Lecture 21

How to count sol'ns w/ multiple, interacting constraints?

Idea: $\mathcal{U}$ is the set of all sol'ns (no constraints)

A is solns w/ 1st prop $\quad$ B is sol'ns w/ 2nd Prop

Observe: $|\mathcal{U} \setminus (A \cup B)| = |\mathcal{U}| - |A| - |B| + |A \cap B|$

$|A \cup B| = |A| + |B| - |A \cap B|$

Examples of Inclusion - Exclusion.

Eg. Arrangements of $0, \dots, 9$ where $d_1 > 1$, $d_{10} < 8$

let $A = \{ s \mid d_1(s) \leq 1 \}$ $\quad$ $B = \{ s \mid d_{10}(s) \geq 8 \}$

$|A| = 2 \cdot 9!$ $\quad$ $|B| = 2 \cdot 9!$

$|A \cup B| = 2 \cdot 8! \cdot 2$ $\quad$ let $\mathcal{U}$ = all arrangements $\Rightarrow |\mathcal{U}| = 10!$

Want $\mathcal{U} \setminus (A \cup B) = A^c \cap B^c$ ← std form for these problems.

$$|A^c \cap B^c| = |\mathcal{U}| - |A| - |B| + |A \cap B| = 10! + 4 \cdot 8! - 4 \cdot 9!$$

$$= 8! \, (90 + 4 - 36) = 58 \cdot 8!$$

Logic Want Solns such that $P_1, P_2, \dots, P_n$ are all true.

Let X_k be all soln's s.t. P_k is FALSE (easier to count)

Then $|\text{Solns w/ } P_1, \dots, P_n| = |\overline{X}_1 \cap \dots \cap \overline{X}_n| := |(\mathcal{U} \setminus X_1) \cap \dots \cap (\mathcal{U} \setminus X_n)|$

where $\mathcal{U}$ is all soln's. $\quad X^c := (\mathcal{U} \setminus \overline{X})$.

Let product of sets denote intersection: $AB := A \cap B$. then,

Let $X_1, \dots, X_n \subseteq \mathcal{U}$ and def $S_0 = |\mathcal{U}|$, $S_{k \geq 1} := \sum_{\{j_1, \dots, j_k\} \subseteq \{1, \dots, n\}} |X_{j_1} \cap \dots \cap X_{j_k}|$

$$\text{P.I.E.} \qquad |\bar{X}_1 \cdots \bar{X}_n| = \sum_{k=0}^{n} (-1)^k S_k \quad , \quad |X_1 \cup \dots \cup X_n| = \sum_{k=1}^{n} (-1)^{k+1} S_k$$

Eg.s

Pie(1) says $|\bar{X}_1| = \sum_{k=0}^{1} (-1)^k S_k = S_0 - S_1 = |U| - \sum_{\{i_1\} \subseteq \{1\}} |X_{i_1}| = |U| - |X_1|$

Pie(2) says $|\bar{X}_1 \cdot \bar{X}_2| = \sum_{k=0}^{2} (-1)^k S_k = S_0 - S_1 + S_2 = |U| - \sum_{\{i_1\} \subseteq \{1,2\}} |X_{i_1}| + \sum_{\{i_1, i_2\} \subseteq \{1,2\}} |X_{i_1} X_{i_2}|$

$$= |U| - (|X_1| + |X_2|) + (|X_1 X_2|)$$

$$= |U| - |X_1| - |X_2| + |X_1 X_2|$$

Pie(3) says : $|\bar{X}_1 \bar{X}_2 \bar{X}_3| = \sum_{k=0}^{3} (-1)^k S^k = S_0 - S_1 + S_2 - S_3$

$$= |U| - \sum_{\{i_1\} \subseteq \{1,2,3\}} |X_{i_1}| + \sum_{\{i_1, i_2\} \subseteq \{1,2,3\}} |X_{i_1} X_{i_2}| - \sum_{\{i_1, i_2, i_3\} \subseteq \{1,2,3\}} |X_{i_1} X_{i_2} X_{i_3}|$$

$$= |U| - \left[|X_1| + |X_2| + |X_3| \right] + \left[|X_1 X_2| + |X_1 X_3| + |X_2 X_3| \right] - |X_1 X_2 X_3|$$

Eg: How many hands of 6 cards w/ one card from each suit? How many with no cards?

U = 6 card hands , $S_0 = |U| = \binom{52}{6}$ $\quad X_1$ = hands w/o spades $\binom{39}{6}$ $\quad X_2$ = hands w/o hearts

$|X_i X_j| = \binom{26}{6}$ $\qquad X_3$ = hands w/o diamds $\quad X_4$ = hands w/o clubs

$|X_i X_j X_k| = \binom{13}{6}$, $|X_i X_j X_k X_\ell| = 0$

$$\text{Ans} = |\bar{X}_1 \bar{X}_2 \bar{X}_3 \bar{X}_4| = S_0 - S_1 + S_2 - S_3 + S_4 = \binom{52}{6} - 4\binom{39}{6} + 6\binom{26}{6} - 4\binom{13}{6} + 0$$

Eg If n people pull hats out of dark closet, what is chance no-one gets their own hat?

U = Perms of hats $\Rightarrow S_0 = n!$ $\quad |X_i| = (n-1)!$ $\quad |X_{i_1 \cdots i_k}| = (n-k)!$

X_i = ith person gets i^{th} hat $\qquad \text{Ans} = |\bar{X}_1 \cdots \bar{X}_n| = \sum_{k=0}^{n} S_k (-1)^n$

$$\underline{\text{Ans}} = n! - \binom{n}{1}(n-1)! + \binom{n}{2}(-1)^k(n-2)! - \dots + (-1)^n\binom{n}{n}(n-n)!$$

$$= \sum_{k=0}^{n} (-1)^n \binom{n}{k}(n-k)! = \sum_{k=0}^{n} (-1)^n \frac{n!}{k!} \approx \frac{n!}{e}$$

$$\underline{\text{Probability}} = \frac{1}{n!} \cdot n! \sum_{k=0}^{n} \frac{(-1)^n}{k!} = \sum_{k=0}^{n} \frac{(-1)^n}{k!} \approx \frac{1}{e}$$

These are the derangement numbers: $D_n = n D_{n-1} + (-1)^n, \; n \geq 2$

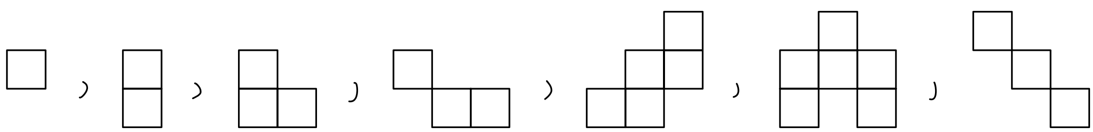
<u>Note</u>:

$$\#\text{ elts in exactly } m \text{ sets} = \sum_{k=m}^{n} (-1)^{k-m}\binom{k}{m} S_k \qquad \#\text{elts in at least } m \text{ sets} = \sum_{k=m}^{n} (-1)^{k-m}\binom{k-1}{m-1} S_k$$

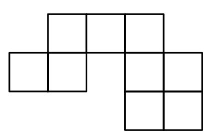
Lecture 22

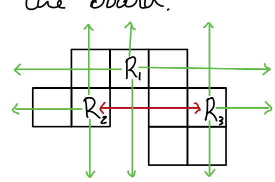
Rook Polynomials : these are a useful tool for PIE problems. They don't seem related to anything at first but are very useful.

Consider a collection of squares on a chess board which make a shape:

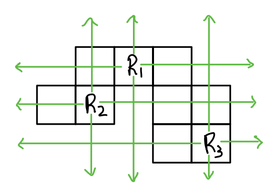
E.g. 

Let B be one of these configurations, we define $R_k(B)$ to be the number of ways to place k rooks on B so none of them can take each other. Note: the rooks are allowed to move through squares not on the board.

Eg. $B =$ 



Illegal b/c R_2, R_3 can see each other



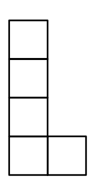
Legal b/c no rooks can see each other

$R_k(B)$ is # ways to put $R_1, \dots, R_k$ on B legally.

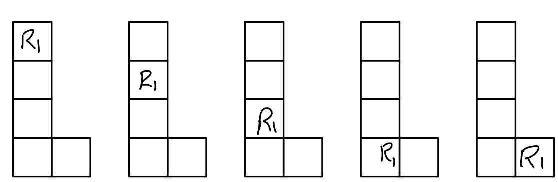
We arrange these #'s into a polynomial:

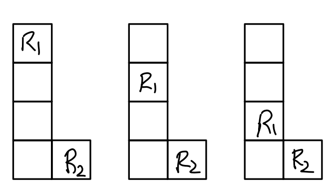
$$R(B,x) = R_0(B) + R_1(B)x + R_2(B)x^2 + \dots = \sum_{n=0}^{\infty} R_n(B)x^n$$

Note: this is finite sum b/c, If $n >$ # of squares in B then $R_n(B) = 0$.

Eg. $B =$ 

$R_0(B) = 1$ (empty board)

$R_1(B) = 5 \rightsquigarrow$ 

$R_2(B) = 3 \rightsquigarrow$ 

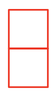
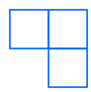
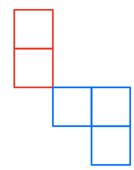
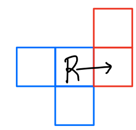
$R_3(B) = R_4(B) = \dots = 0$

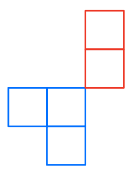
Thus:

$$\boxed{R(B,x) = 1 + 5x + 3x^2}$$

def: we say a board B is the "direct sum" of two smaller boards B_1 & B_2 when $B = B_1 \cup B_2$ AND there is no way for a rook on B_1 to move to B_2 in one step.

We write this $B = B_1 \oplus B_2$.

Eg: $B_1 =$  , $B_2 =$  ; $B_3 =$  $B_4 =$ 

$B_5 =$ 

$B_5 = B_1 \oplus B_2$ b/c No rook on red can get to blue.

$B_3 = B_1 \oplus B_2$ b/c no rook on red can get to blue

$B_4 \neq B_1 \oplus B_2$ b/c the rook drawn on blue can get to red in one move.

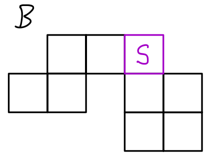
Thm: If $B = B_1 \oplus B_2$ then $R(B,x) = R(B_1,x) \cdot R(B_2,x)$

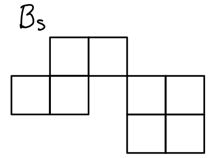
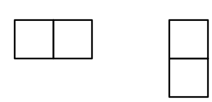
Thus: $R(B_1,x) = 1 + 2x$, $R(B_2,x) = 1 + 3x + x^2$

$$\Rightarrow R(B_3,x) = R(B_5,x) = (1+2x)(1+3x+x^2) = 1 + 3x + x^2 + 2x + 6x^2 + 2x^3$$
$$= 1 + 5x + 7x^2 + 2x^3$$

Reduction Formula:

Pick any square $s \in B$, and let B_s be B without s and B_s^* be B w/o the row and column containing s.

Eg: B 

$\rightsquigarrow$ B_s  & B_s^* 

Claim $R_k(B) = R_k(B_s) + R_{k-1}(B_s^*)$

Pf: Either there's a rook on s or not:

* If so, then there are $k-1$ rooks we need to place in different rows/cols from s $\rightsquigarrow R_{k-1}(B_s^*)$.

* If not, then we place k rooks on any legal squares other than s $\rightsquigarrow R_k(B_s)$.

$$\Longrightarrow \boxed{R(B,x) = R(B_s,x) + x \cdot R(B_s^*,x)}$$ Recursive formula just like Lose-fuse.

Using Rook Polys to solve problems:

<u>Setup</u> you have two sets of objects to pair together, with restrictions.

Eg Match $\{a, b, c, d, e\}$ with $\{1, 2, 3, 4, 5\}$ s.t.

None of the following pairs occur: $\boxed{a \sim 1,\ b \sim 2,3,\ c \sim 3,4,\ e \sim 5}$

First, set up a grid (We're setting this up as a PIE problem)

	1	2	3	4	5
a	X	R			X
b	R	X	X		
c			X	X	R
d			R		
e				R	X

Observe a matching is given by placing 5 non-attacking rooks on Board, avoiding dark sq's

$\{b \sim 1, a \sim 2, d \sim 3, e \sim 4, c \sim 5\}$ Solution

We need to count # of sols. (Use PIE + Rook polys)

let X_k be # of ways to place 5 rooks w/ the k^{th} in a bad position: ↳ clearly $|X_k| = |\text{Bad sq's in } k^{th} \text{ col}| \cdot (5-k!)$

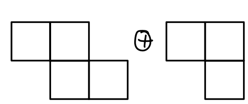
<u>Prop</u>: $\boxed{S_k = R_k(B)\,(n-k)!}$ in this type of problem.

Then by PIE:

$$\boxed{|\text{Solns}| = \sum_{k=0}^{\infty} (-1)^k R_k(B)(n-k)!}$$

Note, by swapping rows/cols we can rearrange the board:

	2	3	4	1	5
a				X	X
e					X
b	X	X			
c		X	X		
d					

B = 

$\Rightarrow R(B, x) = (1 + 4x + 3x^2)(1 + 3x + x^2)$

$= 1 + 3x + x^2$
$\quad 4x + 12x^2 + 4x^3$
$\quad 3x^2 + 9x^3 + 3x^4$

$= \underset{R_0}{1} + \underset{R_1}{7x} + \underset{R_2}{16x^2} + \underset{R_3}{13x^3} + \underset{R_4}{3x^4}$ (B)

By PIE:

$\# \text{Sols} = 5! - 7 \cdot 4! + 16 \cdot 3! - 13 \cdot 2! + 3 \cdot 1!$

$= 120 - 168 + 96 - 26 + 3 = \boxed{25}$