LEC011 Maximum Flow and Linear Program

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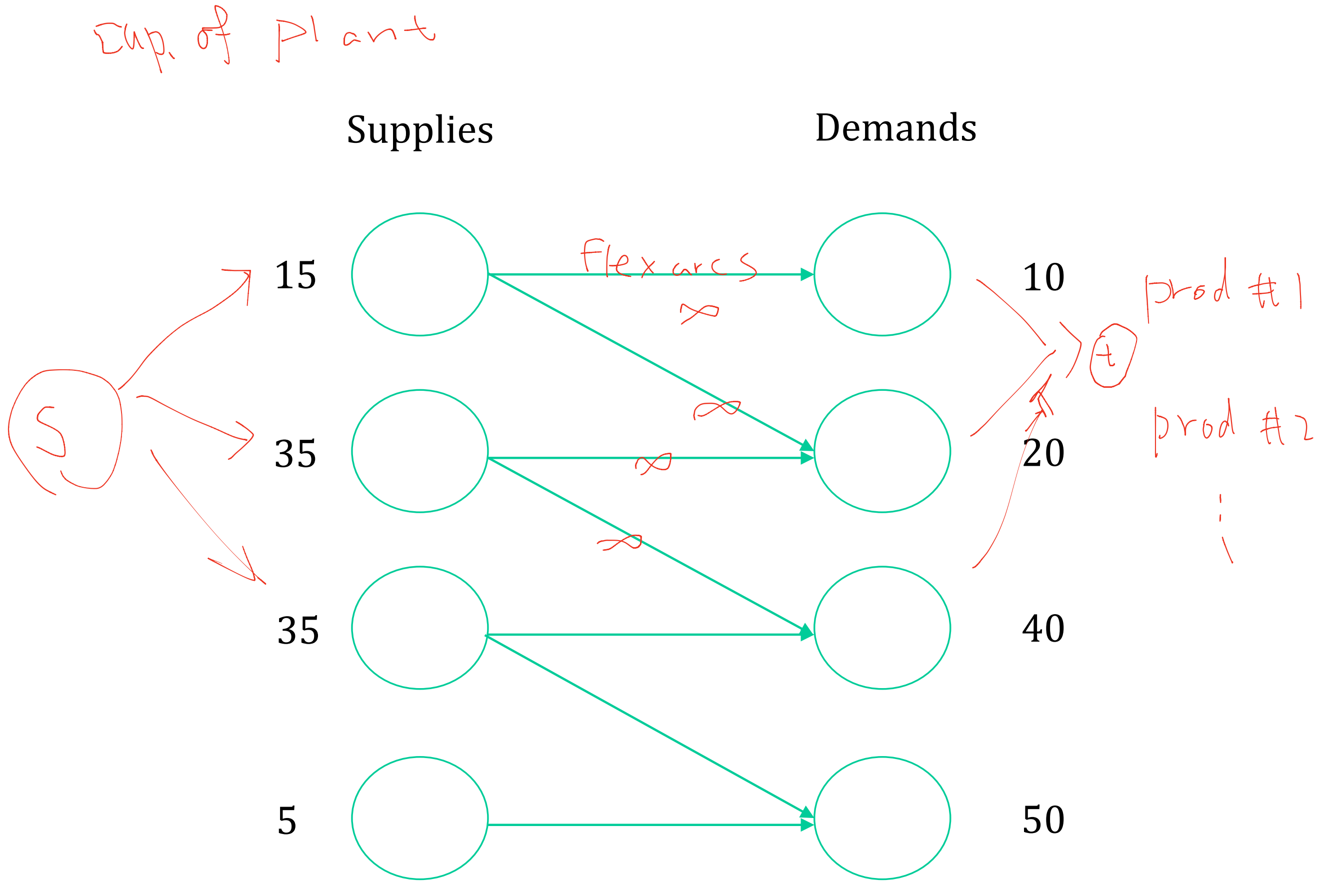
Max Flow Applications



Linear Programming

- Comparison to systems of linear equations

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m
\end{aligned}$$

Gaussian elimination

能解决80%问题

Linear Programming

(LP)

- Ingredients of a Linear Program

$X = (x_1, x_2, \ldots x_n)^T$

1. Decision variables $x_1, \ldots, x_n \in \mathbb{R}$
2. Linear constraints, each of the form

RHS

linear

$$\sum_{j=1}^{n} a_{ij} x_j \quad (*) \quad b_i$$

where $(^*)$ could be $\leq, \geq$, or $=$

3. A linear objective function, of the form

$$\underbrace{\max \sum_{j=1}^{n} c_j x_j \qquad \text{or} \qquad \min \sum_{j=1}^{n} c_j x_j}$$

profit/reward

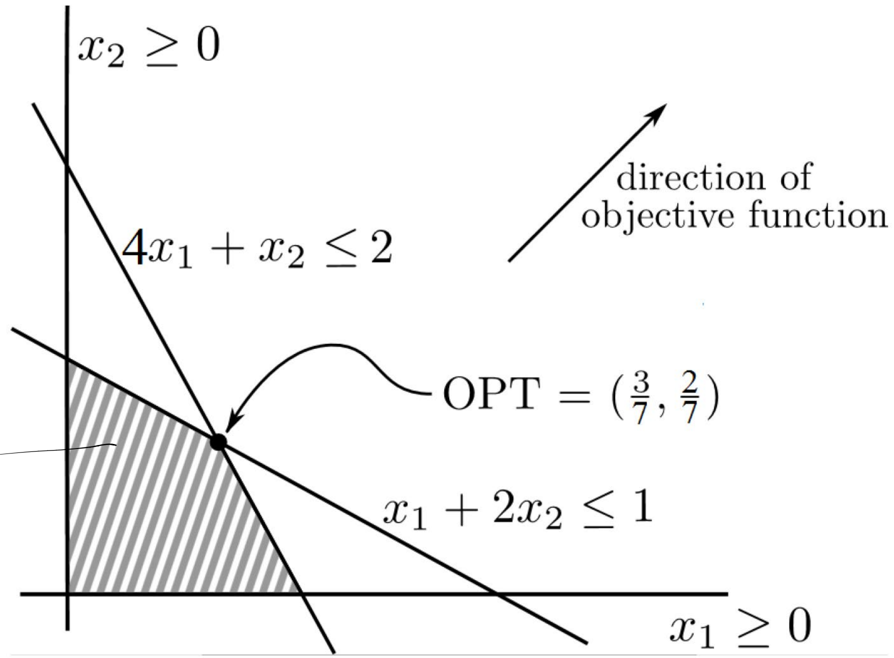
cost/penalty.

A Simple Example

$$\begin{aligned} \max \quad & x_1 + x_2 \\ \text{s.t.} \quad & 4x_1 + x_2 \le 2 \\ & x_1 + 2x_2 \le 1 \\ & x_1, x_2 \ge 0 \end{aligned}$$

2维时
画图即可

feasible
region
‖
polyhedron.



Python + Gurobi Time!

安装 gurobi

加 licence

后一个例子是 numerical example

以前的 slide 上

MaxFlow is a LP

- Decision variables

$$\{f_e\}_{e\in E}$$

flow value on each e.

- Constraints $\underbrace{(2m + n - 2)}$

$G(V,E)$

n m

$$\underbrace{\sum_{e\in\delta^-(v)} f_e}_{\text{flow in}} - \underbrace{\sum_{e\in\delta^+(v)} f_e}_{\text{flow out}} = 0 \quad \forall v \setminus \{s,t\}$$

$n-2$

m — $f_e \le u_e \quad \forall e$

- Objectives

m — $f_e \ge 0 \quad \forall e$

$$\underline{\max \sum_{e\in\delta+(s)} f_e}$$

flow value out of source.

$\underline{\text{Gurobi}}$

[illegible]

Generalization of MaxFlow

- Min-Cost MaxFlow

$$\min \sum_{e \in E} c_e f_e$$

- Easy to change the LP formulation!

for solving MaxFlow

缺点在于

model is not bendable

FF alg

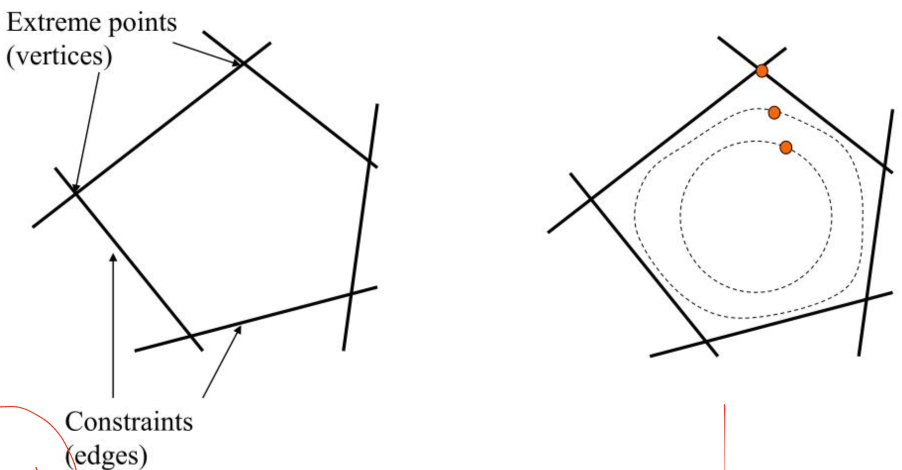
Fatest alg: push & relabel

Solution Approaches

现实中两者接近

- Simplex methods
- Interior point methods

Interior point methods



theoretical worst-case complexity

Simplex: search from vertex to vertex along the edges

not in polynomial time

Interior-point methods: go through the inside of the feasible space

in polynomail time

LP Duality

- Question: given a feasible solution, how can we know whether it is optimal or close to optimal?

$$\boxed{\begin{array}{ll} \max & x_1 + x_2 \\ \text{s.t.} & 4x_1 + x_2 \le 2 \\ & x_1 + 2x_2 \le 1 \\ & x_1, x_2 \ge 0 \end{array}}$$

$$\underbrace{x_1 + x_2}_{\text{objective}} \le 4x_1 + x_2 \le \underbrace{2}_{\text{upper bound}}$$

$$\underbrace{x_1 + x_2}_{\text{objective}} \le x_1 + 2x_2 \le \underbrace{1}_{\text{upper bound}}$$

Can we get an even better upper bound?

$$x_1 + x_2 \le \frac{1}{7}\underbrace{(4x_1 + x_2)}_{\le 2 \text{ by } (2)} + \frac{3}{7}\underbrace{(x_1 + 2x_2)}_{\le 1 \text{ by } (3)} \le \frac{1}{7}\cdot 2 + \frac{3}{7}\cdot 1 = \frac{5}{7}$$

Deriving the Dual LP

Compact form

$$\begin{array}{ll} \max & \mathbf{c}^T\mathbf{x} \\ \text{s.t.} & \mathbf{A}\mathbf{x} \le \mathbf{b} \\ & \mathbf{x} \ge 0 \end{array}$$

dual variables

"Multipliers"

$$\begin{array}{lll} \max & \sum_{j=1}^{n} c_j x_j & \\ \text{s.t.} & \sum_{j=1}^{n} a_{1j} x_j \le b_1 & \longrightarrow \quad y_1 \ge 0 \\ & \sum_{j=1}^{n} a_{2j} x_j \le b_2 & \longrightarrow \quad y_2 \ge 0 \\ & \vdots \le \vdots & \\ & \sum_{j=1}^{n} a_{mj} x_j \le b_m & \longrightarrow \quad y_m \ge 0 \\ & x_1, \ldots, x_n \ge 0 & \end{array}$$

The idea is to "dominates the objective coefficients":

$$\sum_{i=1}^{m} y_i a_{ij} \ge c_j$$

More compactly,

Find $\mathbf{y} \ge 0$ such that $\mathbf{A}^T\mathbf{y} \ge \mathbf{c}$

Deriving the Dual LP

$$
\begin{aligned}
\max \quad & \mathbf{c}^T\mathbf{x} \\
\text{s.t.} \quad & \mathbf{A}\mathbf{x} \le \mathbf{b} \\
& \mathbf{x} \ge 0
\end{aligned}
$$

Find $\mathbf{y} \ge 0$ such that $\mathbf{A}^T\mathbf{y} \ge \mathbf{c}$

$$
\begin{aligned}
\underbrace{\sum_{j=1}^{n} c_j x_j}_{\text{x's obj fn}} \quad & \le \quad \sum_{j=1}^{n} \left(\sum_{i=1}^{m} y_i a_{ij} \right) x_j \\
& = \quad \sum_{i=1}^{m} y_i \cdot \left(\sum_{j=1}^{n} a_{ij} x_j \right) \\
& \le \quad \underbrace{\sum_{i=1}^{m} y_i b_i}_{\text{upper bound}}
\end{aligned}
$$

More compactly, $\mathbf{c}^T\mathbf{x} \le \left(\mathbf{A}^T\mathbf{y}\right)^T \mathbf{x} = \mathbf{y}^T(\mathbf{A}\mathbf{x}) \le \mathbf{y}^t\mathbf{b}$

matrix 轉置

Deriving the Dual LP

$$\boxed{\begin{array}{ll} \max & \mathbf{c}^T\mathbf{x} \\ \text{s.t.} & \mathbf{A}\mathbf{x} \leq \mathbf{b} \\ & \mathbf{x} \geq 0 \end{array}}$$

Find $\mathbf{y} \geq 0$ such that $\mathbf{A}^T\mathbf{y} \geq \mathbf{c}$

$$\mathbf{c}^T\mathbf{x} \leq \left(\mathbf{A}^T\mathbf{y}\right)^T\mathbf{x} = \mathbf{y}^T(\mathbf{A}\mathbf{x}) \leq \mathbf{y}^t\mathbf{b}$$

$$\boxed{\begin{array}{ll} \min & \mathbf{b}^T\mathbf{y} \\ \text{s.t.} & \mathbf{A}^T\mathbf{y} \geq \mathbf{c} \\ & \mathbf{y} \geq 0 \end{array}}$$

A Simple Example

$$\begin{aligned} \max \quad & \mathbf{c}^T\mathbf{x} \\ \text{s.t.} \quad & \mathbf{A}\mathbf{x} \leq \mathbf{b} \\ & \mathbf{x} \geq 0 \end{aligned}$$

$$\begin{aligned} \min \quad & \mathbf{b}^T\mathbf{y} \\ \text{s.t.} \quad & \mathbf{A}^T\mathbf{y} \geq \mathbf{c} \\ & \mathbf{y} \geq 0 \end{aligned}$$

$$\begin{aligned} \max \quad & x_1 + x_2 \\ \text{s.t.} \quad & 4x_1 + x_2 \leq 2 \\ & x_1 + 2x_2 \leq 1 \\ & x_1, x_2 \geq 0 \end{aligned}$$

$$\begin{aligned} \min \quad & 2y_1 + y_2 \\ \text{s.t.} \quad & 4y_1 + y_2 \leq 1 \\ & y_1 + 2y_2 \leq 1 \\ & y_1, y_2 \geq 0 \end{aligned}$$

$$x_1^* = 3/7, x_2^* = 2/7$$

$$y_1^* = 1/7, y_2^* = 3/7$$

Optimal objectives are the same!

Weak and Strong Duality

$$\begin{array}{ll} \max & \mathbf{c}^T\mathbf{x} \\ \text{s.t.} & \mathbf{A}\mathbf{x} \le \mathbf{b} \\ & \mathbf{x} \ge 0 \end{array}$$

$$\begin{array}{ll} \min & \mathbf{b}^T\mathbf{y} \\ \text{s.t.} & \mathbf{A}^T\mathbf{y} \ge \mathbf{c} \\ & \mathbf{y} \ge 0 \end{array}$$

At optimality, we must have

- Weak Duality $\quad \mathbf{c}^T\mathbf{x}^* \le \mathbf{b}^T\mathbf{y}^*$
- Strong Dualiy $\quad \mathbf{c}^T\mathbf{x}^* = \mathbf{b}^T\mathbf{y}^*$ "no gap"

Proof = Separating hyperplance theorem + Farkas's Lemma

不在491范围内，但见canvas

More General Form

Primal	Dual
variables $x_1, \ldots, x_n$	n constraints
m constraints	variables $y_1, \ldots, y_m$
objective function $\mathbf{c}$	right-hand side $\mathbf{c}$
right-hand side $\mathbf{b}$	objective function $\mathbf{b}$
$\max \mathbf{c}^T \mathbf{x}$	$\min \mathbf{b}^T \mathbf{y}$
constraint matrix $\mathbf{A}$	constraint matrix $\mathbf{A}^T$
i-th constraint is " $\leq$ "	$y_i \geq 0$
i-th constraint is " $\geq$ "	$y_i \leq 0$
i-th constraint is " $=$ "	$y_i \in \mathbb{R}$
$x_j \geq 0$	j-th constraint is " $\geq$ "
$x_j \leq 0$	j-th constraint is " $\leq$ "
$x_j \in \mathbb{R}$	j-th constraint is " $=$ "

Back to MaxFlow

- Let the decision variable be flow on an s-t path

flow

$$
\begin{aligned}
\max \quad & \sum_{P \in \mathcal{P}} f_P \\
\text{s.t.} \quad & \underbrace{\sum_{P \in \mathcal{P} : e \in P} f_P \le u_e}_{\text{total flow on } e} \quad \text{for all } e \in E \\
& f_P \ge 0 \quad \text{for all } P \in \mathcal{P}
\end{aligned}
$$

Question: how to write its dual and what is its interpretation?

回家作业！