Multivariate Ito Lemma

2022年6月29日 18:58

Multidimensional BM — n independent BM w/ diff't 方向

$\vec{B} = (B_t^1, B_t^2, B_t^3 \cdots B_t^n)$. 其中 $B_t^i \perp\!\!\!\perp B_t^j \quad \forall i \neq j$

n-dimensional Itô Process

$$\begin{cases} dX_t^1 = a^1(t,\omega)\,dt + \sigma^{11} dB_t^1 + \sigma^{12} dB_t^2 + \cdots + \sigma^{1n} dB_t^n \\ dX_t^2 = a^2\,dt + \sigma^{21} dB_t^1 + \sigma^{22} dB_t^2 + \cdots + \sigma^{2n} dB_t^n \\ \vdots \\ dX_t^m = a^m\,dt + \sigma^{m1} dB_t^1 + \sigma^{m2} dB_t^2 + \cdots + \sigma^{mn} dB_t^n \end{cases}$$

$$\therefore d\vec{X} = \vec{a}\,dt + \Sigma\, d\vec{B_t}$$

$$\underbrace{\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{pmatrix}}_{m \times 1} + \underbrace{\begin{pmatrix} \sigma^{11} & \cdots & \sigma^{1n} \\ \sigma^{21} & \ddots & \vdots \\ \vdots & & \\ \sigma^{m1} & \cdots & \sigma^{mn} \end{pmatrix}}_{m \times n} \underbrace{\begin{pmatrix} B_t^1 \\ \vdots \\ B_t^n \end{pmatrix}}_{n \times 1}$$

$f(t, X_t^1, X_t^2 \cdots X_t^n)$

$$\Rightarrow \quad f(T, \vec{X}_T) = f(0, \vec{X}_0) + \int_0^T f_t(t, \vec{X}_t) + \sum_i \int_0^T f_{x_i}(t, \vec{X}_t)\, dX_t^i \quad \text{共 } n \text{ 个} + \frac{1}{2} \sum_{i,j} \int_0^T f_{x_i x_j}(t, \vec{X}_t)(dX_t^i)(dX_t^j)$$

考虑 $\begin{cases} dB_t^i \cdot dB_t^j = 0 & i \neq j \\ dB_t^i \cdot dB_t^j = t & i = j \end{cases}$