Lecture 13-14

Counting & Enumeration:

Idea

① Given sets $1, \dots, m$ w/ sizes $r_1, \dots, r_m$ and assuming they are disj then there are $r_1 + \dots + r_m$ ways to pick one elt from the union

② Given m successive independent choices w/ r_m ways to make each choice, there are $r_1 \cdot r_2 \cdot \dots \cdot r_m$ ways to make all choices.

Disjointness ① & Independence ② are key!

Eg Ways to do 3 letter words on $a, \dots, e, f$.

— w/ repetition: $\underline{6} \cdot \underline{6} \cdot \underline{6} = 6^3$

— w/o rep: $\underline{6} \cdot \underline{5} \cdot \underline{4} = 6!/3!$ Remove the taken letter to preserve independence.

— w/o rep and w/e:

$$\left.\begin{array}{c} \underline{e}\ \underline{5}\ \underline{4} \\ \underline{5}\ \underline{e}\ \underline{4} \\ \underline{5}\ \underline{4}\ \underline{e} \end{array}\right\} \text{disj} \rightsquigarrow 3 \cdot 20 = \boxed{60}$$

— w/ rep and w/e:

$$\left.\begin{array}{lcll} 1\ e & \rightsquigarrow & \underline{e}\,\underline{5}\,\underline{5} + \underline{5}\,\underline{e}\,\underline{5} + \underline{5}\,\underline{5}\,\underline{e} = & 3 \cdot 25 \\ & & & + \\ 2\ e\text{'s} & \rightsquigarrow & \underline{e}\,\underline{e}\,\underline{5} + \underline{e}\,\underline{5}\,\underline{e} + \underline{5}\,\underline{e}\,\underline{e} = & 3 \cdot 5 \\ & & & + \\ 3\ e\text{'s} & \rightsquigarrow & \underline{e}\,\underline{e}\,\underline{e} = 1 \longrightarrow & 1 \end{array}\right\} = 81$$

- "Disjoint" ⟶ Fixed criterion
- "Independent" ⟶ Remove dependent outcomes.

Main Skill: Model each "state" w/ a math object such as: a number, tuple, matrix, graph, function, etc.

Eg: 6 apples, 9 oranges: How many non-empty fruitbaskets.

Each basket has (a, o): "apples & oranges": $\underbrace{0, ..., 6}_{\text{choices}} \cdot \underbrace{0, ..., 9}_{\text{choices}} = 70 - (0,0)$

$= \underline{69}.$

def **Permutation:** choose r objects one at a time from n-objects and remember the order you pick them.

def **Combination:** choose r objects at once from n-objects No order.

Permutations remembered by tuples: $(2, 7, 1, 0, 5)$

Combinations " " sets: $\{7, 5, 2, 1, 0\}$

Thus $\{2,3\} = \{3,2\}$ but $(2,3) \neq (3,2)$

Same combination different permutation.

Written $P(n,r)$, $C(n,r)$ where r = length & n = possible entries.
also $\binom{n}{r}$.

These count the number of these objects:

Eg. $P(5, 2)$ is the number of pairs $\overset{\text{length } 2.}{\overbrace{(x,y)}}$ where $x, y \in \underbrace{\{1, ..., 5\}}_{\text{5 choices per entry.}}$

(No repetition! $x \neq y$).

Formula: $P(n, r) = n(n-1) \dots (n-r) = \dfrac{n!}{(n-r)!}$

$$C(n,r) = \frac{\overbrace{n(n-1) \dots (n-r)}^{\text{down by } r}}{\underbrace{r(r-1) \dots (1)}_{\text{up by } r}} . \ \frac{n!}{r!(n-r)!}$$

__Eg__ How many ways to arrange letters in $\underset{1}{S}\underset{2}{Y}\underset{3}{S}\underset{4}{T}\underset{5}{E}\underset{6}{M}\underset{7}{S}$.

3 letters the same $\Rightarrow$ pick places for Y, T, E, M. $\Rightarrow (a_1, \dots, a_4) \leq \{1, \dots, 7\}$

$\rightsquigarrow P(7,4) = \underbrace{7 - 6 \cdot 5 \cdot 4}_{4 \,=\, \text{length.}} = 840.$

" Rephrase the problem so that you have numbered positions and you pick positions for each object."

__Eg__ How many binary sequences w/ 6 1's and 2 0's ?

pick positions of 0's from 1, ..., 8. (Order don't matter).

$\rightsquigarrow C(8,2) = \dfrac{8 \cdot 7}{2 \cdot 1} = \boxed{28}$ ($= C(8,6)$ if we pick the 1's !!!)

__Eg__ : How many ways to arrange SYSTEMS w/ E before M ?

__Idea__ pick two spots for E, M, (unordered b/c we make the assignment after picking them).

__Then__ pick spots for T, Y (order matters)!

$\text{Ans} = C(7,2) \cdot P(5,2) = \dfrac{7 \cdot 2}{2 \cdot 1} \cdot 5 \cdot 4 =$

__Eg__ : How many 15 length U–A sequences s.t. the 3rd U in the 12th postn ?

__Ans__ : $\underbrace{\underset{1}{\_} \; \dots \; \underset{11}{\_}}_{2 \text{ are } U} \; \underset{12}{\underline{U}} \; \underset{13}{\underline{2}} \; \underset{14}{\underline{2}} \; \underset{15}{\underline{2}} \rightsquigarrow C(11,2) \cdot 2^3 = \dfrac{11 \cdot 10}{2} \cdot 2^3$

$= 11 \cdot 5 \cdot 2^3$

$= \underline{\underline{440}}.$

__Fun w/ Formulae:__

Won't pick x $\Rightarrow$ pick r from n-1

Will pick x $\Rightarrow$ pick r-1 from n-1.

① $\binom{n}{r} = \binom{n}{n-r}$, ② $\underbrace{\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1}}_{\text{Pascal's Formula}}$ ③ $\binom{n}{r} = \dfrac{P(n,r)}{P(r,r)}$

④ $P(n,n) = n!$

⑤ $P(n,1) = n.$

Perms w/ repetion: "Arrangement" = Order Matters

You have r_1 obj of type 1, r_2 obj of type 2, ---, r_k obj of type k.
and n-obj in total:

$$P(n; r_1, ---, r_k) = C(n, r_1) \cdot C(n-r_1, r_2) \cdot C(n-r_1-r_2, r_3)$$
$$= \frac{n!}{r_1! \cdots r_k!}$$

Eg: How many words on letters BANANA ⇝ 1 B
3 A
2 N

Ans $P(6; 1,3,2) = \frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{1 \cdot 3 \cdot 2 \cdot 1 \cdot 2 \cdot 1}$

Now Selection w/ Repetition: "Selection" Order doesn't matter.

Select r objects from n types w/ repetition:

$$C(r+\underbrace{(n-1)}_{\text{number of partitions for } n \text{ types}}, \overset{\text{pick the objects}}{r})$$

$=$ OR $C(r+(n-1), \underbrace{(n-1)}_{\text{Pick the partitions}})$

Equivalent forms of Selection w/ repetition:

① Choose r objects from n w/ repeats.

② Sort r objects into n boxes

③ non-neg solns to $x_1 + --- + x_n = r$

Eg: Make a word of length 10 w/ 4 a's, b's, c's, d's each letter appears at least twice?

Ans: $10 = \underbrace{2+2+2+4}_{\binom{4}{1} \text{ ways to pick}} = \underbrace{3+3+2+2}_{\binom{4}{2} \text{ ways to pick}}$ no other ways:

$$A = \binom{4}{1} \cdot P(10; 4,2,2,2) + \binom{4}{2} \cdot P(10; 3,3,2,2) = 226{,}800 \text{ ways}.$$

Eg Fill a box w/ 12 bagels so at least one of each kind is picked (5 kinds).

5 are in so pick 7 : $C(7+4 ; 4) = \frac{11 \cdot 10 \cdot 9 \cdot 8}{1 \cdot 2 \cdot 3 \cdot 4} = 11 \cdot 5 \cdot 3 \cdot 2 = \boxed{330}$

Eg: Pick 10 balls from piles of R, B, Purp. so there are at most 5 reds?

<u>Ans</u> Complement : How many ways to pick >5 reds?

$$C\left((10-6)+(3-1), (3-1)\right) = \binom{6}{2} = \frac{6 \cdot 5}{1 \cdot 2} = \underline{15}.$$

$$C(10+(3-1), 2) = \binom{12}{2} = \frac{12 \cdot 11}{2} = \underline{66}$$

$66 - 15 = \boxed{51}.$

<u>Idea</u> "Distributions" ⇝ Sort objects into buckets.

- * distribute differeal obj ⇝ Perms
- * distribute same obj ⇝ Combs.

Distr. r objects to n boxes : $f : R \to N \rightsquigarrow N^R$

<u>Bridge</u> : How many ways for each player to have an ace :

$\exists$ $P(4,4) = 4!$ ways to give each player an ace.
and $P(48; 12,12,12,12)$ ways to deal remaining cards

$$\Rightarrow 4 \cdot 3 \cdot 2 \cdot 1 \quad \frac{48!}{(12!)^4}$$

How many words w/ 5 vowels and 8 const. so no vowels are near each other?

$$\underset{1}{\underline{\quad}} \; v \; \underset{2}{\underset{x}{\underline{\quad}}} \; v \; \underset{3}{\underset{x}{\underline{\quad}}} \; v \; \underset{4}{\underset{x}{\underline{\quad}}} \; v \; \underset{5}{\underset{x}{\underline{\quad}}} \; v \; \underset{6}{\underline{\quad}}$$

Now destr 8 - 4 = 4
const into 6 boxes:

$C(4+6-1 ; 6-1)$
$C(9;5)$

How many binary sequences w/o consecutive 1's : up to length 10.

let k = length $k - w$ = # of zeroes.
let w = # of ones

$\_\_\ 1\ \overset{0}{\_}\ 1\ \overset{0}{\_}\ 1\ \_\_$ use $w-1$ zeroes

Have $k - 2w + 1$ zeroes left.
* distribute into $k+1$ buckets.

get $C(k-2w+1 + k+1 - 1, k+1-1)$

$= C(2(k-w)+1, k)$

$$\boxed{\sum_{k=1}^{10}\left(\sum_{w=0}^{\lceil \frac{k}{2}\rceil} C(2(k-w)+1, k)\right)}$$

Bin Ids :

① $(a+x)^n = \sum_{k=0}^{n}\binom{n}{k}a^{n-k}x^k$ → Expand

② $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$ ⇝ Pascal's Triangle
(Special Elt)

③ $\binom{n}{k}\binom{k}{m} = \binom{n}{m}\binom{n-m}{k-m}$

Caps for sets.

$N \supseteq K \supseteq M$ vs $N \supseteq M$ & $(N \setminus M) \supseteq (K \setminus M)$

④ $k\binom{n}{k} = n\binom{n-1}{k-1}$ or $\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1}$

⑤ $\sum_{k=0}^{r}\binom{n+k}{k} = \binom{n+r+1}{r}$

$\sum_{k=0}^{r}\binom{m}{k}\binom{n}{r-k} = \binom{m+n}{r}$

⑥ $\sum_{k=0}^{n}\binom{r+k}{r} = \binom{r+n+1}{r+1}$

$\sum_{k=0}^{r}\binom{m}{k}\binom{n}{r+k} = \binom{m+n}{m+r}$

⑦ $\sum_{k=0}^{n}\binom{n}{k}^2 = \binom{2n}{n}$

$\sum_{k=n-s}^{m-r}\binom{m-k}{r}\binom{n+k}{s} = \binom{m+n+1}{r+s+1}$

Guidelines for writing Combinatorial Arguments.

① Integer-valued variables represent sizes of sets

Eg. $n \rightsquigarrow$ size of set N.

② $\binom{n}{k}$ means $K \subseteq N$ or $N \supseteq K$ "containment"

③ $(n-m)$ means $N \setminus M$ "compliment"

④ $(n+m)$ means $N \amalg M$ "disjoint union"

⑤ $\underbrace{\binom{\cdots}{\cdots}}_{\alpha} + \underbrace{\binom{\cdots}{\cdots}}_{\beta}$ means either choose α or choose β

⑥ $\underbrace{\binom{\cdots}{\cdots}}_{\alpha}\underbrace{\binom{\cdots}{\cdots}}_{\beta}$ means choose α then choose β.

⑦ $\underbrace{\binom{\cdots}{\cdots}}_{\text{All choices}} - \underbrace{\binom{\cdots}{\cdots}}_{\text{Choices w/ } P}$ means choices with the opposite of property P.

Eg. $$\underbrace{\binom{n}{k}}_{\alpha} = \underbrace{\binom{n-1}{k-1} + \binom{n-1}{k}}_{\beta}$$

Combinatorial Argument:

α describes the # of ways to pick a subset $K \subseteq N$.

β describes picking either a subset

$$(K \setminus \{x\}) \subseteq (N \setminus \{x\})$$

or

$$K \subseteq (N \setminus \{x\})$$

We see these are the same as follows:

let $x \in N$, pick $K \subseteq N$

$\Rightarrow$ either $x \in K$ or $x \notin K$.

If we know $x \in K$, then we just need to pick the other elements of K from the other elements of N which is → $(K \setminus \{x\}) \subseteq (N \setminus \{x\})$

If we know $x \notin K$, then we pick all the elements of K from the other elements of N: $K \subseteq (N \setminus \{x\})$

Thus

$$\left(\#\text{ ways to pick } K \subseteq N\right) = \left(\#\text{ ways to pick } K\setminus\{x\} \subseteq N \setminus \{x\}\right) + \left(\#\text{ ways to pick } K \subseteq N \setminus \{x\}\right)$$

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$