Problem Sets, Honors Theory of Probability

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Disclaimer:

These are the problem sets for the course *Honors Theory of Probability (MATH.UA 238)*, given by professor Eyal Lubetzky at New York University in Spring 2021 and 2022. The main reference of this course is [2]. One may find it useful to read [1] first for an elementary introduction.

If you see any mistakes or think that the presentation is unclear and could be improved, please send an email to: cl5682@nyu.edu. All comments and suggestions are appreciated.

Notations

- $X \perp\!\!\!\perp Y$: X is independent of Y.
- $\mathbb{1}_A(x)$: the indicator function. One has $\mathbb{1}_A(x) = 1$ if $x \in A$, and $\mathbb{1}_A(x) = 0$ otherwise.
- $\mathrm{Cov}(X, Y)$: the covariance between X and Y.
- $\rho(X, Y)$: the correlation between X and Y.
- $\mathrm{Ber}\,(p)$: the Bernoulli distribution with parameter p.
- $\mathrm{Bin}\,(n, p)$: the binomial distribution with parameters n, p.
- $\mathrm{Pois}\,(\lambda)$: the Poisson distribution with parameter λ.
- $\mathrm{Geom}\,(p)$: the geometric distribution with parameter p.
- $\mathrm{NB}\,(n, p)$: the negative binomial distribution with parameters n, p.
- $\mathcal{U}(a, b)$: the uniform distribution with density $f(x) = 1/(b-a) \cdot \mathbb{1}_{[a,b]}(x)$.
- $\mathrm{Exp}\,(\lambda)$: the exponential distribution with parameter λ.
- $\mathcal{N}(\mu, \sigma^2)$: the normal distribution with mean μ and variance σ^2.
- $\Gamma(r, \lambda)$: the gamma distribution with parameters r, λ.
- $\arg\max_S f := \{x \in S : f(s) \leq f(x) \text{ for all } s \in S\}$: arguments of the maxima.
- $[\![A, B]\!]$: the range of integers $\{A, A+1, \ldots, B\}$.
- a.s., a.e.: almost surely, almost everywhere.
- i.o.: infinitely often.

Contents

1 Probability Spaces

1.1 Sets and σ-Fields

W 1.14. Roll six independent fair dice.

(a) Find the probability they all show different faces.

(b) Find the probability they form three pairs.

W 1.16. P.R. de Montmort, writing in 1708, posed a problem on the French game Jeu de Boules. The object of the game is to throw balls close to a target ball. The closest ball wins. Suppose two players, A and B have equal skill. Player A tosses one ball. Player B tosses two. What is the probability that player A wins?

W 1.18. A number is chosen at random from the interval $[0, 1]$. It divides the interval into two parts. Find the probability that the longer part is at least twice as long as the shorter part.

W 1.19. (the question should say "with these two dice") Sicherman dice are like ordinary dice, except that they have different numbers of pips on their faces. One has $1, 3, 4, 5, 6, 8$ on its six faces, and the other has $1, 2, 2, 3, 3, 4$. A pair is thrown. Find the probabilities of rolling the numbers one thru twelve with these three dice. Compare with the usual probabilities.

W 1.20. 100 passengers wait to board an airplane which has 100 seats. Nassif is the first person to board. He had lost his boarding pass, but they allowed him to board anyway. He chose a seat at random. Each succeeding passenger enters and takes their assigned seat if it is not occupied. If it is occupied, they take an empty seat at random. When the final passenger enters, there is only one empty seat left. What is the probability that it is that passenger's correct seat?

W 1.23. Let A, B and C be events. Show that $\mathbb{P}(A \cup B \cup C)$ equals

$$\mathbb{P}(A) + \mathbb{P}(B) + \mathbb{P}(C) - \mathbb{P}(A \cap B) - \mathbb{P}(B \cap C) - \mathbb{P}(C \cap A) + \mathbb{P}(A \cap B \cap C)$$

R 2.31. A 3-person basketball team consists of a guard, a forward, and a center.

(a) If a person is chosen at random from each of three different such teams, what is the probability of selecting a complete team?

(b) What is the probability that all 3 players selected play the same position?

R 2.53. If 4 married couples are arranged in a row, find the probability that no husband sits next to his wife.

R 3.15. An urn initially contains 5 white and 7 black balls. Each time a ball is selected, its color is noted and it is replaced in the urn along with 2 other balls of the same color. Compute the probability that

(a) the first 2 balls selected are black and the next 2 are white;

(b) of the first 4 balls selected, exactly 2 are black.

R 3.30. Suppose an ordinary deck of 52 cards is shuffled, and the cards are turned over one at a time until the first ace appears. Given that the first ace is the 20^{th} card to appear, what is the conditional probability that the card following it is the

(a) ace of spades?

(b) two of clubs?

1.2 Conditional Probability, Independence

W 1.30. A communication channel can increase the probability of successful transmission by using error-correcting codes. One of the simplest of these is called a "parity scheme". In such a scheme, the message is divided into blocks of a fixed number of digits. Then a single bit, called the "parity bit" is added to each block. The parity bit is 0 if there are an even number of 1's in the block, and 1 if there are an odd number of 1's. The receiver compares the block with the parity bit. If they do not agree, the block is retransmitted. If they do agree, the block is accepted as received.

The original block of, say, n bits is transmitted as $n + 1$ bits. Suppose the blocks are three bits long-four, including the parity bit-and that the probability of mistransmitting a single bit is 0.1.

(a) Find the probability of having at least one error in a block of three bits.

(b) Given that there is at least one error in the bit, what is the probability that it will be retransmitted?

Hint. This scheme only detects an odd number of errors. Don't forget that there may also be an error in transmitting the parity bit.

W 1.55. Four shoes are chosen at random from a closet containing five pairs of shoes. What is the probability that there is at least one pair among the four?

W 1.61. Texas Hold'em is a kind of poker in which each player is dealt two cards face-down, called the *hole cards*, and then five cards, called *the board*, are dealt face-up on the table. A player can choose a five card hand from his or her two hole cards and the five cards on the board—so each player has seven cards to choose from.

(a) What is the probability that a player has at least a pair, i.e., that there are at least two cards among the seven of the same denomination?

(b) Suppose a player's hole cards are of different denominations. What is the probability that one of them is paired, that is, what is the conditional probability that at least one of the five cards on the board has the same denomination as one of the hole cards? What is the conditional probability if the hole cards are of the same denomination?

W 1.63. A certain lottery, called 6-49, draws six different numbers at random from 1 to 49, and then draws a seventh, called the bonus number. The player provides a list of six numbers. If the first six numbers drawn match the six numbers on the players list in any order, the player wins the grand prize. If five of the six drawn numbers match numbers on the players list, and the bonus number matches the remaining number, the player wins second prize. The third prize goes to those players whose list contains any five of the six lottery numbers, but not the bonus number. Calculate the probability of getting the grand prize, the second prize, and the third prize.

W 1.69. (G. Slade) A gambler is in desperate need of $\$1000$, but only has $\$\,n$. The gambler decides to play roulette, betting $1 on either red or black each time, until either reaching $\$1000$ (and then stopping) or going broke. The probability of winning a bet on red or black at a US casino is $18/38$.

(a) How large must n be for the gambler to have a probability $1/2$ of success?

(b) Suppose the gambler starts with this stake. How much can the casino possibly lose?

Hint. It is instructive to plot the probability of success as a function of the initial stake n.

W 1.89. (J. L. Doob) Johnny has not studied for tomorrow's test, so his probability of passing it is only $1/4$. However, if he cheats, he will raise his probability of passing to $3/4$. He hasn't yet decided whether or not to cheat, but he is leaning that way: the probability that he will cheat is $3/4$. Suppose he takes the test and passes it. What is the probability that he cheated? If he passes, should he fail?

R 3.43. A deck of cards is shuffled and then divided into two halves of 26 cards each. A card is drawn from one of the halves; it turns out to be an ace. The ace is then placed in the second half-deck. The half is then shuffled, and a card is drawn from it. Compute the probability that this drawn card is an ace.

R 3.35. On rainy days, Joe is late to work with probability 0.3; on non-rainy days he is late with with probability 0.1. With probability 0.7, it will rain tomorrow.

(a) Find the probability that Joe is early tomorrow.

(b) Given that Joe was early, what is the conditional probability that it rained?

R 3.65. The color of a person's eyes is determined by a single pair of genes. If they are both blue-eyedgenes, then the person will have blue eyes; if they are both brown-eyed genes, then the person willhave brown eyes; and if one is a brown-eyed gene and the other is a blue-eyed gene, then the personwill have brown eyes. A newborn child independently receives one eye gene from each of its parents,and the gene it receives from a parent is equally likely to be either one of the genes of that parent.Suppose Smith and both of his parents have brown eyes, but Smith's sister has blue eyes.

(a) What is the probability that Smith possesses a blue-eyed gene?

(b) Suppose that Smith's wife has blue eyes. What is the probability that their first child will have blueeyes?

(c) If their first child has brown eyes, what is the probability that their next child will also have browneyes?

R 3.T.7. (a) An urn contains n white and m black balls. The balls are withdrawn one at a time until only those of the same color are left. Show that with probability $n/(n+m)$ they are all white.

(b) A pond contains 3 distinct species of fish, which we will call the Red, Blue and Green fish. There are r Red, b Blue and g Green fish. Suppose that the fish are removed from the pond in a random order (i.e., each selection is equally likely to be any of the remaining fish). What is the probability that the Red fish are the first species to become extinct in the pond?

2 Random Variables

2.1 Existence and Independence of Random Variables

W 2.22. (a) Choose one of the points $(0,0), (0,1), (1,0)$ and $(1,1)$ at random. Let X be the first coordinate and Y the second. Are X and Y independent?

(b) Do the same for the points $(0,1), (1,0), (-1,0)$ and $(0,-1)$. Are X and Y independent?

W 2.23. Let X be a random variable and f a Borel function. Suppose that $f(X)$ is not a.s. constant. Is it possible for X and $f(X)$ to be independent?

W 2.28. Let X be absolutely continuous with density $f(x) = cx(1-x)$ if $0 \le x \le 1$, $f(x) = 0$ otherwise. Find c. Let $Y = X^4$. Find the density f_Y of Y.

W 2.30. Toss a fair coin four times independently. Let X be the number of heads minus the number of tails. Find the distribution function of X.

W 2.31. Suppose that X has the density $f(x) = 2x$ on the interval $[0,1]$, zero elsewhere. Let $Y = |X - 1/2|$. Find the density of Y.

W 2.45. n players each roll a single die. Any pair of them who roll the same number score one point. Let A_{ij} be the event on which player i and player j roll the same number.

(a) Show that A_{12} and A_{23} are independent.

(b) Find the mean and variance of the total score of the group.

R 3.97. Each member of a population of size n is, independently of other members, female with probability p or male with probability $1-p$. Two individuals of the same sex will, independently of other pairs, be friends with probability α; whereas two individuals of opposite sex will be friends with probability β. Let $A_{k,r}$ be the event that persons k and r are friends.

(a) Find $\mathbb{P}(A_{1,2})$.

(b) Are $A_{1,2}$ and $A_{1,3}$ independent?

(c) Are $A_{1,2}$ and $A_{1,3}$ conditionally independent given the sex of person 1?

(d) Find $\mathbb{P}(A_{1,2} \cap A_{1,3})$.

R 3.T.9. Consider two independent tosses of a fair coin. Let A be the event that the first toss results in heads, let B be the event that the second toss results in heads, and let C be the event that in both tosses the coin lands on the same side. Show that the events A, B, and C are pairwise independent but not independent.

R 3.T.15. Independent trials that result in a success with probability p are successively performed until a total of r successes is obtained. Show that the probability that exactly n trials are required is

$$\binom{n-1}{r-1} p^r (1-p)^{n-r}$$

R 3.T.30. There are $k+1$ coins in a box. When flipped, the i^{th} coin will turn up heads with probability $i/k, i = 0, 1, \ldots, k$. A coin is randomly selected from the box and is then repeatedly flipped. Suppose that the first n flips resulted in r heads and $n-r$ tails. Show that the probability that the $(n+1)$ flip turns up heads is $(r+1)/(n+2)$.

Hint. Prove and use the identity

$$\int_0^1 y^n (1-y)^m dy = \frac{n!m!}{(n+m+1)!}$$

2.2 Expectations I: Discrete Random Variables

W 2.50. Let $X_1, X_2, X_3, \ldots$ be i.i.d. positive-integer-valued random variables. Show that

$$\mathbb{E}\left[\min\{X_1, \ldots, X_m\}\right] = \sum_{n=1}^{\infty} \mathbb{P}(X_1 \geq n)^m$$

W 2.54. Toss a fair coin 10 times. Let N be the number of times the pattern $HHHH$ occurs. (For example, in the sequence $HHHHHHT$ it appears three times, once each for positions 1, 2, and 3.) Find $\mathbb{E}[N]$.

W 2.56. N people live in a neighborhood. Their mean income is m and the mean of the squares of their income is S^2. Of these, n are chosen at random. Find the mean and variance of the total income of these n people.

Hint. The individual incomes are fixed. The only randomness is in the choice of the n people.

W 2.62. Find the probability of k heads in n tosses of a fair coin. Use this to show that $\sum_{k=0}^{n} \binom{n}{k} = 2^n$.

W 2.65. A jar contains n chips, numbered from 1 to n. A chip is drawn at random, its number is noted, and it is replaced in the jar. This continues until a chip is drawn for the second time. Let X be the number of the draw on which this happens. Find the distribution of X.

W 2.67. n couples attend a dance. Each wife has a dance partner chosen at random from the n husbands. Let E_i be the event that wife i dances with her own husband, and let p_n be the probability that none of the n couples dance together.

(a) Find p_2 and p_3 by expressing them in terms of the E_i.

(b) Find $\lim_{n\to\infty} p_n$.

Hint. Calculate the probability that the j^{th} wife dances with her husband, that both the j^{th} and the k^{th} wives do, and so on. The inclusion-exclusion formula gives an answer which links this to the Taylor's expansion of a certain well-known function.

R 4.35. A box contains 5 red and 5 blue marbles. Two marbles are withdrawn randomly. If they are the same color, then you win $\$1.10$; if they are different colors, then you lose $\$1.00$. Calculate the expected value and the variance of the amount you win.

R 4.81. An urn contains 4 white and 4 black balls. We randomly choose 4 balls. If 2 of them are white and 2 are black, we stop. If not, we replace the balls in the urn and again randomly select 4 balls. This continues until exactly 2 of the 4 chosen are white. What is the probability that we shall make exactly selections?

R 4.T.10. Let $X \sim \text{Bin}\,(n, p)$. Show that

$$\mathbb{E}\left[\frac{1}{X+1}\right] = \frac{1-(1-p)^{n+1}}{(n+1)p}$$

R 4.T.32. A jar contains $m+n$ chips, numbered $1, 2, \ldots, n+m$. A set of size n is drawn. If we let X denote the number of chips drawn having numbers that exceed each of the numbers of those remaining, compute the probability mass function of X.

2.3 Special Discrete Distributions

W 2.43. Let $A_1, \ldots, A_n$ be independent events in an experiment, with $\mathbb{P}(A_i) = p_i, i = 1, \ldots, n$. Do the experiment and let N be the number of events A_i that occur. Find the moment generating function of N, and use it to compute $\mathbb{E}\{N\}$. Use the indicator functions of the A_i to give a second, simple derivation of the value of $\mathbb{E}\{N\}$.

W 2.51. It is said that many are called and few are chosen. Suppose that the number called is $\mathrm{Pois}\,(\lambda)$. Each person called is chosen, independently, with probability p. Show that the distribution of the number chosen is $\mathrm{Pois}\,(p\lambda)$.

W 2.52. Let Let X and N be random variables. Suppose that $N \sim \mathrm{Bin}\,(M, q)$ and, given N, X is $\mathrm{Bin}\,(N, p)$. Show X is $\mathrm{Bin}\,(M, pq)$. Do this both analytically and probabilistically.

W 3.3. Let $X \geq 0$ be an integrable random variable. Show that if $\mathbb{E}[X] = 0$, then $X = 0$ a.e.

W 3.4. Let X be a positive random variable with distribution function F. Show that

$$\mathbb{E}[X] = \int_0^\infty (1 - F(x))\,\mathrm{d}x$$

Conclude that if $c > 0$, then $\mathbb{E}[X \wedge c] = \int_0^c (1 - F(x))\,\mathrm{d}x$.

W 3.16. A random variable X satisfies $|X| \leq 1$ a.s. Show that $\mathrm{Var}(X) \leq 1$, and that the maximum is attained.

R 4.T.26. this should be so. As an application of the preceding result, suppose that the number of distinct uranium deposits in a given area is $\mathrm{Pois}\,(10)$. If, in a fixed period of time, each deposit is discovered independently with probability $1/50$, find the probability that (a) exactly 1, (b) at least 1, and (c) at most 1 deposit is discovered during that time.

R 4.T.28. If X is a geometric random variable, show analytically that

$$\mathbb{P}(X = n + k \mid X > n) = \mathbb{P}(X = k)$$

Using the interpretation of a geometric random variable, give a verbal argument as to why the preceding equation is true.

R 4.T.29. Let $X \sim \mathrm{NB}\,(r, p)$ and $Y \sim \mathrm{Bin}\,(n, p)$. Show that $\mathbb{P}(X > n) = \mathbb{P}(Y < r)$.

Hint. Either one could attempt an analytical proof of the preceding equation, which is equivalent to proving the identity

$$\sum_{i=n+1}^{\infty} \binom{i-1}{r-1} p^r (1-p)^{i-r} = \sum_{i=0}^{r-1} \binom{n}{i} p^i (1-p)^{n-i}$$

or one could attempt a proof that uses the probabilistic interpretation of these random variables. That is, in the latter case, start by considering a sequence of independent trials having a common probability p of success. Then try to express the events $\{X > n\}$ and $\{Y < r\}$ in terms of the outcomes of this sequence.

R 4.T.33. A jar contains n chips. Suppose that a boy successively draws a chip from the jar, each time replacing the one drawn before drawing another. The process continues until the boy draws a chip that he has previously drawn. Let X denote the number of draws, and compute its probability mass function.

3 Expectations II: The General Case

3.1 From Discrete to Continuous

W 3.7. Let X be uniform on $[0, L]$. Find the mean and standard deviation of $|X - \mathbb{E}[X]|$.

W 3.11. An airport check-in counter has a single line which feeds into two service counters. Whenever a service counter is empty, the person at the head of the line goes to it. The service time at each counter is exponential with a mean of four minutes, and the two counters are independent.

(a) What is the density of the waiting time until arriving at the service counter for the person at the head of the line? Find the moment generating function of this time.

(b) Beth is the eleventh person in the line. What is the moment generating function for her waiting time?

(c) Find the expectation and variance of the time Beth must wait before arriving at the service counter.

W 3.13. Let $x_1, \ldots, x_n$ be strictly positive numbers. Their arithmetic average is $A := \frac{1}{n}\sum_{i=1}^n x_i$ and their geometric average is $G := \left(\prod_{i=1}^n x_i\right)^{1/n}$. Use Jensen's inequality to show that $G \leq A$: i.e., the geometric average is always less than or equal to the arithmentic average. Show, moreover, that there is equality if and only if $x_1 = \cdots = x_n$.

Hint. For equality, draw the graph-chord picture for the logarithm.

W 3.14. A random variable X has density $f_X(x) = e^{-|x|}/2, -\infty < x < \infty$. Find the moment generating function and the variance of X.

W 3.17. Let X and Y be positive random variables with $\mathbb{E}[X] \leq 1$ and $\mathbb{E}[Y] \leq 1$. Prove or disprove: for each $\varepsilon > 0, \mathbb{P}(X \leq 2+\varepsilon, Y \leq 2+\varepsilon) > 0$. Note that X and Y need not be independent.

W 3.19. Let X and Y be square-integrable random variables, and define the L^2-norm of X to be $\|X\|_2 := \mathbb{E}\left[X^2\right]^{1/2}$. Prove Minkowski's inequality:

$$\|X+Y\|_2 \leq \|X\|_2 + \|Y\|_2.$$

Hint. Compute $\mathbb{E}\left[(X+Y)^2\right]$ and use the Schwarz inequality on the cross term. Then rewrite it in terms of norms.

R 5.T.11. Let $Z \sim \mathcal{N}(0,1)$ and g be a differentiable function.

(a) Calculate $\mathbb{E}\left[Z^4\right]$.

(b) Prove that $\mathbb{E}\left[Z_{n+1}\right] = n\mathbb{E}\left[Z_{n-1}\right]$ for every integer $n \geq 1$.

(c) Prove that $\mathbb{E}\left[g'(Z)\right] = \mathbb{E}\left[Zg(Z)\right]$.

R 5.T.13 & 14. Find the medians and modes for $\mathcal{U}(a,b), \mathcal{N}(\mu,\sigma^2), \mathrm{Exp}\,(\lambda)$ respectively.

R 5.T.31 & 44. Let $X \sim \mathcal{N}(\mu,\sigma^2)$ and $Y = e^X$. Such Y is called a lognormal random variable with parameters (μ,σ^2).

(a) Find the probability density function of Y.

(b) Prove or disprove: for every $c > 0$, cY is a lognormal random variable.

(c) Prove or disprove: for every $c > 0$, $Y + c$ is a lognormal random variable.

R 7.T.7. Tonelli's Theorem implies that

$$\mathbb{E}\left[\int_0^\infty X(t)\,\mathrm{d}t\right] = \int_0^\infty \mathbb{E}[X(t)]\,\mathrm{d}t$$

whenever $X(t), 0 \leq t < \infty$, are all nonnegative random variables. Use it to give another proof of the result that for a nonnegative random variable X,

$$\mathbb{E}[X] = \int_0^\infty \mathbb{P}\{X > t\}\,\mathrm{d}t$$

Hint. Define, for each nonnegative t, the random variable $X(t)$ by

$$X(t) = \begin{cases} 1 & \text{if } t < X \\ 0 & \text{if } t \geq X \end{cases}$$

Now relate $\int_0^\infty X(t)\,\mathrm{d}t$ to X.

3.2 Special Continuous Distributions

W 3.18. Let $U \sim \mathcal{U}(0,1)$. Find the density of U^2 and e^U.

W 3.21. Let $p(x,y)$ be the joint probability mass function of X and Y. Show that the marginal probability mass function of X is $p_X(x) = \sum_y p(x,y)$.

W 3.22. Let X and Y be independent $\mathcal{U}(0,1)$ random variables.

(a) Find the joint density of $\min(X,Y)$ and $\max(X,Y)$.

(b) Divide $(0,1)$ into three line segments, where X and Y are the dividing points. What is the probability that the three line segments can form a triangle?

(c) Show that the lengths of all three segments have the same distribution.

W 3.24. Suppose X and Y have a joint density $f_{XY}(x,y)$. Show that if there exist functions $f(x)$ and $g(y)$ such that $f_{XY}(x,y) = f(x)g(y)$ for all x and y, that X and Y are independent. It is not necessary that f and g be the marginal densities.

W 3.25. Let Y be a random variable with density $f_Y(y) = 2y$ if $0 < y < 1$, and zero otherwise. Let X have a $\mathcal{U}(0,1)$ distribution. Find a function g such that $g(X)$ has the same distribution as Y.

W 3.27. Suppose n darts are thrown at a circular target of radius one. Assume the throws are independent and the landing place of each dart is uniform on the target. Let R_n be the distance from the center to the closest dart. Show that $\mathbb{E}\left[R_n\right] = \int_0^1 \left(1-x^2\right)^n \, \mathrm{d}x$.

R 7.T.8. Let X, Y be nonnegative random variables such that $\mathbb{P}(X > t) \geq \mathbb{P}(Y > t)$ for every t (in this case one says that X stochastically dominates Y). Show that $\mathbb{E}X \geq \mathbb{E}Y$.

R 7.T.24 & 55. Let be a standard Gaussian random variable.

(a) Calculate $\operatorname{Cov}\left(Z, Z^2\right)$.

(b) Show that $\rho\left(Z, a + bZ + cZ^2\right) = b/\sqrt{b^2 + 2c^2}$.

3.3 Joint Distributions and Joint Densities

W 3.26. Let $X_1, X_2, \ldots$ be i.i.d. random variables, and let $N \geq 1$ be an integer-valued random variable which is independent of the X_i. Define $S_N = \sum_{i=1}^N X_i$, i.e., S_N equals $X_1 + \cdots + X_n$ on the set $\{N = n\}$. Show that $\mathbb{E}\left[S_N\right] = \mathbb{E}[N]\mathbb{E}\left[X_1\right]$ and $\operatorname{Var}\left(S_N\right) = \mathbb{E}[N]\operatorname{Var}\left(X_1\right) + \mathbb{E}\left[X_1\right]^2 \operatorname{Var}(N)$.

W 3.30. Let (X, Y) be uniformly distributed on $[0,1]^2$. Let Z be the shortest distance from (X, Y) to the boundary (i.e., the minimum of $X, Y, 1-X$ and $1-Y$.) Find the distribution function, density, and median of Z.

W 3.34. Let U and X be random variables, where U is $U(0,1)$, and $X \mid U = u \sim \operatorname{Exp}(1/u)$.

(a) Find the joint density of U and X.

(b) Find $\mathbb{E}[X]$ and $\mathbb{E}[UX]$.

(c) Are U and X independent?

W 3.38. Let X and Y be independent Gaussian random variables with the same variance. Show that $X + Y$ and $X - Y$ are independent.

W 3.39. (Continuation.) Let U and V be independent $\mathcal{N}(0,1)$ random variables, let $\phi, \theta \in [0, 2\pi)$, and put $X = U \sin\theta + V \cos\theta$ and $Y = U \sin\phi + V \cos\phi$. Express the correlation ρ of X and Y in terms of θ and ϕ, and interpret this in terms of "the angle between X and Y."

W 3.40. Let X and Y be bivariate mean zero Gaussian random variables with variance one and correlation ρ. Show that there are independent Gaussian random variables U and V, and reals a, b, c and d such that $X = aU + bV$ and $Y = cU + dV$.

R 6.T.5. Let X and Y are independent continuous positive random variables. Let $U = XY$ and $V = X/Y$.

(a) Express the probability density function of U and V in terms of the density functions f_X and f_Y.

(b) Evaluate the above two probability density functions in the case where $X \sim \operatorname{Exp}(\lambda)$ and $Y \sim \operatorname{Exp}(\mu)$.

R 6.T.6. If X and Y are jointly continuous with joint density function $f_{X,Y}(x,y)$, show that $X+Y$ is continuous with density function $f_{X+Y}(t) = \int f_{X,Y}(x, t-x)\, \mathrm{d}x$.

R 7.T.28. (a) Show that if random variables X, Y satisfy that $\mathbb{E}[Y \mid X] = \mathbb{E}[Y]$ almost surely, then $\operatorname{Cov}(X,Y) = 0$.

(b) Give a counterexample to the converse statement.

3.4 Conditional Distributions, Densities, and Expectations

W 3.35. Let X be uniform on $[0,1]$ and, given X, let Y be uniform on $\left[0, X^2\right]$. Find the marginal density of Y, and find $\mathbb{E}[X \mid Y]$.

W 3.36. Suppose X has an exponential distribution with parameter λ and, given X, suppose that Y has an exponential distribution with mean X. What is the joint density of X and Y? Find $\mathbb{E}[Y]$.

W 3.37. Two light fixtures hang in different hallways of an apartment building. For each fixture, a given light bulb lasts an exponential time with mean $\lambda > 0$ before it has to be replaced. (λ is the same for both.) There is a stack of n replacement bulbs in a closet in hallway A, and k replacement bulbs in the corresponding closet in hallway B. What is the probability that hallway A runs out of bulbs before B does?

W 3.41. Construct random variables X and Y which have standard normal distributions such that $\mathbb{E}[XY] = 0$ but X and Y are not independent.

W 3.45. Not all continuous distribution functions have densities. Here is an example, called the Cantor function. It is defined on $[0,1]$ as follows. Let $x \in [0,1]$ and let (a_n) be the coefficients in its ternary expansion: $x = \sum_n a_n 3^{-n}$. Let $N = \infty$ if none of the $a_n = 1$, and otherwise, let N be the smallest n for which $a_n = 1$. Set $b_n = a_n/2$ if $n < N$, and set $b_N = 1$. Then set

$$F(x) = \sum_{n=1}^{N} b_n 2^{-n}$$

(a) Show that F is constant on each interval in the complement of the Cantor set, and that it maps the Cantor set onto $[0, 1]$.

(b) Deduce that F (extended to $\mathbb{R}$ in the obvious way) is a continuous distribution function, and the corresponding measure puts all its mass on the Cantor set.

(c) Conclude that F has no density, i.e. there is no function f for which $F(x) = \int_0^x f(y)\,\mathrm{d}y$ for all $x \in [0, 1]$. F is a purely singular distribution function.

W 3.50. Flip a coin with $\mathbb{P}(\text{head}) = p$. Let X_n be the number of flips it takes to get a run of n successive heads. If the first flip is tail and the next n flips are heads, then $X_n = n + 1$.

(a) Consider $\mathbb{E}\left[X_{n+1} \mid X_n = k\right]$ and relate $\mathbb{E}\left[X_n\right]$ and $\mathbb{E}\left[X_{n+1}\right]$.

(b) Show that $\mathbb{E}\left[X_n\right] = \sum_{k=1}^{n} 1/p^k$.

4 Convergence of Random Variables

W 4.7. Show that if $X_n \to X$ in probability and if $X_n \to Y$ in probability, then $X = Y$ a.s.

W 4.10. Let $X_1, X_2, \ldots$ be i.i.d. Cauchy random variables. Show that almost surely

$$\limsup_{n\to\infty} \frac{\log |X_n|}{\log n} = 1$$

W 5.24. Let $X_1, X_2, \ldots$ be i.i.d. $\operatorname{Exp}(\lambda)$. Show that there exists C_0 such that $C < C_0$ implies $\mathbb{P}\left(X_n \geq C \log n \text{ i.o.}\right) = 1$ and $C > C_0$ implies $\mathbb{P}\left(X_n \geq C \log n \text{ i.o.}\right) = 0$. What if $C = C_0$? What is the value of C_0?

W 6.3. Show that X has a symmetric distribution if and only if its characteristic function is real and symmetric.

W 6.22. (a) Show that if (X_n) and (Y_n) converge in probability to X and Y, respectively, that $(X_n + Y_n)$ converges in distribution to $X + Y$.

(b) Give an example of random variables $(X_n), (Y_n), X$ and Y on the same probability space such that X_n and Y_n converge in distribution to X and Y, respectively, but $X_n + Y_n$ does not converge in distribution to $X + Y$.

W 6.35. An astronomer wishes to measure the distance in light years from her observatory to a distant star. Because of atmospheric disturbances and experimental error, each measurement will not be accurate, but will only be approximate. As a result, she will make a series of independent measurements, and use the average value as the final estimate. She believes that she can account for all systematic errors, so that her measurements have a mean d (the true distance) and a standard deviation of four light years. If she wishes to be 95% sure that her estimate is correct to within ± 0.5 light years, approximately how many measurements should she make?

Hint. $\Phi(1.64) = .9495, \Phi(1.65) = .9505, \Phi(1.95) = .9744, \Phi(1.96) = .9750.$

W 6.39. (Close Elections.) In the American presidential election in the year 2000, $5,962,657$ votes were cast in the state of Florida. According to the official tally, $2,912,790$ votes were cast for the Republican candidate, Mr. George W. Bush, and $2,912,253$ votes were cast for the Democratic candidate, Mr. Al Gore. The other $137,614$ votes were cast for other candidates. So, out of almost six million votes, the difference between the top two candidates was less than 550 votes, less than one one-hundredth of one percent of the votes cast. This led to a wrangle in the courts over recounts, allegations of voting irregularities and disputes over "hanging chads" and "butterfly ballots", but that is another story. The interesting question is this: suppose that, in some sense, the voters are equally divided. There are still random elements at work: the rate of voter participation in that election was only 51.3%, and events can cause many people to change their minds. How unlikely is this result?

Suppose that each of the $5,825,043$ voters who voted either Republican or Democratic decides independently to vote Republican with probability $1/2$, and Democratic with probability $1/2$. Let X be the number of Republican votes, and $N - X$ the number of Democratic votes, where $N = 5,825,043$. Find the probability that the absolute value of the difference between the Republican and Democratic votes is 537 or less.

W 6.40. (Continuation: another election model.) In the 2000 federal election, 51.3% of the eligible voters nationwide voted. Suppose there are $12,000,000$ eligible voters in Florida, and suppose that exactly $6,000,000$ of these would vote for Bush and exactly $6,000,000$ would vote for Gore... if they vote. But each voter chooses independently whether or not to vote with probability $1/2$ each. Let B be the number of Bush voters and G the number of Gore voters. Find the approximate probability that $|B - G| \leq 537$.

W 6.41. (Continuation. Getting out the vote: the effect of small biases.) On the day of an important election, political parties do what is called "getting out the vote." They make sure that voters from their own party actually do go to the polls and vote. Consider the situation of the preceding problem, with one difference. Suppose that one of the two parties does nothing, but the other gets out the vote and assures that $6,000$ of their voters (or 0.1%) go to the polls–for sure. The other $5,994,000$ of their voters–and all $6,000,000$ of the opposing party's voters–choose at random between voting or not voting at all. Estimate the probability that the first party wins the election.

5 Markov Chains and Random Walks

W 7.9. Let $Z_1, Z_2, \dots$ be i.i.d. random variables with values ± 1, probability one-half each. Which of the following are Markov chains? Give the transition probabilities of those that are:

(a) $X_n = Z_n Z_{n+1}$,

(b) $X_n = Z_n + Z_{n+1}$,

(c) $X_n = Z_n + \max(Z_1, \dots, Z_n)$,

(d) $X_n = (Z_n, Z_{n+1})$,

(e) $X_n = (Z_1 + \cdots + Z_n)^2$,

(f) $X_n = 10Z_n + Z_{n+1}$.

W 7.11. A transition matrix $\mathbb{P} = (p_{ij})$ is stochastic, i.e., $p_{ij} \geq 0$ for all i, j and $\sum_j p_{ij} = 1$ for all i. It is bi-stochastic if, in addition, $\sum_i p_{ij} = 1$ for all j, i.e., if both row and column sums equal one. Show that if a Markov chain has a bistochastic transition probability matrix, then all of its n-step transition matrices are bi-stochastic.

W 7.13. Let $X_0, X_1, X_2, \dots$ be a simple symmetric random walk with $X_0 = 0$. Let $T_n = \inf\{k : X_k = n\}$. Show that $T_1, T_2 - T_1, T_3 - T_2, \dots$ are i.i.d. random variables.

References

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