Brownian Motion

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课本: Steven Shreve «Stochastic Calc for Finance II»
CTS time models

参考: R. Durrett «Stoch Calc_A Practical Introduction»
I. Karatzas «Brownian Motion & Stoch. Calc»
B. Oksendal «Stoch. Diff. Eq^ns».

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Def^n. 布朗运动

Stoch. Process B_t $t \geq 0$ s.t. — indexed sets of r.v.

① Independence of increments
i.e. r.v. $B_{t_0}, B_{t_1}-B_{t_0}, B_{t_2}-B_{t_1}, B_{t_3}-B_{t_2} \cdots \perp\!\!\!\perp$

② Normal Dist. of increments
$\forall s, t \geq 0$ $\mathbb{P}(B_{s+t} - B_s \in A) = \int_A \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}} dx$ $\boxed{\sim N(0,t)}$.

③ with $\mathbb{P}=1$, $B_0 = 0$, $t \mapsto B_t$ is CTS
可弱化条件.

Symmetric RW 记 $M(n) = M_n$

We start with a simpler stochastic process that is called **Symmetric Random Walk** (SRW). To construct a SRW we repeatedly toss a fair coin: $\mathbb{P}(H) = \mathbb{P}(T) = 1/2$. We define the successive outcomes of the tosses by $w_1, w_2, w_3 \dots \in \{H, T\}$ and define $w = w_1 w_2 w_3 \dots$, in other words, w is the infinite sequence of tosses and w_n is the outcome of the n^{th} toss. Next, let X_j be a random variable defined as

$$X_j = \begin{cases} 1 & \text{if } w_j = \text{H}, \\ -1 & \text{if } w_j = \text{T}. \end{cases}$$

$\mathbb{P}(X_j = 1) = \mathbb{P}(X_j = -1) = 1/2$

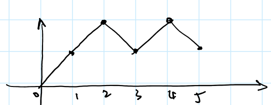
Define discrete-time stochastic process $M(n)$ in the following way:

$$M(0) = 0, M(n) = \sum_{j=1}^{n} X_j.$$

Def of symetric RW

The process $M(n)$ is called a symmetric random walk. With each step it either steps up one unit or down one unit, and each of the two possibilities is equally likely.

eg. H H T H T
1 1 -1 1 -1



例: SRW 离散 (间隔 t=1)
布朗 CTS

用 SRW 构造 Brownian.

1. increments

$0 = n_0 < n_1 < n_2 < \cdots < n_k$ 为正整数

有 $M_{n_1}, M_{n_2} - M_{n_1}, \cdots M_{n_k} - M_{n_{k-1}} \perp\!\!\!\perp$

考虑 $M_{n_1} = \sum_{j=1}^{n_1} X_j$, $1 \leq j \leq n_1$

有 $M_{n_2} - M_{n_1} = \sum_{j=1}^{n_2} X_j - \sum_{j=1}^{n_1} X_j = \sum_{j=n_1+1}^{n_2} X_j$, $n_1+1 \leq j \leq n_2$

考虑 M_{n_1}, $M_{n_2} - M_{n_1}$ 的 j 取值范围互相独立 ⇒ 皆独立

1. 计算 $\mathbb{E}(M_{n_k} - M_{n_{k-1}}) = \mathbb{E}\left(\sum_{j=n_{k-1}+1}^{n_k} X_j\right)$ $\boxed{\text{考虑 Fubini's thm}}$. Absolutely Summable 推出!

$$\begin{matrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ & & \vdots & & & \end{matrix}$$

$\sum_i = 1$ $\sum_j = 0$

$= \sum_{j=n_{k-1}+1}^{n_k} \mathbb{E}X_j$ 考虑 $\mathbb{E}X_j = 1 \times 1/2 + (-1) 1/2 = 0$
$= 0$

2. 计算 $Var(M_{n_k} - M_{n_{k-1}})$
$= Var\left(\sum_{j=n_{k-1}+1}^{n_k} X_j\right)$

$= \sum Var X_j$ (有独立)
考虑 $Var X_j = \frac{1}{2} \cdot 1^2 + \frac{1}{2}(-1)^2 - 0^2 = 1$

$= n_k - n_{k-1}$

II. Martingale Property

Definition 1. *A filtration on $(\Omega, \mathcal{F})$ is a family $\{\mathcal{F}_t\}_{t\geq 0}$ of σ-algebras $\mathcal{F}_t \subset \mathcal{F}$ such that*

$$0 \leq s \leq t \Longrightarrow \mathcal{F}_s \subset \mathcal{F}_t, \text{ i.e., } \{\mathcal{F}_t\}_{t\geq 0} \text{ is increasing.}$$

Thus, informally, one can think about the σ-algebra $\mathcal{F}_t$ in the above definition as a "knowledge" available at time t. The fact that $\{\mathcal{F}_t\}_{t\geq 0}$ is increasing just reflects the fact that as time goes on the amount of available information increases.

Definition 2. *A stochastic process $\{Z_t\}_{t\geq 0}$ is called a martingale with respect to a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ if*

a): *Z_t is $\mathcal{F}_t$-measurable for all t*
b): *$\mathbb{E}|Z_t| < \infty$ for all t*
c): *$\mathbb{E}(Z_t|\mathcal{F}_s) = Z_s$ for all $s \leq t$.*

The most important property in the above definition is property c). Simply put, it states that "the best" guess for X_t given information available at moment s is X_s, and thus process X_t has neither upward nor downward drift.

Filtration : A set of increasing σ-algebras

filtration on $(\Omega, \mathcal{F})$ 为 a family $(\mathcal{F}_t)_{t\geq 0}$

$0 \leq s \leq t \Rightarrow \mathcal{F}_s \subseteq \mathcal{F}_t$

Process Z_t is a martingale w/ respect to $(\mathcal{F}_t)_{t\geq 0}$

1) $Z_t \in \mathcal{F}_t$ ($\mathcal{F}_t$ measurable)

i.e. $\{Z_t < a\} \in \mathcal{F}_t \quad \forall a$ 或 $\{Z_t \in A\} \in \mathcal{F} \quad \forall A$

且 $Y_t = SPX\ (t+30\text{ day}) \notin \mathcal{F}_t$

2) $\mathbb{E}|Z_t| < \infty \quad \forall t$

不会 blow up.

3) $\forall s \leq t$, $\mathbb{E}(Z_t \mid \mathcal{F}_s) = Z_s$

① Claim: SRW 为 Martingale.

i.e. $M_n = \sum_{j=1}^{n} X_j$ wrt $\mathcal{F}_t = \sigma(X_1, X_2, \dots X_t)$

generated by $X_1, X_2 \dots X_t$.

Pf.

1) $M_n \in \mathcal{F}_n = \sigma(X_1, X_2 \cdots X_n)$

2) $|M_n| \leq n \quad \Rightarrow \mathbb{E}|M_n| \leq n < \infty$

3) $\mathbb{E}(M_n \mid \mathcal{F}_s) = \mathbb{E}(M_s + M_n - M_s \mid \mathcal{F}_s)$

$= \mathbb{E}(M_s \mid \mathcal{F}_s) + \mathbb{E}(\underbrace{M_n - M_s}_{\sum_{j=s+1}^{n} X_j} \mid \mathcal{F}_s)$

M_s

$\sum_{j=s+1}^{n} X_j$ 与 $\mathcal{F}_s$ 独立!

$= M_s + \sum_{j=s+1}^{n} \cancel{\mathbb{E}(X_j)}^{0}$

$= M_s$

III

Quadratic variation of the SRW. Quadratic variation of a discrete stochastic process up to time t is defined as

$$\langle M, M \rangle_t = \sum_{j=1}^{t} (M(j) - M(j-1))^2. \qquad (3)$$

The quadratic variation up to time t along a path is computed by taking all the one-step increments $M(j) - M(j-1)$ along that path, squaring these increments and then summing them. Clearly, for the symmetric random walk increments can take values ± 1 and thus

$$\langle M, M \rangle_t = t. \qquad (4)$$

Note that this is computed path by path and the fact that quadratic variation is the same along any path is a special feature of the symmetric random walk we consider. The consequence of the fact that quadratic variation is computed path by path is that for a general stochastic process quadratic variation is a random quantity, depending on the trajectory.

$\langle M_t, M_t \rangle$ 也记作 $\langle M, M \rangle_t = \sum_{j=0}^{t} (M_{j+1} - M_j)^2$ 平方项 ⇒ quadratic

$= t$ 个 1 求和 $= t$. ~ 决定 process 是否 smooth. & $\boxed{\text{volatility}}$.

是 process 性质而非 trajectory 性质!

IV Scaled SRW.

Scaled SRW. To approximate a Brownian motion we speed up time and scale down the step size of a SRW. More precisely, we fix a positive integer n and define the scaled SRW at rational points $\frac{k}{n}$ as

fix n:

$$B^{(n)}\left(\frac{k}{n}\right) = \frac{1}{\sqrt{n}} M(k). \qquad (5)$$

$/ \ B^{n}(t) = \frac{1}{\sqrt{n}} M(nt)$ ~ $n \in \mathbb{Z}^+$

CLT

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in dist

At all other points we define $B^{(n)}(t)$ be linear interpolation between its values at the nearest points of the form $\frac{k}{n}$.

The following properties of the scaled SRW could be easily proved using the corresponding properties of the SRW and we leave their proof as an exercise:

a): independence of increments, i.e., for all rational numbers $0 = t_0 < t_1 < \cdots < t_n$ of the form $\frac{k}{n}$ random variables

$$B^{(n)}(t_1) - B^{(n)}(t_0), B^{(n)}(t_2) - B^{(n)}(t_1), \dots, B^{(n)}(t_n) - B^{(n)}(t_{n-1}) \qquad (6)$$

are independent(because they depend on different coin tosses).

b): $\mathbb{E}(B^{(n)}(t) - B^{(n)}(s)) = 0$, $\text{Var}(B^{(n)}(t) - B^{(n)}(s)) = t - s$.
c): $\mathbb{E}(B^{(n)}(t)|\mathcal{F}_s) = B^{(n)}(s)$.
d): quadratic variation of the scaled SRW:

$$\langle B^{(n)} B^{(n)} \rangle_t = t.$$

当 $n = 1$ 时

$B^{(1)}(t) = \frac{1}{\sqrt{1}} \cdot M(1 \cdot t) = M(t)$

当 $n = 100$ 时

$B^{(100)}(t) = \frac{1}{\sqrt{100}} M(100t) = \frac{1}{10} M(100t)$

当 $t = 1$ 时

$B^{(100)}(1) = \frac{1}{10} M(100)$

考虑 Char. function $\varphi_X(u) = \mathbb{E}(e^{iuX})$

① 傅立叶 ~ uniqueness 唯一 char function ⟺ 唯一 dist.

② $\mathbb{E} e^{iux+ivy} = \mathbb{E} e^{iux} \mathbb{E} e^{ivy} \Leftarrow \perp\!\!\!\perp$

Th^m: Paul-Lévy Th^m

X_n 为 seq. of r.v.

a) if $X_n \xrightarrow{d} X$, 则 $\varphi_{X_n}(u) \to \varphi_X(u) \quad \forall u$ 且 此 convergence 为 uniform.

b) if $\varphi_{X_n}(u) \to \varphi_X(u)$, 则 $X_n \xrightarrow{d} X$

c) if $\varphi_{X_n}(u) \to \varphi(u)$, 且 $\varphi(u)$ CTS at $u = 0$, 则 $X_n \xrightarrow{d} X$ 且 $\varphi_X(u) = \varphi(u)$

考虑 $N(a, \sigma^2)$ w/ density $\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-a)^2}{2\sigma^2}}$

$-(x-a)^2$

考虑 $N(a,\sigma^2)$ w/ density $\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-a)^2}{2\sigma^2}}$

计算 $\phi_X(u) = \int_{\mathbb{R}} e^{iux} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-a)^2}{2\sigma^2}} dx$

$\Downarrow$ 配方 或 contour integral. 或 换元 $y = \frac{x-a}{\sigma}$

$$= \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{1}{2\sigma^2}(2\sigma^2 iux - x^2 + 2ax - a^2)} \, dx$$