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Lecture 1. Basics of Graph Theory.

Q1: Does a town always contain two people with the same number of friends?

Idea: Find comb objects which model the town & friendships, then translate and solve the problem.

Obs: No if town contains one person. Q: if population ≥ 2.

def: A graph is a collection of vertices V and edges E s.t.

— Each edge connects two distinct vertices

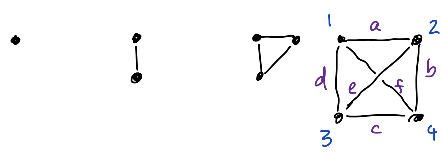
— Each pair of vertices is cctd by 1 or 0 edges.

Variations:

"Multi-graph" — Vertices may be cctd by more than one edge.

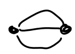
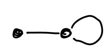
"General graph" — Edges may cct a vertex to itself.

Examples: K_1 K_2 K_3 K_4



$V = \{1,2,3,4\}$

$E = \{a,b,c,d,e,f\}$

Multi-graph:  General Graph: 

Notation: Given $e \in E$, $\partial e := \{v_1, v_2\}$ the set of vertices cctd by e.

Information about graphs:

* Number of Verts, Number of Edges. $\in \mathbb{Z}_{\geq 0}$

* For $v \in V$, how many edges leave $v =: \deg(v)$

Make a function $\deg : V \to \mathbb{Z}_{\geq 0}$

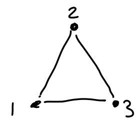
def $v, v' \in V$ are adjacent $\iff$ there is an edge $e \in E$ s.t.
$\partial e = \{v, v'\}$

Already enough to ans Q1 !

Idea: turn town/friendships into a graph.

V = People $\quad$ E = friendships.

Eg. 3 people, 3 friendships:



Person 1 has $\underbrace{\text{2 friends}}_{\text{2 edges leave } p_1}$.

friends = deg.

Q1: $\iff$ Are there 2 vertices w/ same degree?

$\hookrightarrow$ How many values can degree have? (Assume n vertices)

Minimum = 0 edges, Maximum = $n-1$
"Isolated pnt" $\qquad$ "Cctd to all verts except itself."

Thus $\forall v \in V$, $\deg v \in \{0, \dots, n-1\}$.

Obs: If $\deg v_0 = 0$ then v_0 is adj to nothing.
If $\deg v_1 = n-1$ then v_1 is adj to everything.
$\Rightarrow$ There cannot be a deg 0 and deg $n-1$ vertex together.

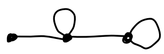
Thus Either:

$\deg v \in \underbrace{\{1, \dots, n-1\}}_{\text{Size } n-1}$ OR $\deg v \in \underbrace{\{0, \dots, n-2\}}_{\text{Size } n-1}$.

$\deg : \underbrace{V}_{\text{Size } n} \longrightarrow \text{Size } n-1 \quad \Rightarrow \exists v_i \neq v_j \mid \deg(v_i) = \deg(v_j)$

$\Rightarrow$ Two people have same # of friends.

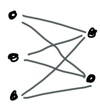
Note:  is a multi-graph w/ degrees $(1,3,2)$ all different

 is a general graph w/ degs $(1,3,2)$.

More examples:

K_n := Graph with n vertices and an edge ccting any pair of verts.

$K_{n,m} := (V = V_n \cup V_m$, Edges cct each vertex of V_n to $V_m)$.

Eg. $K_{3,2} =$ 

def: G is Bipartite $\iff$ V splits as $V_1 \cup V_2$ with $V_1 \cap V_2 = \emptyset$

and $V_1, V_2 \not\supseteq \partial e \quad (\forall e \in E)$.

def: A matching of a graph is a set $M \subseteq E$ s.t.

1. $\forall m_1, m_2 \in M \mid \partial m_1 \cap \partial m_2 = \emptyset$.
2. $\{ v \in \partial m \mid m \in M \} = V$.

def: An independent set is $I \subseteq V$ s.t. $\forall e \in E \mid \partial e \not\subseteq I$.

"p.w. non-adj vertices"

def An Edge Cover is $\mathcal{C} \subseteq V$ s.t. $\forall e \in E$, $\partial e \cap \mathcal{C} \neq \emptyset$.

"has at least one vertex of each edge"

Prop: The smallest E.C. corresps to the largest I.S.

End Lecture 1.