Lecture 15:

def a generating fn for a sequence $a_0, a_1, a_2, \ldots$ is $f(x) = \underbrace{\sum_{i=0} a_i x^i}_{\text{Can often be re-written.}}$

Eg: $(1+x)^n = \sum_{i=0}^{n} \binom{n}{i} x^i \Rightarrow \sum_{k=0}^{n} \binom{n}{k} = 2^n$

More generally: $\underbrace{(1+x+x^2)}_{\text{exps give possible amnt of each type}}{}^{\overbrace{n}^{\#\text{ of types}}} \rightsquigarrow$ Select r objects from n types w/ at most 2 of each type

Qn Select r balls from 3 green, white, blue, gold.

Clearly $g+w+b+d = r$. and we have 4 types so

$(1+x+x^2+x^3)^4$ is the generating fn.

Qn r donuts from 5 choc, 5 strawb, 3 lemon, 3 cherry

$f(x) = (1+x+\ldots+x^5)^2(1+x+x^2+x^3)^2$

* Each factor corresp to a type, exponents corresp to poss. amounts.

W/ 1 of each type: $(x+\ldots+x^5)^2(x+\ldots+x^3)$.

Computing a specific coeff:

Useful: $\dfrac{1-x^{m+1}}{1-x} = 1+x+\ldots+x^m$ $\qquad (1+x)^n = 1+\binom{n}{1}x+\ldots+\binom{n}{n}x^n$

$\dfrac{1}{1-x} = 1+x+x^2+\ldots$ $\qquad (1-x^m)^n = 1-\binom{n}{1}x^m+\binom{n}{2}x^{2m}-\binom{n}{3}x^{3m}+\ldots (-1)^n\binom{n}{n}x^{nm}$

$$\frac{1}{(1-x)^n} = 1+\binom{1+n-1}{1}x+\binom{2+n-1}{2}x^2+\ldots+\binom{r+n-1}{r}x^r$$

$$= \text{Part}(r,n) \qquad \binom{r+n-1}{r} = \binom{r+n-1}{n-1}$$

for $A(x) = a_0 + a_1 x + \cdots$

$B(x) = b_0 + b_1 x + \cdots$

Subscripts add to exp.

$$A(x)\cdot B(x) = a_0 b_0 + (a_0 b_1 + a_1 b_0)x + (a_0 b_2 + a_1 b_1 + a_2 b_0)x^2 + (\cdots)x^3$$

Eg What is a_{16} in $A(x) = (x^2 + x^3 + \cdots)^5$?

$$A(x) = x^{10}(1 + x + x^2 + \cdots)^5 = \frac{x^{10}}{(1-x)^5} \quad \text{so } a_{16} = P(6,5) = \binom{6+5-1}{5-1}$$

$$= \binom{10}{4} = \frac{\overset{5}{\cancel{10}}\cdot\overset{3}{\cancel{9}}\cdot\overset{2}{\cancel{8}}\cdot 7}{1\cdot\cancel{2}\cdot\cancel{3}\cdot\cancel{4}}$$

$$= 5\cdot 3\cdot 2\cdot 7 = \boxed{210}$$

pick 2 of each. 7!

Generally $a_n = \binom{(n-10)+r-1}{r-1}$

Also acheived by factoring x^2. ☺

Eg.
\# of ways to put r balls into 7 boxes w/ ≤ 10 balls in 1st box:

$$A(x) = (1 + \cdots + x^{10})(1 + \cdots)^6 \qquad a_r = ? \qquad \text{" } ?x^r - ?x^{r-11} \text{ "}$$

$$= \frac{1-x^{11}}{1-x}\left(\frac{1}{1-x}\right)^6 = \frac{1-x^{11}}{1-x^7} = \frac{1}{1-x^7} - \frac{x^{11}}{1-x^7}$$

$$a_r = \binom{r+7-1}{7-1} - \binom{r-11+7-1}{7-1} \qquad \binom{n<0}{r} := 0$$

$$= \binom{r+6}{6} - \binom{r-5}{6} \quad \underline{\text{ways}}.$$

Eg Pick 25 toys from 7 types w/ $2 \le - \le 6$ of each type?

$$A(x) = (x^2 + \cdots + x^6)^7 = x^{14}(1 + \cdots + x^4)^7 = x^{14}\cdot\left(\frac{1-x^5}{1-x}\right)^7$$

$$= x^{14}\underbrace{(1-x^5)^7\cdot\frac{1}{(1-x)^7}} \qquad \text{We want coeff of } x^{25}.$$

Thus we need terms from which make $25 - 14 = 11$.

x^0, x^5, x^{10} from $(1-x^5)^7$ ← alt binomial and thus x^{11}, x^6, x from $\frac{1}{(1-x)^7}$. ← partitions

$1, -\binom{7}{1}, \binom{7}{2}$ $\qquad \binom{11+6}{6} \quad \binom{6+6}{6} \quad \binom{1+6}{6}$

$$\binom{11+6}{6} - \binom{7}{1}\binom{12}{6} + \binom{7}{2}\binom{7}{6} = \binom{17}{6} - 7\binom{12}{6} + 21\cdot 7$$

$$= \cdots$$