1. Is $Aut(\mathbb{Z}/8\mathbb{Z})$ isomorphic to $Aut(\mathbb{Z}/10\mathbb{Z})$? Why?

   Solution: $Aut(\mathbb{Z}/8\mathbb{Z}) \cong (\mathbb{Z}/8\mathbb{Z})^{\times} = \{\overline{1}, \overline{3}, \overline{5}, \overline{7}\}$.

   $Aut(\mathbb{Z}/10\mathbb{Z}) \cong (\mathbb{Z}/10\mathbb{Z})^{\times} = \{\overline{1}, \overline{3}, \overline{7}, \overline{9}\}$.

   They are not isomorphic since in $(\mathbb{Z}/8\mathbb{Z})^{\times}$, all the non-identity elements have order 2 but in $(\mathbb{Z}/10\mathbb{Z})^{\times}$, there are elements ($\overline{3}$ and $\overline{7}$) of order 4.

2. $\bar{a} \in \mathbb{Z}/n\mathbb{Z}$, what is the order of $\bar{a}$ in $\mathbb{Z}/n\mathbb{Z}$?

   Solution: $k\bar{a} = \bar{0} \iff \overline{ka} = \bar{0} \iff ka \in n\mathbb{Z} \iff ka \in n\mathbb{Z} \cap a\mathbb{Z} \iff ka \in m\mathbb{Z}$ where m is the least common multiple of a and n. So we see

   $$k \in \frac{m}{a}\mathbb{Z}$$

   $|\bar{a}|$, the smallest positive choice for k, is $k = \frac{m}{a} = \frac{n}{g}$, where g is the greatest common divisor of a and n.

3. Prove that every positive integer a is congruent to the sum of its decimal digits modulo 9.

   Solution: Let the positive number be $a = \sum_{i=0}^{n} a_i \times 10^i$ in decimal expression.

   Then
   $$a = \sum_{i=0}^{n} a_i \times 10^i \equiv \sum_{i=0}^{n} a_i \times 1^i \equiv \sum_{i=0}^{n} a_i (\text{ mod } 9)$$

4. Show that $79^{882} \equiv 11 (\text{mod } 89)$.

   Solution: 89 is a prime and 89 is relatively prime to 79, so the Fermat's Little Theorem implies
   $$79^{88} \equiv 1 (\text{mod } 89)$$

   It follows

   $$79^{882} = (79^{88})^{10} \times 79^2 \equiv 1^{10} \times (-10)^2 \equiv 100 \equiv 11 (\text{mod } 89)$$

5. G is a group. H is a subgroup of G and N is a normal subgroup of G.

   $HN = \{hn \in G | h \in H, n \in N\}$. Prove that $H/(H \cap N) \cong HN/N$.

(Hint: consider $f : H \longrightarrow HN/N$ given by $f(h) = hN$)

Solution:

Consider $f : H \longrightarrow HN/N$ by $f(h) = hN$. This is a homomorphism: $f(h_1h_2) = h_1h_2N = (h_1N)(h_2N) = f(h_1)f(h_2)$. It is also a surjective map: for any $xN \in HN/N$, if $x = hn$ such that $h \in H$ and $n \in N$, we see $xN = hN$, so $xN = f(hN)$.

$\ker(f) = \{h \in H : f(h) = N\} = \{h \in H : hN = N\} = \{h \in H : h \in N\} = H \cap N$. By the First Isommorphism Theorem:

$$H/(H \cap N) = H/\ker(f) \cong HN/N$$

6. How many homomorphisms are there from K_4 to S_3? Prove your answer.

   Solution: Let $f : K_4 \longrightarrow S_3$ be a homomorphism. We know $|\operatorname{Im}(f)|$ divides both $|K_4| = 4$ and $|S_3| = 6$, so $|\operatorname{Im}(f)| = 1$ or 2.

   If $|\operatorname{Im}(f)| = 1$, then f is the trivial homomorphism;

   If $|\operatorname{Im}(f)| = 2$, then $\operatorname{Im}(f) = \{id, (1\ 2)\}$ or $\{id, (1\ 3)\}$ or $\{id, (2\ 3)\}$.

   $|\ker(f)| = \frac{|K_4|}{|\operatorname{Im}(f)|} = 2$, so $\ker(f) = \{1, a\}$ or $\{1, b\}$ or $\{1, c\}$.

   For each of the 3 choices of $\operatorname{Im}(f)$ and 3 choices of $\ker(f)$, there is a unique homomorphism. For example, if $\operatorname{Im}(f) = \{id, (1\ 2)\}$ and $\ker(f) = \{1, a\}$, then $f(1) = f(a) = id$ and $f(b) = f(c) = (1\ 2)$. The same argument can be applied to the other cases. There are in total $3 \times 3 = 9$ homomorphisms when $|\operatorname{Im}(f)| = 2$.

   We conclude there are $1 + 9 = 10$ homomorphisms.

7. Is $\mathbb{Z} \times \mathbb{Z}$ a cyclic group or not? Prove your answer.

   Solution:

   It is not a cyclic group.

   Suppose it is a cyclic group: $\mathbb{Z} \times \mathbb{Z} =< (a, b) >$.

   If $a = 0$, then $(1, 0) \notin \{0\} \times b\mathbb{Z} =< (0, b) >$.

   If $b = 0$, then $(0, 1) \notin a\mathbb{Z} \times \{0\} =< (a, 0) >$.

   If $a \neq 0$ and $b \neq 0$, then $< (a, b) >= \{(ka, kb) \in \mathbb{Z} \times \mathbb{Z} | k \in \mathbb{Z}\}$, which is also strictly smaller than $\mathbb{Z}$ since $(a, 2b) \notin< (a, b) >$.

   So we see there is no possible choice of generator if it is a cyclic group. We conclude it is not a cyclic group.

8. G is a group, N and M are normal subgroups of G, $G = NM$.

$$G \xrightarrow{\sigma} G/N \times G/M$$

$$g \mapsto (gN, gM)$$

Prove σ is surjective.

Solution:

For any $(xN, yM) \in G/N \times G/M$, consider the element $x^{-1}y \in G$. Since $G = NM$, there exists $n \in N$ and $m \in M$ such that $x^{-1}y = nm$, i.e. $xn = ym^{-1}$. Let $g = xn = ym^{-1}$, we see $gN = xnN = xN$ and $gM = ym^{-1}M = yM$, so $\sigma(g) = (xN, yM)$.