BM Properties

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conditional expectation

r.v. X σ-algebra a

$y = \mathbb{E}(X|a)$ if

1) $y \in a$

2) $\forall A \in a$, $\mathbb{E}(X \cdot 1_A) = \mathbb{E}(y \cdot 1_A)$

2. **Paul Lévy Theorem.** Let X_n be a sequence of random variables on $\mathbb{R}$. Then

a) If X_n converges to X is distribution then $\varphi_{X_n}(u) \to \varphi_X(u)$ for every u and in fact this convergence is uniform.

b) If $\varphi_{X_n}(u) \to \varphi_X(u)$ then X_n converges to X is distribution.

c) If $\varphi_{X_n}(u)$ converges to some function $\varphi(u)$ and $\varphi(u)$ is continuous at 0 then there exists a unique random variable X such that φ is the characteristic function of X and X_n converges to X in distribution.

We use part (b) of the theorem to prove that $B^{(n)}(t)$ converges to normal distribution with expectation zero and variance t.

Th^m: Paul-Lévy Th^m

X_n 为 seq. of r.v.

a) if $X_n \xrightarrow{d} X$，则 $\phi_{X_n}(u) \to \phi_X(u)$ $\forall u$ 且此 converge 为 uniform.

b) if $\varphi_{X_n}(u) \to \varphi_X(u)$，则 $X_n \xrightarrow{d} X$

c) if $\varphi_{X_n}(u) \to \varphi(u)$，且 $\varphi(u)$ CTS at $u=0$，则 $X_n \xrightarrow{d} X$ 且 $\varphi_X(u) = \varphi(u)$

下用性质 b

$X_n = B^{(n)}(t)$ ~ rescaled SRW

$X = B(t) \overset{d}{=} N(0,t)$

$B^{(n)}(t) = \frac{1}{\sqrt{n}} M(nt)$, $n \in \mathbb{Z}^+$ ← SRW

有 $\phi_{B^{(n)}_t}(u) = \mathbb{E}\, e^{iu\frac{1}{\sqrt{n}}M(nt)}$

$= \mathbb{E}\, e^{iu\frac{1}{\sqrt{n}}\sum_{j=1}^{nt} X_j}$

$= (\mathbb{E}\, e^{iu\frac{X_1}{\sqrt{n}}}) \cdot \mathbb{E}(e^{iu\frac{X_2}{\sqrt{n}}}) \cdots \mathbb{E}(e^{iu\frac{X_{nt}}{\sqrt{n}}})$

$= (\frac{1}{2}e^{iu/\sqrt{n}} + \frac{1}{2}e^{-iu/\sqrt{n}})^{nt}$

$\boxed{\phi_{X_n}(t)}$

考虑 $\phi_{N(a,\sigma^2)}(u) = e^{iua - \frac{\sigma^2 u^2}{2}}$ $\Longrightarrow$ $\boxed{\phi_X(t)}$

考虑 取对数 $nt \cdot \ln\left[\frac{1}{2}(e^{iu/\sqrt{n}} + e^{-iu/\sqrt{n}})\right] \xrightarrow{?} -\frac{u^2 t}{2}$ $\Longrightarrow$ $\boxed{\phi_{X_n}(t) \to \phi(t)}$

泰勒展开 $e^{iu/\sqrt{n}} = 1 + \frac{iu}{\sqrt{n}} + \frac{1}{2}(\frac{iu}{\sqrt{n}})^2 + O((\frac{iu}{\sqrt{n}})^3)$ → $O(n^{-3/2})$

同理 $e^{-iu/\sqrt{n}} = 1 - \frac{iu}{\sqrt{n}} + \frac{1}{2}(\frac{-iu}{\sqrt{n}})^2 + O(n^{-3/2})$

有 $nt \log(1 - \frac{u^2}{2n} + O(n^{-3/2})) \xrightarrow{?} -u^2 t/2$

再次泰勒展开 $\log(1+a) = a - \frac{a^2}{2} + \frac{a^3}{3} + \frac{(-1)^{n-1}a^n}{n} \approx a + O(a^2)$

有 $nt\left(-\frac{u^2}{2n} + O(n^{-3/2}) + O\left[\left(-\frac{u^2}{2n} + O(n^{-3/2})\right)^2\right]\right) \xrightarrow{?}$ 控制

$= nt\left(-\frac{u^2}{2n} + O(n^{-3/2})\right) \xrightarrow{?}$

$= -\frac{u^2}{2}t + O(\frac{1}{\sqrt{n}}) \to -\frac{u^2 t}{2}$ ✓

从而验证 SRW → 布朗

性质

① Cov of Brownian $(s \le t)$

→ $\mathbb{E} B_s B_t = \mathbb{E} B_s(B_t - B_s + B_s)$

Covariance of BM. Let $0 \le s \le t$. Then

$$\mathbb{E}B(s)B(t) = \mathbb{E}B(s)(B(t) - B(s) + B(s)) = \mathbb{E}B(s)(B(t) - B(s)) + \mathbb{E}B(s)^2.$$

From the independence of increments property one has $\mathbb{E}B(s)(B(t) - B(s)) = \mathbb{E}B(s)\mathbb{E}(B(t) - B(s)) = 0$. From the fact that $B(s)$ is a normally distributed random variable with expectation zero and variance s one has $\mathbb{E}B(s)^2 = s$ and thus

$$\mathbb{E}B(s)B(t) = s. \quad = \min(s,t)$$

② 类比 $\vec{X} = N(\vec{a}, \Sigma)$

$\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \quad \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \quad \begin{pmatrix} \Sigma_{11} & \Sigma_{12} & \cdots & \Sigma_{1n} \\ \Sigma_{21} & & & \end{pmatrix}$

Σ_{11}: X_1 的 Var

Σ_{12}: Cov $X_1 X_2$

② 求此 $\vec{X} = N(\vec{a}, \Sigma)$

$$\begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \begin{pmatrix} \Sigma_{11} & \Sigma_{12} & \cdots & \Sigma_{1n} \\ \Sigma_{21} & - & & \vdots \\ \vdots & & \ddots & \vdots \\ \Sigma_{n1} & \cdots & \cdots & \Sigma_{nn} \end{pmatrix}$$

Σ_{11}: X_1 的 Var

Σ_{12}: Cov $X_1 X_2$

考虑 $(B_{t_1}, B_{t_2}, B_{t_3} \cdots)$

$\quad N(0,t_1) \quad N(0,t_2) \quad \cdots$

求 joint dist.

有 $\vec{X} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \cdot \begin{pmatrix} t_1 & t_1 & t_1 & t_1 & \cdots & t_1 \\ t_1 & t_2 & t_2 & t_2 & \cdots & t_2 \\ t_1 & t_2 & t_3 & t_3 & \cdots & t_3 \\ \vdots & \vdots & \vdots & & \ddots & \\ t_1 & t_2 & t_3 & \cdots & & t_n \end{pmatrix}$

求 $\mathbb{P}(B_{t_1} \in A, B_{t_2} \in B)$

$= \int_A \int_B$ (joint density)

③ Non-Differentiability of Brownian Paths

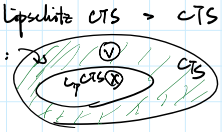
Theorem 1. *With probability one Brownian paths are not Lipschitz continuous (and hence not differentiable) at any point.*

$\exists \lim_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0}$

$|f(x) - f(x_0)| < c|x - x_0|$ as $x \to x_0$

关系: Diff'ble > Lipschitz CTS > CTS

Brownian Path:



Fix some constant C and define event

$[0,1]$ 可推广至 $[1,2]$, $[2,3]$ ⋯ 则有 $[0,+\infty)$ 都成立. ⇒ countable numbers 都成立. (对于 probability measure) i.e. $\mathbb{P}(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mathbb{P}(A_n)$ 对于 countable n 成立.

$A_n = \{w :$ there is an $s \in [0,1]$ s.t. $|B(t) - B(s)| \le C|t-s|$ for $|t-s| \le 3/n\}$.

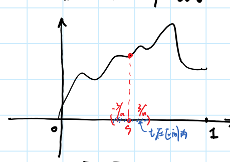
当 n 大时, $3/n$ 小, 则 A_n 为在小区间内 ∃ pt 利 CTS

Clearly, if a path is Lipschitz continuous with constant C at some point s then it belongs to set A_n for some large n. Therefore to prove that BM is not Lipschitz continuous it is enough to prove that $\mathbb{P}(A_n) = 0$. Let us think about the structure of sets A_n. Clearly $A_1 \subset A_2 \subset \cdots \subset A_n \subset \ldots$ Therefore, $\mathbb{P}(A_1) \le \mathbb{P}(A_2) \le \cdots \le \mathbb{P}(A_n) \le \ldots$

increasing events

w: Brownian Path 若 Lip CTS, 有 $w \in A_n$ for some n

A_n是 increasing elements



For $1 \le k \le n-2$ let

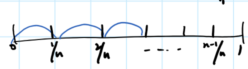
$$Y_{k,n} = \max\left\{\left|B\left(\frac{k+j}{n}\right) - B\left(\frac{k+j-1}{n}\right)\right|, j = 0,1,2\right\} \tag{1}$$

$$= \max\left\{\left|B\left(\frac{k}{n}\right) - B\left(\frac{k-1}{n}\right)\right|, \left|B\left(\frac{k+1}{n}\right) - B\left(\frac{k}{n}\right)\right|, \left|B\left(\frac{k+2}{n}\right) - B\left(\frac{k+1}{n}\right)\right|\right\}.$$

下欲证 $\mathbb{P}(A_n) \to 0$ as $n \to \infty$ 从而 $\mathbb{P}(A_1) \le \mathbb{P}(A_2) \le \cdots \le \mathbb{P}(A_n) \to 0$ 则全为0

再欲构造 $A_n \subseteq G_n$ where $\mathbb{P}(G_n) \to 0$ as $n \to \infty$

and define event $G_n = \{w :$ at least one $Y_{k,n}$ is $\le 5C/n\}$. We claim that $A_n \subset G_n$ and thus it is enough to prove that $\mathbb{P}(G_n) = 0$. Let's explain why $A_n \subset G_n$. Assume that point s is between points $\frac{k}{n}$ and $\frac{k+1}{n}$ and $|B(t) - B(s)| \le C|t-s|$ for $|t-s| \le \frac{3}{n}$. Then clearly

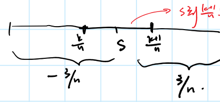


$$\left|B\left(\frac{k+1}{n}\right) - B\left(\frac{k}{n}\right)\right| \le \left|B(s) - B\left(\frac{k}{n}\right)\right| + \left|B\left(\frac{k+1}{n}\right) - B(s)\right|$$
$$\le C\left(s - \frac{k}{n}\right) + C\left(\frac{k+1}{n} - s\right) = C\frac{1}{n}. \tag{2}$$

$y_{k,n} = \max\{|B_{\frac{k}{n}} - B_{\frac{k-1}{n}}|, |B_{\frac{k+1}{n}} - B_{\frac{k}{n}}|, |B_{\frac{k+2}{n}} - B_{\frac{k+1}{n}}|\}$

e.g. $k=1$. $y_{1,n} = \max\{|B_{\frac{1}{n}} - 0|, |B_{\frac{2}{n}} - B_{\frac{1}{n}}|, |B_{3/n} - B_{\frac{2}{n}}|\}$

$y_{1,n}$ = 3个 increments 中最大的. 且是 independent Brownian increments



$$\left|B\left(\frac{k+2}{n}\right) - B\left(\frac{k+1}{n}\right)\right| \le \left|B\left(\frac{k+1}{n}\right) - B(s)\right| + \left|B\left(\frac{k+2}{n}\right) - B(s)\right|$$
$$\le C\left(\frac{k+1}{n} - s\right) + C\left(\frac{k+2}{n} - s\right) = C\frac{3}{n}. \tag{3}$$

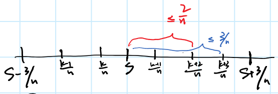
取 $t = \frac{k+1}{n} \Rightarrow |t-s| \le \frac{1}{n}$

此时 $|B_{\frac{k+1}{n}} - B_s| \le c|\frac{k+1}{n} - s| = c(\frac{k+1}{n} - s)$ ①

再 $t = \frac{k}{n} \Rightarrow |t-s| \le \frac{1}{n}$

此时 $|B_{\frac{k}{n}} - B_s| \le c|\frac{k}{n} - s| = c(s - \frac{k}{n})$ ②

$$\left|B\left(\frac{k+3}{n}\right) - B\left(\frac{k+2}{n}\right)\right| \le \left|B\left(\frac{k+2}{n}\right) - B(s)\right| + \left|B\left(\frac{k+3}{n}\right) - B(s)\right|$$
$$\le C\left(\frac{k+2}{n} - s\right) + C\left(\frac{k+3}{n} - s\right) = C\frac{5}{n}. \tag{4}$$



①&② Δ ineq ⇒ $|B_{\frac{k+1}{n}} - B_{\frac{k}{n}}| \le c(\frac{k+1}{n} - \frac{k}{n}) = \frac{c}{n} \le \frac{5c}{n}$ 甲

Thus we conclude that $A_n \subset G_n$ and $\mathbb{P}(A_n) \le \mathbb{P}(G_n) \le n\mathbb{P}(Y_{k,n} \le 5C/n)$. The last inequality follows from the fact that

$$G_n = \bigcup_{1 \le k \le n-2} \{Y_{k,n} \le 5C/n\}.$$

同理 $|B_{\frac{k+2}{n}} - B_s| \le c(\frac{k+2}{n} - s) \le c \cdot 2/n$

$|B_{\frac{k+1}{n}} - B_s| \le c(\frac{k+1}{n} - s) \le c \cdot 1/n$

$\Rightarrow |B_{\frac{k+2}{n}} - B_{\frac{k+1}{n}}| \le c \cdot 3/n$ ②

最优: $|B_s - B_t| \le C|t-s|^{\gamma}$ Hölder CTS

对于 $\gamma > 1/2$ 无 Hölder CTS

对于 $\gamma < 1/2$ 有 Hölder CTS

对于 $\gamma = 1/2$ 特殊.

From the independence of increments property of Brownian motion it follows that random variables $B(\frac{k}{n}) - B(\frac{k-1}{n}), B(\frac{k+1}{n}) - B(\frac{k}{n}), B(\frac{k+2}{n}) - B(\frac{k+1}{n})$ are independent

同理 $|B_{\frac{k+3}{n}} - B_s| \le c(\frac{k+3}{n} - s) \le c \cdot 3/n$

$|B_{\frac{k+2}{n}} - B_s| \le c \, 2/n$

$\Rightarrow |B_{\frac{k+3}{n}} - B_{\frac{k+2}{n}}| \le c \cdot 5/n$ 丙

甲 2 丙 ⇒ $A_n \subseteq G_n$

and thus 考虑 $\mathbb{P}(y_{1,n} \le 5c/n) = \mathbb{P}(\max\{|B_{\frac{1}{n}}|, |B_{2/n} - B_{1/n}|, |B_{3/n} - B_{2/n}|\} \le 5c/n) = [\mathbb{P}(|B_{1/n}| \le 5c/n)] \sim N(0, \frac{1}{n})$

$\cdot [\mathbb{P}(|B_{2/n} - B_{1/n}| < 5c/n)] \cdot [\mathbb{P}(|B_{3/n} - B_{2/n}| < 5c/n)]$

$$n\mathbb{P}(Y_{n,k} \le 5C/n) = n\mathbb{P}\left(\left|B\left(\frac{1}{n}\right)\right| \le 5C/n\right)^3$$

考虑 $|B_{1/n}| \sim N(0, \frac{1}{n})$

$\Rightarrow \mathbb{P}(|B_{1/n}| \le 5c/n) = \int_{-5c/n}^{5c/n} \frac{1}{\sqrt{2\pi \frac{1}{n}}} e^{-\frac{x^2}{2\frac{1}{n}}} dx$

$$= n\mathbb{P}(|B(1)| \le 5C/\sqrt{n})^3 \le n\left(\frac{10C}{\sqrt{n}}\frac{1}{\sqrt{2\pi}}\right)^3. \tag{5}$$

2个积分

Letting $n \to \infty$ shows that $\mathbb{P}(A_n) \to 0$. Notice that $n \to A_n$ is increasing and thus $\mathbb{P}(A_n) = 0$ for all n. □

下证 $\mathbb{P}(G_n) \to 0$ as $n \nearrow \infty$

$\mathbb{P}(G_n) = \mathbb{P}(\bigcup \{y_{k,n} \le 5c/n\}) \le \sum \mathbb{P}\{y_{k,n} \le 5c/n\}$ $\mathbb{P}(A \cup B) \le \mathbb{P}(A) + \mathbb{P}(B)$ 之性质.

$$\Rightarrow \mathbb{P}(|B_{1/n}| \le 5C/n) = \int_{-5C/n}^{5C/n} \frac{1}{\sqrt{2\pi \frac{1}{n}}} e^{-\frac{x^2}{2\frac{1}{n}}}dx$$

$$= n\mathbb{P}(|B(1)| \le 5C/\sqrt{n})^3 \le n\left(\frac{}{\sqrt{n}}\frac{}{\sqrt{2\pi}}\right)^{}. \tag{5}$$

换元积分

let $z = x\sqrt{n}$, $dz = \sqrt{n}\,dx$

$$= \int_{-5C/\sqrt{n}}^{5C/\sqrt{n}} \frac{1}{\sqrt{2\pi\frac{1}{n}}} e^{-\frac{z^2}{2}} \frac{dz}{\sqrt{n}}$$

$$= \int \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz \le \int_{-5C/\sqrt{n}}^{5C/\sqrt{n}} \frac{1}{\sqrt{2\pi}} dz.$$

Letting $n \to \infty$ shows that $\mathbb{P}(A_n) \to 0$. Notice that $n \to A_n$ is increasing and thus $\mathbb{P}(A_n) = 0$ for all n. □

(甲乙丙 ⇒ $A_n \subseteq G_n$

下证 $\mathbb{P}(G_n) \to 0$ as $n \nearrow \infty$

$\mathbb{P}(A \cup B) \le \mathbb{P}(A) + \mathbb{P}(B)$ 之性质.

$$\mathbb{P}(G_n) = \mathbb{P}\left(\bigcup_k \{y_{k,n} \le 5C/n\}\right) \le \sum_k \mathbb{P}\{y_{k,n} \le 5C/n\}$$

考虑 Brownian increment 只取决于 subinterval

$$\Rightarrow \mathbb{P}(G_n) \le n \times \mathbb{P}(y_{k,n} \le 5C/n)$$

④ Scaling Property

Scaling properties of the Brownian Motion. If we fix time t then the distribution of Brownian motion $B(t)$ is normal with zero expectation and variance t. Actually, for any fixed set of times $t_1 < t_2 < \cdots < t_n$ random variables $B(t_1), B(t_2), \ldots, B(t_n)$ are jointly normal with zero mean and covariance given by $\mathbb{E}B(s)B(t) = \min(s, t)$.

1) $B_0 = 0$, $\lambda > 0$ $\Rightarrow \frac{1}{\sqrt{\lambda}}B(\lambda t)$ also a Brownian Motion.

X_t

Theorem 2. *If $B(0) = 0$ then for any $\lambda > 0$ stochastic process $\frac{1}{\sqrt{\lambda}}B(\lambda t), t \ge 0$ is a Brownian motion.*

Proof. First, let us notice that $X(t) = \frac{1}{\sqrt{\lambda}}B(\lambda t)$ is a Gaussian process, i.e. for any set $t_1 < t_2 < \cdots < t_n$ the joint distribution of $X(t_1), X(t_2), \ldots, X(t_n)$ is a multivariate Gaussian distribution. This property is clearly inherited from Brownian motion properties.

Since normal distribution is characterized by its mean and covariance we have to check that the mean and covariance of the process $X(t)$ coincide with those of Brownian motion. It is easy:

$$① \quad \mathbb{E}X(t) = \mathbb{E}\frac{1}{\sqrt{\lambda}}B(\lambda t) = \frac{1}{\sqrt{\lambda}}\mathbb{E}B(\lambda t) = 0, \tag{6}$$

and for $s < t$

$$② \quad \mathbb{E}X(s)X(t) = \mathbb{E}\frac{1}{\sqrt{\lambda}}B(\lambda s)\frac{1}{\sqrt{\lambda}}B(\lambda t) = \frac{1}{\lambda}\mathbb{E}B(\lambda s)B(\lambda t) = \frac{1}{\lambda}\min(\lambda s, \lambda t) = s. \tag{7}$$

Thus $\frac{1}{\sqrt{\lambda}}B(\lambda t)$ has the same distribution as a Brownian motion. □

2) $B_0 = 0$, $X_t = tB(1/t)$ $(X_0 = 0)$ also B.Motion 用于 inverse time.

显然 $B(1/t)$ 仍为 Gaussian

Theorem 3. *If $B(t)$ is a Brownian Motion starting at 0 then so is the process defined by $X(0) = 0$ and $X(t) = tB(1/t)$ for $t > 0$.*

Proof. Fix $t_1 < t_2 < \cdots < t_n$. Then clearly $X(t_1) = t_1B(1/t_1), \ldots, X(t_n) = t_nB(1/t_n)$ has a multivariate Gaussian distribution. We just have to check that it has the same mean and covariance structure. First of all,

$$① \quad \mathbb{E}X(t) = t\mathbb{E}B(1/t) = t \cdot 0 = 0.$$

Also, for $0 < s < t$

$$② \quad \mathbb{E}X(s)X(t) = st\mathbb{E}B(1/s)B(1/t) = st\min(1/s, 1/t) = st \cdot 1/t = s$$

Thus $X(t)$ is a Brownian motion. □