Midterm 1

Chapter 1 First-order differential equations

1.2 First-order linear differential equations

$\frac{dy}{dt} + a(t)y = 0$

$\frac{1}{y}\frac{dy}{dt} = -a(t)$ → $\ln|y| = -\int a(t)dt + c$ → $y = ke^{-\int a(t)dt}$

Given $y(t_0) = y_0$, then $y(t) = y_0 e^{-\int_{t_0}^{t} a(t')dt'}$

$\frac{dy}{dt} + a(t)y = b(t)$

Need $g(t)a(t) = g'(t)$ → integrating factor $g(t) = e^{\int a(t)dt}$ → $\frac{d}{dt}(g(t)f(t)) = g(t)b(t)$

$f(t) = \frac{\int g(t)b(t)dt + c}{g(t)} = \frac{\int e^{\int a(t)dt} b(t)dt + c}{e^{\int a(t)dt}}$

1.4 Separable equations

$\frac{dy}{dt} = \frac{t^3}{y^3}$

$y^3\frac{dy}{dt} = t^3$

$\frac{y^4}{4} = \frac{t^4}{4} + c$ or $y = \pm\sqrt[4]{t^4 + k}$

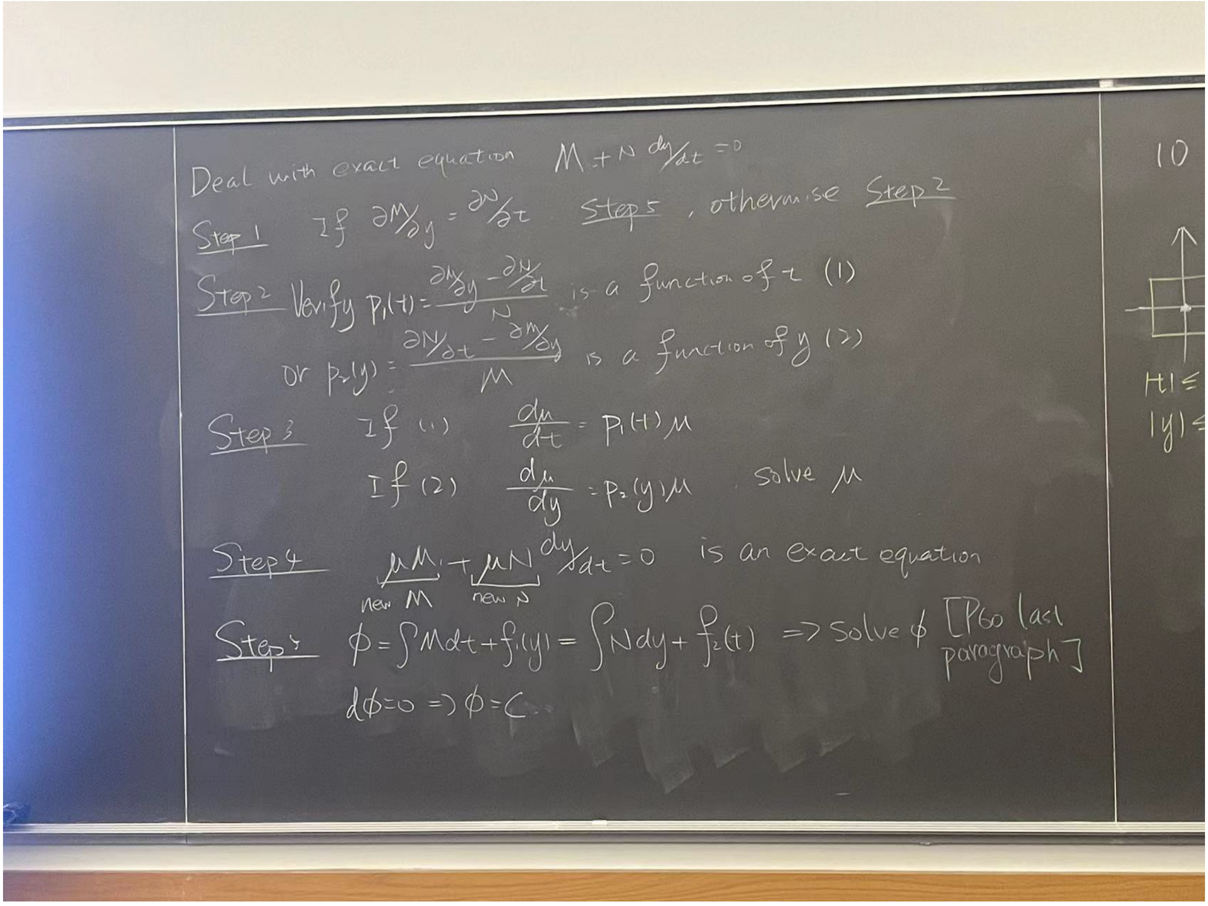
1.5 Population models

1.9 Exact equations, and why we cannot solve very many differential equations

Theorem 1. *Let $M(t,y)$ and $N(t,y)$ be continuous and have continuous partial derivatives with respect to t and y in the rectangle R consisting of those points (t,y) with $a<t<b$ and $c<y<d$. There exists a function $\phi(t,y)$ such that $M(t,y) = \partial\phi/\partial t$ and $N(t,y) = \partial\phi/\partial y$ if, and only if,*

$$\partial M/\partial y = \partial N/\partial t$$

in R.



1.10 The existence-uniqueness theorem; Picard iteration

Picard iterates: $y_{n+1}(t) = y_0 + \int_{t_0}^{t} f(s, y_n(s))ds$

Theorem 2′. *Let f and $\partial f/\partial y$ be continuous in the rectangle $R: t_0 \leqslant t \leqslant t_0 + a$, $|y - y_0| \leqslant b$. Compute*

$$M = \max_{(t,y) \text{ in } R} |f(t,y)|, \quad \textit{and set} \quad \alpha = \min\left(a, \frac{b}{M}\right).$$

Then, the initial-value problem

$$y' = f(t,y), \qquad y(t_0) = y_0 \tag{16}$$

has a unique solution $y(t)$ on the interval $t_0 \leqslant t \leqslant t_0 + \alpha$. In other words, if $y(t)$ and $z(t)$ are two solutions of (16), then $y(t)$ must equal $z(t)$ for $t_0 \leqslant t \leqslant t_0 + \alpha$.

Example 4. Show that the solution $y(t)$ of the initial-value problem

$$\frac{dy}{dt} = e^{-t^2} + y^3, \qquad y(0) = 1$$

exists for $0 \leqslant t \leqslant 1/9$, and in this interval, $0 \leqslant y \leqslant 2$.
Solution. Let R be the rectangle $0 \leqslant t \leqslant \frac{1}{9}$, $0 \leqslant y \leqslant 2$. Computing

$$M = \max_{(t,y) \text{ in } R} e^{-t^2} + y^3 = 1 + 2^3 = 9,$$

we see that $y(t)$ exists for

$$0 \leqslant t \leqslant \min\left(\tfrac{1}{9}, \tfrac{1}{9}\right)$$

and in this interval, $0 \leqslant y \leqslant 2$.