Multi-armed Bandits (MAB)

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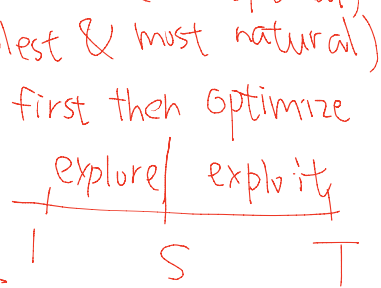
Theory & intuition

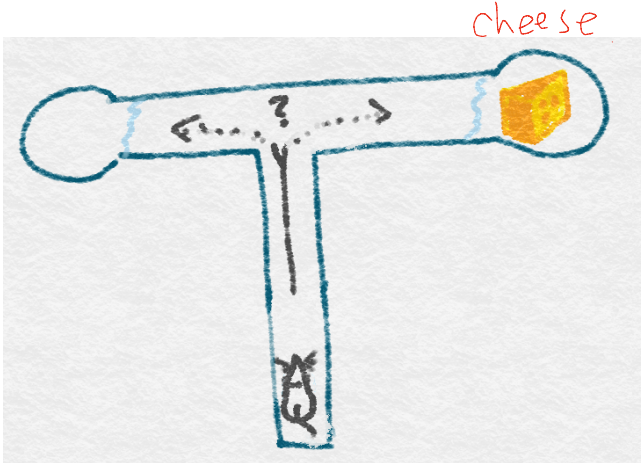
Overview

- What are bandits, and why you should care
- Finite-armed stochastic bandits
  - "Explore-Then-Commit (ETC)" Algorithm
  - Upper Confidence Bound (UCB) Algorithm
  - Lower Bound
- Finite-armed adversarial bandits



What's in a name? A tiny bit of history

First bandit algorithm proposed by Thompson (1933) → clinical trials



→ casino

Bush and Mosteller (1953) were interested in how mice behaved in a T-maze

Applications

- Clinical trials/dose discovery
- Recommendation systems (movies/news/etc)
- Advertisement placement
- A/B testing
- Dynamic pricing (eg., for Amazon products)
- Ranking (eg., for search)
- Resource allocation

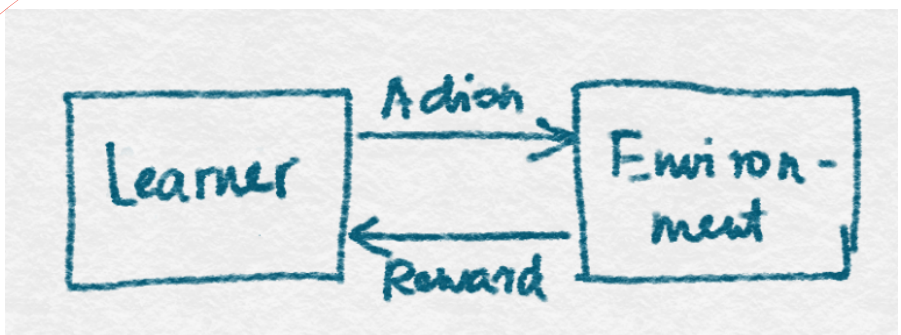
- They isolate an important component of reinforcement learning: exploration-vs-exploitation

Finite-armed bandits

- K actions
- n rounds
- In each round t the **learner** chooses an action

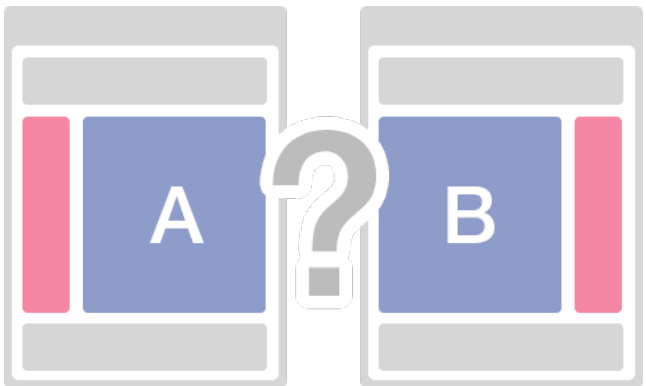
$$\text{Action chosen at time } t = A_t \in \{1, 2, \ldots, K\}.$$

- Observes **reward** $X_t \sim P_{A_t}$ where $P_1, P_2, \ldots, P_K$ are **unknown** distributions (Gaussian or subgaussian)



Example: A/B testing

- Business wants to optimize their webpage
- Actions correspond to ‘A’ and ‘B’ (two arms)
- Users arrive at webpage sequentially
- Algorithm chooses either ‘A’ or ‘B’ (pulling an arm)
- Receives activity feedback (click as the reward)



Measuring performance – the **regret**

$t = 1, 2, \cdots n$

(unknown)

(unknown)

- Let μ_i be the mean reward of distribution P_i
- $\mu^* = \max_i \mu_i$ is the maximum mean (god knows $\mu_1 \sim \mu_k$ ∴ god knows μ^*)
- The (expected) **regret** is

$$\text{regret} = R_n = n\mu^* - \mathbb{E}\left[\sum_{t=1}^{n} X_t\right] \leq o(n)$$

god reward

your reward.

anything Sublinear (越小越好)

Measuring performance – the **regret**

- Let μ_i be the mean reward of distribution P_i
- $\mu^* = \max_i \mu_i$ is the maximum mean
- The (expected) **regret** is

$$R_n = n\mu^* - \mathbb{E}\left[\sum_{t=1}^{n} X_t\right]$$

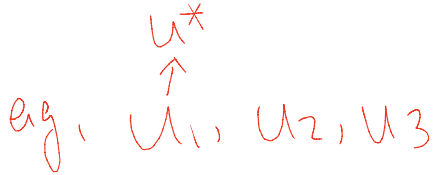
- A reasonable policy for which the regret should be ($R_n = o(n)$)
- Of course we would like to make it as 'small as possible'

Measuring performance – the **regret**

Let $\Delta_i = \mu^* - \mu_i$ be the **suboptimality gap** for the ith arm

Let $T_i(n)$ be the number of times arm i is played over all n rounds

Key decomposition lemma: $R_n = \sum_{i=1}^{K} \Delta_i \mathbb{E}[T_i(n)]$



$\therefore \Delta_1 = 0$

$\Delta_2 = \mu_1 - \mu_2$

$\Delta_3 = \mu_1 - \mu_3$

Measuring performance – the **regret**

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Proof 不重要 233

Proof Let $\mathbb{E}_t[\cdot] = \mathbb{E}[\cdot | A_1, X_1, \ldots, X_{t-1}, A_t]$

$$R_n = n\mu^* - \mathbb{E}\left[\sum_{t=1}^{n} X_t\right] = n\mu^* - \sum_{t=1}^{n} \mathbb{E}[\mathbb{E}_t[X_t]] = n\mu^* - \sum_{t=1}^{n} \mathbb{E}[\mu_{A_t}]$$

$$= \sum_{t=1}^{n} \mathbb{E}[\Delta_{A_t}] = \mathbb{E}\left[\sum_{t=1}^{n}\sum_{i=1}^{K} \mathbb{1}(A_t = i)\Delta_i\right]$$

$$= \mathbb{E}\left[\sum_{i=1}^{K} \Delta_i \sum_{t=1}^{n} \mathbb{1}(A_t = i)\right] = \mathbb{E}\left[\sum_{i=1}^{K} \Delta_i T_i(n)\right] = \sum_{i=1}^{K} \Delta_i \mathbb{E}[T_i(n)]$$

A simple policy: Explore-Then-Commit

1 Choose each action m times

2 Find the empirically best action $I \in \{1, 2, \ldots, K\}$ (i.e., the action I gives the largest average reward over m items)

3 Choose $A_t = I$ for all remaining $(n - mK)$ rounds

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In order to analyse this policy we need to bound the probability of committing to a suboptimal action
Need probability tools: concentration inequalities.

A crash course in concentration

Let $Z_1, Z_2, \ldots, Z_n$ be a sequence of independent and identically distributed random variables with mean $\mu \in \mathbb{R}$ and variance $\sigma^2 < \infty$

$$\text{empirical mean} = \hat{\mu}_n = \frac{1}{n} \sum_{t=1}^{n} Z_t$$

How close is $\hat{\mu}_n$ to μ?

A crash course in concentration

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How close is $\hat{\mu}_n$ to μ?

Classical statistics says:

1. (law of large numbers) $\lim_{n\to\infty} \hat{\mu}_n = \mu$ almost surely
2. (central limit theorem) $\sqrt{n}(\hat{\mu}_n - \mu) \xrightarrow{d} \mathcal{N}(0, \sigma^2)$
3. (Chebyshev's inequality) $\mathbb{P}(|\hat{\mu}_n - \mu| \geq \varepsilon) \leq \frac{\sigma^2}{n\varepsilon^2}$ $(\mathbb{V}(\hat{\mu}_n) = \frac{\sigma^2}{n})$

之前学过的

↓

decay to 0 at rate $\frac{1}{n}$

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Basic probability inequality (R.V. X with finite mean and variance):

1. (Markov's inequality) $\mathbb{P}\left(|X| \geq \varepsilon\right) \leq \frac{\mathbb{E}(|X|)}{\varepsilon}$.
2. (Chebyshev's inequality) $\mathbb{P}\left(|X - \mathbb{E}(X)| \geq \varepsilon\right) \leq \frac{\mathbb{V}(X)}{\varepsilon^2}$

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We need something nonasymptotic and stronger than Chebyshev's (Not possible without assumptions)

A crash course in concentration

Random variable Z is σ-subgaussian if for all $\lambda \in \mathbb{R}$,

$$M_Z(\lambda) \doteq \mathbb{E}[\exp(\lambda Z)] \leq \exp\left(\lambda^2 \sigma^2 / 2\right),$$

where $M_Z(\lambda)$ is known as the moment generating function.

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Definition

Random variable Z is σ-subgaussian if for all $\lambda \in \mathbb{R}$,

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where $M_Z(\lambda)$ is known as the moment generating function.

- Which distributions are σ-subgaussian? Gaussian, Bernoulli, bounded support.
- And not: exponential, power law

heavier tail than normal. (delay rate 更小)

A crash course in concentration

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- And not: exponential, power law

Lemma If $Z, Z_1, \ldots, Z_n$ are independent and σ-subgaussian, then

- aZ is $|a|\sigma$-subgaussian for any $a \in \mathbb{R}$
- $\sum_{t=1}^{n} Z_t$ is $\sqrt{n}\sigma$-subgaussian
- $\hat{\mu}_n$ is $n^{-1/2}\sigma$-subgaussian

A crash course in concentration

Lemma(Tail bound of subgaussian random variable)

- If X is a σ-subgaussian, for any $\varepsilon > 0$, $\mathbb{P}(X \geq \varepsilon) \leq \exp(-\frac{\varepsilon^2}{2\sigma^2})$.

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Lemma(Tail bound of subgaussian random variable)

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Proof We use **Chernoff's method**. Let $\varepsilon > 0$ and $\lambda = \varepsilon/\sigma^2$.

$$\begin{aligned}
\mathbb{P}(X \geq \varepsilon) &= \mathbb{P}(\exp(\lambda X) \geq \exp(\lambda\varepsilon)) \\
&\leq \frac{\mathbb{E}[\exp(\lambda X)]}{\exp(\lambda\varepsilon)} && \text{(Markov's)} \\
&\leq \exp\left(\sigma^2\lambda^2/2 - \lambda\varepsilon\right) && (X \text{ is subgaussian}) \\
&= \exp\left(-\varepsilon^2/(2\sigma^2)\right)
\end{aligned}$$

A crash course in concentration

Theorem If $Z_1, \ldots, Z_n$ are independent and σ-subgaussian, then

$$\mathbb{P}\left(\hat{\mu}_n - \mu \geq \sqrt{\frac{2\sigma^2 \log(1/\delta)}{n}}\right) \leq \delta$$

Proof $\hat{\mu}_n - \mu$ is a $\sigma/\sqrt{n}$-subgaussian random variable and thus

$$\mathbb{P}\left(\hat{\mu}_n - \mu \geq \varepsilon\right) \leq \exp(-\frac{n\varepsilon^2}{2\sigma^2})$$

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Corollary If $Z_1, \ldots, Z_n$ are independent and σ-subgaussian, then

$$\mathbb{P}\left(\hat{\mu}_n - \mu \leq -\sqrt{\frac{2\sigma^2 \log(1/\delta)}{n}}\right) \leq \delta$$

A crash course in concentration

- Comparing Chebyshev's w. subgaussian bound:

$$\textbf{Chebyshev's:} \quad \mathbb{P}\left(\hat{\mu}_n - \mu \geq \sqrt{\frac{\sigma^2}{n\delta}}\right) \leq \delta$$

$$\textbf{Subgaussian:} \quad \mathbb{P}\left(\hat{\mu}_n - \mu \geq \sqrt{\frac{2\sigma^2 \log(1/\delta)}{n}}\right) \leq \delta$$

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- Typically $\delta \ll 1/n$ in our use-cases. Then Chebyshev's inequality is too loose since $\sqrt{\frac{\sigma^2}{n\delta}}$ is too large.

From now on, we will assume that reward distribution associated with each arm is 1-subgaussian (but with different means)

Analysing Explore-Then-Commit

- **Exploration phase:** Chooses each arm m times
- **Exploitation phase:** Then commits to the arm with the largest empirical reward

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- Means that first arm is optimal
- Algorithms are symmetric and do not know this fact

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- We consider only $K = 2$

Analysing Explore-Then-Commit

Step 1 Let $\hat{\mu}_i$ be the average reward of i-th arm (for $i \in \{1, 2\}$) after the exploration phase

The algorithm commits to the wrong arm if

$$\hat{\mu}_2 \geq \hat{\mu}_1 \Leftrightarrow \hat{\mu}_2 - \mu_2 + \mu_1 - \hat{\mu}_1 \geq \Delta = \mu_1 - \mu_2$$

Observation $\underbrace{\hat{\mu}_2 - \mu_2}_{\sqrt{1/m}-\text{subgaussian}} + \underbrace{\mu_1 - \hat{\mu}_1}_{\sqrt{1/m}-\text{subgaussian}}$ is $\sqrt{2/m}$-subgaussian with zero-mean

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Step 2 The regret is

$$\begin{aligned} R_n &= \mathbb{E}\left[\sum_{t=1}^{n} \Delta_{A_t}\right] = \mathbb{E}\left[\sum_{t=1}^{2m} \Delta_{A_t}\right] + \mathbb{E}\left[\sum_{t=2m+1}^{n} \Delta_{A_t}\right] \\ &= m\Delta + (n-2m)\Delta \mathbb{P}\left(\text{commit to the wrong arm}\right) \\ &= m\Delta + (n-2m)\Delta \mathbb{P}\left(\hat{\mu}_2 - \mu_2 + \mu_1 - \hat{\mu}_1 \geq \Delta\right) \\ &\leq m\Delta + n\Delta \exp\left(-\frac{m\Delta^2}{4}\right) \end{aligned}$$

The last inequality is because *if X is a σ-subgaussian, for any $\varepsilon > 0$,* $\mathbb{P}(X \geq \varepsilon) \leq \exp(-\frac{\varepsilon^2}{2\sigma^2})$ ($\varepsilon = \Delta$ and $\sigma = \sqrt{2/m}$).

Analysing Explore-Then-Commit

$$R_n \leq \underbrace{m\Delta}_{\text{(A)}} + \underbrace{n\Delta \exp(-m\Delta^2/4)}_{\text{(B)}}$$

(A) is monotone increasing in m while (B) is monotone decreasing in m

Exploration/Exploitation Trade-off Exploring too much (m large) then (A) is big, while exploring too little makes (B) large

Bound minimised by $m = \left\lceil \frac{4}{\Delta^2} \log\left(\frac{n\Delta^2}{4}\right) \right\rceil$ leading to

$$R_n \leq \Delta + \frac{4}{\Delta} \log\left(\frac{n\Delta^2}{4}\right) + \frac{4}{\Delta},$$

noting that due to ceiling function in m: $(A) \leq (1 + \frac{4}{\Delta^2} \log\left(\frac{n\Delta^2}{4}\right))\Delta$.

Analysing Explore-Then-Commit

Last slide: $R_n \leq \Delta + \frac{4}{\Delta} \log\left(\frac{n\Delta^2}{4}\right) + \frac{4}{\Delta}$

What happens when Δ is very small? (R_n can be unbounded)

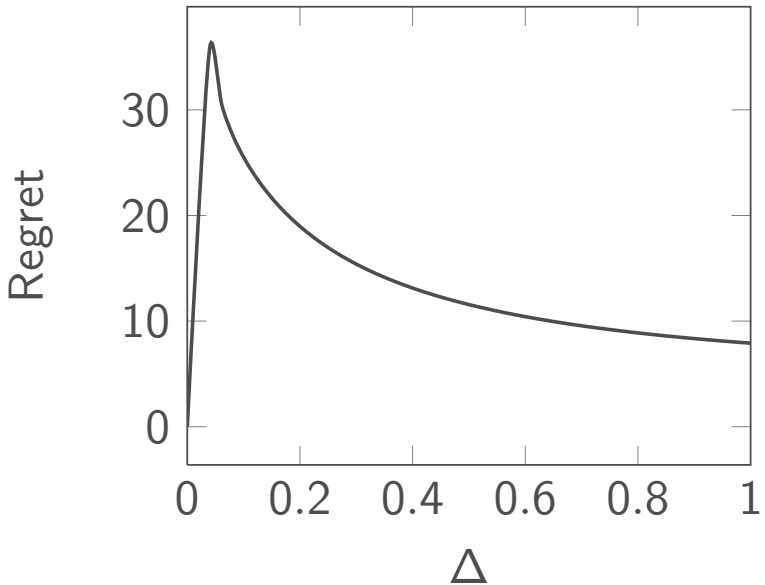
Analysing Explore-Then-Commit

Last slide: $R_n \leq \Delta + \frac{4}{\Delta} \log\left(\frac{n\Delta^2}{4}\right) + \frac{4}{\Delta}$

What happens when Δ is very small? (R_n can be unbounded)
A natural correction:

$$R_n \leq \min\left\{n\Delta,\ \Delta + \frac{4}{\Delta} \log\left(\frac{n\Delta^2}{4}\right) + \frac{4}{\Delta}\right\}$$

Illustration of R_n with $n = 1000$.



Analysing Explore-Then-Commit

Does this figure make sense? Why is the regret largest when Δ is small, but not too small?

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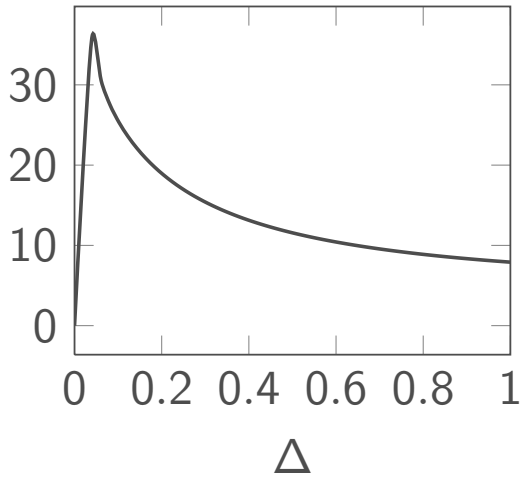

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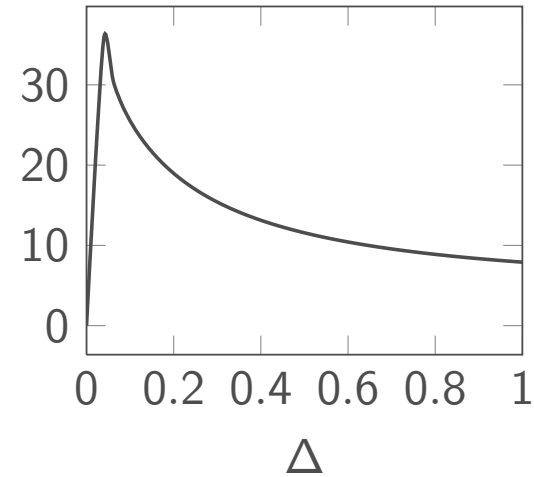

Small Δ makes identification of the best arm hard, but cost of failure (of identification) is low

Large Δ makes the cost of failure high, but identification becomes easy

Worst case is when $\Delta \approx \sqrt{1/n}$ with $R_n \approx \sqrt{n}$

Limitations of Explore-Then-Commit

- Recall that $m = \left\lceil \frac{4}{\Delta^2} \log\left(\frac{n\Delta^2}{4}\right) \right\rceil$
- Need advance knowledge of the unknown horizon length n
- Optimal tuning depends on unknown $\Delta = \mu_1 - \mu_2$

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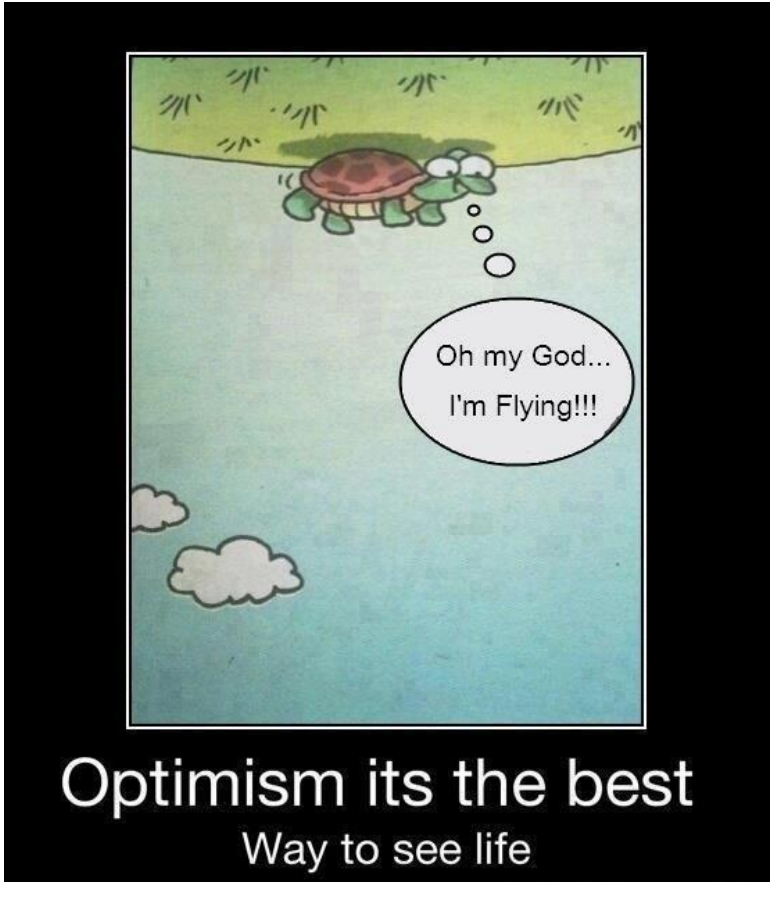
- Need advance knowledge of the unknown horizon length n
- Optimal tuning depends on unknown $\Delta = \mu_1 - \mu_2$
- Better approaches now exist, but Explore-Then-Commit is often a good place to start when analyzing a bandit problem since it captures *exploration-exploitation trade-off*

Optimism principle



Informal illustration

Visiting a new region

Shall I try local cuisine?

Optimist: Yes!

Pessimist: No!

Optimism leads to exploration, pessimism prevents it

Exploration is necessary, but how much?

Optimism principle

- Let $\hat{\mu}_i(t) = \frac{1}{T_i(t)} \sum_{s=1}^{t} \mathbb{1}(A_s = i) X_s$ be the empirical mean reward of i-th arm at time t

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- Optimistic estimate of the mean of arm = 'largest value it could plausibly be'
- Formalise the intuition using confidence intervals ($\sigma = 1$)

$$\mathbb{P}\left(\hat{\mu}_n - \mu \geq \sqrt{\frac{2\sigma^2 \log(1/\delta)}{n}}\right) \leq \delta$$

- Suggests

$$\text{optimistic estimate} = \hat{\mu}_i(t-1) + \sqrt{\frac{2\log(1/\delta)}{T_i(t-1)}}$$

- $\delta \in (0, 1)$ determines the level of optimism

Upper confidence bound algorithm

1. Choose each action once
2. Choose the action maximising

$$A_t = \text{argmax}_i \, \hat{\mu}_i(t-1) + \sqrt{\frac{2\log(t^3)}{T_i(t-1)}}$$

3. **Goto** 2

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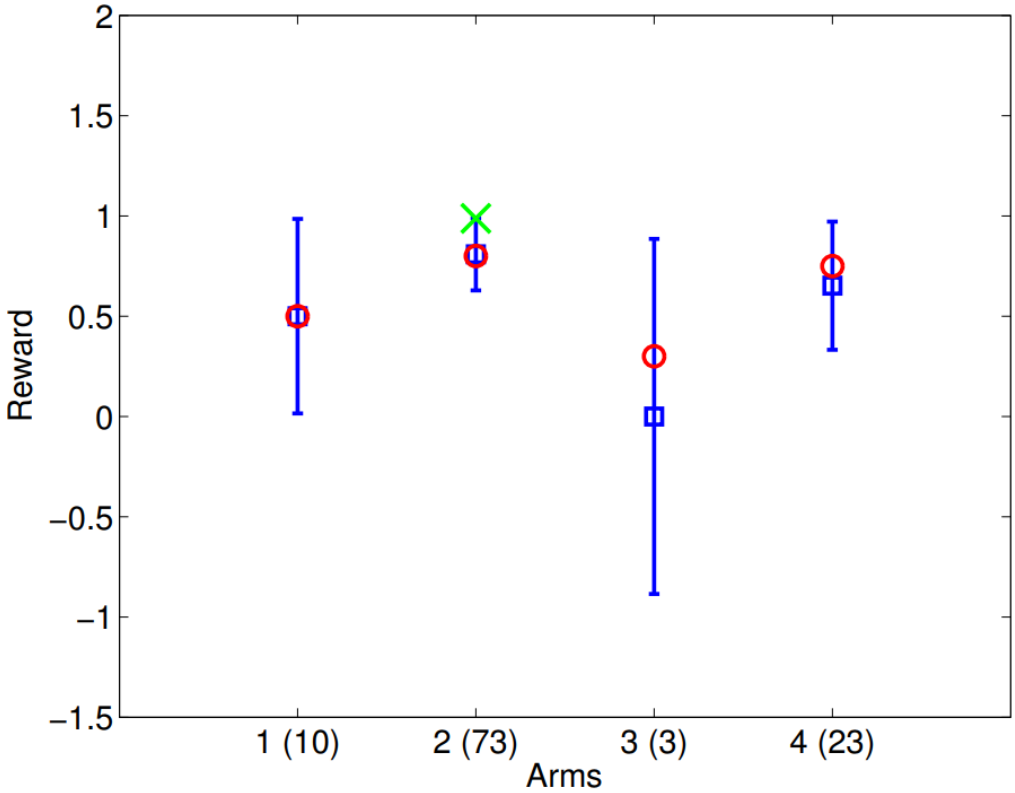
Algorithm does not depend on horizon n (it is **anytime**)

Upper confidence bound algorithm



- Red circle: true mean, Blue rectangle: empirical mean reward.
- (10), (73), (3), (23): number of pulls (a larger number of pulls makes the true and empirical mean closer).

Why UCB?

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- UCB:

$$A_t = \text{argmax}_i \, \hat{\mu}_i(t-1) + \sqrt{\frac{2\log(t^3)}{T_i(t-1)}}$$

- An algorithm should explore arms more often if they are
  1. either promising because $\hat{\mu}_i(t-1)$ is large
  2. or not well explored because $T_i(t-1)$ is small

Regret of UCB

Theorem The regret of UCB is at most

$$R_n = O\left(\sum_{i:\Delta_i>0}\left(\Delta_i + \frac{\log(n)}{\Delta_i}\right)\right)$$

Furthermore,

$$R_n = O\left(\sqrt{Kn\log(n)}\right),$$

where K is the number of arms and n is the time horizon length.

Bounds of the first kind are called **problem dependent** or **instance dependent**, which depends on $\Delta_i = \mu_1 - \mu_i$

Bounds like the second are called **distribution free** or **worst case**

Regret analysis

Rewrite the regret $R_n = \sum_{i=1}^{K} \Delta_i \mathbb{E}[T_i(n)]$

Only need to show that $\mathbb{E}[T_i(n)]$ is not too large for suboptimal arms

Regret analysis

Key insight Arm i is only played if its **index** is larger than the index of the optimal arm

$$\gamma_i(t-1) = \underbrace{\hat{\mu}_i(t-1) + \sqrt{\frac{2\log(t^3)}{T_i(t-1)}}}_{\text{index of arm } i \text{ in round } t}$$

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A suboptimal arm $i \neq 1$ is played implies that

1. either $\gamma_i(t-1) \geq \mu_1$ (index of arm i is larger than the mean of optimal arm)
2. or $\gamma_1(t-1) \leq \mu_1$ (index of arm 1 is smaller than its true mean)

Otherwise, we have $\gamma_i(t-1) \leq \mu_1 \leq \gamma_1(t-1)$: arm 1 should be played since

Both events are unlikely after a sufficiently number of plays.

Regret analysis

To make this intuition a reality we decompose the "pull-count" for the i-th arm ($i \neq 1$)

$$\mathbb{E}[T_i(n)] = \mathbb{E}\left[\sum_{t=1}^{n} \mathbb{1}(A_t = i)\right] = \sum_{t=1}^{n} \mathbb{P}(A_t = i)$$

$$= \sum_{t=1}^{n} \mathbb{P}(A_t = i \text{ and } (\gamma_1(t-1) \leq \mu_1 \text{ or } \gamma_i(t-1) \geq \mu_1))$$

$$\leq \underbrace{\sum_{t=1}^{n} \mathbb{P}(\gamma_1(t-1) \leq \mu_1)}_{\text{index of opt. arm too small?}} + \underbrace{\sum_{t=1}^{n} \mathbb{P}(A_t = i \text{ and } \gamma_i(t-1) \geq \mu_1)}_{\text{index of subopt. arm large?}}$$

Regret analysis

We want to show that $\mathbb{P}(\gamma_1(t-1) \leq \mu_1)$ is small

Tempting to use the concentration theorem...

$$\mathbb{P}(\gamma_1(t-1) \leq \mu_1) = \mathbb{P}\left(\hat{\mu}_1(t-1) + \sqrt{\frac{2\log(t^3)}{T_i(t-1)}} \leq \mu_1\right) \stackrel{?}{\leq} \frac{1}{t^3}$$

What's wrong with this?

Regret analysis

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What's wrong with this? $T_i(t-1)$ is a random variable but not a number! Use union bound $\Pr(\cup_{s=1}^{t-1} A_s) \leq \sum_{s=1}^{t-1} \Pr(A_s)$

$$\begin{aligned}
\mathbb{P}\left(\hat{\mu}_1(t-1) + \sqrt{\frac{2\log(t^3)}{T_i(t-1)}} \leq \mu_1\right) &\leq \mathbb{P}\left(\exists s \leq t-1 : \hat{\mu}_{1,s} + \sqrt{\frac{2\log(t^3)}{s}} \leq \mu_1\right) \\
&\leq \sum_{s=1}^{t-1} \mathbb{P}\left(\hat{\mu}_{1,s} + \sqrt{\frac{2\log(t^3)}{s}} \leq \mu_1\right) \\
&\leq \sum_{s=1}^{t-1} \frac{1}{t^3} \leq \frac{1}{t^2}\,. \qquad (\delta = 1/t^3)
\end{aligned}$$

Regret analysis

$$\sum_{t=1}^{n} \mathbb{P}\left(A_t = i \text{ and } \gamma_i(t-1) \geq \mu_1\right) = \mathbb{E}\left[\sum_{t=1}^{n} \mathbb{1}(A_t = i \text{ and } \gamma_i(t-1) \geq \mu_1)\right]$$

$$= \mathbb{E}\left[\sum_{t=1}^{n} \mathbb{1}(A_t = i \text{ and } \hat{\mu}_i(t-1) + \sqrt{\frac{6\log(t)}{T_i(t-1)}} \geq \mu_1)\right]$$

$$\leq \mathbb{E}\left[\sum_{t=1}^{n} \mathbb{1}(A_t = i \text{ and } \hat{\mu}_i(t-1) + \sqrt{\frac{6\log(n)}{T_i(t-1)}} \geq \mu_1)\right] \qquad (t \leq n)$$

Regret analysis

$$
\begin{aligned}
&\sum_{t=1}^{n} \mathbb{P}\left(A_t = i \text{ and } \gamma_i(t-1) \geq \mu_1\right) \\
\leq &\mathbb{E}\left[\sum_{t=1}^{n} \mathbb{1}\left(A_t = i \text{ and } \hat{\mu}_i(t-1) + \sqrt{\frac{6\log(n)}{T_i(t-1)}} \geq \mu_1\right)\right] \\
\leq &\mathbb{E}\left[\sum_{s=1}^{n} \mathbb{1}\left(\hat{\mu}_{i,s} + \sqrt{\frac{6\log(n)}{s}} \geq \mu_1\right)\right] \\
&\qquad\qquad \text{(For each possible } T_i(t-1) = s \text{ and } s = 1, \ldots, n\text{)} \\
= &\sum_{s=1}^{n} \mathbb{P}\left(\hat{\mu}_{i,s} + \sqrt{\frac{6\log(n)}{s}} \geq \mu_1\right)
\end{aligned}
$$

Regret analysis

Let $u = \frac{24 \log(n)}{\Delta_i^2}$. Then we decompose time periods into $[1, u]$ and $[u+1, n]$:

$$\sum_{s=1}^{n} \mathbb{P}\left(\hat{\mu}_{i,s} + \sqrt{\frac{6\log(n)}{s}} \geq \mu_1\right) \leq u + \sum_{s=u+1}^{n} \mathbb{P}\left(\hat{\mu}_{i,s} + \sqrt{\frac{6\log(n)}{s}} \geq \mu_1\right)$$

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Choose u large enough so that for any $s > u$, $\sqrt{\frac{6\log(n)}{s}} \leq \frac{\Delta_i}{2}$ ($u = \frac{24\log(n)}{\Delta_i^2}$). Then we have

$$\begin{aligned}
\hat{\mu}_{i,s} \geq \mu_1 - \sqrt{\frac{6\log(n)}{s}} \quad &\Rightarrow \quad \hat{\mu}_{i,s} - \mu_i \geq \mu_1 - \mu_i - \sqrt{\frac{6\log(n)}{s}} \\
&\Rightarrow \quad \hat{\mu}_{i,s} - \mu_i \geq \Delta_i - \frac{\Delta_i}{2} \\
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$$\leq u + \sum_{s=u+1}^{n} \mathbb{P}\left(\hat{\mu}_{i,s} \geq \mu_i + \frac{\Delta_i}{2}\right)$$

$$\leq u + \sum_{s=u+1}^{\infty} \exp\left(-\frac{s\Delta_i^2}{8}\right)$$

$$\left(\textstyle\sum_{s=u+1}^{\infty} \exp\left(-\frac{s\Delta_i^2}{8}\right) \leq 1 + \int_{s=u}^{\infty} \exp\left(-\frac{s\Delta_i^2}{8}\right) \mathrm{d}s\right)$$

$$\leq u + 1 + \frac{8}{\Delta_i^2}\,.$$

Regret analysis

Combining the two parts we have

$$\mathbb{E}[T_i(n)] \leq \underbrace{\sum_{t=1}^{n} \mathbb{P}\left(\gamma_1(t-1) \leq \mu_1\right)}_{\text{index of opt. arm too small?}} + \underbrace{\sum_{t=1}^{n} \mathbb{P}\left(A_t = i \text{ and } \gamma_i(t-1) \geq \mu_1\right)}_{\text{index of subopt. arm large?}}$$

$$\leq \sum_{t=1}^{n} \frac{1}{t^2} + 1 + u + \frac{8}{\Delta_i^2}$$

$$\left(u = \tfrac{24\log(n)}{\Delta_i^2},\ \textstyle\sum_{t=1}^{n} \frac{1}{t^2} \leq 1 + \int_{t=1}^{\infty} \frac{1}{t^2} \mathrm{d}t\right)$$

$$\leq 3 + \frac{8}{\Delta_i^2} + \frac{24\log(n)}{\Delta_i^2}$$

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$$\leq 3 + \frac{8}{\Delta_i^2} + \frac{24\log(n)}{\Delta_i^2}$$

So the regret is bounded by (instance dependent bound)

$$R_n = \sum_{i:\Delta_i > 0} \Delta_i \mathbb{E}[T_i(n)] \leq \sum_{i:\Delta_i > 0} \left(3\Delta_i + \frac{8}{\Delta_i} + \frac{24\log(n)}{\Delta_i}\right)$$

$$= O\left(\sum_{i:\Delta_i > 0} \left(\Delta_i + \frac{\log(n)}{\Delta_i}\right)\right)$$

Distribution free bounds

Let $\Delta > 0$ be some constant to be chosen later

$$
\begin{aligned}
R_n &= \sum_{i:\Delta_i \le \Delta} \Delta_i \mathbb{E}[T_i(n)] + \sum_{i:\Delta_i > \Delta} \Delta_i \mathbb{E}[T_i(n)] \\
&\le n\Delta + \sum_{i:\Delta_i > \Delta} \Delta_i \mathbb{E}[T_i(n)] \qquad \left(\textstyle\sum_{i:\Delta_i \le \Delta} T_i(n) \le \sum_i T_i(n) = n\right) \\
&\lesssim n\Delta + \sum_{i:\Delta_i > \Delta} \left(\Delta_i + \frac{\log(n)}{\Delta_i}\right) \\
&\lesssim \underbrace{n\Delta + \frac{K \log(n)}{\Delta}}_{\Delta = \sqrt{K \log(n)/n}} + \sum_{i=1}^{K} \Delta_i \\
&\lesssim \sqrt{nK \log(n)} + \sum_{i=1}^{K} \Delta_i
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&\lesssim \sqrt{nK\log(n)} + \sum_{i=1}^{K} \Delta_i
\end{aligned}
$$

Note that $\sum_{i=1}^{K} \Delta_i$ is unavoidable since each arm needs to be played at least once and this term is negligible when n is large.

Improvements

- The constants in the algorithm/analysis can be improved quite significantly.

$$A_t = \text{argmax}_i \, \hat{\mu}_i(t-1) + \sqrt{\frac{2\log(t)}{T_i(t-1)}}$$

- With this choice:

$$\lim_{n\to\infty} \frac{R_n}{\log(n)} = \sum_{i:\Delta_i>0} \frac{2}{\Delta_i}$$

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- The distribution-free regret is also improvable

$$A_t = \text{argmax}_i \, \hat{\mu}_i(t-1) + \sqrt{\frac{4}{T_i(t-1)} \log\left(1 + \frac{t}{KT_i(t-1)}\right)}$$

- With this index we save a log factor in the distribution free bound

$$R_n = O(\sqrt{nK})$$

Exercise

- Consider different settings of arms (number of arms K, mean gap Δ_i) and different distributions: uniform, Bernoulli, normal
- Compare Explore-Then-Commit with UCB Algorithm in
  1. Regret as a function of horizon n
  2. Frequency of pulling each arm
  3. Tuning the constant ρ:

$$A_t = \text{argmax}_i \, \hat{\mu}_i(t-1) + \sqrt{\frac{\rho \log(t)}{T_i(t-1)}}$$

Lower bounds

Is the bound $R_n = O(\sqrt{nK})$ optimal in n and K?

1. For worst-case regret for a given policy π: $R_n(\pi) = \sup_{\nu \in \mathcal{E}} R_n(\pi, \nu)$, where $\mathcal{E}$ denotes the set of K-armed Gaussian bandits with unit variance and means $\mu \in [0, 1]^K$.

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2. The minimax regret $R_n^*(\mathcal{E}) = \inf_\pi \sup_{\nu \in \mathcal{E}} R_n(\pi, \nu)$

Theorem $R_n^*(\mathcal{E}) \geq \sqrt{(K-1)n}/27$: for every policy π and n and $K \leq n+1$, there exists a K-armed Gaussian bandit ν such that

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UCB with $R_n = O(\sqrt{nK})$ is a rate-optimal policy

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Key idea: reduce the bandit problem into a statistical hypothesis testing problem.

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Select two bandit problem instances (two sets of K distributions) in such a way that the following two conditions hold simultaneously:

- Competition: A sequence of actions that is good for one bandit is not good for the other (choose two instances far away from each other).
- Similarity: The instances are 'close' enough that a policy interacting with either of the two instances cannot statistically identify the true bandit.

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Lower bound: optimize the trade-off between these two opposite goals.

Minimax lower bound

Theorem For every policy π and n and $K \leq n$, there exists a K-armed Gaussian bandit ν such that

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Proof sketch

- Two bandits: $\nu = (P_i)_{i=1}^K$ and $\nu' = (P'_i)_{i=1}^K$, where $P_i = N(\mu_i, 1)$ and $P'_i = N(\mu'_i, 1)$.

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- It suffices to show that for *any policy* π, there exists μ and μ' such that the π incurs regret larger than $\sqrt{Kn}$ on at least one instance:

  $$\max(R_n(\pi, \nu), R_n(\pi, \nu')) \geq c\sqrt{Kn},$$

  or (since $\max(a, b) \geq (a+b)/2$),

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- Choose $\mu = (\Delta, 0, \ldots, 0)$ and

  $$R_n(\pi, \nu) = (n - \mathbb{E}_\nu(T_1(n)))\Delta$$

  (Δ optimized later)

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- $\mu' = (\Delta, 0, \ldots, 2\Delta, 0, \ldots, 0)$ (2Δ at the i-th arm, optimal arm):

$$R_n(\pi, \nu') = \Delta \mathbb{E}_{\nu'}(T_1(n)) + \sum_{j \neq 1, i} 2\Delta \mathbb{E}_{\nu'}(T_j(n)) \geq \Delta \mathbb{E}_{\nu'}(T_1(n)).$$

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- Depend on $T_1(n) \gtrless n/2$,
  - If $T_1(n) \leq n/2$, $R_n(\pi, \nu) = (n - \mathbb{E}_\nu(T_1(n)))\Delta \geq \frac{n\Delta}{2}$. Therefore, $R_n(\pi, \nu) \geq \mathbb{P}_\nu(T_1(n) \leq n/2)\frac{n\Delta}{2}$
  - If $T_1(n) \leq n/2$, $R_n(\pi, \nu') \geq \Delta \mathbb{E}_{\nu'}(T_1(n)) \geq \frac{n\Delta}{2}$. Therefore, $R_n(\pi, \nu') \geq \mathbb{P}_{\nu'}(T_1(n) \geq n/2)\frac{n\Delta}{2}$

Minimax lower bound

$$R_n(\pi, \nu) + R_n(\pi, \nu') \geq \frac{n\Delta}{2}(\mathbb{P}_\nu(T_1(n) \leq n/2) + \mathbb{P}_{\nu'}(T_1(n) \geq n/2))$$

Need to show $\mathbb{P}_\nu(T_1(n) \leq n/2) + \mathbb{P}_{\nu'}(T_1(n) \geq n/2)$ is larger!

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Theorem (Pinsker's inequality) For any two distributions and any event A:

$$P(A) + Q(A^c) \geq \frac{1}{2}\exp(-D(P, Q)),$$

where $D(P, Q) = \int p \log(\frac{p}{q})$ is the Kullback-Leibler (KL) divergence.

Intuition, when P is close to Q, $P(A) + Q(A^c)$ should be large ($P(A) + P(A^c) = 1$)

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Minimax lower bound

$$R_n(\pi, \nu) + R_n(\pi, \nu') \geq \frac{n\Delta}{2}(\mathbb{P}_\nu(T_1(n) \leq n/2) + \mathbb{P}_{\nu'}(T_1(n) \geq n/2))$$

Need to show $\mathbb{P}_\nu(T_1(n) \leq n/2) + \mathbb{P}_{\nu'}(T_1(n) \geq n/2)$ is larger!

Theorem (Pinsker's inequality) For any two distributions and any event A:

$$P(A) + Q(A^c) \geq \frac{1}{2}\exp(-D(P, Q)),$$

where $D(P, Q) = \int p \log(\frac{p}{q})$ is the Kullback-Leibler (KL) divergence.

Intuition, when P is close to Q, $P(A) + Q(A^c)$ should be large ($P(A) + P(A^c) = 1$)

Exercise:

$$D(N(\mu_1, \sigma^2), N(\mu_2, \sigma^2)) = \frac{(\mu_1 - \mu_2)^2}{2\sigma^2}$$

Minimax lower bound

$$
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&\geq \frac{n\Delta}{4}\exp(-D(\mathbb{P}_\nu, \mathbb{P}_{\nu'}))
\end{aligned}
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Theorem (Divergence decomposition)

$$D(\mathbb{P}_\nu, \mathbb{P}_{\nu'}) = \sum_{j=1}^{K} \mathbb{E}_\nu(T_j(n)) D(P_j, P'_j)$$

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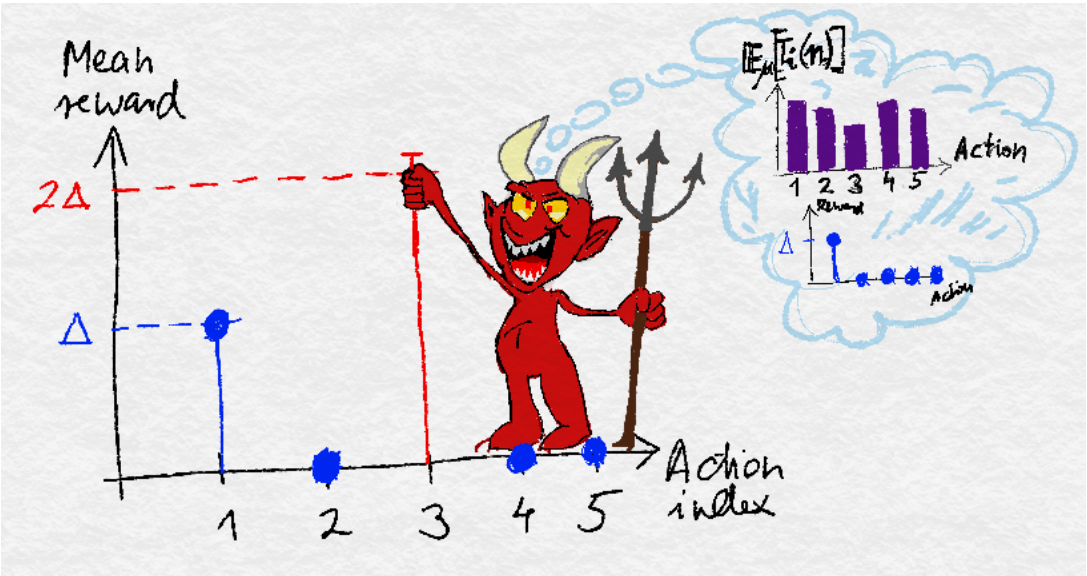
$$R_n(\pi,\nu) + R_n(\pi,\nu') \geq \frac{n\Delta}{4}\exp(-\frac{2n\Delta^2}{K-1})$$

and choose $\Delta = \sqrt{(K-1)/4n}$

Worst case lower bound

Theorem For every policy π and n and $K \leq n+1$, there exists a K-armed Gaussian bandit ν such that

$$R_n(\pi, \nu) \geq \sqrt{(K-1)n}/27$$



What else is there?

- All kinds of variants of UCB for different noise models: Bernoulli, exponential families, heavy tails, Gaussian with unknown mean and variance,...

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- All kinds of variants of UCB for different noise models: Bernoulli, exponential families, heavy tails, Gaussian with unknown mean and variance,...
- Thompson sampling: each round sample mean from posterior for each arm, choose arm with largest
- All manner of twists on the setup: non-stationarity, delayed rewards, playing multiple arms each round, moving beyond expected regret (high probability bounds)

The adversarial viewpoint

- Replace random rewards with an **adversary**
- At the start of the game the adversary secretly chooses **losses** $\ell_1, \ell_2, \ldots, \ell_n$ where $\ell_t \in [0,1]^K$
- Learner chooses actions A_t:
  - observe and suffers the loss ℓ_{tA_t}
- Regret is

$$R_n = \underbrace{\mathbb{E}\left[\sum_{t=1}^{n} \ell_{tA_t}\right]}_{\text{learner's loss}} - \underbrace{\min_i \sum_{t=1}^{n} \ell_{ti}}_{\text{loss of best arm}}$$

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- **Mission** Make the regret small, regardless of the adversary
- There exists an algorithm such that

$$R_n \leq 2\sqrt{Kn}$$

Why this regret definition?

- The regret is with respect to the loss of the best arm

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- The following alternative objective is hopeless

$$R'_n = \underbrace{\mathbb{E}\left[\sum_{t=1}^{n} \ell_{tA_t}\right]}_{\text{learner's loss}} - \underbrace{\sum_{t=1}^{n} \min_i \ell_{ti}}_{\text{loss of best sequence}}$$

- Regret is at least cn for some $c > 1$.

$$\ell = \begin{pmatrix} 1 & \cdots & 1 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 1 & \cdots & 1 \end{pmatrix}$$

Tackling the adversarial bandit

- **Randomisation** is crucial in adversarial bandits
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- How to choose P_t?
- Consider a simpler setting: choose the action A_t and the entire vector ℓ_t is observed (instead of ℓ_{tA_t})
- Online convex optimization with a linear loss

Online convex optimisation (linear losses)

- Domain of x $\mathcal{K} \subset \mathbb{R}^d$ is a convex set
- Adversary secretly chooses
  $\ell_1, \ldots, \ell_n \in \mathcal{K}^\circ = \{u : \sup_{x \in \mathcal{K}} |\langle x, u \rangle| \leq 1\}$ (polar set)
- At each time t, the learner chooses $x_t \in \mathcal{K}$
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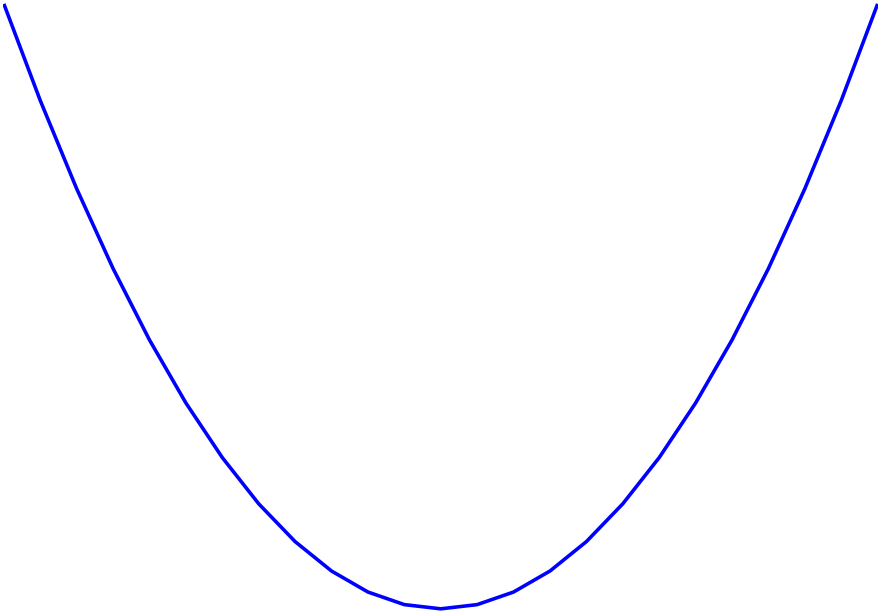
More general online convex optimization

- Learner chooses $x_t \in \mathcal{K}$
- Adversary chooses convex $f_t : \mathcal{K} \to \mathbb{R}$
- Suffer loss in round t is $f_t(x_t)$ and regret is

$$R_n(x) = \sum_{t=1}^{n} (f_t(x_t) - f_t(x))$$

- linear is a special case with $f_t(x) = \langle x, \ell_t \rangle$

Why linear is enough?



- convex function
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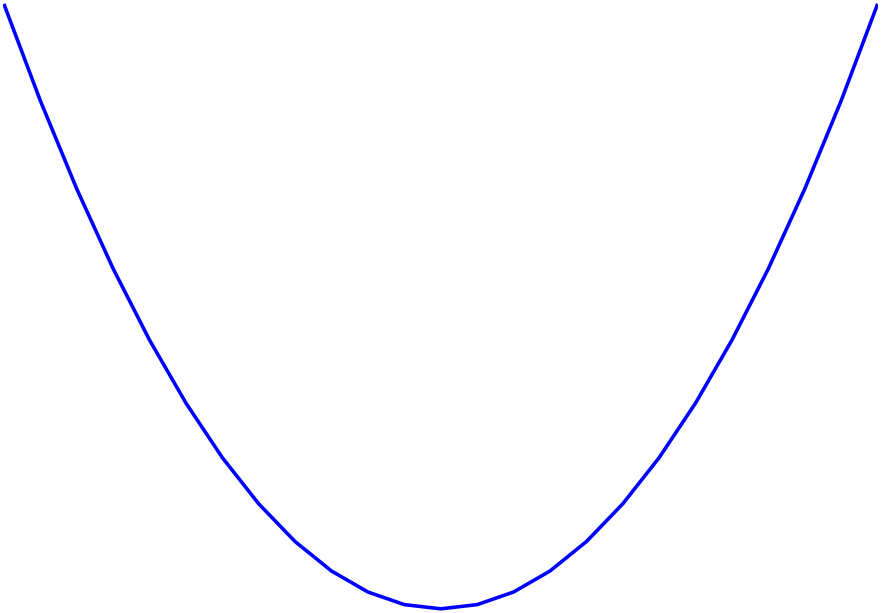
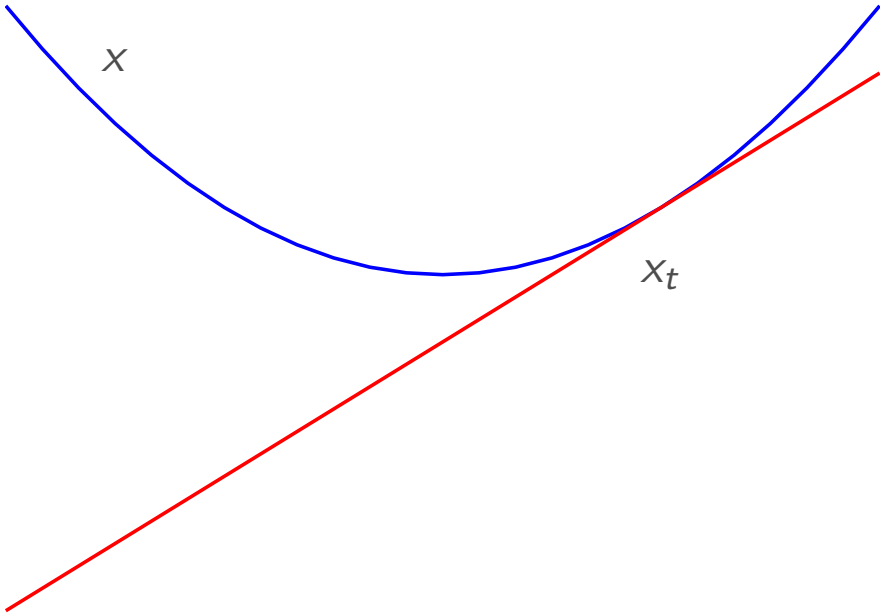


- convex function
- The sum of convex functions is convex
- Strictly convex function has a unique minimizer

Linearisation of a convex function

$$f_t(x) \geq f_t(x_t) + \langle x - x_t, \nabla f_t(x_t) \rangle$$



Rearranging, $f_t(x_t) - f_t(x) \leq \langle x_t - x, \nabla f_t(x_t) \rangle$

Why linear is enough?

- Regret is bounded by

$$\begin{aligned} R_n(x) &= \sum_{t=1}^{n} (f_t(x_t) - f_t(x)) \\ &\leq \sum_{t=1}^{n} \langle x_t - x, \nabla f_t(x_t) \rangle \end{aligned}$$

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- Reduction from nonlinear to linear
- Only uses first order information (the gradient)
- **Linear losses from now on** $f_t(x) = \langle x, \ell_t \rangle$
- Think of $\ell_t = \nabla f_t(x_t)$ for a general convex loss function

Online convex optimisation (linear losses)

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- **How to choose $\mathbf{x}_t$?** Most simple idea 'follow-the-leader'

$$x_{t+1} = \operatorname{argmin}_{x \in \mathcal{K}} \sum_{s=1}^{t} \langle x, \ell_s \rangle \,.$$

- Fails miserably: $\mathcal{K} = [-1, 1]$, $\ell_1 = 1/2$, $\ell_2 = -1$, $\ell_3 = 1$, ...
- $x_1 = ?$, $x_2 = -1$ $(\operatorname{argmin}_{x \in \mathcal{K}} \langle x, \ell_1 \rangle)$, $x_3 = 1$ $(\operatorname{argmin}_{x \in \mathcal{K}} \langle x, \ell_1 + \ell_2 \rangle)$, ...
- $R_n(0) = \sum_{t=1}^{n} \langle x_t, \ell_t \rangle \approx n$.

Follow The regularized Leader (FTRL)

- **New idea** Add **regularization** to stabilize follow-the-leader
- Let F be a strictly convex function and $\eta > 0$ be the **learning rate** and

$$x_{t+1} = \text{argmin}_{x \in \mathcal{K}} \left(F(x) + \eta \sum_{s=1}^{t} \langle x, \ell_s \rangle \right)$$

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- Different choices of F lead to different algorithms.
- One clean analysis.

Example – Gradient descent

- $\mathcal{K} = \mathbb{R}^d$ and $F(x) = \frac{1}{2}\|x\|_2^2$

$$x_{t+1} = \text{argmin}_{x \in \mathcal{K}} \eta \sum_{s=1}^{t} \langle x, \ell_s \rangle + \frac{1}{2}\|x\|_2^2$$

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- Differentiating,

$$0 = \eta \sum_{s=1}^{t} \ell_s + x$$

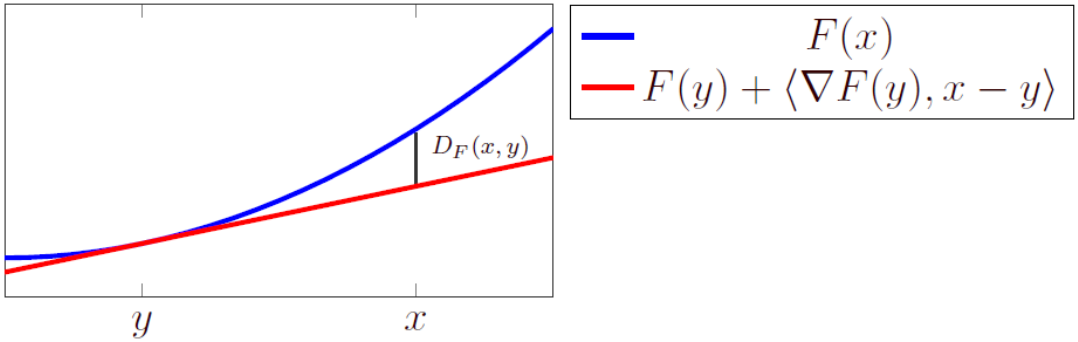
- $x_{t+1} = -\eta \sum_{s=1}^{t} \ell_s = x_t - \eta \ell_t$

A few tools

- Online convex optimization uses many tools from convex analysis
- Bregman divergence
- First-order optimality conditions
- Dual norms

Bregman divergence

For convex F, $D_F(x, y) = F(x) - F(y) - \langle \nabla F(y), x - y \rangle$



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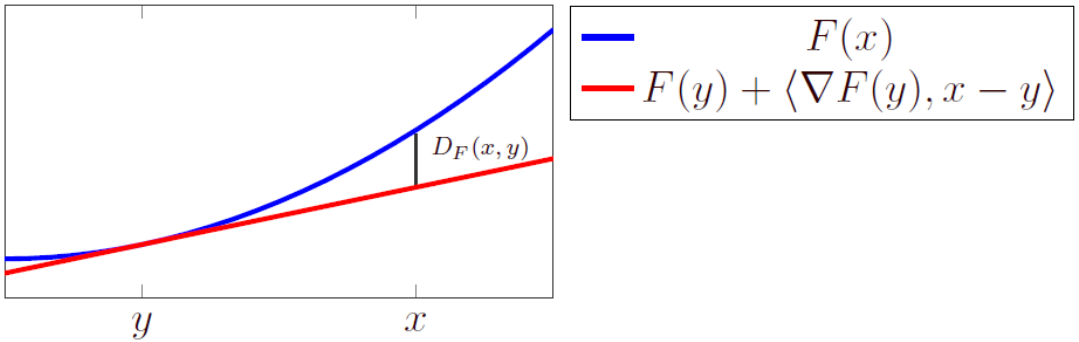

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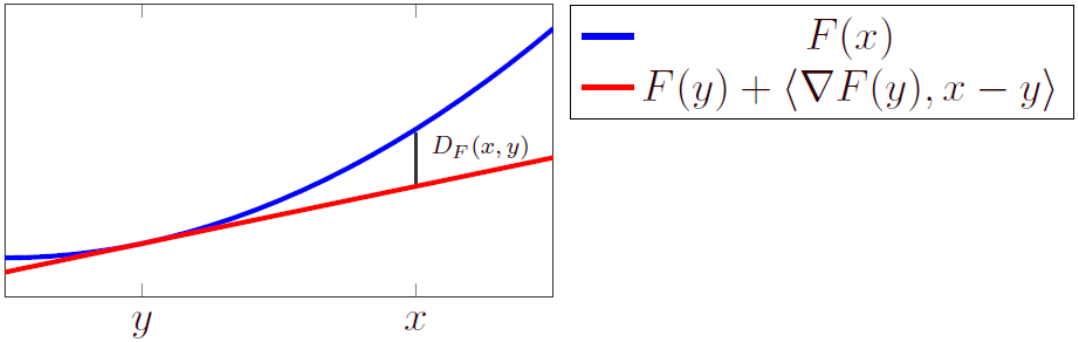

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- Key property: does not change under linear perturbation: For $\widetilde{F}(x) = F + \langle a, x \rangle$, $D_{\widetilde{F}}(x, y) = D_F(x, y)$

Examples

- **Quadratic** $F(x) = \frac{1}{2}\|x\|^2$

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- **Neg-entropy** $F(x) = \sum_{i=1}^d x_i \log(x_i) - x_i$

$$D_F(x, y) = \sum_{i=1}^d x_i \log(\frac{x_i}{y_i}) + \sum_{i=1}^d (y_i - x_i)$$

When $x, y \in \Delta_d$, where $\Delta_d = \{x \in \mathbb{R}^d : x \geq 0, \|x\|_1 = 1\}$ (d-dimensional simplex, usually for modeling a discrete probability distribution):

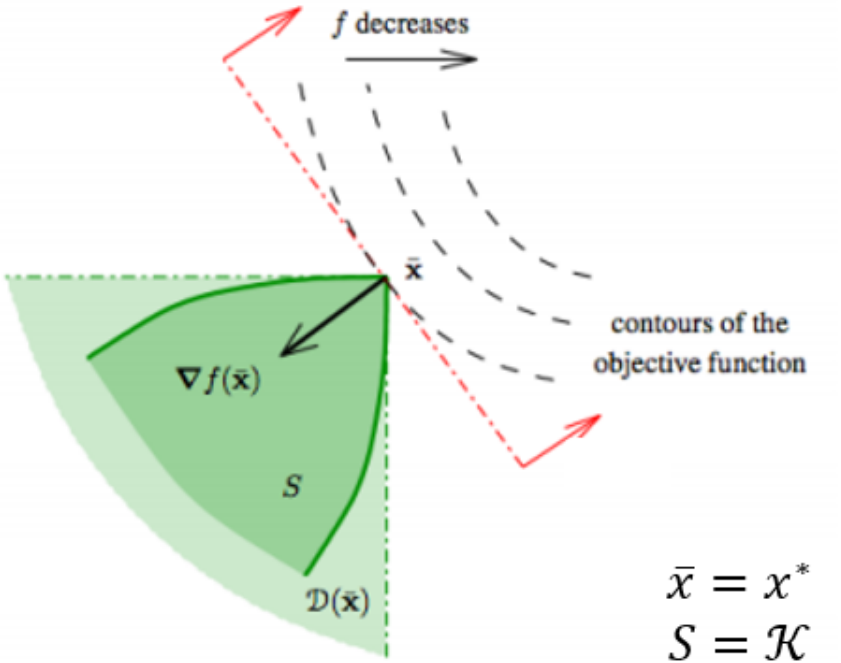
$$D_F(x, y) = \sum_{i=1}^d x_i \log(\frac{x_i}{y_i})$$

First order optimality condition

- Let $\mathcal{K}$ be convex, $f : \mathcal{K} \to \mathbb{R}$ convex, differentiable

$$x^* = \text{argmin}_{x \in \mathcal{K}} f(x) \Leftrightarrow \langle \nabla f(x^*), x - x^* \rangle \geq 0 \ \forall x \in \mathcal{K}$$

- **Interpretation** f is increasing in direction $x - x^*$ for all $x \in \mathcal{K}$



$$\bar{x} = x^*$$
$$S = \mathcal{K}$$

Dual norm

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- The dual norm of $\|\cdot\|_p$ is $\|\cdot\|_q$ (with $\frac{1}{p} + \frac{1}{q} = 1$).
- Hölder's inequality: $\langle z, x\rangle \leq \|x\|_t \|z\|_{t^*}$.

Follow the regularized leader

- $x_{t+1} = \text{argmin}_{x \in \mathcal{K}} \left(F(x) + \eta \sum_{s=1}^{t} \langle x, \ell_s \rangle \right)$
- Equivalent to

$$\begin{aligned} x_{t+1} &= \text{argmin}_{x \in \mathcal{K}} \left(\eta \langle x, \ell_t \rangle + D_F(x, x_t) \right) \\ &= \text{argmin}_{x \in \mathcal{K}} \left(\eta \langle x, \ell_t \rangle + F(x) - F(x_t) - \langle \nabla F(x_t), x - x_t \rangle \right) \end{aligned}$$

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- Assuming the minimizer is achieved in the interior of K.
- The first optimization implies that $\nabla F(x_{t+1}) = -\eta \sum_{s=1}^{t} \ell_s$
- The second optimization implies that $\eta \ell_t + \nabla F(x_{t+1}) - \nabla F(x_t) = 0$ and thus
  $\nabla F(x_{t+1}) = -\eta \ell_t + \nabla F(x_t) = -\eta \sum_{s=1}^{t} \ell_s + \underbrace{\nabla F(x_1)}_{0} = -\eta \sum_{s=1}^{t} \ell_s.$

Regret Analysis: Follow the regularized leader

Theorem For any fixed action x, the regret of follow the regularized leader satisfies

$$\begin{aligned} R_n(x) &:= \sum_{t=1}^{n} \langle x_t - x, \ell_t \rangle \\ &\leq \frac{F(x) - F(x_1)}{\eta} + \sum_{t=1}^{n} \left(\langle x_t - x_{t+1}, \ell_t \rangle - \frac{1}{\eta} D_F(x_{t+1}, x_t) \right) \\ &\leq \frac{F(x) - F(x_1)}{\eta} + \frac{\eta}{2} \sum_{t=1}^{n} \|\ell_t\|_{t*}^2 \end{aligned}$$

Let $z_t \in [x_t, x_{t+1}]$ be such that $D_F(x_{t+1}, x_t) = \frac{1}{2}\|x_t - x_{t+1}\|^2_{\nabla^2 F(z_t)}$ and $\|\cdot\|_t = \|\cdot\|_{\nabla^2 F(z_t)}$ and $\|\cdot\|_{t^*}$ is the dual norm of $\|\cdot\|_t$.

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Choosing $\|\cdot\|_t$ such that $D_F(x_{t+1}, x_t) \geq \frac{1}{2}\|x_t - x_{t+1}\|_t^2$ is also valid.

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$$\le \frac{F(x) - F(x_1)}{\eta} + \frac{\eta}{2} \sum_{t=1}^{n} \|\ell_t\|_{t*}^2$$

$D_F(x_{t+1}, x_t) = \frac{1}{2}\|x_t - x_{t+1}\|^2_{\nabla^2 F(z_t)}$ and $\|\cdot\|_t = \|\cdot\|_{\nabla^2 F(z_t)}$

Proof of the second inequality:

$$\langle x_t - x_{t+1}, \ell_t \rangle - \frac{D_F(x_{t+1}, x_t)}{\eta} \le \|\ell_t\|_{t*} \|x_t - x_{t+1}\|_t - \frac{D_F(x_{t+1}, x_t)}{\eta}$$

$$= \|\ell_t\|_{t*} \sqrt{2 D_F(x_{t+1}, x_t)} - \frac{D_F(x_{t+1}, x_t)}{\eta} \le \frac{\eta}{2} \|\ell_t\|_{t*}^2,$$

The last inequality is due to $ax - bx^2/2 \le a^2/(2b)$ for any $b \ge 0$ with $a = \|\ell_t\|_{t*}$, $x = \sqrt{2 D_F(x_{t+1}, x_t)}$ and $b = \frac{1}{\eta}$

FTRL analysis

- Proof of the first inequality

$$R_n(x) \leq \frac{F(x) - F(x_1)}{\eta} + \sum_{t=1}^{n} \left(\langle x_t - x_{t+1}, \ell_t \rangle - \frac{1}{\eta} D_F(x_{t+1}, x_t) \right)$$

- Rewriting the regret

$$\begin{aligned} R_n(x) &= \sum_{t=1}^{n} \langle x_t - x, \ell_t \rangle \\ &= \sum_{t=1}^{n} \langle x_t - x_{t+1}, \ell_t \rangle + \sum_{t=1}^{n} \langle x_{t+1} - x, \ell_t \rangle \end{aligned}$$

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- Goal: show that

$$\sum_{t=1}^{n} \langle x_{t+1} - x, \ell_t \rangle \leq \frac{F(x) - F(x_1)}{\eta} - \sum_{t=1}^{n} \frac{1}{\eta} D_F(x_{t+1}, x_t)$$

FTRL analysis

- Potential function: $\Phi_t(x) = \frac{F(x)}{\eta} + \sum_{s=1}^{t} \langle x, \ell_s \rangle$
- By FRTL: x_{t+1} minimizes Φ_t in $\mathcal{K}$
-

$$\sum_{t=1}^{n} \langle x_{t+1} - x, \ell_t \rangle$$

$$= \sum_{t=1}^{n} \langle x_{t+1}, \ell_t \rangle - \underbrace{(\sum_{t=1}^{n} \langle x, \ell_t \rangle + \frac{F(x)}{\eta})}_{\Phi_n(x)} + \frac{F(x)}{\eta}$$

$$= \sum_{t=1}^{n} \underbrace{\left(\Phi_t(x_{t+1}) - \Phi_{t-1}(x_{t+1})\right)}_{\left(\frac{F(x_{t+1})}{\eta} + \sum_{s=1}^{t} \langle x_{t+1}, \ell_s \rangle\right) - \left(\frac{F(x_{t+1})}{\eta} + \sum_{s=1}^{t-1} \langle x_{t+1}, \ell_s \rangle\right)} - \Phi_n(x) + \frac{F(x)}{\eta},$$

FTRL analysis

Potential function: $\Phi_t(x) = \frac{F(x)}{\eta} + \sum_{s=1}^{t} \langle x, \ell_s \rangle$ ($\Phi_0(x) = \frac{F(x)}{\eta}$)

Then using: (1) $x_{t+1} = \operatorname{argmin}_x \Phi_t(x)$ and (2) $D_{\Phi_t}(\cdot,\cdot) = \frac{1}{\eta} D_F(\cdot,\cdot)$ (Bregman divergence keeps the same by adding linear functions)

$$\sum_{t=1}^{n} \langle x_{t+1} - x, \ell_t \rangle = \frac{F(x)}{\eta} + \sum_{t=1}^{n} (\Phi_t(x_{t+1}) - \Phi_{t-1}(x_{t+1})) - \Phi_n(x)$$

$$= \frac{F(x)}{\eta} - \Phi_0(x_1) + \underbrace{\Phi_n(x_{n+1}) - \Phi_n(x)}_{\leq 0:\ x_{n+1} = \operatorname{argmin}_x \Phi_n(x)} + \sum_{t=0}^{n-1} (\Phi_t(x_{t+1}) - \Phi_t(x_{t+2}))$$

$$\leq \frac{F(x) - F(x_1)}{\eta} + \sum_{t=0}^{n-1} (\Phi_t(x_{t+1}) - \Phi_t(x_{t+2}))$$

$$= \frac{F(x) - F(x_1)}{\eta} - \sum_{t=0}^{n-1} \left(D_{\Phi_t}(x_{t+2}, x_{t+1}) + \underbrace{\langle \nabla \Phi_t(x_{t+1}), x_{t+2} - x_{t+1} \rangle}_{\geq 0} \right)$$

$$\leq \frac{F(x) - F(x_1)}{\eta} - \frac{1}{\eta} \sum_{t=1}^{n} D_F(x_{t+1}, x_t),$$

where
$D_{\Phi_t}(x_{t+2}, x_{t+1}) = \Phi_t(x_{t+2}) - \Phi_t(x_{t+1}) - \langle \nabla \Phi_t(x_{t+1}), x_{t+2} - x_{t+1} \rangle$.

Final form of the regret

$$
\begin{aligned}
R_n(x) \leq & \frac{F(x) - F(x_1)}{\eta} + \frac{\eta}{2} \sum_{t=1}^{n} \|\ell_t\|_{t^*}^2 \\
\leq & \frac{\mathrm{diam}_F(\mathcal{K})}{\eta} + \frac{\eta}{2} \sum_{t=1}^{n} \|\ell_t\|_{t^*}^2,
\end{aligned}
$$

where $\mathrm{diam}_F(\mathcal{K}) := \max_{a,b \in \mathcal{K}} F(a) - F(b)$.

Final form of the regret

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- Regret depends on distance from start to optimal
- Learning rate needs careful tuning

Application 1: Online gradient descent

Assume $\mathcal{K} = \{x : \|x\|_2 \leq 1\}$ and $\ell_t \in \mathcal{K}$ ($|\langle x, \ell_t \rangle| \leq 1$)

Choose $F(x) = \frac{1}{2}\|x\|_2^2$, $\text{diam}_F(\mathcal{K}) := \max_{a,b \in \mathcal{K}} F(a) - F(b) = \frac{1}{2}$:

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Then by choosing $\eta = \sqrt{1/n}$:

$$R_n(x) \leq \frac{\mathrm{diam}_F(\mathcal{K})}{2\eta} + \frac{\eta}{2} \sum_{t=1}^{n} \|\ell_t\|_2^2 \leq \frac{1}{2\eta} + \frac{\eta n}{2} \leq \sqrt{n}$$

Application 2: Exponential weights

Assume $\mathcal{K} = \Delta_d := \{x \geq 0 : \|x\|_1 = 1\}$ and $\ell_t \in [0,1]^d$ for all t ($|\langle x, \ell_t \rangle| \leq 1$)

$$F(x) = \sum_{i=1}^{d} (x_i \log(x_i) - x_i)$$

$\text{diam}_F(\mathcal{K}) := \max_{a,b \in \mathcal{K}} F(a) - F(b) = \log(d)$. This is because $\max_{x \in \mathcal{K}} F(x) = \log(d) - 1$ (achieving at $(1/d, \ldots, 1/d)$) and $\min_{x \in \mathcal{K}} F(x) = -1$ (achieving $(1, 0, \ldots, 0)$).

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Bregman divergence

$$\begin{aligned} D_F(x,y) &= \sum_{i=1}^{d} x_i \log\left(\frac{x_i}{y_i}\right) && \text{(KL-divergence)} \\ &\geq \frac{1}{2}\|x-y\|_1^2 && \text{(Pinsker's inequality (exercise))} \end{aligned}$$

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FTRL:

$$x_{t+1} = \text{argmin}_{x \in \mathcal{K}} \, \eta \sum_{s=1}^{t} \langle x, \ell_s \rangle + F(x)$$

Optimal action is a standard basis vector e_i, where i is the position that $i = \text{argmin}_{j=1,\ldots,n} \left(\eta \sum_{s=1}^{t} \ell_{s,j} \right)$ (corresponding $F(x)$ is minimized since $F(e_i) = -1$).

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$$\begin{aligned} R_n(x) &\leq \frac{\text{diam}_F(\mathcal{K})}{\eta} + \frac{\eta}{2} \sum_{t=1}^{n} \|\ell_t\|_{t*}^2 \\ &\leq \frac{\log(d)}{\eta} + \frac{\eta}{2} \sum_{t=1}^{n} \|\ell_t\|_{\infty}^2 \\ &\leq \frac{\log(d)}{\eta} + \frac{\eta n}{2} \leq \sqrt{2n \log(d)} \end{aligned}$$

Our Goal: Adversarial bandits

- At the start of the game the **adversary** secretly chooses losses $\ell_1, \ldots, \ell_n$ with $\ell_t \in [0,1]^K$
- In each round the learner chooses the arm $A_t \in \{1, \ldots, K\} \sim P_t$ (from some distribution P_t)
- Suffers and loss ℓ_{t,A_t} (only observe ℓ_{t,A_t})
- Regret is $R_n = \max_a \mathbb{E}\left[\sum_{t=1}^n \ell_{tA_t} - \ell_{ta}\right]$

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- **Surprising result** there exists an algorithm such that $R_n \leq \sqrt{2nK\log(K)}$ for any adversary. How?
- **Key idea**
  - Construct an estimator of the entire loss vector $\ell_t = (\ell_{t,1}, \ldots, \ell_{t,K})$
  - Apply the follow the regularized leader (FTRL) to the estimated loss $\hat{\ell}_t = (\hat{\ell}_{t,1}, \ldots, \hat{\ell}_{t,K})$

Importance-weighted estimators

At time t, our algorithm chooses the arm $A_t = i$ with probability P_{ti} (specify P_{ti} later). Define the estimator of $\ell_{t,i}$

$$\hat{\ell}_{t,i} = \frac{\ell_{t,i} \mathbb{1}(A_t = i)}{P_{ti}}$$

and $\hat{\ell}_t = (\hat{\ell}_{t,1}, \dots \hat{\ell}_{t,K})$.

Unbiased estimator,

$$\begin{aligned} \mathbb{E}\left[\hat{\ell}_{t,i} \mid P_t\right] &= \frac{\ell_{t,i}}{P_{ti}} \mathbb{E}[\mathbb{1}(A_t = i) \mid P_t] = \frac{\ell_{t,i}}{P_{ti}} P_{ti} \\ &= \ell_{t,i} \end{aligned}$$

Second moment: $\mathbb{E}\left[\hat{\ell}_{t,i}^2 \mid P_t\right] = \frac{\ell_{t,i}^2}{P_{ti}}$

Follow the regularized leader for bandits (EXP3)

- Estimate ℓ_t with unbiased **importance-weighted estimator** $\hat{\ell}_t$

$$\hat{\ell}_{t,i} = \frac{\mathbb{1}(A_t = i)\ell_{t,i}}{P_{ti}}$$

- Then the expected regret satisfies

$$\mathbb{E}[R_n] = \max_i \mathbb{E}\left[\sum_{t=1}^{n} \ell_{t,A_t} - \ell_{t,i}\right] = \max_i \mathbb{E}\left[\sum_{t=1}^{n} \langle P_t - e_i, \hat{\ell}_t \rangle\right]$$

This is because

- $\mathbb{E}(\langle P_t, \ell_t \rangle) = \mathbb{E}(\sum_{i=1}^{K} P_{ti}\ell_{t,i}) = \mathbb{E}(\sum_{i=1}^{K} \mathbb{1}(A_t = i)\ell_{t,i}) = \mathbb{E}(\ell_{t,A_t})$
- $\mathbb{E}(\langle e_i, \hat{\ell}_t \rangle) = \mathbb{E}(\hat{\ell}_{t,i}) = \ell_{t,i}$.

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$$P_t = \operatorname{argmin}_{p \in \Delta_K} \frac{F(p)}{\eta} + \sum_{s=1}^{t-1} \langle p, \hat{\ell}_s \rangle$$

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- FTRL:

$$P_t = \operatorname{argmin}_{p \in \Delta_K} \frac{F(p)}{\eta} + \sum_{s=1}^{t-1} \langle p, \hat{\ell}_s \rangle$$

- Since the domain is Δ_K, choose the **negentropy** F

$$F(p) = \sum_{i=1}^{K} p_i \log(p_i) - p_i$$

- Using Lagrange dual, the probability of choosing each arm i:

$$P_{ti} = \frac{\exp\left(-\eta \sum_{s=1}^{t-1} \hat{\ell}_{s,i}\right)}{\sum_{j=1}^{K} \exp\left(-\eta \sum_{s=1}^{t-1} \hat{\ell}_{s,j}\right)}$$

Follow the regularized leader for bandits (EXP3 Algo)

- Using the FTRL regret bound:

$$\begin{aligned}\mathbb{E}[R_n] &\leq \mathbb{E}\left[\frac{F(e_i) - F(P_1)}{\eta} + \frac{\eta}{2}\sum_{t=1}^{n}\|\hat{\ell}_t\|_{t*}^2\right] \\ &\leq \frac{\log(K)}{\eta} + \frac{\eta}{2}\mathbb{E}\left[\sum_{t=1}^{n}\|\hat{\ell}_t\|_{t*}^2\right] \qquad (F(e_i) - F(P_1) \leq \log(K))\end{aligned}$$

  where i is the best arm.

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Follow the regularized leader for bandits

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- How to bound $\|\hat{\ell}_t\|_{t*}^2$?
- Recall that Let $z_t \in [x_t, x_{t+1}]$ be such that $D_F(x_{t+1}, x_t) = \frac{1}{2}\|x_t - x_{t+1}\|^2_{\nabla^2 F(z_t)}$. And $\|\cdot\|_t = \|\cdot\|_{\nabla^2 F(z_t)}$ and $\|\cdot\|_{t^*}$ is the dual norm of $\|\cdot\|_t$.
- For $F(p) = \sum_{i=1}^K p_i \log(p_i) - p_i$,

$$\nabla^2 F(p) = \mathrm{diag}(1/p) \implies \|\hat{\ell}_t\|_{t*}^2 = \|\hat{\ell}_t\|^2_{\nabla^2 F(p)^{-1}} = \sum_{i=1}^K p_i \hat{\ell}_{t,i}^2,$$

for some $p \in [P_t, P_{t+1}]$

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- $\|\hat{\ell}_t\|_{t*}^2 = \|\hat{\ell}_t\|_{\nabla^2 F(p)^{-1}}^2 = \sum_{i=1}^K p_i \hat{\ell}_{t,i}^2$ for some $p \in [P_t, P_{t+1}]$
- $\hat{\ell}_{t,i} = \frac{\mathbb{1}(A_t = i)\ell_{t,i}}{P_{ti}}$ is non-negative and $\hat{\ell}_{t,j} = 0$ for $A_t \neq j$:

$$\|\hat{\ell}_t\|_{t*}^2 = p_{A_t} \hat{\ell}_{t,A_t}^2$$

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- Further note that, $P_{t,A_t} = \frac{\exp\left(-\eta \sum_{s=1}^{t-1} \hat{\ell}_{s,A_t}\right)}{\sum_{j=1}^K \exp\left(-\eta \sum_{s=1}^{t-1} \hat{\ell}_{s,j}\right)} := \frac{\alpha_{A_t}}{\sum_{j=1}^K \alpha_j}$ and

$$P_{t+1,A_t} = \frac{\exp\left(-\eta \sum_{s=1}^{t-1} \hat{\ell}_{s,A_t}\right) \exp(-\eta \hat{\ell}_{t,A_t})}{\sum_{j=1}^K \exp\left(-\eta \sum_{s=1}^{t-1} \hat{\ell}_{s,j}\right) \exp(-\eta \hat{\ell}_{t,j})} = \frac{\alpha_{A_t}}{\alpha_{A_t} + \sum_{j \neq A_t} \alpha_j \exp(\eta \hat{\ell}_{t,A_t})}$$

Since $\exp(\eta \hat{\ell}_{t,A_t}) > 1$, $P_{t+1,A_t} \leq P_{t,A_t}$

$$\|\hat{\ell}_t\|_{t*}^2 = \sum_{j=1}^K p_j \hat{\ell}_{t,j}^2 \leq P_{tA_t} \hat{\ell}_{t,A_t}^2$$

Putting everything together: Regret bound for the follow the regularized leader for bandits

$$
\begin{aligned}
\mathbb{E}[R_n] &\le \frac{\log(K)}{\eta} + \frac{\eta}{2}\mathbb{E}\left[\sum_{t=1}^{n} \|\hat{\ell}_t\|_{t*}^2\right] \\
&\le \frac{\log(K)}{\eta} + \frac{\eta}{2}\mathbb{E}\left[\sum_{t=1}^{n} P_{tA_t}\hat{\ell}_{t,A_t}^2\right] \\
&= \frac{\log(K)}{\eta} + \frac{\eta}{2}\mathbb{E}\left[\sum_{t=1}^{n} \frac{\ell_{t,A_t}^2}{P_{tA_t}}\right] \\
&\le \frac{\log(K)}{\eta} + \frac{\eta}{2}\mathbb{E}\left[\sum_{t=1}^{n} \frac{1}{P_{tA_t}}\right] && (\ell_t \in [0,1]^K) \\
&= \frac{\log(K)}{\eta} + \frac{\eta}{2}\mathbb{E}\left[\sum_{t=1}^{n}\sum_{i=1}^{K} P_{ti}\cdot\frac{1}{P_{ti}}\right] \\
&= \frac{\log(K)}{\eta} + \frac{\eta n K}{2} \le \sqrt{2nK\log(K)} && (\eta = \sqrt{2\log(K)/(nK)})
\end{aligned}
$$

Historical notes

- First paper on bandits is by Thompson (1933). He proposed an algorithm for two-armed Bernoulli bandits and hand-runs some simulations (Thompson sampling)
- Popularized enormously by Robbins (1952)
- Confidence bounds first used by Lai and Robbins (1985) to derive asymptotically optimal algorithm
- UCB by Katehakis and Robbins (1995) and Agrawal (1995). Finite-time analysis by Auer et al. (2002)
- Adversarial bandits: Auer et al. (1995)
- Minimax optimal algorithm by Audibert and Bubeck (2009)

Resources

- Online notes: http://banditalgs.com
- The book "Bandit Algorithms" by Tor Lattimore and Csaba Szepesvari
  https://tor-lattimore.com/downloads/book/book.pdf
- Book by Bubeck and Cesa-Bianchi (2012)
- Book by Cesa-Bianchi and Lugosi (2006)
- The Bayesian books by Gittins et al. (2011) and Berry and Fristedt (1985). Both worth reading.
- Notes by Aleksandrs Slivkins:
  http://slivkins.com/work/MAB-book.pdf

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