Homework 10

Due: Friday Nov. 19, by 11:59pm,
via Gradescope

- Failure to submit homework correctly will result in a zero on the homework.
- Homework must be in LaTeX. Submit the pdf file to Gradescope.
- Problems assigned from the textbook come from the 5th edition.
- No late homework accepted. Lateness due to technical issues will not be excused.

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1. (27 points) Section 6.1 # 6, 15(b)(c), 18.

   Solution:

   # 6 (a). This is False. $2 \in A$ (just set $a = 0$). On the other hand, if $2 = 10b - 3$ then $10b = 6$ and no such integer b exists. Therefore $2 \notin B$.

   # 6(b). This is true. Let's prove it.
   Proof: Let $x \in B$. Then $x = 10b - 3$ for some integer b. Now $10b - 3 = 10b - 5 + 2 = 5(2b - 1) + 2$. Therefore $x = 5a + 2$ where $a = 2b - 1$, i.e. $x \in A$.

   # 6(c). This is true. Let's prove it. Note that we NTS that $B \subseteq C$ and $C \subseteq B$.
   Proof: Let $x \in B$. Then $x = 10b - 3$ for some integer b. Now $10b - 3 = 10b - 10 + 7 = 10(b - 1) + 7$. Therefore $x = 10c + 7$ where $c = b - 1$, i.e. $x \in C$.
   Let $x \in C$. Then $x = 10c + 7$ for some integer c. Now $10c + 7 = 10c + 10 - 3 = 10(c + 1) - 3$. Therefore $x = 10b - 3$ where $b = c + 1$, i.e. $x \in B$ $\square$

   #15. Come to office hours if you need help with the sketch of the Venn diagrams.

   # 18 (a). No. Nothing belongs to the empty set.

   # 18 (b). No. The right hand side is non-empty.

   # 18 (c). Yes. The $\emptyset$ is an object in $\{\emptyset\}$.

   # 18 (d). No. No object belongs to the empty set.

2. (36 points) Section 6.1 # 25, 27.

   Solution:

   # 25(a). There is a typo in the textbook. The sets R_i are defined for $i \geq 1$. Notice that $R_{i+1} \subseteq R_i$. Therefore the answer here is $[1, 2]$.

   # 25(b). $[1, 5/4]$.

   # 25(c). No. $R_1 \cap R_2 = R_2$.

   # 25(d). Because of the nesting, the answer is $[1, 2]$.

   # 25(e). $[1, 1/n]$.

25(f). $[1, 2]$.

25(g). $\{1\}$.

27(a). No. This is because the sets a, d, e and d, f are not disjoint.

27(b). Yes.

27(c). No. The sets $\{5, 4\}$ and $\{1, 3, 4\}$ are not disjoint.

27(d). No. 6 does not belong to any of the given sets in the proposed partition.

27(e). Yes.

3. (15 points) Section 6.1 # 32, 33

Solution:

32(a) Note that $A \times B = \{(1, u), (1, v)\}$. Therefore

$$\mathcal{P}(A \times B) = \{\emptyset, \{(1, u)\}, \{(1, v)\}, A \times B\}$$

32(b). Here $X \times Y = \{(1, x), (1, y), (2, x), (2, y)\}$. The power set has 16 elements.

$$\begin{aligned}\mathcal{P}(X \times Y) = \{&\emptyset, \{(1, x)\},\ \{(1, y)\},\ \{(2, x)\},\ \{(2, y)\},\ \{(1, x), (1, y)\},\\ &\{(1, x), (2, y)\},\ \{(2, x), (1, y)\},\ \{(2, x), (2, y)\},\\ &\{(1, x), (2, x)\},\ \{(1, y), (2, y)\}, \{(1, x), (1, y), (2, x)\},\\ &\{(1, x), (2, x), (2, y)\},\ \{(1, y), (2, y), (1, x)\},\\ &\{(1, y), (2, y), (2, x)\},\ X \times Y\}\end{aligned}$$

33(a) $\mathcal{P}(\emptyset) = \{\emptyset\}$

33(b) $\mathcal{P}^2(\emptyset) = \{\emptyset,\ \{\emptyset\}\}$.

33(c) $\mathcal{P}^3(\emptyset) = \{\ \emptyset,\ \{\emptyset\},\ \{\{\emptyset\}\},\ \{\emptyset,\ \{\emptyset\}\}\ \}$

4. (9 points) Section 6.2 # 6, 11, 15.

Solution:

6(a) $(A \cap B) \cup (A \cap C)$

6(b) A

6(c) C

6(d) $x \in (A \cap B) \cup (A \cap C)$

6(a) and

6(b) and

6(c) $x \in A \cap (B \cup C)$

6(d) subset

11.
Proof: Let $x \in A \cap (B - C)$. Therefore $x \in A$ and $x \in B - C$, i.e. $x \in A$ and $x \in B$ and $x \notin C$. Since $x \in A$ and $x \in B$ it follows that $x \in A \cap B$. Since $x \notin C$, it follows that $x \notin A \cap C$. Therefore $x \in (A \cap B) - (A \cap C)$ $\square$

Remark. *This proof is more in the spirit of Epp. You can however approach it using a Theorem 2.1.1. approach.*

15.
Proof: Let $x \in A \cup \emptyset$. Then

$$\begin{aligned}&(x \in A) \vee (x \in \emptyset)\\ &x \in A \vee \mathbf{c}\\ &x \in A \quad (Identity\ Law)\end{aligned}$$

Let $x \in A$. Then

$$\begin{aligned}&x \in A\\ &x \in A \vee \mathbf{c} \quad (Identity\ Law)\\ &(x \in A) \vee (x \in \emptyset)\end{aligned}$$

$\square$

Remark. *Regarding the second half of the proof. Note that we can substitute the contradiction* **c** *with any false statement. I substituted with* $x \in \emptyset$

5. (6 points) Section 6.2 # 22, 35

22.
Proof: Let $(a, b) \in A \times (B \cup C)$. Then

$$\begin{aligned}&a \in A \wedge (b \in B \vee b \in C)\\ &(a \in A \wedge b \in B) \vee (a \in A \wedge b \in C) \quad (Distribution)\end{aligned}$$

Therefore $(a, b) \in (A \times B) \cup (A \times C)$ $\square$

35. Let's do a contradiction proof. **Proof:** Suppose that $A \cap C$ is non-empty. Then there is an element $x \in A \wedge x \in C$. Since $A \subset B$ it follows that $x \in B \wedge x \in C$. Therefore $B \cap C \neq \emptyset$ and this is a contradiction $\square$

Remark. *If you approached # 35 with a direct proof, note that the empty set is a subset of all sets, so all that we would need to show is* $A \cap C \subseteq \emptyset$*. However, what we see in the contradiction proof is that* $x \in A \cap C$ *is a false statement,therefore* $A \cap C \subseteq \emptyset$ *is vacuously true.*

Remark. *Let* A *and* B *be sets. The set* $A \times B$ *is called the Cartersian product of* A *and* B*. It is defined as*

$$A \times B = \{(a, b) : a \in A \wedge b \in B\}$$

Note that the elements of $A \times B$ *are ordered pairs. That means that* $(a, b) = (c, d)$ *if and only if* $(a = c) \wedge (b = d)$*. See page 12 in the textbook for additional info.*