Homework Solutions

Name: Notes and Solutions

April 9, 2022

1. Homework 3

Exercise 1.1 (1.10.1). *Construct the Picard iterates for the initial-value problem*

$$\begin{cases} y' = 2t(y+1) \\ y(0) = 0 \end{cases}$$

and show that they converge to the solution $y(t) = e^{t^2} - 1$.

Proof. Recall that for $y'(t) = f(t, y)$ the Picard iterates satisfy the formula

$$y_n(t) = y_0 + \int_0^t f(s, y_{n-1}(s))\, ds,$$

which here is given by

$$y_n(t) = \int_0^t 2s(1 + y_{n-1}(s))\, ds.$$

We claim by induction that $y_n(t) = \sum_{k=0}^n \frac{t^{2k}}{k!} - 1$. The base case is clear: the previous would yield $y_0 = 1 - 1 = 0$, which is our initial condition. For the inductive step, we compute from the Picard iteration

$$\begin{aligned} y_{n+1}(t) &= \int_0^t 2s(1 + y_n(s))\, ds \\ &= \int_0^t 2s \sum_{k=0}^n \frac{s^{2k}}{k!}\, ds \\ &= \sum_{k=0}^n \int_0^t \frac{2s^{2k+1}}{k!}\, ds \\ &= \sum_{k=0}^n \frac{2t^{2k+2}}{(2k+2)k!} \\ &= \sum_{k=0}^n \frac{t^{2(k+1)}}{(k+1)!} = \sum_{k=0}^{n+1} \frac{t^{2k}}{k!} - 1, \end{aligned}$$

which is the desired formula. Hence, we conclude that $y_n(t) = \sum_{k=0}^n \frac{t^{2k}}{k!} - 1$ by induction.

To demonstrate the requested convergence, we observe that the nth order Taylor polynomials for $e^x - 1$ evaluated at $x = t^2$ are simply the y_n above. Hence, using the Lagrange form of the error, for every t there is some s with $0 \le s \le t^2$ such that

$$|y_n(t) - y(t)| = \frac{d^{n+1}}{dx^{n+1}}(e^x - 1)\Big|_{x=s} \frac{s^{n+1}}{(n+1)!}$$

$$= e^s \frac{s^{n+1}}{(n+1)!}.$$

We observe that for any real number a, $\frac{a^n}{n!} \to 0$ (this is because eventually $n > a$, and so the successive terms are obtained by multiplication by a factor that vanishes asymptotically). Then, since s is independent of n, we see that $|y_n(t) - y(t)| = e^s \frac{s^{n+1}}{(n+1)!} \to 0$ as $n \to \infty$, obtaining the desired convergence.

One could also bypass all of this work by just observing that the y_n are Taylor polynomials for e^x, and the Taylor series for e^x converges for all x. □

Exercise 1.2 (1.10.5). *Show that the solution $y(t)$ of the initial-value problem*

$$\begin{cases} y' = 1 + y + y^2\cos(t) \\ y(0) = 0 \end{cases}$$

exists on the interval $0 \leq t \leq \frac{1}{3}$.

Sol. The goal is to apply Theorem 2. Notice that here we have a first order ODE of the form $y' = f(t, y)$, where $f(t, y) = 1 + y + y^2\cos(t)$ and $\frac{\partial f}{\partial y} = 1 + 2y\cos(t)$ are continuous for all t and y. Letting a and b be arbitrary now, and letting M be the maximum value of $|f|$ on the rectangle $0 \leq t \leq a$, $|y - y_0| = |y| \leq b$, Theorem 2 guarantees existence up to time α, with

$$\alpha = \min\left(a, \frac{b}{M}\right).$$

We have no restrictions on our choices of a and b, since f and $\frac{\partial f}{\partial y}$ are continuous everywhere. Notice that f obtains its maximum by setting $t = 0$ and making y as large as possible on the rectangle. Hence, $M = 1 + b + b^2\cos(0) = 1 + b + b^2$, and we can rewrite α in terms of a and b as

$$\alpha = \min\left(a, \frac{b}{1 + b + b^2}\right).$$

Notice that if we choose $a = b = 1$, then $\alpha = \frac{1}{3}$ and Theorem 2 guarantees the existence of the solution $y(t)$ on the interval $0 \leq t \leq \frac{1}{3}$.

Exercise 1.3 (1.10.7). *Show that the solution $y(t)$ of the initial-value problem*

$$\begin{cases} y' = e^{-t^2} + y^2 \\ y(0) = 0 \end{cases}$$

exists on the interval $0 \leq t \leq \frac{1}{2}$.

Sol. The goal is to again apply Theorem 2. Notice that here we have a first order ODE of the form $y' = f(t, y)$, where $f(t, y) = e^{-t^2} + y^2$ and $\frac{\partial f}{\partial y} = 2y$ are continuous for all t and y. Letting a and b be arbitrary now, and letting M be the maximum value of $|f|$ on the rectangle $0 \leq t \leq a$, $|y - y_0| = |y| \leq b$, Theorem 2 guarantees existence up to time α, with

$$\alpha = \min\left(a, \frac{b}{M}\right).$$

We have no restrictions on our choices of a and b, since f and $\frac{\partial f}{\partial y}$ are continuous everywhere. Notice that f obtains its maximum by setting $t = 0$ and making y as large as possible on the rectangle. Hence, $M = 1 + b^2$, and we can rewrite α in terms of a and b as

$$\alpha = \min\left(a, \frac{b}{1 + b^2}\right).$$

Notice that if we choose $a = b = 1$, then $\alpha = \frac{1}{2}$ and Theorem 2 guarantees the existence of the solution $y(t)$ on the interval $0 \leq t \leq \frac{1}{2}$.

Exercise 1.4 (1.10.17)**.** *Prove that $y(t) = -1$ is the only solution of the initial-value problem*

$$\begin{cases} y' = t(1+y) \\ y(0) = -1 \end{cases}$$

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Proof. Notice that $y(t) = -1$ solves the ODE since $y' \equiv 0$ and $t(y+1) \equiv 0$. Thus, to prove that this is the only solution, we need only show that the solution to this ODE is unique.

One way to prove uniqueness is to apply Theorem 2' iteratively, tacking on a nonzero α of uniqueness time after each application, until uniqueness for all positive times has been demonstrated. That approach is a bit frustrating to write out rigorously however, so instead in these solutions I intend to mimic the proof of uniqueness in Theorem 2' in the textbook.

Suppose for the sake of a contradiction that there is a second solution $z(t)$ with the initial condition $z(0) = -1$. Then, since $z(t)$ solves the ODE it also solves the integral formulation

$$z(t) = -1 + \int_0^t s(1 + z(s))\ ds.$$

As a result, we can write

$$|y(t) - z(t)| = |-1 - z(t)| = |1 + z(t)| = \left| \int_0^t s(1+z(s))\ ds \right| \le \int_0^t s|1 + z(s)|\ ds$$

and see that

$$|1 + z(t)| \le \int_0^t s|1 + z(s)|\ ds.$$

This is very similar to the setup of Lemma 2 in §1.10, and is an example of a more general inequality called **Grönwall's inequality**. I won't state that in full generality, but those who are interested should look it up! It's a very useful tool in the study of ODE and evolution type PDE.

What does this inequality give us? Set $w(t) = |1 + z(t)|$. Then, the above reads

$$w(t) \le \int_0^t sw(s)\ ds,$$

where we take care to remark that w is nonnegative.

The heuristic is that we can think of $\int_0^t sw(s)\ ds = U(t)$ and for $t > 0$ write

$$\frac{1}{t}\frac{dU}{dt} \le U(t).$$

Rewriting this, we have the inhomogeneous first order linear ODE

$$\frac{dU}{dt} - tU = f(t)$$

for some negative f, which we could solve by introducing the integrating factor $e^{-\frac{1}{2}t^2}$. Instead of dealing with the derivatives, we introduce this integrating factor directly into the integral inequality above.

For those of you who didn't want to read the above heuristic, we define

$$u(t) := e^{-\frac{1}{2}t^2} \int_0^t sw(s)\ ds.$$

Differentiating in t, we have by the product rule and the fundamental theorem of calculus that

$$u'(t) = -te^{-\frac{1}{2}t^2} \int_0^t sw(s)\ ds + e^{-\frac{1}{2}t^2} tw(t) = te^{-\frac{1}{2}t^2} \left(w(t) - \int_0^t sw(s)\ ds \right).$$

By the integral inequality $w(t) \le \int_0^t sw(s)\ ds$, we can conclude that the right hand side is nonpositive for $t \ge 0$, i.e.

$$u'(t) \le 0.$$

Hence, the function u is decreasing, and we have $u(t) \le u(0) = 0$. Since u is positive for all $t \ge 0$, we have that $u(t) = 0$ for all $t \ge 0$. Then, $\int_0^t sw(s)\ ds$ is zero too, and since w is nonnegative we must have that $w(t) = |1 + z(t)| = 0$ for all time. Hence, $z(t) = -1 = y(t)$, and the solution to our ODE is unique! $\square$

Exercise 1.5 (1.10.19)**.** *Find a solution of the initial-value problem*

$$\begin{cases} y' = t\sqrt{1-y^2} \\ y(0) = 1 \end{cases}$$

other than $y(t) = 1$. Does this violate Theorem 2'? Explain.

Sol. Away from $y = 1$, we could solve this as a separable ODE, writing

$$\begin{aligned} \frac{1}{\sqrt{1-y^2}} \frac{dy}{dt} &= t \\ \frac{d}{dt} \left(\arcsin(y) \right) &= t \\ \arcsin(y(t)) &= \frac{1}{2}t^2 + c \\ y(t) &= \sin\left(\frac{1}{2}t^2 + c \right). \end{aligned}$$

Matching to our initial condition $y(0) = 1$, we have $1 = \sin(c)$ and so $c = \frac{\pi}{2}$. Thus, a second solution to our initial value problem is given by

$$y_2(t) = \sin\left(\frac{1}{2}t^2 + \frac{\pi}{2} \right),$$

and we see that our initial value problem has two solutions. This, however, does not violate the uniqueness guaranteed by Theorem 2' because Theorem 2' doesn't apply here. To apply Theorem 2', notice that $y' = f(t, y)$ with $f(t, y) = t\sqrt{1-y^2}$. We require f and $\frac{\partial f}{\partial y}$ to be continuous in a rectangle $0 \le t \le a$, $|y - y_0| = |y - 1| \le b$ for some positive a and b. However, for any choice of $b > 0$, $y = 1$ is included in said rectangle and notice that

$$\frac{\partial f}{\partial y} = \frac{-ty}{\sqrt{1-y^2}}$$

is not continuous there! Hence, no such rectangle exists on which we can apply Theorem 2' and there is no reason to expect uniqueness.

2. Homework 4

Exercise 2.1 (2.1.3). *Show that the operator L defined by*

$$L[y](t) = \int_a^t s^2 y(s)\ ds$$

is linear; that is, $L[cy] = cL[y]$ and $L[y_1 + y_2] = L[y_1] + L[y_2]$.

Proof. We compute for any t, function $y(t)$ and constant c that

$$L[cy](t) = \int_a^t s^2 cy(s)\ ds = c\int_a^t s^2 y(s)\ ds = cL[y](t).$$

Furthermore, for any functions $y_1(t)$ and $y_2(t)$ we have

$$L[y_1 + y_2](t) = \int_a^t s^2(y_1(s) + y_2(s))\ ds = \int_a^t s^2 y_1(s)\ ds + \int_a^t s^2 y_2(s)\ ds = L[y_1](t) + L[y_2](t),$$

establishing the desired linearity. □

Exercise 2.2 (2.1.5).

1. *Show that $y_1(t) = \sqrt{t}$ and $y_2(t) = 1/t$ are solutions of the differential equation*
$$2t^2 y'' + 3ty' - y = 0 \tag{1}$$
on the interval $0 < t < \infty$.
2. *Compute $W[y_1, y_2](t)$. What happens as t approaches zero?*
3. *Show that $y_1(t)$ and $y_2(t)$ form a fundamental set of solutions of (1) on the interval $0 < t < \infty$.*
4. *Solve the initial-value problem*
$$\begin{cases} 2t^2 y'' + 3ty' - y = 0; \\ y(1) = 2, \\ y'(1) = 1. \end{cases}$$

Sol.

1. *Proof.* First, we observe that for any $t > 0$,
$$\begin{aligned} y_1(t) &= \sqrt{t} \\ y_1'(t) &= \frac{1}{2\sqrt{t}} \\ y_1''(t) &= \frac{-1}{4}t^{-3/2} = \frac{-1}{4t^{3/2}}. \end{aligned}$$
Hence,
$$\begin{aligned} 2t^2 y''(t) + 3ty'(t) - y(t) &= 2t^2 \frac{-1}{4t^{3/2}} + 3t\frac{1}{2\sqrt{t}} - \sqrt{t} \\ &= -\frac{1}{2}\sqrt{t} + \frac{3}{2}\sqrt{t} - \sqrt{t} \\ &= 0, \end{aligned}$$
and $y_1(t)$ solves (1) on the interval $0 < t < \infty$. Next, we also have for all $t > 0$ that
$$y_2(t) = \frac{1}{t}$$

$$\begin{aligned} y_2'(t) &= -\frac{1}{t^2} \\ y_2''(t) &= \frac{2}{t^3}. \end{aligned}$$

Hence,

$$\begin{aligned} 2t^2 y''(t) + 3ty'(t) - y(t) &= 2t^2\frac{2}{t^3} + 3t\frac{-1}{t^2} - \frac{1}{t} \\ &= \frac{4}{t} - \frac{3}{t} - \frac{1}{t} \\ &= 0, \end{aligned}$$

and $y_2(t)$ solves (1) on the interval $0 < t < \infty$, as desired. □

2. Using the above computations, we see that

$$\begin{aligned} W[y_1, y_2](t) &= y_1(t)y_2'(t) - y_1'(t)y_2(t) \\ &= \sqrt{t}\frac{-1}{t^2} - \frac{1}{2\sqrt{t}}\frac{1}{t} \\ &= \frac{-1}{t^{3/2}} - \frac{1}{2t^{3/2}} \\ &= \frac{-3}{2t^{3/2}}, \end{aligned}$$

which tends to $-\infty$ as $t \downarrow 0$.

3. *Proof.* Notice that by dividing through by $2t^2$ for $t > 0$, equation (1) is in the form $y''+p(t)y'+q(t)y = 0$ for continuous p and q. Furthermore, $W[y_1, y_2](t)$ vanishes for all positive time. Hence, by Theorem 2 in §2.1, the general solution to (1) is of the form $y(t) = c_1y_1(t)+c_2y_2(t)$, and $\{y_1, y_2\}$ form a fundamental set of solutions. (Note: if you don't feel comfortable using Theorem 2 on an infinite interval, you can just use it on the interval $(0, N)$ and take $N \to \infty$). □

4. Now that we have our general solution, we need only match it to the initial conditions! Since $y(1) = 2$, we have
$$2 = y(1) = c_1y_1(t) + c_2y_2(1) = c_1 + c_2.$$
Since $y'(1) = 1$, we have
$$1 = y'(1) = c_1y_1'(t) + c_2y_2'(t) = \frac{c_1}{2} - c_2.$$
Thus, adding the two equations, we have $\frac{3}{2}c_1 = 3$, so $c_1 = 2$ and $c_2 = 0$. Hence, the solution is
$$y(t) = 2y_1(t) = 2\sqrt{t}.$$

Exercise 2.3 (2.1.7)**.** *Compute the Wronskian of the following pairs of functions.*

1. $\sin(at)$, $\cos(bt)$

2. $\sin^2(t)$, $1 - \cos(2t)$

3. e^{at}, e^{bt}

4. e^{at}, te^{at}

5. $t, t\ln(t)$

6. $e^{at}\sin(bt), e^{at}\cos(bt)$.

Sol. Oof! This is a marathon!

1. We have

$$\begin{aligned} W[\sin(at), \cos(bt)] &= \sin(at)(\cos(bt))' - (\sin(at))' \cos(bt) \\ &= -b \sin(at) \sin(bt) - a \cos(at) \cos(bt). \end{aligned}$$

2. For this one, we see that

$$\begin{aligned} W[\sin^2(t), 1 - \cos(2t)] &= \sin^2(t)(1 - \cos(2t))' - (\sin^2(t))'(1 - \cos(2t)) \\ &= \sin^2(t)(2\sin(2t)) - (2\sin(t)\cos(t))(1 - \cos(2t)) \\ &= 2\sin^2(t)\sin(2t) - \sin(2t)(\sin^2(t) + \cos^2(t) - \cos^2(t) + \sin^2(t)) \\ &= 2\sin^2(t)\sin(2t) - 2\sin(2t)\sin^2(t) \\ &= 0. \end{aligned}$$

   This is unsurprising, since $1 - \cos(2t) = \sin^2(t) + \cos^2(t) - \cos^2(t) + \sin^2(t) = 2\sin^2(t)$, which is a constant multiple of $\sin^2(t)$!

3. Here, we have

$$\begin{aligned} W[e^{at}, e^{bt}] &= e^{at}(e^{bt})' - (e^{at})'e^{bt} \\ &= be^{(a+b)t} - ae^{(a+b)t} \\ &= (b - a)e^{(a+b)t}. \end{aligned}$$

4. For this one, we observe

$$\begin{aligned} W[e^{at}, te^{at}] &= e^{at}(te^{at})' - (e^{at})'te^{at} \\ &= e^{at}(e^{at} + ate^{at}) - ae^{at}te^{at} \\ &= e^{2at} + ate^{2at} - ate^{2at} \\ &= e^{2at}. \end{aligned}$$

5. Here, we compute

$$\begin{aligned} W[t, t\ln(t)] &= t(t\ln(t))' - (t)'t\ln(t) \\ &= t\left(\frac{t}{t} + \ln(t)\right) - t\ln(t) \\ &= t + t\ln(t) - t\ln(t) \\ &= t. \end{aligned}$$

6. Finally, we find

$$\begin{aligned} W[e^{at}\sin(bt), e^{at}\cos(bt)] &= e^{at}\sin(bt)(e^{at}\cos(bt))' - (e^{at}\sin(bt))'(e^{at}\cos(bt)) \\ &= e^{at}\sin(bt)(ae^{at}\cos(bt) - be^{at}\sin(bt)) - (ae^{at}\sin(bt) + be^{at}\cos(bt))e^{at}\cos(bt) \\ &= ae^{2at}\sin(bt)\cos(bt) - be^{2at}\sin^2(bt) - ae^{2at}\sin(bt)\cos(bt) - be^{2at}\cos^2(bt) \\ &= -be^{2at}(\sin^2(bt) + \cos^2(bt)) \\ &= -be^{2at}. \end{aligned}$$

   Slick!

Exercise 2.4 (2.1.9). *1. Let $y_1(t)$ and $y_2(t)$ be solutions of $y'' + p(t)y' + q(t)y = 0$ on the interval $\alpha < t < \beta$, with $y_1(t_0) = 1$, $y_1'(t_0) = 0$, $y_2(t_0) = 0$ and $y_2'(t_0) = 1$. Show that $y_1(t)$ and $y_2(t)$ form a fundamental set of solutions on the interval $\alpha < t < \beta$.*

2. *Show that* $y(t) = y_0 y_1(t) + y_0' y_2(t)$ *is the solution of the initial value problem*

$$\begin{cases} y'' + p(t)y' + q(t)y = 0 \\ y(t_0) = y_0 \\ y'(t_0) = y_0'. \end{cases}$$

Proof. 1. For this first item, Theorem 2 of §2.1 tells us that we need only show that the Wronskian doesn't vanish on the interval (α, β). However, by Theorem 3, it suffices to show that the Wronskian does not vanish at some t_0 with $\alpha < t_0 < \beta$! Choosing the given t_0, we see that

$$W[y_1, y_2](t_0) = y_1(t_0)y_2'(t_0) - y_1'(t_0)y_2(t_0) = 1(1) - 0(0) = 1 \neq 0.$$

Hence, W does not vanish on (α, β) and we see that the general solution is of the form $y(t) = c_1 y_1(t) + c_2 y_2(t)$. Thus, $\{y_1, y_2\}$ form a fundamental set of solutions.

2. For the second item then, we need only match the initial conditions! We have

$$y_0 = y(t_0) = c_1 y_1(t_0) + c_2 y_2(t_0) = c_1(1) = c_1,$$

so $c_1 = y_0$ and

$$y_0' = y'(t_0) = c_1 y_1'(t_0) + c_2 y_2'(t_0) = c_2(1) = c_2,$$

so $c_2 = y_0'$. Hence, the solution of our initial value problem is $y(t) = y_0 y_1(t) + y_0' y_2(t)$, as desired! In a sense, y_1 and y_2 can be thought of as a standard basis for our solution space.

□

3. Homework 5

Exercise 3.1 (2.2.5). *Solve the following initial value problem:*

$$\begin{cases} y'' - 3y' - 4y = 0; \\ y(0) = 1, \\ y'(0) = 0. \end{cases}$$

Sol. Proposing solutions of the form $y = e^{rt}$, we arrive at the characteristic equation

$$r^2 - 3r - 4 = 0 = (r-4)(r+1)$$

with distinct roots $r = 4$ and $r = -1$. Thus, the general solution is of the form $y(t) = ae^{4t} + be^{-t}$. Differentiating, $y'(t) = 4ae^{4t} - be^{-t}$. Matching initial conditions, we have

$$\begin{aligned} 1 &= y(0) = a + b \\ 0 &= y'(0) = 4a - b. \end{aligned}$$

Adding the two equations, we find $5a = 1$ so $a = 1/5$, and hence $b = 4/5$. Thus,

$$y(t) = \frac{1}{5}e^{4t} + \frac{4}{5}e^{-t}.$$

Exercise 3.2 (2.2.9). *Let $y(t)$ be the solution of the initial value problem*

$$\begin{cases} y'' + 5y' + 6y = 0; \\ y(0) = 1, \\ y'(0) = V. \end{cases}$$

For what values of V does $y(t)$ remain nonnegative for all $t \geq 0$?

Sol. We first determine the solution in terms of V. First, proposing solutions of the form $y = e^{rt}$, we arrive at the characteristic equation

$$r^2 + 5r + 6 = 0 = (r+3)(r+2)$$

with distinct roots $r = -3$ and $r = -2$. Thus, the general solution is of the form $y(t) = ae^{-3t} + be^{-2t}$. Differentiating, $y'(t) = -3ae^{-3t} - 2be^{-2t}$. Matching initial conditions, we find

$$\begin{aligned} 1 &= y(0) = a + b \\ V &= y'(0) = -3a - 2b. \end{aligned}$$

Multiplying the first equation by 3 and summing, we find $b = V + 3$, so $a = -(V + 2)$. Hence,

$$y(t) = -(V+2)e^{-3t} + (V+3)e^{-2t}.$$

First, consider the case when the coefficient on e^{-2t} is positive, i.e. $V > -3$ Then, since $e^{-2t} \geq e^{-3t}$ for all $t \geq 0$ (because e^{-3t} decays faster), we have

$$y(t) = (V+3)e^{-2t} - (V+2)e^{-3t} \geq (V+3)e^{-3t} - (V+2)e^{-3t} = (V+3-V-2)e^{-3t} = e^{-3t} \geq 0$$

and see that y is always nonnegative. This also holds for $V = -3$; in that situation, $V + 3 = 0$ and so

$$y(t) = (-V-2)e^{-3t} = (3-2)e^{-3t} = e^{-3t} \geq 0.$$

However, nonnegativity is no longer guaranteed when $V < -3$. To see this, suppose for the sake of a contradiction that $y \geq 0$ for all $t \geq 0$. Then,

$$y(t) = (V+3)e^{-2t} - (V+2)e^{-3t} \geq 0$$

$$\begin{aligned}(V+3)e^{-2t} &\geq (V+2)e^{-3t}\\ (V+3)e^{t} &\geq V+2\\ e^{t} &\leq \frac{V+2}{V+3}\end{aligned}$$

for all $t \geq 0$. The sign of the inequality switched in the last line since $V+3<0$. Notice that since the right hand side is independent of t, this would imply that e^t is bounded from above for $t \geq 0$, which is not true! Hence, $y(t)$ cannot always be nonnegative when $V<-3$ and thus we require $V \geq -3$ to force $y(t) \geq 0$ for all $t \geq 0$.

Exercise 3.3 (2.2.1.3)**.** *Find the general solution of*

$$y'' + 2y' + 3y = 0.$$

Sol. Proposing solutions of the form $y = e^{rt}$, we find the characteristic equation

$$r^2 + 2r + 3 = 0$$

with solutions

$$r = \frac{-2 \pm \sqrt{2^2 - 4(3)}}{2} = \frac{-2 \pm \sqrt{-8}}{2} = -1 \pm i\sqrt{2}.$$

Choosing the root $-1+i\sqrt{2}$, we have

$$e^{rt} = e^{-t+it\sqrt{2}} = e^{-t}\cos(\sqrt{2}t) + ie^{-t}\sin(\sqrt{2}t).$$

Since the real and imaginary parts form linearly independent solutions of the original ODE, we find that our general solution takes the form

$$y(t) = ae^{-t}\cos(\sqrt{2}t) + be^{-t}\sin(\sqrt{2}t)$$

for constants a and b.

Exercise 3.4 (2.2.1.5)**.** *Solve the initial value problem*

$$\begin{cases} y'' + y' + 2y = 0;\\ y(0) = 1,\\ y'(0) = -2.\end{cases}$$

Sol. First, we find the general solution. Proposing solutions of the form $y = e^{rt}$, we find the characteristic equation

$$r^2 + r + 2 = 0$$

with solutions

$$r = \frac{-1 \pm \sqrt{1^2 - 4(2)}}{2} = \frac{-1 \pm i\sqrt{7}}{2}.$$

Choosing the root $\frac{-1+i\sqrt{7}}{2}$, we have

$$e^{rt} = e^{-\frac{1}{2}t + it\frac{\sqrt{7}}{2}} = e^{-\frac{1}{2}t}\cos\left(\frac{\sqrt{7}}{2}t\right) + ie^{-\frac{1}{2}t}\sin\left(\frac{\sqrt{7}}{2}t\right).$$

Since the real and imaginary parts form linearly independent solutions of the original ODE, we find that our general solution takes the form

$$y(t) = ae^{-\frac{1}{2}t}\cos\left(\frac{\sqrt{7}}{2}t\right) + be^{-\frac{1}{2}t}\sin\left(\frac{\sqrt{7}}{2}t\right) = e^{-\frac{1}{2}t}\left(a\cos\left(\frac{\sqrt{7}}{2}t\right) + b\sin\left(\frac{\sqrt{7}}{2}t\right)\right)$$

for constants a and b. Differentiation yields

$$y'(t) = -\frac{1}{2}y(t) + e^{-\frac{1}{2}t}\left(-\frac{a\sqrt{7}}{2}\sin\left(\frac{\sqrt{7}}{2}t\right) + \frac{b\sqrt{7}}{2}\cos\left(\frac{\sqrt{7}}{2}t\right)\right).$$

Matching to the initial conditions given, we find

$$\begin{aligned} 1 &= y(0) = a \\ -2 &= y'(0) = -\frac{1}{2} + \frac{b\sqrt{7}}{2}, \end{aligned}$$

so $a = 1$ and $b = -\frac{3}{\sqrt{7}}$. Hence,

$$y(t) = e^{-\frac{1}{2}t}\left(\cos\left(\frac{\sqrt{7}}{2}t\right) - \frac{3}{\sqrt{7}}\sin\left(\frac{\sqrt{7}}{2}t\right)\right).$$

Exercise 3.5 (2.2.2.3)**.** *Solve the initial value problem*

$$\begin{cases} 9y'' + 6y' + y = 0; \\ y(0) = 1, \\ y'(0) = 0. \end{cases}$$

Sol. Proposing solutions of the form $y = e^{rt}$, we find the characteristic equation

$$9r^2 + 6r + 1 = 0 = 9\left(r + \frac{1}{3}\right)^2,$$

with double root $r = \frac{-1}{3}$. Hence, the general solution takes the form

$$y(t) = (a + bt)e^{-\frac{1}{3}t}.$$

Differentiating,

$$y'(t) = -\frac{1}{3}y(t) + be^{-\frac{1}{3}t}.$$

Matching initial conditions, we find

$$\begin{aligned} 1 &= y(0) = a \\ 0 &= y'(0) = -\frac{1}{3} + b, \end{aligned}$$

so $a = 1$ and $b = \frac{1}{3}$. Thus,

$$y(t) = \left(1 + \frac{t}{3}\right)e^{-\frac{1}{3}t}.$$

Exercise 3.6 (2.2.2.9)**.** *Here is an alternate and very elegant way of finding a second solution $y_2(t)$ of*

$$ay'' + by' + cy = 0$$

when $ar^2 + br + c = 0$ has a double root.

1. *Assume that $b^2 = 4ac$. Show that*

$$L[e^{rt}] = a(e^{rt})'' + b(e^{rt})' + ce^{rt} = a(r - r_1)^2 e^{rt}$$

for $r_1 = -\frac{b}{2a}$.

2. *Show that*
$$\frac{\partial}{\partial r}L[e^{rt}] = L\left[\frac{\partial}{\partial r}e^{rt}\right] = L[te^{rt}] = 2a(r-r_1)e^{rt} + at(r-r_1)^2e^{rt}.$$

3. *Conclude that $L[te^{r_1 t}] = 0$. Hence, $y_2(t) = te^{r_1 t}$ is a second solution of the above constant coefficient equation in the double root case.*

Proof. First, when $b^2 = 4ac$, we can write
$$ar^2 + br + c = a\left(r^2 + \frac{b}{a}r + \frac{c}{a}\right) = a\left(r + \frac{b}{2a}\right)^2,$$
since $\left(\frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} = \frac{c}{a}$. Hence,
$$\begin{aligned}
L[e^{rt}] &= a(e^{rt})'' + b(e^{rt})' + ce^{rt} = a(r-r_1)^2e^{rt} \\
&= ar^2e^{rt} + bre^{rt} + ce^{rt} \\
&= \left(ar^2 + br + c\right)e^{rt} \\
&= a\left(r + \frac{b}{2a}\right)^2 e^{rt} \\
&= a(r-r_1)^2e^{rt},
\end{aligned}$$
as desired. Next, notice that we can interchange the order of partial derivatives; writing the $'$ derivative as $\frac{\partial}{\partial t}$, we see that
$$\begin{aligned}
\frac{\partial}{\partial r}L[e^{rt}] &= \frac{\partial}{\partial r}\left((e^{rt})'' + b(e^{rt})' + ce^{rt} = a(r-r_1)^2e^{rt}\right) \\
&= a\frac{\partial}{\partial r}\frac{\partial^2}{\partial t^2}e^{rt} + b\frac{\partial}{\partial r}\frac{\partial}{\partial t}e^{rt} + c\frac{\partial}{\partial r}e^{rt} \\
&= a\frac{\partial^2}{\partial t^2}\left(\frac{\partial}{\partial r}e^{rt}\right) + b\frac{\partial}{\partial t}\left(\frac{\partial}{\partial r}e^{rt}\right) + c\left(\frac{\partial}{\partial r}e^{rt}\right) \\
&= a\left(\frac{\partial}{\partial r}e^{rt}\right)'' + b\left(\frac{\partial}{\partial r}e^{rt}\right)' + c\left(\frac{\partial}{\partial r}e^{rt}\right) \\
&= L\left[\frac{\partial}{\partial r}e^{rt}\right].
\end{aligned}$$
In particular, we can conclude that
$$\begin{aligned}
L[te^{rt}] = L\left[\frac{\partial}{\partial r}\right] &= \frac{\partial}{\partial r}L[e^{rt}] \\
&= \frac{\partial}{\partial r}\left(a(r-r_1)^2e^{rt}\right) \\
&= 2a(r-r_1)e^{rt} + at(r-r_1)^2e^{rt},
\end{aligned}$$
as desired. Finally, notice that if we set $r = r_1$ we find
$$L[te^{r_1 t}] = 2a(r_1 - r_1)e^{r_1 t} + at(r_1 - r_1)^2e^{r_1 t} = 0,$$
as desired. □

Exercise 3.7 (2.3.1)**.** *Three solutions of a certain second-order nonhomogeneous linear equation are*
$$\psi_1(t) = t^2,\ \psi_2(t) = t^2 + e^{2t}$$
and
$$\psi_3(t) = 1 + t^2 + 2e^{2t}.$$
Find the general solution of this equation.

Sol. By assumption, all of the ψ_i above solve $L[\psi_i] = g$ for some function g. Notice then that

$$L[\psi_2 - \psi_1] = L[\psi_3 - \psi_2] = 0,$$

so that $\psi_2(t) - \psi_1(t) = e^{2t}$ and $\psi_3(t) - \psi_2(t) = e^{2t} + 1$ solve the homogeneous problem. These are linearly independent solutions, since

$$W[e^{2t}, e^{2t} + 1] = 2e^{2t}e^{2t} - 2e^{2t}(e^{2t} + 1) = -2e^{2t} \neq 0.$$

Finally, since any ψ_i gives a particular solution of $L[y] = g$, we can in particular choose ψ_1 and find that he general solution is

$$y(t) = ae^{2t} + b(e^{2t} + 1) + t^2 = c_1 + c_2 e^{2t} + t^2$$

for some constants c_1 and c_2.

Exercise 3.8 (2.4.1)**.** *Find the general solution of*

$$y'' + y = \sec(t), \quad -\frac{\pi}{2} < t < \frac{\pi}{2}.$$

Sol. We use the method of variation of parameters. First, we propose a solution of the form $y = e^{rt}$ to the homogeneous problem

$$y'' + y = 0$$

and find the characteristic equation

$$r^2 + 1 = 0$$

with roots $r = \pm i$. Choosing i, we have

$$e^{rt} = e^{it} = \cos(t) + i\sin(t)$$

and so the general solution to the homogeneous problem is given by $a\cos(t) + b\sin(t)$ for constants a and b.

Then, proposing a particular solution of the form $\psi(t) = u_1(t)y_1(t) + u_2(t)y_2(t) = u_1(t)\cos(t) + u_2(t)\sin(t)$, we know from §2.4 that since the leading coefficient on our second order ODE is 1, u_1 and u_2 are described by

$$u_1'(t) = \frac{-\sec(t)\sin(t)}{W[\cos(t), \sin(t)]} = \frac{-\tan(t)}{\cos^2(t) + \sin^2(t)} = -\tan(t)$$

and

$$u_2'(t) = \frac{\sec(t)\cos(t)}{W[\cos(t), \sin(t)]} = \frac{1}{\cos^2(t) + \sin^2(t)} = 1.$$

Hence, $u_2(t) = t$ and integrating $\tan(t)$ we find

$$u_1(t) = \int -\tan(t)\ dt = \int -\frac{\sin(t)}{\cos(t)}\ dt = \int \frac{du}{u} = \ln|u| = \ln|\cos(t)|$$

with the substitution $u = \cos(t)$. Since $\cos(t) > 0$ for $-\frac{\pi}{2} < t < \frac{\pi}{2}$, we have $u_1(t) = \ln(\cos(t))$. Thus, $\psi(t) = \ln(\cos(t))\cos(t) + t\sin(t)$ and our general solution is

$$y(t) = a\cos(t) + b\sin(t) + \ln(\cos(t))\cos(t) + t\sin(t) = (a + \ln(\cos(t)))\cos(t) + (b + t)\sin(t)$$

for constants a and b.

Exercise 3.9 (2.4.5)**.** *Solve the initial value problem*

$$\begin{cases} 3y'' + 4y' + y = (\sin(t))e^{-t}; \\ y(0) = 1, \\ y'(0) = 0. \end{cases}$$

Sol. First, we solve the homogeneous problem $3y'' + 4y' + y = 0$ by proposing solutions of the form $y = e^{rt}$ and arriving at the characteristic equation

$$3r^2 + 4r + 1 = 0 = (3r+1)(r+1)$$

with roots $r = -\frac{1}{3}$ and $r = -1$. Thus, the general solution of the homogeneous problem is given by $ae^{-\frac{1}{3}t} + be^{-t}$ for constants a and b.

To determine a particular solution, we use the method of variation of parameters. Since we would like to use the formulae already derived in the book, we normalize our ODE to have leading coefficient 1:

$$y'' + \frac{4}{3}y' + \frac{1}{3}y = \frac{1}{3}(\sin(t))e^{-t}.$$

Then, proposing a particular solution of the form $\psi(t) = u_1(t)y_1(t) + u_2(t)y_2(t) = u_1(t)e^{-\frac{1}{3}t} + u_2(t)e^{-t}$, we have that u_1 and u_2 are described by

$$u_1'(t) = \frac{-\frac{1}{3}\sin(t)e^{-t}e^{-t}}{W[e^{-\frac{1}{3}t}, e^{-t}]} = \frac{-\frac{1}{3}\sin(t)e^{-2t}}{-\frac{2}{3}e^{-\frac{4}{3}t}} = \frac{1}{2}\sin(t)e^{-\frac{2}{3}t}$$

and

$$u_2'(t) = \frac{\frac{1}{3}\sin(t)e^{-t}e^{-\frac{1}{3}t}}{W[e^{-\frac{1}{3}t}, e^{-t}]} = \frac{\frac{1}{3}\sin(t)e^{-\frac{4}{3}t}}{-\frac{2}{3}e^{-\frac{4}{3}t}} = -\frac{1}{2}\sin(t).$$

u_2 can be easily integrated to yield $u_2(t) = \frac{1}{2}\cos(t)$. u_1 is trickier, and we use of a general formula for integrals of the form $\int \sin(t)e^{at}\,dt$. Integrating by parts twice, we have

$$\begin{aligned}\int \sin(t)e^{at}\,dt &= \int ae^{at}\cos(t)\,dt - e^{at}\cos(t)\\ &= \int -a^2e^{at}\sin(t)\,dt + a\sin(t)e^{at} - \cos(t)e^{at}.\end{aligned}$$

Rearranging, we have

$$\int \sin(t)e^{at}\,dt = \frac{a\sin(t) - \cos(t)}{1+a^2}e^{at}.$$

Thus,

$$u_1(t) = \frac{1}{2}\int \sin(t)e^{-\frac{2}{3}t}\,dt = \frac{1}{2}\frac{-\frac{2}{3}\sin(t) - \cos(t)}{1+\frac{4}{9}}e^{-\frac{2}{3}t} = -\frac{9}{26}e^{-\frac{2}{3}t}\left(\frac{2}{3}\sin(t) + \cos(t)\right)$$

and

$$\begin{aligned}\psi(t) &= u_1(t)e^{-\frac{1}{3}t} + u_2(t)e^{-t}\\ &= -\frac{3}{13}e^{-t}\sin(t) - \frac{9}{26}e^{-t}\cos(t) + \frac{1}{2}e^{-t}\cos(t)\\ &= e^{-t}\left(\frac{2}{13}\cos(t) - \frac{3}{13}\sin(t)\right).\end{aligned}$$

So, the general solution is of the form

$$y(t) = \left(b + \frac{2}{13}\cos(t) - \frac{3}{13}\sin(t)\right)e^{-t} + ae^{-\frac{1}{3}t}.$$

Differentiating,

$$y'(t) = -\left(b + \frac{2}{13}\cos(t) - \frac{3}{13}\sin(t)\right)e^{-t} + \left(-\frac{2}{13}\sin(t) - \frac{3}{13}\cos(t)\right)e^{-t} - \frac{1}{3}ae^{-\frac{1}{3}t}.$$

Matching to the initial conditions, we have

$$\begin{aligned}1 &= y(0) = b + \frac{2}{13} + a\\ 0 &= y'(0) = -\left(b + \frac{2}{13}\right) - \frac{3}{13} - \frac{1}{3}a.\end{aligned}$$

Summing the two equations yields $\frac{2}{3}a = 1 + \frac{3}{13} = \frac{16}{13}$, or $a = \frac{24}{13}$. Thus, $b = -1$ and

$$y(t) = \left(-1 + \frac{2}{13}\cos(t) - \frac{3}{13}\sin(t)\right)e^{-t} + \frac{24}{13}e^{-\frac{1}{3}t}.$$

Exercise 3.10 (2.5.1)**.** *Find a particular solution to*

$$y'' + 3y = t^3 - 1.$$

Sol. We guess judiciously. Motivated by the polynomial on the right hand side, we propose a solution of the form

$$y(t) = a_3t^3 + a_2t^2 + a_1t + a_0.$$

Differentiating twice yields

$$y''(t) = 6a_3t + 2a_2,$$

and so we must match coefficients in the equation

$$\begin{aligned}t^3 - 1 &= y'' + 3y\\ &= 6a_3t + 2a_2 + 3a_3t^3 + 3a_2t^2 + 3a_1t + 3a_0\\ &= 3a_3t^3 + 3a_2t^2 + (6a_3 + 3a_1)t + (2a_2 + 3a_0).\end{aligned}$$

Thus, $3a_3 = 1$ so $a_3 = \frac{1}{3}$. $3a_2 = 0$, so $a_2 = 0$. $6a_3 + 3a_1 = 2 + 3a_1 = 0$ so $a_1 = -\frac{2}{3}$. Finally, $2a_2 + 3a_0 = 3a_0 = -1$, so $a_0 = -\frac{1}{3}$. Thus,

$$y(t) = \frac{1}{3}t^3 - \frac{2}{3}t - \frac{1}{3}$$

is a particular solution to the above ODE.

Exercise 3.11 (2.5.3)**.** *Find a particular solution to*

$$y'' - y = t^2e^t.$$

Sol. We guess judiciously. Motivated by the polynomial with an exponential on the right hand side, we guess a solution of the form

$$y(t) = (a_2t^2 + a_1t + a_0)e^t.$$

Differentiating twice, we have

$$y''(t) = 2a_2e^t + 2(2a_2t + a_1)e^t + y$$

and observe that

$$y'' - y = 2a_2e^t + 2(2a_2t + a_1)e^t.$$

This cannot be made equal to t^2e^t; due to the structure of the equation, the t^2 term dropped out. To get around this, we try multiplying by t and propose a solution of the form

$$y(t) = (a_2t^2 + a_1t + a_0)te^t = (a_2t^3 + a_1t^2 + a_0t)e^t.$$

Then, differentiating twice yields

$$y'' = (6a_2t + 2a_1)e^t + 2(3a_2t^2 + 2a_1t + a_0)e^t + y$$

and we must match coefficients in

$$\begin{aligned} t^2e^t &= y'' - y \\ &= (6a_2t + 2a_1)e^t + 2(3a_2t^2 + 2a_1t + a_0)e^t + y - y \\ &= (6a_2t^2 + (6a_2 + 4a_1)t + (2a_1 + 2a_0))e^t. \end{aligned}$$

We have $6a_2 = 1$ so $a_2 = \frac{1}{6}$, $6a_2 + 4a_1 = 1 + 4a_1 = 0$ so $a_1 = -\frac{1}{4}$, and $2a_1 + 2a_0 = -\frac{1}{2} + 2a_0 = 0$ so $a_0 = \frac{1}{4}$. Thus,

$$y(t) = \left(\frac{1}{6}t^2 - \frac{1}{4}t + \frac{1}{4}\right) te^t$$

is a particular solution to the above ODE.

Exercise 3.12 (2.5.9). *Find a particular solution to*

$$y'' - 2y' + 5y = 2\cos^2 t.$$

Sol. First, we observe from the trigonometric identity $\cos(2t) = 2\cos^2 t - 1$ that $2\cos^2 t = 1 + \cos(2t)$. Hence, our right hand side is really a linear polynomial in cosine, so we propose a solution of the form

$$y(t) = a + b\sin(2t) + c\cos(2t).$$

Differentiating yields

$$y'(t) = 2b\cos(2t) - 2c\sin(2t)$$

and

$$y''(t) = -4b\sin(2t) - 4c\cos(2t).$$

Hence, we must match coefficients in the equation

$$\begin{aligned} 2\cos^2 t = 1 + \cos(2t) &= y'' - 2y' + 5y \\ &= -4b\sin(2t) - 4c\cos(2t) - 4b\cos(2t) + 4c\sin(2t) + 5a + 5b\sin(2t) + 5c\cos(2t) \\ &= 5a + (b + 4c)\sin(2t) + (c - 4b)\cos(2t). \end{aligned}$$

$5a = 1$, so $a = \frac{1}{5}$. Since $b + 4c = 0$ and $c - 4b = 1$, we can multiply the first equation by 4 and add to find that $17c = 1$, or $c = \frac{1}{17}$. Then, $b = -\frac{4}{17}$ and

$$y(t) = \frac{1}{5} - \frac{4}{17}\sin(2t) + \frac{1}{17}\cos(2t)$$

solves the above ODE.

Exercise 3.13 (2.5.13). *Find a particular solution to*

$$y'' - 3y' + 2y = e^t + e^{2t}.$$

Sol. We guess judiciously. Normally, we would propose a solution of the form $ae^t + be^{2t}$ given the right hand side, but we notice that

$$(e^t)'' - 3(e^t)' + 2e^t = e^t - 3e^t + 2e^t = 0$$

and

$$(e^{2t})'' - 3(e^{2t})' + 2e^{2t} = 4e^{2t} - 6e^{2t} + 2e^{2t} = 0,$$

so that both e^t and e^{2t} are solutions to the homogeneous problem. Hence, we try multiplying by t and propose a solution of the form

$$y(t) = ate^t + bte^{2t}.$$

Differentiating, we find

$$y' = ae^t + ate^t + be^{2t} + 2bte^{2t}$$

and

$$y'' = 2ae^t + ate^t + 4be^{2t} + 4bte^{2t}.$$

Hence, we must match coefficients in the equation

$$\begin{aligned} e^t + e^{2t} &= y'' - 3y' + 2y \\ &= 2ae^t + ate^t + 4be^{2t} + 4bte^{2t} - 3ae^t - 3ate^t - 3be^{2t} - 6bte^{2t} + 2ate^t + 2bte^{2t} \\ &= (2a - 3a)e^t + (at - 3at + 2at)e^t + (4b - 3b)e^{2t} + (4bt - 6bt + 2bt)e^{2t} \\ &= -ae^t + be^{2t}. \end{aligned}$$

Hence, $a = -1$ and $b = 1$, so

$$y(t) = -te^t + te^{2t} = te^t(e^t - 1)$$

solves the above ODE.

4. Homework 6

In all of my solutions for §2.6 problems, I'll elide the units in most of the computations since all problems give standard SI units. We also always choose the downward direction as the positive direction for vertical oscillation, and the rightward direction as positive for horizontal oscillation.

Exercise 4.1 (2.6.1)**.** *It is found experimentally that a* 1 *kg mass stretches a spring* 49/320 *m. If the mass is pulled down an additional* 1/4 *m and released, find the amplitude, period and frequency of the resulting motion, neglecting air resistance (use* $g = 9.8$ *m/s*^{2}*).*

Sol. First, we determine the spring constant. The equilibrium stretch length of the spring tells us where gravity and the restoring force of the spring balance out, i.e.

$$kx = \frac{49}{320}k = mg = 9.8.$$

Hence, $k = 64$. With this, we can right the equation of motion for the spring (neglecting air resistance) as

$$\begin{cases} my'' + cy' + ky = y'' + 64y = 0; \\ y(0) = \frac{1}{4}, \\ y'(0) = 0. \end{cases}$$

Proposing solutions of the form $y = e^{rt}$, we arrive at the equation $r^2 + 64 = 0$, or $r = \pm 8i$. Choosing $r = 8i$, we have $y = e^{8it} = \cos(8t) + i\sin(8t)$. Taking real and imaginary parts, we see that the general solution is given by

$$y(t) = a\cos(8t) + b\sin(8t).$$

Differentiating, $y'(t) = -8a\sin(8t) + 8b\cos(8t)$. Matching the initial conditions, we find

$$\begin{aligned} \frac{1}{4} &= y(0) = a \\ 0 &= y'(0) = 8b. \end{aligned}$$

Hence, $y(t) = \frac{1}{4}\cos(8t)$. The amplitude of the oscillations is $\frac{1}{4}$ m, the period is $\frac{2\pi}{8} = \frac{\pi}{4}$ s, and the frequency is 8 s^{-1}.

Exercise 4.2 (2.6.5)**.** *A small object of mass* 1 *kg is attached to a spring with spring-constant* 1 *N/m and is immersed in a viscous medium with damping constant* 2 *N* $\cdot$ *s/m. At time* $t = 0$*, the mass is lowered* 1/4 *m and given an initial velocity of* 1 *m/s in the upward direction. Show that the mass will overshoot its equilibrium position once, and then creep back to equilibrium.*

Proof. The equation of motion for this mass is given by

$$\begin{cases} my'' + cy' + ky = y'' + 2y' + y = 0; \\ y(0) = \frac{1}{4}, \\ y'(0) = -1. \end{cases}$$

Proposing solutions of the form $y = e^{rt}$, we arrive at the equation $r^2 + 2r + 1 = 0$, or $r = -1$. Since this is a real double root, the general solution is given by

$$y(t) = (a + bt)e^{-t}.$$

Differentiating, $y'(t) = be^{-t} - y(t)$. Matching the initial conditions, we find

$$\frac{1}{4} = y(0) = a$$

$$-1 = y'(0) = b - \frac{1}{4},$$

so that $a = \frac{1}{4}$ and $b = -\frac{3}{4}$. Thus, the solution is given by

$$y(t) = \frac{1}{4}(1 - 3t)e^{-t}.$$

Let's take a qualitative look at this solution. $y(t)$ is positive, and hence lies below its equilibrium point, for $t < \frac{1}{3}$. At $t = \frac{1}{3}$, it crosses back up past its equilibrium point, and $y(t)$ stays negative (i.e. above equilibrium) for all time. However, $e^{-t} \to 0$ as $t \to \infty$, and so $y(t) \to 0$; the mass slowly creeps back to equilibrium over time. □

Exercise 4.3 (2.6.11)**.** *A* 1 *kg mass is attached to a spring with spring constant* $k = 4$ *N/m, and hangs in equilibrium. An external force* $F(t) = (1 + t + \sin(2t)$ *N is applied to the mass beginning at time* $t = 0$*. If the spring is stretched a length* $(1/2 + \pi/4)$ *m or more from its equilibrium position, then it will break. Assuming no damping present, find the time at which the spring breaks.*

Sol. Let's write down the equation of motion of the mass, so that we can determine the break point. The equation of motion is given by

$$\begin{cases} my'' + cy' + ky = y'' + 4y = 1 + t + \sin(2t) \\ y(0) = 0 \\ y'(0) = 0. \end{cases}$$

To find our solution, we first must solve the homogeneous problem $y'' + 4y = 0$. Proposing solutions of the form $y = e^{rt}$, we arrive at the equation $r^2 + 4 = 0$, or $r = \pm 2i$. Choose $r = 2i$, we have $y = e^{2it} = \cos(2t) + i\sin(2t)$. Taking real and imaginary parts, we see that the general solution to the homogeneous problem is given by

$$y(t) = c_1\cos(2t) + c_2\sin(2t).$$

Next, we search for a particular solution, for which we use the method of judicious guessing. To account for $1 + t$, we propose a polynomial of the form $a + bt$. Normally to take care of $\sin(2t)$ we would propose $c\sin(2t) + d\cos(2t)$, but we observe here that $\sin(2t)$ and $\cos(2t)$ solve the homogeneous problem; hence, these parts of our guess would just vanish. To keep the trigonometric terms, we propose $t(c\sin(2t) + d\cos(2t))$ instead. Hence, we search for a solution of the form $\psi(t) = a + bt + t(c\sin(2t) + d\cos(2t))$. Differentiating and plugging ψ into the equation of motion, we find

$$\begin{aligned} \psi'(t) &= b + (c\sin(2t) + d\cos(2t)) + t(2c\cos(2t) - 2d\sin(2t)) \\ \psi''(t) &= 2(2c\cos(2t) - 2d\sin(2t)) + t(-4c\sin(2t) - 4d\cos(2t)) \end{aligned}$$

and so

$$\begin{aligned} 1 + t + \sin(2t) &= \psi''(t) + 4\psi(t) \\ &= 2(2c\cos(2t) - 2d\sin(2t)) + t(-4c\sin(2t) - 4d\cos(2t)) + 4a + 4bt + 4t(c\sin(2t) + d\cos(2t)) \\ &= 4c\cos(2t) - 4d\sin(2t) + 4a + 4bt. \end{aligned}$$

Matching coefficients, we see that $a = \frac{1}{4} = b$, $c = 0$ and $d = -\frac{1}{4}$. Thus, the general solution is given by

$$y(t) = c_1\cos(2t) + c_2\sin(2t) + \frac{1}{4}(1 + t - t\cos(2t)).$$

Differentiating, $y'(t) = -2c_1\sin(2t) + 2c_2\cos(2t) + \frac{1}{4}(1 - \cos(2t) + 2t\sin(2t))$. Matching to our initial conditions, we find

$$0 = y(0) = c_1 + \frac{1}{4}$$

$$0 = y'(0) = 2c_2 + \frac{1}{4} - \frac{1}{4}$$

so that $c_1 = -\frac{1}{4}$ and $c_2 = 0$. Hence,

$$y(t) = \frac{1}{4}(1 + t - t\cos(2t) - \cos(2t)).$$

Now, let's see where the spring will break. Notice that we can control naively

$$|y(t)| \leq \frac{1}{4} + \frac{1}{4}|t| + \frac{1}{4}|t||\cos(2t)| + \frac{1}{4}|\cos(2t)| \leq \frac{1}{2} + \frac{1}{2}|t|.$$

For $t < \frac{\pi}{2}$, we have $|y(t)| < \frac{1}{2} + \frac{\pi}{4}$ and hence the spring will not break before $t = \frac{\pi}{2}$. At $t = \frac{\pi}{2}$, we find

$$y\left(\frac{\pi}{2}\right) = \frac{1}{4} + \frac{1}{4}\frac{\pi}{2} - \frac{1}{4}\frac{\pi}{2}\cos(\pi) - \frac{1}{4}\cos(\pi) = \frac{1}{2} + \frac{\pi}{4},$$

and we see that the spring breaks at $t = \frac{\pi}{2}$.

Exercise 4.4 (2.6.13)**.** *Determine a particular solution $\psi(t)$ of $my'' + cy' + ky = F_0\cos(\omega t)$, of the form $\psi(t) = A\cos(\omega t - \phi)$. Show that the amplitude A is a maximum when $\omega^2 = \omega_0^2 - \frac{1}{2}(c/m)^2$. This value of ω is called the resonant frequency of the system. What happens when $\omega_0^2 < \frac{1}{2}(c/m)^2$?*

Sol. We use the method of judicious guessing, proposing as usual $\psi(t) = a\sin(\omega t) + b\cos(\omega t)$. We will write our solution in the desired form after solving for a and b. Differentiating, we have

$$\begin{aligned}
\psi'(t) &= \omega a\cos(\omega t) - \omega b\sin(\omega t) \\
\psi''(t) &= -\omega^2 a\sin(\omega t) - \omega^2 b\cos(\omega t).
\end{aligned}$$

Substituting into our equation, we find

$$\begin{aligned}
F_0\cos(\omega t) &= m\psi''(t) + c\psi'(t) + k\psi(t) \\
&= m(-\omega^2 a\sin(\omega t) - \omega^2 b\cos(\omega t)) + c(\omega a\cos(\omega t) - \omega b\sin(\omega t)) + k(a\sin(\omega t) + b\cos(\omega t)) \\
&= (-m\omega^2 a - c\omega b + ka)\sin(\omega t) + (-m\omega^2 b + c\omega a + kb)\cos(\omega t).
\end{aligned}$$

Matching coefficients, we have the system of equations

$$\begin{aligned}
(k - m\omega^2)a + (-c\omega)b &= 0 \\
(c\omega)a + (k - m\omega^2)b &= F_0.
\end{aligned}$$

We assume that $\omega \neq 0$ so that our solutions are oscillatory, and $c \neq 0$ because the behavior at resonant frequency is a bit different in the undamped case. Multiplying the second equation through then by $\frac{k-m\omega^2}{c\omega}$ and subtracting the first equation from the second, we find

$$\left(\frac{k - m\omega^2}{c\omega}\right)F_0 = \frac{(k - m\omega^2)^2}{c\omega}b + c\omega b = \frac{(k - m\omega^2)^2 + (c\omega)^2}{c\omega}b,$$

or $b = \frac{F_0(k-m\omega^2)}{(k-m\omega^2)^2+(c\omega)^2}$. Substituting this into the first equation, we find

$$(k - m\omega^2)a = \frac{F_0(c\omega)(k - m\omega^2)}{(k - m\omega^2)^2 + (c\omega)^2},$$

or $a = \frac{F_0 c\omega}{(k-m\omega^2)^2+(c\omega)^2}$. Now, our goal was to write this as $\psi(t) = A\cos(\omega t - \phi) = A\cos(\phi)\cos(\omega t) + A\sin(\phi)\sin(\omega t)$. Matching coefficients with our above solution, we have

$$A\cos(\phi) = \frac{F_0(k - m\omega^2)}{(k - m\omega^2)^2 + (c\omega)^2}, \quad A\sin(\phi) = \frac{F_0 c\omega}{(k - m\omega^2)^2 + (c\omega)^2}.$$

Squaring and summing, we have

$$A^2\cos^2\phi + A^2\sin^2\phi = A^2 = \frac{F_0^2((k-m\omega^2)^2+(c\omega)^2)}{((k-m\omega^2)^2+(c\omega)^2)^2} = \frac{F_0^2}{(k-m\omega^2)^2+(c\omega)^2},$$

so that $A = \frac{F_0}{\sqrt{(k-m\omega^2)^2+(c\omega)^2}}$. Dividing, we have

$$\frac{A\sin(\phi)}{A\cos(\phi)} = \tan(\phi) = \frac{c\omega}{k-m\omega^2}.$$

Hence, $\phi = \arctan\left(\frac{c\omega}{k-m\omega^2}\right)$. (Note: if $A\cos(\phi)=0$ then we require $\phi = \frac{\pi}{2}$ so that $\tan(\phi)=\infty$. This agrees with the above formula if we set $\arctan(\infty)=\frac{\pi}{2}$, since $A\cos(\phi)=0 \implies k-m\omega^2=0 \implies \frac{c\omega}{k-m\omega^2}=\infty$). Thus,

$$\begin{aligned}\psi(t) &= A\cos(\omega t-\phi);\\ A &= \frac{F_0}{\sqrt{(k-m\omega^2)^2+(c\omega)^2}},\\ \phi &= \arctan\left(\frac{c\omega}{k-m\omega^2}\right).\end{aligned}$$

as desired.

Now we can turn to the question of maximizing the amplitude. If we take everything else besides ω to be constant, then A only depends on ω^2; differentiating with respect to ω^2, we find

$$\begin{aligned}\frac{\partial A}{\partial\omega^2} &= -\frac{1}{2}F_0\left((k-m\omega^2)^2+(c\omega)^2\right)^{-3/2}(c^2-2m(k-m\omega^2))\\ &= -\frac{1}{2}F_0\left((k-m\omega^2)^2+(c\omega)^2\right)^{-3/2}(c^2-2km+2m^2\omega^2)).\end{aligned}$$

If $\omega_0^2 > \frac{1}{2}\frac{c^2}{m^2}$, then remembering that $\omega_0 = \sqrt{\frac{k}{m}}$ tells us that $\frac{k}{m} - \frac{c^2}{2m^2} > 0$, or $c^2-2km<0$. In particular, for small values of ω^2, we see that $\frac{\partial A}{\partial\omega^2}>0$, and hence A increases until obtaining a maximum at $c^2-2km+2m^2\omega^2=0$, past which point A decreases. Rearranging using $\omega_0^2 = \frac{k}{m}$, we find that the maximum occurs when

$$\omega^2 = \frac{2km}{2m^2} - \frac{c^2}{2m^2} = \omega_0^2 - \frac{1}{2}\left(\frac{c}{m}\right)^2,$$

as desired. If $\omega_0^2 \le \frac{1}{2}\frac{c^2}{m^2}$, then rearranging as above we have that $c^2-2km\ge 0$. In particular, $c^2-2km+2m^2\omega^2>0$ for all positive ω, and so $\frac{\partial A}{\partial\omega^2}$ is always negative for positive ω. Hence, A is always decreasing and the maximum amplitude is obtained in the $\omega\downarrow 0$ limit.

Exercise 4.5 (2.8.1). *Find the general solution of*

$$\Big\{ y''+ty'+y=0.$$

Sol. We propose a power series solution $y=\sum_{n=0}^\infty a_nt^n$. Differentiating, we find

$$\begin{aligned}y' &= \sum_{n=1}^\infty na_nt^{n-1} \implies ty' = \sum_{n=1}^\infty na_nt^n = \sum_{n=0}^\infty na_nt^n\\ y'' &= \sum_{n=2}^\infty n(n-1)a_nt^{n-2} = \sum_{n=0}^\infty (n+2)(n+1)a_{n+2}t^n.\end{aligned}$$

In the first sum we've reinserted the $n=0$ term since it contributes nothing to the sum, and we've reindexed the second sum to start from $n=0$. Hence,

$$0 = y''+ty'+y$$

$$
\begin{aligned}
&= \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}t^n + \sum_{n=0}^{\infty} n a_n t^n + \sum_{n=0}^{\infty} a_n t^n \\
&= \sum_{n=0}^{\infty}((n+2)(n+1)a_{n+2} + (n+1)a_n)t^n.
\end{aligned}
$$

For the power series to vanish everywhere we require all coefficients to vanish. This yields the recursion

$$(n+2)(n+1)a_{n+2} + (n+1)a_n = 0 \implies a_{n+2} = \frac{-a_n}{n+2}$$

for $n \geq 0$. Notice that we can choose a_0 and a_1 freely, and that the rest of the terms are determined by the recursion relation. Furthermore, if we choose two linearly independent (a_0, a_1) vectors, then we will have obtained two linearly independent solutions $y_1(t)$ and $y_2(t)$ by which we can form our general solution.

Starting with $a_0 = 1$ and $a_1 = 0$, we see that $a_n = 0$ for all odd n. Next, for even n we observe

$$
\begin{aligned}
a_0 &= 1 \\
a_2 &= \frac{-1}{2} \\
a_4 &= \frac{1}{2 \cdot 4} \\
a_6 &= \frac{-1}{2 \cdot 4 \cdot 6} \\
&\vdots \\
a_{2n} &= \frac{(-1)^n}{2 \cdot 4 \cdots 2n} \\
&\vdots
\end{aligned}
$$

so that

$$y_1(t) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2 \cdot 4 \cdots 2n} t^{2n} = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{2^n n!} = \sum_{n=0}^{\infty} \frac{1}{n!}\left(-\frac{t^2}{2}\right)^n = e^{-\frac{1}{2}t^2},$$

where in the last step we've identified the power series expansion of e^x.

Next, turning to $a_0 = 0$ and $a_1 = 1$ we see that $a_n = 0$ for all even n. Next, for odd n we observe

$$
\begin{aligned}
a_1 &= 1 \\
a_3 &= \frac{-1}{3} \\
a_5 &= \frac{1}{3 \cdot 5} \\
a_7 &= \frac{-1}{3 \cdot 5 \cdot 7} \\
&\vdots \\
a_{2n+1} &= \frac{(-1)^n}{3 \cdot 5 \cdots (2n+1)} \\
&\vdots
\end{aligned}
$$

so that

$$y_2(t) = \sum_{n=0}^{\infty} \frac{(-1)^n}{3 \cdot 5 \cdots (2n+1)} t^{2n+1} = \sum_{n=0}^{\infty} \left(\prod_{k=1}^{n} \frac{-1}{2k+1}\right) t^{2n+1}.$$

Our general solution is a linear combination of these two solutions, so

$$y(t) = c_1 e^{-\frac{1}{2}t^2} + c_2 \sum_{n=0}^{\infty} \left(\prod_{k=1}^{n} \frac{-1}{2k+1} \right) t^{2n+1}$$

for some constants c_1 and c_2.

Exercise 4.6 (2.8.5)**.** *Solve the following initial-value problem:*

$$\begin{cases} t(2-t)y'' - 6(t-1)y' - 4y = 0; \\ y(1) = 1, \\ y'(1) = 0. \end{cases}$$

Sol. Since our initial condition is at $t_0 = 1$, we propose a power series solution of the form $y(t) = \sum_{n=0}^{\infty} a_n (t-1)^n$ (this will make our computations for the coefficients easier later on). First though, we rewrite the equation in terms of $t-1$ as

$$\begin{aligned} 0 &= t(2-t)y'' - 6(t-1)y' - 4y \\ &= (2t - t^2)y'' - 6(t-1)y' - 4y \\ &= -(t-1)^2 y'' + y'' - 6(t-1)y' - 4y. \end{aligned}$$

Differentiating our series,

$$\begin{aligned} y'(t) &= \sum_{n=1}^{\infty} n a_n (t-1)^{n-1} \implies 6(t-1)y' = \sum_{n=1}^{\infty} 6n a_n (t-1)^n = \sum_{n=0}^{\infty} 6n a_n (t-1)^n \\ y''(t) &= \sum_{n=2}^{\infty} n(n-1) a_n (t-1)^{n-2} \implies (t-1)^2 y'' = \sum_{n=2}^{\infty} n(n-1) a_n (t-1)^n = \sum_{n=0}^{\infty} n(n-1) a_n (t-1)^n. \end{aligned}$$

In both rightmost sums, we've reinserted the missing terms $n = 0$ and $n = 0, 1$ respectively, since they contribute nothing to their respective sums. We will also need to reindex $y''(t)$ as

$$y''(t) = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} (t-1)^n$$

in order to match coefficients. Hence,

$$\begin{aligned} 0 &= t(2-t)y'' - 6(t-1)y' - 4y \\ &= -(t-1)^2 y'' + y'' - 6(t-1)y' - 4y \\ &= -\sum_{n=0}^{\infty} n(n-1) a_n (t-1)^n + \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} (t-1)^n - \sum_{n=0}^{\infty} 6n a_n (t-1)^n - \sum_{n=0}^{\infty} 4 a_n (t-1)^n \\ &= \sum_{n=0}^{\infty} ((-n(n-1) - 6n - 4) a_n + (n+2)(n+1) a_{n+2})(t-1)^n. \end{aligned}$$

For the power series to vanish everywhere we require all coefficients to vanish. This yields the recursion

$$\begin{aligned} (-n(n-1) - 6n - 4) a_n + (n+2)(n+1) a_{n+2} = 0 \implies a_{n+2} &= \frac{(n(n-1) + 6n - 4) a_n}{(n+2)(n+1)} \\ &= \frac{n^2 + 5n + 4}{(n+2)(n+1)} a_n \\ &= \frac{(n+4)(n+1)}{(n+2)(n+1)} a_n \end{aligned}$$

$$= \frac{n+4}{n+2} a_n$$

for $n \geq 0$. Notice now that $y(1) = 1 = a_0$ and $y'(1) = 0 = a_1$. Hence, all odd terms in our recursion vanish. To determine the even terms, we see that

$$\begin{aligned}
a_0 &= 1 \\
a_2 &= \frac{4}{2} \cdot 1 = 2 \\
a_4 &= \frac{6}{4} \cdot 2 = 3 \\
a_6 &= \frac{8}{6} \cdot 4 = 4 \\
&\vdots \\
a_{2n} &= n + 1 \\
&\vdots
\end{aligned}$$

For the curious reader, we can prove that $a_{2n} = n + 1$ for all n using a technique called induction: mainly, if it's true for $n = 0$, and if we can say that the $n = k$ case implies the $n = k + 1$ case, then it must be true for all n (because $n = 0 \implies n = 1 \implies n = 2 \implies \cdots \implies n = k \implies n = k + 1 \implies \cdots$). Clearly $a_0 = 1 = 0 + 1$, and if $a_{2k} = k + 1$ we have

$$a_{2(k+1)} = a_{2k+2} = \frac{2k+4}{2k+2} a_{2k} = \frac{2(k+2)}{2(k+1)}(k+1) = k + 2 = (k+1) + 1,$$

as desired. Hence, $a_{2n} = n + 1$ for all n and

$$y(t) = \sum_{n=0}^{\infty} (n+1) t^{2n}$$

solves our initial value problem.

Exercise 4.7 (2.8.9)**.** *The equation $y'' - 2ty' + \lambda y = 0$, λ constant, is known as the Hermite differential equation, and it appears in many areas of mathematics and physics.*

- *Find 2 linearly independent solutions of the Hermite equation.*
- *Show that the Hermite equation has a polynomial solution of degree n if $\lambda = 2n$. This polynomial, when properly normalized; that is, when multiplied by a suitable constant, is known as the Hermite polynomial $H_n(t)$.*

Sol. We propose solutions of the form $y = \sum_{n=0}^{\infty} a_n t^n$. Differentiating our series,

$$\begin{aligned}
y'(t) &= \sum_{n=1}^{\infty} n a_n t^{n-1} \implies 2ty' = \sum_{n=1}^{\infty} 2n a_n t^n = \sum_{n=0}^{\infty} 2n a_n t^n \\
y''(t) &= \sum_{n=2}^{\infty} n(n-1) a_n t^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} t^n.
\end{aligned}$$

In the first sum we've reinserted the $n = 0$ term since it contributes nothing to the sum, and we've reindexed the second sum to start from $n = 0$. Hence

$$\begin{aligned}
0 &= y'' - 2ty' + \lambda y \\
&= \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} t^n - \sum_{n=0}^{\infty} 2n a_n t^n + \sum_{n=0}^{\infty} \lambda a_n t^n
\end{aligned}$$

$$= \sum_{n=0}^{\infty} ((n+2)(n+1)a_{n+2} - (2n-\lambda)a_n)t^n.$$

For the power series to vanish everywhere we require all coefficients to vanish. This yields the recursion

$$(n+2)(n+1)a_{n+2} - (2n-\lambda)a_n = 0 \implies a_{n+2} = \frac{2n-\lambda}{(n+2)(n+1)}a_n$$

for $n \geq 0$. Notice that we can choose a_0 and a_1 freely, and that the rest of the terms are determined by the recursion relation. Furthermore, if we choose two linearly independent (a_0, a_1) vectors, then we will obtain two linearly independent solutions $y_1(t)$ and $y_2(t)$.

First we consider $a_0 = 1$ and $a_1 = 0$. Observe that $a_n = 0$ then for all odd n. For even n, we compute

$$\begin{aligned}
a_0 &= 1 \\
a_2 &= \frac{-\lambda}{2} \\
a_4 &= \frac{4-\lambda}{4\cdot 3}\frac{-\lambda}{2} \\
a_6 &= \frac{8-\lambda}{6\cdot 5}\frac{4-\lambda}{4\cdot 3}\frac{-\lambda}{2} \\
&\vdots \\
a_{2n} &= \frac{2(2n-2)-\lambda}{2n(2n-1)}\frac{2(2n-4)-\lambda}{(2n-2)(2n-3)}\cdots\frac{8-\lambda}{6\cdot 5}\frac{4-\lambda}{4\cdot 3}\frac{-\lambda}{2} \\
&\vdots
\end{aligned}$$

and thus we have

$$\begin{aligned}
y_1(t) &= 1 + \sum_{n=1}^{\infty}\left(\frac{2(2n-2)-\lambda}{2n(2n-1)}\frac{2(2n-4)-\lambda}{(2n-2)(2n-3)}\cdots\frac{8-\lambda}{6\cdot 5}\frac{4-\lambda}{4\cdot 3}\frac{-\lambda}{2}\right)t^{2n} \\
&= 1 + \sum_{n=1}^{\infty}\frac{(4(n-1)-\lambda)(4(n-2)-\lambda)\cdots(4-\lambda)(-\lambda)}{(2n)!}t^{2n} \\
&= 1 + \sum_{n=1}^{\infty}\left(\prod_{k=0}^{n-1}(4k-\lambda)\right)\frac{t^{2n}}{(2n)!}.
\end{aligned}$$

Next, we consider $a_0 = 0$ and $a_1 = 1$. Observe that $a_n = 0$ for all even n. For odd n, we compute

$$\begin{aligned}
a_1 &= 1 \\
a_3 &= \frac{2-\lambda}{3\cdot 2} \\
a_5 &= \frac{6-\lambda}{5\cdot 4}\frac{2-\lambda}{3\cdot 2} \\
a_7 &= \frac{10-\lambda}{7\cdot 6}\frac{6-\lambda}{5\cdot 4}\frac{2-\lambda}{3\cdot 2} \\
&\vdots \\
a_{2n+1} &= \frac{2(2n-1)-\lambda}{(2n+1)(2n)}\frac{2(2n-3)-\lambda}{(2n-1)(2n-2)}\cdots\frac{10-\lambda}{7\cdot 6}\frac{6-\lambda}{5\cdot 4}\frac{2-\lambda}{3\cdot 2} \\
&\vdots
\end{aligned}$$

and thus we have

$$y_2(t) = t + \sum_{n=1}^{\infty}\left(\frac{2(2n-1)-\lambda}{(2n+1)(2n)}\frac{2(2n-3)-\lambda}{(2n-1)(2n-2)}\cdots\frac{10-\lambda}{7\cdot 6}\frac{6-\lambda}{5\cdot 4}\frac{2-\lambda}{3\cdot 2}\right)t^{2n+1}$$

$$= t + \sum_{n=1}^{\infty} \frac{((4(n-1)+2)-\lambda)((4(n-2)+2)-\lambda)\cdots(6-\lambda)(2-\lambda)}{(2n+1)!} t^{2n+1}$$
$$= t + \sum_{n=1}^{\infty} \left(\prod_{k=0}^{n-1} (4k+2-\lambda) \right) \frac{t^{2n+1}}{(2n+1)!}.$$

Hence, our two linearly independent solutions of the Hermite equation are

$$y_1(t) = 1 + \sum_{n=1}^{\infty} \left(\prod_{k=0}^{n-1} (4k-\lambda) \right) \frac{t^{2n}}{(2n)!} \quad \text{and} \quad y_2(t) = t + \sum_{n=1}^{\infty} \left(\prod_{k=0}^{n-1} (4k+2-\lambda) \right) \frac{t^{2n+1}}{(2n+1)!}.$$

Now, suppose that there is some $\overline{n}$ such that $\lambda = 2\overline{n}$. We observe from our recursion that for all $k > 0$, $a_{\overline{n}+2k} = 0$ because

$$a_{2\overline{n}+2} = \frac{2\overline{n} - \lambda}{(\overline{n}+2)(\overline{n}+1)} a_{\overline{n}} = 0$$

and thus $a_{2\overline{n}+4} = 0 = a_{2\overline{n}+6}$ and so on. To ensure that $a_{2\overline{n}+k}$ vanishes for odd k as well (and thus constructing our desired Hermite polynomial), we make use of y_1 and y_2.

First, consider the trivial case $n = 0$. Then, $\lambda = 0$ and any constant is a zero degree polynomial solving $y'' - 2ty' + \lambda y = y'' - 2ty' = 0$.

Next, suppose $\overline{n}$ is even, so that $\overline{n} = 2m$ for some m. Consider $y_1(t)$ and observe that for all $n > m$, the product $\prod_{k=0}^{n-1}(4k-\lambda)$ has the term $4m-\lambda = 2\overline{n}-\lambda = 0$. Hence, every coefficient on t^{2n} vanishes for $n > m$, and thus $y_1(t)$ is a polynomial of degree $2m = \overline{n}$: properly normalized, it is the desired Hermite polynomial.

Finally, suppose $\overline{n}$ is odd, so that $\overline{n} = 2m+1$ for some m. Consider $y_2(t)$ and observe that for all $n > m$, the product $\prod_{k=0}^{n-1}(4k+2-\lambda)$ has the term $4m+2-\lambda = 2(2m+1)-\lambda = 2\overline{n}-\lambda = 0$. Hence every coefficient on t^{2n+1} vanishes for $n > m$, and thus $y_2(t)$ is a polynomial of degree $2m+1 = \overline{n}$: properly normalized, it is the desired Hermite polynomial.

So, we see that whenever $\lambda = 2\overline{n}$, the Hermite equation has a polynomial solution of degree $\overline{n}$.

5. Homework 7

Exercise 5.1 (2.8.1.7)**.** *Find the general solution of*

$$t^2y'' + ty' + y = 0.$$

Sol. We observe that this is an Euler equation with $\alpha = \beta = 1$. Proposing solutions of the form $y = t^r$, we observe that $(r(r-1) + r + 1)t^r = 0$, which requires $r(r-1) + r + 1 = 0$. Equivalently,

$$r^2 + 1 = 0,$$

which has solutions $r = \pm i$. Choosing $r = i$ we find

$$t^i = e^{i\ln(t)} = \cos(\ln(t)) + i\sin(\ln(t)).$$

Taking real and imaginary parts, we see that the general solution is

$$y(t) = a\cos(\ln(t)) + b\sin(\ln(t))$$

for constants a and b.

Exercise 5.2 (2.8.1.9)**.** *Solve the initial-value problem*

$$\begin{cases} t^2y'' - ty' - 2y = 0; \\ y(1) = 0, \\ y'(1) = 1 \end{cases}$$

on the interval $0 < t < \infty$.

Sol. We observe that this is an Euler equation with $\alpha = -1$, $\beta = -2$. Proposing solutions of the form $y = t^r$, we observe that $(r(r-1) - r - 2)t^r = 0$, which requires $r(r-1) - r - 2 = 0$. Equivalently,

$$r^2 - 2r - 2 = 0,$$

which has solutions $r = \frac{2\pm\sqrt{12}}{2} = 1 \pm \sqrt{3}$. Thus, the general solution is

$$y(t) = at^{1+\sqrt{3}} + bt^{1-\sqrt{3}}$$

for constants a and b. Differentiating, $y'(t) = (1+\sqrt{3})at^{\sqrt{3}} + (1-\sqrt{3})bt^{-\sqrt{3}}$. Matching our initial conditions, we find

$$\begin{aligned} 0 &= y(1) = a + b \\ 1 &= y'(1) = a(1+\sqrt{3}) + b(1-\sqrt{3}). \end{aligned}$$

Thus, $b = -a$ and $a + a\sqrt{3} - a + a\sqrt{3} = 2a\sqrt{3} = 1$ so $a = \frac{1}{2\sqrt{3}}$. We conclude that

$$y(t) = \frac{1}{2\sqrt{3}}t^{1+\sqrt{3}} - \frac{1}{2\sqrt{3}}t^{1-\sqrt{3}} = \frac{t}{2\sqrt{3}}\left(t^{\sqrt{3}} - \frac{1}{t^{\sqrt{3}}}\right).$$

Exercise 5.3 (2.8.2.1)**.** *Determine whether or not* $t = 0$ *is a regular singular point of the ODE*

$$t(t-2)^2y'' + ty' + y = 0.$$

Sol. First, we renormalize the equation as

$$y'' + \frac{1}{(t-2)^2}y' + \frac{1}{t(t-2)^2}y = 0.$$

Observe that $t\frac{1}{(t-2)^2}$ is analytic at $t = 0$, and $t^2\frac{1}{t(t-2)^2} = \frac{t}{(t-2)^2}$ is also analytic at $t = 0$. Hence, $t = 0$ is in fact a regular singular point.

Exercise 5.4 (2.8.2.5)**.** *Determine whether or not $t = -1$ is a regular singular point of the ODE*

$$(1-t^2)y'' + \frac{1}{\sin(t+1)}y' + y = 0.$$

Sol. We again renormalize the equation as

$$y'' + \frac{1}{(1-t^2)\sin(t+1)}y' + \frac{1}{1-t^2}y = 0.$$

Notice that

$$(t+1)\frac{1}{(1-t^2)\sin(t+1)} = \frac{1}{(1-t)\sin(t+1)}$$

has a singularity at $t = -1$ (since $\sin(-1+1) = \sin(0) = 0$), and is not analytic. Hence, $t = -1$ is not a regular singular point of the equation.

Exercise 5.5 (2.8.2.7)**.** *Find the general solution of*

$$2t^2y'' + 3ty' - (1+t)y = 0.$$

Sol. Because of the t term on the y, we notice that this is not an Euler equation. However, the singular point at $t = 0$ is still regular: normalizing the equation yields

$$y'' + \frac{3}{2t}y' - \frac{1+t}{2t^2}y = 0.$$

Since $t\frac{3}{2t} = \frac{3}{2}$ and $t^2\frac{-(1+t)}{2t^2} = -\frac{1}{2}(1+t)$ are both analytic at $t = 0$, the singular point is regular. As such, we proceed in confidence with the Frobenius method, proposing a solution of the form $y = \sum_{n=0}^{\infty} a_n t^{n+r}$ for some fixed r to be determined. Differentiating, we observe

$$\begin{aligned}
y'(t) &= \sum_{n=0}^{\infty}(n+r)a_n t^{n+r-1} \implies 3ty'(t) = \sum_{n=0}^{\infty} 3(n+r)a_n t^{n+r} \\
y''(t) &= \sum_{n=0}^{\infty}(n+r)(n+r-1)a_n t^{n+r-2} \implies 2t^2y''(t) = \sum_{n=0}^{\infty} 2(n+r)(n+r-1)a_n t^{n+r}.
\end{aligned}$$

Furthermore,

$$(1+t)y(t) = (1+t)\sum_{n=0}^{\infty} a_n t^{n+r} = \sum_{n=0}^{\infty} a_n t^{n+r} + \sum_{n=0}^{\infty} a_n t^{n+r+1} = \sum_{n=0}^{\infty} a_n t^{n+r} + \sum_{n=1}^{\infty} a_{n-1} t^{n+r},$$

where we have reindexed the last sum. Hence, we require

$$\begin{aligned}
0 &= 2t^2y'' + 3ty' - (1+t)y \\
&= \sum_{n=0}^{\infty} 2(n+r)(n+r-1)a_n t^{n+r} + \sum_{n=0}^{\infty} 3(n+r)a_n t^{n+r} - \sum_{n=0}^{\infty} a_n t^{n+r} - \sum_{n=1}^{\infty} a_{n-1} t^{n+r} \\
&= (2r(r-1) + 3r - 1)a_0 t^r + \sum_{n=1}^{\infty}((2(n+r)(n+r-1) + 3(n+r) - 1)a_n - a_{n-1})t^{n+r}.
\end{aligned}$$

Since all of the terms must vanish for the series to vanish everywhere, we first obtain the indicial equation

$$2r(r-1) + 3r - 1 = 2r^2 + r - 1 = (2r-1)(r+1) = 0$$

with solutions $r = -1$ and $r = \frac{1}{2}$. Each choice of r will yield a different linearly independent solution.

Let's start with $r = -1$ and choose $a_0 = 1$. Then, we have

$$(2(n-1)(n-2) + 3(n-1) - 1)a_n - a_{n-1} = 0$$

and so

$$a_n = \frac{a_{n-1}}{2(n-1)(n-2) + 3(n-1) - 1} = \frac{a_{n-1}}{2(n^2 - 3n + 2) + 3n - 3 - 1} = \frac{a_{n-1}}{2n^2 - 3n} = \frac{a_{n-1}}{n(2n-3)}$$

for $n \geq 1$. Thus, we obtain

$$\begin{aligned}
a_1 &= \frac{1}{-1} = -1 \\
a_2 &= \frac{-1}{2 \cdot 1} = -\frac{1}{2} \\
a_3 &= \frac{-1}{2}\frac{1}{3 \cdot 3} = \frac{-1}{2 \cdot 3}\frac{1}{3} \\
a_4 &= \frac{-1}{2 \cdot 3}\frac{1}{3}\frac{1}{4 \cdot 5} = \frac{-1}{2 \cdot 3 \cdot 4}\frac{1}{3 \cdot 5} \\
&\vdots \\
a_n &= \frac{-1}{2 \cdot 3 \cdots (n-1)}\frac{1}{3 \cdot 5 \cdots 2n-5}\frac{1}{n(2n-3)} = \frac{-1}{n!}\frac{1}{3 \cdot 5 \cdots (2n-3)} \\
&\vdots
\end{aligned}$$

and so

$$y_1(t) = \frac{1}{t} - 1 + \frac{1}{t}\sum_{n=2}^{\infty}\frac{-1}{n!}\frac{1}{1 \cdot 3 \cdots (2n-3)}t^n = \frac{1}{t}\left(1 - t - \sum_{n=2}^{\infty}\left(\prod_{k=2}^{n}\frac{1}{2k-3}\right)\frac{t^n}{n!}\right).$$

Next, we consider $r = \frac{1}{2}$ and choose $a_0 = 1$. Then, we have

$$\left(2\left(n + \frac{1}{2}\right)\left(n - \frac{1}{2}\right) + 3\left(n + \frac{1}{2}\right) - 1\right)a_n - a_{n-1} = 0$$

and so

$$a_n = \frac{a_{n-1}}{\left(2\left(n + \frac{1}{2}\right)\left(n - \frac{1}{2}\right) + 3\left(n + \frac{1}{2}\right) - 1\right.} = \frac{a_{n-1}}{2n^2 - \frac{1}{2} + 3n + \frac{3}{2} - 1} = \frac{a_{n-1}}{2n^2 + 3n} = \frac{a_{n-1}}{n(2n+3)}$$

for $n \geq 1$. Thus, we obtain

$$\begin{aligned}
a_1 &= \frac{1}{5} \\
a_2 &= \frac{1}{5}\frac{1}{2 \cdot 7} = \frac{1}{2}\frac{1}{5 \cdot 7} \\
a_3 &= \frac{1}{2}\frac{1}{5 \cdot 7}\frac{1}{3 \cdot 9} = \frac{1}{2 \cdot 3}\frac{1}{5 \cdot 7 \cdot 9} \\
&\vdots \\
a_n &= \frac{1}{2 \cdot 3 \cdots (n-1)}\frac{1}{5 \cdot 7 \cdots 2n+1}\frac{1}{n(2n+3)} = \frac{1}{n!}\frac{1}{5 \cdot 7 \cdots 2n+3} \\
&\vdots
\end{aligned}$$

and so

$$y_2(t) = \sqrt{t} + \sqrt{t}\sum_{n=1}^{\infty}\frac{1}{n!}\frac{1}{5 \cdot 7 \cdots 2n+3}t^n = \sqrt{t}\left(1 + \sum_{n=1}^{\infty}\left(\prod_{k=1}^{n}\frac{1}{2k+3}\right)\frac{t^n}{n!}\right).$$

Thus, the general solution for this equation is given by

$$y(t) = \frac{c_1}{t}\left(1 - t - \sum_{n=2}^{\infty}\left(\prod_{k=2}^{n}\frac{1}{2k-3}\right)\frac{t^n}{n!}\right) + c_2\sqrt{t}\left(1 + \sum_{n=1}^{\infty}\left(\prod_{k=1}^{n}\frac{1}{2k+3}\right)\frac{t^n}{n!}\right).$$

Exercise 5.6 (2.8.2.19)**.** *Consider the equation*

$$t^2y'' + (t^2 - 3t)y' + 3y = 0. \tag{2}$$

1. *Show that $r = 1$ and $r = 3$ are the two roots of the indicial equation of (2).*
2. *Find a power series solution of (2) of the form*

$$y_1(t) = t^3\sum_{n=0}^{\infty}a_nt^n, \quad a_0 = 1.$$

3. *Show that $y_1(t) = t^3e^{-t}$.*
4. *Show that (2) has no solution of the form*

$$t\sum_{n=0}^{\infty}b_nt^n.$$

5. *Find a second solution of (2) using the method of reduction of order. Leave your answer in integral form.*

Proof. Let us consider the first point. This equation has a singular point at $t = 0$. It is not an Euler equation, but if we renormalize it reads

$$y'' + \left(1 - \frac{3}{t}\right)y' + \frac{3}{t^2}y = 0.$$

Since $t\left(1 - \frac{3}{t}\right) = t - 3$ and $t^2\frac{3}{t^2} = 3$ are both analytic at $t = 0$, we conclude that $t = 0$ is a regular singular point. As such, we can proceed with confidence using the method of Frobenius, proposing a solution of the form $y(t) = \sum_{n=0}^{\infty}a_nt^{n+r}$ to find the indicial equation. Differentiating, we observe

$$\begin{aligned}
y'(t) &= \sum_{n=0}^{\infty}(n+r)a_nt^{n+r-1} \implies (t^2-3t)y'(t) = \sum_{n=0}^{\infty}(n+r)a_nt^{n+r+1} - \sum_{n=0}^{\infty}3(n+r)a_nt^{n+r} \\
&= \sum_{n=1}^{\infty}(n-1+r)a_{n-1}t^{n+r} - \sum_{n=0}^{\infty}3(n+r)a_nt^{n+r} \\
y''(t) &= \sum_{n=0}^{\infty}(n+r)(n+r-1)a_nt^{n+r-2} \implies t^2y''(t) = \sum_{n=0}^{\infty}(n+r)(n+r-1)a_nt^{n+r},
\end{aligned}$$

where we have reindexed sums freely. Hence

$$\begin{aligned}
0 &= t^2y'' + (t^2-3t)y' + 3y \\
&= \sum_{n=0}^{\infty}(n+r)(n+r-1)a_nt^{n+r} + \sum_{n=1}^{\infty}(n-1-r)a_{n-1}t^{n+r} - \sum_{n=0}^{\infty}3(n+r)a_nt^{n+r} + \sum_{n=0}^{\infty}3a_nt^{n+r} \\
&= (r(r-1) - 3r + 3)a_0t^r + \sum_{n=1}^{\infty}(((n+r)(n+r-1) - 3(n+r) + 3)a_n + (n-1+r)a_{n-1})t^{n+r}.
\end{aligned}$$

Since all of the terms must vanish for the series to vanish everywhere, we obtain the indicial equation

$$r(r-1) - 3r + 3 = 0 = r^2 - 4r + 3 = (r-1)(r-3)$$

with solutions $r = 1$ and $r = 3$, as desired.

The second point requires us to choose $r = 3$ and $a_0 = 1$ to find a solution of the form $y_1(t) = \sum_{n=0}^{\infty} a_n t^{n+r} = t^r \sum_{n=0} a_n t^n = t^3 \sum_{n=0}^{\infty} a_n t^n$ with $a_0 = 1$. Plugging in $r = 3$, we require for all $n \geq 1$ that

$$((n+3)(n+2) - 3(n+3) + 3)a_n + (n+2)a_{n-1} = 0$$

and so

$$a_n = \frac{-(n+2)a_{n-1}}{(n+3)(n+2) - 3(n+3) + 3} = \frac{-(n+2)a_{n-1}}{n^2 + 5n + 6 - 3n - 9 + 3} = \frac{-(n+2)a_{n-1}}{n^2 + 2n} = \frac{-(n+2)a_{n-1}}{n(n+2)} = \frac{-a_{n-1}}{n}$$

for $n \geq 1$. Thus, we see that

$$\begin{aligned}
a_0 &= 1 \\
a_1 &= -1 \\
a_2 &= \frac{1}{2} \\
a_3 &= \frac{-1}{2 \cdot 3} \\
&\vdots \\
a_n &= \frac{(-1)^{n-1}}{2 \cdot 3 \cdots (n-1)} \frac{-1}{n} = \frac{(-1)^n}{n!} \\
&\vdots
\end{aligned}$$

so that $y_1(t) = t^3 \sum_{n=0}^{\infty} \frac{(-1)^n t^n}{n!} = t^3 \sum_{n=0}^{\infty} \frac{(-t)^n}{n!}$.

For the third point of this question, we recognize the Taylor series for e^x in the above and see that $y_1(t) = t^3 e^{-t}$.

For the fourth point of this question, we need to show that there is no solution of the form $t \sum_{n=0}^{\infty} b_n t^n$. This would correspond to choosing $r = 1$ in $\sum_{n=0}^{\infty} b_n t^{n+r} = t^r \sum_{n=0} b_n t^n = t \sum_{n=0}^{\infty} b_n t^n$. Substituting $r = 1$ into the series expansion above, we would need

$$((n+1)(n) - 3(n+1) + 3)a_n + na_{n-1} = 0$$

and so

$$a_n = \frac{-na_{n-1}}{(n+1)n - 3(n+1) + 3} = \frac{-na_{n-1}}{n^2 + n - 3n - 3 + 3} = \frac{-na_{n-1}}{n^2 - 2n} = \frac{-na_{n-1}}{n(n-2)} = \frac{-a_{n-1}}{n-2}$$

for all $n \geq 1$. Since this recursion relation becomes degenerate at $n = 2$, no such solution is possible and as such the Frobenius method only gives us a solution $y_1(t) = t^3 e^{-t}$.

Towards the fifth point of this question, we use the method of reduction of order to construct a second linearly independent solution $y_2(t)$. Proposing a solution of the form $y_2(t) = y_1(t)v(t)$, we find differentiating that $y_2' = y_1'v + y_1v'$ and so $y_2'' = y_1''v + 2y_1'v' + y_1v''$. Plugging these into the equation and grouping like terms of v, we find

$$\begin{aligned}
0 &= t^2 y_2'' + (t^2 - 3t)y_2' + 3y_2 \\
&= t^2(y_1''v + 2y_1'v' + y_1v'') + (t^2 - 3t)(y_1'v + y_1v') + 3y_1v \\
&= v(t^2y_1'' + (t^2 - 3t)y_1' + 3y_1) + v'(2t^2y_1' + (t^2 - 3t)y_1) + t^2y_1v''
\end{aligned}$$

$$
\begin{aligned}
&= v'(2t^2 y_1' + (t^2 - 3t)y_1) + t^2 y_1 v'' \\
&= w(2t^2 y_1' + (t^2 - 3t)y_1) + w'(t^2 y_1)
\end{aligned}
$$

with $w = v'$, since y_1 solves (2). Recalling $y_1(t) = t^3 e^{-t}$, $y'(t) = 3t^2 e^{-t} - t^3 e^{-t}$ and so the above first order ODE becomes

$$0 = w(6t^4 e^{-t} - 2t^5 e^{-t} + t^5 e^{-t} - 3t^4 e^{-t}) + w'(t^5 e^{-t}) = w(3t^4 e^{-t} - t^5 e^{-t}) + w'(t^5 e^{-t})$$

or

$$w' = w\left(1 - \frac{3}{t}\right)$$

for $t > 0$. This is a separable first order ODE! We have, making all of our integration constants trivial by choice,

$$
\begin{aligned}
\frac{w'}{w} &= 1 - \frac{3}{t} \\
\frac{d}{dt}\left(\ln|w|\right) &= 1 - \frac{3}{t} \\
\ln|w| &= t - 3\ln t \\
w(t) &= e^{t - 3\ln t} = \frac{e^t}{e^{3\ln t}} = \frac{e^t}{t^3}.
\end{aligned}
$$

Hence, $v(t) = \int \frac{e^s}{s^3}\, ds$, and so $y_2(t) = t^3 e^{-t} \int \frac{e^s}{s^3}\, ds = \frac{t^3}{e^t} \int \frac{e^s}{s^3}$ is a second linearly independent solution. $\square$

6. Homework 8

Exercise 6.1 (3.1.1). *Convert the differential equation*

$$\frac{d^3y}{dt^3} + \left(\frac{dy}{dt}\right)^2 = 0$$

into a system of first-order equations.

Sol. Set $x_1 = y$, $x_2 = y'$, and $x_3 = y''$. Then, we observe from the above equation that

$$\begin{aligned}\frac{dx_1}{dt} &= y' = x_2 \\ \frac{dx_2}{dt} &= y'' = x_3 \\ \frac{dx_3}{dt} &= y''' = -(y')^2 = -x_2^2,\end{aligned}$$

which is a system of first-order equations.

Exercise 6.2 (3.1.5). *1. Let $y(t)$ be a solution of the equation $y'' + y' + y = 0$. Show that*

$$\vec{x}(t) = \begin{pmatrix} y(t) \\ y'(t) \end{pmatrix}$$

is a solution of the system of equations

$$\frac{d\vec{x}}{dt} = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} \vec{x}.$$

2. Let

$$\vec{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$$

be a solution of the system of equations

$$\frac{d\vec{x}}{dt} = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} \vec{x}.$$

Show that $y = x_1(t)$ is a solution of the equation $y'' + y' + y = 0$.

Proof. Let us consider the first point first. Notice that with $x_1 = y$, $x_2 = y'$,

$$\begin{aligned}\frac{dx_1}{dt} &= y' = x_2 \\ \frac{dx_2}{dt} &= y'' = -y - y' = -x_1 - x_2.\end{aligned}$$

In matrix form, this system is exactly

$$\frac{d\vec{x}}{dt} = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} \vec{x},$$

as desired.

For the second point, suppose

$$\vec{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$$

is a solution of

$$\frac{d\vec{x}}{dt} = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} \vec{x}.$$

and set $y = x_1$. Then, we observe that $x_2 = \frac{dx_1}{dt} = y'$ and

$$\frac{dx_2}{dt} = y'' = -x_1 - x_2 = -y - y',$$

so that $y'' + y' + y = 0$. Hence, $y = x_1(t)$ solves the desired equation. $\square$

Exercise 6.3 (3.1.7)**.** *Write the system of differential equations and initial values*

$$\begin{cases} \frac{dx_1}{dt} = 5x_1 + 5x_2, & x_1(3) = 0 \\ \frac{dx_2}{dt} = -x_1 + 7x_2, & x_2(3) = 6 \end{cases}$$

in the form $\frac{d\vec{x}}{dt} = \mathbf{A}\vec{x}$*,* $\vec{x}(t_0) = \vec{x}_0$*.*

Sol. We immediately observe that we have

$$\frac{d\vec{x}}{dt} = \begin{pmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \end{pmatrix} = \begin{pmatrix} 5 & 5 \\ -1 & 7 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 5 & 5 \\ -1 & 7 \end{pmatrix} \vec{x}$$

and that our initial condition takes the form

$$\vec{x}(3) = \begin{pmatrix} x_1(3) \\ x_2(3) \end{pmatrix} = \begin{pmatrix} 0 \\ 6 \end{pmatrix}.$$

Exercise 6.4 (3.8.1)**.** *Find all solutions of the system*

$$\frac{d\vec{x}}{dt} = \begin{pmatrix} 6 & -3 \\ 2 & 1 \end{pmatrix} \vec{x}.$$

Sol. We use the eigenmethod, searching for solutions $\vec{x}(t) = e^{\lambda t}\vec{v}$ for eigenvalues λ of the matrix $\mathbf{A}$ and associated eigenvector $\vec{v}$. Observe that the characteristic equation is of the form

$$0 = \det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} 6-\lambda & -3 \\ 2 & 1-\lambda \end{vmatrix} = (6-\lambda)(1-\lambda) + 6 = \lambda^2 - 7\lambda + 12 = (\lambda - 4)(\lambda - 3),$$

with eigenvalue solutions $\lambda = 4$ and $\lambda = 3$. It remains to determine the associated linearly independent eigenvectors.

First, we examine $\lambda = 4$. This requires us to solve

$$\begin{pmatrix} 6 & -3 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 6v_1 - 3v_2 \\ 2v_1 + v_2 \end{pmatrix} = \begin{pmatrix} 4v_1 \\ 4v_2 \end{pmatrix}.$$

The second equation can be written $2v_1 = 3v_2$, or $v_2 = (2/3)v_1$. Sure enough, the first equation takes the same form. Hence, v_1 is free to choose; we pick $v_1 = 3$, so $v_2 = 2$ and our eigenvector is $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$.

Next, we examine $\lambda = 3$. This requires us to solve

$$\begin{pmatrix} 6 & -3 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 6v_1 - 3v_2 \\ 2v_1 + v_2 \end{pmatrix} = \begin{pmatrix} 3v_1 \\ 3v_2 \end{pmatrix}.$$

The second equation gives $2v_1 = 2v_2$, or $v_1 = v_2$. Sure enough, the first equation takes the same form. Hence, v_1 is free to choose; we pick $v_1 = 1$, so $v_2 = 1$ and our eigenvector is $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Thus, our general solution is

$$\vec{x} = c_1 e^{3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{4t} \begin{pmatrix} 3 \\ 2 \end{pmatrix}.$$

Exercise 6.5 (3.8.11)**.** *Solve the initial-value problem*

$$\frac{d\vec{x}}{dt} = \begin{pmatrix} 1 & -3 & 2 \\ 0 & -1 & 0 \\ 0 & -1 & -2 \end{pmatrix} \vec{x}, \quad \vec{x}(0) = \begin{pmatrix} -2 \\ 0 \\ 3 \end{pmatrix}.$$

Sol. We use the eigenmethod, searching for solutions $\vec{x}(t) = e^{\lambda t}\vec{v}$ for eigenvalues λ of the matrix $\mathbf{A}$ and associated eigenvector $\vec{v}$. Observe that the characteristic equation is of the form

$$0 = \det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} 1-\lambda & -3 & 2 \\ 0 & -1-\lambda & 0 \\ 0 & -1 & -2-\lambda \end{vmatrix} = (1-\lambda)(-1-\lambda)(-2-\lambda),$$

with solutions $\lambda = 1$, $\lambda = -1$, and $\lambda = -2$. We could find all of the solutions, but instead we notice a special similarity between the eigenvector for $\lambda = -2$ and the initial conditions. Namely, if we start trying to find an eigenvector for $\lambda = -2$, we observe

$$\vec{0} = (\mathbf{A} + 2\mathbf{I})\vec{v} = \begin{pmatrix} 3 & -3 & 2 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 3v_1 - 3v_2 + 2v_3 \\ v_2 \\ -v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

In particular, $v_2 = 0$ and $3v_1 = -2v_3$, or $v_1 = (-2/3)v_3$. Notice then that v_3 is free to choose, so we might as well set $v_3 = 3$ and find that $\begin{pmatrix} -2 \\ 0 \\ 3 \end{pmatrix} = \vec{x}(0)$ is an eigenvector for $\lambda = -2$! Hence, if we were to write out a general solution

$$\vec{x}(t) = c_1 e^{t}\vec{v}_1 + c_2 e^{-t}\vec{v}_2 + c_3 e^{-2t} \begin{pmatrix} -2 \\ 0 \\ 3 \end{pmatrix}$$

with eigenvector $\vec{v}_1$ for $\lambda = 1$ and $\vec{v}_2$ for $\lambda = -1$, we would find by setting $t = 0$ that

$$\vec{x}(0) = c_1\vec{v}_1 + c_2\vec{v}_2 + c_3 \begin{pmatrix} -2 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \\ 3 \end{pmatrix},$$

so $c_1 = c_2 = 0$ and $c_3 = 1$. Hence, we needn't bother finding other eigenvectors and can see immediately that the solution to this initial value problem is

$$\vec{x}(t) = e^{-2t} \begin{pmatrix} -2 \\ 0 \\ 3 \end{pmatrix}.$$

Exercise 6.6 (3.9.1)**.** *Find the general solution of the system*

$$\frac{d\vec{x}}{dt} = \begin{pmatrix} -3 & 2 \\ -1 & -1 \end{pmatrix} \vec{x}.$$

Sol. We use the eigenmethod, searching for solutions $\vec{x}(t) = e^{\lambda t}\vec{v}$ for eigenvalues λ of the matrix $\mathbf{A}$ and associated eigenvector $\vec{v}$. Observe that the characteristic equation is of the form

$$0 = \det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} -3-\lambda & 2 \\ -1 & -1-\lambda \end{vmatrix} = (3+\lambda)(1+\lambda) + 2 = \lambda^2 + 4\lambda + 5,$$

with solutions $\lambda = \frac{-4 \pm \sqrt{-4}}{2} = -2 \pm i$. These are complex eigenvalues, so we will choose one and find the real and imaginary parts of $e^{\lambda t}\vec{v}$. Choosing $\lambda = -2 + i$, we seek a vector $\vec{v}$ satisfying

$$\begin{pmatrix} -3 & 2 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} -3v_1 + 2v_2 \\ -v_1 - v_2 \end{pmatrix} = \begin{pmatrix} (-2+i)v_1 \\ (-2+i)v_2 \end{pmatrix}.$$

The second equation yields $v_1 = (1-i)v_2$. Similarly, the first equation gives $-3v_1 + 2v_2 = -2v_1 + iv_1$ or $(1+i)v_1 = 2v_2$. In particular, $v_1 = \frac{2}{1+i}v_2 = \frac{2(1-i)}{1+1}v_2 = (1-i)v_2$, which is the same equation we had. Hence, v_2 is free to choose. Choosing $v_2 = 1$, our eigenvector is $\vec{v} = \begin{pmatrix} 1-i \\ 1 \end{pmatrix}$, and we see that

$$\begin{aligned} e^{\lambda t}\vec{v} &= e^{(-2+i)t}\begin{pmatrix} 1-i \\ 1 \end{pmatrix} \\ &= e^{-2t}(\cos(t) + i\sin(t))\begin{pmatrix} 1-i \\ 1 \end{pmatrix} \\ &= e^{-2t}\begin{pmatrix} \cos(t) - i\cos(t) + i\sin(t) + \sin(t) \\ \cos(t) + i\sin(t) \end{pmatrix}. \end{aligned}$$

Taking real and imaginary parts, we find that our general solution is of the form

$$\vec{x}(t) = e^{-2t}\left(c_1\begin{pmatrix} \cos(t) + \sin(t) \\ \cos(t) \end{pmatrix} + c_2\begin{pmatrix} \sin(t) - \cos(t) \\ \sin(t) \end{pmatrix}\right).$$

Exercise 6.7 (3.9.7)**.** *Solve the initial value problem*

$$\frac{d\vec{x}}{dt} = \begin{pmatrix} -3 & 0 & 2 \\ 1 & -1 & 0 \\ -2 & -1 & 0 \end{pmatrix}\vec{x}, \quad \vec{x}(0) = \begin{pmatrix} 0 \\ -1 \\ -2 \end{pmatrix}.$$

Sol. This is...messy. But at least the ideas are straightforward. We again use the eigenmethod, searching for solutions $\vec{x}(t) = e^{\lambda t}\vec{v}$ for eigenvalues λ of the matrix $\mathbf{A}$ and associated eigenvector $\vec{v}$. Observe that the characteristic equation is of the form

$$\begin{aligned} 0 = \det(\mathbf{A} - \lambda\mathbf{I}) &= \begin{vmatrix} -3-\lambda & 0 & 2 \\ 1 & -1-\lambda & 0 \\ -2 & -1 & -\lambda \end{vmatrix} \\ &= (-3-\lambda)(-\lambda(-1-\lambda)) + 2(-1 + 2(-1-\lambda)) \\ &= -\lambda(\lambda^2 + 4\lambda + 3) - 6 - 4\lambda \\ &= -\lambda^3 - 4\lambda^2 - 7\lambda - 6 \\ &= -(\lambda+2)(\lambda^2 + 2\lambda + 3) \end{aligned}$$

with solutions $\lambda = -2$ and $\lambda = \frac{-2\pm\sqrt{-8}}{2} = -1 \pm i\sqrt{2}$.

We first find an eigenvector for $\lambda = -2$. In order to do this, we solve

$$0 = (\mathbf{A} - \lambda\mathbf{I})\vec{v} = \begin{pmatrix} -1 & 0 & 2 \\ 1 & 1 & 0 \\ -2 & -1 & 2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} -v_1 + 2v_3 \\ v_1 + v_2 \\ -2v_1 - v_2 + 2v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

We observe that $v_1 = -v_2$ and $v_3 = \frac{1}{2}v_1$. The third equation vanishes with these substitutions, so v_2 is free to choose Choosing $v_2 = -2$ yields the eigenvector $\vec{v} = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$.

Next we consider the complex eigenvalues. We will choose one and find the real and imaginary parts of $e^{\lambda t}\vec{v}$. Choosing $\lambda = -1 + i\sqrt{2}$, we see that we need to solve

$$\begin{pmatrix} -3 & 0 & 2 \\ 1 & -1 & 0 \\ -2 & -1 & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} -3v_1 + 2v_3 \\ v_1 - v_2 \\ -2v_1 - v_2 \end{pmatrix} = \begin{pmatrix} (-1+i\sqrt{2})v_1 \\ (-1+i\sqrt{2})v_2 \\ (-1+i\sqrt{2})v_3 \end{pmatrix}$$

The second equation yields $v_1 = i\sqrt{2}v_2$. Then, $2v_3 = (2+i\sqrt{2})v_1 = (2+i\sqrt{2})(i\sqrt{2})v_2 = (-2+2i\sqrt{2})v_2$, so $v_3 = (-1+i\sqrt{2})v_2$. With these substitutions the third equation vanishes, so v_2 is free to choose, Choosing $v_2 = 1$, we obtain the eigenvector $\vec{v} = \begin{pmatrix} i\sqrt{2} \\ 1 \\ -1+i\sqrt{2} \end{pmatrix}$. Hence,

$$\begin{aligned} e^{\lambda t}\vec{v} &= e^{(-1+i\sqrt{2})t}\begin{pmatrix} i\sqrt{2} \\ 1 \\ -1+i\sqrt{2} \end{pmatrix} \\ &= e^{-t}(\cos(\sqrt{2}t) + i\sin(\sqrt{2}t))\begin{pmatrix} i\sqrt{2} \\ 1 \\ -1+i\sqrt{2} \end{pmatrix} \\ &= e^{-t}\begin{pmatrix} -\sqrt{2}\sin(\sqrt{2}t) + i\sqrt{2}\cos(\sqrt{2}t) \\ \cos(\sqrt{2}t) + i\sin(\sqrt{2}t) \\ -\cos(\sqrt{2}t) - \sqrt{2}\sin(\sqrt{2}t) - i\sin(\sqrt{2}t) + i\sqrt{2}\cos(\sqrt{2}t) \end{pmatrix}. \end{aligned}$$

Taking real and imaginary parts, it follows that the general solution is given by

$$\vec{x}(t) = c_1 e^{-2t}\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} + e^{-t}\left(c_2 \begin{pmatrix} -\sqrt{2}\sin(\sqrt{2}t) \\ \cos(\sqrt{2}t) \\ -\cos(\sqrt{2}t) - \sqrt{2}\sin(\sqrt{2}t) \end{pmatrix} + c_3 \begin{pmatrix} \sqrt{2}\cos(\sqrt{2}t) \\ \sin(\sqrt{2}t) \\ -\sin(\sqrt{2}t) + \sqrt{2}\cos(\sqrt{2}t) \end{pmatrix} \right).$$

Setting $t = 0$,

$$\begin{pmatrix} 0 \\ -1 \\ -2 \end{pmatrix} = \vec{x}(0) = c_1\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} + \left(c_2\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} + c_3\begin{pmatrix} \sqrt{2} \\ 0 \\ \sqrt{2} \end{pmatrix} \right).$$

So, we have $2c_1 + \sqrt{2}c_3 = 0$, $-2c_1 + c_2 = -1$ and $c_1 - c_2 + \sqrt{2}c_3 = -2$. We see that $\sqrt{2}c_3 = -2c_1$ and $-c_2 = 1 - 2c_1$, so the third equation yields $c_1 =$. By back substitution, we find $c_2 = 1$ and $c_3 = -\sqrt{2}$ so

$$\begin{aligned} \vec{x}(t) &= e^{-2t}\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} + e^{-t}\left(\begin{pmatrix} -\sqrt{2}\sin(\sqrt{2}t) \\ \cos(\sqrt{2}t) \\ -\cos(\sqrt{2}t) - \sqrt{2}\sin(\sqrt{2}t) \end{pmatrix} - \sqrt{2}\begin{pmatrix} \sqrt{2}\cos(\sqrt{2}t) \\ \sin(\sqrt{2}t) \\ -\sin(\sqrt{2}t) + \sqrt{2}\cos(\sqrt{2}t) \end{pmatrix} \right) \\ &= e^{-2t}\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} + e^{-t}\begin{pmatrix} -2\cos(\sqrt{2}t) - \sqrt{2}\sin(\sqrt{2}t) \\ \cos(\sqrt{2}t) - \sqrt{2}\sin(\sqrt{2}t) \\ -3\cos(\sqrt{2}t) \end{pmatrix}. \end{aligned}$$

Exercise 6.8 (3.9.9). *Determine all vectors $\vec{x}_0$ such that the solution of the initial-value problem*

$$\frac{d\vec{x}}{dt} = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 1 & -1 & -1 \end{pmatrix}\vec{x}, \quad \vec{x}(0) = \vec{x}_0$$

is a periodic function of time.

Sol. Via the eigenmethod, periodic contributions to the solution only come from complex eigenvalues; the rest yield either exponential growth or decay. Thus, we want initial conditions that are in the span of the eigenvectors we obtain from the complex eigenvalues. So, we determine the eigenvalues! These solve the characteristic equation

$$0 = \det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} 1-\lambda & 0 & -2 \\ 0 & 1-\lambda & 0 \\ 1 & -1 & -1-\lambda \end{vmatrix} = (1-\lambda)(1-\lambda)(-1-\lambda) + 2(1-\lambda) = (1-\lambda)(\lambda^2 - 1 + 2) = (1-\lambda)(\lambda^2+1),$$

which has solutions $\lambda = 1$ and $\lambda = \pm i$. Since we only care about the span of eigenvectors coming from the complex eigenvalues, we won't compute the eigenvectors associated to $\lambda = 1$ and just refer to them as $c_1\vec{v}_1$.

For the complex eigenvalues, we choose $\lambda = i$ and will compute the real and imaginary parts of $e^{\lambda t}\vec{v}$, for associated eigenvector $\vec{v}$. This vector solves

$$\begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 1 & -1 & -1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} v_1 - 2v_3 \\ v_2 \\ v_1 - v_2 - v_3 \end{pmatrix} = \begin{pmatrix} iv_1 \\ iv_2 \\ iv_3 \end{pmatrix}.$$

It follows that $v_2 = 0$ and $v_1 = (1+i)v_3$. With these substitutions, the first equation vanishes and we see that v_3 is free to choose. Choosing $v_3 = 1$ yields the eigenvector $\begin{pmatrix} 1+i \\ 0 \\ 1 \end{pmatrix}$, and so

$$\begin{aligned} e^{\lambda t}\vec{v} &= e^{it} \begin{pmatrix} 1+i \\ 0 \\ 1 \end{pmatrix} \\ &= (\cos(t) + i\sin(t)) \begin{pmatrix} 1+i \\ 0 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} \cos(t) - \sin(t) + i\cos(t) + i\sin(t) \\ 0 \\ \cos(t) + i\sin(t) \end{pmatrix}. \end{aligned}$$

Taking real and imaginary parts, we find that the general solution is of the form

$$\vec{x}(t) = c_1 e^t \vec{v}_1 + c_2 \begin{pmatrix} \cos(t) - \sin(t) \\ 0 \\ \cos(t) \end{pmatrix} + c_3 \begin{pmatrix} \cos(t) + \sin(t) \\ 0 \\ \sin(t) \end{pmatrix}.$$

Any solution with $c_1 = 0$ will be periodic! When is this the case? Observe that

$$\vec{x}(0) = \vec{x}_0 = c_1 \vec{v}_1 + c_2 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + c_3 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},$$

so $c_1 = 0$ precisely when $\vec{x}_0$ is in the linear span of $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$. Since $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, the linear span is equivalently all vectors of the form $a \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + b \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. Hence, any initial condition of the form $\vec{x}_0 = \begin{pmatrix} a \\ 0 \\ b \end{pmatrix}$ for constants a and b will yield periodic solutions.