MATH-UA 253 - Fall 2022 - Take-home midterm solution

Problem 1.1. First, the gradient of f is:

$$\nabla f(x) = Ax - b, \tag{1}$$

To find x^*, the first-order necessary conditions for optimality require:

$$\nabla f(x^*) = 0 \quad \iff \quad Ax^* - b = 0 \quad \iff x^* = A^{-1}b. \tag{2}$$

Note: since A is a positive definite, all of its eigenvalues are positive. This means that $\det(A) > 0$, implying that A is invertible.

Next, note that the Hessian of f is given by:

$$\nabla^2 f(x) = A. \tag{3}$$

By assumption, A is positive definite. But then our definition of strict convexity tells us that f is strictly convex. The second-order sufficient conditions for optimality then tell us that x^* must be the unique global minimizer of f.

Problem 1.2. For each $k \geq 0$, we have:

$$x_{k+1} = x_k - \alpha\nabla f(x_k) = x_k - \alpha(Ax_k - b) = x_k - \alpha Ax_k + \alpha b = x_k - \alpha Ax_k + \alpha Ax^*, \tag{4}$$

where the second equality follows from (1) and the last equality follows from (2). Now, if we subtract x^* from both sides of (5), we get:

$$x_{k+1} - x^* = x_k - \alpha Ax_k - x^* + \alpha Ax^* = (I - \alpha A)x_k - (I - \alpha A)x^* = (I - \alpha A)(x_k - x^*). \tag{5}$$

Problem 1.3. If $\alpha = \lambda_1^{-1}$, then for $k > 0$:

$$\|x_k - x^*\|_2 = \left\|\left(I - \lambda_1^{-1}A\right)(x_{k-1} - x^*)\right\|_2 \leq \left\|I - \lambda_1^{-1}A\right\|_2 \|x_{k-1} - x^*\|_2 \,. \tag{6}$$

Here, the quantity "$\left\|I - \lambda_1^{-1}A\right\|_2$" is the spectral norm of $I - \lambda_1^{-1}A$. Recall that the spectral norm of a symmetric matrix is just the absolute value of its maximum eigenvalue.

We will show that $1 - \lambda_n/\lambda_1$ is the eigenvalue of $I - \lambda_1^{-1}A$ with the maximum absolute value. First, to compute the eigenvalues of $I - \lambda_1^{-1}A$, let v_i be the eigenvector of A corresponding to λ_i. Then:

$$(I - \lambda_1^{-1}A)v_i = v_i - \lambda_1^{-1}\lambda_i v_i = (1 - \lambda_i/\lambda_1)v_i. \tag{7}$$

Hence, for each i, $1 - \lambda_i/\lambda_1$ is an eigenvalue of $I - \lambda^{-1}A$ with eigenvector v_i. Next, note that since:

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n > 0, \tag{8}$$

we can divide this chain of inequalities by λ_1 to obtain (since $\lambda_1 > 0$, the orientation of the inequalities is unchanged):

$$1 \geq \lambda_2/\lambda_1 \geq \cdots \geq \lambda_n/\lambda_1 > 0. \tag{9}$$

And if we map it under $x \mapsto 1 - x$, we get (since $1 - x$ is a decreasing function, the orientation of the inequalities flips):

$$0 \leq 1 - \lambda_2/\lambda_1 \leq \cdots \leq 1 - \lambda_n/\lambda_1 < 1. \tag{10}$$

But these are exactly the eigenvalues of $I - \lambda_1^{-1}A$, as shown above. We can conclude two things from this: all of the eigenvalues of $I - \lambda_1^{-1}A$ are nonnegative, and the largest eigenvalue is $1 - \lambda_n/\lambda_1$. This proves the claim.

Since $1 - \lambda_n/\lambda_1$ is the eigenvalue of $I - \lambda_1^{-1}A$ with the maximum absolute value, we have:

$$\|x_k - x^*\|_2 \leq \left(1 - \frac{\lambda_n}{\lambda_1}\right) \|x_{k-1} - x^*\|_2 . \tag{11}$$

But notice that we can iterate this inequality:

$$\|x_k - x^*\|_2 \leq \left(1 - \frac{\lambda_n}{\lambda_1}\right)^2 \|x_{k-2} - x^*\|_2 \leq \cdots \leq \left(1 - \frac{\lambda_n}{\lambda_1}\right)^k \|x_0 - x^*\|_2 . \tag{12}$$

Problem 1.4. By iterating (5), we get:

$$x_k - x^* = (I - \alpha A)^k (x_0 - x^*). \tag{13}$$

Since A is symmetric, the matrix:

$$V = \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix} \tag{14}$$

is orthogonal and if we let:

$$\Lambda = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \tag{15}$$

the eigenvalue decomposition of A is written:

$$A = V \Lambda V^\top. \tag{16}$$

Furthermore, if we let $\alpha_k = (\alpha_1^{(k)}, \ldots, \alpha_n^{(k)})$, we can write:

$$x_k - x^* = V \alpha_k. \tag{17}$$

This follows from the definition of each $\alpha_i^{(k)}$. Then, we can rewrite (13) as:

$$V \alpha_k = \left(I - \frac{1}{\lambda_1} V \Lambda V^\top\right)^k V \alpha_0. \tag{18}$$

But note that we can write:

$$\begin{aligned} \left(I - \frac{1}{\lambda_1} V \Lambda V^\top\right)^k &= \left(V V^\top - \frac{1}{\lambda_1} V \Lambda V^\top\right)^k = \left[V \left(I - \frac{1}{\lambda_1}\Lambda\right) V^\top\right]^k \\ &= \underbrace{V \left(I - \frac{1}{\lambda_1}\Lambda\right) V^\top \cdots V \left(I - \frac{1}{\lambda_1}\Lambda\right) V^\top}_{k \text{ times}} = V \left(I - \frac{1}{\lambda_1}\Lambda\right)^k V^\top. \end{aligned} \tag{19}$$

Hence, we have:

$$V \alpha_k = V \left(I - \frac{1}{\lambda_1}\Lambda\right)^k V^\top V \alpha_0 = V \left(I - \frac{1}{\lambda_1}\Lambda\right)^k \alpha_0, \tag{20}$$

and multiplying on the left by $V^\top$ gives:

$$\alpha_k = \left(I - \frac{1}{\lambda_1}\Lambda\right)^k \alpha_0. \tag{21}$$

Equivalently:

$$\alpha_i^{(k)} = \left(1 - \frac{\lambda_i}{\lambda_1}\right)^k \alpha_i^{(0)}. \tag{22}$$

Problem 1.5. Note, from (10), we can see that $(1-\lambda_i/\lambda_1)^k$ will decay faster than $(1-\lambda_j/\lambda_1)^k$ as we increase k if $i < j$. Hence, as we iterate, the components $\alpha_i^{(k)}$ will decay faster if i is smaller—i.e., for components corresponding to larger eigenvalues.

Problem 1.6. First, note that:

$$\|x_k - x^*\|^2 = \|V\alpha_k\|^2 = \alpha_k^\top V^\top V \alpha_k = \alpha_k^\top \alpha_k = \|\alpha_k\|^2. \tag{23}$$

Then:

$$\|x_k - x^*\|^2 = \|\alpha_k\|^2 = \left\| \left(1 - \frac{\lambda_i}{\lambda_1}\right)^k \alpha_0 \right\|^2 = \sum_{i=1}^n \left(1 - \frac{\lambda_i}{\lambda_1}\right)^{2k} \left(v_i^\top (x_0 - x^*)\right)^2, \tag{24}$$

since $\alpha_i^{(0)} = v_i^\top (x_0 - x^*)$.

Problem 2.1. Note that:

$$e(v_0, \ldots, v_n) = \frac{1}{2} \sum_{0 \le i < j \le n} \sigma_{ij}(v_i - v_j)^2 = \sum_{i=0}^n \sum_{j=0}^n \sigma_{ij}(v_i - v_j)^2. \tag{25}$$

Each term in this sum is zero if x_i and x_j are disconnected. Let's write $i \sim j$ to indicate that $\sigma_{ij} > 0$. Then:

$$e(v_0, \ldots, v_n) = \sum_{i=0}^n \sum_{j \sim i} \sigma_{ij}(v_i - v_j)^2. \tag{26}$$

We have:

$$\frac{\partial e}{\partial v_k} = \sum_{i=0}^n \sum_{j \sim i} \sigma_{ij} \frac{\partial}{\partial v_k}(v_i - v_j)^2 = 2 \sum_{i=0}^n \sum_{j \sim i} \sigma_{ij} \Big[(v_i - v_j)\delta_{k-i} - (v_i - v_j)\delta_{j-i}\Big]. \tag{27}$$

Note *very* carefully: in the above, $j \sim i$ implies $j \neq i$! This means that we always have $\delta_{j-i} = 0$. Hence:

$$\frac{\partial e}{\partial v_k} = 2 \sum_{i=0}^n \delta_{k-i} \left[\left(\sum_{j \sim i} \sigma_{ij} \right) v_i - \sum_{j \sim i} \sigma_{ij} v_j \right] = 2 \left(\sum_{j \sim k} \sigma_{kj} \right) v_k - 2 \sum_{j \sim k} \sigma_{kj} v_j. \tag{28}$$

Now, we assume that v_0 and v_n are constant. This is reasonable, since a voltage is applied to x_0 and x_n and held fixed. So, to find $v_1, \ldots, v_{n-1}$, we must solve:

$$\frac{\partial e}{\partial v_i} = 0, \qquad i = 1, \ldots, n-1. \tag{29}$$

We can tell from the form of $\partial e/\partial v_i$ that this is an $(n-1) \times (n-1)$ linear system. To write it out explicitly, we need to rearrange (29) so that terms involving $v_1, \ldots, v_{n-1}$ are on one side of the equation and terms involving v_0 and v_n are on the other side. To this end, we rewrite $\partial e/\partial v_i = 0$ as:

$$\left(\sum_{j \sim i} \sigma_{ij} \right) v_i - \sum_{\substack{j \sim i \\ j \neq 0 \\ j \neq n}} \sigma_{ij} v_j = \delta_{i \sim 0} \sigma_{i0} v_0 + \delta_{i \sim n} \sigma_{in} v_n. \tag{30}$$

If we define $A \in \mathbb{R}^{(n-1)\times(n-1)}$ and $b \in \mathbb{R}^{n-1}$ and $v = (v_1, \ldots, v_{n-1})$ such that:

$$A_{ij} = \begin{cases} \sum_{k \sim i} \sigma_{ki} & \text{if } i = j, \\ -\sigma_{ij} & \text{if } i \sim j, \\ 0 & \text{otherwise,} \end{cases} \tag{31}$$

and:

$$b_i = \delta_{i\sim 0}\sigma_{i0}v_0 + \delta_{i\sim n}\sigma_{in}v_n, \tag{32}$$

where $i = 1, \ldots, n-1$ and $j = 1, \ldots, n-1$, then:

$$Av = b. \tag{33}$$

So, to find the values of v, solve this linear system!

Problem 2.2. The condition $\partial e/\partial v_i = 0$ for $i = 0, \ldots, n$ is equivalent to Kirchoff's law if write $\iota_{ij} = (v_i - v_j)/r_{ij}$.

Problem 3. We have:

$$\nabla \|Ax - b\|^2 = 2A^\top(Ax - b), \qquad \nabla\sigma\|x\|^2 = 2\sigma x. \tag{34}$$

Hence, $\nabla f(x) = 0$ is equivalent to:

$$0 = A^\top(Ax - b) + \sigma x = A^\top Ax - A^\top b + \sigma x \quad \Longrightarrow \quad \left(A^\top A + \sigma I\right)x = A^\top b. \tag{35}$$

This gives the solution:

$$x = \left(A^\top A + \sigma I\right)^{-1} A^\top b. \tag{36}$$

Clearly, if $\sigma = 0$, we recover:

$$x = \left(A^\top A\right)^{-1} A^\top b = A^\dagger b, \tag{37}$$

which is the solution of the standard least squares problem.

Problem 4.1. There are many ways to approach this problem. If what you came up with seemed reasonable, I will give you points. Here is one idea.

First, we define an $(n+1) \times (n+1)$ grid of nodes on $[0,1] \times [0,1]$:

$$(x_i, y_j) = \left(\frac{i}{n}, \frac{j}{n}\right), \qquad i = 0, \ldots, n, \qquad j = 0, \ldots, n. \tag{38}$$

We will represent the solution u indirectly using a grid of nodal values, $u_{ij} = u(x_i, y_j)$. To approximate the partial derivatives at interior grid nodes, we let $h = 1/n$ and approximate the x partial using finite differences as:

$$\left.\frac{\partial u}{\partial x}\right|_{x_i, y_j} = \frac{u_{i+1,j} - u_{i-1,j}}{2h} + O\left(h^2\right), \quad 0 < i < n, \quad 0 < j < n, \tag{39}$$

and similarly for the y partial. For the grid nodes on the boundary of $[0,1] \times [0,1]$, we can use one-sided finite differences. Then, letting g_{ij} be the approximate gradient for the grid node (x_i, y_j), we can approximate the Dirichlet energy as:

$$e(u_{0,0}, u_{1,0}, \ldots, u_{n-1,n}, u_{n,n}) = \frac{1}{2}\sum_{i=0}^{n}\sum_{j=0}^{n}\|g_{ij}\|^2. \tag{40}$$

Problem 4.2. Note that g_{ij} is a function of $u_{i-1,j}$, $u_{i+1,j}$, $u_{i,j-1}$, and $u_{i,j+1}$ if it's in the interior, and fewer nodes if it's on the boundary. In either case, we can proceed as in Problem 2 by taking partial derivative of e defined in 4.1. Note that in this case, it is likely that setting the partials equal to zero will result in a linear system which can be solved directly without needing to use gradient descent.