[Supp.] The Feynman-Kac Formula

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Let X be an Ito diffusion

$$dX_t = a(X_t,t)dt + c(X_t,t)dB_t \tag{167}$$

with generator G_t

$$G_t[v](x) = \sum_i a_i(x,t)\frac{\partial v(x,t)}{\partial x_i} + \frac{1}{2}\sum_i\sum_j (c(x,t)c(x,t)^T)_{i,j}\frac{\partial^2 v(x,t)}{\partial x_i \partial x_j} \tag{168}$$

Let v be the solution to the following pde

$$-\frac{\partial v(x,t)}{\partial t} = G_t[v](x,t) - v(x,t)f(x,t) \tag{169}$$

with a known terminal condition $v(x,T)$, and function f. It can be shown that the solution to the pde (169) is as follows

$$v(x,s) = \mathbb{E}\Big[v(X_T,T)\ \exp\big(-\int_s^T f(X_t)dt\big) \mid X_s = x\Big] \tag{170}$$

We can think of $v(X_T,T)$ as a terminal reward and of $\int_s^T f(X_t)dt$ as a discount factor.

Informal Proof:
Let $s \le t \le T$ let $Y_t = v(X_t,t)$, $Z_t = \exp(-\int_s^t f(X_\tau)d\tau)$, $U_t = Y_t Z_t$. It can be shown (see Lemma below) that

$$dZ_t = -Z_t f(X_t)dt \tag{171}$$

Using Ito's product rule

$$dU_t = d(Y_t Z_t) = Z_t dY_t + Y_t dZ_t + dY_t dZ_t \tag{172}$$

Since dZ_t has a dt term, it follows that $dY_t dZ_t = 0$. Thus

$$dU_t = Z_t dv(X_t,t) - v(X_t,t)Z_t f(X_t)dt \tag{173}$$

Using Ito's rule on dv we get

$$\begin{aligned} dv(X_t,t) =& \nabla_t v(X_t,t)dt + (\nabla_x v(X_t,t))^T a(X_t,t)dt + (\nabla_x v(X_t,t))^T c(X_t,t)dB_t \\ &+ \frac{1}{2}\mathrm{trace}\Big(c(X_t,t)c(X_t,t)^T\nabla_x^2 v(X_t,t)\Big)dt \end{aligned} \tag{174}$$

Thus

$$\begin{aligned} dU_t =& Z_t\Big[\nabla_t v(X_t,t) + (\nabla_x v(X_t,t))^T a(X_t,t) \\ &+ \frac{1}{2}\mathrm{trace}\Big(c(X_t,t)c(X_t,t)^T\nabla_x^2 v(X_t,t)\Big) - v(X_t,t)f(X_t)\Big]dt \\ &+ Z_t(\nabla_x v(X_t,t))^T c(X_t,t)dB_t \end{aligned} \tag{175}$$

and since v is the solution to (169) then

$$dU_t = (\nabla_x v(X_t, t))^T c(X_t, t) dB_t \tag{176}$$

Integrating

$$U_T - U_s = \int_s^T Y_t (\nabla_x v(X_t, t))^T c(X_t, t) dB_t \tag{177}$$

taking expected values

$$\mathbb{E}[U_T \mid X_s = x] - \mathbb{E}[U_s \mid X_s = x] = 0 \tag{178}$$

where we used the fact that the expected values of integrals with respect to Brownian motion is zero. Thus, since $U_s = Y_0 Z_0 = v(X_s, s)$

$$\mathbb{E}[U_T \mid X_s = x] = \mathbb{E}[U_s \mid X_s = x] = v(x, s) \tag{179}$$

Using the definition of U_T we get

$$v(x, s) = \mathbb{E}[v(X_T, T) e^{-\int_s^T f(X_t) dt} \mid X_s = x] \tag{180}$$

We end the proof by showing that

$$dZ_t = -Z_t f(X_t) dt \tag{181}$$

First let $Y_t = \int_s^t f(X_\tau) d\tau$ and note

$$\Delta Y_t = \int_t^{t+\Delta_t} f(X_\tau) d\tau \approx f(X_t) \Delta t \tag{182}$$

$$dY_t = f(X_t) dt \tag{183}$$

Let $Z_t = \exp(-Y_t)$. Using Ito's rule

$$dZ_t = \nabla e^{-Y_t} dY_t + \frac{1}{2} \nabla^2 e^{-Y_t} (dY_t)^2 = -e^{-Y_t} f(X_t) dt = -Z_t f(X_t) dt \tag{184}$$

where we used the fact that

$$(dY_t)^2 = Z_t^2 f(X_t)^2 (dt)^2 = 0 \tag{185}$$

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