Lecture 12

Given the adjacency matrix of G, How do we find a spanning tree?

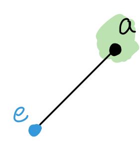
Idea Breadth first algorithm:

Matrix | Tree | Step

$$\begin{array}{c|ccccccccc} & a & b & c & d & e & f & g & h & i \\ \hline a & \bullet & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ b & 0 & \bullet & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ c & 0 & 1 & \bullet & 1 & 0 & 0 & 0 & 0 & 0 \\ d & 0 & 0 & 1 & \bullet & 0 & 1 & 1 & 0 & 1 \\ e & 1 & 0 & 0 & 0 & \bullet & 0 & 0 & 1 & 0 \\ f & 1 & 1 & 0 & 1 & 0 & \bullet & 0 & 0 & 0 \\ g & 1 & 0 & 0 & 1 & 0 & 0 & \bullet & 0 & 1 \\ h & 0 & 1 & 0 & 0 & 1 & 0 & 0 & \bullet & 0 \\ i & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & \bullet \end{array}$$

(1) Start w/ a vertex labeled a and cross out Column a of the matrix

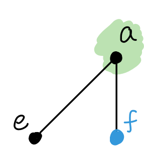
$$\begin{array}{c|ccccccccc} & a & b & c & d & e & f & g & h & i \\ \hline a & \bullet & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ b & 0 & \bullet & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ c & 0 & 1 & \bullet & 1 & 0 & 0 & 0 & 0 & 0 \\ d & 0 & 0 & 1 & \bullet & 0 & 1 & 1 & 0 & 1 \\ e & 1 & 0 & 0 & 0 & \bullet & 0 & 0 & 1 & 0 \\ f & 1 & 1 & 0 & 1 & 0 & \bullet & 0 & 0 & 0 \\ g & 1 & 0 & 0 & 1 & 0 & 0 & \bullet & 0 & 1 \\ h & 0 & 1 & 0 & 0 & 1 & 0 & 0 & \bullet & 0 \\ i & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & \bullet \end{array}$$



(2) Go left to right along row a, stop at the first 1 which is NOT crossed out.

Add that vertex to the tree. Connect it to the vertex of the row you are in.

$$\begin{array}{c|ccccccccc} & a & b & c & d & e & f & g & h & i \\ \hline a & \bullet & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ b & 0 & \bullet & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ c & 0 & 1 & \bullet & 1 & 0 & 0 & 0 & 0 & 0 \\ d & 0 & 0 & 1 & \bullet & 0 & 1 & 1 & 0 & 1 \\ e & 1 & 0 & 0 & 0 & \bullet & 0 & 0 & 1 & 0 \\ f & 1 & 1 & 0 & 1 & 0 & \bullet & 0 & 0 & 0 \\ g & 1 & 0 & 0 & 1 & 0 & 0 & \bullet & 0 & 1 \\ h & 0 & 1 & 0 & 0 & 1 & 0 & 0 & \bullet & 0 \\ i & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & \bullet \end{array}$$

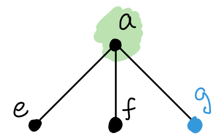


(3) Cross out the column of the vertex you just added

Then keep going left until you hit a 1 (NOT crossed out) OR the end of the row.

If you hit a 1, add that vertex to the tree.

$$\begin{array}{c|ccccccccc} & a & b & c & d & e & f & g & h & i \\ \hline a & \bullet & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ b & 0 & \bullet & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ c & 0 & 1 & \bullet & 1 & 0 & 0 & 0 & 0 & 0 \\ d & 0 & 0 & 1 & \bullet & 0 & 1 & 1 & 0 & 1 \\ e & 1 & 0 & 0 & 0 & \bullet & 0 & 0 & 1 & 0 \\ f & 1 & 1 & 0 & 1 & 0 & \bullet & 0 & 0 & 0 \\ g & 1 & 0 & 0 & 1 & 0 & 0 & \bullet & 0 & 1 \\ h & 0 & 1 & 0 & 0 & 1 & 0 & 0 & \bullet & 0 \\ i & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & \bullet \end{array}$$

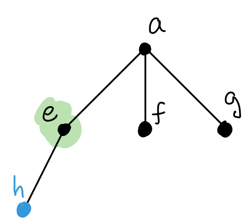


(4) Cross out the column of the vertex you just added (f)

Then proceed to the next available 1 OR the end of the row.

If you hit a 1, add that vertex to the tree (g)

$$\begin{array}{c|ccccccccc} & a & b & c & d & e & f & g & h & i \\ \hline a & \bullet & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ b & 0 & \bullet & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ c & 0 & 1 & \bullet & 1 & 0 & 0 & 0 & 0 & 0 \\ d & 0 & 0 & 1 & \bullet & 0 & 1 & 1 & 0 & 1 \\ e & 1 & 0 & 0 & 0 & \bullet & 0 & 0 & 1 & 0 \\ f & 1 & 1 & 0 & 1 & 0 & \bullet & 0 & 0 & 0 \\ g & 1 & 0 & 0 & 1 & 0 & 0 & \bullet & 0 & 1 \\ h & 0 & 1 & 0 & 0 & 1 & 0 & 0 & \bullet & 0 \\ i & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & \bullet \end{array}$$

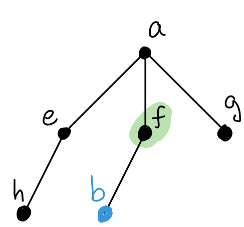


(5) Cross out the col. of the vertex added in the previous step (g) then proceed.

When you hit the end of the row start on the row of the next leaf (e).

When you hit an available 1 connect that vertex to the vertex whose row you are in (h to e)

$$\begin{array}{c|ccccccccc} & a & b & c & d & e & f & g & h & i \\ \hline a & \bullet & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ b & 0 & \bullet & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ c & 0 & 1 & \bullet & 1 & 0 & 0 & 0 & 0 & 0 \\ d & 0 & 0 & 1 & \bullet & 0 & 1 & 1 & 0 & 1 \\ e & 1 & 0 & 0 & 0 & \bullet & 0 & 0 & 1 & 0 \\ f & 1 & 1 & 0 & 1 & 0 & \bullet & 0 & 0 & 0 \\ g & 1 & 0 & 0 & 1 & 0 & 0 & \bullet & 0 & 1 \\ h & 0 & 1 & 0 & 0 & 1 & 0 & 0 & \bullet & 0 \\ i & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & \bullet \end{array}$$

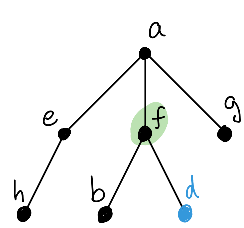


(6) Repeat as before:

Add a vertex when you hit an available 1, Then immediately cross out that column.

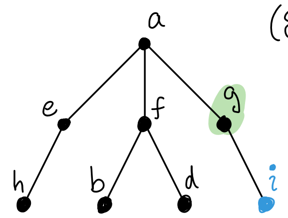
Proceed left to right along the layers of the tree.
Where you are in the tree tells you which row to travel along.

$$\begin{array}{c|ccccccccc} & a & b & c & d & e & f & g & h & i \\ \hline a & \bullet & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ b & 0 & \bullet & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ c & 0 & 1 & \bullet & 1 & 0 & 0 & 0 & 0 & 0 \\ d & 0 & 0 & 1 & \bullet & 0 & 1 & 1 & 0 & 1 \\ e & 1 & 0 & 0 & 0 & \bullet & 0 & 0 & 1 & 0 \\ f & 1 & 1 & 0 & 1 & 0 & \bullet & 0 & 0 & 0 \\ g & 1 & 0 & 0 & 1 & 0 & 0 & \bullet & 0 & 1 \\ h & 0 & 1 & 0 & 0 & 1 & 0 & 0 & \bullet & 0 \\ i & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & \bullet \end{array}$$



(7) Keep going!

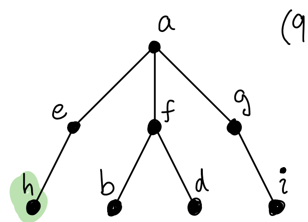
$$\begin{array}{c|ccccccccc} & a & b & c & d & e & f & g & h & i \\ \hline a & \bullet & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ b & 0 & \bullet & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ c & 0 & 1 & \bullet & 1 & 0 & 0 & 0 & 0 & 0 \\ d & 0 & 0 & 1 & \bullet & 0 & 1 & 1 & 0 & 1 \\ e & 1 & 0 & 0 & 0 & \bullet & 0 & 0 & 1 & 0 \\ f & 1 & 1 & 0 & 1 & 0 & \bullet & 0 & 0 & 0 \\ g & 1 & 0 & 0 & 1 & 0 & 0 & \bullet & 0 & 1 \\ h & 0 & 1 & 0 & 0 & 1 & 0 & 0 & \bullet & 0 \\ i & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & \bullet \end{array}$$



(8) The green vertex of the tree tells you what row to look at.

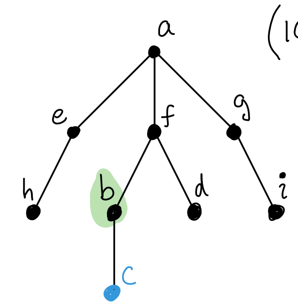
The vertex moves across each layer of the tree Then down to the next one.

$$\begin{array}{c|ccccccccc} & a & b & c & d & e & f & g & h & i \\ \hline a & \bullet & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ b & 0 & \bullet & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ c & 0 & 1 & \bullet & 1 & 0 & 0 & 0 & 0 & 0 \\ d & 0 & 0 & 1 & \bullet & 0 & 1 & 1 & 0 & 1 \\ e & 1 & 0 & 0 & 0 & \bullet & 0 & 0 & 1 & 0 \\ f & 1 & 1 & 0 & 1 & 0 & \bullet & 0 & 0 & 0 \\ g & 1 & 0 & 0 & 1 & 0 & 0 & \bullet & 0 & 1 \\ h & 0 & 1 & 0 & 0 & 1 & 0 & 0 & \bullet & 0 \\ i & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & \bullet \end{array}$$



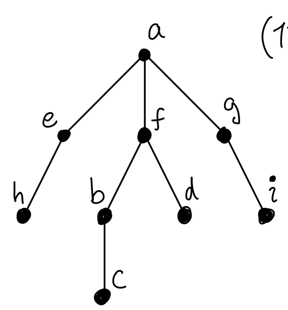
(9) Since there are no available 1's in Row h we don't add any thing to the tree.

$$\begin{array}{c|ccccccccc} & a & b & c & d & e & f & g & h & i \\ \hline a & \bullet & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ b & 0 & \bullet & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ c & 0 & 1 & \bullet & 1 & 0 & 0 & 0 & 0 & 0 \\ d & 0 & 0 & 1 & \bullet & 0 & 1 & 1 & 0 & 1 \\ e & 1 & 0 & 0 & 0 & \bullet & 0 & 0 & 1 & 0 \\ f & 1 & 1 & 0 & 1 & 0 & \bullet & 0 & 0 & 0 \\ g & 1 & 0 & 0 & 1 & 0 & 0 & \bullet & 0 & 1 \\ h & 0 & 1 & 0 & 0 & 1 & 0 & 0 & \bullet & 0 \\ i & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & \bullet \end{array}$$



(10) Finally we hit the last remaining column!

$$\begin{array}{c|ccccccccc} & a & b & c & d & e & f & g & h & i \\ \hline a & \bullet & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ b & 0 & \bullet & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ c & 0 & 1 & \bullet & 1 & 0 & 0 & 0 & 0 & 0 \\ d & 0 & 0 & 1 & \bullet & 0 & 1 & 1 & 0 & 1 \\ e & 1 & 0 & 0 & 0 & \bullet & 0 & 0 & 1 & 0 \\ f & 1 & 1 & 0 & 1 & 0 & \bullet & 0 & 0 & 0 \\ g & 1 & 0 & 0 & 1 & 0 & 0 & \bullet & 0 & 1 \\ h & 0 & 1 & 0 & 0 & 1 & 0 & 0 & \bullet & 0 \\ i & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & \bullet \end{array}$$



(11) Now all the columns are crossed out, so we are done

This is a <u>Spanning Tree</u> for the graph!

Minimal Cost Hamiltonian Circuits

Def: An edge labeled (Weighted OR) graph is one with a fn $\ell : E \to \mathbb{Z}$ (labels to edges).

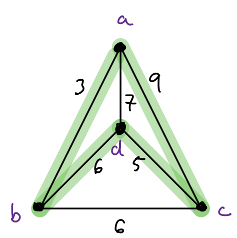
Very Real: Vertices = States ; Edge Labels = Cost.

Problem [Traveling Salesman]

Let $f : E_{K_n} \to \mathbb{Z}$ be a cost labeling, find an H.C. which minimizes the sum of the labels of the edges.

Idea: Weighted Adj Matrix:

K_4



$\rightsquigarrow$

$a \to b$

$$\begin{array}{c|cccc} & a & b & c & d \\ \hline a & \bullet & 3 & 9 & 7 \\ b & 3 & \bullet & 6 & 5 \\ c & 9 & 6 & \bullet & 6 \\ d & 7 & 5 & 6 & \bullet \end{array}$$

$c \to b$

$$\begin{array}{c|cccc} & a & b & c & d \\ \hline a & \bullet & 3 & 9 & 7 \\ b & 3 & \bullet & 6 & 5 \\ c & 9 & 6 & \bullet & 6 \\ d & 7 & 5 & 6 & \bullet \end{array}$$

Bounce off dots

(Works w/ Normal Adj Matrix too!)

A Ham Circuit looks like one entry in each row / col / certain diags
↓
Avoid subcircuits.

Method: "Branch and Bound."

Use a dot in adj to show ∞ cost (can't use that edge)

Key Ideas:

* We must use an entry from each row/col so we subtract the lowest entry from each row/col.

* This doesn't change the MCHC but the sum of what we subtract gives us a lower bound on the original. "LB"

Alg:

1. Get initial LB.
2. Pick 0 somewhere. (c_{ij}).
3. consider w/ c_{ij}: Remove row i, col j and set $c_{ji} = \infty$

   w/o c_{ij}: set $c_{ij} = \infty$

④ Recompute L.B.'s then proceed along lowest branch.

⑤ Continue until HC is found. "• means an entry of ∞".

Eg.

$$C = \begin{array}{c|cccc} & a & b & c & d \\ \hline a & \bullet & 3 & 9 & 7 \\ b & 3 & \bullet & 6 & 5 \\ c & 9 & 6 & \bullet & 6 \\ d & 7 & 5 & 6 & \bullet \end{array}$$

$$\begin{array}{c|cccc|c} & a & b & c & d & \\ \hline a & \bullet & 3 & 9 & 7 & -3 \\ b & 3 & \bullet & 6 & 5 & -3 \\ c & 9 & 6 & \bullet & 6 & -6 \\ d & 7 & 5 & 6 & \bullet & -5 \end{array} = \underset{-1}{\boxed{\begin{matrix} \bullet & 0 & 6 & 4 \\ 0 & \bullet & 3 & 2 \\ 3 & 0 & \bullet & 0 \\ 2 & 0 & 1 & \bullet \end{matrix}}} \quad \underline{17}$$

$$= \underset{18}{\boxed{\begin{matrix} \bullet & 0 & 5 & 4 \\ 0 & \bullet & 2 & 2 \\ 3 & 0 & \bullet & 0 \\ 2 & 0 & 0 & \bullet \end{matrix}}} \xrightarrow{w\ ab} \underset{18}{\boxed{\begin{matrix} \bullet & 0 & 5 & 4 \\ \cancel{0}^{\infty} & \bullet & 2 & 2 \\ 3 & 0 & \bullet & 0 \\ 2 & 0 & 0 & \bullet \end{matrix}}} = \underset{-2}{\boxed{\begin{matrix} \bullet & 2 & 2 \\ 3 & \bullet & 0 \\ 2 & 0 & \bullet \end{matrix}}} \; -2 = \begin{array}{c|ccc} & a & c & d \\ \hline b & \bullet & 0 & 0 \\ c & 1 & \bullet & 0 \\ d & 0 & 0 & \bullet \end{array}$$

$18 + 4 = 20$

Tour: ab, bc, cd, da. $L = \underline{21}$

w/o ab

$$\boxed{\begin{matrix} \bullet & \infty & 5 & 4 \\ 0 & \bullet & 2 & 2 \\ 3 & 0 & \bullet & 0 \\ 2 & 0 & 0 & \bullet \end{matrix}} \; -4 = \underset{20}{\boxed{\begin{matrix} \bullet & \bullet & 1 & 0 \\ 0 & \bullet & 2 & 2 \\ 3 & 0 & \bullet & 0 \\ 2 & 0 & 0 & \bullet \end{matrix}}}$$

Thus Min Cost HC. $\rightsquigarrow$ 20.

Tour: ad, dc, cb, ba. $L = \underline{21}$

Eg: (Note: cost is asymmetric here).

$$\boxed{\begin{matrix} \bullet & 3 & 9 & 7 \\ 3 & \bullet & 6 & 5 \\ 5 & 6 & \bullet & 6 \\ 9 & 7 & 4 & \bullet \end{matrix}} \begin{matrix} -3 \\ -3 \\ -5 \\ -4 \\ \hline 15 \end{matrix} \rightarrow \underset{-1}{\boxed{\begin{matrix} \bullet & 0 & 6 & 4 \\ 0 & \bullet & 3 & 2 \\ 0 & 1 & \bullet & 1 \\ 5 & 2 & 0 & \bullet \end{matrix}}} \rightarrow \underset{16}{\boxed{\begin{matrix} \bullet & 0 & 6 & 3 \\ 0 & \bullet & 3 & 1 \\ 0 & 1 & \bullet & 0 \\ 5 & 2 & 0 & \bullet \end{matrix}}} \rightarrow \cdots$$

$$\cdots \rightarrow \underset{16}{\boxed{\begin{matrix} \bullet & 0 & 6 & 3 \\ 0 & \bullet & 3 & 1 \\ 0 & 1 & \bullet & 0 \\ 5 & 2 & 0 & \bullet \end{matrix}}} \xrightarrow{w\ c_{12}} \underset{16}{\boxed{\begin{matrix} \bullet & 0 & 6 & 3 \\ \cancel{0}^{\infty} & \bullet & 3 & 1 \\ 0 & 1 & \bullet & 0 \\ 5 & 2 & 0 & \bullet \end{matrix}}} \rightarrow \underset{16}{\boxed{\begin{matrix} \bullet & 3 & 1 \\ 0 & \bullet & 0 \\ 5 & 0 & \bullet \end{matrix}}} \; -1 \rightarrow \underset{17}{\begin{array}{ccc|c} 1 & 3 & 4 & \\ \hline \bullet & 2 & 0 & 2 \\ 0 & \bullet & 0 & 3 \\ 5 & 0 & \bullet & 4 \end{array}}$$

Tour: 12, 24, 43, 31

$L = 3 + 5 + 4 + 5 = 17$ ☺

w/o 12

$$\underset{-1 \quad 16}{\boxed{\begin{matrix} \bullet & \bullet & 6 & 3 \\ 0 & \bullet & 3 & 1 \\ 0 & 1 & \bullet & 0 \\ 5 & 2 & 0 & \bullet \end{matrix}}} \; -3 \rightarrow \underset{20}{\boxed{\begin{matrix} \bullet & \bullet & 3 & 0 \\ 0 & \bullet & 3 & 1 \\ 0 & 0 & \bullet & 0 \\ 5 & 1 & 0 & \bullet \end{matrix}}}$$

Too Big!