1. G is a group. H is a normal subgroup of G. If there exists a homomorphism $f : G \longrightarrow H$ such that $f(h) = h$ for any $h \in H$, and $N = \ker(f)$, prove $G = H \times N$.

   Solution: It is given H is a normal subgroup of G. $N = \ker(f)$, so N is also a normal subgroup of G.

   For any $h \in H$, $f(h) = h$, so if $h \in H \setminus \{1\}$, $f(h) = h \neq 1$, we see $H \cap N = H \cap \ker(f) = \{1\}$.

   For any $g \in G$, $f(g) \in H$, so $f(f(g)) = f(g)$, $f(f(g)^{-1}g) = f(f(g)^{-1})f(g) = f(f(g))^{-1}f(g) = 1$, we see $f(g)^{-1}g \in \ker(f) = N$, so there exists $n \in H$ such that $f(g)^{-1}g = n$, $g = f(g)n \in HN$. Thus $G = HN$.

   We conclude $G = H \times N$.

2. Write $(1\ 3\ 7)(2\ 4\ 5\ 6) \in S_7$ as a product of 2-cycles.

   Solution: $(1\ 3\ 7)(2\ 4\ 5\ 6) = (1\ 7)(1\ 3)(2\ 6)(2\ 5)(2\ 4)$

3. Compute the signature of the element $(1\ 2\ 3)(6\ 7)(4\ 5\ 9) \in S_9$.

   Solution:

$$\begin{aligned}\operatorname{Sgn}((1\ 2\ 3)(6\ 7)(4\ 5\ 9)) &= \operatorname{Sgn}((1\ 2\ 3))\operatorname{Sgn}((6\ 7))\operatorname{Sgn}((4\ 5\ 9))\\ &= (+1) \times (-1) \times (+1)\\ &= -1\end{aligned}$$

4. $\tau, \sigma \in S_n$. Prove that $\tau\sigma$ and $\sigma\tau$ have the same cycle type.

   Solution: $\sigma(\tau\sigma)\sigma^{-1} = \sigma\tau$, so $\tau\sigma$ and $\sigma\tau$ are conjugate, hence they have the same cycle type.

5. Complete the following form about S_6:

Cycle Type	Representative	Signature	Number of Elements	Order of Each Element
1+1+1+1+1+1	id	+1	1	1
1+1+1+1+2	(12)	−1	15	2
1+1+1+3	(123)	+1	40	3
1+1+2+2	(12)(34)	+1	45	2
1+1+4	(1234)	−1	90	4
1+2+3	(12)(345)	−1	120	6
2+2+2	(12)(34)(56)	−1	15	2
1+5	(12345)	+1	144	5
24	(12)(3456)	+1	90	4
33	(123)(456)	+1	40	3
6	(123456)	−1	120	6

6. Find all the elements in S_5 that commute with $(1\ 2\ 3)$.

Solution:

$\sigma(1\ 2\ 3)\sigma^{-1} = (\sigma(1)\ \sigma(2)\ \sigma(3))$, so σ commutes with $(1\ 2\ 3)$ if and only if $(1\ 2\ 3) = (\sigma(1)\ \sigma(2)\ \sigma(3))$.

Case 1: $\sigma(1) = 1, \sigma(2) = 2, \sigma(3) = 3$.

There are two possibilities: $\sigma = id$ or $\sigma = (4\ 5)$

Case 2: $\sigma(1) = 2, \sigma(2) = 3, \sigma(3) = 1$.

There are two possibilities: $\sigma = (1\ 2\ 3)$ or $\sigma = (1\ 2\ 3)(4\ 5)$

Case 3: $\sigma(1) = 3, \sigma(2) = 1, \sigma(3) = 2$.

There are two possibilities: $\sigma = (1\ 3\ 2)$ or $\sigma = (1\ 3\ 2)(4\ 5)$

7. H is a subgroup of S_n with $|H|$ an odd number. Prove $H \subseteq A_n$.

(Hint: Consider $[H : H \cap A_n]$)

Solution: Consider the restriction of the signature function to the subgroup H:

$f : H \longrightarrow \{\pm 1\}$ is defined by $f(\sigma) = \mathrm{Sgn}(\sigma)$.

By the definition of A_n, $\ker(f) = \{\sigma \in H \,|\, \mathrm{Sgn}(\sigma) = +1\} = H \cap A_n$.

Suppose $H \not\subseteq A_n$, then there exists $\sigma \in H$ with $sgn(\sigma) = -1$. It follows f is surjective, and then by First Isomorphism Theorem, $H/(H \cap A_n) \cong \{\pm 1\}$.

This means $[H : H \cap A_n] = |\{\pm 1\}| = 2$, so $[H : H \cap A_n]$ is even. By Lagrange Theorem, $|H| = [H : H \cap A_n]|H \cap A_n|$, contradict to $|H|$ is odd.

8. If $n \geq 5$, prove the only proper normal subgroup of S_n is A_n.

   Solution: Let N be a normal subgroup of S_n. Then $N \cap A_n$ is a normal subgroup of A_n. Because A_n is simple when $n \geq 5$, $N \cap A_n = \{id\}$ or $N \cap A_n = A_n$

   Case (1). $N \cap A_n = \{id\}$. Consider the restriction of the signature function to N

$$sgn|_N : N \longrightarrow \{\pm 1\}$$

$$\sigma \mapsto \operatorname{Sgn}(\sigma)$$

   $\ker(\operatorname{Sgn}|_N) = N \cap A_n = \{id\}$, so $\operatorname{Sgn}|_N$ is injective, we see $|N| = |\operatorname{Im}(sgn|_N)|$.

   If $\operatorname{Im}(\operatorname{Sgn}|_N) = \{+1\}$, then $N = \{id\}$.

   If $\operatorname{Im}(sgn|_N) = \{\pm 1\}$, then $|N| = 2$, so we can write $N = \{id, \sigma\}$ for some odd permutation σ. Then N cannot be normal since it is not hard to see that for non-identity $\sigma \in S_n$, its cycle type consists of more than just σ, but those elements who are conjugate to σ are not in N.

   Case (2). $N \cap A_n = A_n$, then $A_n \subseteq N \subseteq S_n$, we get

$$2 = [S_n : A_n] = [S_n : N][N : A_n]$$

   So $[N : A_n] = 1$ or $[N : A_n] = 2$.

   If $[N : A_n] = 1$, then $N = A_n$.

   If $[N : A_n] = 2$, then $[S_n : N] = 1$, so $N = S_n$.

   By Case (1) and (2), we conclude N can be $\{id\}$, A_n or S_n, so the only choice for a proper normal subgroup is A_n.