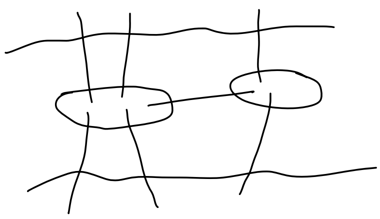
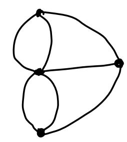
Lecture 8 – 9

Konigsberg on the Pregel

Hamiltonian circuits and Euler cycles.

 $\rightsquigarrow$ 

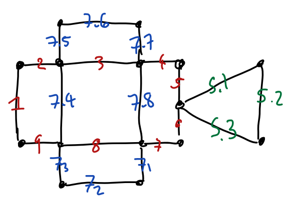
Find a cycle which meets each edge once?

def Euler cycle (repeat verts ok).

Prop "Euler" $\exists$ an Euler cycle $\iff \forall v \in V \mid d(v) \in 2\mathbb{Z}$. & G cctd. (1736)

Pf: Start at v_0, Remove each edge you lose, $\deg \in 2\mathbb{Z}$ guarantees you can keep going until you get back to v_0.

If there are edges left there must be one cctd to one of the verts $\Rightarrow$ Wedge it onto the cycle:



def Euler trail if 2 endpts.

$\exists \iff \exists\, v_1 \neq v_2 \mid d(v_1) \equiv d(v_2) \equiv 1 \bmod 2$

def: Hamilton circuit: Visit each vertex once.

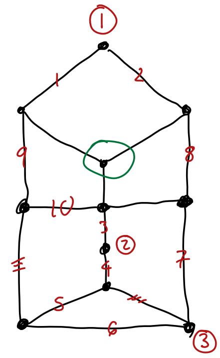
If $\exists \rightsquigarrow$ Find by inspection, else

(1) All $d = 2$ verts must have both edges included.

(2) No proper subcircuit

(3) Use at most two edges from a vertex.

Eg:



* 5 by symmetry
* 6, 7 b/c ③ now has deg 2.
* 8 b/c no subcircuit
* 9, 10 b/c deg 2 $\Rightarrow$ ~~X~~

Theorems about <u>Ham-circuits</u>:

<u>Dirac 1952</u>: If $|G| =: n > 2$ & $d(v) > \frac{n}{2}$ $\forall v \in V_G$ Then $\exists$ HC.

<u>Chvatal 1972</u>: Let G be connected with vertices $(v_1, \dots, v_n)$ such that $d(v_k) \le d(v_{k+1})$. For each $1 \le k < n/2$, if either $[\, k < \deg(v_k)$ OR $n-k \le \deg(v_{n-k}) \,]$ Then G has an H.C.

Eg. Let $(v_1, v_2, v_3, v_4, v_5, v_6) = (2, 3, 3, 3, 3, 6)$.

We check the theorem as follows: $n = 6$ so $1 \le k < 3$ so $k \in \{1, 2\}$

k	$<$	$d(v_k)$	OR	$n-k$	$\le$	$d(v_{n-k})$	
1	$<$	2 ✓	OR	5	$\le$	3	✓
2	$<$	3 ✓	OR	4	$\le$	3 ✗	✓

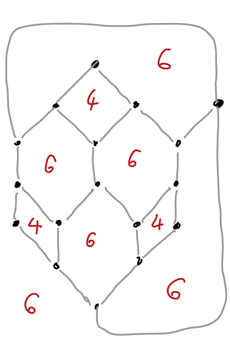
Both rows pass, so any graph with this degree sequence has a H.C.

Thus: If $\sum_{i=3}^{\infty} (i-2)(r_i^{I} - r_i^{O}) \neq 0$, Then P has <u>NO</u> H.C.

<u>Grinberg 1968</u>: If P is a planar Graph with a H.C. then $\sum_{i=3}^{\infty} (i-2)(r_i^{IN} - r_i^{OUT}) = 0$

Here r_i^{IN} means the number of regions of P which have i edges on their boundary and are <u>inside</u> the cycle.
r_i^{OUT} means the same, but the regions must be <u>outside</u> the cycle

Eg:



There are:
- 3 Regions w/ 4 edges $\Rightarrow$ $r_4^{I} + r_4^{O} = 3$ ①
- 6 Regions w/ 6 edges $\Rightarrow$ $r_6^{I} + r_6^{O} = 6$ ②

Can these 3 equations be solved at the same time by non-neg. numbers?

<u>AND</u>: $\underbrace{\sum_{i=3}^{\infty} (i-2)(r_i^{I} - r_i^{O})}_{i} = \underbrace{0}_{i=3} + \underbrace{(4-2)(r_4^{I} - r_4^{O})}_{i=4} + \underbrace{0}_{i=5} + \underbrace{(6-2)(r_6^{I} - r_6^{O})}_{i=6} + \underbrace{0}_{i=7} + \dots$

<u>THUS</u> $2(r_4^{I} - r_4^{O}) + 4(r_6^{I} - r_6^{O}) = 0$ ③

Plug ①, ② into ③:

$2((3 - r_4^{O}) - r_4^{O}) + 4((6 - r_6^{O}) - r_6^{O}) = 0 \Rightarrow 2[3 - 2r_4^{O} + 12 - 4r_6^{O}] = 0$

$\Rightarrow 15 = 2r_4^{O} + 6r_6^{O} \Rightarrow$ Odd number = Even number ~~X~~

Therefore this graph has no <u>H.C.</u>