LEC007 Inventory Management II

VG441 SS2020

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Wagner-Whitin Model

INPUT:

- Deterministic demand (non-stationary) over T periods

$$(d_1, d_2, d_3, \dots, d_{T-1}, d_T)$$

- No stockout is allowed
- Lead time L (setting $L = 0$ WLOG)
- Fixed cost $K > 0$ per order
- Purchase cost c per unit (setting $c = 0$ WLOG)
- Inventory hold cost $h > 0$ per unit per unit of time

OUTPUT:

The optimal ordering strategy

Mixed Integer Linear Program (MILP)

Decision Variables

$q_t =$ the number of units ordered in period
$y_t = 1$ if we order in period $t, 0$ otherwise
$x_t =$ the inventory level at the end of period, with $x_0 \equiv 0$

Inventory-balance constraint

$$
\begin{array}{lll}
\text{minimize} & \sum_{t=1}^{T} (K y_t + h x_t) & \\
\text{subject to} & x_t = x_{t-1} + q_t - d_t & \forall t = 1, \ldots, T \\
& q_t \leq M y_t & \forall t = 1, \ldots, T \\
& x_t \geq 0 & \forall t = 1, \ldots, T \\
& q_t \geq 0 & \forall t = 1, \ldots, T \\
& y_t \in \{0, 1\} & \forall t = 1, \ldots, T
\end{array}
$$

Linking constraint
("a common trick")

ZIO Property

- It is optimal to place orders only in time periods in which the inventory level is zero.

- This suggests that each order is of a size equal to the total demand in an integer number of subsequent periods, i.e., in period t, we either order d_t ,

  or $d_t + d_{t+1}$,

  or $d_t + d_{t+1} + d_{t+2}$,

  and so on.

Dynamic Programming

- Define θ_t to be the optimal cost in periods $[t, T]$ if we place optimal orders over $[t, T]$.

$$\theta_t = \min_{t < s \leq T+1} \left\{ \boxed{K + h \sum_{i=t}^{s-1} (i - t) d_i} + \theta_s \right\}$$

Cost of covering demands of periods $t, t+1, \ldots, s-1$

- Boundary condition:

$$\theta_{T+1} \equiv 0$$

Dynamic Programming

- Backward induction

$$\theta_t = \min_{t<s\leq T+1} \left\{ K + h \sum_{i=t}^{s-1} (i-t) d_i + \theta_s \right\}. \tag{3.39}$$

Algorithm 3.1 Wagner–Whitin algorithm

```
1: θ_{T+1} ← 0                                              ▷ Initialization
2: for t = T, ..., 1 do                                      ▷ Main loop
3:     θ_t ← right-hand side of (3.39)                        ▷ Minimization over s
4:     s(t) ← argmin in right-hand side of (3.39)
5: end for
6: return θ_t, s(t) for all t = 1, ..., T
```

Numerical Example

$K = 500, h = 2$ per period. The demands are 90, 120, 80, and 70.

$$
\begin{aligned}
\theta_5 &= 0 \\
\theta_4 &= K + h\,(0 \cdot d_4) + \theta_5 \\
&= 500 \quad [s(4) = 5] \\
\theta_3 &= \min\left\{K + h\,(0 \cdot d_3) + \theta_4, K + h\,(0 \cdot d_3 + 1 \cdot d_4) + \theta_5\right\} \\
&= \min\{1000, 640\} \\
&= 640 \quad [s(3) = 5] \\
\theta_2 &= \min\left\{K + h\,(0 \cdot d_2) + \theta_3, K + h\,(0 \cdot d_2 + 1 \cdot d_3) + \theta_4\right. \\
&\quad \left. K + h\,(0 \cdot d_2 + 1 \cdot d_3 + 2 \cdot d_4) + \theta_5\right\} \\
&= \min\{1140, 1160, 940\} \\
&= 940 \quad [s(2) = 5] \\
\theta_1 &= \min\left\{K + h\,(0 \cdot d_1) + \theta_2, K + h\,(0 \cdot d_1 + 1 \cdot d_2) + \theta_3\right. \\
&\quad K + h\,(0 \cdot d_1 + 1 \cdot d_2 + 2 \cdot d_3) + \theta_4 \\
&\quad \left. K + h\,(0 \cdot d_1 + 1 \cdot d_2 + 2 \cdot d_3 + 3 \cdot d_4) + \theta_5\right\} \\
&= \min\{1440, 1380, 1560, 1480\} \\
&= 1380 \quad [s(1) = 3]
\end{aligned}
$$

If you think about this…

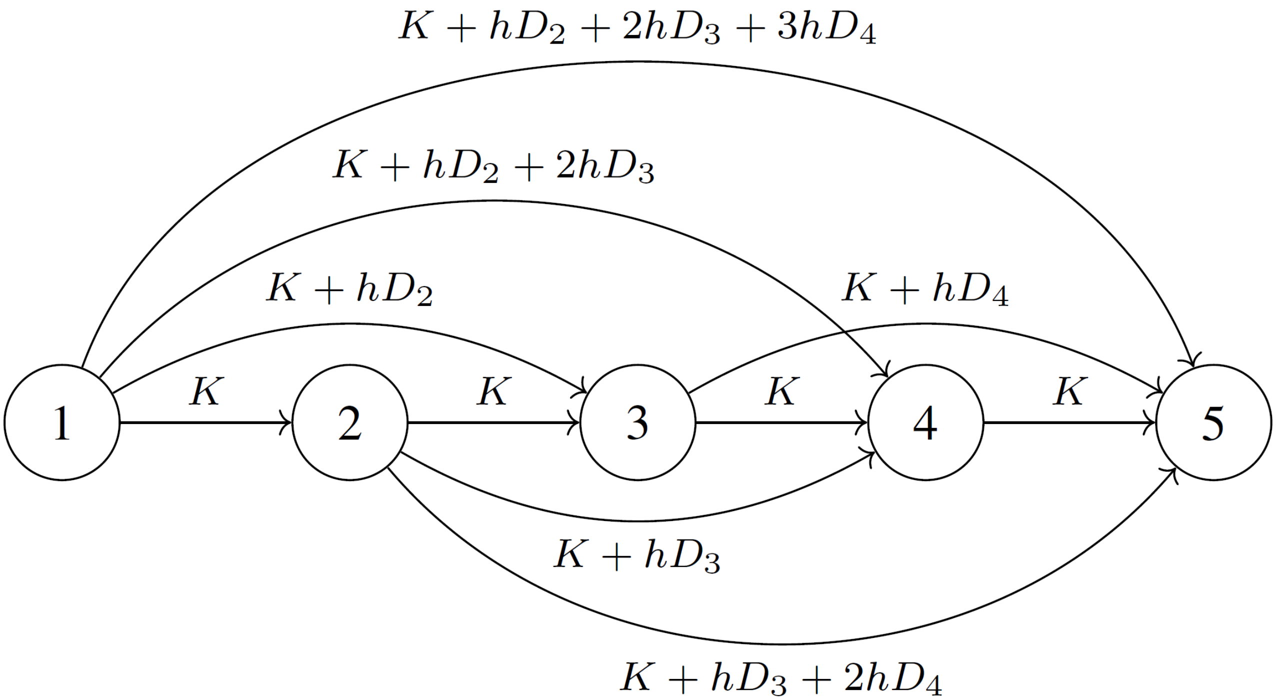


Figure 3.12 Wagner–Whitin network.

Shortest Path Problem

Input:

- Directed Graph $G(V,E)$ with $|V| = n, |E| = m$
- Each edge $e \in E$ has non-negative length $l_e \geq 0$
- Source vertex s

Output:

For each $v \in V$, compute

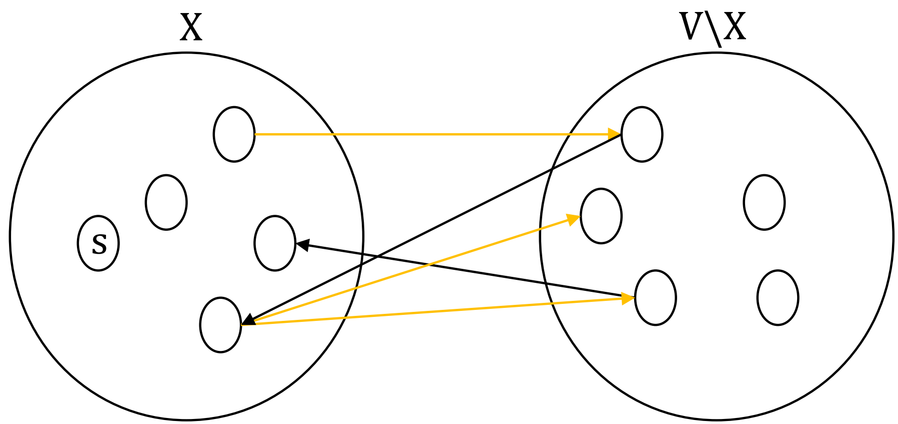
$L(v)$= length of a shortest s-v path in G

Fastest algorithm is called Dijkstra's Algorithm

Caveat: length/weight/travel time $l_e \geq 0$!

Dijkstra's Algorithm

- Initialize $X = \{s\}, A[s] = 0, B[s] = \emptyset$
- Main loop
  - While $X \neq V$

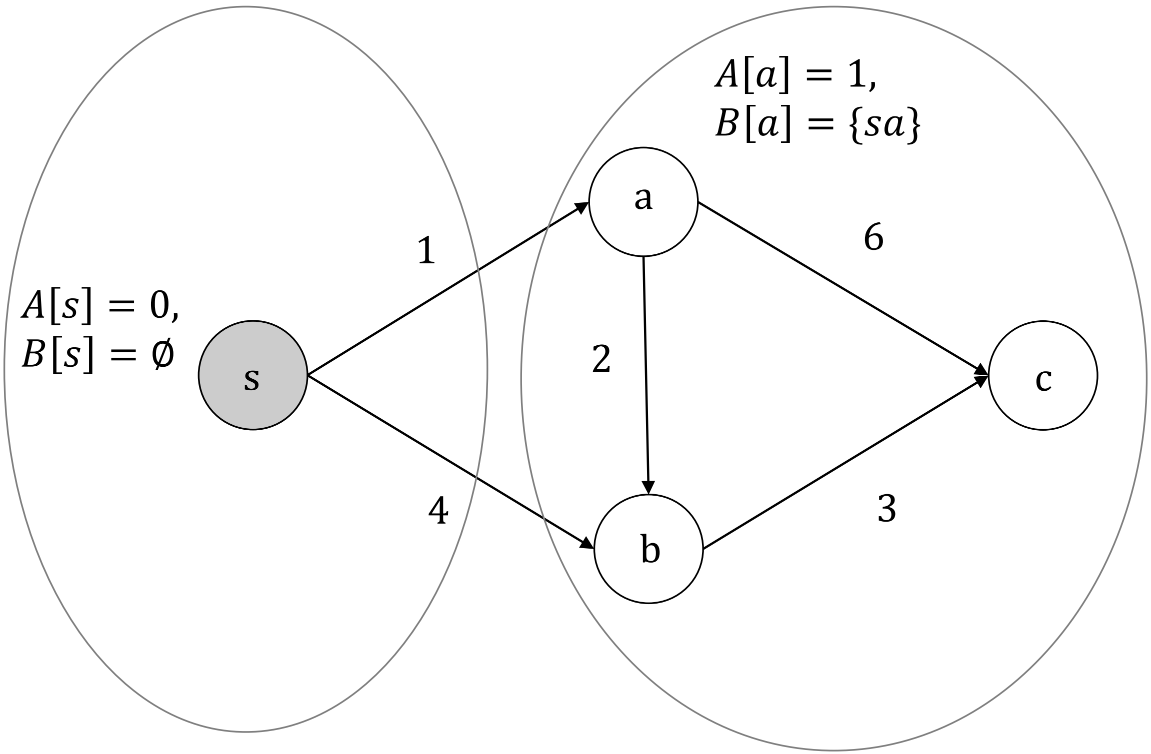


  - Of all edges $(v, w) \in E$ with $v \in$ X and $w \notin X$, pick the one that minimizes $A[v] + l_{vw}$ (Dijkstra’s greedy criterion)
  - Call the minimizing edge (v^*, w^*) and add vertex w^* to X
  - Set $A[w^*] = A[v^*] + l_{v^*w^*}$
  - Set $B[w^*] = B[v^*] \cup (v^*, w^*)$

An Example

$A[v] + l_{vw}$ (Dijkstra's greedy criterion)

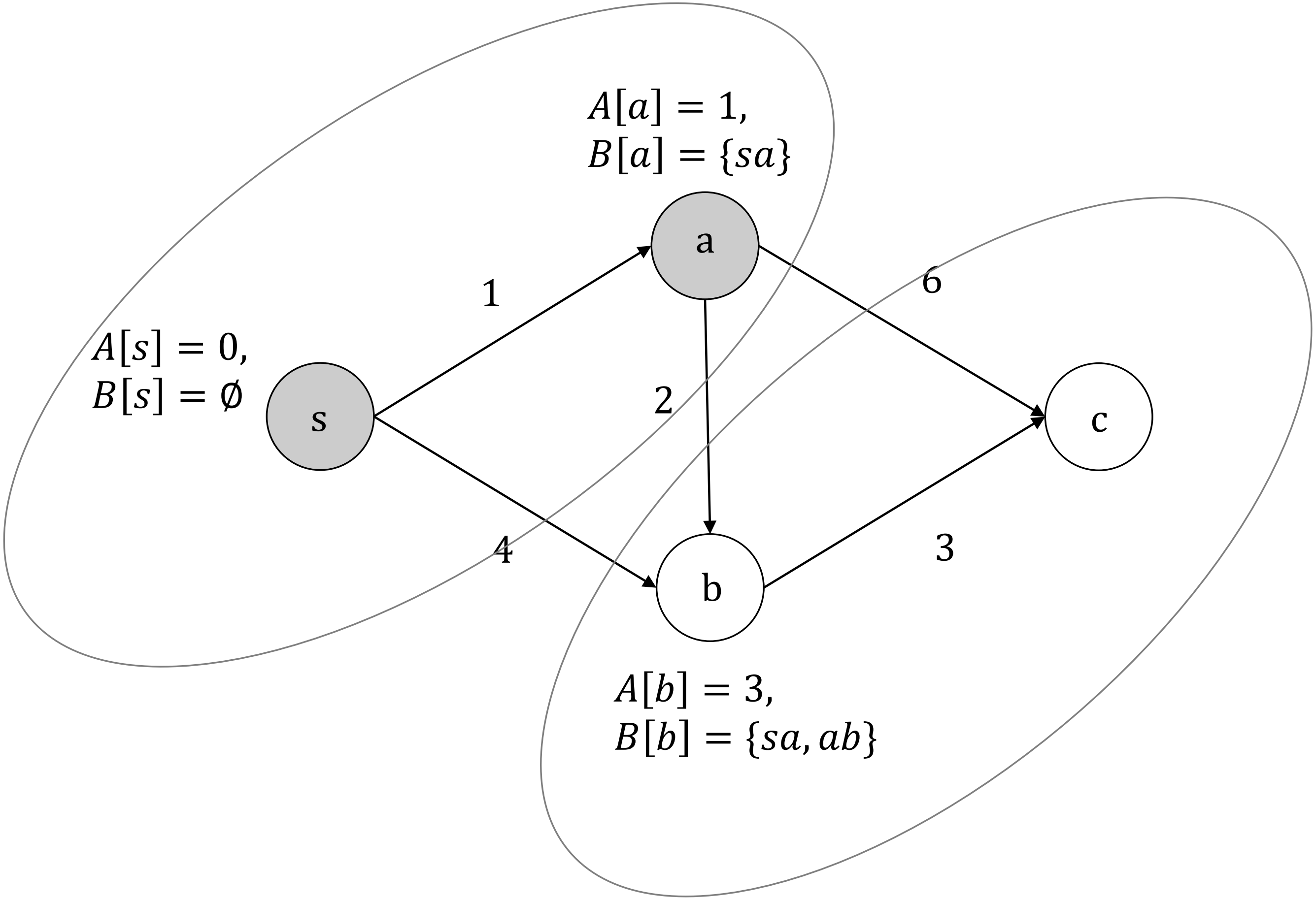
$min\ (A[s] + l_{sa}, A[s] + l_{sb}) = min\ (0 + 1, 0 + 4) = 1$



An Example

$A[v] + l_{vw}$ (Dijkstra's greedy criterion)

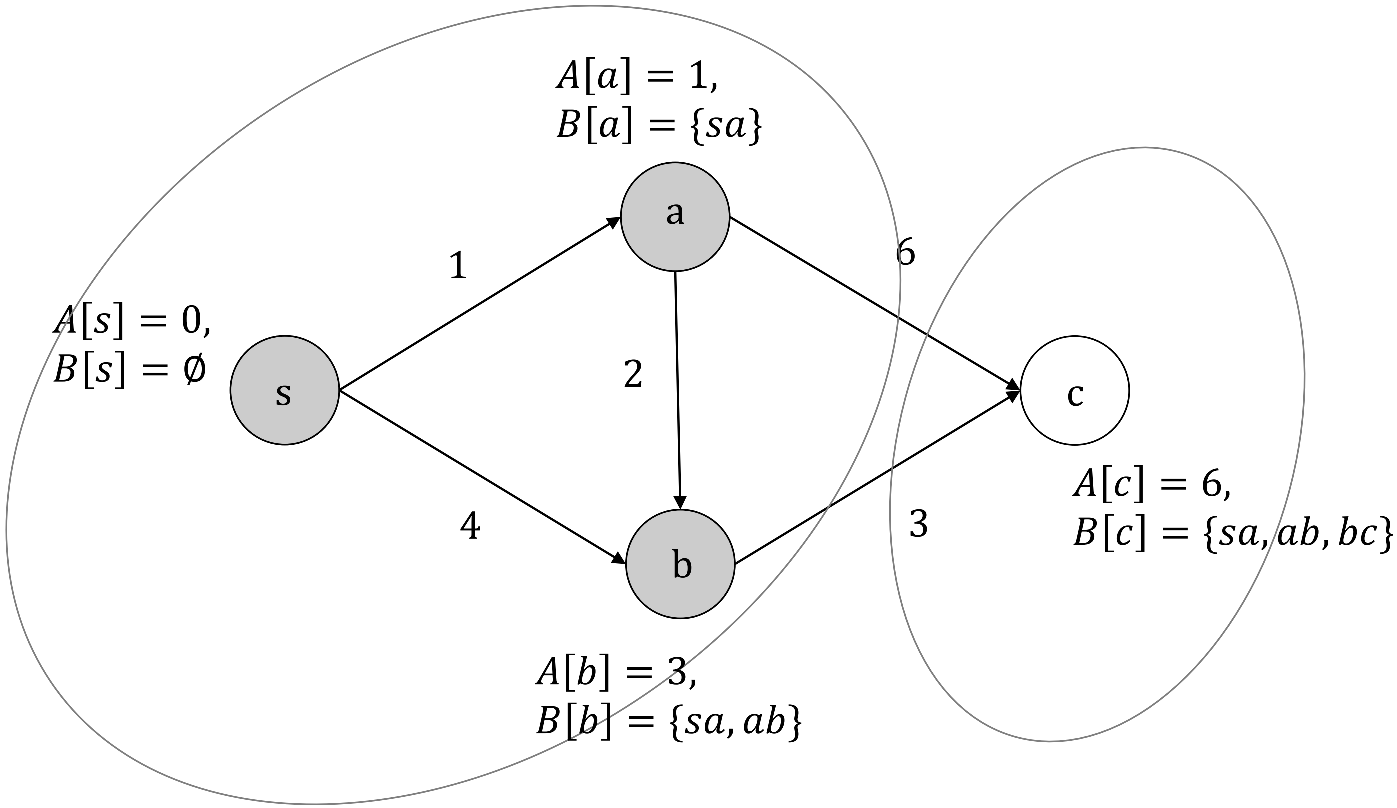
$\min(A[s] + l_{sb}, A[a] + l_{ab}, A[a] + l_{ac}) = \min(0 + 4, 1 + 2, 1 + 6) = 3$



An Example

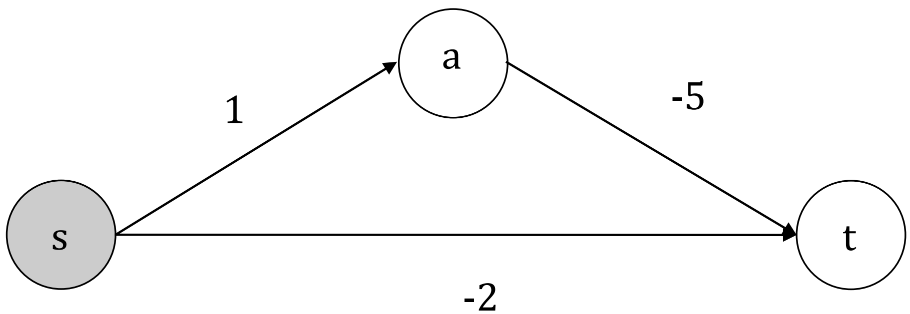
$$A[v] + l_{vw} \text{ (Dijkstra's greedy criterion)}$$

$$\min\,(A[a] + l_{ac}, A[b] + l_{bc}) = \min\,(1 + 6, 3 + 3) = 6$$



Non-Example

- Dijkstra is incorrect on this G



- Use dynamic programming in this case

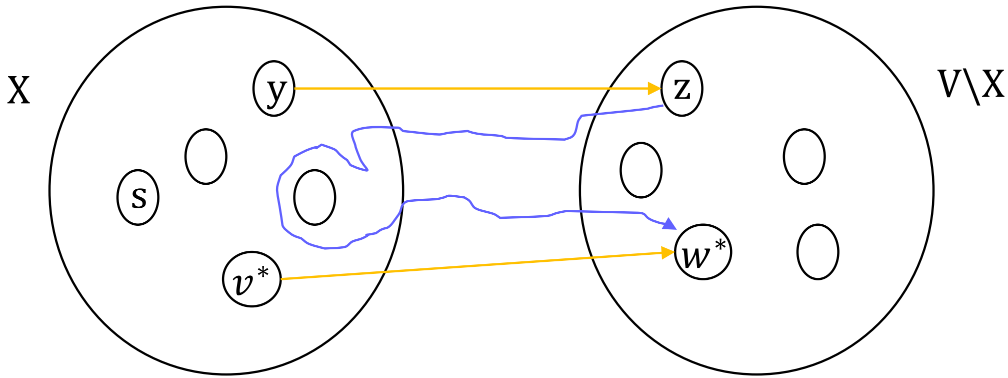
Proof of Correctness

Claim: $A[v] = L[v]$ where A is the output of Dijkstra and L is true shortest distance

Proof: By induction, base case $A[s] = L[s] = 0$ is true.

Inductive Hypothesis (I.H.): all previous iterations are corrrect, i.e.,

$$\forall v \in \mathrm{X}, A[v] = L[v] \text{ and } B[v] \text{ gives the shortest path}$$



In the current iteration, Dijkstra have chosen v^*w^*, we have $A[w^*] = A[v^*] + l_{v^*w^*}$

Now let P be any s $\rightarrow w^*$ path and it must "cross the frontier"

Length of P $\geq L(y) + l_{yz} + 0 = A(y) + l_{yz} + 0$, note that $L(y) = A(y)$ by I.H.

Also, by Dijkstra's greedy criterion,

$$\text{Our length} = A[v^*] + l_{v^*w^*} \leq A(y) + l_{yz} \leq \text{Length of P}$$

General Graph Search

- Let q be an abstract queue object
  - add(node), which adds a node into q
  - popFirst(), which pops the first node from q

- General graph search

  While q is not empty:
  - $v \leftarrow q.popFirst()$
  - For all neighbors u of v such that $u \notin q$:
    - add(u)

General Graph Search

- General graph search

  While q is not empty:

  - $v \leftarrow q.popFirst()$
  - For all neighbors u of v such that $u \notin q$:
    - add(u)

- If q is a standard LIFO stack, then DFS
- If q is a standard FIFO queue, then BFS
- If q is a priority queue, then Dijkstra
- If q is a priority queue with a heuristic, then A*

Priority Queue Implementation

- $A[s] \leftarrow 0$, and $A[v] \leftarrow \infty$ for all $v \in V \backslash \{s\}$
- $q.add(s)$
- While q is not empty:
  - $v \leftarrow q.popFirst()$ Extract min-q
  - For all neighbors u of v such that $A[v] + l_{vu} \leq A[u]$
    - $A[u] \leftarrow A[v] + l_{vu}$
    - *q.update(u, A[u])* Decrease-key(q)

Runtime: If using binary minheap, $O\big((\lvert E\rvert + \lvert V\rvert)logV\big)$ If using Fibonacci minheap, $O(\lvert E\rvert + \lvert V\rvert logV)$

Summary

- Wagner-Whitin model
- Dynamic Programming
- Shortest Path (Dijkstra's Algorithm)

- Next Up: Stochastic Inventory Model