4. Derivatives

4.1 The derivative

Let I be an interval, let $f : I \to \mathbb{R}$ and $c \in I$. If the limit

$L = \lim_{x \to c} \frac{f(x)-f(c)}{x-c}$

exists, then we say f is differentiable at c, L is the derivative of f at c, $f'(c) = L$.

If f is differentiable at all $c \in I$, we say f is differentiable, and we get $f' : I \to \mathbb{R}$, also written as $\frac{df}{dx}$, $\frac{d}{dx}(f(x))$.

Prop. Let $f : I \to \mathbb{R}$ be differentiable at $c \in I$, then it is continuous at c.

Pf. $\lim_{x \to c} f(x) - f(c) = \lim_{x \to c} \frac{f(x)-f(c)}{x-c} \lim_{x \to c}(x-c) = f'(c) \cdot 0 = 0$

→ $\lim_{x \to c} f(x) = f(c)$ → continous at c

Prop. Let I be an interval, let $f, g : I \to \mathbb{R}$, $c \in I$, and $\alpha \in \mathbb{R}$.

1. $h : I \to \mathbb{R}$, $h(x) = f(x) + g(x)$ is differential at c, with $h'(c) = f'(c) \pm g'(c)$
2. $h(x) \coloneqq f(x)g(x)$, $h'(c) = f(c)g'(c) + f'(c)g(c)$
3. $h(x) \coloneqq \frac{f(x)}{g(x)}$, $h'(c) = \frac{f'(c)g(c)-f(c)g'(c)}{(g(c))^2}$
4. Chain rule: $h(x) \coloneqq (f \circ g)(x)$, $h'(x) = f'(g(c))g'(c)$

4.2 Mean value theorem

Let $S \subset \mathbb{R}$, $f : S \to \mathbb{R}$. We say f has a relative maximum at c if there exists $\delta > 0$ s.t. for all $x \in S$, $|x - c| < \delta$, $f(c) \geq f(x)$.

Lemma. Suppose $f : [a, b] \to \mathbb{R}$ is differentiable at $c \in (a, b)$, and f has a relative min/max at c. Then $f'(c) = 0$.

Rolle's Theorem. Let $f : [a, b] \to \mathbb{R}$ be a continuous function differentialable on $[a, b]$ such that $f(a) = f(b)$. Then there exists $c \in (a, b)$ s.t. $f'(c) = 0$.

Pf. $K = f(a) = f(b)$. If $\exists x$ s.t. $f(x) > K$, c = abs max. If $\exists x$ s.t. $f(x) < K$, c = abs min. Else, $\forall x, f(x) = K$, any $c \in (a, b)$.

Mean Value Theorem. Let $f : [a, b] \to \mathbb{R}$ be a continuous function differentialable on $[a, b]$ such that $f(a) = f(b)$. Then there exists $c \in (a, b)$ s.t. $f'(c) = \frac{f(b)-f(a)}{b-a}$.

Pf. $g(x) \coloneqq f(x) - f(b) - \frac{f(b)-f(a)}{b-a}(x - b)$, since $g(a) = g(b) = 0$, by Rolle's Theorem, $\exists c \in (a, b)$ s.t. $0 = g'(c) = f'(c) - \frac{f(b)-f(a)}{b-a}$.

Prop. Let $f : I \to \mathbb{R}$ be a differentiable function s.t. $f'(x) = 0$ for all $x \in I$. Then f is a constant.

Pf. By Mean Value Theorem, $\forall x, y \in I$, $f(y) - f(x) = f'(c)(y - x) = 0$.

Prop. (Sign of derivative as inc/dec) Let $f : I \to \mathbb{R}$ be a differentiable function. Then:

1. f is increasing $\iff f'(x) \geq 0, \forall x \in I$
2. f is strictly increasing $\iff f'(x) > 0, \forall x \in I$

Pf. f is increasing → if $x > c$, then $f(x) \geq f(c)$ → $\frac{f(x)-f(c)}{x-c} \geq 0$ → $f(c) \geq 0$

$f'(x) \geq 0, \forall x \in I$ → take $x, y \in I, x > y$, by MVT, $\exists c \in (x, y)$ s.t. $f(y) = f(x) + f'(c)(y - x) \geq f(x)$

4.3 Taylor's theorem

If $f : I \to \mathbb{R}$ is differentiable, $f' : I \to \mathbb{R}$ is the first derivative of f.

If $f' : I \to \mathbb{R}$ is differentiable, $f'' : I \to \mathbb{R}$ is the second derivative of f.

We similarly obtain the nth derivative of f — $f^{(n)}$.

If f possesses n derivatives, we say f is n times differentiable.

For an n times differentiable function f defined near a point $x_0 \in \mathbb{R}$, define the nth order Taylor polynomial for f at x_0 as

$$\begin{aligned} P_n^{x_0}(x) &:= \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k \\ &= f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2}(x - x_0)^2 + ... + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n \end{aligned}$$

Taylor Theorem. Suppose $f : [a, b] \to \mathbb{R}$ is a function with n continuous derivatives on $[a, b]$ and such that $f^{(n+1)}$ exists on (a, b). Given distinct points x_0 and x in $[a, b]$, we can find a point c strictly between x_0 and x ($c \in (x_0, x)$ or $c \in (x, x_0)$) such that

$f(x) = P_n^{x_0}(x) + \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}$

where $R_n^{x_0}(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}$ is called the remainder term.

Two ways to read the equation:

- $f(x) = \text{Taylor polynomial} + O((x - x_0)^{n+1})$
- There exists a solution c that depends on $x, x_0, f, f', ..., f^{(n+1)}$: $\frac{f^{(n+1)}(c)}{(n+1)!} = \frac{f(x)-P_n^{x_0}(x)}{(x-x_0)^{n+1}}$

Pf. Similar to pf of MVT, define $M := \frac{f(x)-P_n^{x_0}(x)}{(x-x_0)^{n+1}}$, $g(s) := f(s) - P_n^{x_0}(s) - M(s - x_0)^{n+1}$

→ $g(x_0) = g'(x_0) = ...g^{(n)}(x_0) = 0$

In particular, $g(x) = g(x_0) = 0$, by MVT, $\exists x_1$ between x and x_0 s.t. $g'(x_1) = 0$

Similarly, $\exists x_2$ between x_0 and x_1 s.t. $g''(x_2) = 0$

$\exists x_{n+1}$ between x_0 and x_n s.t. $g^{(n+1)}(x_{n+1}) = 0$

let $c = x_{n+1}$, so c between x and x_0, $0 = g^{(n+1)}(c) = f^{(n+1)}(c) - (n+1)!M$

→ $M = \frac{f^{(n+1)}(c)}{(n+1)!}$ Q.E.D.

Example. $f(x) = \sin x$, $n = 3$, $\forall x \in [-1, 1]$, $x_0 = 0$,

$\exists c$ s.t. $\sin(x) = x - \frac{x^3}{3!} + \sin(c)\frac{x^4}{4!}$ where we know $|\sin(c)| \leq 1$

Prop. (Second derivative test) Suppose $f : (a, b) \to \mathbb{R}$ is twice continously differentiable, $x_0 \in (a, b)$, $f'(x_0) = 0$, $f''(x_0) > 0$. Then f has a strict relative minimum at x_0.

Pf. f'' is continous → $\exists\delta > 0$ s.t. $f''(c) > 0, \forall c \in (x_0 - \delta, x_0 + \delta)$

take $x \in (x_0 - \delta, x_0 + \delta)$, $x \neq x_0$, by Taylor Theorem, $\exists c$, $f(x) = f(x_0) + \frac{f''(c)}{2}(x - x_0)^2 > f(x_0)$