MA-UY 2114
Extra Practice Worksheet
Calculus III, Fall 2021

Some extra problems to practice before the exam are given below. This is not a comprehensive list. We recommend going over homework problems and problems covered in class to prepare for the exam.

1. Integrate $f(x, y, z) = x + y + z$ over the straight line segment from $(1, 2, 3)$ to $(0, -1, 1)$.
2. Evaluate $\int_C y^2 dx + x^2 dy$ where C is the circle $x^2 + y^2 = 4$.
3. Find the area of the surface given by

$$\mathbf{r}(u, v) = < u + v, u - v, v >$$

$$0 \leq u \leq 1, 0 \leq v \leq 1$$

4. Determine if any of the following vector fields are conservative or not
   (a) $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$
   (b) $\mathbf{F} = \frac{\mathbf{r}}{\|\mathbf{r}\|^3}, \mathbf{r} = < x, y, z >$
   (c) $\mathbf{F} = xe^y\mathbf{i} + ye^z\mathbf{j} + ze^x\mathbf{k}$
   (d) $\mathbf{F} = < 2, 2y + z, y + 1 >$

use Curl test.

Curl $\vec{F} = \vec{0}$, Domain $\mathbb{R}^3$ Conservati.

Now find potential.

5. Find the work done by $\mathbf{F} = \frac{x\mathbf{i} + y\mathbf{j}}{(x^2 + y^2)^{3/2}}$ over the plane curve $\mathbf{r}(t) = < e^t \cos t, e^t \sin t >$ from $(1, 0)$ to $(e^{2\pi}, 0)$
6. Show that $\oint_C \ln x \sin y dy - \frac{\cos y}{x} dx = 0$ for any closed curve C to which Green's Theorem applies.
7. Find the outward flux of $\mathbf{F} = 2xy\mathbf{i} + 2yz\mathbf{j} + 2xz\mathbf{k}$ across the boundary of the cube cut from the first octant by the planes $x = a, y = a, z = a$ where $a > 0$.
8. Use Green's Theorem to evaluate $\int_C x^2 y dx - xy^2 dy$ where C is the circle $x^2 + y^2 = 4$ with counterclockwise orientation.
9. Use the Divergence Theorem to calculate the flux of $\mathbf{F} = (3z + 1)\mathbf{k}$ upward across the hemisphere $x^2 + y^2 + z^2 = a^2$, where $z \geq 0$.
10. Give an example of a vector field that has value $\mathbf{0}$ at only one point and such that curl $\mathbf{F}$ is nonzero everywhere.

Fix this

11. Use Stokes' Theorem to evaluate $\iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S}$ where $\mathbf{F} = < x^2yz, yz^2, z^3e^{xy} >$

~~12.~~ If curl $\mathbf{F}$ = curl $\mathbf{G}$ then is it true that $\mathbf{F} = \mathbf{G}$? $\vec{F} = \vec{G} + \vec{\nabla} f$ NAH

13. If $\mathbf{a}$ is a constant vector, $\mathbf{r} =< x, y, z >$ and S is an oriented, smooth surface with a simple, closed, smooth, positively oriented boundary curve C, show

$$\iint_S 2\mathbf{a} \cdot d\mathbf{S} = \int_C (\mathbf{a} \times \mathbf{r}) \cdot d\mathbf{r}$$

14. If f is a harmonic function (it satisfies Laplace's equation), show that the line integral $\int_C f_y dx - f_x dy$ is independent of path in any simple region D.

Quick comments

Scalar potential f s.t. $\vec{\nabla} f = \vec{F}$

Vector potential $\vec{F}$ s.t. $\vec{\nabla} \times \vec{F} = \vec{G}$

(a) $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = \nabla f = f_x \hat{i} + f_y \hat{j} + f_z \hat{k}$

$$f_x = x \xrightarrow{\int f_x dx} \frac{x^2}{2} + h(y,z)$$

$$\downarrow \frac{\partial}{\partial y}$$

$$f_y = y = \frac{\partial}{\partial y} h(y,z) \Rightarrow h(y,z) = \frac{y^2}{2} + g(z)$$

$$f_z = z$$

$$f(x,y,z) = \frac{x^2}{2} + \frac{y^2}{2} + g(z)$$

$$f_z = g'(z) = z$$

$$g = \frac{z^2}{2} + C$$

$$f(x,y,z) = \frac{x^2+y^2+z^2}{2} + C.$$

if not cons. we would get something like

$g(y) = f(x)$

contradiction.

(c) $\mathbf{F} \overset{\overset{\nabla f}{=}}{=} \boxed{xe^y}^{\,= f_x}\,\mathbf{i} + \boxed{ye^z}^{\,= f_y}\,\mathbf{j} + \boxed{ze^x}^{\,= f_z}\,\mathbf{k}$ - not conservative.

$f_x = xe^y \xrightarrow{\int f_x dx} = \frac{x^2 e^y}{2} + h(y,z)$

$\downarrow \frac{\partial}{\partial y}$

$f_y = ye^z = \frac{x^2}{2}e^y + \frac{\partial}{\partial y}\boxed{h(y,z)}$

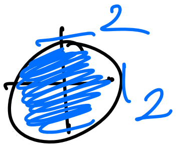
contradiction.

$f_z = ze^x$

8. Use Green's Theorem to evaluate $\int_C x^2y\,dx - xy^2\,dy$ where C is the circle $x^2+y^2=4$ with counterclockwise orientation.

$\int_C \vec{F}\cdot d\vec{r} = \iint_D \text{curl}\,\vec{F}\cdot\hat{k}\,dA$

upwards normal.



$0 \le r \le 2$

$0 \le \theta \le 2\pi$

C is counterclockwise.

$\vec{F} = \langle \underset{P}{x^2y}, \underset{Q}{-xy^2}, \underset{R}{0} \rangle$

if C is clockwise: $\hat{k} \mapsto -\hat{k}$

need: $Q_x - P_y$

$-y^2 - x^2$

$\int_C x^2y\,dx - xy^2\,dy = \int_C \vec{F}\cdot d\vec{r}$

$= \iint_D -y^2 - x^2\,dA$

$$= - \iint_D x^2+y^2 \, dA$$

$$= - \int_0^2 \int_0^{2\pi} r^2 r \, d\theta \, dr$$

7. Find the outward flux of $\mathbf{F} = 2xy\mathbf{i} + 2yz\mathbf{j} + 2xz\mathbf{k}$ across the boundary of the cube cut from the first octant by the planes $x = a, y = a, z = a$ where $a > 0$.

Use div Thm.

Solid: E: $0 \le x \le a$
$0 \le y \le a$
$0 \le z \le a$

$$\operatorname{div} \vec{F} = 2y + 2z + 2x$$

$$\iint_S \vec{F} \cdot d\vec{S} = \iiint_E 2y+2z+2x \, dV$$

$$= \int_0^a \int_0^a \underbrace{\int_0^a (2y+2z+2x)}_{} \, dy \, dx \, dz$$

$$= \int_0^a \int_0^a (y^2+2yz+2yx)\Big|_{y=0}^{y=a} \, dx \, dz$$

$$= \int_0^a \underbrace{\int_0^a a^2+2az+2ax \, dx}_{} \, dz$$

$$= \int_0^a \left(a^2x+2axz+ax^2 \Big|_{x=0}^{x=a} \right) dz$$

$$= \int_0^a 2a^3+2a^2z \, dz$$

$$= 2a^3z + a^2z^2 \Big|_{z=0}^{z=a}$$

$$= 2a^4 + a^4 = 3a^4$$

1. Integrate $f(x, y, z) = x + y + z$ over the straight line segment from $(1, 2, 3)$ to $(0, -1, 1)$.

$\vec{r}(t) = (1-t)\vec{r}_0 + t\vec{r}_1$ $0 \le t \le 1$

$\vec{r}_0 = \langle 1, 2, 3 \rangle$

$r_1 = \langle 0, -1, 1 \rangle$

$\vec{r}(t) = \langle (1-t), (2-2t), (3-3t) \rangle + \langle 0, -t, t \rangle$

$= \langle (1-t), (2-3t), (3-2t) \rangle$

$\vec{r}\,'(t) = \langle -1, -3, -2 \rangle$

$\|\vec{r}\,'(t)\| = \sqrt{1^2 + 3^2 + 2^2} = \sqrt{14}$

$$\int_C f(x,y)\, ds = \int_a^b f(r(t))\, \|r'(t)\|\, dt$$

$f(x, y) = x + y + z$ $0 \le t \le 1$

$f(r(t)) = (1-t) + (2-3t) + (3-2t)$

$\sqrt{14}\int_0^1 6 - 6t\, dt = 3\sqrt{14}$

easy!!

make sure
$\|r'(t)\|$ is
const. before
taking it out
the integral

2. Evaluate $\underbrace{\int_C \overset{P}{y^2dx} + \overset{Q}{x^2dy}}$ where C is the circle $\underline{x^2+y^2=4}$. $\leftarrow r \neq 4$

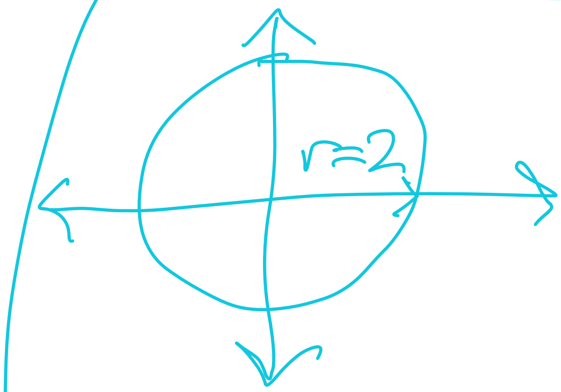
$r=2$

$$\int_C P dx + Q dy = \iint_D (Q_x - P_y)\, dA \qquad \text{G.T.}$$

$$P = y^2 \xrightarrow{\frac{d}{dy}} P_y = 2y$$

$$Q = x^2 \xrightarrow{\frac{d}{dx}} Q_x = 2x$$

$$\int_C y^2 dx + x^2 dy = \iint_D (2x - 2y)\, dA$$



$x = r\cos\theta$

$y = r\sin\theta$

$0 \le r \le 2$

$0 \le \theta \le 2\pi$

$$\int_0^{2\pi}\int_0^{2} (2r\cos\theta - 2r\sin\theta)\, r\, dr\, d\theta$$

$$= 2\int_0^{2\pi}\int_0^2 r^2 \underbrace{(\cos\theta - \sin\theta)}\underbrace{dr\,d\theta}$$

$$= 2\cancel{\int_0^{2\pi} \overset{0}{(\cos\theta - \sin\theta)} d\theta} \cancel{\int_0^2 r^2 dr}$$

$$= 0$$

3. Find the area of the surface given by

$$\mathbf{r}(u, v) = < u + v, u - v, v >$$

$$0 \le u \le 1, 0 \le v \le 1$$

$$A(S) = \iint_D \|\vec{r}_u \times \vec{r}_v\| \, dA$$

$$\vec{r}_u = \langle 1, 1, 0 \rangle$$

$$\vec{r}_v = \langle 1, -1, 1 \rangle$$

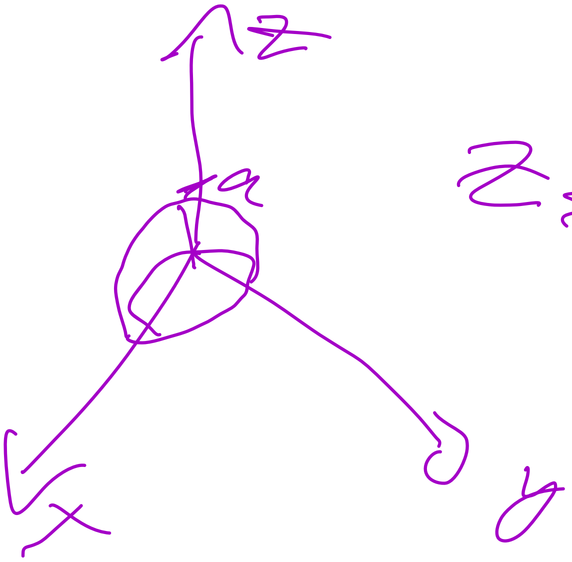
$$\vec{r}_u \times \vec{r}_v = \langle 1, -1, -2 \rangle$$

$$\|\vec{r}_u \times \vec{r}_v\| = \sqrt{1^2 + 1^2 + 2^2} = \sqrt{6}$$

$$A(S) = \int_0^1 \int_0^1 \sqrt{6}\, dv\, du$$

$$= \cancel{\int_0^1} = \sqrt{6}$$

9. Use the Divergence Theorem to calculate the flux of $\mathbf{F} = (3z+1)\mathbf{k}$ upward across the hemisphere $x^2 + y^2 + z^2 = a^2$, where $z \geq 0$.



E

$z \geq 0$

$\operatorname{div} \vec{F} = \vec{\nabla} \cdot \vec{F}$

$P_x + Q_y + R_z$

$\uparrow \quad \uparrow \quad \uparrow$

$0 \quad 0 \quad 3$

$\leq x \leq$

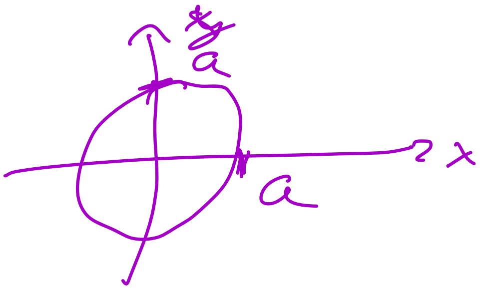
$$\iiint_E 3\, dV$$

$$3 \iiint dV$$

$$\frac{1}{2} \frac{4}{3} \pi a^3$$

$$\frac{12}{6} \pi a^3$$

$$2\pi a^3 - \iint_S \vec{F} \cdot d\vec{S}$$



normal vector $= -\hat{k}$

$F = (3z+1)\hat{k}$

$F = \hat{k}$

$F \cdot -k = -1$

$\iint_S F \cdot dS = \iint_S F \cdot \hat{n}\, dS$

$= \iint -1\, dS$

$= -\iint dS$

$= -\pi a^2$

$2\pi a^3 - (-\pi a^3) = \boxed{2\pi a^3 + \pi a^2}$ ✓