6. Sequences of Functions

6.1 Pointwise and uniform convergence

For every $n \in \mathbb{N}$, let $f_n : S \to \mathbb{R}$. The sequence $\{f_n\}_{n=1}^{\infty}$ converges pointwise to $f : S \to \mathbb{R}$ if for every $x \in S$, we have

$\lim_{n\to\infty} f_n(x) = f(x)$

- Limit of $\{f_n\}_{n=1}^{\infty}$ converges, so f is unique.
- If $f_n : S \to \mathbb{R}$ converges to f on $T \subset S$ for some $f : T \to \mathbb{R}$, we say $\{f_n\}$ converges to T.

Example. $f_n : (-1,1) \to \mathbb{R}$, $f_n = \frac{1-x^n}{1-x}$, $f(x) = \frac{1}{1-x}$. $\{f_n\}$ converges pointwise to f on $(-1,1)$ since $\lim_{n\to\infty} \frac{1-x^n}{1-x} = \frac{1}{1-x} = f(x)$.

Let $f_n : S \to \mathbb{R}$, $f : S \to \mathbb{R}$. The sequence $\{f_n\}_{n=1}^{\infty}$ converges uniformly to $f : S \to \mathbb{R}$ if for every $\epsilon > 0$, there exists an $M \in \mathbb{N}$ such that for all $n \geq M$ and $x \in S$, we have

$|f_n(x) - f(x)| < \epsilon$

Example. $f_n : [0,1] \to \mathbb{R}$, $f_n(x) = x^n$, $f(x) = \begin{cases} 1 & x = 1 \\ 0 & x \neq 1 \end{cases}$.

It converges pointwise to f but not uniformly since

$\exists M \in \mathbb{N}$: $\forall x \in [0,1]$, $\forall n \geq M$, $|f_n(x) - f(x)| < \epsilon = \frac{1}{2}$ → $\forall x \in [0,1)$, $|x^M - 0| < \frac{1}{2}$.

Let $\{x_k\}$ be a sequence s.t. $x_k \in [0,1)$, $x_k \to 1$ as $k \to \infty$ (such as $x_k = 1 - \frac{1}{k}$). Then $\lim_{k\to\infty} |x_k^M| = 1^M = 1 > \frac{1}{2}$ → contradiction

Let $f_n : S \to \mathbb{R}$ be bounded functions. Define

$||f||_u = \sup\{|f(x)| : x \in S\}$

$||\cdot||_u$ is called the uniform norm.

Prop. converge uniformly $\iff \lim_{n\to\infty} ||f_n - f||_u = 0$

Let $f_n : S \to \mathbb{R}$ be bounded functions. The sequence is Cauchy in the uniform norm or uniformly Cauchy if for every $\epsilon > 0$, there exists an $M \in \mathbb{N}$ such that for all $n, k \geq M$,

$||f_n - f_k|| < \epsilon$

Remark. Definition of convergence needs f to be given, but Cauchy doesn't need f to be given.

Prop. $\{f_n\}$ is Cauchy in the uniform norm $\iff$ converges uniformly to some f

Pf. If Cauchy → fix x, $f_n(x)$ is Cauchy → $f_n(x)$ converges, defines $f = \lim_{n\to\infty} f_n(x)$ →

Given $\epsilon > 0$, find N s.t. for all $n, k \geq N$, $||f_n - f_k||_u < \frac{\epsilon}{2}$, which means for any x, $||f_n(x) - f_k(x)||_u < \frac{\epsilon}{2}$ → $\lim_{k\to\infty} ||f_n(x) - f_k(x)|| = ||f_n(x) - f(x)|| \leq \frac{\epsilon}{2} < \epsilon$

If converges uniformly, given $\epsilon > 0$, find N s.t. $\forall n \geq N$, $x \in S$, $|f_n(x) - f(x)| < \frac{\epsilon}{4}$ → $\forall n, k \geq N$, $|f_n(x) - f_k(x)| = |f_n(x) - f(x) + f(x) - f_k(x)| \leq \frac{\epsilon}{4} + \frac{\epsilon}{4}$ →

take supremum over all x to obtain $||f_m - f_k||_u \leq \frac{\epsilon}{2} < \epsilon$

Summary: $f_n \to f$

Cauchy in $||\cdot||_u \iff$ convergence in $||\cdot||_u \iff$ uniform convergence $\to$ pointwise

6.2 Interchange of limits

Thm. (uniform convergence preserves continuity) Let $\{f_n\}$ be a sequence of continuous functions $f_n : S \to \mathbb{R}$ converging uniformly to $f : S \to R$. Then f is continuous.

Pf. Let $\{x_n\}$ be a sequence in S converging to x.

Let $\epsilon > 0$ be given. As $f_n \to f$ uniformly and f_n is continous,

$\exists M \in \mathbb{N}$: $\forall n \geq M$, $\forall y \in S$, $|f_n(y) - f(y)| < \frac{\epsilon}{3}$

$\exists k \in \mathbb{N}$: $\forall k \in K$, $|f_M(x_k) - f_M(x)| < \frac{\epsilon}{3}$

Thus $\forall k \geq K$, $|f(x_k) - f(x)| \leq |f(x_k) - f_M(x_k)| + |f_M(x_k) - f_M(x)| + |f_M(x) - f(x)| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon$

$\to \lim_{k\to\infty} f(x_k) = f(x)$ → f is continous at x → as x is arbitrary, f is continuous

Thm. (uniform convergence preserves Riemann integrability) Let $\{f_n\}$ be a sequence of Riemann integrable functions $f_n : S \to \mathbb{R}$ converging uniformly to $f : S \to R$. Then f is Riemann integrable and $\lim_{n\to\infty} \int_a^b f_n = \int_a^b f$

Pf. Let $\epsilon > 0$ be given. As f_n is Riemann integrable,

$\exists M \in \mathbb{N}$: $\forall n \geq M$, $\forall x \in [a, b]$, $|f_n(x) - f(x)| \leq \frac{\epsilon}{2(b-a)}$

$|f(x)| \leq \frac{\epsilon}{2(b-a)} + |f_n(x)|$, $\forall x \in [a, b]$, since f_n is bounded, f is bounded

$$\begin{aligned}
\overline{\int_a^b} f - \underline{\int_a^b} f &= \overline{\int_a^b}(f(x) - f_n(x) + f_n(x))dx - \underline{\int_a^b}(f(x) - f_n(x) + f_n(x))dx \\
&\leq \overline{\int_a^b}(f(x) - f_n(x))dx + \overline{\int_a^b} f_n(x)dx - \underline{\int_a^b}(f(x) - f_n(x))dx - \underline{\int_a^b} f_n(x)dx \\
&= \overline{\int_a^b}(f(x) - f_n(x))dx - \underline{\int_a^b}(f(x) - f_n(x))dx \\
&\leq \frac{\epsilon}{2(b-a)}(b-a) + \frac{\epsilon}{2(b-a)}(b-a) \\
&= \epsilon
\end{aligned}$$

$\forall n \geq M$, $|\int_a^b f - \int_a^b f_n| = |\int_a^b (f(x) - f_n(x))dx| \leq \frac{\epsilon}{2(b-a)}(b-a) < \epsilon$

Thm. (uniform convergence preserves differentiability) Let I be a bounded interbal and $f_n : I \to \mathbb{R}$ be continously differentiable functions. Suppose $\{f_n'\}$ converges uniformly to $g : I \to \mathbb{R}$, and $\{f_n(c)\}$ is a convergent sequence for some $c \in I$. Then $\{f_n\}$ converges uniformly to a continously differentiable function $f : I \to \mathbb{R}$, and $f' = g$.

Pf. Define $f(c) = \lim_{n\to\infty} f_n(c)$.

As f_n' are continous and hence Riemann integrable, by FTC, $\forall x \in I$, $f_n(x) = f_n(c) + \int_c^x f_n'$.

Since $f'(n) \to g$ uniformly, $\lim_{n\to\infty} f_n(x) = f(c) + \lim_{n\to\infty} f_n' = f(c) + \int_c^x g$. Define this to be $f(x)$, so $f_n(x) \to f(x)$, and by 2nd form of FTC, $f'(x) = g(x)$.

Let ϵ be given.

$\exists M \in \mathbb{N}$: $\forall n \geq M$, $|f(c) - f_n(c)| < \frac{\epsilon}{2}$ since $f_n(c) \to f(c)$

$|f_n'(x) - g(x)| < \frac{\epsilon}{2(b-a)}$ since $f_n' \to g$ uniformly

$$\begin{aligned}
|f_n(x) - f(x)| &= |f_n(c) + \int_c^x f_n' - f(c) - \int_c^x g| \\
&\leq |f_n(c) - f(c)| + |\int_c^x f_n' - \int_c^x g| \\
&< \frac{\epsilon}{2} + |\int_c^x (f_n' - g)| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2(b-a)}(b-a) \\
&= \epsilon
\end{aligned}$$

6.3 Picard's theorem

Limit characterization of continuity in two variables. Let $U \subset \mathbb{R}^2$ be a set, let $F : U \to \mathbb{R}$, $(x, y) \in U$. We say F is continous at (x, y) if for every sequence $\{x_n, y_n\}_{n=1}^{\infty} \in U$ such that $x_n \to x$ and $y_n \to y$ as $n \to \infty$, we have

$\lim_{n\to\infty} F(x_n, y_n) = F(\lim_{n\to\infty} x_n, \lim_{n\to\infty} y_n) = F(x, y)$

We say F is continous if it is continous at every point $(x, y) \in U$.

Picard's theorem. (Existence and uniqueness of first order ODEs) Let $I, J \in \mathbb{R}$ be closed and bounded intervals, let I°, J° be their interiors, and $(x_0, y_0) \in I^\circ \times J^\circ$. Suppose $F : I \times J \to \mathbb{R}$ is continuous and Lipschitz in the second variable, that is, $\exists L \in \mathbb{R}$: $\forall y, z \in J$, $\forall x \in I$, $|F(x, y) - F(x, z)| \leq L|y - z|$.

Then there exists an $h > 0$ and a unique differentiable function $f : [x_0 - h, x_0 + h] \to J \subset \mathbb{R}$ such that $f'(x) = F(x, f(x))$ and $f(x_0) = y_0$.

Idea of proof:

- Construct Picards iterates — a sequence of functions $\{f_n\}$ that approximate solutions to the ODE
  - Find $h > 0$
    - F is bounded since it is continuous on $I \times J$
    - Take $M \in \mathbb{R}$ s.t. $|F(x, y)| \leq M, \forall (x, y) \in I \times J$
    - Take $\alpha > 0$ s.t. $[x_0 - \alpha, x_0 + \alpha] \subset I$, $[y_0 - \alpha, y_0 + \alpha] \subset J$
    - Define $h = \min\{\alpha, \frac{\alpha}{M + L\alpha}\}$, note that $[x_0 - h, x_0 + h] \subset I$
  - Define sequence $\{f_n\}$ inductively
    - Let $f_0(x) = y_0$
    - Induction step: assume $f_{k-1}([x_0 - h, x_0 + h]) \subset [y_0 - \alpha, y_0 + \alpha]$, f_{k-1} is continous, then $F(t, f_{k-1}(t))$ is well defined and continuous

        - So $f_k(x) = y_0 + \int_0^x F(t, f_{k-1}(t))$ is well-defined, and continuous by FTC
        - Show $f_k([x_0 - h, x_0 + h]) \subset [y_0 - \alpha, y_0 + \alpha]$
- Show $\{f_n\}$ is convergent
    - Since F is Lipschitz, for all $x \in [x_0 - h, x_0 + h]$,
      $|f_n(x) - f_k(x)| = |\int_{x_0}^x F(t, f_{n-1}(t))dt - \int_{x_0}^x F(t, f_{k-1}(t))dt| \leq L||f_{n-1} - f_{k-1}||_u \cdot |x - x_0| \leq \frac{L\alpha}{M+L\alpha}||f_{n-1} - f_{k-1}||_u$
    - Let $C = \frac{L\alpha}{M+L\alpha} < 1$, then $||f_n - f_k||_u \leq C^k||f_{n-k} - f_0||_u \leq C^k\alpha$
    - $\lim_{k\to\infty} C^k\alpha = 0$, so $\{f_n\}$ is Cauchy in the uniform norm, and therefore converges uniformly to a function $f : [x_0{-}h, x_0 + h]$
- Take the limit of $\{f_n\}$ to define f, then show f solves the ODE
    - $F(t, f_k(t))$ converges uniformly to $F(t, f(t))$ since
        - $|F(t, f_k(t)) - F(t, f(t))| \leq L|f_k(t) - f(t)| \leq L||f_k - f||_u$
        - $\lim_{k\to\infty} L||f_k - f||_u = 0$ → $\lim_{k\to\infty} ||F(t, f_k(t)) - F(t, f(t))||_u = 0$
    - For $x \in [x_0 - h, x_0 + h]$,
      $$\begin{aligned} f(x) = \lim_{k\to\infty} f_{k+1}(x) = f(x) = \lim_{k\to\infty}(y_0 + \int_0^x F(t, f_k(t))dt) \\ = y_0 + \int_0^x F(t, \lim_{k\to\infty} f_k(t))dt \\ = y_0 + \int_0^x F(t, f(t))dt \end{aligned}$$
    - By FTC, f is differentiable, $f'(x) = F(x, f(x))$ and $f(x_0) = y_0$
- Show f is unique
    - Suppose $g : [x_0{-}h, x_0 + h] \to J \subset \mathbb{R}$ is another solution
      $|f(x) - g(x)| = |\int_{x_0}^x F(t, f(t))dt - \int_{x_0}^x F(t, g(t))dt| \leq L||f - g||_u \cdot |x - x_0| \leq \frac{L\alpha}{M+L\alpha}||f_{n-1} - f_{k-1}||_u$
    - Let $C = \frac{L\alpha}{M+L\alpha} < 1$, then $||f_n - f_k||_u \leq C||f_n - f_k||_u$
    - Can only be true if $||f - g||_u = 0$ → $f = g$