Ornstein–Uhlenbeck process

2022年6月29日 19:34

O-U SDE

即 $dX_t = (\beta - X_t)\alpha\,dt + \sigma\,dB_t$, $\alpha, \beta, \sigma \in \mathbb{R}$.

β - level of process revert in the long term (MR level)

α - MR speed. >0

σ - controls volatility

即 $\begin{cases} a(t,x) = (\beta - x)\alpha \\ \sigma(t,x) = \sigma \end{cases}$

$X_t > \beta$ 时 X_t ↘ → β

$X_t < \beta$ 时 β ← X_t ↗

令 $dX_t = \alpha\beta\,dt - \alpha X_t\,dt + \sigma\,dB_t$ ← 两边同乘 $e^{\alpha t}$: $e^{\alpha t}dX_t + \alpha e^{\alpha t}X_t\,dt$

take function $f(t,x) = e^{\alpha t}x$

有 $\begin{cases} f_t = \alpha e^{\alpha t}x \\ f_x = e^{\alpha t} \\ f_{xx} = 0 \end{cases}$.

代入 Itô ⇒ $df(t,X_t) = \alpha e^{\alpha t}x\,dt + e^{\alpha t}dX_t + \frac{1}{2}\cdot 0\cdot (dX_t)^2$

∴ SDE $\Leftrightarrow$ $e^{\alpha t}(dX_t + \alpha X_t dt) = e^{\alpha t}(\alpha\beta dt + \sigma dB_t)$.

$\Leftrightarrow$ $d(e^{\alpha t}X_t) = \alpha\beta e^{\alpha t}dt + \sigma e^{\alpha t}dB_t$

积分 $\Leftrightarrow$ $e^{\alpha T}X_T - e^{\alpha 0}X_0 = \beta\int_0^T \alpha e^{\alpha t}dt + \sigma\int_0^T e^{\alpha t}dB_t$

$\Leftrightarrow$ $e^{\alpha T}X_T - X_0 = \beta(e^{\alpha T} - 1) + \sigma\int_0^T e^{\alpha t}dB_t$.

$\Leftrightarrow$ $X_T = X_0 e^{-\alpha T} + \beta(1 - e^{-\alpha T}) + \sigma e^{-\alpha T}\int_0^T e^{\alpha t}dB_t$

X_T is driftless !

$\Rightarrow$ $X_T \sim$ Normal dist.

计算 $\mathbb{E}(X_T) = X_0 e^{-\alpha T} + \beta(1 - e^{-\alpha T})$ 收敛于 β as $T \nearrow \infty$ ← mean reversion 于 β

$\mathrm{Var}(X_T) = \sigma^2 e^{-2\alpha T}\cdot\left(\mathrm{Var}\int_0^T e^{\alpha t}dB_t\right)$

$\overset{\text{Itô Isometry}}{=} \sigma^2 e^{-2\alpha T}\cdot\frac{1}{2\alpha}[e^{2\alpha T} - 1]$ ← $\left(\int_0^T e^{2\alpha t}dt\right)$.

as $T \nearrow \infty$

有 $\mathrm{Var}(X_T) \to \frac{\sigma^2}{2\alpha}$ 为常数! (对比 BM var = T 趋向 $+\infty$ as $T \nearrow$)