Midterm 2

Chapter 2 Second-order linear differential equations

2.1 Algebraic properties of solutions

$W[y_1, y_2] = y_1(t)y_2'(t) - y_1'(t)y_2(t)$

$W[y_1, y_2] \neq 0 \quad \rightarrow \quad y(t) = c_1y_1(t) + c_2y_2(t)$

$W[y_1, y_2] = 0 \quad \rightarrow \quad y_1(t) = cy_2(t)$ linealy dependent

2.2 Linear equations with constant coefficients

2.2.1 Complex roots
2.2.2 Equal roots; reduction of order

$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = 0$

$ar^2 + br + c = 0$

Case 1: distinct real roots r_1, r_2

$y(t) = c_1e^{r_1t} + c_2e^{r_2t}$

Case 2: complex roots $r = \alpha \pm \beta i$

$y(t) = e^{\alpha t}(c_1cos(\beta t) + c_2sin(\beta t))$

Case 3: equal roots r; reduction of order

$y(t) = (c_1 + c_2t)e^{rt}$

2.3 The nonhomogeneous equation

2.4 The method of variation of parameters

To solve $L[y] = g(t)$, know $y_1(t)$, $y_2(t)$

$\psi(t) = u_1(t)y_1(t) + u_2(t)y_2(t)$

$u_1(t) = \int -\frac{g(t)y_2(t)}{W[y_1,y_2]}dt$, $u_2(t) = \int \frac{g(t)y_1(t)}{W[y_1,y_2]}dt$, where $W[y_1, y_2] = y_1(t)y_2'(t) - y_1'(t)y_2(t)$

2.5 The method of judicious guessing

$L[y] = t^3 + 2t^2 + 1 \quad \rightarrow \quad \psi(t) = p(t)$ polynomial of degree 3

$L[y] = (t^3 + 2t^2 + 1)e^{rt} \quad \rightarrow \quad \psi(t) = p(t)e^{rt}$

$L[y] = (t^3 + 2t^2 + 1)cos(wt) \quad \rightarrow \quad \psi(t) = p(t)(Acos(wt) + Bsin(wt))$

2.6 Mechanical vibrations

$my'' + cy' + ky = F_0cos(wt)$

Case 1: Free vibrations $my'' + ky = 0$

$y(t) = acos(w_0t) + bsin(w_0t)$, where $w_0 = \sqrt{\frac{k}{m}}$

$\quad = Rcos(w_0t - \delta)$, where $R = \sqrt{a^2 + b^2}, \delta = tan^{-1}\frac{b}{a}$

Case 2: Damped free vibrations $my'' + cy' + ky = 0$

$mr^2 + cr + k = 0$

distinct real roots r_1, r_2 → $y(t) = ae^{r_1 t} + be^{r_2 t}$

equal roots r → $y(t) = (a + bt)e^{rt}$

complex roots $r = \alpha \pm \beta i$ → $y(t) = e^{\alpha t}(acos(\beta t) + bsin(\beta t))$

Case 3: Damped forced vibrations $my'' + cy' + ky = F_0 cos(wt)$

$y(t) = \phi(t) + \psi(t)$, where $\phi(t)$ is solution of $my'' + cy' + ky = 0$, $\psi(t) = \frac{F_0 cos(wt-\delta)}{[(k-mw^2)^2+c^2w^2]^{\frac{1}{2}}}$

Case 4: Forced free vibrations $my'' + ky = F_0 cos(wt)$, $y'' + w_0 y = \frac{F_0}{m} cos(wt)$

$w \neq w_0$: $y(t) = c_1 cos(w_0 t) + c_2 sin(w_0 t) + \frac{F_0}{m(w_0^2 - w^2)} cos(wt)$

$w = w_0$: $y(t) = c_1 cos(w_0 t) + c_2 sin(w_0 t) + \frac{F_0 t}{2mw_0} sin(wt)$

force in resonance with the natural frequency of the system → oscillation with increasing amplitude

2.8 Series solutions

2.8.1 Singular points, Euler equations
2.8.2 Regular singular points, the method of Frobenius
2.8.3 Equal roots, and roots differing by an integer

Euler equation: $L[y] = t^2 \frac{d^2y}{dt^2} + \alpha t \frac{dy}{dt} + \beta y = 0$

Plug in $y = t^r$ → $[r^2 + (\alpha - 1)r + \beta]t^r = 0$

Case 1 two real roots: $y(t) = c_1 t^{r_1} + c_2 t^{r_2}$

Case 2 double roots: $y(t) = (c_1 + c_2 \ln t)t^r$

Case 3 imagenary roots: $y(t) = c_1 t^a \cos(b \ln t) + c_2 t^a \sin(b \ln t)$

$L[y] = P(t)y'' + Q(t)y' + R(t)y = 0$

If $P(t) \neq 0$, no singular points,

$y(t) = \Sigma_{i=0}^{n} a_n t^n$

$y'(t) = \Sigma_{i=0}^{n} n a_n t^{n-1}$

$y''(t) = \Sigma_{i=0}^{n} n(n-1) a_n t^{n-2}$

solve for $L[y] = 0$

plug in linearly independent $y_1(t), y_2(t)$ ($a_0 = 1, a_1 = 0; a_0 = 0, a_1 = 1$)

or plug in $a_0 = y_0, a_1 = y_0'$

Singular points $\iff$ $P(t) = 0$ at $t = t_0$

Regular singular points $\iff$ $(t - t_0)p(t), (t - t_0)^2 q(t)$ analytic at $t = t_0$

$y(t) = \Sigma_{i=0}^{n} a_n t^{n+r}$

$y'(t) = \Sigma_{i=0}^{n} (n + r) a_n t^{n+r-1}$

$y''(t) = \Sigma_{i=0}^{n} (n + r)(n + r - 1) a_n t^{n+r-2}$

solve for $L[y] = 0$

$y_1(t) = t^{r_1} \Sigma_{i=0}^{n} a_n t^n$, $y_2(t) = t^{r_2} \Sigma_{i=0}^{\infty} b_n t^n$

$r_1 = r_2 \quad \rightarrow \quad y_1(t) = t^r \Sigma_{i=0}^{n} a_n t^n$, $y_2(t) = y_1(t) lnt + t^{r_1} \Sigma_{i=0}^{\infty} b_n t^n$

$r_1 = r_2 + N \quad \rightarrow \quad y_1(t) = t^r \Sigma_{i=0}^{n} a_n t^n$, $y_2(t) = a y_1(t) lnt + t^{r_2} \Sigma_{i=0}^{\infty} b_n t^n$

Chapter 3 Systems of differential equations

3.1 Algebraic properties of solutions of linear systems

E.g. $y''' + 2y'' - y' + y = 0$

Let $x_1 = y$, $x_2 = y'$, $x_3 = y''$

$\frac{dx_1}{dt} = 0 \cdot x_1 + 1 \cdot x_2 + 0 \cdot x_3$

$\frac{dx_2}{dt} = 0 \cdot x_1 + 0 \cdot x_2 + 1 \cdot x_3$

$\frac{dx_3}{dt} = -1 \cdot x_1 + 1 \cdot x_2 - 2 \cdot x_3$

$\rightarrow \frac{d}{dt}\vec{x} = A\vec{x}$, where $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ -1 & 1 & -2 \end{bmatrix}$

3.8 The eigenvalue-eigenvector method of finding solutions

$\det(A - \lambda x) = 0$

Case 1: distinct roots λ_1, λ_2

$(A - \lambda I)\vec{v} = \vec{0} \quad \rightarrow \quad \vec{v}$

$\vec{x}(t) = c_1 e^{\lambda_1 t}\vec{v}_1 + c_2 e^{\lambda_2 t}\vec{v}_2 + \dots$

3.9 Complex roots

Case 2: complex roots $\lambda = \alpha \pm \beta i$

$\lambda = \alpha + \beta i$, $(A - \lambda I)\vec{v} = \vec{0} \quad \rightarrow \quad \vec{v}$

$\vec{x}(t) = e^{(\alpha + \beta i)t}\vec{v} = e^{\alpha t}(cos(\beta t) + isin(\beta t))\vec{v} = e^{\alpha t} x_1(t) + i e^{\alpha t} x_2(t)$

$\vec{x}(t) = c_1 e^{\alpha t} x_1(t) + c_2 e^{\alpha t} x_2(t) + \dots$

3.10 Equal roots

Case 3: equal roots λ

$(A - \lambda I)\vec{v}_1 = \vec{0} \quad \rightarrow \quad \vec{v}_1$

$(A - \lambda I)^2\vec{v}_2 = \vec{0}$, $(A - \lambda I)\vec{v}_2 \neq \vec{0} \quad \rightarrow \quad \vec{v}_2$

$\vec{x}(t) = c_1 e^{\lambda t}\vec{v}_1 + c_2 e^{\lambda t}[I + t(A - \lambda I)]\vec{v}_2 + \dots$

3.11 Fundamental matrix solutions; e^{At}

$X(t)$ — fundamental matrix solution of the differential equation $\dot{x} = Ax$

$e^{At} = X(t)X^{-1}(0)$