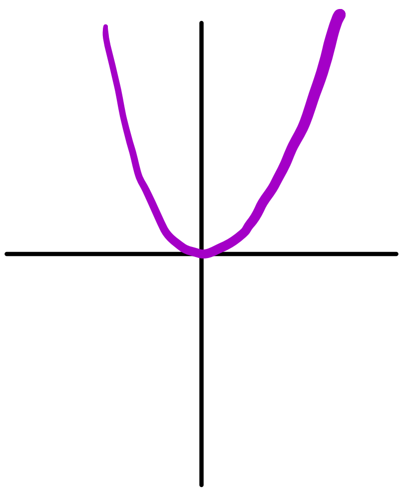
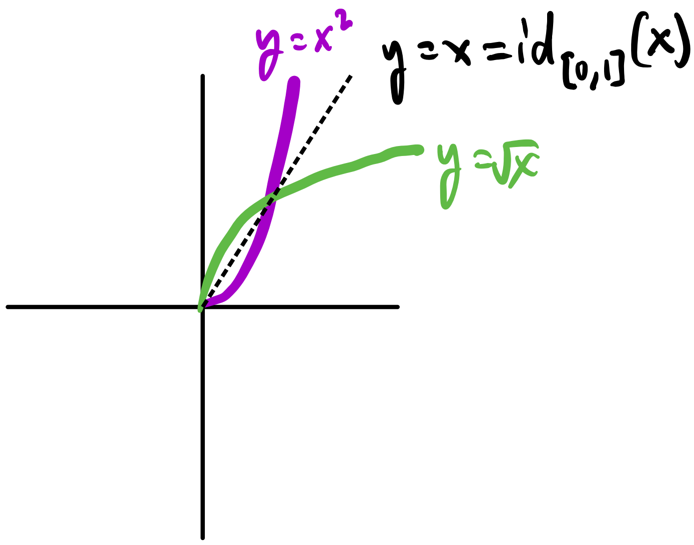
Today: $\boxed{\text{Ken}}$

$\boxed{7.3}$ Composition of Functions

Last time:

$\boxed{7.2}$ One-to-one, Onto, and Inverse Functions





$\boxed{7.3}$ Composition of Functions

Definition

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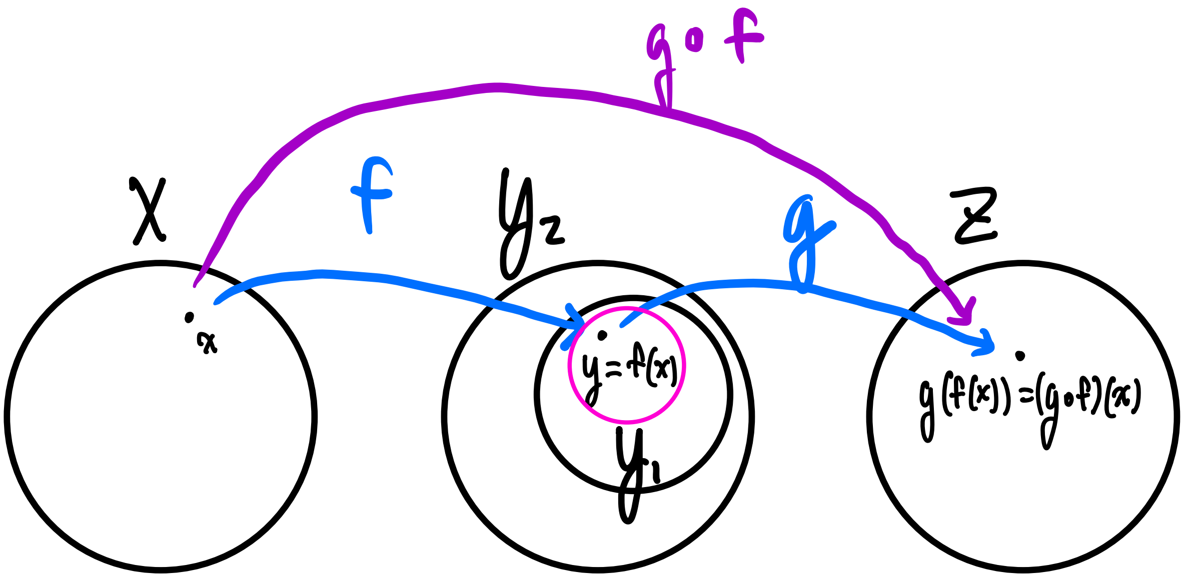
Let $f : X \to Y_1$ and $g : Y_2 \to Z$ such that $f(X) \subset Y_2$. Define $g \circ f : X \to Z$ such that, for any $x \in X$, $(g \circ f)(x) = g(f(x))$.

The function $g \circ f$ is called the **composition of f and g**.



Theorem 7.3.1

Let $f : X \to Z$, $id_X : X \to X$, and $id_Z : Z \to Z$ such that, for any $x \in X$, $id_X(x) = x$ and, for any $z \in Z$, $id_Z(z) = z$ are the identity functions on X and Z respectively. Then

$$f \circ id_X = f \quad \text{and} \quad id_Z \circ f = f.$$

$f : 2\mathbb{Z} \to (\mathbb{Z} - 2\mathbb{Z})$

$id_{2\mathbb{Z}} : 2\mathbb{Z} \to 2\mathbb{Z}$, $id_{2\mathbb{Z}}(z) := z$

$id_{\mathbb{Z}-2\mathbb{Z}} : \mathbb{Z}-2\mathbb{Z} \to \mathbb{Z}-2\mathbb{Z}$, $id_{\mathbb{Z}-2\mathbb{Z}}(w) = w$

$\cancel{id_{2\mathbb{Z}} \circ f}$ $\qquad$ $\cancel{(id_{2\mathbb{Z}} \circ f)(x) = id_{2\mathbb{Z}}(f(x))}$

$f(x) \in \mathbb{Z} - 2\mathbb{Z}$

Let $f: A \to B$. Consider $id_A: A \to A$ such that $id_A(x) = x$ for any $x \in A$.

Suppose $id_A': A \to A$ such that $id_A'(a) = a$ for any $a \in A$. id_A and id_A' have the same domain and codomain.

$id_A(x) = x = id_A'(x)$ for any $x \in A$ by definition. Let $a \in A$.

$$(f \circ id_A)(a) = f(id_A(a)) = f(a)$$
$$= f(id_A'(a)) = (f \circ id_A')(a)$$

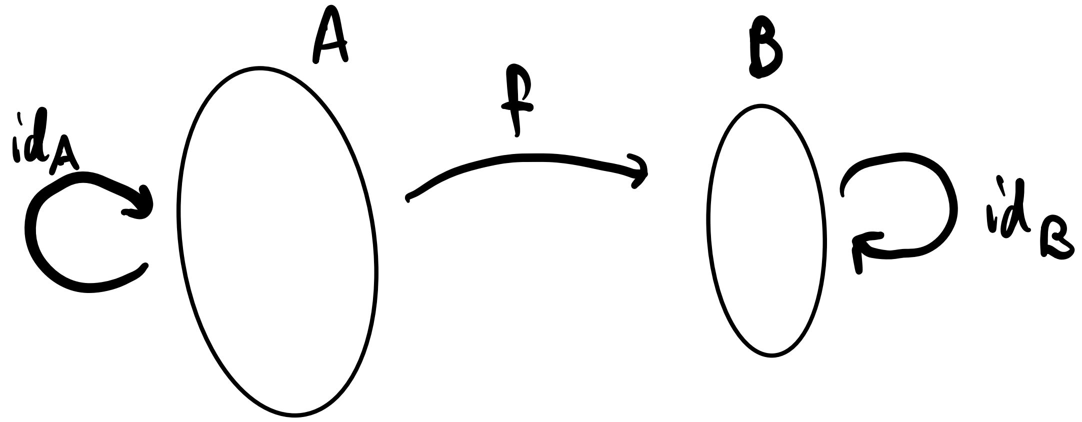
Consider $id_B: B \to B$ such that, for any $b \in B$, $id_B(b) := b$. Let $z \in A$. Then $f(z) \in B$, $y := f(z)$.

$$(id_B \circ f)(z) = id_B(f(z)) = id_B(y)$$
$$= y = f(z)$$

also $id_B \circ f : A \longrightarrow B$ and $f: A \to B$

have the same domain and codomain.

so $id_B \circ f = f$



How to show two functions $f: A \to B$ and $g: S \to T$ are equal:

① $A = S$ domains must be equal

② $B = T$ codomains must be equal

③ $f(x) = g(x)$ for any $x \in A = S$.

Theorem 7.3.2

If $f: X \to Z$ is a bijection and $f^{-1}: Z \to X$ its inverse, then

① $f^{-1} \circ f = id_X$

② $f \circ f^{-1} = id_Z$

$(\arctan \circ \tan)(x) = \arctan(\tan(x)) = x \in (-\pi/2, \pi/2)$

$(\tan \circ \arctan)(x) = \tan(\arctan(z)) = z \in (-\infty, \infty)$

Suppose $f: X \to Z$ is a bijection and $f^{-1}: Z \to X$ is its inverse.

① $f^{-1} \circ f : X \to X$

$id_X : X \to X$

$f^{-1}(f(a))$

$a \in \cancel{Z}\ X$

so $f^{-1} \circ f$ and id_X have the same domain and codomain.

Recall the property:
For any $x \in X$, $f(x) = z$ if and only if $f^{-1}(z) = x$.

Let $b \in X$. Consider
$f^{-1}(f(b)) =: y$.

Make a substitution,
$w := f(b)$. So
$y = f^{-1}(f(b)) = f^{-1}(w)$
if and only if
$f(y) = w$.

But $w = f(b)$ so
$f(y) = f(b)$. Since f
is one-to-one, i.e.

$\forall x_1, x_2 \in X \, (f(x_1) = f(x_2) \rightarrow x_1 = x_2)$,
$y = b$. Finally, recall that

$y = f^{-1}(f(b))$ so

$f^{-1}(f(b)) = b = id_X(b)$.

Theorem 7.3.3

If $f: X \to Y$ and $g: Y \to Z$ are both injective functions, then $g \circ f$ is injective.

Suppose $f: X \to Y$ and $g: Y \to Z$ are both injective. By def. of injective:

$\forall x_1, x_2 \in X \, (f(x_1) = f(x_2) \longrightarrow x_1 = x_2)$

$\forall y_1, y_2 \in Y \, (g(y_1) = g(y_2) \longrightarrow y_1 = y_2)$.

Recall $g \circ f : X \to Z$.

Let $a_1, a_2 \in X$.

Suppose $(g \circ f)(a_1) = (g \circ f)(a_2)$.

$$g(f(a_1)) = g(f(a_2))$$

Let $f(a_1) = b_1$ and $f(a_2) = b_2$.

So $g(b_1) = g(b_2)$ and by injectivity $b_1 = b_2$.

Then $f(a_1) = b_1 = b_2 = f(a_2)$.

Therefore $a_1 = a_2$, since f is injective.