Final Review

SVD

$A = U \sum V^T$, U, V are orthogonal, $U^{-1} = U^T, V^{-1} = V^T$

$A = \sum_{i=1}^{r} \sigma_i u_i v_i^T$, $r = rank(A)$

Low Rank Approximation: $A_k = \sum_{i=1}^{k} \sigma_i u_i v_i^T$, $k \le r$

Eigenvalue algorithms

$Au = \lambda u \rightarrow \det(A - \lambda I) = 0.$

Eigendecomposition of A: $A = Q\Lambda Q^{-1}$

Eigendecomposition of symmetric A: $A = Q\Lambda Q^T$ where Q is orthogonal.

For a symmetric A, Rayleigh Quotient $R(A,u) = \frac{u^T A u}{u^T u} = \frac{u^T A u}{||u||_2 ||u||_2} = \frac{u^T}{||u||_2} A \frac{u}{||u||_2}$.

The Power Method (to find a dominant eigenvalue & eigenvector)

Idea: $Au = \lambda u \rightarrow u = \frac{1}{\lambda} Au$. This is a fixed point iteration. The key is multiplication by A.

1. Choose $u^{(0)}$
2. While not converged:
    - set $u^{(k+1)} = Au^{(k)}$, then normalize $u^{(k+1)}$
    - set $\lambda^{(k+1)} = R(A, u^{(k+1)}) = u^{(k+1)T} A u^{(k+1)}$

convergence: $|\frac{\lambda_2}{\lambda_1}|$

Shifted Power Method: If (λ_i, μ_i) are A's eigenpairs, then $(\lambda_i - \mu_i, \mu_i)$ are $(A - \mu I)$'s eigenpairs.

Inverse Power Method: $Au = \lambda u \Rightarrow A^{-1}u = \frac{1}{\lambda}u$. If (λ_i, u_i) are A's eigenpairs, then $(1/\lambda_i, u_i)$ are A^{-1}'s eigenpairs.

Discretizing the derivative

$$\begin{bmatrix} f'(x_0) \\ \vdots \\ f'(x_i) \\ \vdots \\ f'(x_n) \end{bmatrix} = \begin{bmatrix} * \\ \vdots \\ \frac{f(x_{i+1}) - f(x_{i-1})}{2h} \\ \vdots \\ * \end{bmatrix} = \frac{1}{2h} \begin{bmatrix} 0 & -2 & 2 & \cdots & & & 0 \\ -1 & 0 & 1 & 0 & \cdots & & 0 \\ 0 & -1 & 0 & 1 & \cdots & & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & & \cdots & & -1 & 0 & 1 \\ 0 & & & \cdots & -2 & 2 & 0 \end{bmatrix} \begin{bmatrix} f(x_0) \\ f(x_1) \\ \\ \vdots \\ \\ f(x_n) \end{bmatrix}$$

Lagrange interpolation

There exists a unique polynomial $p_n(x)$ of degree n which assumes prescribed values at $n+1$ disctinct real numbers $x_0 < ... < x_n$. Let $w(x) = (x - x_0)...(x - x_n)$.

$p_n(x) = \Sigma_{i=i}^n f(x_i) l_i(x)$

where $l_i(x) = \frac{w(x)}{(x-x_i)w'(x_i)} = \frac{(x-x_0)...(x-x_{i-1})(x-x_{i+1})...(x-x_n)}{(x_i-x_0)...(x_i-x_{i-1})(x_i-x_{i+1})...(x_i-x_n)}$.

Note that $l_i(x_j) = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$.

Error: $f(x) - p(x) = \frac{f^{n+1}(\xi)}{(n+1)!}(x - x_0)(x - x_1)...(x - x_n)$ for some $\xi \in (a, b)$.

$||f - p|| \leq \frac{||f^{(n+1)}||h^{(n+1)}}{4(n+1)}$ where $h = \max_{0 \leq i \leq n-1}(x_{i+1} - x_i)$.

Piecewise Lagrange interpolation

Function approximation: L^∞

$$\underbrace{T_n(x) = \cos(n \cos^{-1} x).}_{\text{This is called the degree } n \text{ Chebyshev poly.}}$$

T_n is degree n, have n zeros: $x_j = \cos\left(\frac{(2j-1)\pi}{2n}\right)$ for $j = 1, \ldots, n$.

If we use the zeros of the T_n's as Lagrange interpolation nodes over $[-1, 1]$, the error is close to optimal.

$$\begin{aligned} (T_m, T_n)\,\omega &= \int_{-1}^{1} T_m(x) T_n(x) \omega(x) dx \\ &= \begin{cases} 0 & \text{if } m \neq n \\ s\pi & \text{if } m = n = 0 \\ \frac{\pi}{2} & \text{if } m = n \neq 0 \end{cases} \end{aligned}$$

Chebyshev polys are orthogonal with $\omega(x) = \frac{1}{\sqrt{1-x^2}}$.

Function approximation: L^2

Hermite interpolation

Example: $p = 2$, find $p \in \mathbb{P}_{2*2+1} = \mathbb{P}_5$ such that
$p(x_0) = f(x_0), p(x_1) = f(x_1)$
$p'(x_0) = f'(x_0), p'(x_1) = f'(x_1)$
$p''(x_0) = f''(x_0), p''(x_1) = f''(x_1)$
this is called a “quintic Hermite interpolation”.

$$\begin{bmatrix} 1 & x_0 & {x_0}^2 & {x_0}^3 & {x_0}^4 & {x_0}^5 \\ 1 & x_1 & {x_1}^2 & {x_1}^3 & {x_1}^4 & {x_1}^5 \\ 0 & 1 & 2x_0 & 3x_0^2 & 4x_0^3 & 5x_0^4 \\ 0 & 1 & 2x_1 & 3x_1^2 & 4x_1^3 & 5x_1^4 \\ 0 & 0 & 2 & 6x_0 & 12x_0^2 & 20x_0^3 \\ 0 & 0 & 2 & 6x_1 & 12x_1^2 & 20x_1^3 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} = \begin{bmatrix} f(x_0) \\ f(x_1) \\ f'(x_0) \\ f'(x_1) \\ f''(x_0) \\ f''(x_1) \end{bmatrix}$$

Cardinal basis:

$p(x) = f(x_0)H_{00}(x) + f(x_1)H_{01}(x) + f'(x_0)H_{10}(x) + f'(x_1)H_{11}(x) + f''(x_0)H_{20}(x) + f''(x_1)H_{21}(x)$

such that

$H_{ij}^{(l)}(x_k) = \delta_{jk}\delta_{il}$
for $i = 0, 1, 2$; $j = 0, 1$; $k = 0, 1$; $l = 0, 1, 2$.
i is the order of the derivative, and j is the endpoint.

$H_{00}(x_0) = 1$, $H_{00}(x_1) = H'_{00}(x_0) = H'_{00}(x_1) = H''_{00}(x_0) = H''_{00}(x_1) = 0$

$H_{01}(x_1) = 1$

$H'_{10}(x_0) = 1$

$p(a) = f(a)$, $p'(a) = f'(a)$, $p(b) = f(b)$, $p'(b) = f'(b)$.

$p(t) = p(a)\phi_1(t) + p(b)\phi_2(t) + p'(a)\psi_1(t) + p'(b)\psi_2(t)$

where

$\phi_1(t) = \frac{(t-b)^2[(a-b)+2(a-t)]}{(a-b)^3}$

$\phi_2(t) = \frac{(t-a)^2[(b-a)+2(b-t)]}{(a-b)^3}$

$\psi_1(t) = \frac{(t-a)(t-b)^2}{(a-b)^2}$

$\psi_1(t) = \frac{(t-a)^2(t-b)}{(a-b)^2}$

Numerical integration: Newton-Cotes integration

1. pick Lagrange interpolation of degree n with uniformly spaced nodes
2. compute quadrature weights $w_i = \int_a^b L_i(x)dx$, where $L_i(x) = \Pi\frac{x-x_j}{x_i-x_j}$
3. $\int_a^b f(x)dx \approx \Sigma_{i=0}^n w_i f(x_i)$

Example: $n = 2$

$w_0 = \int_a^b L_0(x)dx = \int_a^b \frac{x-\frac{a+b}{2}}{a-\frac{a+b}{2}}\frac{x-b}{a-b}dx = \frac{2}{(b-a)^2}\int_a^b(x^2 - \frac{a+3b}{2}x + \frac{ab+b^2}{2})dx = \frac{2}{(b-a)^2}\frac{(b-a)^3}{12} = \frac{b-a}{6}$

$w_1 = \int_a^b L_1(x)dx = \int_a^b \frac{x-a}{\frac{a+b}{2}-a}\frac{x-b}{\frac{a+b}{2}-b}dx = \frac{-4}{(b-a)^2}\int_a^b[x^2 - (a+b)x + ab]dx = \frac{-4}{(b-a)^2}\frac{-(b-a)^3}{6} = \frac{2(b-a)}{3}$

$\int_a^b f(x)dx \approx w_0 f(a) + w_1 f(\frac{a+b}{2}) + w_2 f(b) = \frac{b-a}{6}[f(a) + 4f(\frac{b-a}{2}) + f(b)]$

— Simpson's rule

Divided differences

$f(x) = f[x_0] + f[x_0, x_1](x - x_0) + ... + f[x_0, ..., x_n](x - x_0)...(x - x_{n-1})$

$f[x_i] = f(x_i)$, $f[x_i, x_{i+1}, ..., x_{i+k+1}] = \frac{f[x_{i+1},...,x_{i+k+1}] - f[x_i,...,x_{i+k}]}{x_{i+k+1} - x_i}$

$f[a, ..., a] = \frac{f^{(p)}(a)}{p!}$

Runge's phenomenon

Runge's phenomenon is a problem of oscillation at the edges of an interval that occurs when using polynomial interpolation with polynomials of high degree over a set of equispaced interpolation points. When the degree increases, the error doesn't go down/converge.

Cause: uniformly spaced nodes. Solution: use Chebyshev nodes.

Error Approximation

How do we approximate f''(x) using f? Taylor expansion.

$f(x_i + h) = f(x_i) + hf'(x_i) + \frac{h^2}{2}f''(x_i) + \frac{h^3}{6}f^{(3)}(x_i) + \frac{h^4}{24}f^{(4)}(x_i)$

$f(x_i - h) = f(x_i) - hf'(x_i) + \frac{h^2}{2}f''(x_i) - \frac{h^3}{6}f^{(3)}(x_i) + \frac{h^4}{24}f^{(4)}(x_i)$

$f(x_i + h) + f(x_i - h) = 2f(x_i) + h^2 f''(x_i)$ Error: $\frac{1}{12}h^4 f^{(4)}(x_i)$

$f''(x_i) = \frac{f(x_i+h) - 2f(x_i) + f(x_i - h)}{h^2}$ Error: $O(h^2)$

Convergence of Power Method

$\lambda = \frac{u^T A u}{u^T u}$

$u^{(i)} = A^i u^{(0)}$

$A = Q\Lambda Q^{-1}$ → $u^{(i)} = Q\Lambda^i Q^{-1} u^{(0)}$

How to interpret $Q^T u^{(0)}$?

vector of coefficients of $u^{(i)}$ expanded in the Q basis

This means that: $u^{(i)} = c^{(i)} q_1 + ... + c^{(i)} q_n = Q(Q^T u^{(i)})$