Dot Product

$\vec{u}, \vec{v} \in \mathbb{R}^n$

$\vec{u}\cdot\vec{v} = u_1v_1 + u_2v_2 + \cdots + u_nv_n = \|\vec{u}\|\,\|\vec{v}\| \cos\theta$

$\vec{u}\cdot\vec{v} = 0 \iff \vec{u} \perp \vec{v}$

if $\vec{u} \parallel \vec{v}$ then $\vec{u} = \alpha\vec{v}$

Scalar.

$\vec{v} = \langle v_1, \ldots, v_n \rangle \quad \alpha\vec{v} = \langle \alpha v_1, \ldots, \alpha v_n \rangle$

$\vec{u}\cdot\vec{v} = (\alpha\vec{v})\cdot\vec{v} = \langle \alpha v_1, \ldots, \alpha v_n \rangle \cdot \langle v_1, \ldots, v_n \rangle$

$= \alpha v_1^2 + \alpha v_2^2 + \cdots + \alpha v_n^2$

$= \alpha(v_1^2 + \cdots + v_n^2)$

$= \alpha\|\vec{v}\|^2$

① $(\vec{a}\cdot\vec{b})\cdot\vec{c}$ = (Scalar) · vector ✗

$\|\vec{a}\|(\vec{b}\cdot\vec{c})$ = (Scalar)(Scalar) = Scalar ✓

$\vec{a}\cdot\vec{b} + \vec{c}$ = Scalar + vector ✗

$(\vec{a}\cdot\vec{b})\vec{c}$ ✓ Scalar(vector) = Vector.

⑥ $\vec{a} = \langle p, -p, 2p \rangle \qquad \vec{b} = \langle 2q, q, -q \rangle$

$\vec{a}\cdot\vec{b} = 2pq - pq - 2pq = -pq.$

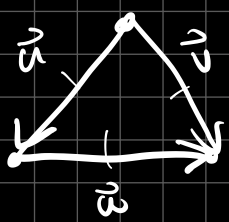
⑧ $\vec{a} = 3\hat{i} + 2\hat{j} - \hat{k} \qquad \vec{b} = 4\hat{i} + 5\hat{k} = 4\hat{i} + 0\hat{j} + 5\hat{k}$

$\vec{a}\cdot\vec{b} = (3)(4) + (2)(0) + (-1)(5) = 7$

Note: We know if $\vec{a}\cdot\vec{b}=0$ $\vec{a}\perp\vec{b}$

$\vec{a}\cdot\vec{b}>0 \Rightarrow$ $\vec{a},\vec{b}$ point in same direction.

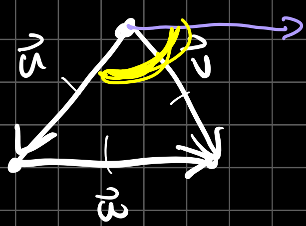
$\vec{a}\cdot\vec{b}<0 \Rightarrow$ " " point in opp. direction.

(11) 

$\|\vec{u}\|=1$

$\vec{u}\cdot\vec{v}$
$=\|\vec{u}\|\,\|\vec{v}\|\cos\theta$
$=\frac{1}{2}$

$\vec{u}\cdot\vec{w} = \|\vec{u}\|\,\|\vec{w}\|\cos(120^\circ) = -\frac{1}{2}$



(15ish) $\vec{a}=\langle 4,3\rangle$ $\vec{b}=\langle 5,12\rangle$

find angle b/w $\vec{a}$ & $\vec{b}$.

$\vec{a}\cdot\vec{b} = \|\vec{a}\|\,\|\vec{b}\|\cos\theta$

$\vec{a}\cdot\vec{b} = 20+36 = 56$

$\|\vec{a}\| = 5$

$\|\vec{b}\| = 13$

$56 = (5)(13)\cos\theta$ $\quad \frac{56}{65} = \cos\theta \Rightarrow \theta = \cos^{-1}\left(\frac{56}{65}\right)$

(24) $\vec{u} = \langle c,c,c\rangle$ $\quad \vec{v} = \langle c,0,-c\rangle$, $c\in\mathbb{R}$.

parallel, $\perp$, or neither.

$\vec{u}\cdot\vec{v} = c^2+0-c^2 = 0 \Rightarrow \perp$.

$\vec{u} = \langle -5, 4, -2 \rangle$ $\vec{v} = \langle 3, 4, -1 \rangle$

$\vec{u} \cdot \vec{v} \neq 0$ not $\perp$

check if $\vec{u} = \alpha \vec{v} = \langle 3\alpha, 4\alpha, -\alpha \rangle$

$-5 = 3\alpha$ $\alpha = -5/3$

$4 = 4\alpha$ $\alpha = 1$

$-2 = -\alpha$ $\alpha = 2$

Contradiction! Not $\parallel$.

26 $\langle 2, 1, -1 \rangle$

$\langle 1, x, 0 \rangle$ find x s.t angle b/w vectors $= 45°$

$\langle 2, 1, -1 \rangle \cdot \langle 1, x, 0 \rangle = 2 + x = \| \langle 2, 1, -1 \rangle \| \| \langle 1, x, 0 \rangle \| \cos(45°)$

$2 + x = \sqrt{6} \sqrt{1 + x^2} \left(\frac{1}{\sqrt{2}} \right)$

$4 + 4x + x^2 = 3 + 3x^2 \Rightarrow 2x^2 - 4x - 1 = 0$

$$x = \frac{4 \pm \sqrt{16 - 4(2)(-1)}}{2(2)}$$

$$= \frac{4 \pm \sqrt{24}}{4} = 1 \pm \frac{1}{2}\sqrt{6}$$

Projections. $\vec{a}, \vec{b}$

unit vector in direction of $\vec{a}$

$$\text{proj}_{\vec{a}} \vec{b} = \frac{\vec{a} \cdot \vec{b}}{\|\vec{a}\|} \frac{\vec{a}}{\|\vec{a}\|} = \frac{\vec{a} \cdot \vec{b}}{\|a\|^2} \vec{a}$$

└ component of proj of $\vec{b}$ onto $\vec{a}$

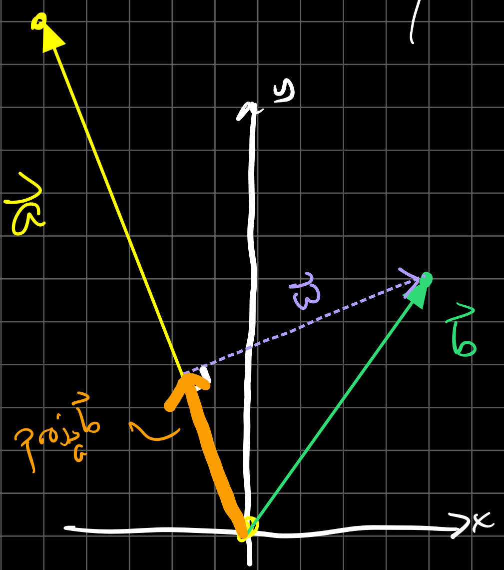
Scalar proj

(39) $\vec{a} = \langle -5, 12 \rangle$ $\quad$ $\vec{b} = \langle 4, 6 \rangle$

$\vec{a} \cdot \vec{b} = -20 + 72 = 52$

$\operatorname{proj}_{\vec{a}} \vec{b} = \dfrac{52}{169} \langle -5, 12 \rangle$

$\|\vec{a}\| = 13$



$\operatorname{proj}_{a}\vec{b} + \vec{w} = \vec{b}$

$\vec{w} = \vec{b} - \operatorname{proj}_{a}\vec{b}$

$\operatorname{orth}_{a}\vec{b}$

$\vec{b} = \operatorname{proj}_{a}\vec{b} + \operatorname{orth}_{a}\vec{b}$

(54) $\vec{r} = \langle x, y, z \rangle$

$\vec{a} = \langle a_1, a_2, a_3 \rangle$

$\vec{b} = \langle b_1, b_2, b_3 \rangle$

what does $(\vec{r} - \vec{a}) \cdot (\vec{r} - \vec{b}) = 0$ represent? Sphere.

$(\vec{r} - \vec{a}) \cdot \vec{r} - (\vec{r} - \vec{a}) \cdot \vec{b} = 0$

$\vec{r} \cdot \vec{r} - \vec{a} \cdot \vec{r} - \vec{r} \cdot \vec{b} + \vec{a} \cdot \vec{b} = 0$

$$\vec a\cdot\vec b=\vec b\cdot\vec a$$

$$\|\vec r\|^2$$

$$x^2+y^2+z^2-(a_1x+a_2y+a_3z)-(b_1x+b_2y+b_3z)+(a_1b_1+a_2b_2+a_3b_3)=0$$

$$\left(\frac{-(a_1+b_1)}{2}\right)^2$$

$$x^2-(a_1+b_1)x+y^2-(a_2+b_2)y+z^2-(a_3+b_3)z+\underbrace{(a_1b_1+a_2b_2+a_3b_3)}_{\vec a\cdot\vec b}=0.$$

$$+\frac{(a_1+b_1)^2}{4} \qquad\qquad +\frac{(a_1+b_1)^2}{4}$$

$$\left(x-\frac{(a_1+b_1)}{2}\right)^2+\left(y-\left(\frac{a_2+b_2}{2}\right)\right)^2+\left(z-\left(\frac{a_3+b_3}{2}\right)\right)^2=\frac{(a_1+b_1)^2}{4}+\frac{(a_2+b_2)^2}{4}+\frac{(a_3+b_3)^2}{4}-\vec a\cdot\vec b$$

center: $\left(\frac{a_1+b_1}{2},\frac{a_2+b_2}{2},\frac{a_3+b_3}{2}\right)$

radius $=\sqrt{\frac{(a_1+b_1)^2}{4}+\frac{(a_2+b_2)^2}{4}+\frac{(a_3+b_3)^2}{4}-\vec a\cdot\vec b}$ we could probably simplify.

$$=\frac{1}{2}\sqrt{(a_1+b_1)^2+(a_2+b_2)^2+(a_3+b_3)^2-4\vec a\cdot\vec b}$$

$$=\frac{1}{2}\sqrt{a_1^2+2a_1b_1+b_1^2+a_2^2+2a_2b_2+b_2^2+a_3^2+2a_3b_3+b_3^2-4a_1b_1-4a_2b_2-4a_3b_3}$$

$$=\frac{1}{2}\sqrt{(a_1-b_1)^2+(a_2-b_2)^2+(a_3-b_3)^2}$$

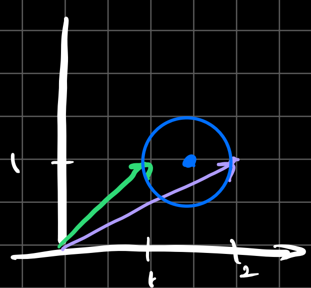
– half distance from (a_1,a_2,a_3) to (b_1,b_2,b_3).

look in 2-D

$$(\vec r-\langle 1,1\rangle)\cdot(\vec r-\langle 2,1\rangle)=0$$

center: $\left(\frac{1+2}{2},\frac{1+1}{2}\right)$
$=\left(\frac{3}{2},1\right)$

radius $=\frac{1}{2}\sqrt{(2-1)^2+(1-1)^2}$
$=\frac{1}{2}.$



Cauchy-Schwartz ineq.

(61) $|\vec{a}\cdot\vec{b}| \leq \|\vec{a}\|\|\vec{b}\|$ prove true.

$$|\vec{a}\cdot\vec{b}| = \Big|\|\vec{a}\|\|\vec{b}\|\cos\theta\Big| = \|\vec{a}\|\|\vec{b}\|\underbrace{|\cos\theta|}_{\leq 1}$$

$$\underline{|\vec{a}\cdot\vec{b}|} \leq \underline{\|\vec{a}\|\|\vec{b}\|}$$

$$\sqrt{\vec{a}\cdot\vec{a}} = \|\vec{a}\|$$

norm is induced by inner product