# Boundary Value Problems and Exit Times

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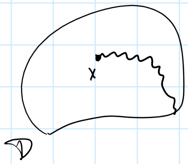
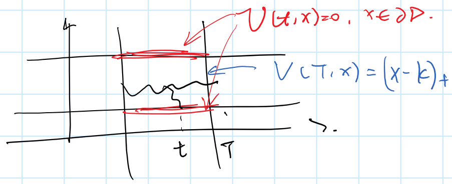



$\tau$: exit time $= \inf\{t : X_t \notin D\}$

(Borel-Cantelli Lemma to rescue!)

$$u(t,x) = \mathbb{E}_{X_t=x} V(\tau, X_\tau)$$

eg. KO option.




**Lemma 3.** *Let $\mathbb{E}\tau < \infty$. Then $\mathbb{E}(V(X_\tau)|\mathcal{F}_t)$ is a martingale with respect to filtration $(\mathcal{F}_{t\wedge\tau})_{t\geq 0}$.*

Applying Itô's lemma we get

$$du(t,x) = (u_t + au_x + \frac{1}{2}\sigma^2 u_{xx})dt + u_x\sigma dB_t. \tag{17}$$

By lemma 3 function $u(t,x)$ from (16) is a martingales and therefore there is no drift term in (17). Thus $u(t,x)$ solves the following PDE

$$u_t + au_x + \frac{1}{2}\sigma^2 u_{xx} = 0 \tag{18}$$

with boundary conditions $u(t,x) = V(t,x)$ for $x \in \partial D$ and $u(t,x) = V(T,x)$ for $x \in D$.

$$\begin{cases} u_t + \mathcal{L}u = 0 \\ u(T,x) = V(T,x) \\ u(t,x) = V(t,x), & x \in \partial D. \end{cases}$$ Type 1

$$\begin{cases} u_t + \mathcal{L}u - bu = 0 \\ u(T,x) = V(T,x) \\ u(t,x) = V(t,x) & x \in \partial D \end{cases}$$ Type 2

$$\begin{cases} u_t + \mathcal{L}u + b = 0 \\ u(T,x) = V(T,x) \\ u(t,x) = V(t,x) & x \in \partial D \end{cases}$$ Type 3

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问 As $T \to \infty$, $\tau = \min(T, \inf\{t, X_t \notin D\}) = \inf\{t, X_t \in D\}$

考虑 Type 3, let $b = 1$

$$u(t,x) = \mathbb{E}_{X_t=x} \int_t^\tau \cancel{b(s,X_s)}^{\,1} ds = \mathbb{E}_{X_t=x}(\tau - t)$$

final 考 ~ BVD. & stopping times

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再 $T \to \infty$.

考虑 $\vec{a}$, $\Sigma$ don't depend on $t$.

即 $\begin{cases} a(t,x) = a(x) \\ \sigma(t,x) = \sigma(x) \end{cases}$ ~ stationary process

则 只关心 where starts, 而非 when!

elliptic PDEs

则 type 1 $\Rightarrow \begin{cases} \mathcal{L}u = 0 \\ u(x) = V(x) & x \in \partial D \end{cases}$ type 2 $\Rightarrow \begin{cases} \mathcal{L}u - bu = 0 \\ u(x) = V(x) & x \in \partial D \end{cases}$ type 3 $\Rightarrow \begin{cases} \mathcal{L}u + b = 0 \\ u(x) = 0 & x \in \partial D \end{cases}$

elliptic PDEs

TW) type 1 $\Rightarrow$ $\begin{cases} \mathcal{L}u = 0 \\ u(x) = V(x), & x \in \partial D. \end{cases}$

type 2 $\Rightarrow$ $\begin{cases} \mathcal{L}u - bu = 0 \\ u(x) = V(x), & x \in \partial D \end{cases}$

type 3 $\Rightarrow$ $\begin{cases} \mathcal{L}u + b = 0 \\ u(x) = 0, & x \in \partial D. \end{cases}$

**Application: distribution of the first arrivals.** As the first application let us consider $\mathbb{E}_{X_t=x}\mathbf{1}_{\tau<T} = \mathbb{P}_{X_t=x}(\tau < T)$. According to the above we have to solve PDE

$$u_t + au_x + \frac{1}{2}\sigma^2 u_{xx} = 0 \tag{19}$$

with boundary condition $u = 1$ at $x \in \partial D$.