Today: $\boxed{\text{Ken}}$

$\boxed{7.3}$ Composition of Functions

$\boxed{9.4}$ The Pigeonhole Principle

Last time:

$\boxed{7.3}$ Composition of Functions

$\boxed{9.4}$ The Pigeonhole Principle

#16 How many integers from 1 through 100 must you pick in order to be sure of getting one that is divisible by 5?

The subset of integers divisible by 5 is

$A := \{5, 10, 15, 20, \ldots, 95, 100\}$,

of the total collection

$\mathcal{U} = \{1, 2, 3, \ldots, 99, 100\}$.

$N(A) = 20$

How many choices of integers can we make (maximally) from $\mathcal{U}$ such that none are divisible by 5? Answer: $N(A^c) = 80$

i.e. $N(\mathcal{U} - A) = 80$. So we must choose $N(\mathcal{U}) - N(A) + 1 = 81$

integers from $\mathcal{U}$ to guarantee at least one is divisible by 5.

#24 Show that within any set of thirteen integers chosen from 2 through 40, there are at least two integers with a common divisor greater than 1.

$A := \{2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37\}$

$B := \{x_{13}\}$ $\quad N(A \cup B) = 13$

$\{2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37\} \cup \{x_{13}\}$

Since there are 12 prime numbers from 2 through 40, all of which are relatively prime, the subsequent (or thirteenth, i.e. pigeon) choice of an integer will share a factor with at least one of those twelve primes, at worst.

#24 $\{2, 3, 4, \ldots, 39, 40\}$

$X = \{x_1, x_2, x_3, \ldots, x_{12}, x_{13}\}$

$Y = \{2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37\}$

$N(X) = 13$ & $N(Y) = 12$

Let $f: X \to Y$ such that $f(x_i)$ is the smallest prime that divides x_i.

$f(x_i) = f(x_j)$ for some $i \neq j$ since f is not injective. So there are at least two integers with a common divisor greater than 1.

#26 In a group of 30 people, must at least 4 have been born in the same month? Why?

No. For example, consider the case where a function f assigns 3 distinct people ("pigeons") per month ("pigeonholes") from January through October.
Then there is no month where 4 people share a birthday.

Generalized Pigeonhole Principle

For any function f from a finite set X with n elements to a finite set Y with m elements and, for any positive integer k, if $km < n$, then there is some $y \in Y$ such that y is the image of at least $k+1$ distinct elements of X.

Generalized Pigeonhole Principle

For any function f from a finite set X with n elements to a finite set Y with m elements and, for any positive integer k, if for each $y \in Y$, $f^{-1}(y)$ has at most k elements, then X has at most km elements, i.e. $n \leq km$.

Let f, g be functions such that

(i) $f: A \to B$ and $g: B \to A$

(ii) $f \circ g = id_B$ and $g \circ f = id_A$

(a) Prove that f is injective and surjective.

(b) Prove that $g = f^{-1}$.

Proof:

Let f, g be functions such that

(i) $f: A \to B$ and $g: B \to A$

(ii) $f \circ g = id_B$ and $g \circ f = id_A$

(a) (I) Goal: $\forall x_1, x_2 \in A \left(f(x_1) = f(x_2) \longrightarrow x_1 = x_2 \right)$

Let $a_1, a_2 \in A$. Suppose $f(a_1) = f(a_2)$.

Apply g to the equation

$g(f(a_1)) = g(f(a_2))$ (since g is well-defined)

$(g \circ f)(a_1) = (g \circ f)(a_2)$ (def. of composition)

$id_A(a_1) = id_A(a_2)$ (from the problem statement)

$a_1 = a_2$ (by def. of identity map)

So f is one-to-one (injective).

(II) Goal: $\forall y \in B \, \exists x \in A \, (y = f(x))$

Let $b \in B$. Since $g: B \to A$, there exists $a \in A$ such that $g(b) = a$ because g is a function and so is defined on all its domain B. Apply f such that

$$f(g(b)) = f(a)$$

but $f(a) = f(g(b)) = (f \circ g)(b)$

$$= id_B(b) = b.$$

So f is surjective.

(b) Goal: Show $f^{-1} = g$.

Note that since $f: A \to B$ is (just previously shown) bijective, there exists $f^{-1}: B \to A$, its inverse under composition. Meanwhile $g: B \to A$ so f^{-1} & g have identical domain & codomain.

Let $b \in B$. Since f is surjective, there exists $a \in A$, such that $f(a) = b$. By def. of inverse, $f(a) = b$ implies $f^{-1}(b) = a$. So

$$f^{-1}(b) = a = id_A(a) = (g \circ f)(a)$$
$$= g(f(a)) = g(b).$$

Therefore $f^{-1} = g$.