Final

Chapter 4 Qualitative theory of differential equations

4.1 Introduction

$$\begin{cases} \frac{dx}{dt} = f_1(t,x,y) \\ \frac{dy}{dt} = f_2(t,x,y) \end{cases}$$

equilibrium point → $f_1(t,x,y) = f_2(t,x,y) = 0$

4.2 Stability of linear systems

Consider the real parts of eigenvalues

$\forall j, Re(\lambda_j) < 0$ → asymptotically stable

$\exists j, Re(\lambda_j) > 0$ → unstable

$Re(\lambda_j) = 0$

n linearly independent v_j , or we can solve v_j from only $(A - \lambda I)v = 0$ → stable

We need to solve v_j from $(A - \lambda I)v = 0$ and $(A - \lambda I)v^2 = 0$ → unstable

4.3 Stability of equilibrium solutions

First solve for equilibrium point → $f_1(t,x,y) = f_2(t,x,y) = 0$

Then find Jacobian matrix

$$\mathbf{F}(\vec{x}) = \begin{pmatrix} f_1(x,y) \\ f_2(x,y) \end{pmatrix} = \begin{pmatrix} \cos y - \sin x - 1 \\ x - y - y^2 \end{pmatrix}.$$

Taylor expanding the solution about $(0,0)$, we have with $\vec{z} = \vec{x} - (0,0)$ that

$$\begin{aligned} \frac{d}{dt}\vec{z} = \frac{d}{dt}\vec{x} &= \mathbf{F}\begin{pmatrix} 0 \\ 0 \end{pmatrix} + \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix}\Big|_{(0,0)}\vec{z} + \mathbf{g}(\vec{z}) \\ &= \begin{pmatrix} -\cos x & -\sin y \\ 1 & -1-2y \end{pmatrix}\Big|_{(0,0)}\vec{z} + \mathbf{g}(\vec{z}) \\ &= \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix}\vec{z} + \mathbf{g}(\vec{z}). \end{aligned}$$

Finally determine stability

4.4 The phase-plane

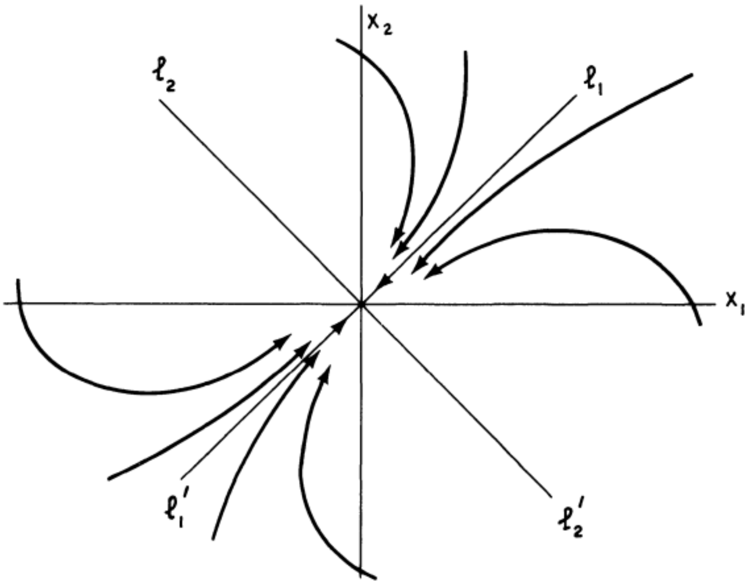
$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}}$

$\frac{dy}{dx} = \frac{-x}{y}$ → circle

$\frac{dy}{dx} = \frac{-M}{N}$ → $M + N\frac{dy}{dx} = 0, \phi = \int Mdx + f_1(y) = \int Ndy + f_2(x)$

4.7 Phase portraits of linear system

$\lambda_1 < \lambda_2 < 0$

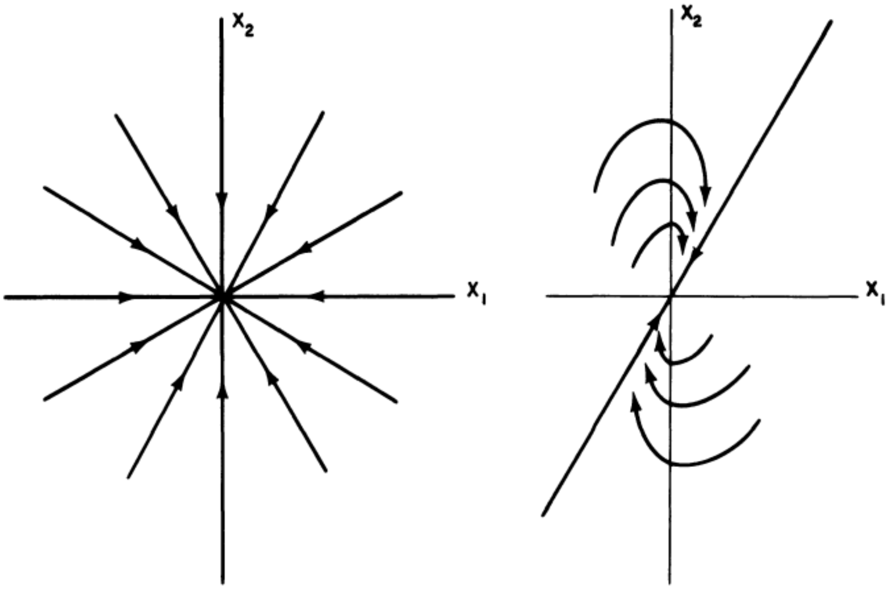


$0 < \lambda_1 < \lambda_2$

same as above, reversed the direction

$\lambda_1 = \lambda_2 < 0$

two linearly independent eigenvectors   v.s.   one linearly independent eigenvectors

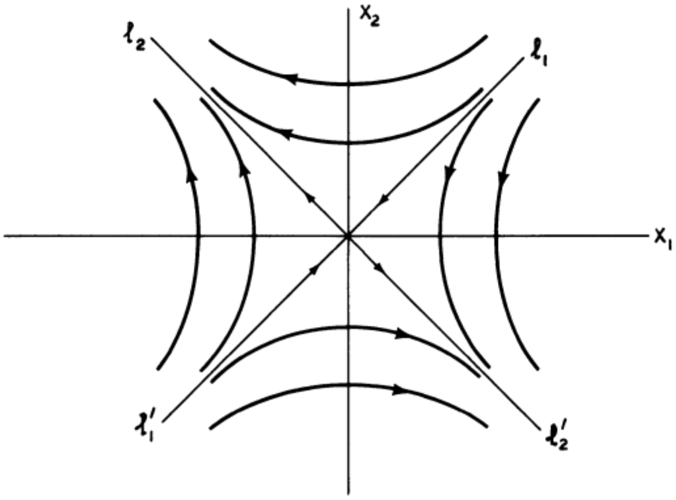


(a)   (b)

$\lambda_1 = \lambda_2 > 0$

same as above, reversed the direction

$\lambda_1 < 0 < \lambda_2$



$\lambda_1 = \alpha + i\beta, \lambda_2 = \alpha - i\beta$

$\alpha = 0$ v.s. $\alpha < 0$ v.s. $\alpha > 0$

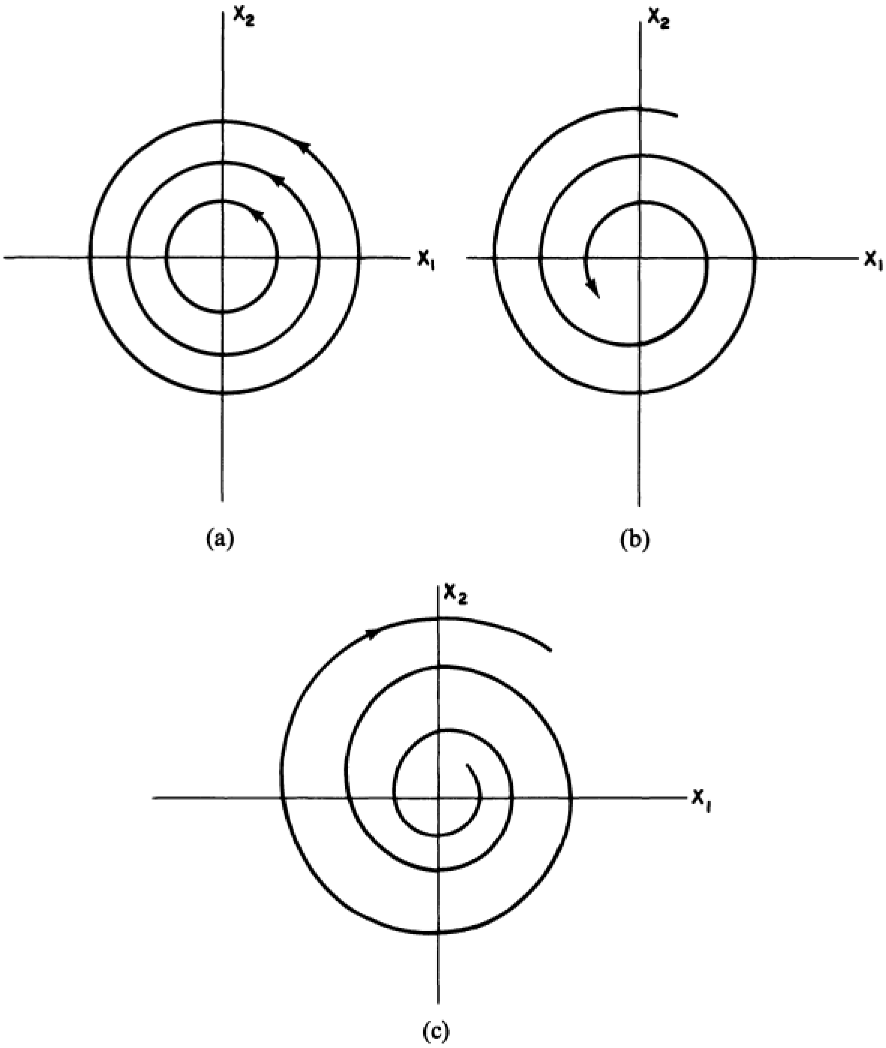


Figure 6. (a) $\alpha = 0$; (b) $\alpha < 0$; (c) $\alpha > 0$

Chapter 5 Separation of variables and Fourier series

5.1 Two point boundary-value problems

$\frac{d^2y}{dx^2} + \lambda y = 0$, $y(0) = 0$, $y(l) = 0$

nontrivial y → $\lambda = \frac{n^2\pi^2}{l^2}, n = 1, 2, ...; y(x) = c\sin\frac{n\pi x}{l}$

Case 1: $\lambda = 0$ → $y = c_1 x + c_2$

Case 2: $\lambda < 0$ → $y = c_1 e^{\sqrt{-\pi}x} + c_2 e^{-\sqrt{-\pi}x}$

Case 3: $\lambda > 0$ → $y = c_1\cos(\sqrt{\lambda}x) + c_2\sin(\sqrt{\lambda}x)$