# VE414 Assignment 2

Due 10/24/2021

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## Problem 1

Mushroom can be poisonous. One of the keys for mushroom identification is the spore deposit. Measurements of 14 spores are given below:

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|---|---|---|---|---|---|---|
| 7.96 | 6.73 | 8.52 | 8.68 | 7.25 | 9.59 | 9.58 |
| 8.36 | 8.72 | 8.57 | 8.21 | 9.12 | 7.81 | 10.58 |

The proposed model for the measurements is $X|\theta, \sigma^2 \sim N(\theta, \sigma^2)$ with both $\theta$ and $\sigma^2$ unknown. We are interested in parameter $\theta$.

(a) Find the frequentist 95% confidence interval for the unknown mean. Assume noninformative prior on $(\theta, \sigma^2)$, i.e, $\pi(\theta, \sigma^2) \propto 1/\sigma^2$.

(b) Find the 95% credible set for $\theta$ if the prior is noninformative. Compare the solutions in (a) and (b).

(c) Assume now that the prior is normal-inverse-chi square($N - Inv\chi^2$):

$$\theta|\sigma^2 \sim N(8.35, \sigma^2/10)$$

$$\sigma^2 \sim Inv\chi^2(4, 1.5)$$

Find the 95% credible set for $\theta$ and compare it with interval estimators in (a) and (b).

## Problem 2

Calculate the following distributions

(a) Suppose that $X_1, X_2, \ldots, X_n \overset{iid}{\sim} Poisson(\lambda)$ and $\lambda$ has a $\Gamma(\alpha, \beta)$ prior. Find the posterior distribution for $\lambda$.

(b) Suppose that $X_1, X_2, \ldots, X_n \overset{iid}{\sim} Bernoulli(\theta)$ and $\theta$ has a $Beta(\alpha, \beta)$ prior. Find the posterior mean of $\theta$. Show that it is a weighted average of the sample mean and the prior mean.

# Problem 3

Suppose that we have n independent trials of an experiment where each trial can result in one of two possible outcomes labeled "success" (S) or "failure" (F). Suppose that the probability of success remains the same from trial to trial. Let $\theta = P(success)$ for each trial. Let X be the number of successes in n trials. We know that X has a binomial distribution with parameters n and (We write $X \sim bin(n, \theta)$.) and that X has pdf

$$P(X = x) = \binom{n}{k} \theta^x (1-\theta)^{n-x} = \frac{n!}{k!(n-k)!} \theta^x (1-\theta)^{n-x}$$

If we generalize this to k possible outcomes/categories with probabilities $\theta_1, \theta_2, \ldots, \theta_k$, respectively ($\Sigma_{i=1}^k \theta_i = 1$) and let $X_i$ be the number (in n trials) of outcomes in category $i$ ($i = 1, 2, \ldots, k$), we can collect the counts as a vector $(X_1, X_2, \ldots, X_k)$. Note that we must have $\Sigma_{i=1}^k X_i = n$.
The vector is said to have a multinomial distribution and the pdf is

$$P(X_1 = x_1, X_2 = x_2, \ldots, X_k = x_k) = \frac{n!}{x_1! x_2! \ldots x_k!} \theta_1^{x_1} \theta_2^{x_2} \ldots \theta_k^{x_k}$$

Here, the $x_i$ are non-negative integers that must sum to n. Suppose $(\theta_1, \theta_2, \ldots, \theta_k)$ follows a Dirichlet distribution which has pdf

$$f(\theta_1, \theta_2, \ldots, \theta_k) = \frac{\Gamma(\Sigma_{i=1}^k \alpha_i)}{\Pi_{i=1}^k \Gamma(\alpha_i)} \theta_1^{\alpha_1 - 1} \theta_2^{\alpha_2 - 1} \ldots \theta_k^{\alpha_k - 1}$$

where the $\alpha_i > 0$ are parameters of Dirichlet distribution.

(a) Show that the Dirichlet distribution is a conjugate prior for the multinomial distribution

(b) Does every distribution have a conjugate prior? Prove or give a counterexample and explain.

# Problem 4

Let $y_i | \theta \overset{iid}{\sim} Bernoulli(\theta)$ for $i \in \{1, ..., n\}$, and let $p(\theta) = I_{[0,1]}(\theta)$, i.e. $\theta$ is uniform on $[0, 1]$.

(a) Calculate the marginal distribution of $(y_1, ..., y_n)$.

(b) Calculate the predictive probability that $y_{n+1} = 1$ given that $y_1 = \cdots = y_n = 1$. Simplify the formula you get using the fact that $\Gamma(a+1) = a\Gamma(a)$ and thus establish that for $n = 1000$, $y_{n+1} = 1 = \frac{1001}{1002}$.

## Problem 5

Simulate 10,000 values from an exponential distribution with rate $\lambda$ for your choice of $\lambda$. Provide a histogram of your results with the exponential density superimposed. (Do not use a built-in function for exponential random variable generation. Do use a built-in function for uniform random variable generation!) Include your code when you turn in this homework.