# MA-UY 2114
# Extra Practice Worksheet
# Calculus III, Fall 2021

Some extra problems to practice before the exam are given below. This is not a comprehensive list. We recommend going over homework problems and problems covered in class to prepare for the exam.

1. Integrate $f(x, y, z) = x + y + z$ over the straight line segment from $(1, 2, 3)$ to $(0, -1, 1)$.
2. Evaluate $\int_C y^2 dx + x^2 dy$ where $C$ is the circle $x^2 + y^2 = 4$.
3. Find the area of the surface given by
$$\mathbf{r}(u, v) = < u + v, u - v, v >$$
$$0 \le u \le 1, 0 \le v \le 1$$
4. Determine if any of the following vector fields are conservative or not
   (a) $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$
   (b) $\mathbf{F} = \dfrac{\mathbf{r}}{\|\mathbf{r}\|^3}, \mathbf{r} = < x, y, z >$
   (c) $\mathbf{F} = xe^y\mathbf{i} + ye^z\mathbf{j} + ze^x\mathbf{k}$
   (d) $\mathbf{F} = < 2, 2y + z, y + 1 >$
5. Find the work done by $\mathbf{F} = \dfrac{x\mathbf{i} + y\mathbf{j}}{(x^2 + y^2)^{3/2}}$ over the plane curve $\mathbf{r}(t) = < e^t \cos t, e^t \sin t >$ from $(1, 0)$ to $(e^{2\pi}, 0)$
6. Show that $\oint_C \ln x \sin y dy - \dfrac{\cos y}{x} dx = 0$ for any closed curve $C$ to which Green's Theorem applies.
7. Find the outward flux of $\mathbf{F} = 2xy\mathbf{i} + 2yz\mathbf{j} + 2xz\mathbf{k}$ across the boundary of the cube cut from the first octant by the planes $x = a, y = a, z = a$ where $a > 0$.
8. Use Green's Theorem to evaluate $\int_C x^2 y dx - xy^2 dy$ where $C$ is the circle $x^2 + y^2 = 4$ with counterclockwise orientation.
9. Use the Divergence Theorem to calculate the flux of $\mathbf{F} = (3z + 1)\mathbf{k}$ upward across the hemisphere $x^2 + y^2 + z^2 = a^2$, where $z \ge 0$.
10. Give an example of a vector field that has value $\mathbf{0}$ at only one point and such that curl $\mathbf{F}$ is nonzero everywhere.

11. Use Stokes' Theorem to evaluate $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$ where $\mathbf{F} = < x^2yz, yz^2, z^3e^{xy} >$

12. If curl $\mathbf{F}$ = curl $\mathbf{G}$ then is it true that $\mathbf{F} = \mathbf{G}$? possible for $\vec{F} = \vec{G} + \nabla f$

13. If $\mathbf{a}$ is a constant vector, $\mathbf{r} = < x, y, z >$ and $S$ is an oriented, smooth surface with a simple, closed, smooth, positively oriented boundary curve C, show

$$\iint_S 2\mathbf{a} \cdot d\mathbf{S} = \int_C (\mathbf{a} \times \mathbf{r}) \cdot d\mathbf{r}$$

14. If $f$ is a harmonic function (it satisfies Laplace's equation), show that the line integral $\int_C f_y dx - f_x dy$ is independent of path in any simple region $D$.

4. Determine if any of the following vector fields are conservative or not

(a) $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

(b) $\mathbf{F} = \dfrac{\mathbf{r}}{\|\mathbf{r}\|^3}, \mathbf{r} = < x, y, z >$

(c) $\mathbf{F} = xe^y\mathbf{i} + ye^z\mathbf{j} + ze^x\mathbf{k}$

(d) $\mathbf{F} = < 2, 2y+z, y+1 >$

(a) find $f(x,y,z)$ s.t $\nabla f = x\hat{i} + y\hat{j} + z\hat{k}$

$f_x = x \rightarrow f(x,y,z) = \frac{x^2}{2} + g(y,z)$

$\downarrow \frac{\partial}{\partial y}$

$f_y = y \quad = \quad \frac{\partial}{\partial y} g(y,z) \Rightarrow g(y,z) = \frac{y^2}{2} + h(z)$

$f(x,y,z) = \frac{x^2}{2} + \frac{y^2}{2} + h(z)$

$\downarrow \frac{\partial}{\partial z}$

$f_z = z \quad = \quad h'(z) \Rightarrow h(z) = \frac{z^2}{2} + C$

$f(x,y,z) = \frac{x^2+y^2+z^2}{2} + C.$

OK

$$\text{Curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = (0)\hat{i} - (0)\hat{j} + (0)\hat{k}$$
$$= \vec{0}$$

Domain of $\vec{F} = x\hat{i} + y\hat{j} + z\hat{k}$ $\quad \mathbb{R}^3$

By curl test conservative.

2. Evaluate $\int_C y^2 dx + x^2 dy$ where $C$ is the circle $x^2 + y^2 = 4$.

$r^2 = 4$

$r = 2$

$x = r\cos t = 2\cos t$

$y = 2\sin t$

$0 \leq t \leq 2\pi$

$\frac{dx}{dt} = -2\sin t$

$dx = -2\sin t\, dt$

$dy = 2\cos t\, dt$

$$\int_0^{2\pi} (2\sin t)^2(-2\sin t)\,dt + (2\cos t)^2(2\cos t)\,dt$$

$$= 8\int_0^{2\pi} \cos^3 t - \sin^3 t\, dt$$

$$= 8\int_0^{2\pi} \cos^3 t\, dt - 8\int_0^{2\pi} \sin^3 t\, dt$$

$\cos^2 t \cos t$ $\quad$ $\sin^2 t \sin t$

$1 - \sin^2 t$ $\quad$ $1 - \cos^2 t$

$$= 8\int_0^{2\pi} (1-\sin^2 t)\cos t\, dt - 8\int_0^{2\pi} (1-\cos^2 t)\sin t\, dt$$

$u = \sin t$ $\quad$ $u = \cos t$

$du = \cos t\, dt$ $\quad$ $du = -\sin t\, dt$

$$= \underbrace{8\int_0^0 1-u^2\, du}_{0} - \underbrace{8\int_1^1 (1-u^2)\,du}_{0} = 0.$$

3. Find the area of the surface given by

$$\mathbf{r}(u,v) = < u+v, u-v, v >$$

$$0 \le u \le 1, 0 \le v \le 1$$

$$\iint_S \|\vec{r}_u \times \vec{r}_v\| \, dS$$

$$= \int_0^1 \int_0^1 \sqrt{6} \, du \, dv$$

$$= \sqrt{6}.$$

$\vec{r}_u = \langle 1, 1, 0 \rangle$

$\vec{r}_v = \langle 1, -1, 1 \rangle$

$\vec{r}_u \times \vec{r}_v = \langle 1, -1, -2 \rangle$

$\|\vec{r}_u \times \vec{r}_v\| = \sqrt{1+1+4} = \sqrt{6}$

5. Find the work done by $\mathbf{F} = \dfrac{x\mathbf{i} + y\mathbf{j}}{(x^2+y^2)^{3/2}}$ over the plane curve $\mathbf{r}(t) = < e^t \cos t, e^t \sin t >$ from $(1,0)$ to $(e^{2\pi}, 0)$ $\sim t = 2\pi$

$t = 0$

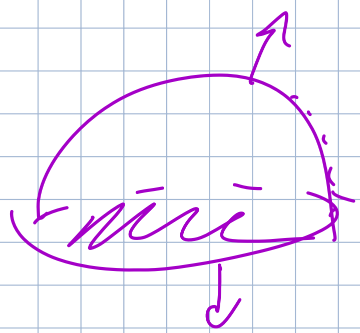
$$\int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t) \, dt$$

$$= \int_0^{2\pi} \frac{e^t \cos t \, \hat{\imath} + e^t \sin t \, \hat{\jmath}}{(\underbrace{e^{2t}\cos^2 t + e^{2t}\sin^2 t}_{e^{2t}})^{3/2}} \cdot \left[ \left(e^t \cos t - e^t \sin t\right)\hat{\imath} + \left(e^t \sin t + e^t \cos t\right)\hat{\jmath} \right] dt$$

$$= \int_0^{2\pi} \frac{e^{2t}\cos^2 t \cancel{- e^{2t}\sin t \cos t} + e^{2t}\sin^2 t \cancel{+ e^{2t}\sin t \cos t}}{e^{3t}} \, dt$$

$$= \int_0^{2\pi} \frac{e^{2t}}{e^{3t}} \, dt = \int_0^{2\pi} e^{-t} \, dt = -e^{-t} \Big|_{t=0}^{t=2\pi} = \boxed{1 - e^{-2\pi}}.$$

9. Use the Divergence Theorem to calculate the flux of $\mathbf{F} = (3z+1)\mathbf{k}$ upward across the hemisphere $x^2 + y^2 + z^2 = a^2$, where $z \geq 0$.



if bottom exists we can use div thm.

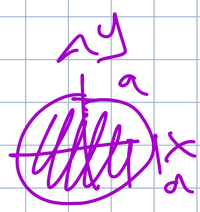
$\operatorname{div} \vec{F} = 3$

div thm $\iint_S \vec{F} \cdot d\vec{S} = \iiint_E \operatorname{div}\vec{F}\, dV$

$= 3 \underbrace{\iiint_E dV}_{\text{volume of hemisphere.}}$

$= 3\left(\frac{1}{2}\right)\frac{4}{3}\pi (a)^3 = \pi a^3$

Flux out of bottom



$z = 0$ $\quad 0 \leq r \leq a$

$0 \leq \theta \leq 2\pi$

normal vector $-\hat{k}$

$\vec{F}$ on surface $\vec{F} = (3z+1)\hat{k}$

$z = 0$

$\vec{F} = \hat{k}$

$\iint_S \vec{F} \cdot d\vec{S} = \iint_S \vec{F} \cdot \hat{n}\, dS = -1 \iint_S dS = -1(\pi a^2) = -\pi a^2$

Subtract from 1st integral

$\pi a^3 - (-\pi a^2) = \pi(a^3 + a^2)$