7. Metric Spaces

7.1 Metric spaces

Let X be a set, and let $d : X \times X \to \mathbb{R}$ be a function such that for all $x, y, z \in X$,

1. $d(x,y) \geq 0$ (nonnegativity)
2. $d(x,y) = 0 \iff x = y$ (identity of indiscernibles)
3. $d(x,y) = d(y,x)$ (symmetry)
4. $d(x,z) \leq d(x,y) + d(y,z)$ (triangle inequality)

(X, d) is called a metric space.

If d is clear from constant, we write X as the metric space.

Example: $(\mathbb{R}, d)$: $d(x,y) = |x-y|$, $x, y \in \mathbb{R}$

Example: $(C^0([a,b],\mathbb{R}), d)$: $C^0([a,b],\mathbb{R})$ be the set of continuous real-valued functions on $[a,b]$

$d(f,g) = ||f-g||_u$

Example: $(\mathbb{R}, d_n)$: d_n be the evaluation norm — $x \in \mathbb{R}^n$, $||x||_p = \sqrt{\sum_{i=1}^n |x_i|^p}$

7.2 Open and closed sets & 7.3 Sequences and convergence

A sequence in a metric space (X,d) is a function $x : \mathbb{N} \to X$.

As before, we write $x_n = x(n)$ as the elements of the sequence.

$\{x_n\}$ denotes the sequence.

We say a sequence $\{x_n\}$ in a metric space (X,d) is bounded if there exists a point $p \in X$ and $B \in \mathbb{R}$ such that $d(p, x_n) \leq B$, $\forall n \in \mathbb{N}$.

A sequence converges to some $p \in X$ if for all $\epsilon > 0$, there exists $M \in \mathbb{N}$ s.t. for all $n \geq M$, $d(p, x_n) < \epsilon$.

Prop. A convergent sequence is bounded.

Let $A \subset X$ be a subset of a metric space X.

Define the open ball of radius $\delta > 0$ around $p \in X$ as

$B(p,\delta) = \{x \in X : d(p,x) < \delta\}$.

We say $p \in X$ is an interior point of A (we write $x \in A^\circ$) if there exists $\delta > 0$ such that

$B(p,\delta) \subset A$

We say $p \in X$ is a limit point of A if there exists a sequence $\{x_n\}$ in A converging to p.

Prop. A is open $\iff$ every $p \in A$ is an interior point of A

A is closed $\iff$ A contains all of its limit points