Ito Formula

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Let us step back for a second and take a look at the Riemann integral. Even though it is defined as the limit of Riemann sums in practice one never does this. Instead, one uses the fundamental theorem of calculus and the chain rule. For instance, to compute $I(t) = \int_0^t se^{-s^2/2}ds$ we notice that $(-e^{-s^2/2})' = se^{-s^2/2}$ and thus

$$I(t) = \int_0^t se^{-s^2/2}ds = -e^{-s^2/2}\Big|_0^t = 1 - e^{-t^2/2}. \qquad (3)$$

而 $f(g(x))' = f'(g(x)) \cdot g'(x)$

In general, it is desirable to have some analog of the chain rule in the case of Itô integral (to avoid taking the limit of $I(\varphi_n)$. The analog for the chain rule is the Itô formula.

Itô's Formula. $f(t,x)$, f_t, f_x, f_{xx} are defined & cts

$$\forall\ T \geq 0 \quad f(T, B_T) = f(0, B_0) + \int_0^T f_t(s, B_s)ds + \int_0^T f_x(s, B_s)dB_s + \frac{1}{2}\int_0^T f_{xx}(s, B_s)\, d\cancel{\langle B_s, B_s\rangle}^{ds}$$

Theorem 1. *Let $f(t,x)$ be a function for which the partial derivatives $f_t(t,x)$, $f_x(t,x)$ and $f_{xx}(t,x)$ are defined and continuous, and let B_t be a Brownian motion. Then for every $T \geq 0$*

$$f(T, B_T) = f(0, B_0) + \int_0^T f_t(t, B_t)dt + \int_0^T f_x(t, B_t)dB_t + \frac{1}{2}\int_0^T f_{xx}(t, B_t)dt. \quad (5)$$

Remark 2. One often writes Itô formula in the differential form:

$$df(t, B_t) = f_t(t, B_t)dt + f_x(t, B_t)dB_t + \frac{1}{2}f_{xx}(t, B_t)dt. \qquad (13)$$

例 $f(t,x) = x^2$ ($B_s = x$) 而 $f_t = 0$ $\quad f_x = 2x$, $f_{xx} = 2$

$$\therefore B_T^2 = B_0^2 + \int_0^T 2B_s dB_s + \frac{1}{2}\int_0^T 2\,ds$$

$$\therefore B_T^2 = 0 + \int_0^T 2B_s dB_s + T \quad \Rightarrow \int_0^T B_s dB_s = \frac{1}{2}(B_T^2 - T)$$

泰勒展开 (t_j, x_j)

$$f(t_{j+1}, x_{j+1}) = f(t_j, x_j) + f_t(t_j, x_j)(t_{j+1} - t_j) + f_x(t_j, x_j)(x_{j+1} - x_j)$$
$$+ \frac{1}{2}f_{tt}(t_j, x_j)(t_{j+1} - t_j)^2 + \frac{1}{2}f_{xx}(t_j, x_j)(x_{j+1} - x_j)^2 + f_{xt}(t_j, x_j)(t_{j+1} - t_j)\cdot(x_{j+1} - x_j)$$
$$+ \text{higher order terms}$$

推广至BM

$$\sum_j f(t_{j+1}, B_{j+1}) = \sum_j f(t_j, B_j) + f_t(t_j, B_j)(t_{j+1} - t_j) + f_x(t_j, B_j)(B_{j+1} - B_j)$$
$$+ \frac{1}{2}f_{tt}(t_j, B_j)(t_{j+1} - t_j)^2 + \frac{1}{2}f_{xx}(t_j, B_j)(B_{j+1} - B_j)^2 + f_{xt}(t_j, x_j)(t_{j+1} - t_j)\cdot(x_{j+1} - x_j)$$
$$+ \text{higher order terms}$$

Apply this formula with $x_{j+1} = B(t_{j+1}), x_j = B(t_j)$ and sum over j:

$$\begin{aligned} f(T, B_T) - f(0, B_0) &= \sum_j \Big(f(t_{j+1}, B(t_{j+1})) - f(t_j, B(t_j))\Big) \\ &= \sum_j f_t(t_j, B(t_j))(t_{j+1} - t_j) \\ &+ \sum_j f_x(t_j, B(t_j))(B(t_{j+1}) - B(t_j)) \\ &+ \sum_j \frac{1}{2}f_{xx}(t_j, B(t_j))(B(t_{j+1}) - B(t_j))^2 \\ &+ \sum_j f_{tx}(t_j, B(t_j))(B(t_{j+1}) - B(t_j))(t_{j+1} - t_j) \\ &+ \sum_j \frac{1}{2}f_{tt}(t_j, B(t_j))(t_{j+1} - t_j)^2 + \text{ higher order terms.} \end{aligned} \qquad (7)$$

As we take the limit $||\Pi|| \to 0$ then the first term on the right-hand side converges to an ordinary Riemann integral

$$\sum_j f_t(t_j, B(t_j))(t_{j+1} - t_j) \to \int_0^T f_t(t, B_t)dt. \qquad (8)$$

As $||\Pi|| \to 0$ the second term converges to an Itô integral

$$\sum_j f_x(t_j, B(t_j))(B(t_{j+1}) - B(t_j)) \to \int_0^T f_x(t, B_t)dB_t. \qquad (9)$$

Let us study the third sum. To simplify notation put $a_j = f_{xx}(t_j, B_j), \Delta B_j =$

考虑 $\sum_j f_t(t_j, B_j)\Delta t_j \to \int_0^T f_t(s, B_s)ds$

$\sum_j f_x(t_j, B_j)\Delta B_j \to \int_0^T f_x(s, B_s)dB_s$

$\left|\frac{1}{2}\sum_j f_{tt}(t_j, B_j)(\Delta t_j)^2\right| \to 0$

$\frac{1}{2}\sum_j f_{xt}(t_j, B_j)(\Delta B_j)(\Delta t_j) \to 0$ as $\langle B_t, t\rangle = 0$ 之前证过

$\frac{1}{2}\sum_j f_{xx}(t_j, B_j)(\Delta B_j)^2 \to \frac{1}{2}\int_0^T f_{xx}(s, B_s)ds$

证明: $\sigma^2(\text{LHS} - \text{RHS}) \to 0$

$$\sum_j f_x(t_j, B(t_j))(B(t_{j+1}) - B(t_j)) \to \int_0^T f_x(t, B_t)dB_t. \tag{9}$$

Let us study the third sum. To simplify notation put $a_j = f_{xx}(t_j, B_j), \Delta B_j = B(t_{j+1}) - B(t_j)$. Then

$$\sum_j \frac{1}{2} f_{xx}(t_j, B(t_j))(B(t_{j+1}) - B(t_j))^2 = \sum_j a_j \left(\Delta B_j\right)^2. \tag{10}$$

Consider

$$\mathbb{E}\left[\left(\sum_j a_j(\Delta B_j)^2 - \sum_j a_j \Delta t_j\right)^2\right] = \sum_{i,j} \mathbb{E}\left[a_i a_j((\Delta B_j)^2 - \Delta t_j)((\Delta B_i)^2 - \Delta t_i)\right].$$

If $i < j$ then $a_i a_j((\Delta B_i)^2 - \Delta t_i)$ and $(\Delta B_j)^2 - \Delta t_j$ are independent and so the terms vanish in this case. Similarly for $i > j$. So we are left with

$$\begin{aligned}
\sum_j \mathbb{E}\left[a_j^2((\Delta B_j)^2 - \Delta t_j)^2\right] &= \sum_j \mathbb{E}\left[a_j^2\right] \mathbb{E}\left[(\Delta B_j)^4 - 2(\Delta B_j)^2 \Delta t_j + (\Delta t_j)^2\right] \\
&= \sum_j \mathbb{E}\left[a_j^2\right] \left[3(\Delta t_j)^2 - 2\Delta t_j \Delta t_j + (\Delta t_j)^2\right] \\
&= \sum_j \mathbb{E}\left[a_j^2\right] (\Delta t_j)^2 \to 0 \text{ as } \Delta t_j \to 0. \qquad (11)
\end{aligned}$$

Thus the third term converges to

$$\frac{1}{2} \int_0^T f_{xx}(t, B_t)dt. \tag{12}$$

□