Office 878

F.T. Line Integrals.

C : p.w. smooth.
$\vec{r}(t)$
$a \le t \le b$

$$\boxed{\vec{F} = \vec{\nabla} f}$$

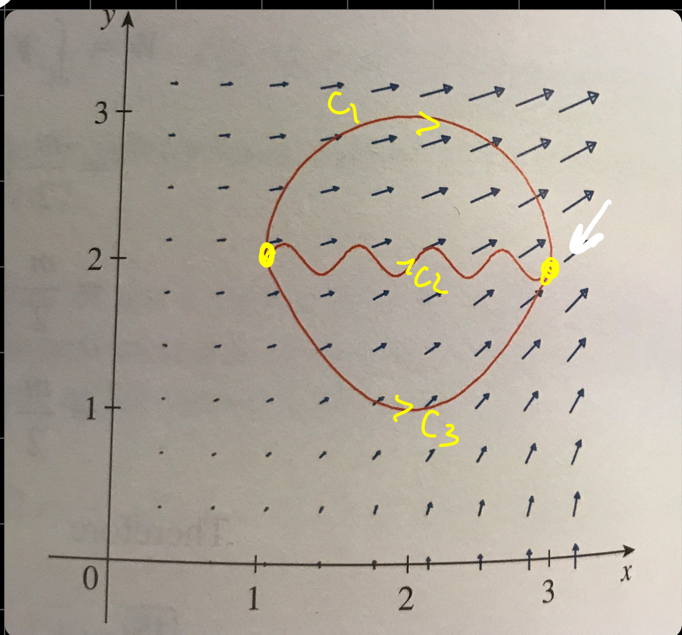
scalar potential.

$\vec{F}$ is conservative

$$\int_a^b f'(x)\,dx = f(b) - f(a)$$

$$\int_C \vec{\nabla} f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$$

①



$\vec{F} = \langle 2xy, x^2 \rangle$

Curves start at $(1,2)$
end at $(3,2)$

$$\int_{C_1} \vec{F} \cdot d\vec{r} \text{ vs. } \int_{C_2} \vec{F} \cdot d\vec{r} \text{ vs. } \int_{C_3} \vec{F} \cdot d\vec{r}$$

Is $\vec{F}$ conservative? Can $\vec{F} = \vec{\nabla} f$ for some f?

$$f_x = 2xy \xrightarrow{\int f_x\,dx} \boxed{x^2 y + g(y) = f(x,y)}$$

$$\downarrow \partial_y$$

$$f_y = x^2 \longleftrightarrow x^2 + g'(y) \Rightarrow g'(y) = 0$$

$g(y) =$ constant.

choose 0 as our constant.

Potential: $f(x,y) = x^2 y$

$$\int_{C_1} \vec{F} \cdot d\vec{r} = f(3,2) - f(1,2) = (3)^2(2) - (1)^2(2) = 16.$$

all same.

$$\int_{C_2} \vec{F} \cdot d\vec{r} =$$

$$\int_{C_3} \vec{F} \cdot d\vec{r} =$$

13. $\vec{F}(x,y) = x^2 y^3 \hat{i} + x^3 y^2 \hat{j}$

$C: \vec{r}(t) = \langle \underbrace{t^3 - 2t}_{x}, \underbrace{t^3 + 2t}_{y} \rangle \quad 0 \le t \le 1$

evaluate $\displaystyle\int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$

$$= \int_0^1 \left[(t^3-2t)^2 (t^3+2t)^3 \hat{i} + (t^3-2t)^3 (t^3+2t)^2 \hat{j} \right] \cdot \left[(3t^2-2)\hat{i} + (3t^2+2)\hat{j} \right] dt$$

$$= \int_0^1 (t^3-2t)^2 (t^3+2t)^3 (3t^2-2) + (t^3-2t)^3 (t^3+2t)^2 (3t^2+2)\, dt$$

Lots of algebra to do here.

Check if F is conservative.

$\vec{F}(x,y) = x^2 y^3 \hat{i} + x^3 y^2 \hat{j}$

$f_x = x^2 y^3 \xrightarrow{\int f_x \, dx} \frac{x^3 y^3}{3} + g(y) = f(x,y)$

$\downarrow \frac{\partial}{\partial y}$

$f_y = x^3 y^2 \quad = \quad x^3 y^2 + g'(y) \Rightarrow g'(y) = 0$

$g =$ constant

pick $g = 0$.

$$f(x,y) = \frac{x^3 y^3}{3}$$

$\vec{r}(t) = \langle t^3 - 2t, \; t^3 + 2t \rangle \quad 0 \le t \le 1$

need $\vec{r}(1) = \langle -1, 3 \rangle$

$\vec{r}(0) = \langle 0, 0 \rangle$

$\int_C \vec{F} \cdot d\vec{r} = f(-1,3) - f(0,0)$

$= -9.$

To find if $\vec{F}$ is conservative, find $f(x,y)$ that satisfies $\vec{F} = \vec{\nabla} f$

Shortcut: if $\vec{F} = P\hat{i} + Q\hat{j}$ defined over open & simply connected domain AND P_y, Q_x are continuous AND $P_y = Q_x$ Then $\vec{F}$ is conservative.

Ex)

$$\vec{F}(x,y) = \underbrace{x^2 y^3}_{P} \hat{i} + \underbrace{x^3 y^2}_{Q} \hat{j}$$

$P_y = 3x^2y^2 \qquad Q_x = 3x^2y^2$

$\vec{F}$ cons.

<u>Ex)</u> 3 $\quad \vec{F}(x,y) = (\underbrace{xy+y^2}_{P})\hat{i} + (\underbrace{x^2+2xy}_{Q})\hat{j}$

$P_y = x+2y \qquad Q_x = 2x+2y$

$P_y \neq Q_x \Rightarrow$ not conservative.

or

$f_x = xy+y^2 \xrightarrow{\int f_x\,dx} \frac{x^2y}{2} + xy^2 + g(y) = f(x,y)$

$\downarrow \frac{\partial}{\partial y}$

$f_y = x^2+2xy \quad = \quad \frac{x^2}{2} + 2xy + g'(y)$

$x^2 + \cancel{2xy} = \frac{x^2}{2} + \cancel{2xy} + g'(y) \Rightarrow g'(y) = \frac{x^2}{2}$

Not conservative.

No potential.

<u>Exam Questions</u>

max | min

$f(x,y) = e^{-x^2-y^2}(x^2+2y^2)$ find max/min on $x^2+y^2 \le 4$

$$f_x = -2xe^{-x^2-y^2}(x^2+2y^2) + e^{-x^2-y^2}(2x)$$
$$f_y = -2ye^{-x^2-y^2}(x^2+2y^2) + e^{-x^2-y^2}(4y)$$

$$f_x = \underline{e^{-x^2-y^2}}\left(-2x(x^2+2y^2) + 2x\right)$$

$$f_y = \underline{e^{-x^2-y^2}}\left(-2y(x^2+2y^2) + 4y\right)$$

$x^2+y^2<4$

find when

$f_x = f_y = 0$

$$-2x(x^2+2y^2) + 2x = 0$$
$$-2y(x^2+2y^2) + 4y = 0$$

solve for x, y.

$\underset{\sim\sim}{x^2+y^2}=4$ use L.M.

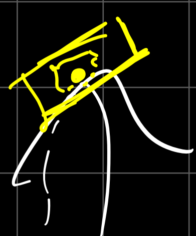
$g(x,y)=c$

$\vec{\nabla} f = \lambda \vec{\nabla} g$

$g_x = 2x \quad g_y = 2y$

$$e^{-x^2-y^2}\left(-2x(x^2+2y^2)+2x\right) = \lambda 2x$$
$$e^{-x^2-y^2}\left(-2y(x^2+2y^2)+4y\right) = \lambda 2y$$

Tangent plane vs. Linear approx.



Tan plane. $z - z_0 = f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$

Linear approx $L(x,y) = z_0 + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$

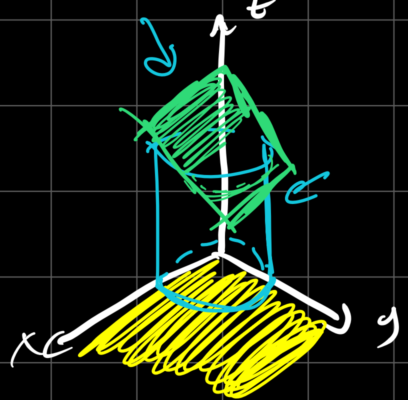
Near (x_0,y_0) $f(x,y) \approx L(x,y)$.

Directional Deriv. $D_{\vec{u}} f = \vec{\nabla} f \cdot \vec{u}$

Must be a unit vector.

Volume enclosed in a region

find volume enclosed by $x^2+y^2=4$ $z=0$ and $y=3-z$

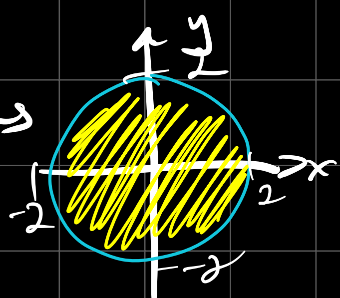


$0 \le z \le 3-y$

$0 \le r \le 2$

$0 \le \theta \le 2\pi$

$0 \le z \le 3 - r\sin\theta$



$-2 \le x \le 2$

$-\sqrt{4-x^2} \le y \le \sqrt{4-x^2}$

$$\text{Volume} = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{0}^{3-y} dz\,dy\,dx$$

$$= \int_0^{2\pi}\int_0^{2}\int_0^{3-r\sin\theta} r\,dz\,dr\,d\theta$$