Divergence Thm

$F = P\hat{i} + Q\hat{j} + R\hat{k}$

$$\operatorname{div}\vec{F} = \vec{\nabla}\cdot\vec{F} = P_x + Q_y + R_z$$

E is a simple solid region with surface boundary S positively oriented outwards.

F vector field with cont. p.d. on an open region containing E.

$$\text{flux}\quad \iint_S \vec{F}\cdot d\vec{S} = \iiint_E \underbrace{\operatorname{div}\vec{F}}_{\vec{\nabla}\cdot\vec{F}}\, dV$$

① $\vec{F}(x,y,z) = 3x\hat{i} + xy\hat{j} + 2xz\hat{k}$

E is the cube bounded by $x=0,\ x=1$, $y=0,\ y=1$, $z=0,\ z=1$

oriented out.

Can we use div Thm? Yes!

E: ✓

$\vec{F}$: ✓

Now find Flux of $\vec{F}$ through S.

6 surface integrals OR 1 triple integral.

E: $0 \le x \le 1$, $0 \le y \le 1$, $0 \le z \le 1$

$$\underbrace{\operatorname{div}\vec{F}}_{\vec{\nabla}\cdot\vec{F}} = \frac{\partial}{\partial x}(3x) + \frac{\partial}{\partial y}(xy) + \frac{\partial}{\partial z}(2xz) = 3x+3$$

$$\text{Flux} = \iint_S \vec{F}\cdot d\vec{S} = \int_0^1\int_0^1\int_0^1 (3x+3)\,dx\,dy\,dz = 3\int_0^1 (x+1)\,dx = 3\left(\frac{x^2}{2}+x\right)\Big|_{x=0}^{x=1}$$

$$= 3\left(\frac{1}{2}+1\right) = \frac{9}{2}.$$

③ $\vec{F} = \langle z, y, x \rangle$

E = solid ball $x^2+y^2+z^2 \le 16$. - S = surface of E oriented out.

find flux of $\vec{F}$ through S

$\operatorname{div} \vec{F} = 1$

$$\iint_S \vec{F} \cdot d\vec{S} = \iiint_E 1 \, dV = \frac{4}{3}\pi(4)^3$$

⑤ S = surface of box bounded by coordinate planes and $x=3$, $y=2$, $z=1$
oriented out.

$$\vec{F} = \langle \underline{xye^z}, \underline{xy^2z^3}, \underline{-ye^z} \rangle$$

find flux of $\vec{F}$ through S.

$$\operatorname{div}\vec{F} = \cancel{ye^z} + 2xyz^3 - \cancel{ye^z}$$

E: $0 \le x \le 3$
$0 \le y \le 2$
$0 \le z \le 1$

Flux of $\vec{F}$ through S: $\displaystyle\iint_S \vec{F} \cdot d\vec{S} = \int_0^3\int_0^2\int_0^1 2xyz^3 \, dz\,dy\,dx$

easy to do!! ☺

⑦ $\vec{F}(x,y,z) = 3xy^2\hat{\imath} + xe^z\hat{\jmath} + z^3\hat{k}$

S = surface of solid bounded by $y^2+z^2=1$ and $x=-1$, $x=2$.
oriented out.

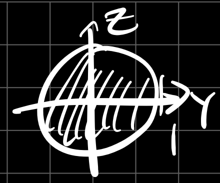
find flux of $\vec{F}$ through S.

$$\operatorname{div}\vec{F} = 3y^2 + 0 + 3z^2 = 3(y^2+z^2)$$

$E: -1 \le x \le 2$

$-1 \le y \le 1$

$-\sqrt{1-y^2} \le z \le \sqrt{1-y^2}$



$0 \le r \le 1 \quad y = r\cos\theta$

$0 \le \theta \le 2\pi \quad z = r\sin\theta$

$$\iint_S \vec{F}\cdot d\vec{S} = \int_{-1}^{2}\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} 3(y^2+z^2)\,dz\,dy\,dx = 9\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} (y^2+z^2)\,dz\,dy$$

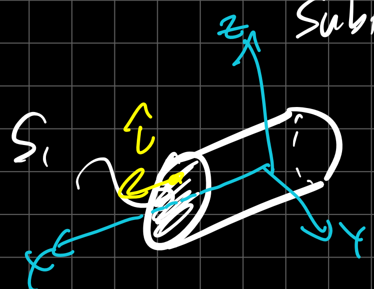
$$= 9\int_0^1\int_0^{2\pi} r^2\, r\,d\theta\,dr$$

$$= 18\pi\left(\frac{r^4}{4}\right)\Big|_{r=0}^{r=1} = \frac{18\pi}{4} = \frac{9\pi}{2}$$

(7^*) Same problem but remove disk at $x=2$.

find flux of $\vec{F}$ through S.

To use div thm pretend top included and then subtract the flux through top from $\frac{9\pi}{2}$.



on S_1 $\quad x=2$

$y^2+z^2 \le 1$

$\hat{n} = \hat{\imath}$

$\vec{F}$ on S_1

$\vec{F}(x,y,z) = 3xy^2\hat{\imath} + xe^z\hat{\jmath} + z^3\hat{k}$

$\vec{F}(2,y,z) = 6y^2\hat{\imath} + 2e^z\hat{\jmath} + z^3\hat{k}$

$$\iint_{S_1} \vec{F}\cdot d\vec{S} = \iint_{S_1} \vec{F}\cdot\hat{n}\,dS = \iint_{S_1} 6y^2\,dS =$$

$y = r\cos\theta \quad 0 \le r \le 1$

$z = r\sin\theta \quad 0 \le \theta \le 2\pi$

Flux through $S = \frac{9\pi}{2} - \iint_{S_1} \vec{F}\cdot d\vec{S}$

13 $\vec{F} = \|\vec{r}\|\vec{r}$, $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

S = hemisphere $z = \sqrt{1-x^2-y^2}$ and $x^2+y^2 \le 1$ in xy plane

oriented out.

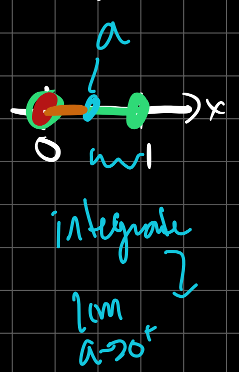
Find flux of $\vec{F}$ through S.

$$\vec{F} = \sqrt{x^2+y^2+z^2}\,(x\hat{i}+y\hat{j}+z\hat{k}) = x\sqrt{x^2+y^2+z^2}\,\hat{i} + \cdots$$

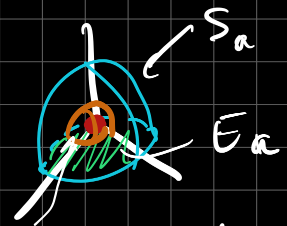
$$\operatorname{div}\vec{F} = \underbrace{\sqrt{x^2+y^2+z^2} + x\frac{x}{\sqrt{x^2+y^2+z^2}}}_{P_x} + \underbrace{\sqrt{x^2+y^2+z^2} + y\frac{y}{\sqrt{x^2+y^2+z^2}}}_{Q_y} + \underbrace{\sqrt{x^2+y^2+z^2} + z\frac{z}{\sqrt{x^2+y^2+z^2}}}_{R_z}$$

$$= 3\sqrt{x^2+y^2+z^2} + \frac{x^2+y^2+z^2}{\sqrt{x^2+y^2+z^2}} = 4\sqrt{x^2+y^2+z^2}$$

Thought process



$$\int_0^1 \frac{1}{x}\,dx$$



hemisphere of radius = a.

Now describe region in spherical

$0 \le \theta \le 2\pi$

$0 \le \phi \le \frac{\pi}{2}$

$a \le \rho \le 1$

$$\iint_{S_a} \vec{F}\cdot d\vec{S} = \iiint_{E_a} \operatorname{div} F\, dV = \int_0^{2\pi}\int_0^{\pi/2}\int_a^1 4\rho\,\rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$$

$$= 8\pi \int_0^{\pi/2} \sin\phi\, d\phi \int_a^1 \rho^3\, d\rho$$

$$\underbrace{\left(-\cos\phi\Big|_0^{\pi/2}\right)}_{1}$$

$$= 8\pi\left(\frac{\rho^4}{4}\right)\Bigg|_{\rho=a}^{\rho=1}$$

$$= 8\pi\left(\tfrac{1}{4}\right) - 8\pi\frac{a^4}{4}$$

$$= 2\pi - 2\pi a^4$$

Flux of $\vec{F}$ through S

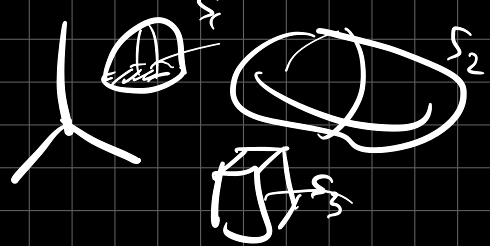
$$= \lim_{a\to 0^+}\left(2\pi - 2\pi a^4\right) = 2\pi.$$

Ex) (13^*) $\vec{F} = \vec{r}$ Same surface.

Much easier.

$(13)^{**}$ $\vec{F} = \dfrac{\vec{r}}{\|\vec{r}\|}$ may cause some issues. at $(0,0,0)$.

if S is moved:



now if S is moved AND

$\vec{F} = \dfrac{\vec{r}}{\|\vec{r}\|^3}$ $\quad \operatorname{div}\vec{F} = 0$ $\quad \displaystyle\iint_{S_{1,2,3}} \vec{F}\cdot d\vec{S} = 0$ by div Thm.

$$\left\langle \frac{x}{(x^2+y^2+z^2)^{3/2}},\ \frac{y}{(x^2+y^2+z^2)^{3/2}},\ \frac{z}{(x^2+y^2+z^2)^{3/2}} \right\rangle$$

$$\operatorname{div}\vec{F} = \frac{\partial}{\partial x}\left(\quad\right) + \frac{\partial}{\partial y}\left(\quad\right) + \frac{\partial}{\partial z}\left(\quad\right)$$

$$= \frac{(x^2+y^2+z^2)^{3/2}(1) - x\left(\frac{3}{2}\right)(x^2+y^2+z^2)^{1/2}\,2x}{(x^2+y^2+z^2)^3} + \cdots$$

$$= \frac{\|\vec{r}\|^3 - 3x^2\|\vec{r}\|}{\|\vec{r}\|^6} + \frac{\|\vec{r}\|^3 - 3y^2\|\vec{r}\|}{\|\vec{r}\|^6} + \frac{\|\vec{r}\|^3 - 3z^2\|\vec{r}\|}{\|\vec{r}\|^6}$$

$$= \frac{3\|\vec{r}\|^3 - 3\|\vec{r}\|\overbrace{(x^2+y^2+z^2)}^{\|\vec{r}\|^2}}{\|\vec{r}\|^6} = 0.$$

Also, div curl $\vec{F}$ for some $\vec{F}$ as long as 2^{nd} p.d. are defined

$$= \vec{\nabla}\cdot(\vec{\nabla}\times\vec{F})$$

$$\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

$$= \vec{\nabla}\cdot\left[(R_y - Q_z)\hat{\imath} - (R_x - P_z)\hat{\jmath} + (Q_x - P_y)\hat{k}\right]$$

$$= R_{xy} - Q_{xz} - (R_{xy} - P_{zy}) + Q_{xz} - P_{yz}$$

$$= 0.$$

then what is flux of a curl field through a surface that satisfies div Thm?

$\vec{F} = \vec{\nabla}\times\vec{G}$

Flux $= 0$.