Lecture 20 The Catalan #'s

Eg's Balanced parentheses n pairs $\rightsquigarrow C_{n-1}$, Mountain Ranges n of each stroke $\rightsquigarrow C_n$, binary trees on n verts, etc. $\rightsquigarrow C_n$

Placing Parentheses:

Given n objects in order: $k_1\ k_2 \cdots k_n$ and a binary operation $(x \cdot y) = z$, how many ways are there to place parentheses around $k_1 \cdots k_n$ to get a single output?

Ans $a_n = C_{n-1}$ $(n \geq 1)$

Eg. a_0 = parentheses on no objects $= 0$

a_1 = parentheses on 1 object $= (k_1) = 1$

a_2 = p's on $k_1\ k_2 = (k_1 \cdot k_2) = 1$

a_3 = p's on $k_1\ k_2\ k_3 = ((k_1 \cdot k_2)\cdot k_3),\ (k_1 \cdot (k_2 \cdot k_3)) = 2$

$a_4 = (k_1)\cdot((k_2\cdot k_3)\cdot k_4),\ (k_1)\cdot(k_2\cdot(k_3\cdot k_4)),\ (k_1\cdot k_2)\cdot(k_3\cdot k_4),\ (k_1(k_2\cdot k_3))\cdot(k_4),\ ((k_1\cdot k_2)\cdot k_3)\cdot(k_4) = 5.$

Idea: look at outermost first:

$$\underbrace{(k_1 \cdot k_2 \cdot \ldots \cdot k_i)}_{a_i} \cdot \underbrace{(k_{i+1} \cdot \ldots \cdot k_n)}_{a_{n-i}} \qquad a_i \cdot a_{n-i}$$

Sum only works when $n \geq 2$

$$\Rightarrow \boxed{a_n = \sum_{k=1}^{n-1} a_k a_{n-k} \qquad a_0 = 0,\ a_1 = 1}$$

$$A(x) - a_0 - a_1 x = \sum_{n=2}^{\infty} a_n x^n = \sum_{n=2}^{\infty}\left(\sum_{k=1}^{n-1} a_k a_{n-k}\right)x^n = \left((a_1 a_1)x^2 + (a_1a_2 + a_2a_1)x^3 + \cdots\right)$$

$$\sum_{k=1}^{n-1} a_k a_{n-k} = \text{coeff of } x^n \text{ in } (a_1x + a_2x^2 + a_3x^3 + \cdots)\cdot(a_1x + a_2x^2 + a_3x^3 + \cdots) = (A(x) - a_0)(A(x) - a_0)$$

$$A(x) - a_0 - a_1 x = (A(x) - a_0)^2 \qquad (a_0 = 0,\ a_1 = 1)$$

$$A(x) - x = A(x)^2 \Rightarrow A^2 - A + x = 0$$

$$\Rightarrow A = \frac{1 \pm \sqrt{1-4x}}{2} = \frac{1}{2}(1 \pm \sqrt{1-4x}) = A_+,\ A_-$$

$$1 = a_0 = A_\pm(0) = \frac{1}{2}(1 \pm \sqrt{1}) \Rightarrow A(x) = A_-(x) = \frac{1}{2}(1 + \sqrt{1-4x})$$

$$A(x) = \frac{1}{2}(1 - \sqrt{1-4x}) = \frac{1}{2} - \frac{1}{2}\sqrt{1-4x} = \frac{1}{2} - \frac{1}{2}(1+y)^{1/2} \qquad y = -4x$$

Now: def Let $q \in \mathbb{R}$, $k \in \mathbb{Z}_{\geq 0}$ then define $\binom{q}{n} = \frac{1}{n!}\prod_{k=0}^{n-1}(q-k)$

Eg. $\binom{1/2}{3} = \frac{(1/2)(1/2 - 1)(1/2 - 2)}{3 \cdot 2 \cdot 1}$, $\binom{3}{5} = \frac{3 \cdot 2 \cdot 1 \cdot 0 \cdot (-1)}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}$

$$\binom{-5/3}{2} = \frac{(-5/3)(-5/3 - 1)}{2 \cdot 1}$$

Prop $(1+x)^q = \sum_{n=0}^{\infty} \binom{q}{n} x^n$

$$A(x) = \frac{1}{2} - \frac{1}{2}(1+y)^{1/2} = \frac{1}{2} - \frac{1}{2}\sum_{n=0}^{\infty}\binom{1/2}{n} y^n = \frac{1}{2} - \frac{1}{2}\sum_{n=0}^{\infty}\binom{1/2}{n}(-4)^n x^n$$

$$= \frac{1}{2} - \frac{1}{2}\underbrace{\binom{1/2}{0}(-4)^0 x^0}_{=1} - \sum_{n=1}^{\infty}(-1)^n 2^{2n-1}\binom{1/2}{n} x^n$$

Thus $A(x) = 1 - \sum_{n=1}^{\infty}(-1)^{n+1}2^{2n-1}\binom{1/2}{n}x^n$ so $a_n = (-1)^n 2^{2n-1}\binom{1/2}{n}$, $n \geq 1$

but lets unpack $\binom{1/2}{n}$:

$$\binom{1/2}{n} = \frac{1}{n!}\prod_{k=0}^{n-1}(1/2 - k) = \frac{1}{n!}\prod_{k=1}^{n}(3/2 - k) = \frac{1}{n!}\prod_{k=1}^{n}\frac{1}{2}(3-2k) = \frac{(-1)^n}{2^n \cdot n!}\prod_{k=1}^{n}(2k-3)$$

but $\prod_{k=1}^{n}(2k-3) = -\prod_{k=1}^{n-1}(2k-1) =$ − product of first $(n-1)$ odd #'s

How can we write the product of the first n odd #'s w/ factorials?

n=3: $1\cdot 3\cdot 5 = \frac{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6}{2\cdot 4\cdot 6} = \frac{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6}{2^3\cdot 1\cdot 2\cdot 3} = \frac{(2n)!}{2^n \cdot n!}$

Fact $\prod_{k=1}^{n}(2k-1) = \frac{(2n)!}{2^n\cdot n!}$ Thus, $\prod_{k=1}^{n-1}(2k-1) = \frac{(2n-2)!}{2^{n-1}\cdot(n-1)!}$

We plug in to get: $\binom{1/2}{n} = \frac{(-1)^{n+1}}{2^n\cdot n!}\cdot\frac{(2n-2)!}{2^{n-1}\cdot(n-1)!} = \frac{(-1)^{n+1}}{2^{2n-1}\cdot n}\cdot\frac{(2n-2)!}{(n-1)!\,(n-1)!} = \frac{(-1)^{n+1}}{2^{n-1}\cdot n}\cdot\binom{2n-2}{n-1}$

Finally,

$$a_n = (-1)^{n+1}2^{2n-1}\binom{1/2}{n} = (-1)^{n+1}2^{2n-1}\cdot\frac{(-1)^{n+1}}{2^{n-1}\cdot n}\cdot\binom{2n-2}{n-1} = \frac{1}{n}\binom{2n-2}{n-1}$$

which are the **Catalan #'s** (shifted down by one)

Memorize: $A(x) = \frac{1}{2}\left(1 - \sqrt{1-4x}\right) = \sum_{n=0}^{\infty} \text{Catalan}(n-1)\, x^n$

$\text{Catalan}(n) = \frac{1}{n+1}\binom{2n}{n}$, so $\text{catalan}(n-1) = \frac{1}{n}\binom{2n-2}{n-1}$

$$\text{Catalan}(n) = \sum_{k=1}^{n-1}\text{Catalan}(k)\cdot\text{Catalan}(n-k)$$