1. Prove that an isometry on a straight line $\mathbb{R}$ is either a translation along the line or a reflection about some point on the line.

   Solution:

   $O_1(\mathbb{R}) = \{\begin{bmatrix} b \end{bmatrix} \in GL_1(\mathbb{R}) | \begin{bmatrix} b \end{bmatrix}^t = \begin{bmatrix} b \end{bmatrix}^{-1}\} = \{b \in \mathbb{R}^* | b = b^{-1}\} = \{\pm 1\}$. So the orthogonal linear operators on $\mathbb{R}^1$ are $\phi_+(x) = x$ and $\phi_-(x) = -x$.

   We know each $f \in M_1$ can be written as $f = t_a\phi$ with $a \in \mathbb{R}$ and ϕ an orthogonal linear operator. So $f(x) = t_a\phi_+(x) = x + \vec{a}$ or $f(x) = t_a\phi_-(x) = -x + a = a - x$. The former is translation by a units, the latter is reflection with respect to $\frac{a}{2}$.

2. Prove that every matrix in $SO_3(\mathbb{R})$ has an eigenvalue $\lambda = 1$. Is it true for $SO_2(\mathbb{R})$?

   Solution: If $A \in SO_3$, then $AA^T = I$ and $det(A) = 1$.

   $\det(I - A) = det(AA^T - A) = (detA)(det(A - I)^T) = det(A - I) = -det(I - A)$

   So $det(I - A) = 0$, i.e. 1 is an eigenvalue.

   It is not true for $SO_2(\mathbb{R})$. For example, the matrix $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \in SO_2(\mathbb{R})$ has no eigenvalue.

3. Let s be the rotation of the plane with angle $\frac{\pi}{2}$ about the point $(1, 1)$. Write the formula for s as a product $t_{\vec{a}}\rho_\theta$.

   Solution: Let $\vec{p} = (1, 1)$.

   $s = t_{\vec{p}}\rho_{\frac{\pi}{2}}t_{-\vec{p}} = t_{\vec{p}}t_{\rho_{\frac{\pi}{2}}(-\vec{p})}\rho_{\frac{\pi}{2}} = t_{(1,1)+(1,-1)}\rho_{\frac{\pi}{2}} = t_{(2,0)}\rho_{\frac{\pi}{2}}$

4. Let s be the reflection along the line $y = x + 1$ followed by a translation along the vector $\vec{v} = (1, 1)$. Write s in the form $t_{\vec{a}}\rho_\theta r$

   Solution: $y = x + 1$ is parallel to $y = x$. $y = x$ forms angle $\frac{\pi}{4}$ with x-axis, so reflection along $y = x$ is $\rho_{\frac{\pi}{2}}r$. $y = x + 1$ is obtained from $y = x$ by translation along $(-\frac{1}{2}, \frac{1}{2})$ that is perpendicular to $y = x+1$, we get $\vec{a}_2 = 2(-\frac{1}{2}, \frac{1}{2}) = (-1, 1)$. $\vec{a}_1 = (1, 1)$ is given, so $\vec{a} = \vec{a}_1 + \vec{a}_2 = (1 - 1, 1 + 1) = (0, 2)$. We conclude $s = t_{(0,2)}\rho_{\frac{\pi}{2}}r$.

5. Let $H = \{t_{\vec{a}}\rho_\theta \in M_2 | \vec{a} \in \mathbb{Z}\times\mathbb{Z}, \theta = \frac{\pi k}{2}, k \in \mathbb{Z}\}$. Prove H is a subgroup of M_2.

   Solution: For any $t_{\vec{a}}\rho_\theta, t_{\vec{b}}\rho_\eta \in H$,

   $(t_{\vec{a}}\rho_\theta)^{-1}t_{\vec{b}}\rho_\eta = \rho_{-\theta}t_{-\vec{a}}t_{\vec{b}}\rho_\eta = \rho_{-\theta}t_{\vec{b}-\vec{a}}\rho_\eta = t_{\rho_{-\theta}(\vec{b}-\vec{a})}\rho_{\eta-\theta}$.

   Note that $\rho_{\frac{\pi}{2}}(x, y) = (-y, x)$, and θ is an integer multiple of $\frac{\pi}{2}$, so $\rho_{-\theta}$ sends $\mathbb{Z}\times\mathbb{Z}$ to $\mathbb{Z}\times\mathbb{Z}$. In particular, $\rho_{-\theta}(\vec{b} - \vec{a}) \in \mathbb{Z}\times\mathbb{Z}$, so $t_{\rho_{-\theta}(\vec{b}-\vec{a})}\rho_{\eta-\theta} \in H$.

6. Prove that $\rho_\theta r^k = \rho_\omega r^l$ in O_2 if and only if $\theta - \omega = 2\pi m$ for some $m \in \mathbb{Z}$ and $\bar{k} \equiv \bar{l}$ (mod 2)

   Solution: $\rho_\theta r^k = \rho_\omega r^l \iff \rho_\omega^{-1}\rho_\theta = r^l r^k \iff \rho_{\theta-\omega} = r^{l-k}$. We know $< r >= \{1, r\}$, and $< r > \cap SO_2 = \{id\}$, so

   $$\rho_{\theta-\omega} = r^{l-k} = id$$

   which implies $\theta - \omega = 2\pi m$ for some $m \in \mathbb{Z}$ and $\bar{k} \equiv \bar{l}$ (mod 2)

7. Define a map

   $$\Psi : M_2 \longrightarrow \{\pm 1\}$$
   $$t_{\vec{a}}\rho_\theta r^k \mapsto (-1)^k$$

   Prove Ψ is a well-defined homomorphism.

   (Remark: This provides an algebraic way to define the orientation of an isometry. Those corresponding to $+1$ are called orientation preserving, and those corresponding to -1 are called orientation reversing.)

   Solution: If $t_{\vec{a}}\rho_\theta r^k = t_{\vec{b}}\rho_\omega r^l$, by the unique decomposition of isometry into translation and orthogonal linear operator, we get

   $$t_{\vec{a}} = t_{\vec{b}} \text{ and } \rho_\theta r^k = \rho_\omega r^l$$

   Then by the previous question, $\rho_\theta r^k = \rho_\omega r^l$ if and only if $\theta - \omega = 2\pi m$ for some $m \in \mathbb{Z}$ and $\bar{k} \equiv \bar{l}$ (mod 2), so

   $$\Psi(t_{\vec{a}}\rho_\theta r^k) = (-1)^k = (-1)^l = \Psi(t_{\vec{b}}\rho_\omega r^l)$$

   We see Ψ is well-defined.

   Ψ is a homomorphism because for any isometries $t_{\vec{a}}\rho_\theta r^k$ and $t_{\vec{c}}\rho_\alpha r^n$:

$$
\begin{aligned}
&\Psi((t_{\vec{a}}\rho_\theta r^k)(t_{\vec{c}}\rho_\alpha r^n))\\
=&\Psi(t_{\vec{a}}\rho_\theta r^k t_{\vec{c}}\rho_\alpha r^n))\\
=&\Psi(t_{\vec{a}}t_{\rho_\theta r^k(\vec{c})}\rho_\theta r^k \rho_\alpha r^n))\\
=&\Psi(t_{\vec{a}+\rho_\theta r^k(\vec{c})}\rho_\theta \rho_{(-1)^k\alpha} r^k r^n))\\
=&\Psi(t_{\vec{a}+\rho_\theta r^k(\vec{c})}\rho_{\theta+(-1)^k\alpha} r^{k+n}))\\
=&(-1)^{k+l}\\
=&(-1)^k(-1)^l\\
=&\Psi(t_{\vec{a}}\rho_\theta r^k)\Psi(t_{\vec{c}}\rho_\alpha r^n)
\end{aligned}
$$