Lecture 17

Why do exp gen fn's work?

Idea: $\binom{n}{k} = P(n,k)/k!$, # of selections = $\frac{\text{\# of arrangements}}{\text{length!}}$

Arrangements of n objects: $n!$

Ways to select n objects from n objects: 1.

$\Rightarrow$ Selections = arrangements $\cdot \frac{1}{n!}$

How many arrangements are there of n identical objects: 1.
so there should be $1/n!$ selections.

Principle there are $\frac{1}{k!}$ ways to select k objects from n identical objects.

(There is 1 way to arrange k objects from n identical objects)

$\Rightarrow$ taking 1, 2, 3, etc in order: $\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots\right)$

All are identical so there's one arrangement

$\Rightarrow$ the coeff is the selection $1/n!$

A slightly more rigorous approach:

Show that dividing by $\frac{1}{r!}$ works for a single factor: obvious b/c there's only one arrangement.

Show multiplication works:

let $E_1(x) = a_0 + a_1 x + \frac{a_2 x^2}{2!} + \frac{a_3}{3!}x^3 + \frac{a_4}{4!}x^4 + \dots$

$E_2(x) = b_0 + b_1 x + \frac{b_2}{2!}x^2 + \frac{b_3}{3!}x^3 + \frac{b_4}{4!}x^4 + \dots$

rth term of $E_1 \cdot E_2 = \left(\sum_{k=0}^{r} \frac{a_k}{k!} \cdot \frac{b_{r-k}}{(r-k)!}\right) x^r \Rightarrow r$th coeff is $\sum_{k=0}^{r} \frac{r!}{k!(r-k)!} a_k b_{r-k}$

$\frac{r!}{k!(r-k)!} a_k b_{r-k} \rightsquigarrow$ choose k places out of r for the a_k sequence.

the rest is b_{r-k}, summed over all $0 \le k \le r$.

Eg $\underbrace{\_\ \_\ \_\ \_\ \_}_{r=5}$ $a_2 = \underline{2}\ \underline{6}$ $b_3 = \underline{p}\ \underline{c}\ \underline{c}$

$\rightsquigarrow$ p2c6c OR 2pc6c OR pcc26 OR etc.

Eg: Put r distinct objects into n boxes w/o empty box:

$$A(x) = (e^x - 1)^n = \sum_{k=0}^{n} \binom{n}{k} e^{kx} (-1)^{n-k} = \sum_{k=0}^{n} \sum_{r=0}^{\infty} (-1)^{n-k} \binom{n}{k} \frac{k^r x^r}{r!}$$

$$\frac{a_r}{r!} = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \frac{k^r}{r!} \Rightarrow \boxed{a_r = \sum_{k=0}^{n} (-1)^{n-k} k^r \binom{n}{k}}$$

Qn how to build ordinary generating fn from a formula for the sequence?

Eg: $a_r = 2r^2$ start w $a_r \equiv 1 \rightsquigarrow A(x) = \frac{1}{1-x}$

Notice $a^*_r = r a_r$ gives $A^*(x) = x \frac{d}{dx}(A(x))$.

We do this twice then multiply by 2: $2r^2 = 2 \cdot r \cdot r \cdot 1$ and $a_r \equiv 1 \rightsquigarrow A(x) = \frac{1}{1-x}$ so

$$a_r = 2 \cdot r \cdot r \cdot 1 \rightsquigarrow 2 \cdot x \frac{d}{dx}\left[x \frac{d}{dx} \left[\frac{1}{1-x} \right]\right] = \frac{2x(1+x)}{(1-x)^3}$$

Eg. $a_r = (r+1)(r-1)(r+1) \rightsquigarrow (r+3)(r+2)(r+1) \rightsquigarrow 3! \binom{r+3}{3}$

$\rightsquigarrow A(x) = 3! \cdot \frac{1}{(1-x)^4}$ want to shift r down, mult by x^2

$$\boxed{A(x) = \frac{3!\, x^2}{(1-x)^4}}$$

Thm: Let $h^*(x)$ denote the ord. gen fn for the partial sums of the gen fn $h(x)$.

Then $h^*(x) = \frac{h(x)}{1-x}$ Pf: $h(x) \stackrel{?}{=} h^*(x) - x h^*(x)$

rth coeff $= (a_0 + a_1 + \dots + a_r) - (a_0 + \dots + a_{r-1}) = \underline{a_r}$

Eg. Evaluate $\sum_{k=1}^{n} 2k^2$

$h(x) = \frac{2x(1+x)}{(1-x)^3} \rightsquigarrow h^*(x) = \frac{2x(1+x)}{(1-x)^4} = \frac{2x}{(1-x)^4} + \frac{2x^2}{(1-x)^4}$

$a_r = 2r^2$

$$a_r = 2\binom{(r-1)+3}{3} + 2\binom{(r-2)+3}{3}$$

$$\boxed{\text{Recall } \frac{1}{(1-x)^n} = \sum_{r=0}^{\infty} \binom{n+r-1}{r-1} x^r}$$