Homework 8

Due: Friday Nov 5, by 11:59pm,
via Gradescope

- Failure to submit homework correctly will result in a zero on the homework.
- Homework must be in LaTeX. Submit the pdf file to Gradescope.
- Problems assigned from the textbook come from the 5th edition.
- No late homework accepted. Lateness due to technical issues will not be excused.

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1. (3 points) We can apply mathematical induction on $\mathbb{Z}^+ \times \mathbb{Z}^+$. Suppose that you want to prove

$$P(b,d) = T \text{ for all } (b,d) \in \mathbb{Z}^+ \times \mathbb{Z}^+$$

The proof goes as follows

Proof:

Base Case: Show that $P(b,1) = T$ for all $b \geq 1$. Notice that your base case is an induction argument.

Induction Step: Assume b and k are any positive integers such that

$$P(b,k) = T$$

You need to show

$$P(b,k+1) = T$$

With that said, use the induction method outlined above to prove

$$\sum_{a=1}^{b}\sum_{c=1}^{d}(a+c) = \frac{bd(b+d+2)}{2} \text{ for all } (b,d) \in \mathbb{Z}^+ \times \mathbb{Z}^+.$$

Solution:

Base Case: $\sum_{a=1}^{b}\sum_{c=1}^{1}(a+c) = \sum_{a=1}^{b}(a+1)$. Now we must show that this equals $b(b+3)/2$ using induction.

Base Case: When $b = 1$, we have $\sum_{a=1}^{b}(a+1) = 2$. and $b(b+3)/2 = 2$.

Induction Step: Assume that k is any integer ≥ 1 such that

$$\sum_{a=1}^{k}(a+1) = k(k+3)/2$$

Let's add $k+2$ to both sides. Then we have

$$\begin{aligned}\sum_{a=1}^{k+1}(a+1) &= k(k+3)/2+k+2\\ &= \frac{k(k+3)+2(k+2)}{2}\\ &= \frac{k^2 5k+4}{2}\\ &= \frac{(k+1)(k+4)}{2}\end{aligned}$$

Induction Step. Assume that k is any integer ≥ 1 such that

$$\sum_{a=1}^{b}\sum_{c=1}^{k}(a+c) = \frac{bk(b+k+2)}{2}$$

for all $b \geq 1$. Let's add

$$\begin{aligned}\sum_{a=1}^{b}\sum_{c=1}^{k+1}(a+c) &= \sum_{a=1}^{b}\sum_{c=1}^{k}(a+c) + \sum_{a=1}^{b}(a+k+1)\\ &= \frac{bk(b+k+2)}{2} + \sum_{a=1}^{b}(a+k+1)\\ &= \frac{bk(b+k+2)}{2} + \frac{b(b+1)}{2} + (k+1)b\\ &= \frac{bk(b+k+2)}{2} + \frac{b(b+1)}{2} + \frac{2(k+1)b}{2}\\ &= \frac{b\left[k(b+k+2)+b+1+2(k+1)\right]}{2}\\ &= \frac{b\left[kb+k^2+2k+b+1+2(k+1)\right]}{2}\\ &= \frac{b\left[kb+(k+1)^2+b+2(k+1)\right]}{2}\\ &= \frac{b\left[(k+1)b+(k+1)^2+2(k+1)\right]}{2}\\ &= \frac{b(k+1)\left[b+k+1+2\right]}{2}\end{aligned}$$

2. (6 points) Section 5.2 # 14, 18.

   Solution:

   #14. **Proof:**

   Base Case: When $n=0$, the LHS (left hand side) reads $1 \cdot 2^1 = 2$ while the RHS (right hand side) reads $0 \cdot 2^2 + 2 = 2$. Therefore $P(0)$ is T.

Induction Step: Assume that k is any non-negative integer such that

$$\sum_{i=1}^{k+1} i \cdot 2^i = k \cdot 2^{k+2} + 2 \tag{1}$$

We NTS (need to show)

$$\sum_{i=1}^{k+2} i \cdot 2^i = (k+1) \cdot 2^{k+3} + 2$$

Let's add $(k+2) \cdot 2^{k+2}$ to both side of the induction hypothesis (line (1). We obtain

$$\begin{aligned}\sum_{i=1}^{k+2} i \cdot 2^i &= k \cdot 2^{k+2} + 2 + (k+2) \cdot 2^{k+2} \\ &= (k+k+2)2^{k+2} + 2 \\ &= (2k+2)2^{k+2} + 2 \\ &= (k+1)2^{k+3} + 2\end{aligned}$$

$\square$

#18. **Proof:**

Base Case: When $n = 2$, the LHS (left hand side) reads $1 - \frac{1}{2} = \frac{1}{2}$ while the RHS (right hand side) reads $\frac{1}{2}$. Therefore $P(2)$ is T.

Induction Step: Assume that k is any integer ≥ 2 such that

$$\prod_{i=2}^{k} \left(1 - \frac{1}{i}\right) = \frac{1}{k} \tag{2}$$

We NTS (need to show)

$$\prod_{i=2}^{k+1} \left(1 - \frac{1}{i}\right) = \frac{1}{k+1}$$

We multiple both sides of the induction hypothesis (line (2) by $\left(1 - \frac{1}{k+1}\right)$. We obtain

$$\begin{aligned}\prod_{i=2}^{k+1} \left(1 - \frac{1}{i}\right) &= \frac{1}{k}\left(1 - \frac{1}{k+1}\right) \\ &= \frac{1}{k} \cdot \frac{k}{k+1} \\ &= \frac{1}{k+1}\end{aligned}$$

3. (3 points) Section 5.2 # 40.

40. Let $x_1, x_2, \dots x_p$ be any collection of p consecutive integers. Let's assume that they are listed in increasing order so that $x_i = x_1 + i$ for $i = 2, 3, \dots p-1$. Then

$$\begin{aligned} \sum_{i=1}^{p} x_i^2 &= \sum_{i=0}^{p-1} (x_1 + i)^2 \\ &= \sum_{i=0}^{p-1} x_1^2 + 2x_1 i + i^2 \\ &= px_1^2 + 2x_1 \frac{(p-1)(p)}{2} + \frac{(p-1)(p)(2p-1)}{6} \\ & p\left(x_1^2 + x_1(p-1) + \frac{(p-1)(2p-1)}{6}\right) \end{aligned} \tag{3}$$

Note that we've used the formulas

$$\begin{aligned} \sum_{i=1}^{n} i &= \frac{n(n+1)}{2} \\ \sum_{i=1}^{n} i^2 &= \frac{n(n+1)(2n+1}{6} \end{aligned}$$

We need to ensure that the terms

$$\frac{(p-1)(2p-1)}{6} \in \mathbb{Z}$$

Claim: Assume $6|ps$ where p is prime. Then 6 must divide s. To see this we use Quotient-Remainder Theorem to obtain $s = 6q + r$ where q and r are unique and $0 \leq r \leq 5$. It follows that $ps = 6qp + rp$, hence $6m = 6qp + rp$ for some integer m. We find that

$$6[m - qp] = rp$$

Case 1: $r = 1$. Then 6 is a factor of $p \Rightarrow\Leftarrow$ (since p is prime).

Case 2: $r = 2$. Then 3 is a factor of $p \Rightarrow\Leftarrow$ (since $p \geq 5$)

Case 3: $r = 3$. Then 2 is a factor of $p \Rightarrow\Leftarrow$ (since $p \geq 5$)

Case 4: $r = 4$. Notice that $4p = 6p - 2p$, therefore

$$6[m - qp - p] = -2p$$

and 3 is a factor of $p \Rightarrow\Leftarrow$ (since $p \geq 5$)

Case 4: $r = 5$. Notice that $5p = 6p - p$, therefore

$$6[m - qp - p] = -p$$

and 6 is a factor of $p \Rightarrow\Leftarrow$

Now the expression

$$\frac{p(p-1)(2p-1)}{6}$$

is an integer and moreover 6 does not divide p. It follows that

$$\frac{(p-1)(2p-1)}{6} \in \mathbb{Z}$$

$\square$

4. (12 points) Section 5.3 # 3, 12, 21.

Solution:

#3 I will leave part (a) to you. Let's tackle (b).

Base Case: Notice that $28 = 4 \cdot 5 + 1 \cdot 8$, therefore $P(28)$ is T.

Induction Step: Assume that k is any integer ≥ 28 such that

$$k = 5a + 8b$$

for some non-negitive integers a and b. Note that a and b will depend on k. Notice that

$$k + 1 = 5a + 8b + 1 = 5a + 8b + 2 \cdot 8 - 3 \cdot 5 = 5(a-3) + 8(b+2)$$

Case 1: $a \geq 3$. Then we're done.
Case 2: $0 \leq a \leq 2$. Then $5a \leq 10$. Since $k \geq 28$, then we must have $b \geq 3$. Now

$$k + 1 = 5a + 8b + 1 = 5a + 8(b-3) + 3 \cdot 8 + 1 = 5(a+5) + 8(b-3)$$

Therefore $P(k+1)$ is T $\square$
21.

Base Case: Set $n = 0$. Then $7^n - 2^n = 0$ and $5|0$. Therefore $P(0)$ is T.
Induction Step: Assume k is any integer ≥ 0 such that $5|(7^k - 2^k)$. We NTS that $5|(7^{k+1} - 2^{k+1})$. We have

$$7^k - 2^k = 5m$$

for some integer m. Let's multiple both sides by 7. We have

$$7^{k+1} - 7 \cdot 2^k = 7 \cdot 5m$$

Notice that $7 \cdot 2^k = (2+5)2^k = 2^{k+1} + 5 \cdot 2^k$. Therefore

$$7^{k+1} - 2^{k+1} = 7 \cdot 5m + 5 \cdot 2^k$$

i.e. $P(k+1)$ is T. $\square$

#21.

i. *Base Case:* When $n = 2$, the LHS is $\sqrt{2}$. The RHS is $1 + \frac{1}{\sqrt{2}}$. I would work backwards to establish that $P(2)$ is T. Indeed,

$$\sqrt{2} \leq 1 + \frac{1}{\sqrt{2}} \longrightarrow 2 \leq \sqrt{2} + 1$$

and $\sqrt{2} \sim 1.14$. Therefore $P(2)$ is T.

ii. *Induction Step:* Assume k is any integer ≥ 2 such that

$$\sqrt{k} < 1 + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{k}}$$

NTS

$$\sqrt{k+1} < 1 + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{k+1}}$$

Let's add $\frac{1}{\sqrt{k+1}}$ to both sides of the induction hypothesis. We obtain

$$\sqrt{k} + \frac{1}{\sqrt{k+1}} < 1 + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{k+1}}$$

The proof will follow provided we can prove

$$\sqrt{k+1} \leq \sqrt{k} + \frac{1}{\sqrt{k+1}}$$

Let's work backwards. Let's multiply both sides by $\sqrt{k+1}$. We obtain

$$k + 1 \leq \sqrt{k(k+1)} + 1$$

which implies that

$$k \leq \sqrt{k^2 + k}$$

and we know this is true.

□

5. (6 points) Section 5.3 # 27, 28.

Solution:

27.

Base Case: For $n = 1$, $\frac{2}{2!} = 1$ and this is equal to d_1. Therefore $P(1)$ is T.

Induction Step: Assume k is any integer ≥ 1 such that

$$d_k = \frac{2}{k!}$$

NTS

$$d_{k+1} = \frac{2}{(k+1)!}$$

Now

$$d_{k+1} = \frac{d_k}{k+1} = \frac{2}{k!} \cdot \frac{1}{k+1} = \frac{2}{(k+1)!}$$

□

28.

Base Case: For $n = 1$, notice that $(2n + 2n - 1) = 3$, therefore the RHS is $\frac{1}{3}$. That is, $P(1)$ is T.

Induction Step: Assume that k is any integer ≥ 1 such that

$$\frac{1}{3} = \frac{1 + 3 + 5 + \cdots + (2k-1)}{(2k+1) + (2k+3) + \cdots (2k + (2k-1))}$$

We NTS

$$\frac{1}{3} = \frac{1 + 3 + 5 + \cdots + (2k-1) + (2(k+1)-1)}{(2k+1) + (2k+3) + \cdots (2k + (2k-1)) + (2(k+1) + 2(k+1) - 1)}$$

6. (6 points) Section 5.3 # 33, 45, 46.

Solution:

33. I will leave the drawing to you. Please come by office hours for a solution.

45. Tricky one indeed! We know $k \geq 1$ so that $k + 1 \geq 2$. The constructions of sets B and C imply that $k + 1 \geq 3$. For us to legally construct sets B and C we would need $k \geq 2$, hence a second base case. Of course a second base case is not possible.

46. Nothing is wrong with the induction step. However there is no base case. In fact, the base case is not possible. Hence, one could never iterate the induction step over and over to conclude $3^n - 2$ is even.

Remark. *We really need to understand problems # 45, 46. Please stop by during office hours if you have questions.*