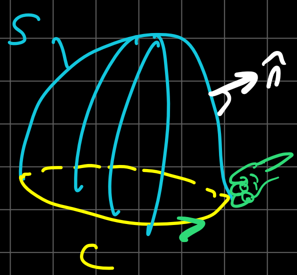
# Stokes Theorem

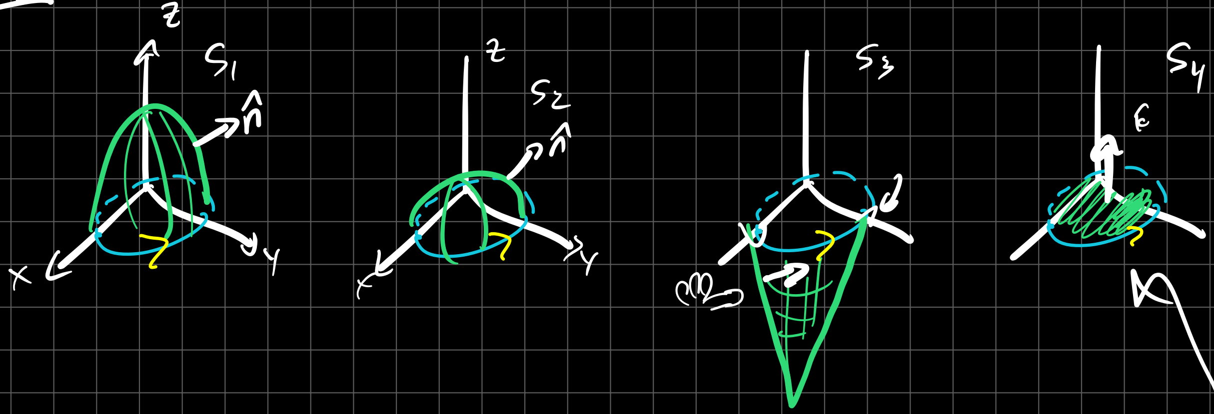



for "Nice enough" $\vec{F}$

$$\iint_S \underbrace{\text{curl}\,\vec{F}}_{\vec{\nabla}\times\vec{F}} \cdot d\vec{S} = \underbrace{\int_C \vec{F}\cdot d\vec{r}}_{\text{Circulation of } \vec{F} \text{ around } C.}$$

## 16.8

① 




Compare $\iint_{S_1} \vec{\nabla}\times\vec{F}\cdot d\vec{S}$, $\iint_{S_2} \vec{\nabla}\times\vec{F}\cdot d\vec{S}$, $\iint_{S_3} \vec{\nabla}\times\vec{F}\cdot d\vec{S}$, $\iint_{S_4} \vec{\nabla}\times\vec{F}\cdot d\vec{S}$

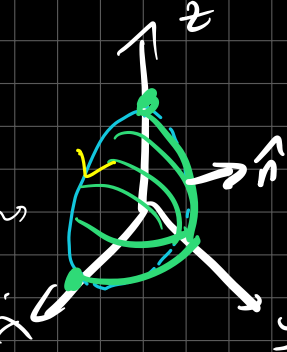
all equal to $\int_C \vec{F}\cdot d\vec{r}$

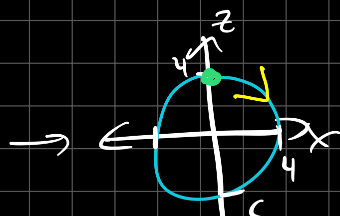
③ $\vec{F}(x,y,z) = ze^y\,\hat{i} + x\cos y\,\hat{j} + xz\sin y\,\hat{k}$

S is hemisphere $x^2+y^2+z^2=16$ $y\geq 0$ oriented pos y direction

evaluate $\iint_S \text{curl}\,\vec{F}\cdot d\vec{S}$

boundary in xz plane.




need to param. C.

$\vec{r}(t) = \langle 4\cos t, 0, -4\sin t \rangle$ , $0 \le t \le 2\pi$.

$\vec{r}\,'(t) = \langle -4\sin t, 0, -4\cos t \rangle$

$\vec{F}(\vec{r}(t)) = -4\sin t\,\hat{i} + 4\cos t\,\hat{j} + 0\hat{k}$

$$\int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt$$

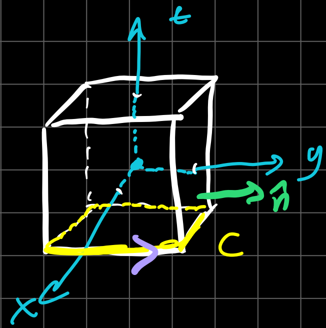
$$\iint_S \operatorname{curl}\vec{F} \cdot d\vec{S} = \int_0^{2\pi} 16\sin^2 t\,dt = 16\int_0^{2\pi} \frac{1-\cos(2t)}{2}\,dt$$

$$= 8(2\pi) = 16\pi.$$

(5) $\vec{F}(x,y,z) = xyz\,\hat{i} + xy\,\hat{j} + x^2yz\,\hat{k}$

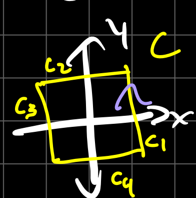
S is a box with an open bottom & vertices at $(\pm1, \pm1, \pm1)$ oriented outwards.

Eval. $\iint_S \operatorname{curl}\vec{F} \cdot d\vec{S}$




$\Rightarrow$

$z = -1$




5 surface int + curl calculation

4 line int

$C_1$: $\vec{r}(t) = \langle 1, t, -1 \rangle$ , $-1 \le t \le 1$

$\vec{r}\,'(t) = \langle 0, 1, 0 \rangle$

$\vec{F}(\vec{r}) = \langle -t, t, -t \rangle$

$\vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t) = t$

$\vec{F}(x,y,z) = xyz\,\hat{i} + xy\,\hat{j} + x^2yz\,\hat{k}$

$C_2$: $\vec{r}(t) = \langle -t, 1, -1 \rangle$ , $-1 \le t \le 1$

$\vec{r}\,'(t) = \langle -1, 0, 0 \rangle$

$\vec{F}(\vec{r}(t)) = \langle t, -t, -t^2 \rangle$

$\vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t) = -t$

$C_3: \vec{r}(t) = \langle -1, -t, -1\rangle$, $-1 \le t \le 1$

$\vec{r}\,'(t) = \langle 0, -1, 0\rangle$

$\vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t) = -t$

$C_4: \vec{r}(t) = \langle t, -1, -1\rangle$, $-1 \le t \le 1$

$\vec{r}\,'(t) = \langle 1, 0, 0\rangle$

$\vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t) = t$

$$\int_C \vec{F}\cdot d\vec{r} = \int_{-1}^{1} t\,dt + \int_{-1}^{1} -t\,dt + \int_{-1}^{1} -t\,dt + \int_{-1}^{1} t\,dt = 0.$$

$$\iint_S \operatorname{curl}\vec{F}\cdot d\vec{S} = 0.$$

Comment: if $\vec{F} = \vec{\nabla} f$ then $\operatorname{curl}\vec{F} = \vec{0} \Rightarrow \iint_S \operatorname{curl}\vec{F}\cdot d\vec{S} = \int_C \vec{F}\cdot d\vec{r} = 0.$

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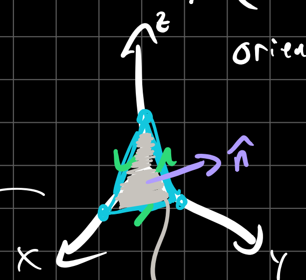
(7) $\vec{F}(x,y,z) = (x+y^2)\,\hat{i} + (y+z^2)\,\hat{j} + (z+x^2)\,\hat{k}$

$C$ is triangle with vertices $(1,0,0)$, $(0,1,0)$, $(0,0,1)$

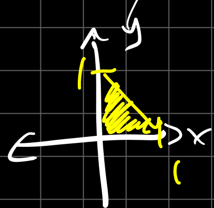
oriented counterclockwise when viewed from above.

evaluate $\int_C \vec{F}\cdot d\vec{r}$




plane: $x+y+z=1$

$$\int_C \vec{F}\cdot d\vec{r} = \iint_S \operatorname{curl}\vec{F}\cdot d\vec{S}$$




$z = \underbrace{1-x-y}_{g(x,y)}.$

$0 \le x \le 1$

$0 \le y \le 1-x$

$$\operatorname{curl}\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+y^2 & y+z^2 & z+x^2 \end{vmatrix}$$

$$= (0-2z)\,\hat{i} - (2x-0)\,\hat{j} + (0-2y)\,\hat{k} = \underbrace{-2z}_{P}\,\hat{i}\ \underbrace{-2x}_{Q}\,\hat{j}\ \underbrace{-2y}_{R}\,\hat{k}$$

Recall: $\iint_S \vec{F}\cdot d\vec{S} = \iint_D -P g_x - Q g_y + R\,dA$

for up. orientation.

$g(x,y) = 1-x-y \quad , g_x = -1 \qquad g_y = -1$

if we see $z$, replace with $1-x-y$.

$$\iint_S \vec{F}\cdot d\vec{S} = \int_0^1\int_0^{1-x} \underbrace{-(-2)(1-x-y)(-1) - (-2x)(-1) + (-2y)}_{\text{simplifies nicely}} \, dy\, dx$$

$$= \int_0^1\int_0^{1-x} -2+2x+2y-2x-2y \, dy\, dx = -2\underbrace{\int_0^1\int_0^{1-x} dy\, dx}_{A(\text{Triangle})} = -2\left(\tfrac{1}{2}\right) = -1$$

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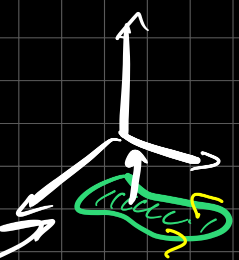
Relationship b/w G.T. & Stokes

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_D Q_x - P_y \, dA$$

$\vec{F} = P\hat{i} + Q\hat{j} + R\hat{k}$, $\operatorname{curl}\vec{F}\cdot\hat{k} = Q_x - P_y$.

if $D$ is a domain in $xy$ plane.

$\hat{n} = \hat{k}$



So stokes thm on

$= G.T.$

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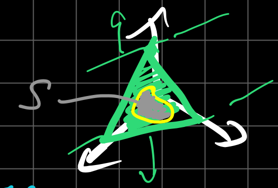
(16) $C$ is a simple closed smooth curve in plane $x+y+z=1$

Show $\int_C z\,dx - 2x\,dy + 3y\,dz$ depends only on area of region enclosed by $C$ and not on shape or location of $C$.

$\vec{F}(x,y,z) = \langle z, -2x, 3y\rangle$

$$\int_C z\,dx - 2x\,dy + 3y\,dz = \int_C \vec{F}\cdot d\vec{r}$$




Choose orientation upwards.

$\hat{n} = \left\langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right\rangle$

$$= \iint_S \operatorname{curl}\vec{F}\cdot d\vec{S}$$

$\operatorname{curl}\vec{F} = \vec{\nabla}\times\vec{F}$ $\qquad$ $\vec{\nabla} = \frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}$

$$= \left(\frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}\right)\times\left(z\hat{\imath} - 2x\hat{\jmath} + 3y\hat{k}\right) = 3\hat{\imath} + \hat{\jmath} - 2\hat{k}$$

$$= \iint_S \left(3\hat{\imath} + \hat{\jmath} - 2\hat{k}\right)\cdot d\vec{S} = \iint_S \operatorname{curl}\vec{F}\cdot\hat{n}\,dS = \iint_S \left(\frac{3}{\sqrt{3}} + \frac{1}{\sqrt{3}} - \frac{2}{\sqrt{3}}\right)dS$$

$$= \iint_S \frac{2}{\sqrt{3}}\,dS$$

$$= \frac{2}{\sqrt{3}}\underbrace{\iint_S dS}_{A(S)}$$

By Stoke's Thm:

$$\int_C z\,dx - 2x\,dy + 3y\,dz = \frac{2}{\sqrt{3}}\underbrace{A(S)}_{\text{Area of Surface enclosed by } C.}$$