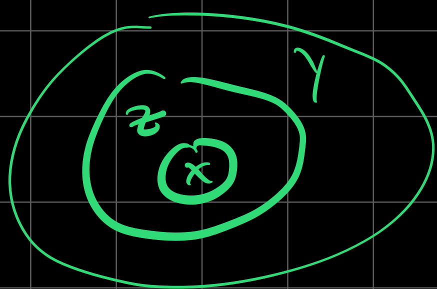
15.6

$$\iiint_E f(x,y,z)\,dV$$

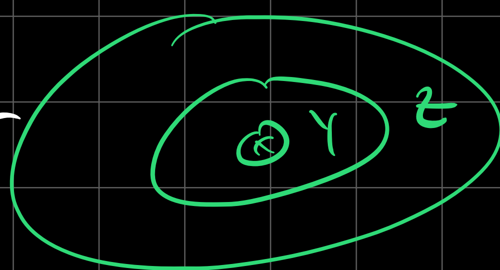
$dV \to dx\,dy\,dz$

⑥ $$\int_0^1\int_0^1\int_0^{\sqrt{1-z^2}} \frac{z}{y+1}\,dx\,dz\,dy$$

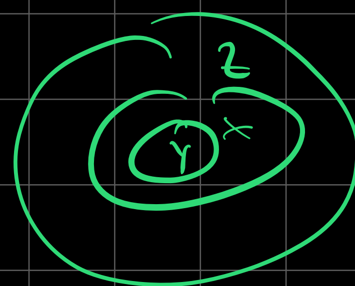
$0 \le x \le \sqrt{1-z^2}$, $0 \le z \le 1$, $0 \le y \le 1$



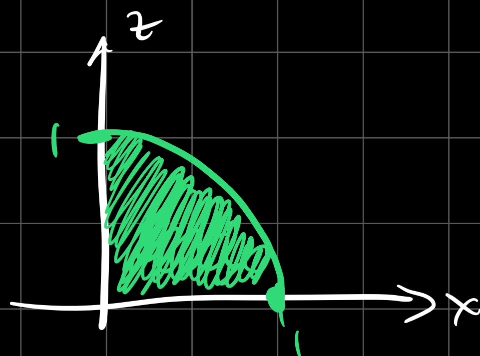
or



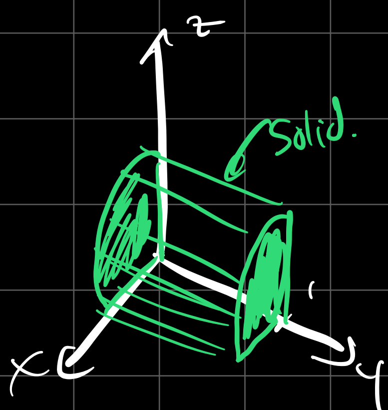
or



what is the region of integration?



now let $0 \le y \le 1$



now evaluate integral.

$$\int_0^1\int_0^1\int_0^{\sqrt{1-z^2}} \frac{z}{y+1}\,dx\,dz\,dy = \int_0^1\int_0^1 \frac{z}{y+1}\int_0^{\sqrt{1-z^2}} dx\,dz\,dy$$

$$= \int_0^1 \frac{1}{y+1}\left(\int_0^1 z\underbrace{\left(\int_0^{\sqrt{1-z^2}} dx\right)}_{\sqrt{1-z^2}}dz\right)dy$$

$$= \int_0^1 \frac{1}{y+1}\left(\int_0^1 z\sqrt{1-z^2}\,dz\right)dy = \underbrace{\int_0^1 \frac{1}{y+1}dy}_{\ln|y+1|}\int_0^1 z\sqrt{1-z^2}\,dz$$

Substitution

$u = 1-z^2$

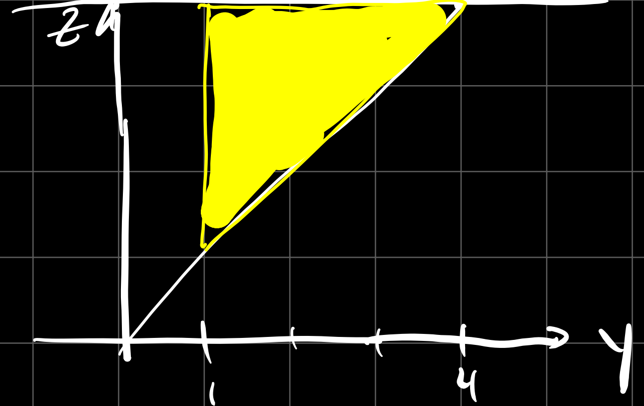
$du = -2z\,dz \rightarrow -\frac{1}{2}du = z\,dz$

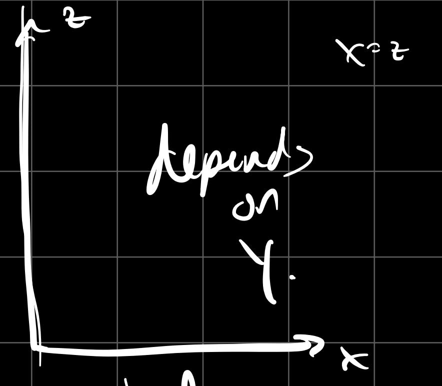
$$= (\ln 2)\left(-\frac{1}{2}\right)\int_1^0 \sqrt{u}\,du = \frac{\ln 2}{2}\int_0^1 u^{\frac{1}{2}}du = \frac{\ln 2}{2}\left(\frac{2}{3}\right)u^{3/2}\Big|_{u=0}^{u=1}$$

$$= \frac{\ln 2}{3}.$$

(11) $\iiint_E \frac{z}{x^2+z^2}\,dV \quad E=\{(x,y,z) \mid 1\le y\le 4,\ y\le z\le 4,\ 0\le x\le z\}$

y z x





not for right now.

$$\iiint_E \frac{z}{x^2+z^2}\,dV = \int_1^4 \int_y^4 \int_0^z \frac{z}{x^2+z^2}\,dx\,dz\,dy$$

want $\int \frac{1}{u^2+1}\,du$

$$= \int_1^4 \int_y^4 \frac{1}{z^2} \int_0^z \frac{z}{\left(\frac{x}{z}\right)^2+1}\,dx\,dz\,dy$$

$u = \frac{x}{z}$

$du = \frac{1}{z}\,dx$

$z\,du = dx$

$$= \int_1^4 \int_y^4 \frac{1}{\cancel{z^2}} \int_0^1 \frac{\cancel{z^2}\,du}{u^2+1}\,dz\,dy$$

$$= \int_1^4 \int_y^4 \underbrace{\int_0^1 \frac{1}{u^2+1}\,du}_{\arctan(1)-\arctan(0).}\,dz\,dy$$

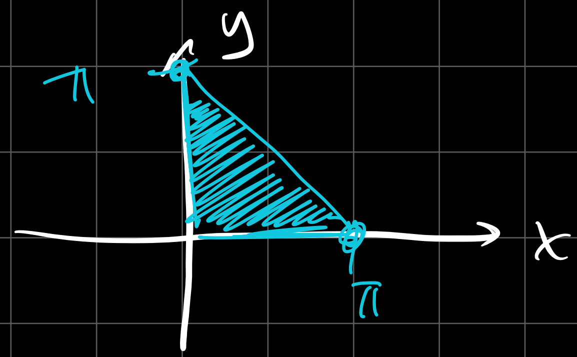
$$= \frac{\pi}{4} \int_1^4 \int_y^4 dz\,dy$$

$$= \frac{\pi}{4} \int_1^4 (4-y)\,dy \quad \text{easy...}$$

(12) $\iiint_E \sin y \, dV$

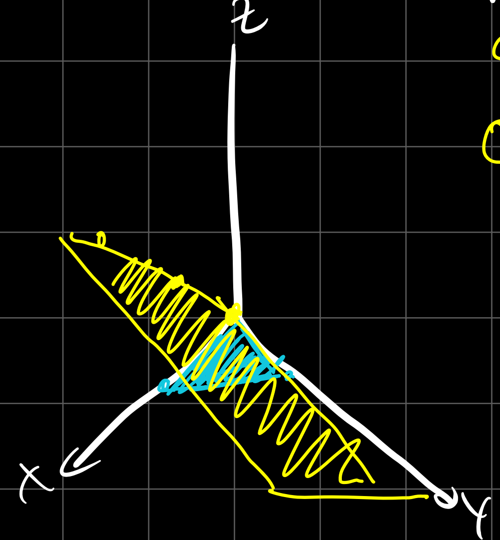
E ↪ below $z=x$ and above triangular region with vertices: $(0,0,0)$, $(0,\pi,0)$, $(\pi,0,0)$

$0 \le z \le x$



$0 \le x \le \pi$

$0 \le y \le \pi - x$ $\quad z = x \Leftrightarrow x - z = 0$



$$\iiint_E \sin y \, dV = \int_0^{\pi} \int_0^{\pi - x} \int_0^{x} \sin y \, dz \, dy \, dx$$

(53) Find Avg value of $f(x,y,z) = xyz$ over the cube with sides $= L$ in 1^{st} octant, one vertex at $(0,0,0)$ edges parallel to coordinate axes

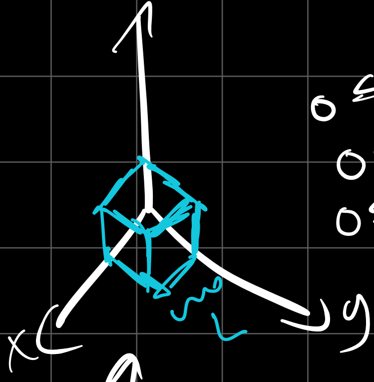
note:

$f_{avg} = \dfrac{\int_a^b f \, dx}{b-a}$

$f_{avg} = \dfrac{\iint_D f \, dA}{A(D)}$

$\iint 1 \, dA$

z



$0 \le x \le L$
$0 \le y \le L$
$0 \le z \le L$

$f_{avg} = \dfrac{\iiint_E f\,dV}{V(E)}$

$\iiint_E dV$

$V(E) = L^3$

$$f_{avg} = \frac{1}{L^3}\int_0^L\int_0^L\int_0^L xyz\,dx\,dy\,dz$$

$$= \frac{1}{L^3}\left(\int_0^L x\,dx\right)^3 = \frac{1}{L^3}\left(\frac{L^2}{2}\right)^3 = \frac{L^6}{8L^3} = \frac{L^3}{8}$$

(55) $\iiint_E (1 - x^2 - 2y^2 - 3z^2)\,dV$

find E s.t. integral is a max.

Think:

$\int_I 1 - x^2\,dx$ want $1 - x^2 \ge 0 \Rightarrow -1 \le x \le 1$

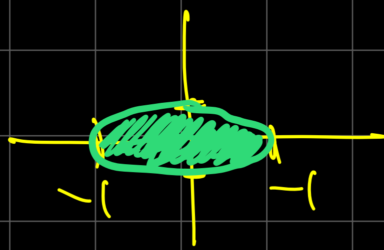
$\iint_D 1 - x^2 - 2y^2\,dA$ want $1 - x^2 - 2y^2 \ge 0$

$x^2+2y^2 \le 1 \to 2y^2 \le 1-x^2$

$y^2 \le \frac{1-x^2}{2}$

$-1 \le x \le 1$

$-\sqrt{\frac{1-x^2}{2}} \le y \le \sqrt{\frac{1-x^2}{2}}$



$$\iiint_E (1-x^2-2y^2-3z^2)\,dV$$

need: $1-x^2-2y^2-3z^2 \ge 0$

$x^2+2y^2+3z^2 \le 1$

$3z^2 \le 1-x^2-2y^2$

$z^2 \le \frac{1-x^2-2y^2}{3}$

Same x,y now:

$$-\sqrt{\frac{1-x^2-2y^2}{3}} \le z \le \sqrt{\frac{1-x^2-2y^2}{3}}$$

Optimal value:

$$\int_{-1}^{1}\int_{-\sqrt{\frac{1-x^2}{2}}}^{\sqrt{\frac{1-x^2}{2}}}\int_{-\sqrt{\frac{1-x^2-2y^2}{3}}}^{\sqrt{\frac{1-x^2-2y^2}{3}}} (1-x^2-2y^2-3z^2)\,dz\,dy\,dx$$