Ito Integral for general functions

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$I_t(f) = \int_0^t f(s,\omega)\, dB_s$

integrand

integrator is a stochastic process (usually BM)

a) $f(s,\omega) \in \mathcal{F}_t$

b) $\mathbb{E}\int_0^T f^2(s,\omega)\, ds < \infty$

性质

1) Martingale

2) Itô Isometry $\mathbb{E}(I_t(f))^2 = \mathbb{E}\left[\int_0^t f^2(s,\omega)\,ds\right]$

L^2空间理解

Def RV $I_t(f)$ 有 $\|I_t(f)\|_2 = \sqrt{\mathbb{E}(I_t(f)^2)}$ 在 Space Ω 上

且 $\|f\|_2$ 在空间 $(\Omega \times [0,t])$ 有 $\|f\|_2 = \sqrt{\mathbb{E}\int_0^t f^2(s,\omega)ds}$

measure of integral = measure of f. } in L_2

3) QV of Itô Integral.

$\langle I_t, I_t \rangle_{[0,T]} = \int_0^T f^2(s,\omega)\, ds$

推广至 $\forall f$

·先论.

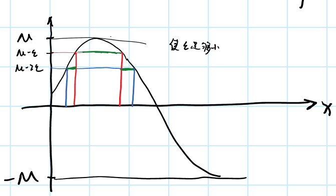
$\forall f$ 都可被一个 sequence of simple functions

$\varphi_n(s,\omega) \to f(s,\omega)$ 意义: $\mathbb{E}\left(\int_0^t (\varphi_n - f)^2 ds\right) \to 0$ as $n \nearrow$

simple functions

calculate $I_t(\varphi_n) \to$ 有 limit $I_t(f)$

FACT 1. bounded CTS function 可被 approxi. by simple functions.



$$\int_0^t f(s,\omega)\, dB_s = I(f) \tag{1}$$

for stochastic processes $f(t,w)$ that are $\mathcal{F}_t$-adapted and square-integrable. We defined $I(f)$ to be the limit in $L^2(\Omega, \mathcal{F}, \mathbb{P})$ of $I(\varphi_n)$ for any sequence of approximating simple stochastic processes φ_n. Approximating sequence was defined as satisfying

$$\mathbb{E}\int_0^t (\phi_n - f)^2\, dt \to 0 \text{ when } n \to \infty. \tag{2}$$

Thus the procedure of calculating Itô integral (1) from the definition is rather work consuming.

Let us step back for a second and take a look at the Riemann integral. Even though it is defined as the limit of Riemann sums in practice one never does this. Instead, one uses the fundamental theorem of calculus and the chain rule. For instance, to compute $I(t) = \int_0^t se^{-s^2/2}ds$ we notice that $(-e^{-s^2/2})' = se^{-s^2/2}$ and thus

$$I(t) = \int_0^t se^{-s^2/2}ds = -e^{-s^2/2}\Big|_0^t = 1 - e^{-t^2/2}. \tag{3}$$

FACT 2: Bounded function 可被 approxi. by CTS bounded functions

FACT 3: General function 可被 approxi. w/ bounded functions

$f \cdot \mathbb{1}_{|f| \le n}$

cut the tails

FACT 4: $\{a_n\}$ convergent if $\exists\, a$ s.t $\|a_n - a\| \to 0$ as $n \nearrow$ 其中 a_n 可为 functional space 中元素

$\{a_n\}$ is Cauchy if $\|a_n - a_m\| \to 0$ as $n \nearrow$

Cauchy 收敛

FACT 5: Complete Space := every Cauchy seq 收敛

且 $\mathcal{L}^2(\Omega)$ is complete

现考虑. $\{I_t(\varphi_1), I_t(\varphi_2), I_t(\varphi_3)\cdots\}$

$\Cap$ $\mathcal{L}^2$ (square integrable)

Claim: $\{I_t(\varphi_1), I_t(\varphi_2)\cdots\}$ 为 Cauchy.

Pf. $\|I_t(\varphi_n)-I_t(\varphi_m)\|_2^2 = E\left(I_t(\varphi_n)-I_t(\varphi_m)\right)^2$

$= E\left(\int_0^t(\varphi_n-\varphi_m)\,dB_s\right)^2$

$\overset{\text{Itô isometry}}{=} E\left(\int_0^t(\varphi_n-\varphi_m)^2ds\right)$

考虑 $(a-b)^2\le 2a^2+2b^2$

let $a=\varphi_n-f \quad b=\varphi_m-f$

有 $E\left(\int_0^t(\varphi_n-\varphi_m)^2ds\right)\le E\int_0^t 2(\varphi_n-f)^2+2(\varphi_m-f)^2ds$

$=2E\int_0^t \underbrace{(\varphi_n-f)^2}_{\to 0}+\underbrace{(\varphi_m-f)^2}_{\to 0}ds$

$\to 0$

考虑漏洞 $\varphi_n\to f \quad I_t(\varphi_n)\to A$ 且 $A\ne B$

$\psi_n\to f \quad I_t(\psi_n)\to B$

下证 $A=B$.

反证法. 考虑 $\{\varphi_1\ \psi_1\ \varphi_2\ \psi_2\ \varphi_3\ \psi_3\cdots\}$

有 $\{I_t(\varphi_1), I_t(\psi_1)\cdots\}$ 有极限

Yet $\{\varphi,\psi\cdots\}$ 无 limit 矛盾!

例 let $f(t,\omega)=B_t$

$\int_0^t B_s\,dB_s$

a) $f(t,\omega)\in\mathcal{F}_t=\sigma(B_s,\ s\le t)$

b) $E\int_0^T f^2(s,\omega)ds = E\int_0^T B_s^2ds = \int_0^T E(B_s^2)ds = \frac12 s^2\Big|_0^T = \frac12 T^2<\infty$

$0 \quad t_1=\frac{t}{n} \quad t_2=\frac{2t}{n} \quad t_3=\frac{3t}{n} \quad \cdots \quad t_{n-1}=\frac{n-1}{n}t \quad t$

let $\varphi_n(t,\omega)=B(t_i)$, $t_i\le t<t_{i+1}$

下证 $\varphi_n\to f$

即证 $E\int_0^t(\varphi_n-f)^2ds\to 0$

拆分 $=E\int_0^{t_1}(\varphi_n-f)^2ds+E\int_{t_1}^{t_2}(\varphi_n-f)^2ds+\cdots+\int_{t_{n-1}}^{t_n}(\varphi_n-f)^2ds$

一般地, 考虑 $E\int_{t_i}^{t_{i+1}}(\varphi_n-f)^2ds$

$=E\int_{t_i}^{t_{i+1}}(B_{t_i}-B_s)^2ds$

$=\int_{t_i}^{t_{i+1}}E(B_{t_i}-B_s)^2ds$

$=\int_{t_i}^{t_{i+1}}(t_i-s)\,ds$

$=\frac12(t_{i+1}-t_i)^2$

累和 $E\int_0^t(\varphi_n-f)^2ds=\sum_i\frac12(t_{i+1}-t_i)^2$ By QV性质 $=0$ □

∴有 $I_t(\varphi_n)=\sum_i\int_{t_i}^{t_{i+1}}\varphi_n(s)\,dB_s$

$=B(t_0)\cdot(B_{t_1}-B_{t_0})+B(t_1)\cdot(B_{t_2}-B_{t_1})+\cdots+B(t_{n-1})\cdot(B_{t_n}-B_{t_{n-1}})$

即 $I_t(\varphi_n)=\sum_i B_{t_i}\Delta B_{t_i}$ ← $B_{t_{i+1}}-B_{t_i}$

考虑 $b^2-a^2=(b-a)^2+2a(b-a)$

let $b^2=B_{t_{i+1}}^2$ 且 $a^2=B_{t_i}^2$

有 $B_{t_{i+1}}^2-B_{t_i}^2=(\ \ )^2 \cdots$

有 $B_{t_{i+1}}^2 - B_{t_i}^2 = (\Delta B_{t_i})^2 + 2B_{t_i} \cdot \Delta B_{t_i}$

$\therefore \sum_i \left(B_{t_{i+1}}^2 - B_{t_i}^2\right) = \sum_i (\Delta B_{t_i})^2 + 2\sum_i B_{t_i} \Delta B_{t_i}$

$\therefore B_t^2 - B_0^2 = t + 2 I_t(\psi_n)$

$\therefore \lim I_t(\psi_n) = \frac{1}{2}(B_t^2 - t)$