Lecture 19

def inhomogeneous recurrence: $a_n = c\,a_{n-1} + f(n)$.

Note: we can add any soln to $b_n = c\,b_{n-1}$ to a_n and it still satisfies the inhomog eqn.

Eg: If $c=1$, then $a_n = a_{n-1} + f(n)$ and $\boxed{a_n = a_0 + \sum_{k=1}^{n} f(k)}$.

Eg: Generic lines cut the plane into how many regions?

$a_n = a_{n-1} + n \qquad a_1 = 2 \quad (\Leftrightarrow \underbrace{a_0 = 1}_{\hookrightarrow \text{ No lines } \rightsquigarrow \text{ one region.}})$

$a_n = 1 + \sum_{k=1}^{n} k = 1 + \frac{1}{2}n(n+1)$.

Solving Recurrence relations w/ generating functions:

Key Idea: let $A(x)$ be a gen. fn for $a_0, a_1, \dots$ then we can expand each a_n in terms of a rec. reln! to obtain an equation for $A(x)$

Eg: $a_n = a_{n-1} + n$ for $n \geq 1$, $a_0 = 1$.

$$A(x) - a_0 = \sum_{n=1}^{\infty} a_n x^n = \sum_{n=1}^{\infty} (a_{n-1} + n)\, x^n$$

$$= \sum_{n=1}^{\infty} a_{n-1} x^{n-1} + \sum_{n=1}^{\infty} n x^n = x \sum_{n=1}^{\infty} a_{n-1} x^{n-1} + \sum_{n=1}^{\infty} n x^n$$

$$= x \sum_{n=0}^{\infty} a_n x^n + \sum_{n=1}^{\infty} n x^n = x\,A(x) + \sum_{n=1}^{\infty} n x^n$$

$$= x\,A(x) + x \sum_{n=1}^{\infty} n x^{n-1} = x\,A(x) + x \frac{d}{dx}\left(\sum_{n=0}^{\infty} x^n\right)$$

$$= x\,A(x) + x \frac{d}{dx}\left(\frac{1}{1-x}\right) = x\,A(x) + \frac{x}{(1-x)^2}$$

Thus $A(x) - a_0 = x A(x) + \frac{x}{(1-x)^2}$, $a_0 = 1$

$= A(x) - 1 = x A(x) + \frac{x}{(1-x)^2} \Rightarrow A(x) = \frac{1}{1-x} + \frac{x}{(1-x)^3}$

$\boxed{a_n = 1 + \binom{n+1}{2} = 1 + \frac{n(n+1)}{2}}$ Coeff 1 Coeff $\binom{(n-1)+(3-1)}{(3-1)}$

Eg: Fibbonacci: $a_n = a_{n-1} + a_{n-2}$ $a_1 = 1$, $a_2 = 2$ $\iff$ $a_0 = a_1 = 1$

$$A(x) - a_0 - a_1 x = \sum_{n=2}^{\infty} a_n x^n = \sum_{n=2}^{\infty} (a_{n-1} + a_{n-2}) x^n =$$

$$\sum_{n=2}^{\infty} a_{n-1} x^n + \sum_{n=2}^{\infty} a_{n-2} x^n = x \sum_{n=1}^{\infty} a_n x^n + x^2 \sum_{n=0}^{\infty} a_n x^n$$

$$= x\left(A(x) - a_0\right) + x^2 A(x)$$

$$\Rightarrow A(x) - 1 - x = x\left(A(x) - 1\right) + x^2 A(x)$$

$\Rightarrow A(x) = \frac{1}{1 - x - x^2}$ Not a fn we recongize, but we can factor and do P.F.D.

We need to factor $(1 - x - x^2)$ into a product of the form $(1-\alpha x)\cdot(1-\beta x)$

so we can decompose $A(x)$ as $(?)\cdot\frac{1}{1-\alpha x} + (?)\frac{1}{1-\beta x}$

$$(1-\alpha x)(1-\beta x) = 1 - (\alpha+\beta)x + \alpha\beta x^2 = 1 - x - x^2$$

Thus $1 = \alpha + \beta$, $\alpha\cdot\beta = -1$ so $\beta = -1/\alpha$ and $1 = \alpha - 1/\alpha$

so $\alpha = \alpha^2 - 1 \Rightarrow \alpha^2 - \alpha - 1 = 0 \Rightarrow \alpha = \frac{1}{2}(1+\sqrt{5})$, $\beta = \frac{1}{2}(1-\sqrt{5})$

Quadratic formula, one solution is α, the other must be β.

Now

$$A(x) = \frac{1}{(1-\alpha x)(1-\beta x)} = \frac{Z_0}{(1-\alpha x)} + \frac{Z_1}{(1-\beta x)} = \frac{Z_0(1-\beta x) + Z_1(1-\alpha x)}{(1-\alpha x)(1-\beta x)}$$

Thus $Z_0(1-\beta x) + Z_1(1-\alpha x) = 1$

$\Rightarrow Z_0 + Z_1 - (Z_0\beta + Z_1\alpha)X = 1 \Rightarrow Z_0 + Z_1 = 1 \;,\; Z_0\beta + Z_1\alpha = 0$

$Z_0 = -Z_1\frac{\alpha}{\beta} \Rightarrow Z_1\left(1-\frac{\alpha}{\beta}\right) = 1 \Rightarrow Z_1 = \frac{1}{1-\frac{\alpha}{\beta}} = \frac{\beta}{\beta-\alpha} = \frac{\beta}{\sqrt{5}}$

$Z_0 = -\frac{\alpha}{\sqrt{5}}$

Thus: $A(x) = \dfrac{\beta/\sqrt{5}}{1-\beta x} - \dfrac{\alpha/\sqrt{5}}{1-\alpha x} = \frac{\beta}{\sqrt{5}}\sum_{n=0}^{\infty}\beta^n x^n - \frac{\alpha}{\sqrt{5}}\sum_{n=0}^{\infty}\alpha x^n$

$$= \sum_{n=0}^{\infty}\left(\frac{\beta^{n+1}-\alpha^{n+1}}{\sqrt{5}}\right)x^n \Rightarrow \boxed{a_n = \frac{1}{2^{n+1}\sqrt{5}}\left((1-\sqrt{5})^{n+1}-(1+\sqrt{5})^{n+1}\right)}$$

General observations:

1. $A(x) - a_0 - a_1 x - a_2 x^2 - \dots - a_{p-1}x^{p-1} = \sum_{n=p}^{\infty} a_n x^n$

2. If a reccurence Rel'n requires p initial conditions then we can apply it to the RHS of 1.