Nature of Exam

FTC Integrals onwards

Similar in format to Exam 1 & 2

one less m/c one more show work.

MA-UY 2114
Extra Practice Worksheet
Calculus III, Fall 2021

Some extra problems to practice before the exam are given below. This is not a comprehensive list. We recommend going over homework problems and problems covered in class to prepare for the exam.

1. Integrate $f(x, y, z) = x + y + z$ over the straight line segment from $(1, 2, 3)$ to $(0, -1, 1)$.

2. Evaluate $\int_C y^2 dx + x^2 dy$ where C is the circle $x^2 + y^2 = 4$.

   $x = 2\cos t \to dx = -2\sin t\,dt$
   $y = 2\sin t \to dy = 2\cos t\,dt$
   $0 \le t \le 2\pi$

3. Find the area of the surface given by

   $A(S) = \iint_D \|\vec{r}_u \times \vec{r}_v\|\,dA$

$$\mathbf{r}(u, v) =< u + v, u - v, v >$$

$$0 \le u \le 1, 0 \le v \le 1$$

4. Determine if any of the following vector fields are conservative or not

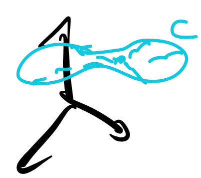
   use curl test or check if potential exists.

   (a) $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

   (b) $\mathbf{F} = \dfrac{\mathbf{r}}{\|\mathbf{r}\|^3}, \mathbf{r} =< x, y, z >$

   (c) $\mathbf{F} = xe^y\mathbf{i} + ye^z\mathbf{j} + ze^x\mathbf{k}$ $\operatorname{curl}\vec{F} \ne \vec{0}$

   (d) $\mathbf{F} =< 2, 2y + z, y + 1 >$ $\operatorname{curl}\vec{F} = \vec{0}$ domain $\mathbb{R}^3$ by curl test cons.



5. Find the work done by $\mathbf{F} = \dfrac{x\mathbf{i} + y\mathbf{j}}{(x^2 + y^2)^{3/2}}$ over the plane curve $\mathbf{r}(t) =< e^t \cos t, e^t \sin t >$ from $(1, 0)$ to $(e^{2\pi}, 0)$

6. Show that $\oint_C \ln x \sin y dy - \dfrac{\cos y}{x} dx = 0$ for any closed curve C to which Green's Theorem applies.

7. Find the outward flux of $\mathbf{F} = 2xy\mathbf{i} + 2yz\mathbf{j} + 2xz\mathbf{k}$ across the boundary of the cube cut from the first octant by the planes $x = a, y = a, z = a$ where $a > 0$.

8. Use Green's Theorem to evaluate $\int_C x^2 y dx - xy^2 dy$ where C is the circle $x^2 + y^2 = 4$ with counterclockwise orientation.

9. Use the Divergence Theorem to calculate the flux of $\mathbf{F} = (3z + 1)\mathbf{k}$ upward across the hemisphere $x^2 + y^2 + z^2 = a^2$, where $z \ge 0$.

10. Give an example of a vector field that has value $\mathbf{0}$ at only one point and such that curl $\mathbf{F}$ is nonzero everywhere.

11. Use Stokes' Theorem to evaluate $\iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S}$ where $\mathbf{F} = < x^2yz, yz^2, z^3e^{xy} >$

12. If curl $\mathbf{F}$ = curl $\mathbf{G}$ then is it true that $\mathbf{F} = \mathbf{G}$? No! Recall $\text{curl } \nabla f = \vec{0} \Rightarrow \vec{F} = \vec{G} + \vec{\nabla} f$

$\text{curl } \vec{F} = \text{curl } \vec{G}$

13. If $\mathbf{a}$ is a constant vector, $\mathbf{r} = < x, y, z >$ and S is an oriented, smooth surface with a simple, closed, smooth, positively oriented boundary curve C, show

$$\iint_S 2\mathbf{a} \cdot d\mathbf{S} = \int_C (\mathbf{a} \times \mathbf{r}) \cdot d\mathbf{r}$$

14. If f is a harmonic function (it satisfies Laplace's equation), show that the line integral $\int_C f_y dx - f_x dy$ is independent of path in any simple region D.

1. Integrate $f(x, y, z) = x + y + z$ over the straight line segment from $(1, 2, 3)$ to $(0, -1, 1)$.

$\vec{r}_0 = \langle 1, 2, 3 \rangle$, $\vec{r}_1 = \langle 0, -1, 1 \rangle$

$\boxed{\vec{r}(t) = (1-t)\vec{r}_0 + t\vec{r}_1}$ $\quad 0 \le t \le 1$

$= \langle 1-t, 2-2t, 3-3t \rangle + \langle 0, -t, t \rangle$

$= \langle 1-t, 2-3t, 3-2t \rangle$, $\vec{r}'(t) = \langle -1, -3, -2 \rangle$, $\|\vec{r}'\| = \sqrt{1+9+4} = \sqrt{14}$

$x = 1-t$, $y = 2-3t$, $z = 3-2t$ $\quad x+y+z = 1-t+2-3t+3-2t = 6-6t$

$$\int_C f(x,y,z)\,dS = \int_0^1 6-6t \; \|\vec{r}'(t)\| \, dt = \int_0^1 (6-6t)\sqrt{14}\, dt$$

$$= \sqrt{14}\left(6t - 3t^2\right)\Big|_{t=0}^{t=1}$$

$$= 3\sqrt{14}.$$

4. Determine if any of the following vector fields are conservative or not

(a) $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

(b) $\mathbf{F} = \dfrac{\mathbf{r}}{\|\mathbf{r}\|^3}, \mathbf{r} =< x, y, z >$

(c) $\mathbf{F} = xe^y\mathbf{i} + ye^z\mathbf{j} + ze^x\mathbf{k}$

(d) $\mathbf{F} =< 2, 2y + z, y + 1 >$

$$\text{curl}\,\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix}$$

$$= (0)\hat{i} - (0)\hat{j} + (0)\hat{k}$$

$$= \vec{0}$$

Conservative by curl test.
every closed curve in domain can be contracted to a pt.

(a) $f_x = x \xrightarrow{\text{int.}} \frac{x^2}{2} + h(y,z)$
$\downarrow \frac{\partial}{\partial y}$

$f_y = y \quad = \quad \frac{\partial}{\partial y} h(y,z) \Rightarrow h(y,z) = \frac{y^2}{2} + g(z)$
$\downarrow \frac{\partial}{\partial z}$

$f_z = z$

$g'(z) = z \qquad g(z) = \frac{z^2}{2} + C$

$f(x,y,z) = \dfrac{x^2+y^2+z^2}{2} + C$ is a potential

$\therefore \vec{F} = x\hat{i} + y\hat{j} + z\hat{k}$ is conservative.

(b) $\vec{F} = \dfrac{\langle x, y, z\rangle}{(x^2+y^2+z^2)^{3/2}}$

Domain satisfies curl test.
Check $\text{curl}\,\vec{F} \overset{?}{=} \vec{0}$

2. Evaluate $\int_C y^2 dx + x^2 dy$ where C is the circle $x^2 + y^2 = 4$.

$x = 2\cos t \quad , y = 2\sin t \qquad 0 \le t \le 2\pi$

$dx = -2\sin t\, dt, dy = 2\cos t\, dt$

$$\int_C y^2 dx + x^2 dy = \int_0^{2\pi} 4\sin^2 t\,(-2\sin t)\,dt + 4\cos^2 t\,(2\cos t)\,dt$$

$$= 8\int_0^{2\pi} \cos^3 t - \sin^3 t\, dt$$

$$= 8\int_0^{2\pi} \cos^3 t\,dt - 8\int_0^{2\pi} \sin^3 t\,dt$$

$\cos^2 t \cos t$ $\qquad$ $\sin^2 t \sin t$

$(1-\sin^2 t)\cos t$ $\qquad$ $(1-\cos^2 t)(\sin t)$

$u = \cos t$

$u = \sin t$ $\qquad$ $du = -\sin t\,dt$

$du = \cos t\,dt$

$$= \underbrace{8\int_0^0 (1-u^2)\,du}_{0} + \underbrace{8\int_1^1 (1-u^2)\,du}_{0} = 0.$$

3. Find the area of the surface given by

$$\mathbf{r}(u,v) = < u+v, u-v, v >$$

$$0 \le u \le 1, 0 \le v \le 1$$

$\vec{r}_u = \langle 1, 1, 0 \rangle$ $\qquad$ $\vec{r}_v = \langle 1, -1, 1 \rangle$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 0 \\ 1 & -1 & 1 \end{vmatrix} = (1-0)\hat{i} - (1-0)\hat{j} + (-1-1)\hat{k}$$

$$= \hat{i} - \hat{j} - 2\hat{k}$$

$$\|\vec{r}_u \times \vec{r}_v\| = \sqrt{1+1+4} = \sqrt{6}$$

$$A(S) = \iint_D \|\vec{r}_u \times \vec{r}_v\|\,dA = \int_0^1\int_0^1 \sqrt{6}\,dA = \sqrt{6}.$$

5. Find the work done by $\mathbf{F} = \dfrac{x\mathbf{i} + y\mathbf{j}}{(x^2+y^2)^{3/2}}$ over the plane curve $\mathbf{r}(t) = < \underbrace{e^t \cos t}_{x}, \underbrace{e^t \sin t}_{y} >$ from $(1,0)$ to $(e^{2\pi}, 0)$

$t = 2\pi$

$t = 0$

$\vec{r}'(t) = \langle -e^t \sin t + e^t \cos t, e^t \cos t + e^t \sin t \rangle$

$$W = \int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t)\, dt$$

$$= \int_0^{2\pi} \frac{e^t \cos t\, \hat{i} + e^t \sin t\, \hat{j}}{(\underbrace{(e^t \cos t)^2 + (e^t \sin t)^2}_{e^{2t}})^{3/2}} \cdot \left[(-e^t \sin t + e^t \cos t)\hat{i} + (e^t \cos t + e^t \sin t)\hat{j}\right] dt$$

$$= \int_0^{2\pi} \frac{\cancel{-e^{2t} \cos t \sin t} + e^{2t} \cos^2 t + \cancel{e^{2t} \sin t \cos t} + e^{2t} \sin^2 t}{e^{3t}}\, dt$$

$$= \int_0^{2\pi} e^{-t}\, dt = -e^{-t}\Big|_{t=0}^{t=2\pi} = -e^{-2\pi} + 1$$

6. Show that $\boxed{\oint_C \ln x \sin y\, dy - \frac{\cos y}{x} dx = 0}$ for any closed curve C to which Green's Theorem applies.

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_D (Q_x - P_y)\, dA = 0.$$

$$\vec{F} = \langle \underbrace{\ln x \sin y}_{Q}, \underbrace{-\frac{\cos y}{x}}_{P} \rangle$$

$$Q_x = \frac{\sin y}{x} \qquad P_y = \frac{\sin y}{x}$$

$$Q_x - P_y = 0$$