§3.4 Joint Prob. Dist.
(only Discrete Case)

Ex. Throw a pair of 4-sided dice.

Y = the sum $\quad X$ = the larger one.

$f(x,y) = P(X=x, Y=y)$ — joint p.m.f.

$X \backslash Y$	2	3	4	5	6	7	8	$f_X(x)$
1	$\frac{1}{16}$	-	-	-	-	-	-	$\frac{1}{16}$
2	-	$\frac{2}{16}$	$\frac{1}{16}$	-	-	-	-	$\frac{3}{16}$
3	-	-	$\frac{2}{16}$	$\frac{2}{16}$	$\frac{1}{16}$	-	-	$\frac{5}{16}$
4	-	-	-	$\frac{2}{16}$	$\frac{2}{16}$	$\frac{2}{16}$	$\frac{1}{16}$	$\frac{7}{16}$
$f_Y(y)$	$\frac{1}{16}$	$\frac{2}{16}$	$\frac{3}{16}$	$\frac{4}{16}$	$\frac{3}{16}$	$\frac{2}{16}$	$\frac{1}{16}$	

marginal p.m.f. $\quad f_X(x) = \sum_y f(x,y)$

$$f_Y(y) = \sum_x f(x,y)$$

X & Y are independent $\Longleftrightarrow$

$$f(x,y) = f_X(x)\, f_Y(y)$$

Conditional dist of X given $Y=y$

$$f_{X|Y}(x|Y=y) = \frac{P(X=x, Y=y)}{P(Y=y)} = \frac{f(x,y)}{f_Y(y)}$$

Chapter 4. Mathematical Expectation.

(expected value, expectation, mean)

X, $f(x)$

$$E(X) = \begin{cases} \sum_x x f(x) & \text{discrete} \\ \int_{-\infty}^{\infty} x f(x)\,dx & \text{continuous} \end{cases}$$

Ex. $f(x) = \begin{cases} 0.5 & x=1 \\ 0.3 & x=2 \\ 0.2 & x=3 \end{cases}$

$E(X) = 1*0.5 + 2*0.3 + 3*0.2 = 1.7$

$E(2X+3) = 5*0.5 + 7*0.3 + 9*0.2 = 6.4$

$E(X^2-1) = 0*0.5 + 3*0.3 + 8*0.2 = \underline{\underline{2.5}}$

For any function $g(x)$,

$$E(g(X)) = \begin{cases} \sum_x g(x) \cdot f(x) & \text{discrete} \\ \int_{-\infty}^{\infty} g(x) f(x)\,dx & \text{continuous} \end{cases}$$

Ex. $f(x) = 2-2x \quad 0<x<1.$

Find $E(X)$, $E(2X)$ $E(X^2)$

$$E(X) = \int_0^1 x \cdot (2-2x)\, dx$$

$$= \int_0^1 2x - 2x^2\, dx = x^2 - \frac{2x^3}{3}\Big|_0^1$$

$$= \frac{1}{3}$$

$$E(2X) = \int_0^1 2x \cdot (2-2x)\, dx = \frac{2}{3}$$

$$E(X^2) = \int_0^1 x^2 \cdot (2-2x)\, dx = \int_0^1 2x^2 - 2x^3\, dx$$

$$= \frac{2}{3} - \frac{1}{2} = \frac{1}{6}.$$

$$E(2X-1) = -\frac{1}{3}$$

$$= \int_0^1 (2x-1)(2-2x)\, dx$$

Prop: For r.v. X, const a & b.

$$E(aX+b) = aE(X) + b$$

Pf: $E(aX+b) = \int_{-\infty}^{\infty} (ax+b) f(x)\, dx$

$$= \int_{-\infty}^{\infty} ax f(x)\, dx + \int_{-\infty}^{\infty} b f(x)\, dx$$

$= a\,E(X) + b$ $\quad \mu = E(X)$

Variance of a r.v. X.

$$\sigma^2 = Var(X) = E\left((X-\mu)^2\right) = E(X^2) - \mu^2$$

expected value of the squared deviation

$$\sigma = \sqrt{\sigma^2}$$

standard dev.

Ex: $f(x) = \begin{cases} 0.5 & x=1 \\ 0.3 & x=2 \\ 0.2 & x=3 \end{cases}$ $\quad \mu = 1.7.$

$$\sigma^2 = E\left((X-1.7)^2\right) = 0.49 * 0.5 + 0.09 * 0.3 + 1.69 * 0.2$$

$$= 0.245 + 0.027 + 0.338$$

$.7$ $\quad 0.272$

$$= 0.61.$$

$$\sigma = \sqrt{0.61} \approx 0.78$$

$$\sigma^2 = E(X^2) - \mu^2 = 3.5 - 1.7^2 = 0.61$$

$f(x) = 2 - 2x \quad 0 \leq x \leq 1 \qquad \mu = \frac{1}{3}$

$$\sigma^2 = E\left(\left(X - \frac{1}{3}\right)^2\right) = \int_0^1 \left(x - \frac{1}{3}\right)^2 (2-2x)\, dx$$

$$= \frac{2}{9} \int_0^1 (3x-1)^2 (1-x)\, dx$$

$$= \frac{2}{9} \int_0^1 (9x^2 + 1 - 6x)(1-x)\, dx$$

$$= \frac{2}{9} \int_0^1 (-9x^3 + 15x^2 - 7x + 1)\, dx$$

$$= \frac{2}{9} \left(-\frac{9}{4} + \frac{15}{3} - \frac{7}{2} + 1\right)$$

$$\frac{-27 + 60 - 42 + 12}{12}$$

$$72 - 69$$

$$= \frac{1}{18}$$

$$\sigma^2 = E((X-\mu)^2) = E\left[X^2 - 2\mu X + \mu^2\right]$$

$$= E(X^2) - E(2\mu X) + \underbrace{E(\mu^2)}_{+\mu^2}$$

$$= E(X^2) \quad - 2\mu^2 \qquad + \mu^2$$

$$\sigma^2 = E(X^2) - \mu^2$$

Prop. For a r.v. X, and const a & b.

$$Var(aX+b) = a^2 Var(X)$$

Pf: $Var(aX+b) = E\left[\left(aX+\cancel{b}-a\mu-\cancel{b}\right)^2\right]$

$= E\left[a^2(X-\mu)^2\right]$

$= a^2 Var(X)$.

$$E(X+Y) = E(X) + E(Y)$$

$Var(X+Y) = E\left[\left(X+Y-(\mu_X+\mu_Y)\right)^2\right]$

$= E\left[(X-\mu_X)^2 + (Y-\mu_Y)^2 + 2(X-\mu_X)(Y-\mu_Y)\right]$

$= Var(X) + Var(Y) + 2E\left[(X-\mu_X)(Y-\mu_Y)\right]$

Def: $\mathrm{Cov}(X, Y) = E\left[(X-\mu_X)(Y-\mu_Y)\right]$

$\mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y) + 2\,\mathrm{Cov}(X, Y)$

When X & Y are independent,

$\mathrm{Cov}(X, Y) = 0$. and

$\mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y)$

$$E\left((X-\mu_X)(Y-\mu_Y)\right) = \sum_{x,y} (x-\mu_X)(y-\mu_Y) f(x,y)$$

$$\overset{\text{indep}}{=} \sum_y \sum_x (x-\mu_X)(y-\mu_Y) f_X(x) f_Y(y)$$

$$= \sum_y (y-\mu_Y) f_Y(y) \sum_x (x-\mu_X) f_X(x)$$

$$= E(Y-\mu_Y)\, E(X-\mu_X) = 0$$

$X \backslash Y$	2	3	4	5	6	7	8	$f_X(x)$
1	$\frac{1}{16}$	-	-	-	-	-	-	$\frac{1}{16}$
2	-	$\frac{2}{16}$	$\frac{1}{16}$	-	-	-	-	$\frac{3}{16}$
3	-	-	$\frac{2}{16}$	$\frac{2}{16}$	$\frac{1}{16}$	-	-	$\frac{5}{16}$
4	-	-	-	$\frac{2}{16}$	$\frac{2}{16}$	$\frac{2}{16}$	$\frac{1}{16}$	$\frac{7}{16}$
$f_Y(y)$	$\frac{1}{16}$	$\frac{2}{16}$	$\frac{3}{16}$	$\frac{4}{16}$	$\frac{3}{16}$	$\frac{2}{16}$	$\frac{1}{16}$	

Find $E(X+Y)$ & $Var(X+Y)$.

$E(X)$, $E(Y) = 2 * \frac{1}{16} + 3 * \frac{2}{16} + 4 * \frac{3}{16} + 5 * \frac{4}{16} + 6 * \frac{3}{16} + 7 * \frac{2}{16} + 8 * \frac{1}{16}$

$= 5$

$E(X) = 1 * \frac{1}{16} + 2 * \frac{3}{16} + 3 * \frac{5}{16} + 4 * \frac{7}{16}$

$= \frac{25}{8}$

$E(X+Y) = \frac{65}{8}$

$W = X + Y$

w	3	4	5	6	7	8	9	10	11	12
$f(w)$	$\frac{1}{16}$		$\frac{2}{16}$	$\frac{1}{16}$	$\frac{2}{16}$	$\frac{2}{16}$	$\frac{3}{16}$	$\frac{2}{16}$	$\frac{2}{16}$	$\frac{1}{16}$

$$Var(W) = E(W^2) - \left(\frac{65}{8}\right)^2$$

$$= ----$$

$$Cov(X,Y) = E\big((X-\mu_x)(Y-\mu_y)\big)$$

$$= E(XY) - \mu_x \mu_y$$

$$E[XY - X\mu_y - Y\mu_x + \mu_x\mu_y]$$

$$= E(XY) - \mu_x\mu_y \cancel{- \mu_x\mu_y} \cancel{+ \mu_x\mu_y}$$

Prop: If X & Y are indep,

$$E(XY) = E(X) \cdot E(Y)$$

$$\Rightarrow \operatorname{cov}(X, Y) = 0$$

$$E(aX + bY + c) = aE(X) + bE(Y) + c$$

$$\operatorname{Var}(aX + bY + c) = \operatorname{Var}(aX + bY)$$

$$= a^2 \operatorname{Var}(X) + b^2 \operatorname{Var}(Y) + 2ab \operatorname{Cov}(X, Y)$$

when indep $a^2 \operatorname{Var}(X) + b^2 \operatorname{Var}(Y)$