_{\vec{a}} \vec{b}$

$$\text{proj}_{\vec{a}} \vec{b} = \frac{\vec{a} \cdot \vec{b}}{\|\vec{a}\|^2} \vec{a} \qquad \vec{b} = \alpha \vec{a}$$

$$\vec{a} \cdot \vec{b} = \alpha (\vec{a} \cdot \vec{a}) = \alpha \|\vec{a}\|^2$$

$$\text{proj}_{\vec{a}} \vec{b} = \frac{\alpha \|\vec{a}\|^2}{\|\vec{a}\|^2} \vec{a} = \alpha \vec{a} = \vec{b}$$

(45) orthogonal proj $\qquad \text{orth}_{\vec{a}} \vec{b} = \vec{b} - \text{proj}_{\vec{a}} \vec{b}$

show $\text{orth}_{\vec a}\vec b \perp \vec a$

$$\vec a\cdot \text{orth}_{\vec a}\vec b = \vec a\cdot(\vec b - \text{proj}_{\vec a}\vec b)$$

$$= \vec a\cdot\vec b - \vec a\cdot \underbrace{\text{proj}_{\vec a}\vec b}_{\underbrace{\frac{\vec a\cdot\vec b}{\|\vec a\|^2}\vec a}_{\vec a\cdot\vec b}}$$

$\vec a\cdot(\text{scaler})\vec a$

$(\text{scaler})(\vec a\cdot\vec a)$

$\vec a\cdot(2\vec a)$
$= 2(\vec a\cdot\vec a)$

$\vec a\cdot\vec a = \|\vec a\|^2$

$$= \vec a\cdot\vec b - \vec a\cdot\vec b = 0.$$

Comment:

$$\text{orth}_{\vec a}\vec b = \vec b - \text{proj}_{\vec a}\vec b$$

$$\vec b = \text{proj}_{\vec a}\vec b + \text{orth}_{\vec a}\vec b$$