Reflection Principle

2022年6月8日　　18:09

Running maximum and first passage time. For a stochastic process $X_t, t \geq 0$ we define the running maximum as

$$M_t = \max_{0 \leq s \leq t} X_s. \tag{1}$$

let X_t has CTS trajectories

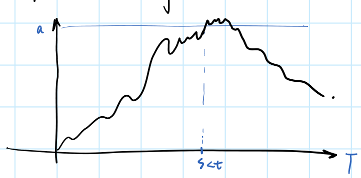
For this process to be well defined we assume that the process X_t is has continuous trajectories(in fact, we need only with probability one). Closely related to running maximum is the $\boxed{\text{first passage time}}$ which is defined as

$$T_a = \inf\{t > 0 : X_t = a\}, \quad \text{第一次 hits level } a. \tag{2}$$

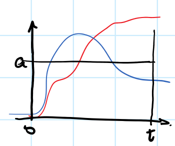
that is for a fixed a time T_a is the first time when X_t reaches level a. From the definitions it is clear that events $\{T_a < t\}$ and $\{M_t \geq a\}$ coincide, i.e.,

$$w \in \{T_a < t\} \iff w \in \{M_t > a\}. \tag{3}$$

reach a before t



Brownian motion. In the case when stochastic process X_t is a Brownian motion we can explicitly calculate the distribution of the running maximum and hitting time.



Theorem 1 (Reflection Principle.). *Let $a > 0$. Then $\mathbb{P}(T_a < t) = 2\mathbb{P}(B_t > a)$.*

Remark 2. Before the start of the proof let us rewrite the above equation as

$$\mathbb{P}(T_a < t) = 2\int_a^\infty \frac{e^{-x^2/(2t)}}{\sqrt{2\pi t}} dx. \tag{4}$$

$\frac{x^2}{t} = \frac{a^2}{s}$

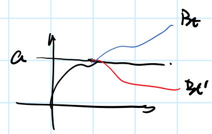
To find the probability density of T_a we change variables $x = \frac{\sqrt{t}a}{\sqrt{s}}$. Then $dx = -\frac{\sqrt{t}a}{2s^{3/2}}$ and

$$\mathbb{P}(T_a < t) = \int_0^t \frac{a}{\sqrt{2\pi s^3}} e^{-a^2/2s} ds, \tag{5}$$

and thus the density of T_a is $\frac{a}{\sqrt{2\pi s^3}} e^{-a^2/2s}$.

Proof. Assume that B_t hits level a at some time $s < t$. From the independence of increments property it follows that $B_t - B_{T_a}$ is independent of what happened before time T_a. Moreover, the increment $B_t - B_{T_a}$ is normally distributed. Since normal distribution is symmetric and probability of B_t being equal to a is zero we obtain

$$\mathbb{P}(T_a < t, B_t > a) = \frac{1}{2}\mathbb{P}(T_a < t). \tag{6}$$



每一條 above B_t 都有 B_t' below 且关于 a 对称

i.e., $\mathbb{P}(T_a < t, B_t > a) = \mathbb{P}(T_a < t, B_t < a)$

We multiply by 2 and notice that event $\{B_t > a\}$ is a subset of the event $\{T_a < t\}$, thus

$$\mathbb{P}(T_a < t) = 2\mathbb{P}(T_a < t, B_t > a) = 2\mathbb{P}(B_t > a). \tag{7}$$

□