3 Types of SDE

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$dX_t = a(t,X_t)\,dt + \sigma(t,X_t)\,dB_t$ 形式

Given function $V(x)$

$u(t,x) = \mathbb{E}_{X_t=x} V(X_T)$ 为 martingale w.r.t. $\{\mathcal{F}_t = \sigma(X_{s\le t})\}$

Type 1: Expected Payoff

形如 $u(t,x) = \mathbb{E}_{X_t=x} V(X_T)$ solves $\begin{cases} u_t + a(t,x)u_x + \frac{1}{2}\sigma^2(t,x)\cdot u_{xx} = 0 \\ u(T,x) = V(x) \end{cases}$

e.g. $u(t,x) = \mathbb{E}_{B_t=x} B_T^2$

step 1: 找出 stoch. process. 即 $X_t = B_t$

$\Rightarrow dX_t = \underbrace{0}_{a}\,dt + \underbrace{1}_{\sigma}\,dB_t$

step 2: 代入 公式 $\begin{cases} u_t + \frac{1}{2}u_{xx} = 0. & ① \\ u(T,x) = V(x) = x^2 & ② \end{cases}$

step 3 解 PDE: 猜测 $u(t,x) = a(t)x^2 + b(t)x + c(t)$

则有 $u(T,x) = a(T)x^2 + b(T)x + c(T)$

② 代入 I.C. 解得 $a(T)=1$, $b(T)=c(T)=0$

考虑 $\begin{cases} u_t = a'x^2 + b'x + c' \\ u_x = 2ax + b \\ u_{xx} = 2a \end{cases}$

代入① $\Rightarrow a'x^2 + b'x + c' + \frac{1}{2}\cdot 2a = 0$

$\Rightarrow \begin{cases} a' = 0 & a(T) = 1 \\ b' = 0 & b(T) = 0 \\ c' = -a & c(T) = 0 \end{cases}$

解得 $a(t) = 1$

$b(t) = 0$

$c(t) = -t + T$

从而有 $u(t,x) = x^2 + T - t$

另证

step 4: $u(t,x) = \mathbb{E}_{B_t=x}(B_T - B_t + B_t)^2$

$= \mathbb{E}_{B_t=x}\left[(B_T - B_t)^2 + 2B_t(B_T - B_t) + B_t^2\right]$

$= T - t + 2\cdot x\cdot 0 + x^2$

Type 2 Feynman-Kac formula.

$u(t,x) = \mathbb{E}_{X_t=x}\, e^{-\int_t^T b(s,X_s)\,ds}\cdot V(X_T)$ solves $\begin{cases} u_t + a(t,x)u_x + \frac{1}{2}\sigma^2(t,x)\cdot u_{xx} - b(t,x)\cdot u = 0 \\ u(T,x) = V(x) \end{cases}$

Pf.

考虑 $\underbrace{u(t,X_t)\,e^{-\int_0^t b\,ds}}_{\text{martingale!}} = \mathbb{E}_{X_t=x}\, e^{-\int_0^T b\,ds}\, V(X_T)$

$$\boxed{\text{注意 } d(z_t y_t) = z_t\,dy_t + y_t\,dz_t + dz_t\cdot dy_t}$$

let $z_t = u(t,X_t)$, $y_t = e^{-\int_0^t b\,ds}$

有 $\begin{cases} dz_t = u_t\,dt + u_x\,dX_t + \frac{1}{2}u_{xx}(dX_t)^2 \quad \boxed{\text{Itô}} = u_t\,dt + u_x\cdot(a(t,X_t)\,dt + \sigma(t,X_t)\,dB_t) + \frac{1}{2}u_{xx}\sigma^2(t,X_t)\cdot dt \\ dy_t = -b\,e^{-\int_0^t b\,ds}\,dt \end{cases}$

$\therefore dz_t = (u_t + au_x + \sigma^2 u_{xx})\,dt + \sigma u_x\,dB_t$

注意 $dy_t\cdot dz_t = 0$ 因 dy_t 无 oscillatory term.

则 $d(y_t z_t) = u(t,X_t)\cdot(-b)\cdot y_t\,dt + y_t(u_t + au_x + \frac{1}{2}\sigma^2 u_{xx})\,dt + y_t(\sigma u_x)\cdot dB_t$

有 $d(y_t z_t) = y_t\left[-bu + u_t + au_x + \frac{1}{2}\sigma^2 u_{xx}\right]dt$ martingale $= 0$! $\boxed{\text{QED}}$

$+ y_t\,[\sigma u_x]\,dB_t$.

Type 3 Running Payoff

$u(t,x) = \mathbb{E}_{X_t=x}\int_t^T b(s,X_s)\,ds$ solves $\begin{cases} u_t + au_x + \frac{1}{2}\sigma^2 u_{xx} + b(t,x) = 0 \\ u(T,x) = 0 \end{cases}$

Pf. construct. $u(t,X_t) + \int_0^t b(s,X_s)\,ds$ 为 martingale $= \mathbb{E}_{X_t=x}\int_0^T b(s,X_s)\,ds$