13.3 Arclength

$\vec{r}(t) \quad a \le t \le b$

$$\text{Length} = \int_a^b \|\vec{r}\,'(t)\|\,dt$$

$$\cancel{S(x) = \int_0^x f(x)\,dx}$$

$$\cancel{S(1) = \int_0^1}$$

Arc length function $\quad t \ge a$

$$S(t) = \int_a^t \|\vec{r}\,'(u)\|\,du$$

not t

③ $\vec{r}(t) = \sqrt{2}\,t\,\hat{\imath} + e^t\,\hat{\jmath} + e^{-t}\,\hat{k} \qquad 0 \le t \le 1$

find arc length.

$$\vec{r}\,'(t) = \sqrt{2}\,\hat{\imath} + e^t\,\hat{\jmath} - e^{-t}\,\hat{k}$$

$$\|\vec{r}\,'(t)\| = \sqrt{(\sqrt{2})^2 + (e^t)^2 + (-e^{-t})^2} = \sqrt{2 + e^{2t} + e^{-2t}}$$

$$\int_0^1 \sqrt{2 + e^{2t} + e^{-2t}}\,dt$$

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

$$\sinh(x) = \frac{e^x - e^{-x}}{2}$$

$$= \int_0^1 \sqrt{2 + 2\cosh(2t)}\,dt$$

$$= \sqrt{2}\int_0^1 \sqrt{1 + \cosh(2t)}\,dt$$

much better way

b/c we need $1 + \cosh^2$

to get an Identity.

Don't use this here.

Try to factor

$$(e^t)^2 + (e^{-t})^2 + 2e^te^{-t}$$

$$= (e^t + e^{-t})^2$$

$$\sqrt{(e^t+e^{-t})^2} = e^t + e^{-t}$$

$$= \int_0^1 e^t + e^{-t}\,dt \qquad \text{or} \qquad 2\int_0^1 \cosh t\,dt = 2\sinh t\Big|_{t=0}^{t=1}$$

$$= (e^t - e^{-t})\Big|_{t=0}^{t=1}$$

(5) $\vec{r}(t) = \hat{i} + t^2\hat{j} + t^3\hat{k} \qquad 0 \le t \le 1$

$$\vec{r}\,'(t) = 2t\hat{j} + 3t^2\hat{k}$$

$$\|\vec{r}\,'(t)\| = \sqrt{4t^2 + 9t^4} = t\sqrt{4+9t^2}$$

$$\int_0^1 t\sqrt{4+9t^2}\,dt = \frac{1}{18}\int_4^{13} \sqrt{u}\,du = \frac{1}{18}\left(\frac{2}{3}u^{3/2}\right)\Big|_{u=4}^{u=13}$$

$u^{1/2}$

$u = 4+9t^2$

$du = 18t\,dt$

(11) let C be the intersection of $x^2 = 2y$ & $3z = xy$

find length of C from origin to $(6, 18, 36)$.

let $t=x$ $\quad y=\frac{t^2}{2}$ $\quad 3z=\frac{t^3}{2} \Rightarrow z=\frac{t^3}{6}$

$$\vec{r}(t)=t\hat{\imath}+\frac{t^2}{2}\hat{\jmath}+\frac{t^3}{6}\hat{k}$$

at $(0,0,0)$ $\quad t=0$ $\qquad$ at $(6,18,36)$ $\quad t=6.$

$$\vec{r}\,'(t)=\hat{\imath}+t\hat{\jmath}+\frac{t^2}{2}\hat{k}$$

$$\|\vec{r}\,'(t)\|=\sqrt{1+t^2+\frac{t^4}{4}}$$

$$\frac{t^4}{4}+t^2+1$$

$$\frac{t^2}{2}=a \rightarrow t^2=2a$$

$$a^2+2a+1$$
$$=(a+1)^2$$
$$=\left(\frac{t^2}{2}+1\right)^2$$

$$\int_0^6 \sqrt{1+t^2+t^4/4}\,dt$$

$$=\int_0^6 \sqrt{\left(\frac{t^2}{2}+1\right)^2}\,dt$$

$$=\int_0^6 \frac{t^2}{2}+1\,dt$$

$$=\left(\frac{t^3}{6}+t\right)\Big|_{t=0}^{t=6}$$

$$=36+6-0=42.$$

(14) $\vec{r}(t)=e^t\sin t\,\hat{\imath}+e^t\cos t\,\hat{\jmath}+\sqrt{2}e^t\hat{k}$ $\quad P(0,1,\sqrt{2})$

(a) find arclength function.

what is t at P? $\quad \vec{r}(0)=0\hat{\imath}+\hat{\jmath}+\sqrt{2}\hat{k}$ $\quad \underline{t=0}$

$$\vec{r}\,'(t)=\left(\underline{e^t\cos t+e^t\sin t}\right)\hat{\imath}+\left(\underline{e^t(-\sin t)+e^t\cos t}\right)\hat{\jmath}+\underline{\sqrt{2}e^t}\hat{k}$$

$$\|\vec r\,'(t)\| = \sqrt{e^{2t}\cos^2 t + \cancel{2e^{2t}\sin t\cos t} + e^{2t}\sin^2 t + e^{2t}\sin^2 t - \cancel{2e^{2t}\sin t\cos t} + e^{2t}\cos^2 t + 2e^{2t}}$$

$$= \sqrt{2e^{2t}\cos^2 t + 2e^{2t}\sin^2 t + 2e^{2t}}$$

$$= \sqrt{2}e^t\sqrt{\underbrace{\cos^2 t + \sin^2 t + 1}_{\sqrt{2}}}$$

$$= 2e^t$$

Arc length: $s(t) = \int_0^t 2e^u\,du = 2(e^t - e^0) = 2(e^t - 1)$

$$s = 2(e^t - 1)$$

(b) reparameterize the curve with respect to arc length

find $t(s)$ and put into $\vec r$

$$\frac{s}{2} = e^t - 1 \Rightarrow \frac{s}{2} + 1 = e^t \Rightarrow \underline{t = \ln\left(\frac{s}{2} + 1\right)}.$$

use: $e^{\ln\left(\frac{s}{2}+1\right)} = \frac{s}{2} + 1$

position vector after t units of time

$$\vec r(t) = e^t\sin t\,\hat\imath + e^t\cos t\,\hat\jmath + \sqrt{2}e^t\,\hat k$$

$$\vec r(s(t)) = \left(\frac{s}{2}+1\right)\sin\left[\ln\left(\frac{s}{2}+1\right)\right]\hat\imath + \left(\frac{s}{2}+1\right)\cos\left[\ln\left(\frac{s}{2}+1\right)\right]\hat\jmath + \sqrt{2}\left(\frac{s}{2}+1\right)\hat k$$

position vector after s units of length.

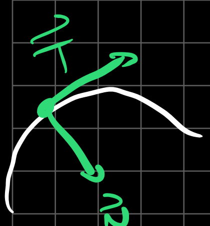
or $\int_0^t \|\vec r\,'(u)\|\,du =$ fixed s. Solve for t.

18 $\vec{r}(t) = \langle t^2, \sin t - t\cos t, \cos t + t\sin t \rangle \qquad t \geq 0$

(a) find $\vec{T}$, $\vec{N} = \dfrac{\vec{T}'}{\|\vec{T}'\|}$

unit tangent

unit normal



$$\vec{r}\,'(t) = \langle 2t, \cancel{\cos t} - \cancel{\cos t} + t\sin t, \cancel{-\sin t} + \cancel{\sin t} + t\cos t \rangle$$

$$= \langle 2t, t\sin t, t\cos t \rangle$$

$$\|\vec{r}\,'(t)\| = \sqrt{4t^2 + t^2\sin^2 t + t^2\cos^2 t} = \sqrt{5}\,t$$

$$\vec{T}(t) = \frac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|} = \left\langle \frac{2}{\sqrt{5}}, \frac{\sin t}{\sqrt{5}}, \frac{\cos t}{\sqrt{5}} \right\rangle$$

$$\vec{T}\,'(t) = \left\langle 0, \frac{\cos t}{\sqrt{5}}, \frac{-\sin t}{\sqrt{5}} \right\rangle$$

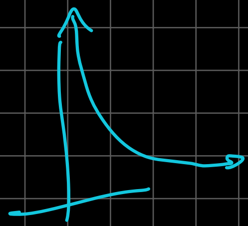
$$\|\vec{T}\,'(t)\| = \sqrt{0^2 + \frac{\cos^2 t}{5} + \frac{\sin^2 t}{5}} = \frac{1}{\sqrt{5}}$$

$$\vec{N}(t) = \frac{\vec{T}\,'}{\|\vec{T}\,'\|} = \langle 0, \cos t, -\sin t \rangle$$

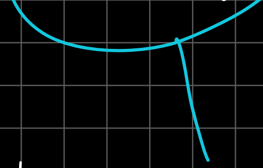
direction of bending of our curve

(b) find $\kappa(t)$ = curvature.

$$= \left\| \frac{d\vec{T}}{ds} \right\| = \frac{\|\vec{T}\,'(t)\|}{\|\vec{r}\,'(t)\|} = \frac{\|\vec{r}\,'(t) \times \vec{r}\,''(t)\|}{\|\vec{r}\,'(t)\|^3}$$



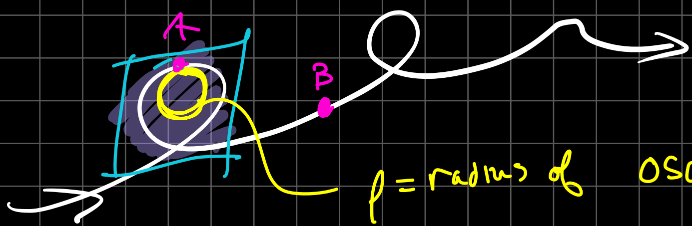
deriv w.r.t. arclength



$$\kappa = \frac{\frac{1}{\sqrt{5}}}{\sqrt{5}\,t} = \frac{1}{5t}$$

Osculating plane + circle.

κ at A bigger than κ at B



ρ = radius of osculating circle.

$$\kappa = \frac{1}{\rho}$$

Try 59 later if you can.