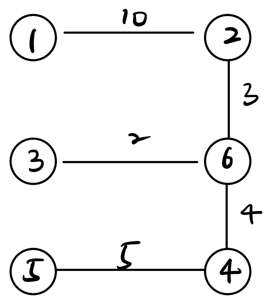
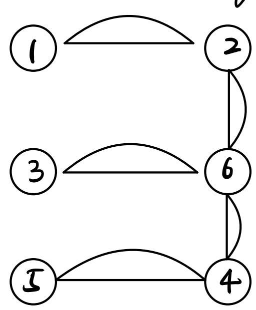
Problem 1.

Task 1. Given the data. we can get the MST.



Double the edge. we can get.



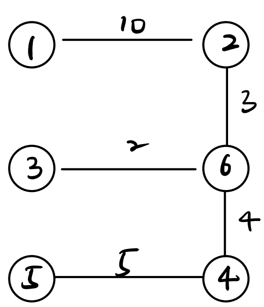
$\Rightarrow 1 \to 2 \to 6 \to 3 \to 6 \to 4 \to 5 \to 4 \to 6$
$\to 2 \to 1$

After shortcutting. we can get:

$1 \to 2 \to 6 \to 3 \to 4 \to 5 \to 1$.

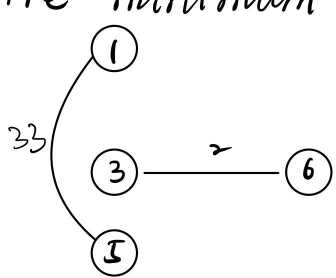
The ~~total~~ cost is: $10+3+2+6+5+33 = 59$.

Task 2. Given the data. we can get the MST.

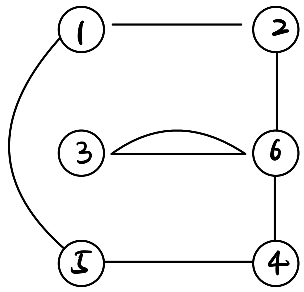


The set of odd degree vertices is: $U = \{1, 3, 5, 6\}$.

The minimum cost matching is:



Add back to MST. we can get:



$\Rightarrow 1 \to 2 \to 6 \to 3 \to 6 \to 4 \to 5 \to 1$

After shortcutting. we can get:

$1 \to 2 \to 6 \to 3 \to 4 \to 5 \to 1$.

The total cost is 59.

Problem 2.

Task 1.

$V_{max} = 6$.

Let $T[i, w]$ be the min-size of subset $S \subseteq \{1, 2, \cdots, i\}$

s.t. $\sum_{k \in S} V_k = w$

Since $n \times V_{max} = 4 \times 6 = 24$

$T[1,0] = 0$. $T[1,1] = \infty$. $T[1,2] = \infty$. $T[1,3] = \infty$

$T[1,4] = 3$. $T[1,w] = \infty$ for $5 \leq w \leq 24$.

$T[2,0] = 0$. $T[2,1] = \infty$ $T[2,2] = \infty$ $T[2,3] = \infty$

$T[2,4] = 3$. $T[2,5] = \infty$ $T[2,6] = \infty$ $T[2,7] = \infty$

$T[2,8] = 6$ $T[2,w] = \infty$ for $9 \leq w \leq 24$.

$T[3,0] = 0$ $T[3,1] = \infty$ $T[3,2] = \infty$ $T[3,3] = \infty$

$T[3,4] = 3$. $T[3,5] = \infty$ $T[3,6] = 8$ $T[3,7] = \infty$

$T[3,8] = 6$ $T[3,9] = \infty$ $T[3,10] = 11$ $T[3,11] = \infty$

$T[3,12] = \infty$ $T[3,13] = \infty$ $T[3,14] = 14$ $T[3,w] = \infty$ for $15 \leq w \leq 24$.

$T[4,0]=0$ $T[4,1]=\infty$ $T[4,2]=\infty$ $T[4,3]=\infty$

$T[4,4]=3$ $T[4,5]=5$ $T[4,6]=8$ $T[4,7]=\infty$

$T[4,8]=6$ $T[4,9]=8$ $T[4,10]=11$ $T[4,11]=13$

$T[4,12]=\infty$ $T[4,13]=11$ $T[4,14]=14$ $T[4,15]=16$

$T[4,16]=\infty$ $T[4,17]=\infty$ $T[4,18]=\infty$ $T[4,19]=19$

$T[4,w]=\infty$ for $20 \le w \le 24$.

For $v \le 8$. the maximum value is 9 with the size of 8.

Task 2.

i	V_i	S_i	V_i/S_i
1	4	3	4/3
2	4	3	4/3
3	8	6	3/4
4	5	5	1

By ranking. $\frac{V_1}{S_1} \geq \frac{V_2}{S_2} > \frac{V_4}{S_4} > \frac{V_3}{S_3}$.

According to greedy algorithm, we will choose item #1, item #2 first. After that, there's no room for other items.

However. if choose item #1 and item #4 or item #2 and item #4, we get volume = 9. size = 8

Therefore. greedy algorithm is not optimal.

Problem 3.

Task 1.

Round #1

$R_1 = \frac{cost}{num} = \frac{6}{5} = 1.2$

$R_2 = \frac{15}{5} = 3$ $\Rightarrow$ choose set 3.

$R_3 = \frac{7}{7} = 1$ ✓ uncovered $= \{e_1, e_2, e_3, e_6, e_7\}$

Round #2.

$R_1 = \frac{6}{3} = 2$ ✓ $\Rightarrow$ choose set 1

$R_2 = \frac{15}{5} = 3$ uncovered $= \{e_6, e_7\}$

Round #3

$R_2 = \frac{15}{2} = 7.5$ ✓ $\Rightarrow$ choose set 2.

$Cost\_tot = 6 + 15 + 7 = 28$

Task 2.

Obviously, the greedy algorithm is not optimal.

Simply choose set 2 and set 3 can cover the whole ground set.

$cost\_tot' = 15 + 7 = 22$.

Bonus

Task 1:

$$\max \sum_{j \in D} \alpha_j$$

$$\begin{aligned}
\text{s.t.} \quad & \alpha_j - \beta_{ij} \leq d_{ij}, && \forall i \in F, j \in D \\
& \textstyle\sum_{j \in D} \beta_{ij} \leq f_i && \forall i \in F. \\
& \alpha_j \geq 0 && \forall j \in D \\
& \beta_{ij} \geq 0 && \forall i \in F, j \in D.
\end{aligned}$$

Task 2.

- $\sum_{j\in D} \beta_{ij} \le f_i \quad \forall i \in F.$

  The total money needed to open facility #i is at least the total contribution from every client towards opening facility #i.

- $\alpha_j - \beta_{ij} \le d_{ij} \quad \forall i \in F, j \in D$

  This is equivalent to $\alpha_j \le \beta_{ij} + d_{ij} \quad \forall i \in F, j \in D.$

  The total payment by client j is at most the sum of contribution it makes towards opening facility #i and the distance in between.