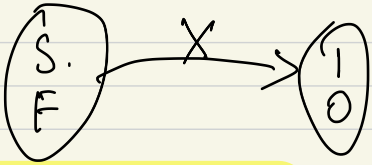
Chapter 5. Some Discrete Prob. Dist.

Ex. coin tossing. $P(\text{heads}) = p$

$P(\text{tails}) = 1-p = q$

$P(\text{Success}) = p \quad P(\text{failure}) = q \quad (p+q=1)$



① Bernoulli r.v. $\quad f(x) = \begin{cases} p & x=1 \\ q & x=0 \end{cases}$

$\mu = p, \quad \sigma^2 = pq,$

$E(X) = \sum x f(x) = 1 * p + 0 * q$
$= p.$

$E(X^2) = \sum_x x^2 f(x) = 1^2 * p + 0^2 * q$
$= p$

$\sigma^2 = E(X^2) - \mu^2 = p - p^2 = p(1-p)$

② Repeat a Bernoulli ($\underline{p}$) $\underline{n}$ times indep.

Let X be the number of successes.

$X \sim \text{Binomial}(n, p). \quad \text{Bin}(n,p)$

$f(x) = \binom{n}{x} p^x q^{n-x} \quad x = 0, 1, 2, \dots n$

$E(X) = np \qquad Var(X) = npq$

Ex: $P(\text{heads}) = 0.6$ $P(\text{Tails}) = 0.4$. Toss 5 times.

a. $P(HHHTT) = 0.6^3 \cdot 0.4^2$

$P(THTHH) = 0.6^3 0.4^2$

b. $P(\text{exactly 3 heads}) = \binom{5}{3} 0.6^3 0.4^2$

$10 = \frac{5!}{3!2!}$ ✓

HHHT
HHTH
HTHH
THHH

$\ominus \ominus - \ominus -$ ✓

Ex: $P(\text{passes the ~~red~~ test}) = 0.7$.

10 people. ① $P(\text{exactly 8 pass}) = \binom{10}{8} 0.7^8 0.3^2$

② $P(\text{at least 8 pass})$

$= \binom{10}{8} 0.7^8 0.3^2 + \binom{10}{9} 0.7^9 0.3^1 + 0.7^{10}$

③ $P(\text{more than 8 pass})$

Let $X_1, X_2, \cdots X_n$ be indep Ber(p)

X_1: 0, 1 $\quad X_2$: 0, 1 $\quad X_n$: 0, 1

Let $Y = X_1 + X_2 + \cdots + X_n$

$\sim$ # of successes in n indep Bernoulli

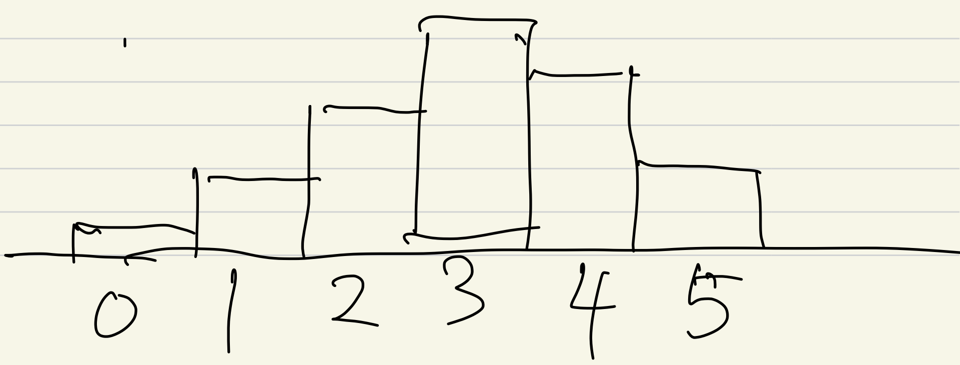
$\sim \text{Bin}(n, p)$

$$E(Y) = E(X_1 + X_2 + \cdots + X_n)$$
$$= E(X_1) + E(X_2) + \cdots + E(X_n)$$
$$= p + p + \cdots + p = np$$

$$Var(Y) = Var(X_1 + \cdots + X_n)$$
(indep)
$$= Var(X_1) + \cdots + Var(X_n)$$
$$= pq + \cdots + pq = npq$$

Ex: Toss a coin 5 times, P(heads) = 0.6.
X = # of heads. Write out $f(x)$.

x	0	1	2	3	4	5
$f(x)$	0.0102	0.0768	0.2304	0.3456	0.2592	0.0778



Ex: P(light last > 800 hr) = 0.9 . indep.

$n = 20$ lights.

① P(at least 18 will last > 800 hrs)

$$= \binom{20}{18} 0.9^{18} 0.1^{2} + \binom{20}{19} 0.9^{19} 0.1^{1} + 0.9^{20}$$

③ Geometric (p).

Repeat a Bernoulli (p), stop when the 1st success occurs.

X = # of trials required to see the 1st success. $\sim Geo(p)$

$f(x) = q^{x-1} \cdot p$ $\quad x = 1, 2, 3, \ldots$

$E(X) = \frac{1}{p}$ $\quad Var(X) = q/p^2$

$P(X > x) = q^x$

Ex: P(heads) = 0.6.

P(get the 1st heads on 3rd toss)

$$= 0.4^2 \, 0.6$$

P(get the 1st heads on 6th toss)

$$= 0.4^5 \, 0.6$$

(4) Repeat a Bernoulli (p) indep.

X = # of trials required to observe the r^{th} success. $(r \geq 1)$

$\sim$ Negative Binomial (r, p)

$$f(x) = \binom{x-1}{r-1} p^r q^{x-r}$$

$$x = r, r+1, r+2, \ldots$$

$$E(X) = \frac{r}{p} \qquad Var(X) = rq/p^2$$

$P(\text{heads}) = 0.6$

$P(\text{the } 4^{th} \text{ heads occurred on } 10^{th} \text{ toss})$

$$= \binom{9}{3} \quad 0.6^4\, 0.4^6$$

$$\underbrace{\_\ \_\ \_\ \_\ \_\ \_\ \_\ \_\ \_}_{\text{9 tosses}} \mid \underset{\uparrow\ 4^{th}\ H.}{\overset{S}{\_}}$$

3 H, 6 T.

$>3, \quad \underline{4, 5, 6 \ldots}$

P(it takes more than 3 tosses to see the 1st heads) $= 0.064$

$= 1 - P(X = 1, 2, 3)$

$= 1 - 0.6 - 0.4 \cdot 0.6 - 0.4^2 \, 0.6 = 0.064$

or:

$$P(X>3) = \sum_{x=4}^{\infty} P(X=x)$$

$$= \sum_{x=4}^{\infty} 0.4^{x-1} \cdot 0.6$$

$$= 0.4^3 \, 0.6 + 0.4^4 \, 0.6 + 0.4^5 \, 0.6 + \cdots$$

$$= \frac{0.4^3 \, \cancel{0.6}}{\cancel{1 - 0.4}} = 0.4^3 = 0.064$$

or: $P(X>3) = q^3 = 0.4^3 = 0.064$

$$S = a + ar + ar^2 + ar^3 + \cdots$$

$$rS = ar + ar^2 + ar^3 + \cdots$$

$$(1-r)S = a \qquad S = \frac{a}{1-r}$$

⑤ Multinomial dist.

$E_1, E_2, \cdots E_K$ outcomes — mutually excl.

$p_1 + p_2 + \cdots p_K = 1$

repeat n times, what's the prob

E_1 occurs n_1 times,

E_2 — n_2 —

$\vdots$

E_K — n_K times?

$n_1 + n_2 + \cdots + n_K = n$.

$$f(n_1, n_2, \cdots n_K, p_1, p_2 \cdots p_K, n)$$

$$= \frac{n!}{n_1!\, n_2! \cdots n_K!} p_1^{n_1} p_2^{n_2} \cdots p_K^{n_K}$$

$$= \binom{n}{n_1, n_2, \cdots n_K} p_1^{n_1} p_2^{n_2} \cdots p_K^{n_K}$$