Lecture 16

def "partitions" soln to $e_1 + 2e_2 + 3e_3 + 4e_4 + \dots + ne_n = n$
$e_1, \dots, e_n \in \mathbb{Z}_{\geq 0}$.

get gen fn: $g(x) = \prod_{k=1}^{\infty} \frac{1}{(1-x^k)}$

Eg parts of $4 = \{1+1+1+1,\ 2+1+1,\ 3+1,\ 2+2,\ 4\}$

Think: pile of singles, pile of doubles, pile of triples. ...

Eg Write r as sum of distinct $\mathbb{Z}$.

You can use each # one or zero times: $\prod_{k=1}^{\infty} (1+x^k)$

Eg: Make $r¢$ from $3¢, 2¢, 5¢$: $A(x) = \frac{1}{(1-x^2)(1-x^3)(1-x^5)}$

Eg: Prove every $n \in \mathbb{Z}_{\geq 0}$ has a unique rep as a Σ powers of 2.

$G(x) = \prod_{k=0}^{\infty} (1+x^{2^k})$ STS $G(x) = 1 + x + x^2 + \dots = \frac{1}{1-x}$

$\iff (1-x)\,G(x) = 1$.

$$(1-x)\,G(x) = (1-x)(1+x)\prod_{k=1}^{\infty}(1+x^{2^k})$$

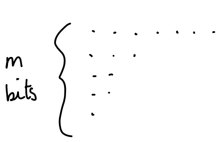
$$= (1-x^2)\prod_{k=1}^{\infty}(1+x^{2^k}) = (1-x^2)(1+x^2)\prod_{k=2}^{\infty}(1+x^{2^k})$$

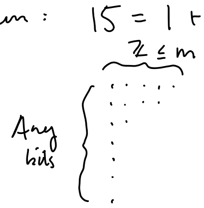
$$= \dots = 1.$$ QED.

Eg # of Parts of r into m pieces is equal to parts of r into $\mathbb{Z} \leq m$

into m bits = # into bits of size at most m.

Use Ferrer diagram: $15 = 1+2+2+3+7$





* Take transpose to get 1-1 corresp.

Def: an exponential gen fn for sequence $a_0, a_1, \ldots$ is

$$\sum_{i=0}^{\infty} \frac{a_i}{i!} x^i = a_0 + a_1 x + \frac{a_2 x^2}{2} + \frac{a_3 x^3}{3!} + \ldots$$

$\rightsquigarrow$ Solves the problem of finding arrangements (order matters) for the problem of picking r objects from n types in certain fixed amounts.

Notice we need to adjust coeffs for exp gen fnc's:

If $A(x) = 1 + 2x + 4x^2 + 8x^3$ is an exp generating fn

Then $a_0 = 1 \quad a_1 = 2 \quad a_2 = 4 \cdot 2! = 8 \quad a_3 = 8 \cdot 3! = 48$

so $A(x) = 1 + 2x + 8\frac{x^2}{2!} + 48\frac{x^3}{3!}$

Eg words on letters a, b, c w/ 2 a's:

$$\underbrace{\left(\frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\right)}_{a}\underbrace{\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\right)}_{b}\underbrace{\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\right)}_{c}$$

at least 2.

Find a_4:

$$\frac{a_4}{4!} = \left(\frac{1}{2!}\cdot 1 \cdot 1\right) + \left(\frac{1}{2!}\cdot\frac{1}{2!}\cdot 1\right) + \left(\frac{1}{2!}\cdot 1\cdot\frac{1}{2!}\right) + \left(\frac{1}{3!}\cdot 1\cdot 1\right) + \left(\frac{1}{3!}\cdot 1\cdot 1\right) + \left(\frac{1}{4!}\cdot 1\cdot 1\right)$$

$$\Rightarrow a_4 = \frac{4!}{2!} + \frac{2\cdot 4!}{2!\,2!} + \frac{2\ 4!}{3!} + 1 = 4\cdot 2 + 4\cdot 3 + 4\cdot 2 + 1 = 16 + 12 + 1 = \boxed{29}$$

Eg Find exp gen for arrangements of length r from n obj w/o repetition:

$A(x) = \underbrace{(1+x)\ldots(1+x)}_{n\text{-times}} = (1+x)^n \rightsquigarrow \frac{a_r}{r!} = \binom{n}{r} \Rightarrow a_r = r!\binom{n}{r} = \underline{P(n,r)}$

which is what it should be.

Useful Formula's $\quad 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^r}{r!} + \ldots = e^x$

$1 - x + \frac{x^2}{2!} - \ldots + \frac{x^r}{r!}(-1)^r + \ldots = e^{-x}$

$\frac{1}{2}[e^x + e^{-x}] = $ Sum of evens, $\frac{1}{2}[e^x - e^{-x}] = $ Sum of odds.

Eg: How many length r sequences of $0,1,2,3$ have even # of 0's and odd # of 1's ?

$$A(x) = \underbrace{\tfrac{1}{2}(e^x + e^{-x})}_{0\text{'s}} \cdot \underbrace{\tfrac{1}{2}(e^x - e^{-x})}_{1\text{'s}} \cdot \underbrace{e^x}_{2\text{'s}} \cdot \underbrace{e^x}_{3\text{'s}}$$

$$= \frac{1}{4}(e^{2x}+1)(e^{2x}-1) = \frac{1}{4}(e^{4x}-1) = -\frac{1}{4} + \sum_{r=0}^{\infty} \frac{4^{r-1}x^r}{r!}$$

$$\frac{a_r}{r!} = \frac{4^{r-1}}{r!} \Rightarrow a_r = 4^{r-1}, \quad r \geq 1. \qquad a_0 = 0.$$