Homework 7

Due: Friday Oct. 29, by 11:59pm,
via Gradescope

- Failure to submit homework correctly will result in a zero on the homework.
- Homework must be in LaTeX. Submit the pdf file to Gradescope.
- Problems assigned from the textbook come from the 5th edition.
- No late homework accepted. Lateness due to technical issues will not be excused.

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1. (3 points) Use contradiction to prove the following statement

   If x is a real number such that $|x| < \varepsilon$ for any $\varepsilon > 0$, then x must be zero.

   Solution:

   Proof: Assume that x is a real number such that $|x| < \varepsilon$ for any $\varepsilon > 0$ and $x \neq 0$. Take $\varepsilon = \frac{|x|}{2}$. Then $|x| < |x|/2 \Rightarrow\Leftarrow \square$

2. (9 points) Section 4.7 # 9, 18.

   Solution:

   # 9(a) . If you negate the beginning of the students proof, you have

   There exists an irrational number and there exists a irrational number such that their difference is irrational

   However, this is not the statement you wish to prove.

   Remark. *Let's recall that*

   $$\begin{aligned} &\sim p \to \mathbf{c} \\ &\therefore p \end{aligned}$$

   is a valid argument. If p is the statement **The difference of any irrational number and any rational number is irrational**, *then students proof should begin*

   Proof: *Assume there exists a rational number and there exists an irrational number such that their difference is rational.*

   #9(b).

   Proof: Assume there exists a rational number r and there exists an irrational number s such that their difference is rational. Then $s - r = \frac{a}{b}$ where a and b are integers and $b \neq 0$. Since r is a rational number, we know that $r = \frac{c}{d}$ where c and d are integers and $d \neq 0$. Therefore

   $$\begin{aligned} s &= \frac{c}{d} + \frac{a}{b} \\ &= \frac{cb + ad}{db} \end{aligned}$$

Set $x = cb + ad$ and $y = db$. Then x and y are integers and $y \neq 0$ by the Zero Property. Therefore $s = \frac{x}{y}$ is a rational number $\Rightarrow\Leftarrow$ $\square$

18.

Proof: Assume there exists rational numbers a and b, $b \neq 0$, and an irrational number r such that $a + br$ is rational. Then $a + br = \frac{s}{t}$ where s and t are integers and $t \neq 0$. Therefore $br = \frac{s}{t} - a \rightarrow r = \frac{s}{bt} - \frac{a}{b} \rightarrow r = \frac{sb-abt}{b^2t}$. Since $b^2t \neq 0$ by the zero property and $sb - abt$ is an integer it follows that r is a rational $\Rightarrow\Leftarrow$ $\square$

3. (9 points) Section 4.7 # 22, 24.

   Solution:

   # 22(a)

   Proof: Assume there exists a real number r such that r^2 is irrational and r is rational. You NTS (need to show) that this yields a contradiction. For instance, may you can show that r is irrational to obtain the contradiction.

   #22(b)

   Proof: Assume that r is rational. You NTS that r^2 is rational.

   #24.

   Proof by Contradiction: Assume there exists an irrational number x such that it's reciprocal, $\frac{1}{x}$, is rational. It follows that $\frac{1}{x} = \frac{a}{b}$ where a and b are integers and $b \neq 0$. Notice that $\frac{1}{x} \neq 0$, therefore $a \neq 0$. It follows that $x = \frac{b}{a}$ $\Rightarrow\Leftarrow$ $\square$

   Let's now do contrapositive. Note that you will have to write the given statement in if-then form. In doing so, we obtain

   If x is irrational, then $\frac{1}{x}$ is irrational

   It follows that we want to prove

   If $\frac{1}{x}$ is rational, then x is rational

   Proof by Contrapositive: Assume that $\frac{1}{x}$ is rational. Then $\frac{1}{x} = \frac{a}{b}$ where a and b are integers and $b \neq 0$. Notice that $\frac{1}{x} \neq 0$, therefore $a \neq 0$. It follows that $x = \frac{b}{a}$. That is, x is rational $\square$

4. (3 points) Section 4.7 # 28. Prove this using both contradiction and the contrapositve.

   Solution:

   Proof by Contradiction: Assume there exists integers a, b and c such that $a|b$ and $a \nmid c$ and $a|(b + c)$. Since $a|b$ we have $b = ka$ for some integer k. Since $a|(b + c)$ we have $b + c = ma$ for some integer m. Therefore, $ka + c = ma \rightarrow c = (m - k)a$ $\Rightarrow\Leftarrow$ $\square$

   Proof by Contrapositive: Assume that $a|(b + c)$. Therefore $b + c = ak$ for some integer k.

   Case 1: $a \nmid b$ is T. Then we're done (note that the negation of the hypothesis is $a \nmid b \vee a|c$).

Case 2: $a \nmid b$ is F. Therefore $b = ma$ for some integer m. The equation $b + c = ak$ now reads $ma + c = ka \to c = ka - ma \to c = a(k - m)$, i.e. $a|c$

$\square$

5. (9 points) Section 4.7 # 31.

   (a) **Proof:** Assume n, r and s be any positive integers such that $r > \sqrt{n}$ and $s > \sqrt{n}$. We multiply $r > \sqrt{n}$ by s and obtain $rs > s\sqrt{n}$. Note that $s\sqrt{n} > \sqrt{n}\sqrt{n} = n$. Therefore $rs > n$ $\square$

   (b) **Proof:** Let n be any integer larger than 1 that is not prime. Then $n = rs$ where $1 < s < n$ and $1 < r < n$. By part (a) we know that $r \le \sqrt{n} \vee s \le \sqrt{n}$. By Theorem 4.4.4 we know that r has a prime divisor, call it p_1 and s has a prime divisor p_2.

   Case 1: $r \le \sqrt{n}$: By Theorem 4.4.4. we know that there exists a prime divisor of r, call it p_1. That is, $r = k_1 p_1$ for some integer k_1. Now we have $n = k_1 p_1 s$ so that $p_1|n$. By part (a), we must have $p_1 \le \sqrt{r}$. Since $r > 1$, we have $\sqrt{r} \le r$. Therefore $p_1 \le \sqrt{n}$.

   Case 2: $r > \sqrt{n}$: Then $s \le \sqrt{n}$. Now repeat the argument as in case 1. Let's just do it. By Theorem 4.4.4. we know that there exists a prime divisor of s, call it p_2. That is, $s = k_2 p_2$ for some integer k_2. Now we have $n = k_2 p_2 r$ so that $p_2|n$. By part (a), we must have $p_2 \le \sqrt{s}$. Since $s > 1$, we have $\sqrt{s} \le s$. Therefore $p_2 \le \sqrt{n}$.

   (c) For each integer $n > 1$, if for any prime number p, $p > \sqrt{n}$ or $p \nmid n$ then n is prime.

6. (3 points) Section 5.1 # 79.

   This problem is quite tricky (IMO). With that said, let's prove the following.

   $\forall a, b \in \mathbb{Z}^+$, $\forall$ primes p, if $p|ab$, then $p|a \vee p|b$.

   Proof: Assume that a and b are any positive integers and p is any prime with the property $p|ab \wedge p \nmid a \wedge p \nmid b$. Then $ab = pk$ for some integer k Note that p must be a part of the prime factorization of ab.

   Note that p cannot be a part of the prime factorization of a or b. Otherwise p would divide a or b. It follows that prime factorization of ab cannot contain a factor of p. Indeed, if for some reason p were to show up in the prime factorization, then there are a product of terms in the prime factorization of a and of b that multiply to p. However, this is not possible since p is prime.

   $\Rightarrow\Leftarrow$

   $\square$

   Now $p! = \binom{p}{r} \cdot r! \cdot (p - r)!$. Therefore p divides $\binom{p}{r} \cdot r! \cdot (p - r)!$. Now the prime factorization of $r!$ cannot have p since $r < p$. Similarly, the prime factorization of $(p - r)!$ cannot have p since $p - r < p$. Therefore p cannot divide $r!$ or $(p - r)!$. It follows that p must divide $\binom{p}{r}$.

   Proof: