CHAPTER 3

DETERMINISTIC INVENTORY MODELS

3.1 INTRODUCTION TO INVENTORY MODELING

3.1.1 Why Hold Inventory?

Think about some of the products you bought the last time you went to the grocery store. How much of each did you buy? Why did you choose these quantities?

Here are some possible reasons:

1. You bought a gallon of milk but only a pint of cream because you drink much more milk than cream in a week.

2. You bought a six-pack of soda, rather than a single bottle, because you don't want to have to go to the store every time you want to drink a bottle of soda.

3. You bought a "family size" box of cereal, rather than a small box, because larger boxes are more cost-effective (cheaper per ounce) than smaller ones.

4. Although you usually eat one bag of potato chips per week, you bought three bags in case your hungry friends show up unexpectedly one night this week.

5. You asked the store to special-order your favorite brand of gourmet mustard (which it doesn't normally stock), even though you already have a half jar at home, because you know it will take a few weeks before the mustard is delivered.

Fundamentals of Supply Chain Theory, Second Edition. Lawrence V. Snyder and Zuo-Jun Max Shen.

Companion website: www.wiley.com/go/Snyder/SupplyChainTheory

6. Although it would be more cost-effective and convenient to buy 12 rolls of paper towels, you only bought 3, because you don't have enough space to store 12 rolls at home.

7. You bought four boxes of pasta, even though you only eat one box per week, because they were on sale for a greatly reduced price.

8. Even though grapes were on sale, you bought one pound instead of two because you knew the second pound would spoil before you had a chance to eat them.

9. You bought a pound of butter (four sticks), even though you probably won't use more than one stick before your next trip to the store, because butter only comes in 1-pound packages.

All of these decisions affected the amount of inventory of groceries that you have in your home. Aside from the cost you paid to purchase these items, you are also paying a cost simply to hold the inventory (as opposed to buying a single item each time you need it and using it immediately). For example, if you used your credit card to make your purchase, then you are paying a little more interest by buying a six-pack of soda today rather than buying individual bottles throughout the week. If you paid cash, then you are tying up your cash in groceries rather than using it for some other purpose, such as going to the movies, or putting your money in an interest-earning savings account. You are also paying for the physical space required to store your groceries (as part of your rent or mortgage), the energy required to keep refrigerated items cold, and the insurance to protect your grocery investment if your house is burglarized or damaged in a fire.

Companies, too, would prefer not to hold any inventory, since inventory is expensive (even more than it is for you). However, most companies hold some inventory, for the same reasons that you hold inventory of your groceries:

1. Different products are purchased at different rates—the *demand rate*—and therefore require different levels of inventory.

2. There is an inconvenience, and often an expense, associated with placing an order with a supplier (analogous to your trip to the grocery store). For example, there may be an administrative cost to process the order and transmit it to the supplier, or there may be a cost to rent a truck to deliver the products. These are *fixed costs* since they are (roughly) independent of the size of the order, and they make it impractical to place an order each time a single item is needed.

3. Firms often receive *volume discounts* for placing large orders with their suppliers. Volume discounts and fixed costs are both types of *economies of scale*, which make it more cost-effective to order in bulk; that is, to place fewer, larger orders.

4. Demand for most products is random, and often so are lead times and other supply factors, and this *uncertainty* requires firms to hold inventory to ensure that they can satisfy the demand (at least most of the time).

5. After a firm places an order, the products do not arrive until after a (typically nonzero) *lead time*. Since the firm's own customers usually don't want to wait for this lead time, especially in retail settings, the firm must place a replenishment order even when it is still holding some inventory.

6. Warehouses have only a finite amount of *storage capacity*, and this may constrain the size of the firm's order. A related type of capacity (which is less relevant for the grocery example) is *production capacity*: If demand is highly seasonal (e.g., for snowblowers) but production capacity is limited, then the firm may need to produce more in off-peak times (summer) in order to meet the demand during peak times (winter).

7. Suppliers often offer sales and temporary discounts, just like retail stores do, and prices for many products (especially commodities) vary constantly. In response to both types of *price fluctuations*, firms buy large quantities when prices are low and hold goods in inventory until they're needed.

8. Some inventory is *perishable*, so firms must limit the quantity they buy to avoid being saddled with unusable inventory.

9. Many products are available only in fixed *batch sizes* such as cases or pallets, and the firm is forced to order in increments of those units.

These are all reasons that firms plan to hold inventory. In addition, firms may hold *unplanned* inventory—for example, inventory of products that have become obsolete sooner than expected.

Firms may hold inventory of goods at all stages of production—raw materials, components, work-in-process, and finished goods. The latter types of inventory are usually made by the firm, rather than ordered from a supplier, but similar issues still arise—for example, there may be a fixed cost to initiate a production run, it may be cheaper per unit to produce large batches, the processing time may be uncertain, and so on. In fact, although we tend to discuss inventory models as though the firm is buying a product from an outside supplier, most inventory models apply equally well to production systems, in which case we are deciding how much to produce, rather than how much to order, and the "ordering" costs are really production costs.

3.1.2 Classifying Inventory Models

Mathematical inventory models can be classified along a number of different dimensions:

- *Demand.* Is demand deterministic or stochastic? Does the rate stay the same all the time or does it vary over time—say, from season to season?

- *Lead time.* Is production or delivery instantaneous, or is there a positive lead time? Is the lead time deterministic or stochastic?

- *Review type.* Is inventory assessed *continuously* or *periodically*? In continuous-review models, the inventory is constantly monitored, and an order is placed whenever a certain condition is met (for example, the inventory level falls below a given value). In periodic-review models, the inventory is only checked every time period (say, every week), and an order is placed if the reorder condition is met. In periodic-review models, we usually assume that demands occur at a single instant during the period, even though they may really occur continuously throughout it.

- *Planning horizon. Finite-horizon* models consider a finite number of periods or time units, while *infinite-horizon* models assume the planning horizon extends forever.

Although it is unrealistic to assume that the firm will continue operating the same system, under the same conditions, forever, infinite-horizon models are often more tractable than finite-horizon ones and are therefore quite common.

- *Stockout type.* If demand exceeds supply, how is the excess demand handled? Most models consider either *backorders*, in which case excess demand stays on the books until it can be satisfied from a future shipment, or *lost sales*, in which case excess demands are simply lost—the customer takes her business elsewhere. In retail settings, it is usually more accurate to assume lost sales, whereas backorders are more common in business-to-business settings.

- *Ensuring good service.* Some models ensure that not too many stockouts occur by including a penalty in the cost function for each stockout. Others include a constraint on the allowable percentage of demands that may be stocked out. The former approach often leads to more tractable models, but it can be very difficult to quantify the cost of a stockout; therefore, service-level constraints are common in practice.

- *Fixed cost.* Some inventory models include a fixed cost to place an order, while others do not. The presence and magnitude of a fixed cost determines whether the firm places many small orders or few large orders. Moreover, inventory models with fixed costs are often more difficult to analyze and solve than those without, so we often ignore the fixed cost in modeling an inventory system even if one is present in the real system.

- *Perishability.* Can inventory be held across multiple time periods, or is it perishable? Perishable items include not just foods, but also fresh flowers and medicine (which will spoil), high-tech products (which will become obsolete), and newspapers and airline tickets (which have a deadline after which they can't be sold).

Like all mathematical models, inventory models must balance two competing factors—realism and tractability. In many cases, it is more accurate to assume one thing but easier to assume the opposite. For example, many inventory models are much more mathematically tractable if we assume backorders, so we might do so even if we are modeling inventory at a retail store, for which the lost-sales assumption is more accurate. Similarly, it is often convenient to assume lead times are zero even though they rarely are in practice. If the lead time is short compared to the order cycle—for example, if the firm places monthly orders and the lead time is 2 days—this assumption may not hurt the model's accuracy too much. Modeling is as much an art as a science, and part of modeling process involves determining both the cost (in terms of realism) and the benefit (in terms of tractability) of "assuming away" a given real-life factor.

3.1.3 Costs

The goal of most inventory models is to minimize the cost (or maximize the profit) of the inventory system. Four types of costs are most common:

- *Holding cost.* This represents the cost of actually keeping the inventory on hand. Like the costs associated with storing your groceries, the holding cost includes the

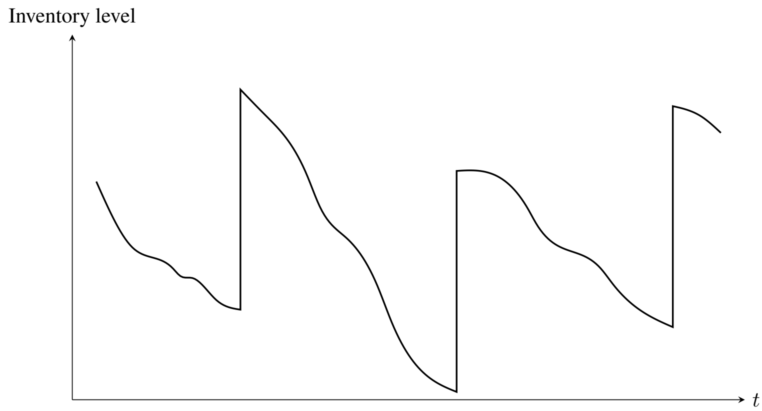


Figure 3.1 Inventory level curve.

cost of storage space, taxes, insurance, breakage, theft, and, most significantly, opportunity cost—the money the firm could be earning if it didn't have its capital tied up in inventory. The holding cost is often expressed as a percentage of the value of the product per year. For example, the holding cost might be 25% per year. If the item costs \$100, then it costs \$1562.50 to hold 250 items for 3 months ($1562.50 = 0.25 \cdot 100 \cdot 250 \cdot (3/12)$). We will usually use h to represent the holding cost per item per unit time.

In reality, the inventory level is not constant but fluctuates over time, as pictured in Figure 3.1. Here, the holding cost is the area under the curve times h, so we would use integration to compute it. In some of the inventory models discussed in this book, the inventory "curve" is made up of straight lines, so computing the area is easy.

- *Fixed cost.* This is the cost to place an order, independent of the size of the order. It is sometimes called the *setup cost*, and we will usually denote it by K. The fixed cost accounts for the administrative cost of placing an order, the cost of using a truck to deliver the product, and so on.

- *Purchase cost.* This is the cost per unit to buy and ship the product, generally denoted by c. (It is also sometimes known as the *variable cost* or *per-unit cost*.) Therefore, the total order cost (fixed + purchase) to order x units is given by

$$\begin{cases} 0, & \text{if } x = 0 \\ K + cx, & \text{if } x > 0. \end{cases}$$

One picky but worthwhile note: If there is a nonzero lead time, then we typically assume that the firm pays the purchase cost c when the order arrives, not when it is placed. This assumption doesn't affect the total purchase cost per year (unless we're modeling the time value of money), but it does affect the holding cost if h is a function of c: If the firm was to pay the purchase cost when the order is placed, its capital would be tied up during the lead time, but this would not be accurately reflected in the holding cost.

- *Stockout cost.* This is the cost of not having sufficient inventory to meet demand, also called the *penalty cost* or *stockout penalty*, and is denoted by p. If excess demand is backordered, the penalty cost includes bookkeeping costs, delay costs, fines for missing promised delivery dates, and—most significantly—*loss of goodwill* (the potential loss of future business since the customer is unhappy). If excess demand is lost, the penalty cost also includes the lost profit from the missed sale. The penalty is generally charged per unit of unmet demand. If excess demand is backordered, the penalty may be proportional to the amount of time the backorder is on the books before it is filled, or (less commonly) it may be a one-time penalty charged when the demand is backordered.

3.1.4 Inventory Level and Inventory Position

There are several measures that we use to assess the amount of inventory in the system at any given time. *On-hand inventory* (OH) refers to the number of units that are actually available at the stocking location. *Backorders* (BO) represent demands that have occurred but have not been satisfied. Generally, it's not possible for the on-hand inventory *and* the backorders to be positive at the same time.

The *inventory level* (IL) is equal to the on-hand inventory minus backorders:

$$IL = OH - BO.$$

If $IL > 0$, we have on-hand inventory, and if $IL < 0$, we have no units on hand but we do have backorders. Therefore, we can write

$$\begin{aligned} OH &= IL^+ \\ BO &= IL^-, \end{aligned}$$

where $x^+ = \max\{x, 0\}$ and $x^- = |\min\{x, 0\}|$. (Be warned: Some authors use $x^- = \min\{x, 0\}$.)

It seems reasonable to think of IL as the relevant measure to consider when making ordering decisions—we look at the shelves, see how much inventory we have, and place an order if there's not enough. But IL by itself does not give us enough information to make good ordering decisions. For instance, suppose the inventory level is 5, you're expecting a demand of 50 next week, and there's a lead time of 4 weeks. How much should you order? The answer depends on how much you've already ordered—i.e., how much is "in the pipeline," ordered but not received. Such items are called *on order* (OO). Therefore, we usually make ordering decisions based on the *inventory position* (IP), which equals the inventory level plus items on order:

$$IP = OH - BO + OO.$$

The distinction between inventory level and inventory position is subtle but important. Typically, we use inventory position to make ordering decisions, but holding and backorder costs are assessed based on inventory level. If the lead time is zero, then $OO = 0$ and $IL = IP$.

3.1.5 Roadmap

In this chapter and the next three, we will explore some classical inventory models and a few of their variants. This chapter discusses deterministic models—first a continuous-review model, the economic order quantity (EOQ) model, perhaps the oldest and best-known mathematical inventory model (Section 3.2), and some of its extensions; and then a periodic-review model, the Wagner–Whitin model (Section 3.7). Then, Chapters 4 and 5 discuss stochastic models. The models in all three of these chapters make inventory decisions for a single stage (location). Multistage models are considered in Chapter 6.

The models discussed in this chapter are sometimes known as *economic lot size problems*. In fact, there is some inconsistency about how this term is used in the literature. Some authors refer to the EOQ model (Section 3.2) as *the* economic lot size model. Other authors refer to the Wagner–Whitin model (Section 3.7) as *the* economic lot size model. More generally, the term can be used to refer to any model in which an optimal lot size must be determined, typically under deterministic demand. To avoid confusion, we will avoid this term and instead use the names of the individual models discussed.

3.2 CONTINUOUS REVIEW: THE ECONOMIC ORDER QUANTITY PROBLEM

3.2.1 Problem Statement

The *economic order quantity* (EOQ) problem is one of the oldest and most fundamental inventory models; it was first introduced by Harris (1913). The goal is to determine the optimal amount to order each time an order is placed to minimize the average cost per year. (We'll express everything per year, but the model could just as easily be per month or any other time period.)

We assume that demand is deterministic and constant with a rate of λ units per year. Stockouts are not allowed—we must always order enough so that demand can be met. Since demand is deterministic, this is a plausible assumption. The lead time is 0—orders are received instantaneously. There is a fixed cost K per order, a purchase cost c per unit ordered, and an inventory holding cost h per unit per year. There is no stockout penalty since stockouts are not allowed.

The inventory level[1] evolves as follows. Assume that the on-hand inventory is 0 at time 0; we place an order at time 0, and it arrives instantaneously. The inventory level then decreases at a constant rate λ until the next order is placed, and the process repeats.

Any optimal solution for the EOQ model has two important properties:

- *Zero-inventory ordering (ZIO) property.* Since the lead time is 0, it never makes sense to place an order when there is a positive amount of inventory on hand—we only place an order when the inventory level is 0.

- *Constant order sizes.* If Q is the optimal order size at time 0, it will also be the optimal order size every other time we place an order since the system looks the same every time the inventory level hits 0. Therefore, the order size is the same every time an order is placed.

(You should convince yourself that these properties are indeed optimal.) The inventory level is pictured as a function of time in Figure 3.2. T is called the *cycle length*—the amount

[1] Since the lead time is 0, the inventory position is equal to the inventory level at all times.

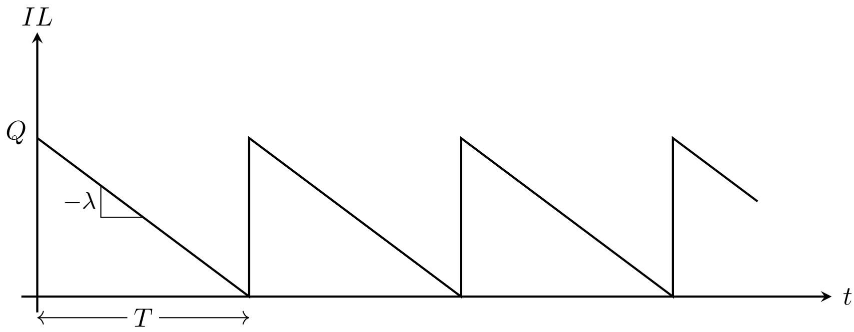


Figure 3.2 EOQ inventory level curve.

of time between orders—and it relates to the order quantity Q and λ by the equation

$$T = \frac{Q}{\lambda}.$$

3.2.2 Cost Function

We want to find the optimal Q to minimize the average annual cost. (We say "average" annual cost since the actual cost in any given year may fluctuate a bit as the sawtooth pattern falls slightly differently across the start of each year.) Note that minimizing the annual cost is not the same as minimizing the cost per cycle; minimizing the cost per cycle would mean choosing very tiny order quantities. The key trade-off is between fixed cost and holding cost: If we use a large Q, we'll place fewer orders and hold more inventory (small fixed cost but large holding cost), whereas if we use a small Q, we'll place more orders and hold less inventory (large fixed cost but small holding cost).

The strategy for solving the EOQ is to express the average annual cost as a function of Q, then minimize it to find the optimal Q.

Order Cost: Each order incurs a fixed cost of K. It also incurs a purchase cost of c per unit ordered, but this cost is irrelevant for the optimization problem at hand—that is, the optimal value of Q does not depend on c. (Why?) Therefore, we'll ignore the per-unit cost c in our analysis. Since the time between orders is T years, the order cost per year is

$$\frac{K}{T} = \frac{K\lambda}{Q}. \tag{3.1}$$

Holding Cost: The average inventory level in a cycle is $Q/2$, so the average amount of inventory per year is $Q/2 \cdot 1$ year = $Q/2$. (Another way to think about this is that the area of a triangle in the inventory curve in Figure 3.2 is $QT/2$, and there are $1/T$ cycles per year, so the total area under the inventory curve for 1 year is $QT/2 \cdot 1/T = Q/2$.) Therefore, the average annual holding cost is

$$\frac{hQ}{2}. \tag{3.2}$$

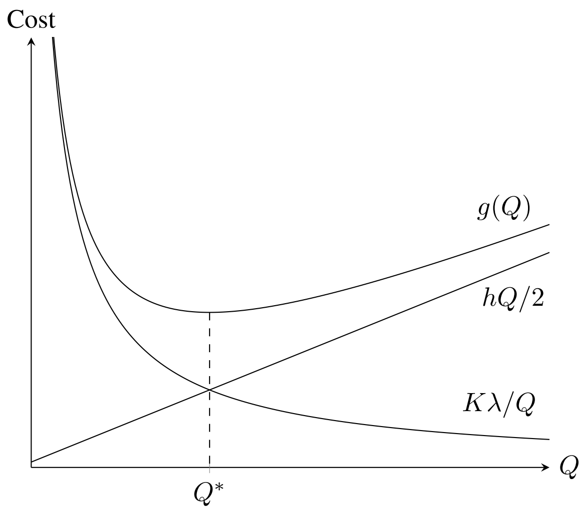


Figure 3.3 Fixed, holding, and total costs as a function of Q.

Total Cost: Combining (3.1) and (3.2), we get the total average annual cost, denoted $g(Q)$:

$$g(Q) = \frac{K\lambda}{Q} + \frac{hQ}{2}. \tag{3.3}$$

The fixed, holding, and total cost curves are plotted as a function of Q in Figure 3.3.

3.2.3 Optimal Solution

The optimal Q can be obtained by taking the derivative of $g(Q)$ and setting it to 0:

$$\begin{aligned} \frac{dg(Q)}{dQ} &= -\frac{K\lambda}{Q^2} + \frac{h}{2} = 0 \\ &\implies Q^2 = \frac{2K\lambda}{h} \\ &\implies Q^* = \sqrt{\frac{2K\lambda}{h}}. \end{aligned} \tag{3.4}$$

Q^* is known as the *economic order quantity*. ("Economic" is just another word for "optimal.") We should also take a second derivative to verify that $g(Q)$ is convex (and thus the first-order condition yields a minimum, not a maximum):

$$\frac{d^2 g(Q)}{dQ^2} = \frac{2K\lambda}{Q^3} > 0,$$

as desired.

Note that in Figure 3.3, we drew the optimal order quantity Q^* at the intersection of the fixed and holding cost curves. This was not an accident. Of course, in general, it is not true that the minimum of the sum of two functions occurs where the two functions intersect, but it happens to be true for the EOQ. Why? The curves intersect when

$$\frac{K\lambda}{Q} = \frac{hQ}{2} \implies \frac{K\lambda}{Q^2} = \frac{h}{2}.$$

This is exactly the condition obtained by setting the first derivative to 0. Thus, the fixed and holding costs should always be balanced. If the fixed cost $K\lambda/Q$ is greater than the holding cost $hQ/2$, then Q is not optimal; we should be ordering less frequently and holding more inventory. (And vice versa.)

Another way to see that the fixed and holding costs are equal in the optimal solution is to note that the product of the two terms in (3.3) is

$$\frac{K\lambda}{Q} \cdot \frac{hQ}{2} = \frac{K\lambda h}{2},$$

a constant. In general, when two quantities multiply to a constant, their sum is minimized when the quantities are equal. Another non-calculus-based proof is given in Problem 3.21.

It should also be noted that, although we ignored the per-unit cost c in this analysis, c does influence Q^* indirectly if h is a function of c.

The optimal cost can be expressed as a function of the parameters by plugging the optimal Q^* into $g(Q)$:

$$\begin{aligned} g(Q^*) &= \frac{K\lambda}{\sqrt{\frac{2K\lambda}{h}}} + \frac{h}{2}\sqrt{\frac{2K\lambda}{h}} \\ &= \sqrt{\frac{K\lambda h}{2}} + \sqrt{\frac{K\lambda h}{2}} \\ &= \sqrt{2K\lambda h}. \end{aligned} \tag{3.5}$$

It's nice that the optimal cost has such a convenient form. This is not true for many other problems. The ability to express $g(Q^*)$ in closed form allows us to learn about structural properties of the EOQ and related models, such as the power-of-two policies discussed in Section 3.3, as well as to embed the EOQ into other, richer models, such as the location model with risk pooling (LMRP) in Section 12.2.

The optimal EOQ solution and its cost are summarized in the next theorem, whose proof follows from arguments already made above.

Theorem 3.1 *The optimal order quantity in the EOQ model is given by*

$$Q^* = \sqrt{\frac{2K\lambda}{h}} \tag{3.6}$$

and its cost is given by

$$g(Q^*) = \sqrt{2K\lambda h}. \tag{3.7}$$

Using Theorem 3.1, we can make some statements about how the solution changes as the parameters change:

- As h increases, Q^* decreases, since larger holding cost $\Longrightarrow$ it's more expensive to hold inventory $\Longrightarrow$ order smaller quantities more frequently
- As K increases, Q^* increases, since it's more expensive to place orders $\Longrightarrow$ we place fewer of them, with larger quantities
- As c increases, Q^* decreases if h is proportional to c (and stays the same if they are independent)

- As λ increases, Q^* increases

Obviously, if any of the costs increase, then $g(Q^*)$ will increase. If λ increases, $g(Q^*)$ will increase, as well. This does not mean that the firm prefers small demand, however. Remember that the EOQ only reflects costs, not revenues; the increased cost of large λ would be outweighed by the increased revenue.

☐ **EXAMPLE 3.1**

Joe's Corner Store sells 1300 candy bars per year. It costs \$8 to place an order to the candy bar supplier. Each candy bar costs the store \$0.75. Holding costs are estimated to be 30% per year. What is the optimal order quantity?

We have $h = 0.3 \cdot 0.75 = 0.225$, so

$$Q^* = \sqrt{\frac{2K\lambda}{h}} = \sqrt{\frac{2 \cdot 8 \cdot 1300}{0.225}} = 304.1.$$

The optimal cycle time is

$$T^* = \frac{Q^*}{\lambda} = \frac{304.1}{1300} = 0.23.$$

So the store should order 304.1 candy bars every 0.23 years, or approximately four times per year. The optimal cost is

$$\sqrt{2K\lambda h} = \sqrt{2 \cdot 8 \cdot 1300 \cdot 0.225} = 68.41.$$

If we must order in integer quantities, then we need to round Q^* down and up and check the cost of each:

$$\begin{aligned} g(304) &= \frac{8 \cdot 1300}{304} + \frac{0.225 \cdot 304}{2} = 68.4105 \\ g(305) &= \frac{8 \cdot 1300}{305} + \frac{0.225 \cdot 305}{2} = 68.4108, \end{aligned}$$

so we should order 304. ☐

3.2.4 Sensitivity Analysis

Suppose the firm did not want to order Q^* exactly. For example, it might need to order in multiples of 10 ($Q = 10n$), or it might want to order every month ($T = 1/12$). How much more expensive is a suboptimal solution? It turns out that the answer is "not much," and that we can determine the exact percentage increase in cost using a very simple formula.

Theorem 3.2 *Suppose Q^* is the optimal order quantity in the EOQ model. Then for any $Q > 0$,*

$$\frac{g(Q)}{g(Q^*)} = \frac{1}{2}\left(\frac{Q^*}{Q} + \frac{Q}{Q^*}\right). \tag{3.8}$$

Proof.

$$\frac{g(Q)}{g(Q^*)} = \frac{\frac{K\lambda}{Q} + \frac{hQ}{2}}{\sqrt{2K\lambda h}}$$

$$
\begin{aligned}
&= \frac{K\lambda}{Q\sqrt{2K\lambda h}} + \frac{hQ}{2\sqrt{2K\lambda h}} \\
&= \frac{1}{Q}\sqrt{\frac{K\lambda}{2h}} + \frac{Q}{2}\sqrt{\frac{h}{2K\lambda}} \\
&= \frac{1}{2Q}\underbrace{\sqrt{\frac{2K\lambda}{h}}}_{=Q^*} + \frac{Q}{2}\underbrace{\sqrt{\frac{h}{2K\lambda}}}_{=1/Q^*} \\
&= \frac{1}{2}\left(\frac{Q^*}{Q} + \frac{Q}{Q^*}\right)
\end{aligned}
$$

■

The right-hand side of (3.8) grows slowly as Q deviates more from Q^*, meaning that the EOQ is not very sensitive to errors in Q. For example, if we order twice as much as we should ($Q = 2Q^*$), the error is 1.25 (25% more expensive than optimal). If we order half as much ($Q = Q^*/2$), the error is also 1.25.

Theorem 3.2 ignores the per-unit cost c. If we include the annual cost $c\lambda$ in the numerator and denominator of (3.8), then the percentage increase in cost would be even smaller (and the expressions would not simplify as nicely).

☐ **EXAMPLE 3.2**

Suppose Joe's Corner Store (Example 3.1) ordered 250 candy bars per order instead of the optimal 304.1. How much would the cost increase as a result of this suboptimal solution?

$$\frac{g(Q)}{g(Q^*)} = \frac{1}{2}\left(\frac{304.1}{250} + \frac{250}{304.1}\right) = 1.019$$

So this solution would cost 1.9% more than the optimal solution. (You can also confirm this by calculating $g(250)$ explicitly and comparing it to $g(Q^*)$.) ☐

3.2.5 Order Lead Times

We assumed the lead time is 0. What if the lead time was positive—say, L years? The optimal solution doesn't change—we just place our order L years before it's needed. For example, if $L = 1$ month = 1/12 years, then the order should be placed 1/12 years before the inventory level reaches 0. It's generally more convenient to express this in terms of the *reorder point* (r). When the inventory level reaches r, an order is placed. How do we compute r? Well, r should be equal to the amount of product demanded during the lead time, or

$$r = \lambda L. \tag{3.9}$$

☐ **EXAMPLE 3.3**

In Example 3.1, if $L = 1/12$, the store should place an order whenever the inventory level reaches $r = 1300 \cdot (1/12) = 108.3$. ☐

3.3 POWER-OF-TWO POLICIES

From Section 3.2.3, we know that the optimal solution to the EOQ model is $Q^* = \sqrt{2K\lambda/h}$. We also know that the order interval T is given by $T = Q/\lambda$, so the optimal order interval is $T^* = \sqrt{2K/\lambda h}$. But what if T^* is some inconvenient number? How can we place an order, for example, every $\sqrt{10}$ weeks? In this section, we discuss *power-of-two policies*, in which the order interval is required to be a power-of-two multiple of some *base period*. The base period may be any time period—week, day, work shift, etc. If the base period is a day (say), then the power-of-two restriction says that orders can be placed every 1 day, or every 2 days, or every 4 days, or every 8 days, and so on, or every 1/2 day, or every 1/4 day, and so on. Policies based on a convenient base period such as days or months are more convenient to implement than those involving base periods like $\sqrt{10}$. We already know that the EOQ model is relatively insensitive to deviations from the optimal solution from Theorem 3.2. Our goal is to determine exactly how much more expensive a power-of-two policy is than the optimal policy.

Power-of-two policies have another advantage over the optimal EOQ policy: They make coordination easier at a central warehouse. If retailers each order according to their own EOQ policies, the warehouse will see a chaotic mess of order times. If, instead, each retailer follows a power-of-two policy with the same base period, the warehouse will see orders line up nicely, making its own inventory planning easier. The problem of finding optimal order intervals in this setting is one version of a problem known as the *one warehouse, multiretailer (OWMR) problem*. The optimal policy for the OWMR problem is not known, but it has been shown that power-of-two policies are very close to optimal (Roundy 1985, Muckstadt and Roundy 1993).

3.3.1 Analysis

The problem statement is exactly as in the EOQ model (see Section 3.2.1). In addition, we assume there is some base planning period T_B. The actual reorder interval chosen must be of the form

$$T = T_B 2^k \tag{3.10}$$

for some $k \in \{\ldots, -2, -1, 0, 1, 2, \ldots\}$. We need to determine (1) the best power-of-two policy, i.e., the best value of k, and (2) how far from optimal this policy is.

From the EOQ model, we know that the optimal order interval is

$$T^* = \sqrt{\frac{2K}{\lambda h}}. \tag{3.11}$$

Let $f(T)$ be the EOQ cost if an order interval of T is chosen, ignoring the per-unit cost; that is,

$$f(T) = \frac{K}{T} + \frac{h\lambda T}{2}. \tag{3.12}$$

(This follows from substituting $Q = T\lambda$ in the EOQ cost function (3.3).) One can easily verify that f is convex, so the optimal k in (3.10) is the smallest integer k satisfying

$$f(T_B 2^k) \leq f(T_B 2^{k+1}), \tag{3.13}$$

that is,

$$\begin{aligned}\frac{K}{T_B 2^k} + \frac{h\lambda}{2} T_B 2^k &\le \frac{K}{T_B 2^{k+1}} + \frac{h\lambda}{2} T_B 2^{k+1} \\ \iff \frac{K}{T_B 2^{k+1}} &\le \frac{h\lambda}{2} T_B 2^k \\ \iff \frac{K}{h\lambda} &\le (T_B 2^k)^2 \\ \iff \frac{1}{\sqrt{2}} T^* = \sqrt{\frac{K}{h\lambda}} &\le T_B 2^k. \end{aligned} \tag{3.14}$$

Therefore, the optimal power-of-two order interval is $\hat{T} = T_B 2^k$, where k is the smallest integer satisfying (3.14).

3.3.2 Error Bound

Theorem 3.3 *If $\hat{T}$ is the optimal power-of-two order interval and T^* is the optimal (not necessarily power-of-two) order interval, then*

$$\frac{f(\hat{T})}{f(T^*)} \le \frac{3}{2\sqrt{2}} \approx 1.06.$$

In other words, the cost of the optimal power-of-two policy is no more than 6% greater than the cost of the optimal (non-power-of-two) policy. This holds for any choice of the base period T_B.

Proof. Since k is the smallest integer satisfying (3.13), we have

$$\begin{aligned} & f(T_B 2^{k-1}) > f(T_B 2^k) \\ \iff & \frac{K}{T_B 2^k} > \frac{h\lambda}{2} T_B 2^{k-1} \\ \iff & \sqrt{\frac{4K}{h\lambda}} > T_B 2^k, \end{aligned}$$

or

$$\hat{T} < \sqrt{2} T^*. \tag{3.15}$$

Together, (3.14) and (3.15) imply that the optimal power-of-two order interval $\hat{T}$ must be in the interval $[\frac{1}{\sqrt{2}}T^*, \sqrt{2}T^*)$. Note that this is true for *any* base period T_B. Now, using (3.11) and (3.12),

$$\begin{aligned} f\left(\frac{1}{\sqrt{2}} T^*\right) &= \frac{\sqrt{2}K}{T^*} + \frac{h\lambda}{2} \frac{1}{\sqrt{2}} T^* \\ &= \frac{\sqrt{2}K}{\sqrt{\frac{2K}{\lambda h}}} + \frac{h\lambda}{2} \frac{1}{\sqrt{2}} \sqrt{\frac{2K}{\lambda h}} \end{aligned}$$

$$= \frac{3}{2\sqrt{2}}\sqrt{2K\lambda h}$$
$$= \frac{3}{2\sqrt{2}}f(T^*).$$

Similarly,

$$\begin{aligned} f(\sqrt{2}T^*) &= \frac{K}{\sqrt{2}T^*} + \frac{h\lambda}{2}\sqrt{2}T^* \\ &= \frac{1}{\sqrt{2}}\sqrt{\frac{K\lambda h}{2}} + \frac{\sqrt{2}}{2}\sqrt{2K\lambda h} \\ &= \frac{3}{2\sqrt{2}}\sqrt{2K\lambda h} \\ &= \frac{3}{2\sqrt{2}}f(T^*). \end{aligned}$$

Since f is convex and the optimal $\hat{T}$ lies somewhere between $\frac{1}{\sqrt{2}}T^*$ and $\sqrt{2}T^*$,

$$\frac{f(\hat{T})}{f(T^*)} \le \frac{3}{2\sqrt{2}} \approx 1.06.$$

■

Since we don't know precisely where $\hat{T}$ falls in the range $[\frac{1}{\sqrt{2}}T^*, \sqrt{2}T^*)$, this is only a worst-case bound that occurs on the endpoints of the range. If $\hat{T}$ falls somewhere in the middle of the range, the power-of-two policy may be even better than 6% above optimal. In fact, if we assume that $\hat{T}$ is uniformly distributed in the range, we get an expected bound of only 2%:

Theorem 3.4 *Assuming that the optimal power-of-two order interval $\hat{T}$ is uniformly distributed in the range $[\frac{1}{\sqrt{2}}T^*, \sqrt{2}T^*)$,*

$$\frac{\mathbb{E}[f(\hat{T})]}{f(T^*)} \le \frac{1}{\sqrt{2}}\left(\ln 2 + \frac{3}{4}\right) \approx 1.02. \tag{3.16}$$

Proof. Omitted. ■

☐ **EXAMPLE 3.4**

Suppose Joe (owner of Joe's Corner Store, from Example 3.1) must order candy bars in power-of-two multiples of 1 month. What is the optimal power-of-two order interval, and what is the cost ratio versus the optimal (non-power-of-two) solution?

We have $T_B = 1/12$ years. You can confirm that

$$\begin{aligned} f\left(T_B 2^0\right) &= f(0.0833) = 108.19 \\ f\left(T_B 2^1\right) &= f(0.1667) = 72.38 \\ f\left(T_B 2^2\right) &= f(0.3333) = 72.75 \end{aligned}$$

By the convexity arguments above, the optimal power-of-two order interval is $\hat{T} = 0.1667$ years, or every 2 months. The cost ratio is $72.38/68.41 = 1.0580$, within the bound of 1.06. □

3.4 THE EOQ WITH QUANTITY DISCOUNTS

It is common for suppliers to offer discounts based on the quantity ordered. The larger the order, the lower the purchase cost per item. (You may have observed something similar when you shop for groceries. When you buy in bulk, you pay less per unit.) The specific structure for the discounts can take many forms, but two types are most common: *all-units discounts* and *incremental discounts*. Both discount structures use *breakpoints* to determine the purchase price. For example, the supplier may charge \$1 per unit if the firm orders 0–100 units, \$0.90 per unit if the firm orders 100–250 units, and \$0.85 per unit if the firm orders more than 250 units. The two discount structures differ based on how the total purchase cost is determined.

We assume there are n *breakpoints*, denoted $b_1, \ldots, b_n$. For convenience, we also define $b_0 \equiv 0$ and $b_{n+1} \equiv \infty$. The interval $[b_j, b_{j+1})$ is called the *region* for breakpoint j, or simply region j for short. Each region j, $j = 0, \ldots, n$, is associated with a purchase price c_j. The costs are decreasing in j: $c_0 > c_1 > \cdots > c_n$. The total purchase cost, denoted $c(Q)$, is calculated in each of the discount structures as follows:

- *All-units discounts.* All units in the order incur the price determined by the breakpoint. That is, if $Q \in [b_j, b_{j+1})$, then the total purchase cost is $c(Q) = c_j Q$.

- *Incremental discounts.* The units in each region incur the purchase price for that region. That is, if $Q \in [b_j, b_{j+1})$, then the total purchase cost is

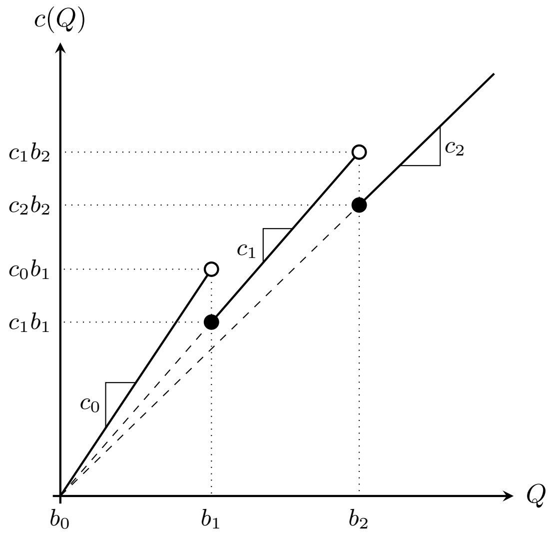
$$c(Q) = \sum_{i=0}^{j-1} c_i(b_{i+1} - b_i) + c_j(Q - b_j). \tag{3.17}$$

(Note that $c(Q)$ does not include the fixed ordering cost.) Figure 3.4 plots $c(Q)$ as a function of Q for both all-units and incremental discounts.

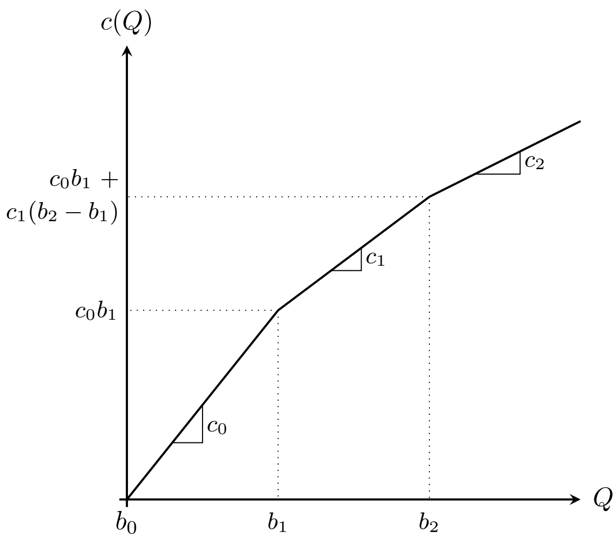
□ **EXAMPLE 3.5**

Suppose that Joe's candy supplier (from Example 3.1) charges \$0.75 per candy bar if Joe orders 0–400 candy bars, \$0.72 each for 401–800, and \$0.68 each for 800 or more. That is, $b_1 = 400$, $b_2 = 800$, $c_0 = 0.75$, $c_1 = 0.72$, and $c_2 = 0.68$. Figures 3.5(a) and 3.5(b) depict the total purchase cost, $c(Q)$, for the all-units and incremental discount structures, respectively. □

We will formulate models to determine the optimal order quantity under both discount structures. In both cases, the approach will amount to solving multiple EOQ problems, one for each region, and using their solutions to determine the solution to the original problem.

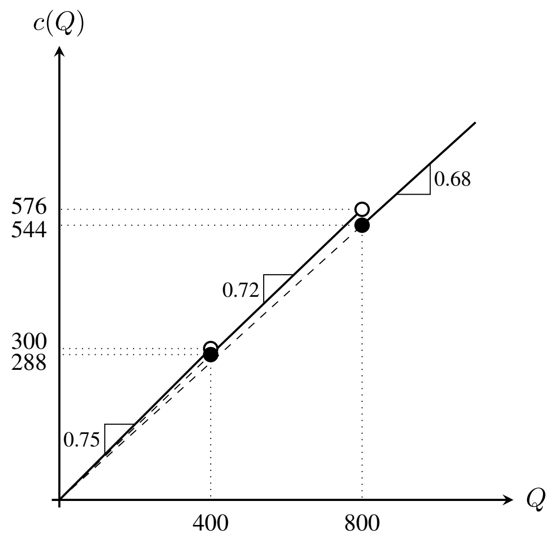


(a) All-units discounts.

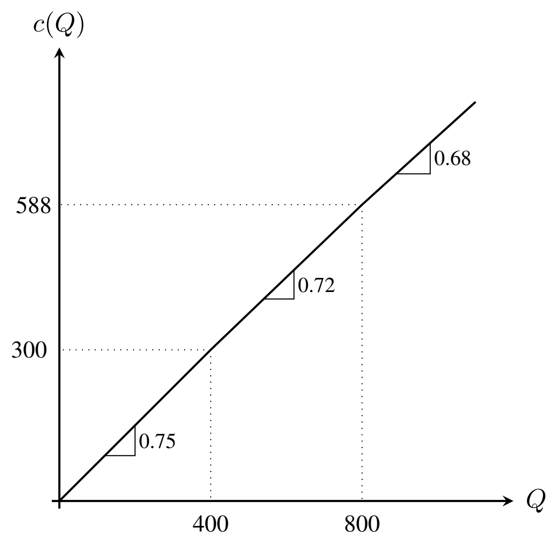


(b) Incremental discounts.

Figure 3.4 Total purchase cost $c(Q)$ under quantity discounts.



(a) All-units discounts.



(b) Incremental discounts.

Figure 3.5 Total purchase cost $c(Q)$ for Example 3.5.

3.4.1 All-Units Discounts

We can no longer ignore the purchase cost as we did in (3.3). In fact, not only do we need to include the purchase cost itself, but we must also account for the fact that the holding cost typically depends on the purchase cost, as discussed in Section 3.1.3. Let i be the annual holding cost rate expressed as a percentage of the purchase cost. That is, if $i = 0.25$ and $c = 100$, then $h = 25$ per year.

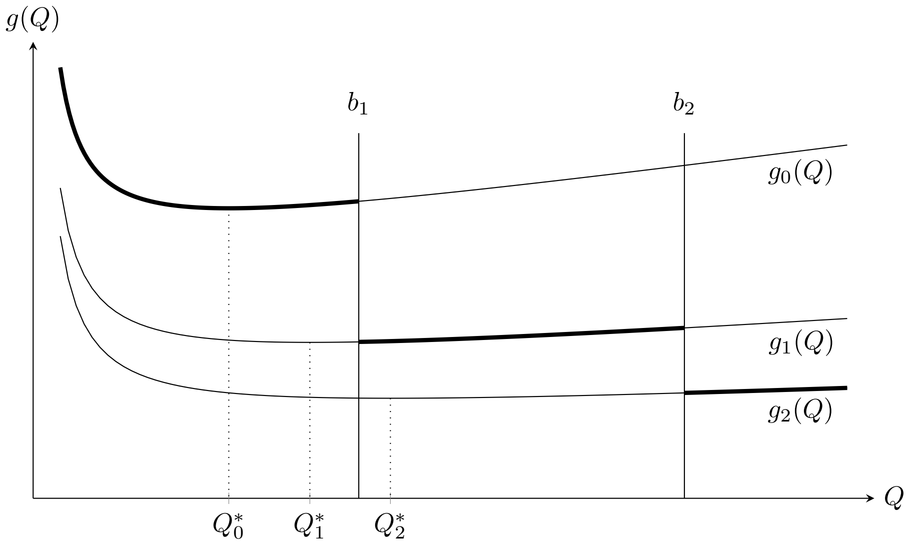


Figure 3.6 Total cost curves for all-units quantity discount structure.

Suppose we knew that the optimal order quantity lies in region j. Then we would simply need to find the Q that minimizes the EOQ cost function for region j:

$$g_j(Q) = c_j\lambda + \frac{K\lambda}{Q} + \frac{ic_jQ}{2}. \tag{3.18}$$

As j increases, c_j decreases, $g_j(Q)$ shifts down and becomes flatter, and its minimum point moves to the right; see Figure 3.6. The heavy segments of the cost curves identify the "active" cost function in each region. Our objective is to minimize $g(Q)$, the discontinuous function defined by the heavy segments.

The function $g_j(Q)$ has the same structure as $g(Q)$ in (3.3) except for the additional constant. Therefore, its minimizer is given by

$$Q_j^* = \sqrt{\frac{2K\lambda}{ic_j}}. \tag{3.19}$$

Of course, if Q_j^* falls outside of region j, then if the firm orders Q_j^*, it will incur a cost other than $g_j(Q_j^*)$. Q_j^* is meaningless in this case. We say that Q_j^* is *realizable* if it lies in region j. In Figure 3.6, only Q_0^* is realizable. Does this mean that Q_0^* is necessarily the optimal solution? No: The breakpoints to the right of Q_0^* are also candidates. The optimal order quantity always equals either the largest realizable Q_j^* or one of the breakpoints to its right. (Why?)

Therefore, we can determine Q^* as follows. First, we calculate Q_j^* for each j. Let Q_i^* be the largest realizable Q_j^*, and $g_i(Q_i^*)$ its cost. We then evaluate $g_j(b_j)$ for each b_j greater than Q_i^*. Finally, we set Q^* to the quantity with the lowest cost (Q_i^* if $g_i(Q_i^*)$ is the lowest cost, and b_j if $g_j(b_j)$ is the lowest cost for some j).

Since Q_j^* increases as j increases, if we start in region n when we calculate Q_j^* and work backward, we can stop as soon as we find one realizable Q_j^*; this is necessarily the largest realizable Q_j^*.

☐ **EXAMPLE 3.6**

Recall from Example 3.1 that $\lambda = 1300$, $K = 8$, and $i = 0.3$. If candy purchases follow the quantity discount structure in Example 3.5, what is Joe's optimal order quantity?

We first determine the largest realizable Q_j^* by working backward from segment 2:

$$Q_2^* = \sqrt{\frac{2 \cdot 8 \cdot 1300}{0.3 \cdot 0.68}} = 319.3$$
$$Q_1^* = \sqrt{\frac{2 \cdot 8 \cdot 1300}{0.3 \cdot 0.72}} = 310.3$$
$$Q_0^* = \sqrt{\frac{2 \cdot 8 \cdot 1300}{0.3 \cdot 0.75}} = 304.1$$

Only Q_0^* is realizable, and it has cost

$$0.75 \cdot 1300 + \sqrt{2 \cdot 8 \cdot 1300 \cdot 0.3 \cdot 0.75} = 1043.4.$$

Next, we calculate the cost of the breakpoints to the right of Q_0^*:

$$g_1(400) = 0.72 \cdot 1300 + \frac{8 \cdot 1300}{400} + \frac{0.3 \cdot 0.72 \cdot 400}{2} = 1005.2$$
$$g_2(800) = 0.68 \cdot 1300 + \frac{8 \cdot 1300}{800} + \frac{0.3 \cdot 0.68 \cdot 800}{2} = 978.6$$

Therefore, the optimal order quantity is $Q = 800$, which incurs a purchase cost of \$0.68 and a total annual cost of \$978.60. ☐

3.4.2 Incremental Discounts

We now turn our attention to incremental discounts. The total cost function for region j is given by

$$g_j(Q) = \frac{c(Q)}{Q}\lambda + \frac{K\lambda}{Q} + \frac{i\frac{c(Q)}{Q}Q}{2},$$

where $c(Q)$ is given by (3.17). Note that the purchase cost term is no longer a constant with respect to Q, even within a given segment: As Q increases, so does the number of "cheap" units, and the average cost per unit decreases.

We can rewrite $g_j(Q)$ as

$$\begin{aligned} g_j(Q) =& \frac{1}{Q}\left[\sum_{i=0}^{j-1} c_i(b_{i+1} - b_i) - c_j b_j\right]\lambda + c_j\lambda + \frac{K\lambda}{Q} \\ &+ \frac{i}{2}\left[\sum_{i=0}^{j-1} c_i(b_{i+1} - b_i) - c_j b_j\right] + \frac{ic_jQ}{2} \\ =& c_j\lambda + \frac{i\bar{c}_j}{2} + \frac{(K + \bar{c}_j)\lambda}{Q} + \frac{ic_jQ}{2}, \end{aligned} \tag{3.20}$$

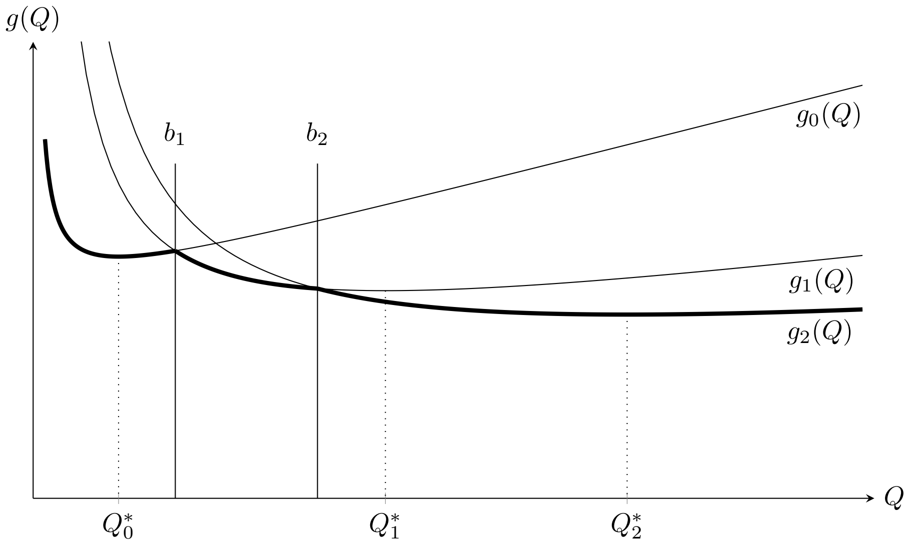


Figure 3.7 Total cost curves for incremental quantity discount structure.

where

$$\bar{c}_j = \sum_{i=0}^{j-1} c_i(b_{i+1} - b_i) - c_j b_j.$$

The right-hand side of (3.20) is structurally identical to the EOQ cost function; therefore, its minimizer is given by

$$Q_j^* = \sqrt{\frac{2(K + \bar{c}_j)\lambda}{ic_j}} \tag{3.21}$$

with cost

$$g_j(Q_j^*) = c_j\lambda + \frac{i\bar{c}_j}{2} + \sqrt{2(K + \bar{c}_j)\lambda i c_j}. \tag{3.22}$$

Figure 3.7 plots $g_j(Q)$ for a two-breakpoint problem. As a rule, $g_j(Q)$ is always the lowest curve in region j because the functions are convex and are equal at the breakpoints. On the other hand, Q_j^* is not always realizable. (In the figure, Q_1^* is not realizable.) Our objective is to minimize $g(Q)$, the continuous, piecewise function defined by the heavy segments.

If Q_j^* is not realizable, then clearly it cannot be optimal for $g(Q)$, and moreover, its breakpoints cannot be optimal either. (Why?) Therefore, the optimal order quantity is equal to the realizable Q_j^* that has the lowest cost.

□ **EXAMPLE 3.7**

Return to Example 3.6 and suppose now that Joe faces an incremental quantity discount structure with the same breakpoints and purchase costs. What is Joe's optimal order quantity?

We first determine $\bar{c}_j$ for each j:

$$\bar{c}_0 = 0$$

$$\bar{c}_1 = 0.75 \cdot 400 - 0.72 \cdot 400 = 12$$
$$\bar{c}_2 = 0.75 \cdot 400 + 0.72 \cdot 400 - 0.68 \cdot 800 = 44$$

Next, we calculate Q_j^* for each j:

$$Q_0^* = \sqrt{\frac{2(8+0)1300}{0.3 \cdot 0.75}} = 304.1$$
$$Q_1^* = \sqrt{\frac{2(8+12)1300}{0.3 \cdot 0.72}} = 490.7$$
$$Q_2^* = \sqrt{\frac{2(8+44)1300}{0.3 \cdot 0.68}} = 814.1$$

All three solutions are realizable. Using (3.22), these solutions have the following costs:

$$g_0(Q_0^*) = 0.75 \cdot 1300 + \frac{0.3 \cdot 0}{2} + \sqrt{2(8+0)1300 \cdot 0.3 \cdot 0.75} = 1043.4$$
$$g_1(Q_1^*) = 0.72 \cdot 1300 + \frac{0.3 \cdot 12}{2} + \sqrt{2(8+12)1300 \cdot 0.3 \cdot 0.72} = 1043.8$$
$$g_2(Q_2^*) = 0.68 \cdot 1300 + \frac{0.3 \cdot 44}{2} + \sqrt{2(8+44)1300 \cdot 0.3 \cdot 0.68} = 1056.7$$

Therefore, the optimal order quantity is $Q = 304.1$, which incurs a total annual cost of \$1043.40. □

3.4.3 Modified All-Units Discounts

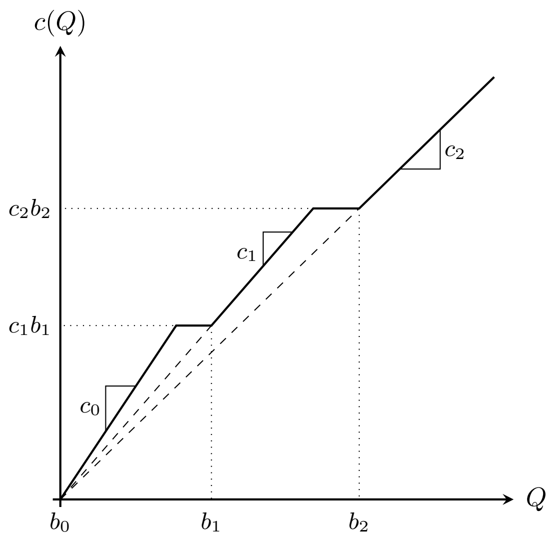
All-units discounts are somewhat problematic because, for order quantities Q just to the left of breakpoint j, it is cheaper to order b_j than to order Q, even though $Q < b_j$. For example, under the cost structure in Example 3.5, it costs \$292.50 to purchase 390 units but \$288.00 to purchase 400 units. (See Figure 3.5(a).)

In practice, suppliers usually allow the buying firm to pay the lower price—\$288.00 in the example above—for order quantities that fall into this awkward zone. This is especially true for transportation costs, since all-units discounts are common in shipping, with the cost determined based on the weight shipped. If a shipment totals, say, 390 kg but it is cheaper to ship 400 kg, the firm could add 10 kg worth of bricks to the shipment, but a solution that is preferable for both the shipper and the transportation company is for the firm to "ship x, declare y"—for example, ship 390 kg, declare 400 kg.

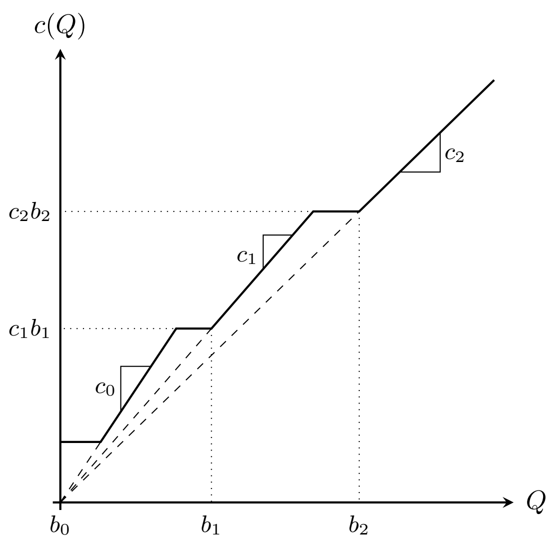
This structure is sometimes known as the *modified all-units discount structure*. Its $c(Q)$ curve is displayed in Figure 3.8(a). The flat portions of the curve represent the regions in which the firm orders or ships one quantity but declares a greater quantity.

Sometimes, there is also a minimum charge for each order or shipment, in which case there is an additional horizontal segment at the start of the $c(Q)$ curve; see Figure 3.8(b).

A special case of the modified all-units discount structure is the *carload discount structure*, in which the b_j are equally spaced and c_j is the same for all j. This structure arises from rail or truck carload shipments, in which the transportation company charges a per-unit cost c for each unit shipped, up to some maximum cost for each car. Once the capacity of a car is exceeded, a new car begins, at a cost of c per unit, and so on.



(a) Without minimum cost.



(b) With minimum cost.

Figure 3.8 Total purchase cost $c(Q)$ for modified all-units discounts structure.

Unfortunately, modified all-units discount structures are much more difficult to analyze than the discount structures discussed above. (See, for example, Chan et al. (2002).) We omit further discussion here.

3.5 THE EOQ WITH PLANNED BACKORDERS

We assumed in Section 3.2.1 that backorders are not allowed. In this section, we discuss a variant of the EOQ problem in which backorders are allowed. Since demand is determin-

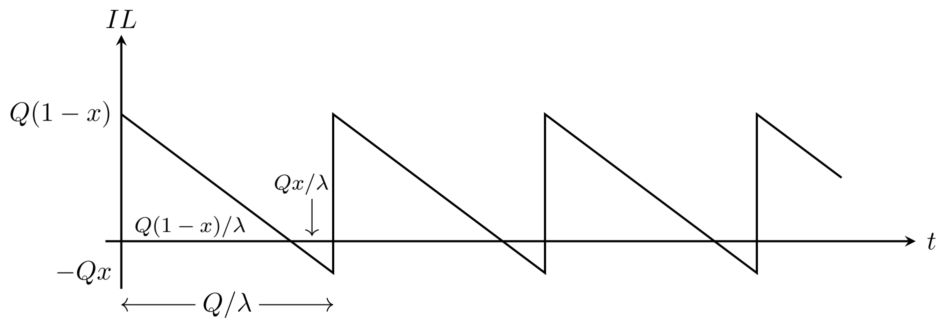


Figure 3.9 EOQB inventory curve.

istic, we have the same number of backorders in every order cycle—they are "planned" backorders. (See Figure 3.9.) We'll call this model the *EOQ with backorders* (EOQB).

Let p be the backorder penalty per item per year, and let x be the fraction of demand that is backordered. Both Q and x are decision variables. The holding cost is charged based on on-hand inventory; the average on-hand inventory is given by

$$\frac{Q(1-x)^2}{2}.$$

Similarly, the backorder cost is charged based on the number of backorders; the average backorder level is given by

$$\frac{Qx^2}{2}.$$

(Compute the area under the triangle, then divide by the length of an order cycle.) Finally, the number of orders per year is given by λ/Q, just like in the EOQ model.

Therefore, the total average cost per year in the EOQB is given by

$$g(Q,x) = \frac{hQ(1-x)^2}{2} + \frac{pQx^2}{2} + \frac{K\lambda}{Q}. \tag{3.23}$$

Note that g is a function of both Q and x. Therefore, to minimize it, we need to take partial derivatives with respect to both variables and set them equal to 0.

$$\frac{\partial g}{\partial x} = -hQ(1-x) + pQx = 0 \tag{3.24}$$

$$\frac{\partial g}{\partial Q} = \frac{h(1-x)^2}{2} + \frac{px^2}{2} - \frac{K\lambda}{Q^2} = 0 \tag{3.25}$$

Let's first look at (3.24):

$$\begin{aligned} -hQ(1-x) + pQx &= 0 \\ \iff h(1-x) &= px \\ \iff x^* &= \frac{h}{h+p} \end{aligned} \tag{3.26}$$

Interestingly, x^* does not depend on Q; even if we choose a suboptimal Q, the optimal x to choose is still $h/(h+p)$. At this point, we could substitute $h/(h+p)$ for x in (3.25) and solve for Q, but instead we'll plug x^* into $g(Q,x)$:

$$\begin{aligned} g(Q,x^*) &= \frac{hQ}{2}\left(\frac{p}{h+p}\right)^2 + \frac{pQ}{2}\left(\frac{h}{h+p}\right)^2 + \frac{K\lambda}{Q} \\ &= \frac{Q}{2}\left(\frac{p^2h+h^2p}{(h+p)^2}\right) + \frac{K\lambda}{Q} \\ &= \frac{hp}{h+p}\frac{Q}{2} + \frac{K\lambda}{Q} \end{aligned}$$

This is exactly the same form as the EOQ cost function (3.3) with the holding cost h replaced by $hp/(h+p)$. In other words, the EOQB cost function (assuming x is set optimally) is equivalent to the EOQ cost function with the holding cost h scaled by $p/(h+p)$. Therefore we can use (3.6) and (3.7) to obtain the optimal Q and the optimal cost for the EOQB, as stated in the next theorem.

Theorem 3.5 *In the EOQ model with backorders, the optimal solution and cost are given by*

$$Q^* = \sqrt{\frac{2K\lambda(h+p)}{hp}} \tag{3.27}$$

$$x^* = \frac{h}{h+p} \tag{3.28}$$

$$g(Q^*,x^*) = \sqrt{\frac{2K\lambda hp}{h+p}} \tag{3.29}$$

How do the optimal solution and cost in Theorem 3.5 compare to the analogous quantities from the EOQ model? First, comparing (3.29) and (3.7), we can see that the optimal cost is smaller in the EOQB than in the EOQ. This makes sense, since the EOQ is a special case of the EOQB in which the constraint $x = 0$ has been added. From (3.27), we can see that the optimal order quantity is greater in the EOQB than in the EOQ. This is because placing larger orders in the EOQB does not require us to carry quite as much inventory as it does in the EOQ, and therefore, the extra flexibility offered by the backorder option allows us to place larger orders.

As $p \to \infty$, Q^* approaches the optimal EOQ order quantity, x^* approaches 0, and the optimal cost approaches the EOQ optimal cost.

Note also that x is strictly greater than 0, provided that h is. Therefore, it is *always* optimal to allow some backorders. To see why, suppose we set $x = 0$—then the EOQB inventory curve in Figure 3.9 collapses to the EOQ curve in Figure 3.2. Now, if we increase x slightly, we create a tiny negative triangle at the end of each cycle in Figure 3.9, incurring a tiny backorder cost. (See Figure 3.10.) But we also reduce the height of the positive part of the inventory curve throughout the rest of the cycle, resulting in a substantial savings in holding cost. As we continue to increase the number of backorders, the marginal savings in holding cost decreases and the marginal increase in backorder cost increases. At some point, the marginal cost of adding a backorder will outweigh the marginal savings in holding cost, so we will have an x^* somewhere between 0 and 1.

What if we consider the same model but assume that unmet demands are lost, rather than backordered? It turns out that in this case, it is optimal either to meet every demand ($x = 0$) or to meet no demands ($x = 1$)—see Problem 3.16.

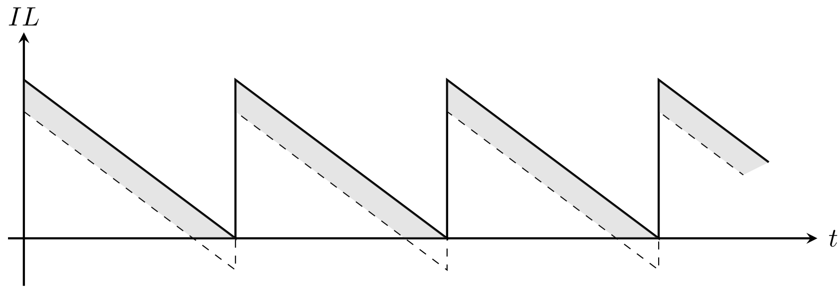


Figure 3.10 Inventory–backorder trade-off in EOQB.

□ **EXAMPLE 3.8**

Recall Example 3.1. Suppose Joe is willing to stock out occasionally and estimates that each backorder costs the store \$5 in lost profit and loss of good will. What is the optimal order quantity, the optimal fill rate (fraction of demand met from stock), and the optimal cost?

$$
\begin{aligned}
Q^* &= \sqrt{\frac{2K\lambda(h+p)}{hp}} = \sqrt{\frac{2 \cdot 8 \cdot 1300(0.225+5)}{0.225 \cdot 5}} = 310.81 \\
x^* &= \frac{h}{h+p} = 0.0431 \\
g(Q^*, x^*) &= \sqrt{\frac{2K\lambda hp}{h+p}} = \sqrt{\frac{2 \cdot 8 \cdot 1300 \cdot 0.225 \cdot 5}{0.225+5}} = 66.92
\end{aligned}
$$

The fill rate is $1 - x^* = 0.9569$. The cost has decreased by 2.2% versus the cost without backorders. □

3.6 THE ECONOMIC PRODUCTION QUANTITY MODEL

In a manufacturing environment, the amount of time required to produce a batch of items usually depends on how large the batch is—producing more items requires more time. The EOQ model cannot handle this feature, since it assumes that orders are received after a deterministic (possibly zero) lead time, regardless of the order quantity. In other words, the EOQ assumes that the production rate is infinite—an arbitrary number of items can be produced in a fixed amount of time. This assumption may be reasonable in settings in which the firm is placing orders to an outside supplier that holds finished goods in inventory, or whose capacity is much larger than the firm's order quantity, so that the production time is negligible. In this section, we discuss a variant of the EOQ model that allows the production rate to be finite and is therefore more applicable to manufacturing settings. It is known as the *economic production quantity* (EPQ) model. The EPQ was introduced by Taft (1918, as cited by Erlenkotter (1990)). It is sometimes known as the economic production lot (EPL) problem.

Let μ be the production rate, i.e., the firm can produce μ items per year. We assume $\mu > \lambda$ (otherwise the manufacturing process cannot keep up with the demand). The

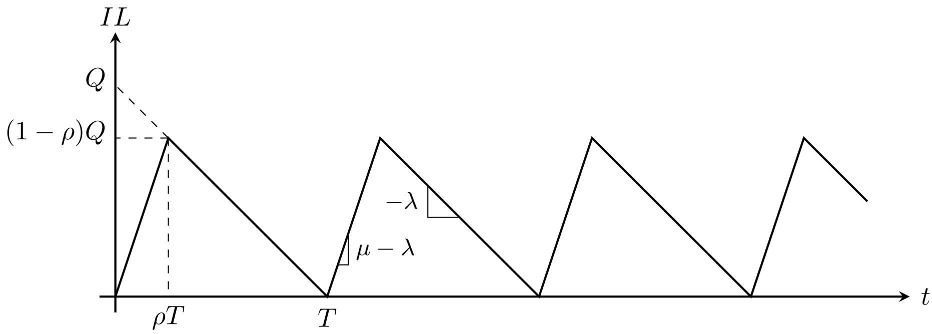


Figure 3.11 EPQ inventory level curve.

manufacturing process is active during a portion of the time (called *active intervals*) and is idle otherwise, and during active intervals, the process adds finished goods to inventory at a rate of μ. Meanwhile, the demand process is ongoing, reducing the inventory at a rate of λ. Let $\rho = \lambda/\mu$ be the *utilization ratio*, which indicates the portion of time the system is active. Q is now interpreted as a production batch size rather than an order quantity.

The process is depicted in Figure 3.11. Note that during active intervals, the inventory increases at a rate $\mu - \lambda$ since items are being added to inventory by the manufacturing process and withdrawn from it by the demand process simultaneously. Since we still initiate the replenishment process after exactly Q items have been demanded, the order interval T still equals Q/λ years. Moreover, since we produce exactly Q units in an active interval, the active interval must last $Q/\mu = \rho T$ years. This means that the maximum inventory level, which occurs ρT years into each cycle, is $\rho T(\mu - \lambda) = (1-\rho)Q$.

The fixed cost per year is still $K\lambda/Q$, as in the EOQ model, since $T = Q/\lambda$. The average inventory level is $(1-\rho)Q/2$, so the average annual holding cost is $h(1-\rho)Q/2$. Therefore, the total annual cost is

$$g(Q) = \frac{K\lambda}{Q} + \frac{h(1-\rho)Q}{2}. \tag{3.30}$$

We could find the Q that minimizes this cost function by differentiating, as we did for the EOQ, but it is simpler to recognize that (3.30) differs from (3.3) only by the constant $(1-\rho)$ in the second term. In other words, the EPQ is equivalent to the EOQ with the holding cost parameter h scaled by $1-\rho$. Therefore, the optimal solution to the EPQ, and its cost, are as given in the next theorem.

Theorem 3.6 *In the EPQ model, the optimal solution and cost are given by*

$$Q^* = \sqrt{\frac{2K\lambda}{h(1-\rho)}} \tag{3.31}$$

$$g(Q^*) = \sqrt{2K\lambda h(1-\rho)}. \tag{3.32}$$

Proof. Follows from replacing h with $h(1-\rho)$ in Theorem 3.1. ■

Since $\rho < 1$, the optimal EPQ solution is larger than that of the EOQ, while the optimal EOQ cost is smaller. Both results are justified by the fact that items arrive later after the replenishment order in the EPQ than they do in the EOQ, and therefore, the holding cost for a given Q is smaller. Note also that as $\mu \to \infty$, the EPQ reduces to the EOQ.

3.7 PERIODIC REVIEW: THE WAGNER–WHITIN MODEL

3.7.1 Problem Statement

We now shift our attention to a periodic-review model known as the *Wagner–Whitin model* (Wagner and Whitin 1958). Similar to the EOQ model, the Wagner–Whitin model assumes that the demand is deterministic, there is a fixed cost to place an order, and stockouts are not allowed. The objective is to choose order quantities to minimize the total cost. However, unlike the EOQ model, the Wagner–Whitin model allows the demand to change over time—to be different in each period. This model is sometimes referred to as the dynamic economic lot-sizing (DEL) model or the uncapacitated lot-sizing (ULS) model.

Because of the fixed cost, it may not be optimal to place an order in every time period. However, we will show that, as in the EOQ, optimal solutions have the zero-inventory ordering (ZIO) property. Therefore, the problem boils down to deciding how many whole periods' worth of demand to order at once.

Unlike the infinite-horizon EOQ model, the Wagner–Whitin model considers a finite horizon, consisting of T periods. In each period, we must decide whether to place a replenishment order, and if so, how large an order to place. The demand in period t is given by d_t, and stockouts are not allowed. The lead time is 0. As in the EOQ model, there is a fixed cost K per order and an inventory holding cost h per unit per period. (Note that h represents the holding cost per year in the EOQ model but per period here.) One could also include a purchase cost c, but since the total number of units ordered throughout the horizon is constant (independent of the ordering pattern), it is safe to ignore this cost.

Assume that the on-hand inventory is 0 at time 0. In each time period, the following events occur, in the following order:

1. The replenishment order, if any, is placed and is received instantly.

2. Demand occurs and is satisfied from inventory.

3. Holding costs are assessed based on the on-hand inventory.

(This type of timeline is known as a *sequence of events*. It is important to specify the sequence of events clearly in periodic-review models. For example, the holding costs would be very different if events 2 and 3 were reversed.)

We first formulate this model as a mixed-integer optimization problem (MIP). We will then discuss a dynamic programming (DP) algorithm for solving it.

3.7.2 MIP Formulation

Our formulation will use the following decision variables:

q_t = the number of units ordered in period t
y_t = 1 if we order in period t, 0 otherwise
x_t = the inventory level at the end of period t

We also define $x_0 \equiv 0$. Then the Wagner–Whitin model can be formulated as follows:

$$\text{minimize} \quad \sum_{t=1}^{T} (Ky_t + hx_t) \tag{3.33}$$

$$\text{subject to} \quad x_t = x_{t-1} + q_t - d_t \quad \forall t = 1, \ldots, T \tag{3.34}$$

$$q_t \leq My_t \quad \forall t = 1, \ldots, T \tag{3.35}$$

$$x_t \geq 0 \quad \forall t = 1, \ldots, T \tag{3.36}$$

$$q_t \geq 0 \quad \forall t = 1, \ldots, T \tag{3.37}$$

$$y_t \in \{0, 1\} \quad \forall t = 1, \ldots, T \tag{3.38}$$

The objective function (3.33) calculates the fixed cost (for each period in which we place an order) plus the cost of holding inventory at the end of each period. Constraints (3.34) are the *inventory-balance constraints*: They say that the ending inventory in period t is equal to the starting inventory, plus the new units ordered, minus the demand. Constraints (3.35) prohibit q_t from being positive unless y_t is 1. Here, M is a large number; it could be set to $\sum_{s=t}^{T} d_s$, for example. Constraints (3.36)–(3.37) are nonnegativity constraints. In particular, (3.36) also prohibits stockouts by requiring every period to end with nonnegative inventory. Finally, constraints (3.38) are integrality constraints on the y variables.

This problem can be interpreted as a simple supply chain network design problem (to be more precise, an arc design problem; see Section 8.7.2). It can be solved as an MIP, but it is more common to solve it using DP or as a shortest path problem, as we discuss in the next section. See Pochet and Wolsey (1995, 2006) for thorough discussions of mathematical programming formulations for this and other lot-sizing models. See also Case Study 3.1 for an alternate formulation approach for a similar problem.

3.7.3 Dynamic Programming Algorithm

The DP algorithm depends on the following result:

Theorem 3.7 *Every optimal solution to the Wagner–Whitin model has the ZIO property; that is, it is optimal to place orders only in time periods in which the initial inventory is zero.*

Proof. Suppose (for a contradiction) there is an optimal solution in which an order is placed in period t even though the inventory level at the beginning of period t is positive; i.e., $x_{t-1} > 0$. The x_{t-1} units in inventory were ordered in a period before t and incurred a holding cost to be held from period $t-1$ to t. If these items had instead been ordered in period t, then (1) the holding cost would decrease since fewer units are held in inventory, and (2) the fixed cost would stay the same since the number of orders would not change, only the size of each order. This contradicts the assumption that the original policy is optimal; hence, every optimal solution must have the ZIO property. ■

Theorem 3.7 and its proof assume that $h > 0$; if h may equal 0, then the theorem would read "There exists an optimal solution..."

As a corollary to Theorem 3.7, each order is of a size equal to the total demand in an integer number of subsequent periods; that is, in period t we either order d_t, or $d_t + d_{t+1}$, or $d_t + d_{t+1} + d_{t+2}$, and so on. The problem then boils down to deciding in which periods to order. We formulate this problem as a DP.

Let θ_t be the optimal cost in periods $t, t+1, \ldots, T$ if we place an order in period t (and act optimally thereafter). We can define θ_t recursively in terms of θ_s for later periods s. First define $\theta_{T+1} \equiv 0$. Then

$$\theta_t = \min_{t < s \leq T+1} \left\{ K + h \sum_{i=t}^{s-1} (i-t) d_i + \theta_s \right\}. \tag{3.39}$$

The minimization determines the next period s in which we will place an order, assuming that we order in period t. (Setting $s = T+1$ means we never order again; the order in period t is the last order.) A given choice of s is evaluated using the expression inside the braces. The first two terms calculate the cost incurred in periods t through $s-1$: the order cost of K, plus the holding cost for the items that will be held until future periods. (The d_t units demanded in period t will be held for 0 periods; d_{t+1} units will be held for 1 period; $\ldots$; and d_{s-1} units will be held for $s-1-t$ periods.) A new order will be placed in period s, and θ_s includes the cost in period s and all future periods.

The DP algorithm for the Wagner–Whitin problem is summarized in Algorithm 3.1. At the conclusion of the algorithm, θ_1 equals the cost of the optimal solution. The optimal solution itself is obtained by "backtracking"—we place orders in period 1, period $s(1)$, period $s(s(1))$, and so on.

Algorithm 3.1 Wagner–Whitin algorithm

1: $\theta_{T+1} \leftarrow 0$ ▷ Initialization
2: **for** $t = T, \ldots, 1$ **do** ▷ Main loop
3: $\quad \theta_t \leftarrow$ right-hand side of (3.39) ▷ Minimization over s
4: $\quad s(t) \leftarrow \text{argmin}$ in right-hand side of (3.39)
5: **end for**
6: **return** θ_t, $s(t)$ for all $t = 1, \ldots, T$

The complexity of the algorithm is $O(T^2)$ since step 2 requires $O(T)$ operations and must be performed $O(T)$ times. Faster algorithms, which run in $O(T)$ time, have been developed for this problem but will not be discussed here (Federgruen and Tzur 1991, Wagelmans et al. 1992). Despite the efficiency of this algorithm, a number of heuristics have been introduced and are still popular in practice. These include Silver–Meal, part period balancing, least unit cost, and other heuristics (Silver et al. 1998). One explanation for the persistent use of these approximate methods is that they tend to be less sensitive to changes in the data, so that as demand forecasts change for several periods into the future, the current production plan doesn't change much.

The Wagner–Whitin model can equivalently be represented by a network with $T+1$ nodes in which each node represents a time period and an arc from period t to period s represents ordering in period t to satisfy the demands of periods $t, t+1, \ldots, s-1$. The cost of this arc is

$$K + h \sum_{i=t}^{s-1} (i-t) d_i. \tag{3.40}$$

Solving the Wagner–Whitin problem is equivalent to finding a shortest path through this network (which is, in turn, equivalent to solving the DP given above). Figure 3.12 depicts the network for a 4-period problem. Note that there is one extra node, node 5, called the

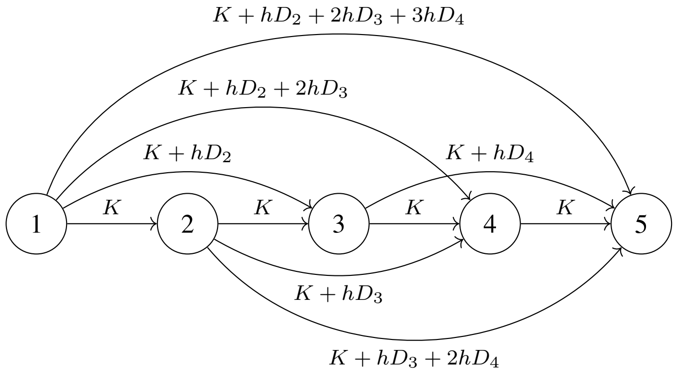


Figure 3.12 Wagner–Whitin network.

"dummy node," that serves as a sink for arcs representing ordering from the current time period until the end of the horizon.

☐ **EXAMPLE 3.9**

A garden center sells bags of organic compost for vegetable gardens. Compost is heavy, and special trucks must be used to transport it, so shipping is expensive; each order therefore incurs a fixed cost of \$500. The holding cost for each cubic meter of compost is \$2 per period. We consider a 4-period planning horizon. The demand for compost in periods 1–4 is 90, 120, 80, and 70 cubic meters, respectively. Find the optimal order quantity in each period and the total cost.

From (3.39), we have the following:

$$
\begin{aligned}
\theta_5 =& 0\\
\theta_4 =& K + h(0 \cdot d_4) + \theta_5\\
=& 500 \quad [s(4) = 5]\\
\theta_3 =& \min\{K + h(0 \cdot d_3) + \theta_4, K + h(0 \cdot d_3 + 1 \cdot d_4) + \theta_5\}\\
=& \min\{1000, 640\}\\
=& 640 \quad [s(3) = 5]\\
\theta_2 =& \min\{K + h(0 \cdot d_2) + \theta_3, K + h(0 \cdot d_2 + 1 \cdot d_3) + \theta_4,\\
& \quad K + h(0 \cdot d_2 + 1 \cdot d_3 + 2 \cdot d_4) + \theta_5\}\\
=& \min\{1140, 1160, 940\}\\
=& 940 \quad [s(2) = 5]\\
\theta_1 =& \min\{K + h(0 \cdot d_1) + \theta_2, K + h(0 \cdot d_1 + 1 \cdot d_2) + \theta_3,\\
& \quad K + h(0 \cdot d_1 + 1 \cdot d_2 + 2 \cdot d_3) + \theta_4,\\
& \quad K + h(0 \cdot d_1 + 1 \cdot d_2 + 2 \cdot d_3 + 3 \cdot d_4) + \theta_5\}\\
=& \min\{1440, 1380, 1560, 1480\}\\
=& 1380 \quad [s(1) = 3]
\end{aligned}
$$

Therefore, we order in periods 1 and $s(1) = 3$; the optimal order quantities are $Q_1 = d_1 + d_2 = 210$ and $Q_3 = d_3 + d_4 = 150$ cubic meters; and the total cost is 1380. □

3.7.4 Extensions

Many of the assumptions made in Section 3.7.1 can be relaxed without making the problem substantially harder. For example, period-specific costs (h_t, K_t, c_t) can easily be accommodated. Similarly, nonzero lead times can be handled, provided the lead time is still fixed and constant. Positive initial inventories can be handled with appropriate modifications to the cost function in period 1.

Other extensions are considerably more difficult. For example, we assumed implicitly that there were no capacity constraints—an order can be placed of any size, and any amount of inventory can be carried over. Capacitated versions of the Wagner–Whitin model turn out to be NP-hard (Florian et al. 1980). Backlogging and concave order costs (instead of linear) are considered by Zangwill (1966); the model is still polynomially solvable, but the solution approach is less tractable than the DP presented here.

CASE STUDY 3.1 Ice Cream Production and Inventory at Scotsburn Dairy Group

Scotsburn Dairy Group is one of Canada's largest producers of ice cream and other dairy products. Its factory in Truro, Nova Scotia produces nearly 30 million liters of ice cream per year. Scotsburn collaborated with the industrial engineering department at Dalhousie University to optimize the production and inventory of ice cream at the Truro facility. The collaboration first began as an undergraduate design project, then a Master's project. The approach is described by Gunn et al. (2014).

The team developed a hierarchical planning process that includes a monthly model for setting inventory targets and staffing levels over a 1-year horizon; a weekly model to determine how much of each stock-keeping unit (SKU) to produce per week; and a daily model to optimize the production schedule. All three were formulated as integer programming (IP) models. We focus on the weekly model, which is an extension of the Wagner–Whitin model discussed in Section 3.7.

The Truro facility produces over 300 SKUs of ice cream, which the researchers aggregated into just over 100 product families. The weekly model determines how much of each family to produce in each week over a 13-week horizon. The model is used on a rolling-horizon basis, meaning that the company only implements next week's plan; it then solves the model again for another 13-week horizon.

Let F be the set of product families. Let a_t^+ and a_t^- be the maximum and minimum number of production hours that may be used in week t, respectively. (These are outputs from the monthly planning model.) Let u_{f,t_1,t_2} be the number of production hours required to produce family $f \in F$ in week t_1 to cover the demand in weeks $t_1, \ldots, t_2$, and let c_{f,t_1,t_2} be the cost (including both fixed and holding costs) to do so. Similar to (3.40),

$$c_{f,t_1,t_2} = K_f + h_f \sum_{t=t_1}^{t_2} (t - t_1) d_{tf},$$

where the parameters are as in Section 3.7 but are now also indexed by the product family, f. The decision variable x_{f,t_1,t_2} equals 1 if family f is produced in week t_1 in order to cover the demand in weeks $t_1, \ldots, t_2$, and 0 otherwise. Note that this is a different type of formulation than that used in Section 3.7.2 since the decision variables determine how many periods' of demand to produce rather than modeling the production and inventory levels explicitly.

The Scotsburn weekly model can be formulated as follows[2]:

$$\text{minimize} \quad \sum_{f \in F} \sum_{t_1=1}^{T} \sum_{t_2=t_1}^{T} c_{f,t_1,t_2} x_{f,t_1,t_2} \tag{3.41}$$

$$\text{subject to} \quad \sum_{t_1=1}^{t} \sum_{t_2=t}^{T} x_{f,t_1,t_2} = 1 \qquad \forall f \in F, \forall t = 1, \ldots, T \tag{3.42}$$

$$a_t^- \leq \sum_{f \in F} \sum_{t_2 \geq t} u_{f,t,t_2} x_{f,t,t_2} \leq a_t^+ \qquad \forall t = 1, \ldots, T \tag{3.43}$$

$$x_{f,t_1,t_2} \in \{0,1\} \qquad \forall f \in F, \forall t_1, t_2 = 1, \ldots, T \tag{3.44}$$

The objective function (3.41) calculates the total production and inventory costs. Constraints (3.42) ensure that the demand for each product family f in each week t is produced in some production run that includes period t. Constraints (3.43) require the total number of production hours used in period t to be within the allowable range. Constraints (3.44) are integrality constraints.

Scotsburn solves this model using CPLEX, which can solve a typical instance—roughly 10,000 variables and 2,000 constraints—to 2% optimality within a few minutes. The company reports that the full project—including the monthly, weekly, and daily planning models—helped to improve the fill rate (fraction of demand met from stock) from 90.2% to 96.2%; it also improved the production rate (units produced per hour) by 3% as a result of having fewer time-consuming production setups.

PROBLEMS

3.1 (EOQ for Steel) An auto manufacturer uses 500 tons of steel per day. The company pays \$1100 per ton of steel purchased, and each order incurs a fixed cost of \$2250. The holding cost is \$275 per ton of steel per year. Using the EOQ model, calculate the optimal order quantity, cycle length, and average cost per year.

3.2 (EOQ for MP3s) Suppose that your favorite electronics store maintains an inventory of a certain brand and model of MP3 player. The store pays the manufacturer \$165 for each MP3 player ordered. Each order incurs a fixed cost of \$40 in order processing, shipping, etc. and requires a 2-week lead time. The store estimates that its cost of capital is 17% per year, and it estimates its other holding costs (warehouse space, insurance, etc.) at \$1 per MP3 player per month. The demand for MP3 players is steady at 40 per week.

[2]The real model includes multiple production lines and allows for overtime, but we omit these aspects for the sake of simplicity and instead assume that the factory has a single production line with hard constraints on the production hours available.

a) Using the EOQ model, calculate the optimal order quantity, reorder point (r), and average cost per year.

b) Now suppose that backorders are allowed, and that each backorder incurs a stockout penalty of \$60 per stockout per year. Using the EOQ model with planned backorders, calculate the optimal order quantity, stockout percentage (x), reorder point (r), and average cost per year. How much money would the store save per year by allowing stockouts, expressed as a percentage?

3.3 (EOQ for Cat Toys) Mason's Meows is a company that makes cat toys. The company sells 1200 toys per year. The firm incurs a fixed cost of \$150 in labor each time it starts up the manufacturing process to begin a new batch of toys. Each toy costs Mason's Meows \$9 to produce. The company's accountant recommends using a holding cost equal to 20% of the cost of the toy, per year.

a) What is the optimal batch size, Q^*? If the company uses batches of size Q^*, how many times per year, on average, will it start up the manufacturing process?

b) After careful analysis, the inventory team at Mason's Meows realized that the per-unit production cost is smaller if the batch size is larger. In particular, the production cost is \$9 per unit for batches of fewer than 400 units and \$7.50 per unit for batches of 400 or more units. Now what is the optimal batch size?

3.4 (EOQ for Vaccines) A medical clinic dispenses vaccines at a steady rate of 520 doses per month. Each order placed to the vaccine manufacturer incurs a fixed cost of \$140. Each vaccine dose held in inventory incurs a holding cost of \$3 per year.

a) Using the EOQ model, calculate the optimal order quantity, Q^*, and the optimal average cost per year, $g(Q^*)$.

b) Suppose that the fixed cost K increases. Will Q^* increase, decrease, or stay the same? Briefly explain your answer.

3.5 (EOQ for Automobile Components) An automobile manufacturing plant uses exactly 8 power-lock mechanisms per hour. Each replenishment order to the supplier of the power-lock mechanisms incurs a fixed cost of \$85. Each mechanism stored in inventory incurs a holding cost of \$1.50 per week.

a) Using the EOQ model, calculate the optimal order quantity, Q^*, and the optimal average cost per year, $g(Q^*)$.

b) Suppose that the plant must order in power-of-two multiples of 1 week. (That is, the plant can place an order every week, or every 2 weeks, or every 4 weeks, ..., or every $\frac{1}{2}$ week, or every $\frac{1}{4}$ week, ...) What is the optimal power-of-two order interval, and what is the cost ratio versus the optimal (non-power-of-two) solution?

3.6 (Snack Bar Inventory Management, Part 1) A snack bar at a certain theme park sees a (constant, deterministic, continuous) demand of 150 cases per day. (We are aggregating the various products sold by the snack bar into a single product and expressing its demand in terms of number of cases.) Replenishment orders are placed to a central warehouse located within the theme park, with negligible lead time, and it costs \$10 in labor costs to deliver an order to the snack bar from the warehouse. It costs \$1.20 per case per day in refrigeration costs and other holding costs to hold cases of food in inventory at the snack bar.

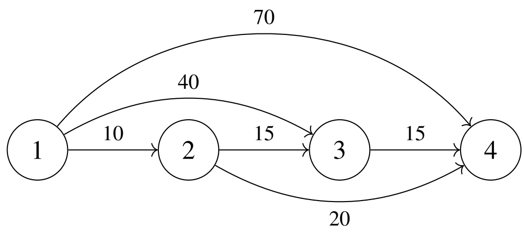


Figure 3.13 Shortest path network for Problem 3.7.

a) Calculate the optimal order quantity, Q^*, for the snack bar.

b) If the snack bar uses Q^* as its order quantity, how often will it order?

c) Suppose the snack bar must order in multiples of 20 cases. (That is, it must order 20 cases, or 40 cases, or 60 cases, or ....) Do you think the snack bar's costs will increase significantly due to this restriction? Briefly explain your answer.

3.7 (Snack Bar Inventory Management, Part 2) For the snack bar in Problem 3.6, suppose now that the demand is different on different days of the week, as given in the following table. Replenishment orders can only be placed at the start of each day. The fixed and holding costs are as given in Problem 3.6.

Day (#)	Day (Name)	Demand
1	Sunday	220
2	Monday	155
3	Tuesday	105
4	Wednesday	90
5	Thursday	170
6	Friday	210
7	Saturday	290

a) Assume that the snack bar uses a 7-day planning horizon, beginning on Sunday. Let c_{ts} be the cost to place an order on day t that will last through the end of day $s-1$, including both the fixed ordering cost and the holding cost. Calculate c_{12}, c_{47}, and c_{68}.

b) Suppose instead that the snack bar uses a 3-day planning horizon and that the shortest path network representing fixed and holding costs is as given in Figure 3.13. (The numbers in this figure come from different data than those in part (a).) On which day(s) should the snack bar place orders?

3.8 (EOQ with Nonzero Lead Time) Consider the EOQ model with fixed lead time $L > 0$ (Section 3.2.5). Prove that the average amount of inventory on order is equal to the lead-time demand.

3.9 (Change in Optimal EOQ Cost) Suppose we have two instances of the EOQ problem, h_1, K_1, λ_1 and h_2, K_2, λ_2, such that $\sqrt{2K_1\lambda_1 h_1} < \sqrt{2K_2\lambda_2 h_2}$. True, false, or indeterminate: The holding cost component (i.e., the $hQ/2$ part) of the optimal objective function value is greater under instance 2 than under instance 1. Briefly explain your answer.

3.10 (EOQ with Fixed Batch Sizes) Suppose that in the EOQ model we can only order batches that are an integer multiple of some number Q_B; that is, we can order a batch of size Q_B, $2Q_B$, $3Q_B$, etc.

a) Prove that, for the optimal order quantity $\hat{Q} = mQ_B$,

$$\sqrt{\frac{m-1}{m}} \le \frac{Q_E}{\hat{Q}} \le \sqrt{\frac{m+1}{m}},$$

where $Q_E = \sqrt{2K\lambda/h}$ is the optimal (non-integer-multiple) EOQ quantity.

b) Suppose that $m \ge 2$ for $\hat{Q}$. Using the result in part (a), prove that $g(\hat{Q}) \le 1.32g(Q_E)$, where $g(\cdot)$ is the EOQ cost function.

c) *Bonus*: Prove that $g(\hat{Q}) \le 1.06g(Q_E)$ (still assuming $m \ge 2$).

3.11 (Tightness of Power-of-2 Bound) Prove that the bound given in Theorem 3.3 is tight by developing an instance of the problem such that

$$\frac{f(\hat{T})}{f(T^*)} = \frac{3}{2\sqrt{2}}.$$

Hint: You should be able to come up with a suitable value of T_B in terms of the problem parameters. That is, you should not need to pick values for λ, h, and K; instead, you should be able to leave the values of these parameters unspecified and to express T_B in terms of the parameters to achieve the desired result.

3.12 (Quantity Discounts for Steel) Return to Problem 3.1 and suppose that the steel supplier offers the auto manufacturer a price of \$1490 per ton of steel if $Q < 1200$ tons; \$1220 per ton if $1200 \le Q < 2400$, and \$1100 per ton if $Q \ge 2400$. The annual holding cost rate, i, is 0.25.

a) Calculate Q^* and $g(Q^*)$ for the all-units discount structure.

b) Calculate Q^* and $g(Q^*)$ for the incremental discount structure.

3.13 (Sequence of Q_j^*) In the EOQ model with incremental quantity discounts, prove that $Q_{j-1}^* < Q_j^*$ for all $j = 1, \ldots, n$.

3.14 (Sensitivity Analysis for EOQB: Q) Prove that a result analogous to Theorem 3.2 also describes the sensitivity of the EOQB model with respect to Q; that is, prove that, for any Q:

$$\frac{g(Q, x^*)}{g(Q^*, x^*)} = \frac{1}{2}\left(\frac{Q^*}{Q} + \frac{Q}{Q^*}\right).$$

3.15 (Sensitivity Analysis for EOQB: x) In this problem, you will explore the EOQB model's sensitivity to x, the fraction of demand that is backordered.

a) Let $Q(x)$ be the optimal Q for a given x. Derive an expression for $g(Q(x), x)$, the cost that results from choosing an arbitrary value of x and then setting Q optimally.

b) Prove that for any $0 \le x \le 1$,

$$\frac{g(Q(x), x)}{g(Q^*, x^*)} = \sqrt{\frac{(1-x)^2 h + x^2 p}{x^* p}}.$$

c) Prove that if $h < p$, then for all x,

$$\frac{g(Q(x),x)}{g(Q^*,x^*)} \le \frac{1}{\sqrt{x^*}}.$$

3.16 (EOQ with Planned Lost Sales) Suppose that we are allowed to stock out in the EOQ model, but instead of excess demands being backordered (as in Section 3.5), they are lost. Let x be the fraction of demand that is lost, and let p be the cost per lost sale. Let c be the cost to order each unit. In the standard EOQ and the EOQ with backorders, we could ignore c because each year we order exactly λ items per year on average, regardless of the order quantity Q. But if some demands are lost, we will not order items to replenish those demands; therefore, the total per-unit ordering cost per year *does depend* on the solution we choose.

a) Formulate the total cost per year as a function of Q and x.

b) Prove that

$$x^* = \begin{cases} 0, & \text{if } \lambda(p-c) > \sqrt{2K\lambda h} \\ 1, & \text{if } \lambda(p-c) < \sqrt{2K\lambda h} \\ \text{anything in } [0,1], & \text{if } \lambda(p-c) = \sqrt{2K\lambda h} \end{cases}$$

c) Give an interpretation of the condition $\lambda(p-c) > \sqrt{2K\lambda h}$ and explain in words why the optimal value of x^* follows the rule given in part (b).

d) Part (b) implies that either we meet *every* demand or we stock out on *every* demand—x^* is never strictly between 0 and 1 (except in the special case in which $\lambda(p-c) = \sqrt{2K\lambda h}$). This is not the case in the EOQ with backorders. Explain in words why the two models give different results.

3.17 (EOQ with Nonlinear Holding Costs) We assumed that the holding cost for one item in the EOQ model equals ht, where t is the amount of time the item is in inventory. Suppose instead the holding cost for one item is given by he^{bt}, for $b > 0$.

a) Write the average annual cost as a function of Q, $g(Q)$. (Your answer should not include integrals.)

b) Write the first-order condition (i.e., $dg/dQ = 0$) for the function you derived in part (a).

c) The first-order condition cannot be solved explicitly for Q—we can't write an expression like Q^* = [something or other]. Instead, $g(Q)$ must be optimized numerically. Using a nonlinear programming solver, find the Q that minimizes $g(Q)$ using the following parameter values: $\lambda = 500$, $K = 100$, $h = 1$, $b = 0.5$. Report both Q^* and $g(Q^*)$.

Note: As part (e) establishes, $g(Q)$ is quasiconvex everywhere; therefore, you may use a nonlinear solver that relies on this property.

d) Prove that $g(Q)$ is convex at $Q = Q^*$.

Hint: We know the first-order condition says $dg/dQ = 0$ at $Q = Q^*$. Write the second-order condition in such a way that you can make use of the first-order condition.

e) A function f is said to be *unimodal* if there exists some point x^* such that f is increasing on the range $x \le x^*$ and decreasing on the range $x \ge x^*$. A function

f is said to be *quasiconvex* if $-f$ is unimodal. Prove that $g(Q)$ is quasiconvex for all $Q > 0$.

f) *Bonus*: Prove that $g(Q)$ is convex for all $Q > 0$.

3.18 (EOQ with Batch Demands) Consider an inventory system in which each order is for Q units. Instead of the demand occurring continuously over time (as in the EOQ model), the customer purchases exactly half of the inventory exactly halfway through the order cycle and the remaining half exactly at the end of the order cycle. At that point, a new order is placed, and it arrives instantly. (Therefore, there is no time at which the inventory level equals 0.) The total demand per year is λ, just as in the EOQ model, which means that each order cycle has the same length as in the EOQ model.

a) Write an expression for the average annual total cost.

b) What is the optimal order quantity, Q^*?

3.19 (EOQ vs. EOQB Costs)

a) Prove that the optimal annual holding plus backorder costs in the EOQB model is strictly less than the optimal annual holding cost in the EOQ model.

b) Use part (a) to prove that the total cost (including fixed costs) decreases when we allow backorders.

3.20 (EOQ Generalization) Consider an EOQ-like inventory model whose cost function is given by

$$g(Q) = \frac{aQ^2 + b}{cQ + d} \tag{3.45}$$

for constants $a,\ b,\ c$, and d with $a, c > 0$ and $b, d \geq 0$.

Note that the classical EOQ problem is a special case, since the EOQ cost function (3.3) can be obtained by setting

$$\begin{aligned} a &= h \\ b &= 2K\lambda \\ c &= 2 \\ d &= 0. \end{aligned} \tag{3.46}$$

In this problem you will prove some properties of the cost function (3.45).

a) Prove that

$$Q^* = \frac{\sqrt{a^2d^2 + abc^2} - ad}{ac}.$$

Then show that the classical EOQ model is a special case, i.e., that for the appropriate values of the constants, we get the classical EOQ order quantity.

b) Prove that

$$(Q^*)^2 = \frac{bc - 2adQ^*}{ac}.$$

c) Use part (b) to prove that

$$g(Q^*) = \frac{2a}{c}Q^*.$$

Then show that the classical EOQ model is a special case, i.e., that for the appropriate values of the constants, we have $g(Q^*) = hQ^*$ (Theorem 3.1).

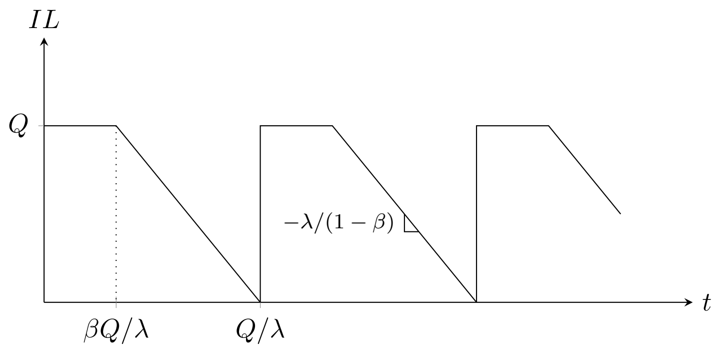


Figure 3.14 Inventory level curve for Problem 3.23.

d) Calculate Q^* and $g(Q^*)$ assuming $a = 20$, $b = 125$, $c = 1.2$, $d = 2.7$.

e) *Bonus*: Prove that

$$\frac{g(Q)}{g(Q^*)} = \frac{1}{2}\left(\frac{Q}{Q^*} + \frac{Q^*}{Q}\right) - [\text{a nonnegative constant}]$$

(analogous to Theorem 3.2), and indicate what the nonnegative constant is. Then show that the classical EOQ model is a special case.

3.21 **(Alternate EOQ Proof)** Prove that the EOQ cost function can be rewritten as

$$g(Q) = \frac{h}{2\lambda Q}\left(Q - \sqrt{\frac{2K\lambda}{h}}\right)^2 + \sqrt{\frac{2Kh}{\lambda}}.$$

Use this to prove (3.4) *without* using calculus. (Thus, this method provides a proof of the EOQ formula using algebra only.)

3.22 **(EPQ for Laundry)** A restaurant uses 80 cloth napkins per hour. The napkins are washed by hand at a rate of 110 per hour. Each time the laundry process is started, the restaurant incurs a fixed cost of \$4.00. Napkins in inventory incur a holding cost of \$0.08 per napkin per hour. Stockouts are not allowed. How many napkins should the restaurant have in circulation?

3.23 **(EOQ with Zero-Demand Sub-Cycles)** Consider the following modification to the EOQ problem. Suppose that, each time an order is placed, the demand is initially 0 for a fraction β of the cycle, and then the demand occurs at a rate of $\lambda/(1-\beta)$ for the duration of the cycle. One can show (you need not) that the total cycle length is still Q/λ, just like in the original EOQ model, and the cycle is divided as shown in Figure 3.14. Calculate the optimal order quantity, Q^*.

3.24 **(EOQ with Cycle-Length Costs)** Suppose that the inventory ordered in the EOQ problem must be stored in a special piece of storage equipment, and the cost of the equipment depends on the amount of time the inventory will be stored, i.e., the amount of time between replenishment orders. (For example, the product might be perishable; the longer it will be stored in inventory, the more insulation is required in the container.) The storage equipment is leased from a material-handling company. The lease cost per year is

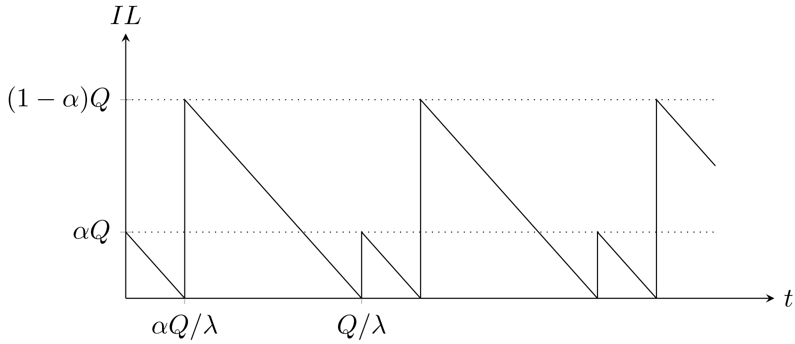


Figure 3.15 Inventory level curve for Problem 3.26.

given by $w \ln T$, where w is a constant, T is the time between consecutive orders, and $\ln$ is the natural log function. Holding and fixed costs are still incurred, as in the original EOQ problem. (You can continue to ignore the per-unit purchase cost.)

a) Write the total cost function, $g(Q)$.

b) Write an expression for the optimal order quantity, Q^*.

c) Suppose $h = 2$, $\lambda = 150$, $K = 700$, and $w = 100$. What is Q^*?

d) If $w > 0$, is the optimal order quantity for this model less than, greater than, or equal to that for the original EOQ model?

3.25 (EOQ with Random Half-Orders) Suppose that, in the EOQ model, some orders randomly arrive at only half the requested size. That is, if the order quantity is Q, then the quantity delivered is Q with probability α and $\frac{1}{2}Q$ with probability $1-\alpha$, for some constant α ($0 \le \alpha \le 1$). The remaining parameters and assumptions are as in the standard EOQ model.

a) Determine a closed-form expression for the optimal order quantity, Q^*, as a function of the problem parameters.

b) Will the optimal order quantity in this model be greater than, less than, or equal to that of the classical EOQ? Briefly explain why. (Provide a logical explanation based on the problem, not a mathematical answer based on part (a).)

3.26 (EOQ with Two Deliveries) Consider a variant of the EOQ model in which each order arrives in two separate deliveries. In particular, if we place an order of size Q, then a quantity αQ arrives instantly, and the remaining quantity, $(1-\alpha)Q$, arrives $\alpha Q/\lambda$ years later, for a fixed constant $0 < \alpha < 1$. Thus, the inventory curve looks like the curve pictured in Figure 3.15.

The fixed cost K is incurred once per order cycle, even though there are two deliveries. As in the standard EOQ, the holding cost is given by h per item per year.

Calculate the optimal order quantity, Q^*.

3.27 (Wagner–Whitin for Aircraft Engines) The Pratt & Whitin Company, which manufactures aircraft engines, needs to decide how many units of a particular bolt to order in order to build engines over the next 4 months. Orders for engines are placed over a year in advance, so the company knows its near-term demand exactly; in particular, the number of engines to produce in the next 4 months will be 150, 100, 80, and 200 in months 1 through 4, respectively. Each engine requires a single bolt. Orders for bolts incur a fixed cost of \$120, and bolts held in inventory incur a holding cost of \$0.80 per bolt per month. Find the optimal order quantities in each period and the optimal total cost.

3.28 (Wagner–Whitin for Sunglasses) The file `sunglasses.xlsx` contains forecast demand (measured in cases) for sunglasses at a major retailer for each of the next 52 weeks. Each order placed to the supplier incurs a fixed cost of \$1100. One case of sunglasses held in inventory for one period incurs a holding cost of \$2.40. Find the optimal order quantities in each period and the optimal total cost.

3.29 (Wagner–Whitin for Glass) A small maker of art glass has orders to make paperweights, vases, and so on over the course of the coming 5 weeks. Based on these orders, it has projected its requirements for its primary raw material—glass rods—over these 5 weeks to be 730, 580, 445, 650, and 880 kg, respectively. Each order to the glass rod supplier incurs a fixed cost of \$100, and each kg of glass rods held in inventory incurs a holding cost of \$0.10 per week.

a) Determine the optimal order quantity in each week, as well as the optimal total cost.

b) Let $\hat{t}$ be the first period in which there is *no* order in your optimal solution from part (a). Suppose the raw material inventory is destroyed at the beginning of period $\hat{t}$ so that the workshop *must* order in period $\hat{t}$. How much should it order in each remaining period of the horizon, and what will be the resulting cost for the entire horizon?

3.30 (Wagner–Whitin Solution from DP #1) Consider the Wagner–Whitin problem with $h = 2$, $K = 50$, $T = 4$, and $(d_1, \ldots, d_4) = (20, 12, 17, 23)$. Suppose you have performed the calculations for $t \geq 2$ and found the following values for θ_t and $s(t)$:

	t			
	2	3	4	5
θ_t	134	96	50	0
$s(t)$	4	5	5	—

Determine which periods to order in, how much to order in each of those periods, and the corresponding optimal cost.

3.31 (Wagner–Whitin Solution from DP #2) Follow the instructions for Problem 3.30 for an instance with $h = 1$, $K = 20$, $T = 4$, and $(d_1, \ldots, d_4) = (25, 15, 15, 30)$, using the following values for θ_t and $s(t)$:

	t			
	2	3	4	5
θ_t	55	40	20	0
$s(t)$	4	4	5	—

3.32 (Wagner–Whitin with Randomly Perishable Goods) Suppose that in the Wagner–Whitin model, all of the items currently held in inventory will perish (be destroyed) with some probability q at the *end* of each time period. For example, if we order 4 periods' worth of demand in period 1, the demand for period 1 will be satisfied for sure, but the inventory consisting of the demand for periods 2–4 will perish with probability q; if it survives (with probability $1 - q$), the inventory for periods 3–4 will perish at the end of period 2 with probability q; and so on. Once the initial ordering schedule is set, no additional orders may be placed.

Obviously, we can no longer require that all demand be satisfied. We will assume that unmet demand is lost (not backordered), and that lost demands incur a penalty cost of p per unit. As in the standard Wagner–Whitin model, we will assume a holding cost of h per unit per time period and a setup cost of K per order.

The sequence of events in each period is as follows:

1. The replenishment order, if any, is placed and is received immediately.
2. Demand occurs and is satisfied from inventory if possible.
3. Remaining inventory either perishes or does not.
4. Holding and stockout costs are incurred based on remaining inventory and lost sales.

a) Show how the arc costs can be computed to capture the new cost function so that the Wagner–Whitin DP algorithm can still be used. Simplify your answer as much as possible.

Hint: The formulas in Section C.5 may come in handy.

b) Illustrate your method by finding the optimal solution for the following 4-period instance: $h = 0.2$, $K = 200$, $p = 3$, $q = 0.25$, and the demands in periods 1–4 are 200, 125, 250, 175. Indicate the optimal solution (order schedule) and the cost of that solution.

c) Do you think the optimal solution to the problem with perishability will involve *more* orders, *fewer* orders, or *the same number* of orders than the optimal solution to the normal Wagner–Whitin problem (without perishability)? Explain your answer.

3.33 (Wagner–Whitin $\rightarrow$ EOQ?) Does the Wagner–Whitin model approach the EOQ model as the length of a time period gets shorter (keeping the total time horizon fixed)? Conduct a small numerical experiment to confirm your answer.