

**Algebra, Chapter 0**

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**Chapter I. Preliminaries: Set theory and categories****§1. Naive Set Theory**

**1.6** Define a relation  $\sim$  on the set  $\mathbb{R}$  of real numbers, by setting  $a \sim b \iff b - a \in \mathbb{Z}$ . Prove that this is an equivalence relation, and find a ‘compelling’ description for  $\mathbb{R}/\sim$ . Do the same for the relation  $\approx$  on the plane  $\mathbb{R} \times \mathbb{R}$  defined by declaring  $(a_1, a_2) \approx (b_1, b_2) \iff b_1 - a_1 \in \mathbb{Z}$  and  $b_2 - a_2 \in \mathbb{Z}$ . [§II.8.1, II.8.10]

Imaginatively,  $\mathbb{R}/\sim$  can be viewed as a ring of length 1 by bending the real line  $\mathbb{R}$ . Then we can rotate a ring around an axis of rotation to get  $\mathbb{R} \times \mathbb{R}/\approx$ , which makes a torus. ■

## §2. Functions between sets

**2.1** How many different bijections are there between a set  $S$  with  $n$  elements and itself? [§II.2.1]

There are  $n!$  different bijections  $S \rightarrow S$ . ■

## §3. Categories

**3.1** Let  $\mathbf{C}$  be a category. Consider a structure  $\mathbf{C}^{op}$  with:

- $\text{Obj}(\mathbf{C}^{op}) := \text{Obj}(\mathbf{C})$ ;
- for  $A, B$  objects of  $\mathbf{C}^{op}$  (hence, objects of  $\mathbf{C}$ ),  $\text{Hom}_{\mathbf{C}^{op}}(A, B) := \text{Hom}_{\mathbf{C}}(B, A)$

Show how to make this into a category (that is, define composition of morphisms in  $\mathbf{C}^{op}$  and verify the properties listed in §3.1). Intuitively, the 'opposite' category  $\mathbf{C}^{op}$  is simply obtained by 'reversing all the arrows' in  $\mathbf{C}$ . [5.1, §VIII.1.1, §IX.1.2, IX.1.10]

- For every object  $A$  of  $\mathbf{C}$ , there exists one identity morphism  $1_A \in \text{Hom}_{\mathbf{C}}(A, A)$ . Since  $\text{Obj}(\mathbf{C}^{op}) := \text{Obj}(\mathbf{C})$  and  $\text{Hom}_{\mathbf{C}^{op}}(A, A) := \text{Hom}_{\mathbf{C}}(A, A)$ , for every object  $A$  of  $\mathbf{C}^{op}$ , the identity on  $A$  coincides with  $1_A \in \mathbf{C}$ .
- For  $A, B, C$  objects of  $\mathbf{C}^{op}$  and  $f \in \text{Hom}_{\mathbf{C}^{op}}(A, B) = \text{Hom}_{\mathbf{C}}(B, A)$ ,  $g \in \text{Hom}_{\mathbf{C}^{op}}(B, C) = \text{Hom}_{\mathbf{C}}(C, B)$ , the composition laws in  $\mathbf{C}$  determines a morphism  $f * g$  in  $\text{Hom}_{\mathbf{C}}(C, A)$ , which deduces the composition defined on  $\mathbf{C}^{op}$ :

$$\begin{aligned} \text{Hom}_{\mathbf{C}^{op}}(A, B) \times \text{Hom}_{\mathbf{C}^{op}}(B, C) &\longrightarrow \text{Hom}_{\mathbf{C}^{op}}(A, C) \\ (f, g) &\longmapsto g \circ f := f * g \end{aligned}$$

- Associativity. If  $f \in \text{Hom}_{\mathbf{C}^{op}}(A, B)$ ,  $g \in \text{Hom}_{\mathbf{C}^{op}}(B, C)$ ,  $h \in \text{Hom}_{\mathbf{C}^{op}}(C, D)$ , then

$$f \circ (g \circ h) = f \circ (h * g) = (h * g) * f = h * (g * f) = (g * f) \circ h = (f \circ g) \circ h.$$

- Identity. For all  $f \in \text{Hom}_{\mathbf{C}^{op}}(A, B)$ , we have

$$f \circ 1_A = 1_A * f = f, \quad 1_B \circ f = f * 1_B = f.$$

Thus we get the full construction of  $\mathbf{C}^{op}$ . ■

## §4. Morphisms

**4.2** In Example 3.3 we have seen how to construct a category from a set endowed with a relation, provided this latter is reflexive and transitive. For what types of relations is the corresponding category a groupoid (cf. Example 4.6)? [§4.1]

For a reflexive and transitive relation  $\sim$  on a set  $S$ , define the category  $\mathbf{C}$  as follows:

- Objects:  $\text{Obj}(\mathbf{C}) = S$ ;
- Morphisms: if  $a, b$  are objects (that is: if  $a, b \in S$ ) then let

$$\text{Hom}_{\mathbf{C}}(a, b) = \begin{cases} (a, b) \in S \times S & \text{if } a \sim b \\ \emptyset & \text{otherwise} \end{cases}$$

In Example 3.3 we have shown the category. If the relation  $\sim$  is endowed with symmetry, we have

$$(a, b) \in \text{Hom}_{\mathbf{C}}(a, b) \implies a \sim b \implies b \sim a \implies (b, a) \in \text{Hom}_{\mathbf{C}}(b, a).$$

Since

$$(a, b)(b, a) = (a, a) = 1_a, \quad (b, a)(a, b) = (b, b) = 1_b,$$

in fact  $(a, b)$  is an isomorphism. From the arbitrariness of the choice of  $(a, b)$ , we show that  $\mathbf{C}$  is a groupoid. Conversely, if  $\mathbf{C}$  is a groupoid, we can show the relation  $\sim$  is symmetric. To sum up, the category  $\mathbf{C}$  is a groupoid if and only if the corresponding relation  $\sim$  is an equivalence relation. ■

## §5. Universal properties

**5.1** Prove that a final object in a category  $\mathbf{C}$  is initial in the opposite category  $\mathbf{C}_{op}$  (cf. Exercise 3.1).

An object  $F$  of  $\mathbf{C}$  is final in  $\mathbf{C}$  if and only if

$$\forall A \in \text{Obj}(\mathbf{C}) : \text{Hom}_{\mathbf{C}}(A, F) \text{ is a singleton.}$$

That is equivalent to

$$\forall A \in \text{Obj}(\mathbf{C}_{op}) : \text{Hom}_{\mathbf{C}_{op}}(F, A) \text{ is a singleton,}$$

which means  $F$  is initial in the opposite category  $\mathbf{C}_{op}$ . ■

## Chapter II. Groups, first encounter

### §1. Definition of group

**1.1** Write a careful proof that every group is the group of isomorphisms of a groupoid. In particular, every group is the group of automorphisms of some object in some category.

Assume  $G$  is a group. Define a category  $\mathbf{C}$  as follows:

- Objects:  $\text{Obj}(\mathbf{C}) = \{*\}$ ;
- Morphisms:  $\text{Hom}_{\mathbf{C}}(*, *) = \text{End}_{\mathbf{C}}(*) = G$ .

The composition of homomorphism is corresponding to the multiplication between two elements in  $G$ . The identity morphism on  $*$  is  $1_* = e_G$ , which satisfies for all  $g \in \text{Hom}_{\mathbf{C}}(*, *)$ ,

$$ge_G = e_Gg = g,$$

and

$$gg^{-1} = e_G, g^{-1}g = e_G.$$

Thus any homomorphism  $g \in \text{Hom}_{\mathbf{C}}(*, *)$  is an isomorphism and accordingly  $\mathbf{C}$  is a groupoid. Now we see  $G = \text{End}_{\mathbf{C}}(*)$  is the group of isomorphisms of a groupoid. Moreover, supposing that  $*$  is an object in some category  $\mathbf{D}$ ,  $G$  would be the group of automorphisms of  $*$ , which is denoted as  $\text{Aut}_{\mathbf{D}}(*)$ . ■

**1.4** Suppose that  $g^2 = e$  for all elements  $g$  of a group  $G$ ; prove that  $G$  is commutative.

For all  $a, b \in G$ ,

$$abab = e \implies a(abab)b = ab \implies (aa)ba(bb) = ab \implies ba = ab.$$

■

## §2. Examples of groups

**2.1** One can associate an  $n \times n$  matrix  $M_\sigma$  with a permutation  $\sigma \in S_n$ , by letting the entry at  $(i, \sigma(i))$  be 1, and letting all other entries be 0. For example, the matrix corresponding to the permutation

$$\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \in S_3$$

would be

$$M_\sigma = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

Prove that, with this notation,

$$M_{\sigma\tau} = M_\sigma M_\tau$$

for all  $\sigma, \tau \in S_n$ , where the product on the right is the ordinary product of matrices.

By introducing the Kronecker delta function

$$\delta_{i,j} = \begin{cases} 0 & \text{if } i \neq j, \\ 1 & \text{if } i = j, \end{cases}$$

the entry at  $(i, j)$  of the matrix  $M_{\sigma\tau}$  can be written as

$$(M_{\sigma\tau})_{i,j} = \delta_{\tau(\sigma(i)),j}$$

and the entry at  $(i, j)$  of the matrix  $M_\sigma M_\tau$  can be written as

$$(M_\sigma M_\tau)_{i,j} = \sum_{k=1}^n (M_\sigma)_{i,k} (M_\tau)_{k,j} = \sum_{k=1}^n \delta_{\sigma(i),k} \cdot \delta_{\tau(k),j} = \sum_{k=1}^n \delta_{\sigma(i),k} \cdot \delta_{k,\tau^{-1}(j)} = \delta_{\sigma(i),\tau^{-1}(j)},$$

where the last but one equality holds by the fact

$$\tau(k) = j \iff k = \tau^{-1}(j).$$

Noticing that

$$\tau(\sigma(i)) = j \iff \sigma(i) = \tau^{-1}(j),$$

we see  $M_{\sigma\tau} = M_\sigma M_\tau$  for all  $\sigma, \tau \in S_n$ . ■

**2.2** Prove that if  $d \leq n$ , then  $S_n$  contains elements of order  $d$ .

The cyclic permutation

$$\sigma = (1 \ 2 \ 3 \cdots d)$$

is an element of order  $d$  in  $S_n$ . ■

**2.3** For every positive integer  $n$  find an element of order  $n$  in  $S_{\mathbb{N}}$ .

The cyclic permutation

$$\sigma = (1 \ 2 \ 3 \cdots n)$$

is an element of order  $d$  in  $S_n$ . ■

**2.4** Define a homomorphism  $D_8 \rightarrow S_4$  by labeling vertices of a square, as we did for a triangle in §2.2. List the 8 permutations in the image of this homomorphism.

The image of  $n$  rotations under the homomorphism are

$$\sigma_1 = e_{D_8}, \sigma_2 = (1 \ 2 \ 3 \ 4), \sigma_3 = (1 \ 3)(2 \ 4), \sigma_4 = (1 \ 4 \ 3 \ 2).$$

The image of  $n$  reflections under the homomorphism are

$$\sigma_5 = (1 \ 3), \sigma_6 = (2 \ 4), \sigma_7 = (1 \ 2)(3 \ 4), \sigma_8 = (1 \ 4)(3 \ 2).$$

■

**2.11** Prove that the square of every odd integer is congruent to 1 modulo 8.

Given an odd integer  $2k + 1$ , we have

$$(2k + 1)^2 = 4k(k + 1) + 1,$$

where  $k(k + 1)$  is an even integer. So  $(2k + 1)^2 \equiv 1 \pmod{8}$ . ■

**2.12** Prove that there are no integers  $a, b, c$  such that  $a^2 + b^2 = 3c^2$ . (Hint: studying the equation  $[a]_4^2 + [b]_4^2 = 3[c]_4^2$  in  $\mathbb{Z}/4\mathbb{Z}$ , show that  $a, b, c$  would all have to be even. Letting  $a = 2k, b = 2l, c = 2m$ , you would have  $k^2 + l^2 = 3m^2$ . What's wrong with that?)

$$a^2 + b^2 = 3c^2 \implies [a]_4^2 + [b]_4^2 = 3[c]_4^2.$$

Noting that  $[0]_4^2 = [0]_4, [1]_4^2 = [1]_4, [2]_4^2 = [0]_4, [3]_4^2 = [1]_4$ , we see  $[c]_4^2$  must be  $[0]_4$  and so do  $[a]_4^2$  and  $[b]_4^2$ . Hence  $[a]_4, [b]_4, [c]_4$  can only be  $[0]_4$  or  $[2]_4$ , which justifies letting  $a = 2k_1, b = 2l_1, c = 2m_1$ . After substitution we have  $k^2 + l^2 = 3m^2$ . Repeating this process  $n$  times yields  $a = 2^n k_n, b = 2^n l_n, c = 2^n m_n$ . For a sufficiently large number  $N$ , the absolute value of  $k_N, l_N, m_N$  must be less than 1. Thus we conclude that  $a = b = c = 0$  is the unique solution to the equation  $a^2 + b^2 = 3c^2$ . ■

**2.13** Prove that if  $\gcd(m, n) = 1$ , then there exist integers  $a$  and  $b$  such that  $am + bn = 1$ . (Use Corollary 2.5.) Conversely, prove that if  $am + bn = 1$  for some integers  $a$  and  $b$ , then  $\gcd(m, n) = 1$ . [2.15, §V.2.1, V.2.4]

Applying corollary 2.5, we have  $\gcd(m, n) = 1$  if and only if  $[m]_n$  generates  $\mathbb{Z}/n\mathbb{Z}$ . Hence

$$\gcd(m, n) = 1 \iff a[m]_n = [1]_n \iff [am]_n = [1]_n \iff am + bn = 1.$$

■

**2.15** Let  $n > 0$  be an odd integer.

- Prove that if  $\gcd(m, n) = 1$ , then  $\gcd(2m + n, 2n) = 1$ . (Use Exercise 2.13.)
- Prove that if  $\gcd(r, 2n) = 1$ , then  $\gcd(\frac{r+n}{2}, n) = 1$ . (Ditto.)
- Conclude that the function  $[m]_n \rightarrow [2m + n]_{2n}$  is a bijection between  $(\mathbb{Z}/n\mathbb{Z})^*$  and  $(\mathbb{Z}/2n\mathbb{Z})^*$ .

The number  $\phi(n)$  of elements of  $(\mathbb{Z}/n\mathbb{Z})^*$  is Euler's  $\phi(n)$ -function. The reader has just proved that if  $n$  is odd, then  $\phi(2n) = \phi(n)$ . Much more general formulas will be given later on (cf. Exercise V.6.8). [VII.5.11]

- According to Exercise 2.13,

$$\gcd(m, n) = 1 \implies am + bn = 1 \implies \frac{a}{2}(2m + n) + \left(b - \frac{a}{2}\right)n = 1.$$

If  $a$  is even, we have shown  $\gcd(2m + n, 2n) = 1$ . Otherwise we can let  $a' = a + n$  be an even integer and  $b' = b - m$ . Then it holds that

$$\frac{a'}{2}(2m + n) + \left(b' - \frac{a'}{2}\right)n = 1,$$

which also indicates  $\gcd(2m + n, 2n) = 1$ .

- If  $\gcd(r, 2n) = 1$ , then  $r$  must be an odd integer and accordingly

$$\gcd(2r + 2n, 4n) = 1 \implies a(2r + 2n) + b(4n) = 1 \implies 4a\frac{r + n}{2} + 4bn = 1,$$

which is  $\gcd(\frac{r+n}{2}, n) = 1$ .

- It is easy to check that the function  $f : (\mathbb{Z}/n\mathbb{Z})^* \rightarrow (\mathbb{Z}/2n\mathbb{Z})^*$ ,  $[m]_n \mapsto [2m + n]_{2n}$  is well-defined. The fact

$$\begin{aligned} f([m_1]_n) = f([m_2]_n) &\implies f([2m_1 + n]_{2n}) = f([2m_2 + n]_{2n}) \\ &\implies (2m_1 + n) - (2m_2 + n) = 2kn \\ &\implies m_1 - m_2 = kn \\ &\implies [m_1]_n = [m_2]_n \end{aligned}$$

indicates that  $f$  is injective. For any  $[r]_{2n} \in (\mathbb{Z}/2n\mathbb{Z})^*$ , we have

$$\gcd(r, 2n) = 1 \implies \gcd\left(\frac{r + n}{2}, n\right) = 1 \implies \left[\frac{r + n}{2}\right]_n \in (\mathbb{Z}/n\mathbb{Z})^*,$$

and

$$f\left(\left[\frac{r + n}{2}\right]_n\right) = [r + 2n]_{2n} = [r]_{2n},$$

which indicates that  $f$  is surjective. Thus we show  $f$  is a bijection. ■

**2.16** Find the last digit of  $1238237^{18238456}$ . (Work in  $\mathbb{Z}/10\mathbb{Z}$ .)

$$1238237^{18238456} \equiv 7^{18238456} \equiv (7^4)^{4559614} \equiv 2401^{4559614} \equiv 1 \pmod{10},$$

which indicates that the last digit of  $1238237^{18238456}$  is 1. ■

**2.17** Show that if  $m \equiv m' \pmod{n}$ , then  $\gcd(m, n) = 1$  if and only if  $\gcd(m', n) = 1$ . [§2.3]

Assume that  $m - m' = kn$ . If  $\gcd(m, n) = 1$ , for any common divisor  $d$  of  $m'$  and  $n$

$$d|m', d|n \implies d|(m' + kn) \implies d|m \implies d = 1,$$

which means  $\gcd(m', n) = 1$ . Likewise, we can show  $\gcd(m', n) = 1 \implies \gcd(m, n) = 1$  ■

### §3. The category Grp

**3.1** Let  $\varphi : G \rightarrow H$  be a morphism in a category  $\mathbf{C}$  with products. Explain why there is a unique morphism

$$(\varphi \times \varphi) : G \times G \longrightarrow H \times H.$$

(This morphism is defined explicitly for  $\mathbf{C} = \mathbf{Set}$  in §3.1.)

By the universal property of product in  $\mathbf{C}$ , there exist a unique morphism  $(\varphi \times \varphi) : G \times G \longrightarrow H \times H$  such that the following diagram commutes.

$$\begin{array}{ccc} G & \xrightarrow{\varphi} & H \\ \pi_G \uparrow & & \uparrow \pi_H \\ G \times G & \xrightarrow{\varphi \times \varphi} & H \times H \\ \pi_G \downarrow & & \downarrow \pi_H \\ G & \xrightarrow{\varphi} & H \end{array}$$

■

**3.2** Let  $\varphi : G \rightarrow H, \psi : H \rightarrow K$  be morphisms in a category with products, and consider morphisms between the products  $G \times G, H \times H, K \times K$  as in Exercise 3.1. Prove that

$$(\psi\varphi) \times (\psi\varphi) = (\psi \times \psi)(\varphi \times \varphi).$$

(This is part of the commutativity of the diagram displayed in §3.2.)

By the universal property of product in  $\mathbf{C}$ , there exists a unique morphism

$$(\psi\varphi) \times (\psi\varphi) : G \times G \rightarrow K \times K$$

such that the following diagram commutes.

$$\begin{array}{ccc} G & \xrightarrow{\psi\varphi} & H \\ \pi_G \uparrow & & \uparrow \pi_H \\ G \times G & \xrightarrow{(\psi\varphi) \times (\psi\varphi)} & H \times H \\ \pi_G \downarrow & & \downarrow \pi_H \\ G & \xrightarrow{\psi\varphi} & H \end{array}$$



As the following commutative diagram tells us the composition

$$(\psi \times \psi)(\varphi \times \varphi) : G \times G \rightarrow K \times K$$

can make the above diagram commute,

$$\begin{array}{ccccc}
 & & \psi\varphi & & \\
 & \curvearrowright & & \curvearrowleft & \\
 G & \xrightarrow{\varphi} & H & \xrightarrow{\psi} & K \\
 \pi_G \uparrow & & \pi_H \uparrow & & \pi_K \uparrow \\
 G \times G & \xrightarrow{\varphi \times \varphi} & H \times H & \xrightarrow{\psi \times \psi} & K \times K \\
 \pi_G \downarrow & & \pi_H \downarrow & & \pi_K \downarrow \\
 G & \xrightarrow{\varphi} & H & \xrightarrow{\psi} & K \\
 & \curvearrowleft & & \curvearrowright & \\
 & \psi\varphi & & & 
 \end{array}$$

there must be  $(\psi\varphi) \times (\psi\varphi) = (\psi \times \psi)(\varphi \times \varphi)$ . ■

**3.3** Show that if  $G, H$  are abelian groups, then  $G \times H$  satisfies the universal property for coproducts in **Ab**.

Define two monomorphisms:

$$i_G : G \longrightarrow G \times H, \quad a \longmapsto (a, 0_H)$$

$$i_H : H \longrightarrow G \times H, \quad b \longmapsto (0_G, b)$$

We are to show that for any two homomorphisms  $g : G \rightarrow M$  and  $h : H \rightarrow M$  in **Ab**, the mapping

$$\begin{aligned}
 \varphi : G \times H &\longrightarrow M, \\
 (a, b) &\longmapsto g(a) + h(b)
 \end{aligned}$$

is a homomorphism and makes the following diagram commute.

$$\begin{array}{ccc}
 G & & \\
 i_G \downarrow & \searrow g & \\
 G \times H & \xrightarrow{\varphi} & M \\
 i_H \uparrow & \nearrow h & \\
 H & & 
 \end{array}$$

Exploiting the fact that  $g, h$  are homomorphisms and  $M$  is an abelian group, it is easy to

check that  $\varphi$  preserves the addition operation

$$\begin{aligned}
\varphi((a_1, b_1) + (a_2, b_2)) &= \varphi((a_1 + a_2, b_1 + b_2)) \\
&= g(a_1 + a_2) + h(b_1 + b_2) \\
&= (g(a_1) + g(a_2)) + (h(b_1) + h(b_2)) \\
&= (g(a_1) + h(b_1)) + (g(a_2) + h(b_2)) \\
&= g(a_1 + b_1) + h(a_2 + b_2) \\
&= \varphi((a_1, b_1)) + \varphi((a_2, b_2))
\end{aligned}$$

and the diagram commutes

$$\begin{aligned}
\varphi \circ i_G(a) &= \varphi((a, 0_H)) = g(a) + h(0_H) = g(a) + 0_M = g(a), \\
\varphi \circ i_H(b) &= \varphi((0_G, b)) = g(0_G) + h(b) = 0_M + h(b) = h(b).
\end{aligned}$$

To show the uniqueness of the homomorphism  $\varphi$  we have constructed, suppose a homomorphism  $\varphi'$  can make the diagram commute. Then we have

$$\varphi'((a, b)) = \varphi'((a, 0_H) + (0_G, b)) = \varphi'(i_G(a)) + \varphi'(i_H(b)) = g(a) + h(b) = \varphi((a, b)),$$

that is  $\varphi' = \varphi$ . Hence we show that there exist a unique homomorphism  $\varphi$  such that the diagram commutes, which amounts to the universal property for coproducts in **Ab**. ■

**3.4** Let  $G, H$  be groups, and assume that  $G \cong H \times G$ . Can you conclude that  $H$  is trivial? (Hint: No. Can you construct a counterexample?)

Consider the function

$$\begin{aligned}
\varphi : \mathbb{Z} \times \mathbb{Z}[x] &\longrightarrow \mathbb{Z}[x] \\
(n, f(x)) &\longmapsto n + xf(x)
\end{aligned}$$

Firstly, we can show  $\varphi$  is a homomorphism as follows

$$\begin{aligned}
\varphi((n_1, f_1(x)) + (n_2, f_2(x))) &= \varphi((n_1 + n_2, f_1(x) + f_2(x))) \\
&= (n_1 + n_2) + x(f_1(x) + f_2(x)) \\
&= (n_1 + xf_1(x)) + (n_2 + xf_2(x)) \\
&= \varphi((n_1, f_1(x))) + \varphi((n_2, f_2(x))).
\end{aligned}$$

Secondly, we are to show  $\varphi$  is a monomorphism. It follows by

$$\varphi((n, f(x))) = n + xf(x) = 0 \implies n = 0, f(x) = 0 \implies \ker \varphi = \{(0, 0)\}.$$

Lastly, since the cardinal numbers of both  $\mathbb{Z} \times \mathbb{Z}[x]$  and  $\mathbb{Z}[x]$  are  $\aleph_0$ ,  $\varphi$  is indeed an isomorphism. Therefore, as a counterexample we have  $\mathbb{Z}[x] \cong \mathbb{Z} \times \mathbb{Z}[x]$ . ■

**3.5** Prove that  $\mathbb{Q}$  is not the direct product of two nontrivial groups.

Consider the additive group of rationals  $(\mathbb{Q}, +)$ . Assume that  $\varphi$  is an isomorphism between the product  $G \times H = \{(a, b) | a \in G, b \in H\}$  and  $(\mathbb{Q}, +)$ . Note that  $\{e_G\} \times H$  and  $G \times \{e_H\}$  are subgroups in  $G \times H$  and their intersection is the trivial group  $\{(e_G, e_H)\}$ . It is easy to check that bijection  $\varphi$  satisfies  $\varphi(A \cap B) = \varphi(A) \cap \varphi(B)$ . So applying the fact we have

$$\varphi(\{(e_G, e_H)\}) = \varphi(\{e_G\} \times H \cap G \times \{e_H\}) = \varphi(\{e_G\} \times H) \cap \varphi(G \times \{e_H\}) = \{0\}.$$

Suppose both  $\varphi(\{e_G\} \times H)$  and  $\varphi(G \times \{e_H\})$  are nontrivial groups. If  $\frac{p}{q} \in \varphi(\{e_G\} \times H) - \{0\}$  and  $\frac{r}{s} \in \varphi(G \times \{e_H\}) - \{0\}$ , there must be

$$rp = rq \cdot \frac{p}{q} = ps \cdot \frac{r}{s} \in \varphi(\{e_G\} \times H) \cap \varphi(G \times \{e_H\}),$$

which implies  $rp = 0$ . Since both  $\frac{p}{q}$  and  $\frac{r}{s}$  are non-zero, it leads to a contradiction. Thus without loss of generality we can assume  $\varphi(\{e_G\} \times H)$  is a trivial group  $\{0\}$ . Since  $\varphi$  is isomorphism, we see that for all  $h \in H$ ,

$$\varphi(e_G, h) = \varphi(e_G, e_H) = 0 \iff h = e_H.$$

That is,  $H$  is a trivial group. Therefore, we have shown  $(\mathbb{Q}, +)$  will never be isomorphic to the direct product of two nontrivial groups. ■

**3.6** Consider the product of the cyclic groups  $C_2, C_3$  (cf. §2.3):  $C_2 \times C_3$ . By Exercise 3.3, this group is a coproduct of  $C_2$  and  $C_3$  in **Ab**. Show that it is not a coproduct of  $C_2$  and  $C_3$  in **Grp**, as follows:

- find injective homomorphisms  $C_2 \rightarrow S_3, C_3 \rightarrow S_3$ ;
- arguing by contradiction, assume that  $C_2 \times C_3$  is a coproduct of  $C_2, C_3$ , and deduce that there would be a group homomorphism  $C_2 \times C_3 \rightarrow S_3$  with certain properties;
- show that there is no such homomorphism.

- Monomorphisms  $g : C_2 \rightarrow S_3, h : C_3 \rightarrow S_3$  can be constructed as follows:

$$g([0]_2) = e, g([1]_2) = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}.$$

$$h([0]_3) = e, h([1]_3) = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}, h([2]_3) = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}.$$

- Supposing that  $C_2 \times C_3$  is a coproduct of  $C_2, C_3$ , there would be a unique group

homomorphism  $\varphi : C_2 \times C_3 \rightarrow S_3$  such that the following diagram commutes

$$\begin{array}{ccc}
 C_2 & & \\
 i_{C_2} \downarrow & \searrow g & \\
 C_2 \times C_3 & \xrightarrow{\varphi} & S_3 \\
 i_{C_3} \uparrow & \nearrow h & \\
 C_3 & & 
 \end{array}$$

In other words, for all  $a \in C_2, b \in C_3$ ,

$$\begin{aligned}
 \varphi(a, b) &= \varphi([0]_2, b) + (a, [0]_3) = \varphi([0]_2, b)\varphi(a, [0]_3) = \varphi(i_{C_3}(b))\varphi(i_{C_2}(a)) = h(b)g(a) \\
 &= \varphi(a, [0]_3) + ([0]_2, b) = \varphi(a, [0]_3)\varphi([0]_2, b) = \varphi(i_{C_2}(a))\varphi(i_{C_3}(b)) = g(a)h(b).
 \end{aligned}$$

• Since

$$\begin{aligned}
 g([1]_2)h([1]_3) &= \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}, \\
 h([1]_3)g([1]_2) &= \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix},
 \end{aligned}$$

we see  $g(a)h(b) \neq h(b)g(a)$  not always holds. The derived contradiction shows that  $C_2 \times C_3$  is not a coproduct of  $C_2, C_3$  in **Grp**. ■

**3.7** Show that there is a surjective homomorphism  $Z * Z \rightarrow C_2 * C_3$ . ( $*$  denotes coproduct in **Grp**.)

Consider the mapping

$$\begin{aligned}
 \varphi : \mathbb{Z} * \mathbb{Z} &\longrightarrow C_2 * C_3 \\
 x^{m_1}y^{n_1} \dots x^{m_k}y^{n_k} &\longmapsto x^{[m_1]_2}y^{[n_1]_3} \dots x^{[m_k]_2}y^{[n_k]_3}
 \end{aligned}$$

Since

$$\begin{aligned}
 &\varphi(x^{m_1}y^{n_1} \dots x^{m_k}y^{n_k}x^{m'_1}y^{n'_1} \dots x^{m'_{k'}}y^{n'_{k'}}) \\
 &= x^{[m_1]_2}y^{[n_1]_3} \dots x^{[m_k]_2}y^{[n_k]_3}x^{[m'_1]_2}y^{[n'_1]_3} \dots x^{[m'_{k'}]_2}y^{[n'_{k'}]_3}, \\
 &= \varphi(x^{m_1}y^{n_1} \dots x^{m_k}y^{n_k})\varphi(x^{m'_1}y^{n'_1} \dots x^{m'_{k'}}y^{n'_{k'}})
 \end{aligned}$$

$\varphi$  is a homomorphism. It is clear that  $\varphi$  is surjective. Thus we show there exists a surjective homomorphism  $Z * Z \rightarrow C_2 * C_3$ . ■

**3.8** Define a group  $G$  with two generators  $x, y$ , subject (only) to the relations  $x^2 = e_G, y^3 = e_G$ . Prove that  $G$  is a coproduct of  $C_2$  and  $C_3$  in **Grp**. (The reader will obtain an even more concrete description for  $C_2 * C_3$  in Exercise 9.14; it is called the modular group.) [§3.4, 9.14]

Given the maps  $i_1 : C_2 \rightarrow G, [m]_2 \mapsto x^m$  and  $i_2 : C_3 \rightarrow G, [n]_3 \mapsto y^n$ , we can check that  $i_1, i_2$  are homomorphisms. We are to show that for every group  $H$  endowed with two homomorphisms  $f_1 : C_2 \rightarrow H, f_2 : C_3 \rightarrow H$ , there would be a unique group homomorphism  $\varphi : G \rightarrow H$  such that the following diagram commutes

$$\begin{array}{ccc} C_2 & & \\ i_1 \downarrow & \searrow f_1 & \\ G & \xrightarrow{\varphi} & H \\ i_2 \uparrow & \nearrow f_2 & \\ C_3 & & \end{array}$$

or

$$\varphi(i_1([m]_2)) = \varphi(x^m) = \varphi(x)^m = f_1([m]_2),$$

$$\varphi(i_2([n]_3)) = \varphi(y^n) = \varphi(y)^n = f_2([n]_3).$$

Define  $\phi : G \rightarrow H$  as  $\phi(x^m y^n) = f_1([m]_2) f_2([n]_3)$ ,  $\phi(y^n x^m) = f_2([n]_3) f_1([m]_2)$ . It is clear to see  $\phi$  makes the diagram commute. Moreover, if  $\varphi$  makes the diagram commute, it follows that for all  $x^m y^n, y^n x^m \in G$ ,

$$\varphi(x^m y^n) = \varphi(x^m) \varphi(y^n) = f_1([m]_2) f_2([n]_3),$$

$$\varphi(y^n x^m) = \varphi(y^n) \varphi(x^m) = f_2([n]_3) f_1([m]_2),$$

which implies  $\varphi = \phi$ . Thus we can conclude  $G$  is the coproduct of  $C_2$  and  $C_3$  in  $\mathbf{Grp}$ . ■

## §4. Group homomorphisms

**4.1** Check that the function  $\pi_m^n$  defined in §4.1 is well-defined, and makes the diagram commute. Verify that it is a group homomorphism. Why is the hypothesis  $m|n$  necessary? [§4.1]

In §4.1 the function  $\pi_m^n$  is defined as

$$\begin{aligned} \pi_m^n : \mathbb{Z}/n\mathbb{Z} &\longrightarrow \mathbb{Z}/m\mathbb{Z} \\ [a]_n &\longmapsto [a]_m \end{aligned}$$

with the condition  $m|n$ . We can check that  $\pi_m^n$  is well-defined as

$$[a_1]_n = [a_2]_n \iff a_1 - a_2 = kn = (kl)m \implies [a_1]_m = [a_2]_m \iff \pi_m^n([a_1]_n) = \pi_m^n([a_2]_n).$$

Note  $\pi_m^n(\pi_n(a)) = \pi_m^n([a]_n) = [a]_m = \pi_m(a)$ . The diagram in §4.1 must commute.

$$\begin{array}{ccc} \mathbb{Z} & & \\ \pi_n \downarrow & \searrow \pi_m & \\ \mathbb{Z}/n\mathbb{Z} & \xrightarrow{\pi_m^n} & \mathbb{Z}/m\mathbb{Z} \end{array}$$

Since

$$\pi_m^n([a]_n + [b]_n) = [a + b]_m = [a]_m + [b]_m = \pi_m^n([a]_n) + \pi_m^n([b]_n),$$

it follows that  $\pi_m^n$  is a group homomorphism. Actually we have shown that without the hypothesis  $m|n$ ,  $\pi_m^n$  may not be well-defined. ■

**4.2** Show that the homomorphism  $\pi_2^4 \times \pi_2^4 : C_4 \rightarrow C_2 \times C_2$  is not an isomorphism. In fact, is there any nontrivial isomorphism  $C_4 \rightarrow C_2 \times C_2$ ?

Let calculate the order of each non-zero element in both  $C_4$  and  $C_2 \times C_2$ . For the group  $C_4$ ,

$$|[2]_4| = 2, \quad |[1]_4| = |[3]_4| = 4.$$

For the group  $C_2 \times C_2$ ,

$$|([1]_2, [0]_2)| = |([0]_2, [1]_2)| = |([1]_2, [1]_2)| = 2.$$

Since isomorphism must preserve the order, we can assert that there is no such isomorphism  $C_4 \rightarrow C_2 \times C_2$ . ■

**4.3** Prove that a group of order  $n$  is isomorphic to  $\mathbb{Z}/n\mathbb{Z}$  if and only if it contains an element of order  $n$ . [§4.3]

Assume some group  $G$  is isomorphic to  $\mathbb{Z}/n\mathbb{Z}$ . Since  $|[1]_n| = n$  and isomorphism preserves the order, we can affirm that there is an element of order  $n$  in  $G$ .

Conversely, assume there is a group  $G$  of order  $n$  in which  $g$  is an element of order  $n$ . By definition we see  $g^0, g^1, g^2 \dots g^{n-1}$  are distinct pairwise. Noticing group  $G$  has exactly  $n$  elements,  $G$  must consist of  $g^0, g^1, g^2 \dots g^{n-1}$ . We can easily check that the function

$$\begin{aligned} f : G &\longrightarrow \mathbb{Z}/n\mathbb{Z} \\ g^k &\longmapsto [k]_n \end{aligned}$$

is an isomorphism. ■

**4.4** Prove that no two of the groups  $(\mathbb{Z}, +)$ ,  $(\mathbb{Q}, +)$ ,  $(\mathbb{R}, +)$  are isomorphic to one another. Can you decide whether  $(\mathbb{R}, +)$ ,  $(\mathbb{C}, +)$  are isomorphic to one another? (Cf. Exercise VI.1.1.)

Suppose there exists an isomorphism  $f : \mathbb{Z} \rightarrow \mathbb{Q}$ . Let  $f(1) = p/q$  ( $p, q \in \mathbb{Z}$ ). If  $p = 1$ , for all  $n \in \mathbb{Z}$ , we have

$$f(n) = \frac{n}{q} \neq \frac{1}{2q}.$$

If  $p \neq 1$ , for all  $n \in \mathbb{Z}$ , we have

$$f(n) = \frac{np}{q} \neq \frac{p+1}{q}.$$

In both cases, it implies  $f(\mathbb{Z}) \not\subseteq \mathbb{Q}$ . Hence we see  $f$  is not a surjection, which contradicts the fact that  $f : \mathbb{Z} \rightarrow \mathbb{Q}$  is an isomorphism. Compare the cardinality of  $\mathbb{Z}$ ,  $\mathbb{Q}$ ,  $\mathbb{R}$

$$|\mathbb{Z}| = |\mathbb{Q}| < |\mathbb{R}|$$

and we show there exists no such isomorphisms like  $f : \mathbb{Z} \rightarrow \mathbb{R}$  or  $f : \mathbb{Q} \rightarrow \mathbb{R}$ .

We can prove  $(\mathbb{R}, +)$ ,  $(\mathbb{C}, +)$  are isomorphic, if considering the both as vector spaces over  $\mathbb{Q}$ . ■

**4.5** Prove that the groups  $(\mathbb{R} \setminus \{0\}, \cdot)$  and  $(\mathbb{C} \setminus \{0\}, \cdot)$  are not isomorphic.

Suppose  $f : \mathbb{R} \rightarrow \mathbb{C}$  is an isomorphism. Then there exists a real number  $x$  such that  $f(x) = i$ .

$$f(x^4) = f(x)^4 = i^4 = 1.$$

Since isomorphism preserves the identity, we have

$$f(1) = 1 = f(x^4).$$

which indicates  $x^4 = 1$ . Noticing that  $x \in \mathbb{R}$ , there must be  $x^2 = 1$ . Now we see

$$f(1) = f(x^2) = f(x)^2 = i^2 = -1,$$

which derives a contradiction. Thus we can conclude that groups  $(\mathbb{R} \setminus \{0\}, \cdot)$  and  $(\mathbb{C} \setminus \{0\}, \cdot)$  are not isomorphic. ■

**4.6** We have seen that  $(\mathbb{R}, +)$  and  $(\mathbb{R}_{>0}, \cdot)$  are isomorphic (Example 4.4). Are the groups  $(\mathbb{Q}, +)$  and  $(\mathbb{Q}_{>0}, \cdot)$  isomorphic?

Suppose  $f : \mathbb{Q} \rightarrow \mathbb{Q}_{>0}$  is an isomorphism. Since isomorphism preserves the multiplication, we have

$$f(1) = f\left(n \cdot \frac{1}{n}\right) = f\left(\frac{1}{n}\right)^n \quad (n \in \mathbb{Z}_{>0}),$$

which implies

$$f\left(\frac{1}{n}\right) = f(1)^{\frac{1}{n}}.$$

Assume

$$f(1) = \frac{p}{q} = \frac{p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k}}{q_1^{s_1} q_2^{s_2} \cdots q_l^{s_l}}$$

where  $p_i, q_i$  are pairwise distinct positive prime numbers. Then let

$$M = \max\{p, q\} + 1 > \max\{r_1, \dots, r_k, s_1, \dots, s_l\}.$$

Thus we assert

$$f\left(\frac{1}{M}\right) = \left(\frac{p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k}}{q_1^{s_1} q_2^{s_2} \cdots q_l^{s_l}}\right)^{\frac{1}{M}} \notin \mathbb{Q},$$

which can be proved by contradiction. In fact, Suppose

$$\left(\frac{p}{q}\right)^{\frac{1}{M}} = \frac{a}{b} \in \mathbb{Q}$$

or say

$$pb^M = qa^M,$$

where  $a, b$  are coprime. Note that  $b^M, a^M$  are also coprime and that the prime factorization of  $a^M$  can be written as  $a_1^{Mt_1} a_2^{Mt_2} \cdots a_j^{Mt_j}$  where  $a_i$  are pairwise distinct positive prime numbers. That forces

$$p = p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k} = N \cdot a_1^{Mt_1} a_2^{Mt_2} \cdots a_j^{Mt_j}.$$

Noticing that  $a_i$  must coincide with one number in  $\{p_1, p_2, \dots, p_k\}$ , we can assume  $a_1 = p_1$  without loss of generality. However, since  $M > \max\{r_1, \dots, r_k\}$ , we see the exponent of  $p_1$  is distinct from that of  $a_1$ , which violates the unique factorization property of  $\mathbb{Z}$ . Hence we get a contradiction and verify  $f\left(\frac{1}{M}\right) \notin \mathbb{Q}$ . Moreover, it contradicts our assumption that  $f : \mathbb{Q} \rightarrow \mathbb{Q}_{>0}$  is an isomorphism. Eventually we show that the groups  $(\mathbb{Q}, +)$  and  $(\mathbb{Q}_{>0}, \cdot)$  are not isomorphic. ■

**4.7** Let  $G$  be a group. Prove that the function  $G \rightarrow G$  defined by  $g \mapsto g^{-1}$  is a homomorphism if and only if  $G$  is abelian. Prove that  $g \mapsto g^2$  is a homomorphism if and only if  $G$  is abelian.

Given the function

$$\begin{aligned} f : G &\longrightarrow G \\ g &\longmapsto g^{-1} \end{aligned}$$

we have

$$f(g_1 g_2) = (g_1 g_2)^{-1} = g_2^{-1} g_1^{-1}, \quad f(g_1) f(g_2) = g_1^{-1} g_2^{-1}.$$

If  $G$  is abelian, it is clear to see  $f(g_1 g_2) = f(g_1) f(g_2)$ . If  $f$  is a homomorphism,  $\forall h_1, h_2 \in G$ ,

$$h_1 h_2 = (h_2^{-1} h_1^{-1})^{-1} = f(h_2^{-1} h_1^{-1}) = f(h_2^{-1}) f(h_1^{-1}) = h_2 h_1.$$



Given the function

$$\begin{aligned} h : G &\longrightarrow G \\ g &\longmapsto g^2 \end{aligned}$$

we have

$$h(g_1g_2) = (g_1g_2)^2 = g_1g_2g_1g_2, \quad h(g_1)h(g_2) = g_1^2g_2^2 = g_1g_1g_2g_2.$$

If  $G$  is abelian, it is clear to see  $h(g_1g_2) = h(g_1)h(g_2)$ . If  $h$  is a homomorphism, by cancellation we have

$$h(g_1g_2) = h(g_1)h(g_2) \implies g_2g_1 = g_1g_2.$$

■

**4.8** Let  $G$  be a group, and  $g \in G$ . Prove that the function  $\gamma_g : G \rightarrow G$  defined by  $(\forall a \in G) : \gamma_g(a) = gag^{-1}$  is an automorphism of  $G$ . (The automorphisms  $\gamma_g$  are called ‘inner’ automorphisms of  $G$ .) Prove that the function  $G \rightarrow \text{Aut}(G)$  defined by  $g \mapsto \gamma_g$  is a homomorphism. Prove that this homomorphism is trivial if and only if  $G$  is abelian.

Since

$$\gamma_g(ab) = gabg^{-1} = gag^{-1}gbg^{-1} = \gamma_g(a)\gamma_g(b),$$

$\gamma_g$  is an automorphism of  $G$ . For all  $a \in G$ , we have

$$\gamma_{g_1g_2}(a) = g_1g_2ag_2^{-1}g_1^{-1} = \gamma_{g_1}(g_2ag_2^{-1}) = (\gamma_{g_1} \circ \gamma_{g_2})(a),$$

which implies  $\gamma_{g_1g_2} = \gamma_{g_1} \circ \gamma_{g_2}$  and  $g \mapsto \gamma_g$  is a homomorphism. If  $G$  is abelian, for all  $g$  the homomorphism

$$\gamma_g(a) = gag^{-1} = gg^{-1}a = a$$

is the identity in  $\text{Aut}(G)$ . That is, the homomorphism  $g \mapsto \gamma_g$  is trivial. If the homomorphism  $g \mapsto \gamma_g$  is trivial, we have for all  $g, a \in G$ ,

$$gag^{-1} = a,$$

which implies for all  $a, b \in G$ ,

$$ab = bab^{-1}b = ba.$$

Thus we show the homomorphism  $g \mapsto \gamma_g$  is trivial if and only if  $G$  is abelian. ■

**4.9** Prove that if  $m, n$  are positive integers such that  $\gcd(m, n) = 1$ , then  $C_{mn} \cong C_m \times C_n$ .

Define a function

$$\begin{aligned} \varphi : C_m \times C_n &\longrightarrow C_{mn} \\ ([a]_m, [b]_n) &\longmapsto [anp + bmq]_{mn} \end{aligned}$$

where  $[pn]_m = [1]_m$  and  $[qm]_n = [1]_n$ , as  $\gcd(m, n) = 1$  guarantees the existence of  $p, q$  (see textbook p56). First of all, we have to check whether  $\varphi$  is well-defined. Note that

$$[(anp_1 + bmq_1) - (anp_2 + bmq_2)]_m = [a(p_1n - p_2n) + b(q_1m - q_2m)]_m = [0]_m$$

$$[(anp_1 + bm q_1) - (anp_2 + bm p_2)]_n = [a(p_1n - p_2n) + b(q_1m - q_2m)]_n = [0]_n$$

and  $\gcd(m, n) = 1$ . Thus we have

$$[(anp_1 + bm q_1) - (anp_2 + bm p_2)]_{mn} = [0]_{mn},$$

or

$$[anp_1 + bm q_1]_{mn} = [anp_2 + bm p_2]_{mn}.$$

Then we show  $\varphi$  is a homomorphism.

$$\begin{aligned} \varphi([a_1]_m, [b_1]_n) + ([a_2]_m, [b_2]_n) &= \varphi([a_1 + a_2]_m, [b_1 + b_2]_n) \\ &= [(a_1 + a_2)np + (b_1 + b_2)m q]_{mn} \\ &= [a_1np + b_1m q]_{mn} + [a_2np + b_2m q]_{mn} \\ &= \varphi([a_1]_m, [b_1]_n) + \varphi([a_2]_m, [b_2]_n). \end{aligned}$$

In order to show  $\varphi$  is a monomorphism, we can check

$$\begin{aligned} \varphi([a_1]_m, [b_1]_n) &= \varphi([a_2]_m, [b_2]_n) \\ \implies [a_1np + b_1m q]_{mn} &= [a_2np + b_2m q]_{mn} \\ \implies [(a_1 - a_2)np + (b_1 - b_2)m q]_{mn} &= [0]_{mn} \\ \implies [(a_1 - a_2)np + (b_1 - b_2)m q]_m &= [a_1 - a_2]_m = [0]_m, \\ [(a_1 - a_2)np + (b_1 - b_2)m q]_n &= [b_1 - b_2]_n = [0]_n \\ \implies [a_1]_m &= [a_2]_m, [b_1]_n = [b_2]_n. \end{aligned}$$

Since  $|C_m \times C_n| = |C_{mn}| = mn$ , we can conclude  $\varphi$  is an isomorphism. Thus we complete proving  $C_{mn} \cong C_m \times C_n$ . ■

## §5. Free groups

**5.1** Does the category  $\mathcal{F}^A$  defined in §5.2 have final objects? If so, what are they?

Yes, they are functions from  $A$  to any trivial group, for example  $T = \{t\}$ .

$$\begin{array}{ccc} G & \xrightarrow{\exists! \varphi} & \{t\} \\ j \uparrow & \nearrow e & \\ A & & \end{array}$$

For any object  $(j, G)$  in  $\mathcal{F}^A$ , the trivial homomorphism  $\varphi : g \mapsto t$  is the unique homomorphism such that the diagram commutes. That is,  $\text{Hom}((j, G), (e, T)) = \{\varphi\}$ . ■

**5.2** Since trivial groups  $T$  are initial in  $\mathbf{Grp}$ , one may be led to think that  $(e, T)$  should be initial in  $\mathcal{F}^A$ , for every  $A$ :  $e$  would be defined by sending every element of  $A$  to the (only) element in  $T$ ; and for any other group  $G$ , there is a unique homomorphism  $T \rightarrow G$ . Explain why  $(e, T)$  is not initial in  $\mathcal{F}^A$  (unless  $A = \emptyset$ ).

Let  $G = C_2 = \{[0]_2, [1]_2\}$ . Note that  $\varphi \circ e(A)$  must be the trivial subgroup  $\{[0]_2\}$ . If  $x \in A$  and  $j(x) = [1]_2$ , we see  $\varphi \circ e \neq j$  and the following diagram does not commute.

$$\begin{array}{ccc} T & \xrightarrow{\varphi} & G \\ e \uparrow & \nearrow j & \\ A & & \end{array}$$

That implies  $(e, T)$  is not initial in  $\mathcal{F}^A$  unless  $A = \emptyset$ . ■

**5.3** Use the universal property of free groups to prove that the map  $j : A \rightarrow F(A)$  is injective, for all sets  $A$ . (Hint: it suffices to show that for every two elements  $a, b$  of  $A$  there is a group  $G$  and a set-function  $f : A \rightarrow G$  such that  $f(a) \neq f(b)$ . Why? and how do you construct  $f$  and  $G$ ?) [§III.6.3]

Let  $G = S_A$  be the symmetric group over  $A$ . Define functions  $g_a : A \rightarrow A$ ,  $x \mapsto a$  sending every element of  $A$  to  $a$ . Since  $g_a \in S_A$ , we can define an injection

$$\begin{aligned} f : A &\longrightarrow S_A \\ a &\longmapsto g_a \end{aligned}$$

In light of the commutative diagram

$$\begin{array}{ccc} F(A) & \xrightarrow{\varphi} & S_A \\ j \uparrow & \nearrow f & \\ A & & \end{array}$$

we have  $\forall a, b \in A$ ,

$$j(a) = j(b) \implies \varphi(j(a)) = \varphi(j(b)) \implies f(a) = f(b) \implies a = b.$$

■

**5.4** In the ‘concrete construction of free groups, one can try to reduce words by performing cancellations in any order; the ‘elementary reductions’ used in the text(that is, from left to right) is only one possibility. Prove that the result of iterating cancellations on a word is independent of the order in which the cancellations are performed. Deduce the associativity of the product in  $F(A)$  from this. [§5.3]

We use induction on the length of  $w$ . If  $w$  is reduced, there is nothing to show. If not, there must be some pair of symbols that can be cancelled, say the underlined pair

$$w = \cdots \underline{xx}^{-1} \cdots$$

(Let's allow  $x$  to denote any element of  $A'$ , with the understanding that if  $x = a^{-1}$  then  $x^{-1} = a$ .) If we show that we can obtain every reduced form of  $w$  by cancelling the pair  $xx^{-1}$  first, the proposition will follow by induction, because the word  $w^* = \cdots \cancel{xx}^{-1} \cdots$  is shorter.

Let  $w_0$  be a reduced form of  $w$ . It is obtained from  $w$  by some sequence of cancellations. The first case is that our pair  $xx^{-1}$  is cancelled at some step in this sequence. If so, we may as well cancel  $xx^{-1}$  first. So this case is settled. On the other hand, since  $w_0$  is reduced, the pair  $xx^{-1}$  can not remain in  $w_0$ . At least one of the two symbols must be cancelled at some time. If the pair itself is not cancelled, the first cancellation involving the pair must look like

$$\cdots x^{-1} \underline{xx}^{-1} \cdots \quad \text{or} \quad \cdots \underline{xx}^{-1} x \cdots$$

Notice that the word obtained by this cancellation is the same as the one obtained by cancelling the pair  $xx^{-1}$ . So at this stage we may cancel the original pair instead. Then we are back in the first case, so the proposition is proved. ■

**5.5** Verify explicitly that  $H^{\oplus A}$  is a group.

Assume the  $A$  is a set and  $H$  is an abelian group.  $H^{\oplus A}$  are defined as follows

$$H^{\oplus A} := \{\alpha : A \rightarrow H \mid \alpha(a) \neq e_H \text{ for only finitely many elements } a \in A\}.$$

Now that  $H^{\oplus A} \subset H^A := \text{Hom}_{\text{Set}}(A, H)$ , we can first show  $(H^A, +)$  is a group, where for all  $\phi, \psi \in H^A$ ,  $\phi + \psi$  is defined by

$$(\forall a \in A) : (\phi + \psi)(a) := \phi(a) + \psi(a).$$

Here is the verification:

- Identity: Define a function  $\varepsilon : A \rightarrow H, a \mapsto e_H$  sending all elements in  $A$  to  $e_H$ . Then for any  $\alpha \in H^A$  we have

$$(\forall a \in A) : (\alpha + \varepsilon)(a) = \alpha(a) + \varepsilon(a) = \alpha(a),$$

which is  $\alpha + \varepsilon = \alpha$ . Because of the commutativity of the operation  $+$  defined on  $H^A$ ,  $\varepsilon$  is the identity indeed.

- Associativity: This follows by the associativity in  $H$ :

$$(\forall a \in A) : ((\alpha + \beta) + \gamma)(a) = (\alpha + \beta)(a) + \gamma(a) = \alpha(a) + (\beta + \gamma)(a) = (\alpha + (\beta + \gamma))(a).$$

- Inverse: Every function  $\phi \in H^A$  has inverse  $-\phi$  defined by

$$(\forall a \in A) : (-\phi)(a) = -\phi(a).$$

Thus  $H^A$  makes a group.

Then it is time to show  $H^{\oplus A}$  is a subgroup of  $H^A$ . For all  $\alpha, \beta \in H^{\oplus A}$ , let  $N_\alpha = \{a \in A \mid \alpha(a) \neq e_H\}$ ,  $N_\beta = \{a \in A \mid \beta(a) \neq e_H\}$ ,  $N_{\alpha-\beta} = \{a \in A \mid (\alpha - \beta)(a) \neq e_H\}$ . Since

$$(\forall a \in A) : (\alpha - \beta)(a) = \alpha(a) - \beta(a),$$

we have

$$(\alpha - \beta)(a) \neq e_H \implies \alpha(a) \neq e_H \text{ or } \beta(a) \neq e_H,$$

which implies  $N_{\alpha-\beta} \subset N_\alpha \cup N_\beta$ . Note that  $N_\alpha, N_\beta$  are both finite sets, which forces  $N_{\alpha-\beta}$  to be finite. So there must be  $\alpha - \beta \in H^{\oplus A}$ . Now we see  $H^{\oplus A}$  is closed under additions and inverses. And  $e_{H^A} = \varepsilon \in H^{\oplus A}$  means that  $H^{\oplus A}$  is nonempty. Finally we can conclude  $H^{\oplus A}$  is a subgroup of  $H^A$ . ■

**5.6** Prove that the group  $F(\{x, y\})$  (visualized in Example 5.3) is a coproduct  $\mathbb{Z} * \mathbb{Z}$  of  $\mathbb{Z}$  by itself in the category **Grp**. (Hint: with due care, the universal property for one turns into the universal property for the other.) [§3.4, 3.7, 5.7]

Define two homomorphisms

$$\begin{aligned} i_1 : \mathbb{Z} &\longrightarrow F(\{x, y\}), & n &\longmapsto x^n, \\ i_2 : \mathbb{Z} &\longrightarrow F(\{x, y\}), & n &\longmapsto y^n. \end{aligned}$$

We need to show that for any group  $G$  with two homomorphisms  $f_1, f_2 : \mathbb{Z} \rightarrow G$ , there exists a unique homomorphism  $\varphi$  such that the following diagram commutes.

$$\begin{array}{ccc} \mathbb{Z} & & \\ \downarrow i_1 & \searrow f_1 & \\ F(\{x, y\}) & \xrightarrow{\varphi} & G \\ \uparrow i_2 & \nearrow f_2 & \\ \mathbb{Z} & & \end{array}$$

Given the notation of indicator function

$$\mathbf{1}_A(x) := \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A, \end{cases}$$

we can define a function

$$\begin{aligned}\varphi : F(\{x, y\}) &\longrightarrow G, \\ z_1^{n_1} \cdots z_k^{n_k} &\longmapsto f_1(n_1)^{\mathbf{1}_{\{x\}}(z_1)} f_2(n_1)^{\mathbf{1}_{\{y\}}(z_1)} \cdots f_1(n_k)^{\mathbf{1}_{\{x\}}(z_k)} f_2(n_k)^{\mathbf{1}_{\{y\}}(z_k)}, \quad z_i \in \{x, y\}\end{aligned}$$

and check that it is a homomorphism indeed. For all  $n \in \mathbb{Z}$ , we have

$$\begin{aligned}(\varphi \circ i_1)(n) &= \varphi(x^n) = f_1(n), \\ (\varphi \circ i_2)(n) &= \varphi(y^n) = f_2(n),\end{aligned}$$

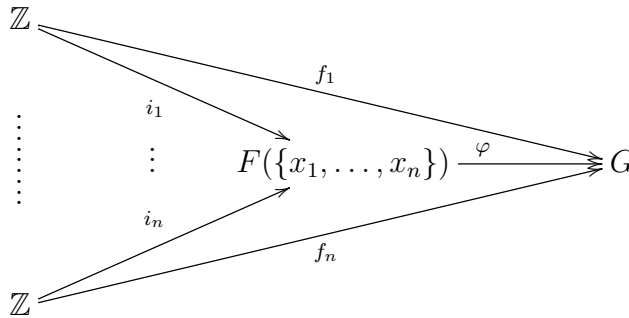
that is, the diagram commutes. Now we see  $\varphi$  exists. For the uniqueness of  $\varphi$ , let  $\varphi^*$  be another homomorphism that makes diagram commute. For all  $z_1^{n_1} \cdots z_k^{n_k} \in F(\{x, y\})$ ,  $z_i \in \{x, y\}$ , we have

$$\begin{aligned}\varphi^*(z_1^{n_1} \cdots z_k^{n_k}) &= \varphi^*(z_1^{n_1}) \cdots \varphi^*(z_k^{n_k}) \\ &= \varphi^*(i_1(n_1))^{\mathbf{1}_{\{x\}}(z_1)} \varphi^*(i_2(n_1))^{\mathbf{1}_{\{y\}}(z_1)} \cdots \varphi^*(i_1(n_k))^{\mathbf{1}_{\{x\}}(z_k)} \varphi^*(i_2(n_k))^{\mathbf{1}_{\{y\}}(z_k)} \\ &= f_1(n_1)^{\mathbf{1}_{\{x\}}(z_1)} f_2(n_1)^{\mathbf{1}_{\{y\}}(z_1)} \cdots f_1(n_k)^{\mathbf{1}_{\{x\}}(z_k)} f_2(n_k)^{\mathbf{1}_{\{y\}}(z_k)} \\ &= \varphi(z_1^{n_1} \cdots z_k^{n_k}).\end{aligned}$$

To sum up, we have shown that the group  $F(\{x, y\})$  is a coproduct  $\mathbb{Z} * \mathbb{Z}$  of  $\mathbb{Z}$  by itself in the category **Grp**. ■

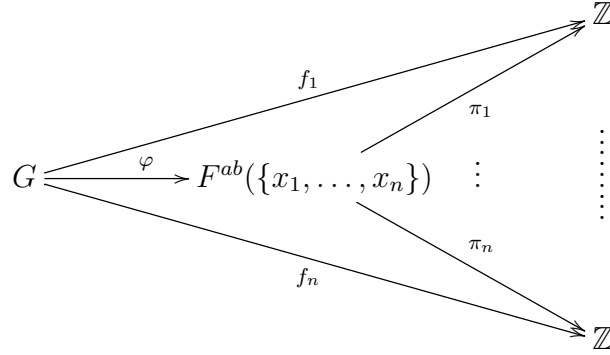
**5.7** Extend the result of Exercise 5.6 to free groups  $F(\{x_1, \dots, x_n\})$  and to free abelian groups  $F^{ab}(\{x_1, \dots, x_n\})$ . [3.4, 5.4]

Let  $*$  be coproduct. Then we have  $\underbrace{\mathbb{Z} * \mathbb{Z} * \cdots * \mathbb{Z}}_{n \text{ times}} \cong F(\{x_1, \dots, x_n\})$ , as the following diagram demonstrates:



Dually, let  $\times$  be product. Then we have  $\underbrace{\mathbb{Z} \times \mathbb{Z} \times \cdots \times \mathbb{Z}}_{n \text{ times}} \cong F^{ab}(\{x_1, \dots, x_n\})$ , as the fol-

following diagram demonstrates:



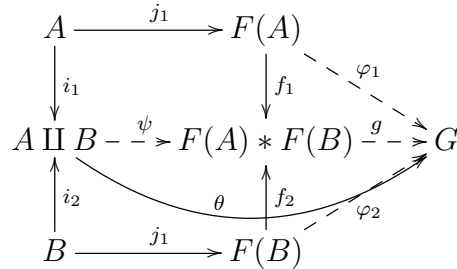
■

**5.8** Still more generally, prove that  $F(A \amalg B) = F(A) * F(B)$  and that  $F^{ab}(A \amalg B) = F^{ab}(A) \oplus F^{ab}(B)$  for all sets  $A, B$ . (That is, the constructions  $F, F^{ab}$  'preserve coproducts'.)

In order to show  $F(A) * F(B)$  is a free group generated by  $A \amalg B$ , we should first set an appropriate function  $\psi : A \amalg B \rightarrow F(A) * F(B)$  and then prove that given any  $(\theta, G)$  there exists a unique group homomorphism  $g$  such that the following diagram commutes.

$$\begin{array}{ccccc}
 A \amalg B & \xrightarrow{\psi} & F(A) * F(B) & \xrightarrow{\exists! g} & G \\
 & \searrow \theta & & & \nearrow
 \end{array}$$

The complete proof can be divided into three steps, by decomposing the following diagram into parts.



**Step 1. Construct  $\psi : A \amalg B \rightarrow F(A) * F(B)$ .**

Define injective functions

$$\begin{aligned}
 i_1 : A &\rightarrow A \amalg B, & a &\mapsto (a, 1), \\
 i_2 : B &\rightarrow A \amalg B, & b &\mapsto (b, 2), \\
 j_1 : A &\rightarrow F(A), & a &\mapsto a, \\
 j_2 : B &\rightarrow F(B), & b &\mapsto b.
 \end{aligned}$$

Let  $f_1, f_2$  be the homomorphisms specified by the coproduct in **Grp**. Since  $A \amalg B$  is a coproduct in **Set**, the universal property guarantees a unique mapping  $\psi : A \amalg B \rightarrow F(A) *$

$F(B)$  such that the following diagram commutes

$$\begin{array}{ccc}
 A & \xrightarrow{j_1} & F(A) \\
 \downarrow i_1 & & \downarrow f_1 \\
 A \amalg B & \xrightarrow{\exists! \psi} & F(A) * F(B) \\
 \uparrow i_2 & & \uparrow f_2 \\
 B & \xrightarrow{j_1} & F(B)
 \end{array}$$

That is,

$$\exists! \psi : A \amalg B \longrightarrow F(A) * F(B) \quad (\psi \circ i_1 = f_1 \circ j_1) \wedge (\psi \circ i_2 = f_2 \circ j_2).$$

**Step 2. Prove the existence of  $g$ .**

$$\begin{array}{ccc}
 A & \xrightarrow{j_1} & F(A) \\
 \downarrow i_1 & & \searrow \exists! \varphi_1 \\
 A \amalg B & \xrightarrow{\theta} & G \\
 \uparrow i_2 & & \nearrow \exists! \varphi_2 \\
 B & \xrightarrow{j_1} & F(B)
 \end{array}$$

Given some  $(\theta, G)$ , according to the universal property of free groups  $F(A)$ ,  $F(B)$ , we have

$$\begin{aligned}
 \exists! \varphi_1 : F(A) &\longrightarrow G & (\varphi_1 \circ j_1 = \theta \circ i_1), \\
 \exists! \varphi_2 : F(B) &\longrightarrow G & (\varphi_2 \circ j_2 = \theta \circ i_2).
 \end{aligned}$$

$$\begin{array}{ccc}
 F(A) & & \\
 \downarrow f_1 & \searrow \varphi_1 & \\
 F(A) * F(B) & \xrightarrow{\exists! g} & G \\
 \uparrow f_2 & \nearrow \varphi_2 & \\
 F(B) & &
 \end{array}$$

Then according to the universal property of coproduct  $F(A) * F(B)$  in  $\mathbf{Grp}$ , we have

$$\exists! g : F(A) * F(B) \longrightarrow G \quad (g \circ f_1 = \varphi_1) \wedge (g \circ f_2 = \varphi_2).$$

The commutative diagram tells us

$$\begin{aligned}
 g \circ \psi \circ i_1 &= g \circ f_1 \circ j_1 = \varphi_1 \circ j_1 = \theta \circ i_1, \\
 g \circ \psi \circ i_2 &= g \circ f_2 \circ j_2 = \varphi_2 \circ j_2 = \theta \circ i_2.
 \end{aligned}$$



Note that  $A \amalg B = i_1(A) \cup i_2(B)$ . For all  $x \in A \amalg B$ ,  $x$  must be either  $i_1(a)$  or  $i_2(b)$ . If  $x = i_1(a)$ , then

$$g \circ \psi(x) = g \circ \psi \circ i_1(a) = \theta \circ i_1(a) = \theta(x).$$

If  $x = i_2(b)$ , then

$$g \circ \psi(x) = g \circ \psi \circ i_2(b) = \theta \circ i_2(b) = \theta(x).$$

Hence we show that given some  $(\theta, G)$  there exists  $g : F(A) * F(B) \longrightarrow G$  such that  $g \circ \psi = \theta$ .

### Step 3. Prove the uniqueness of $g$ .

Assume there exists another homomorphism  $h$  such that  $h \circ \psi = \theta$ . We have

$$h \circ f_1 \circ j_1 = h \circ \psi \circ i_1 = \theta \circ i_1,$$

$$h \circ f_2 \circ j_2 = h \circ \psi \circ i_2 = \theta \circ i_2.$$

Since

$$\exists! \varphi_1 : F(A) \longrightarrow G \quad (\varphi_1 \circ j_1 = \theta \circ i_1),$$

$$\exists! \varphi_2 : F(B) \longrightarrow G \quad (\varphi_2 \circ j_2 = \theta \circ i_2),$$

there must be

$$h \circ f_1 = \varphi_1,$$

$$h \circ f_2 = \varphi_2.$$

Again by universal property

$$\exists! g : F(A) * F(B) \longrightarrow G \quad (g \circ f_1 = \varphi_1) \wedge (g \circ f_2 = \varphi_2)$$

we get  $h = g$ , which implies  $g$  is unique.

### Conclusion.

To sum up, we prove that there exists a unique group homomorphism  $g$  such that the first diagram in this proof commutes. As a result, we have  $F(A \amalg B) = F(A) * F(B)$ . Note that if **Grp** turns into **Ab**, the method of diagram chasing applied here also works. In the light of the following diagram, we can get  $F^{ab}(A \amalg B) = F^{ab}(A) \oplus F^{ab}(B)$  step by step.

$$\begin{array}{ccccc}
 A & \xrightarrow{j_1} & F^{ab}(A) & & \\
 \downarrow i_1 & & \downarrow f_1 & \searrow \varphi_1 & \\
 A \amalg B & \xrightarrow{\psi} & F^{ab}(A) \oplus F^{ab}(B) & \xrightarrow{g} & G \\
 \uparrow i_2 & \searrow \theta & \uparrow f_2 & \nearrow \varphi_2 & \\
 B & \xrightarrow{j_2} & F^{ab}(B) & & 
 \end{array}$$

■

**5.9** Let  $G = \mathbb{Z}^{\oplus \mathbb{N}}$ . Prove that  $G \times G \cong G$ .

Define a function

$$\begin{aligned}\varphi : G \times G &\longrightarrow G \\ ((a_1, a_2, \dots), (b_1, b_2, \dots)) &\longmapsto (a_1, b_1, a_2, b_2, \dots)\end{aligned}$$

It plain to check that  $\varphi$  is a homomorphism

$$\begin{aligned}&\varphi[((a_1, a_2, \dots), (b_1, b_2, \dots)) + ((a'_1, a'_2, \dots), (b'_1, b'_2, \dots))] \\ &= \varphi[((a_1 + a'_1, a_2 + a'_2, \dots), (b_1 + b'_1, b_2 + b'_2, \dots))] \\ &= (a_1 + a'_1, b_1 + b'_1, a_2 + a'_2, b_2 + b'_2, \dots) \\ &= (a_1, b_1, a_2, b_2, \dots) + (a'_1, b'_1, a'_2, b'_2, \dots) \\ &= \varphi[((a_1, a_2, \dots), (b_1, b_2, \dots))] + \varphi[((a'_1, a'_2, \dots), (b'_1, b'_2, \dots))].\end{aligned}$$

Since  $\ker \varphi = \{(0, 0, \dots)\}$  and  $|G \times G| = |G| = \aleph_0$ , we can conclude that  $\varphi$  is an isomorphism and accordingly  $G \times G \cong G$ . ■

## §6. Subgroups

**6.1**  $\neg$  (If you know about matrices.) The group of invertible  $n \times n$  matrices with entries in  $\mathbb{R}$  is denoted  $\mathrm{GL}_n(\mathbb{R})$  (Example 1.5). Similarly,  $\mathrm{GL}_n(\mathbb{C})$  denotes the group of  $n \times n$  invertible matrices with complex entries. Consider the following sets of matrices:

- $\mathrm{SL}_n(\mathbb{R}) = \{M \in \mathrm{GL}_n(\mathbb{R}) \mid \det(M) = 1\}$ ;
- $\mathrm{SL}_n(\mathbb{C}) = \{M \in \mathrm{GL}_n(\mathbb{C}) \mid \det(M) = 1\}$ ;
- $\mathrm{O}_n(\mathbb{R}) = \{M \in \mathrm{GL}_n(\mathbb{R}) \mid MM^t = M^t M = I_n\}$ ;
- $\mathrm{SO}_n(\mathbb{R}) = \{M \in \mathrm{O}_n(\mathbb{R}) \mid \det(M) = 1\}$ ;
- $\mathrm{U}_n(\mathbb{C}) = \{M \in \mathrm{GL}_n(\mathbb{C}) \mid MM^\dagger = M^\dagger M = I_n\}$ ;
- $\mathrm{SU}_n(\mathbb{C}) = \{M \in \mathrm{U}_n(\mathbb{C}) \mid \det(M) = 1\}$ .

Here  $I_n$  stands for the  $n \times n$  identity matrix,  $M^t$  is the transpose of  $M$ ,  $M^\dagger$  is the conjugate transpose of  $M$ , and  $\det(M)$  denotes the determinant of  $M$ . Find all possible inclusions among these sets, and prove that in every case the smaller set is a subgroup of the larger one.

These sets of matrices have compelling geometric interpretations: for example,  $\mathrm{SO}^3(\mathbb{R})$  is the group of rotations in  $\mathbb{R}^3$ . [8.8, 9.1, III.1.4, VI.6.16]

The following diagram commutes, where all arrows are inclusions.

$$\begin{array}{ccc}
\mathrm{GL}_n(\mathbb{R}) & \longrightarrow & \mathrm{GL}_n(\mathbb{C}) \\
\uparrow & & \uparrow \\
\mathrm{SL}_n(\mathbb{R}) & \longrightarrow & \mathrm{SL}_n(\mathbb{C}) \\
\uparrow & & \uparrow \\
\mathrm{O}_n(\mathbb{R}) & \longrightarrow & \mathrm{U}_n(\mathbb{C}) \\
\uparrow & & \uparrow \\
\mathrm{SO}_n(\mathbb{R}) & \longrightarrow & \mathrm{SU}_n(\mathbb{C})
\end{array}$$

■

**6.2**  $\neg$  Prove that the set of  $2 \times 2$  matrices

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$$

with  $a, b, d$  in  $\mathbb{C}$  and  $ad \neq 0$  is a subgroup of  $\mathrm{GL}_2(\mathbb{C})$ . More generally, prove that the set of  $n \times n$  complex matrices  $(a_{ij})_{1 \leq i, j \leq n}$  with  $a_{ij} = 0$  for  $i > j$ , and  $a_{11} \cdots a_{nn} \neq 0$ , is a subgroup of  $\mathrm{GL}_n(\mathbb{C})$ . (These matrices are called 'upper triangular', for evident reasons.) [IV.1.20]

Let  $A, B$  be  $n \times n$  upper triangular matrices. If  $i > j$ ,

$$(AB)_{ij} = \sum_{k=1}^n a_{ik}b_{kj} = \sum_{k=1}^{i-1} a_{ik}b_{kj} + \sum_{k=i}^n a_{ik}b_{kj} = \sum_{k=1}^{i-1} 0b_{kj} + \sum_{k=i}^n a_{ik}0 = 0,$$

which means the set of upper triangular matrices is closed with respect to the matrix multiplication. Thus it is a subgroup of  $\mathrm{GL}_n(\mathbb{C})$ . ■

**6.3**  $\neg$  Prove that every matrix in  $\mathrm{SU}_2(\mathbb{C})$  may be written in the form

$$\begin{pmatrix} a + bi & c + di \\ -c + di & a - bi \end{pmatrix}$$

where  $a, b, c, d \in \mathbb{R}$  and  $a^2 + b^2 + c^2 + d^2 = 1$ . (Thus,  $\mathrm{SU}_2(\mathbb{C})$  may be realized as a three-dimensional sphere embedded in  $\mathbb{R}^4$ ; in particular, it is simply connected.) [8.9, III.2.5]

Let

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in \mathrm{SU}_2(\mathbb{C})$$

and we have

$$AA^\dagger = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} \overline{a_{11}} & \overline{a_{21}} \\ \overline{a_{12}} & \overline{a_{22}} \end{pmatrix} = \begin{pmatrix} |a_{11}|^2 + |a_{12}|^2 & a_{11}\overline{a_{21}} + a_{12}\overline{a_{22}} \\ a_{21}\overline{a_{11}} + a_{22}\overline{a_{12}} & |a_{21}|^2 + |a_{22}|^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21} = 1$$

Note

$$\begin{aligned} \overline{a_{11}a_{12}} &= \overline{a_{11}a_{12}} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = \begin{vmatrix} |a_{11}|^2 & |a_{12}|^2 \\ a_{21}\overline{a_{11}} & a_{22}\overline{a_{12}} \end{vmatrix} = \begin{vmatrix} |a_{11}|^2 & |a_{11}|^2 + |a_{12}|^2 \\ a_{21}\overline{a_{11}} & a_{21}\overline{a_{11}} + a_{22}\overline{a_{12}} \end{vmatrix} = \begin{vmatrix} |a_{11}|^2 & 1 \\ a_{21}\overline{a_{11}} & 0 \end{vmatrix} = -a_{21}\overline{a_{11}} \\ &\implies \overline{a_{11}}(\overline{a_{12}} + a_{21}) = 0, \end{aligned}$$

and

$$\begin{aligned} \overline{a_{21}a_{22}} &= \overline{a_{21}a_{22}} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = \begin{vmatrix} a_{11}\overline{a_{21}} & a_{12}\overline{a_{22}} \\ |a_{21}|^2 & |a_{22}|^2 \end{vmatrix} = \begin{vmatrix} a_{11}\overline{a_{21}} & a_{11}\overline{a_{21}} + a_{12}\overline{a_{22}} \\ |a_{21}|^2 & |a_{21}|^2 + |a_{22}|^2 \end{vmatrix} = \begin{vmatrix} a_{11}\overline{a_{21}} & 0 \\ |a_{21}|^2 & 1 \end{vmatrix} = a_{11}\overline{a_{21}} \\ &\implies \overline{a_{21}}(\overline{a_{11}} - a_{22}) = 0. \end{aligned}$$

If  $\overline{a_{11}} \neq 0$ , it must be  $\overline{a_{12}} + a_{21} = 0$ . If  $\overline{a_{11}} = 0$ , then  $|a_{12}|^2 = 1$ ,  $a_{12}\overline{a_{22}} = 0$  and accordingly  $a_{22} = 0$ . Since  $-a_{12}a_{21} = 1 = a_{12}\overline{a_{12}}$ , we also have  $\overline{a_{12}} + a_{21} = 0$ , that is  $a_{12} = c + di$ ,  $a_{21} = -c + di$ . Likewise, we can show  $\overline{a_{11}} - a_{22} = 0$  and  $a_{11} = a + bi$ ,  $a_{22} = a - bi$ . And we have

$$|a_{11}|^2 + |a_{12}|^2 = a^2 + b^2 + c^2 + d^2 = 1.$$

■

**6.4** Let  $G$  be a group, and  $g \in G$ . Verify that the image of the exponential map  $\epsilon_g : \mathbb{Z} \rightarrow G$  is a cyclic group (in the sense of Definition 4.7).

If  $|g| = \infty$ , then  $g^i \neq g^j$  ( $i \neq j$ ). Define

$$\varphi : \mathbb{Z} \longrightarrow \epsilon_g(\mathbb{Z}), n \longmapsto g^n$$

and we can check it is an isomorphism.

If  $|g| = k$ , then  $e_G, g, g^2, \dots, g^{k-1}$  are distinct. Define

$$\varphi : \mathbb{Z}/k\mathbb{Z} \longrightarrow \epsilon_g(\mathbb{Z}), [n]_k \longmapsto g^n$$

and we can check it is an isomorphism.

Since  $\epsilon_g(\mathbb{Z})$  is isomorphic to  $\mathbb{Z}$  or  $\mathbb{Z}/k\mathbb{Z}$ , we show  $\epsilon_g(\mathbb{Z})$  is a cyclic group.

■

**6.6** Prove that the union of a family of subgroups of a group  $G$  is not necessarily a subgroup of  $G$ . In fact:

- Let  $H, H'$  be subgroups of a group  $G$ . Prove that  $H \cup H'$  is a subgroup of  $G$  only if  $H \subseteq H'$  or  $H' \subseteq H$ .
- On the other hand, let  $H_0 \subseteq H_1 \subseteq H_2 \subseteq \dots$  be subgroups of a group  $G$ . Prove that  $\cup_{i \geq 0} H_i$  is a subgroup of  $G$ .

- Let  $H \cup H'$  be a subgroup of  $G$ . Suppose neither  $H \subseteq H'$  nor  $H' \subseteq H$  hold. Let  $a \in H - H', b \in H' - H, h = ab^{-1} \in H \cup H'$ . In the case of  $h \in H$ , we have  $b = h^{-1}a \in H$ , contradiction! In the case of  $h \in H'$ , we have  $a = hb \in H'$ , contradiction again! Therefore, there must be  $H \subseteq H'$  or  $H' \subseteq H$ .
- For all  $a, b \in \cup_{i \geq 0} H_i$ , we can suppose  $a \in H_j, b \in H_k$  and we have  $a, b \in H_{\max\{j, k\}}$ . Then  $ab \in H_{\max\{j, k\}} \subseteq \cup_{i \geq 0} H_i$ , implies that  $\cup_{i \geq 0} H_i$  is closed and that  $\cup_{i \geq 0} H_i$  is a subgroup of  $G$ .

■

**6.7**  $\neg$  Show that inner automorphisms (cf. Exercise 4.8) form a subgroup of  $\text{Aut}(G)$ ; this subgroup is denoted  $\text{Inn}(G)$ . Prove that  $\text{Inn}(G)$  is cyclic if and only if  $\text{Inn}(G)$  is trivial if and only if  $G$  is abelian. (Hint: Assume that  $\text{Inn}(G)$  is cyclic; with notation as in Exercise 4.8, this means that there exists an element  $a \in G$  such that  $\forall g \in G \exists n \in \mathbb{Z} \gamma_g = \gamma_a^n$ . In particular,  $gag^{-1} = a^n aa^{-n} = a$ . Thus  $a$  commutes with every  $g$  in  $G$ . Therefore...) Deduce that if  $\text{Aut}(G)$  is cyclic then  $G$  is abelian. [7.10, IV.1.5]

With notation as in Exercise 4.8, we assume  $\gamma_g \in \text{Inn}(G)$  is defined by

$$\forall h \in G \quad (\gamma_g(h) = ghg^{-1}).$$

We have

$$\begin{aligned} & \text{Inn}(G) \text{ is cyclic} \\ \iff & \exists \gamma_a \in \text{Inn}(G), \text{Inn}(G) = \langle \gamma_a \rangle \\ \iff & \exists a \in G \forall g \in G \exists n \in \mathbb{Z} (\gamma_g = \gamma_a^n) \\ \implies & \exists a \in G \forall g \in G \exists n \in \mathbb{Z} (\gamma_g(a) = gag^{-1} = \gamma_a^n(a) = a^n aa^{-n} = a) \\ \implies & \exists a \in G \forall g \in G (ga = ag) \\ \implies & \forall h \in G, \gamma_a(h) = aha^{-1} = haa^{-1} = h \\ \implies & \text{Inn}(G) = \langle \text{id} \rangle \\ \implies & \text{Inn}(G) \text{ is trivial} \end{aligned}$$

$$\begin{aligned}
& \text{Inn}(G) \text{ is trivial} \\
\implies & \forall g \in G \forall h \in G (\gamma_g(h) = ghg^{-1} = h) \\
\implies & \forall g \in G \forall h \in G (gh = hg) \\
\iff & G \text{ is abelian}
\end{aligned}$$

$$\begin{aligned}
& G \text{ is abelian} \\
\implies & \forall g \in G \forall h \in G (\gamma_g(h) = ghg^{-1} = h) \\
\implies & \text{Inn}(G) = \{\text{id}\} \\
\implies & \text{Inn}(G) \text{ is cyclic}
\end{aligned}$$

If  $\text{Aut}(G)$  is cyclic, its subgroup  $\text{Inn}(G)$  is also cyclic. As we have shown, that means  $G$  is abelian. ■

**6.8** Prove that an abelian group  $G$  is finitely generated if and only if there is a surjective homomorphism

$$\underbrace{\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}_{n \text{ times}} \twoheadrightarrow G$$

for some  $n$ .

Given any set  $H \subseteq G$ , there exists a unique homomorphism  $\varphi_H$  such that the following diagram commutes.

$$\begin{array}{ccc}
F^{ab}(H) & \xrightarrow{\exists! \varphi} & G \\
j \uparrow & \nearrow i & \\
H & & 
\end{array}$$

The homomorphism image  $\varphi_H(F^{ab}(H)) \leq G$  is called the subgroup generated by  $H$  in  $G$ , denoted by  $\langle H \rangle$ .

If  $G$  is finitely generated, there is a finite subset  $G_n \subseteq G$  with  $n$  elements such that  $\varphi_H(F^{ab}(G_n)) = \varphi_H(\mathbb{Z}^{\oplus n}) = G$ . And  $\varphi_H$  is exactly the surjective homomorphism that we need.

If there is a surjective homomorphism  $\psi : \mathbb{Z}^{\oplus n} \twoheadrightarrow G$  for some  $n$ . Suppose

$$\psi : \mathbf{1}_i = (0, \dots, 0, \underset{i\text{-th place}}{1}, 0, \dots, 0) \mapsto g_i$$

and  $G_n = \{g_1, g_2, \dots, g_n\}$ . Then define

$$j : G_n \longrightarrow \mathbb{Z}^{\oplus n}, \quad g_i \longmapsto \mathbf{1}_i.$$

We can check the following diagram commutes

$$\begin{array}{ccc} \mathbb{Z}^{\oplus n} & \xrightarrow{\psi} & G \\ j \uparrow & \nearrow i & \\ G_n & & \end{array}$$

which means  $\langle G_n \rangle = \psi(\mathbb{Z}^{\oplus n})$ . Since  $\psi$  is surjective, we have  $\langle G_n \rangle = G$ . Hence we show  $G$  is finitely generated. ■

**6.9** Prove that every finitely generated subgroup of  $\mathbb{Q}$  is cyclic. Prove that  $\mathbb{Q}$  is not finitely generated.

Given any two rationals

$$a_1 = \frac{p_1}{q_1} \in \mathbb{Q}, (p_1, q_1) = 1,$$

$$a_2 = \frac{p_2}{q_2} \in \mathbb{Q}, (p_2, q_2) = 1,$$

there exists  $r = \frac{1}{q_1 q_2} \in \mathbb{Q}$  such that  $\langle a_1, a_2 \rangle \leq \langle r_1 \rangle$ . Then for some  $a_3$  we have  $\langle a_1, a_2, a_3 \rangle \leq \langle r_1, a_3 \rangle \leq \langle r_2 \rangle$ . In general, let's set  $B_n = \{a_1, a_2, \dots, a_n\}$ . If  $\langle B_n \rangle \leq \langle r_{n-1} \rangle$ . we have  $\langle B_{n+1} \rangle = \langle B_n, a_{n+1} \rangle \leq \langle r_{n-1} a_{n+1} \rangle \leq \langle r_n \rangle$ . By induction we can prove  $\langle a_1, a_2, \dots, a_n \rangle \leq \langle r_{n-1} \rangle$  for  $n \in \mathbb{N}_+$ . ■

**6.10**  $\neg$  The set of  $2 \times 2$  matrices with integer entries and determinant 1 is denoted  $\text{SL}_2(\mathbb{Z})$ :

$$\text{SL}_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ such that } a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\}.$$

Prove that  $\text{SL}_2(\mathbb{Z})$  is generated by the matrices:

$$s = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad t = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Let  $H$  be the subgroup generated by  $s$  and  $t$ . We can check that

$$P = \begin{pmatrix} 1 & -p \\ 0 & 1 \end{pmatrix} = t^{-p} \quad \text{and} \quad Q = \begin{pmatrix} 1 & 0 \\ -q & 1 \end{pmatrix} = s^{-1} t^q s.$$

are in  $H$ . Given an arbitrary matrix

$$m = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}),$$

it suffices to show that we can obtain the identity  $I_2$  by multiplying  $m$  by matrices in  $H$ . Note that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & -p \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b - pa \\ c & d - pc \end{pmatrix}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -q & 1 \end{pmatrix} = \begin{pmatrix} a - qb & b \\ c - qd & d \end{pmatrix},$$

and  $c, d$  cannot be nonzero simultaneously. Without loss of generality, we can assume that  $0 < c < d$  and perform Euclidean algorithm. Let  $p_1 = \lfloor \frac{d}{c} \rfloor, d_1 = d - p_1c < c$ . Multiplying  $m$  by  $P_1 = \begin{pmatrix} 1 & -p_1 \\ 0 & 1 \end{pmatrix}$  on the right yields

$$m_1 = mP_1 \begin{pmatrix} a & b - p_1a \\ c & d_1 \end{pmatrix}.$$

Then let  $q_1 = \lfloor \frac{c}{d_1} \rfloor, c_1 = c - q_1d_1 < d_1$  and right multiplying  $m$  by  $Q_1 = \begin{pmatrix} 1 & 0 \\ -q_1 & 1 \end{pmatrix}$  yields

$$m_2 = mP_1Q_1 \begin{pmatrix} a - q_1(b - p_1a) & b - p_1a \\ c_1 & d_1 \end{pmatrix}.$$

We can repeat this procedure until some  $d_i$  or  $c_i$  reduce to 0. The Euclidean algorithm generates a sequence

$$d > c > d_1 > c_1 > d_2 > c_2 > \dots.$$

If  $c_i, d_i$  never reduce to 0, we will get an infinite decreasing positive sequence, which is impossible. Suppose  $d_N$  is the first number reducing to 0. Then

$$m_{2N-1} = mP_1Q_1 \cdots P_N = \begin{pmatrix} a_N & b_N \\ c_{N-1} & 0 \end{pmatrix} \in \text{SL}_2(\mathbb{Z}),$$

which implies

$$m_{2N-1} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and  $m_{2N-1}s^{-1} = I_2$ . Suppose  $c_N$  is the first number reducing to 0. Then

$$m_{2N} = mP_1Q_1 \cdots P_NQ_N = \begin{pmatrix} a_N & b_N \\ 0 & d_N \end{pmatrix} \in \text{SL}_2(\mathbb{Z}),$$

which implies

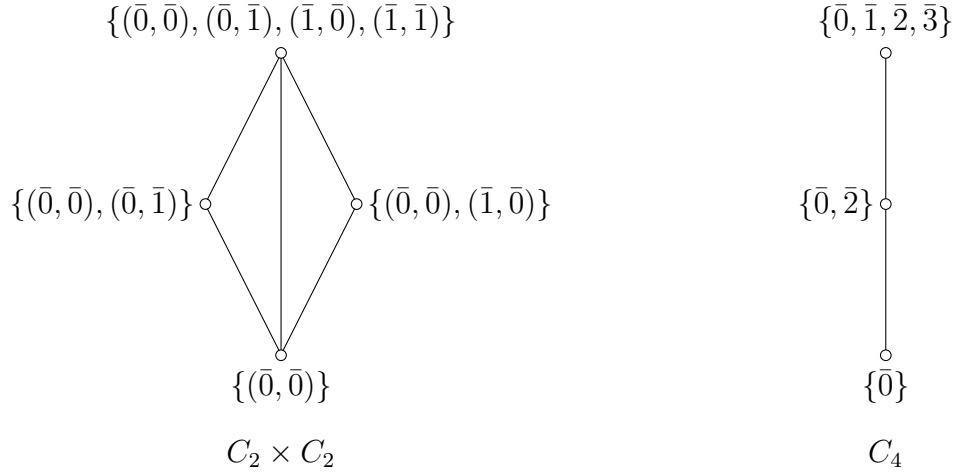
$$m_{2N} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2.$$

We have shown that we can obtain the identity  $I_2$  by multiplying  $m$  by matrices in  $H$ , that is,  $m$  can be represented as a product of matrices in  $H$ . Thus we can conclude  $\text{SL}_2(\mathbb{Z})$  is generated by  $s$  and  $t$ . ■

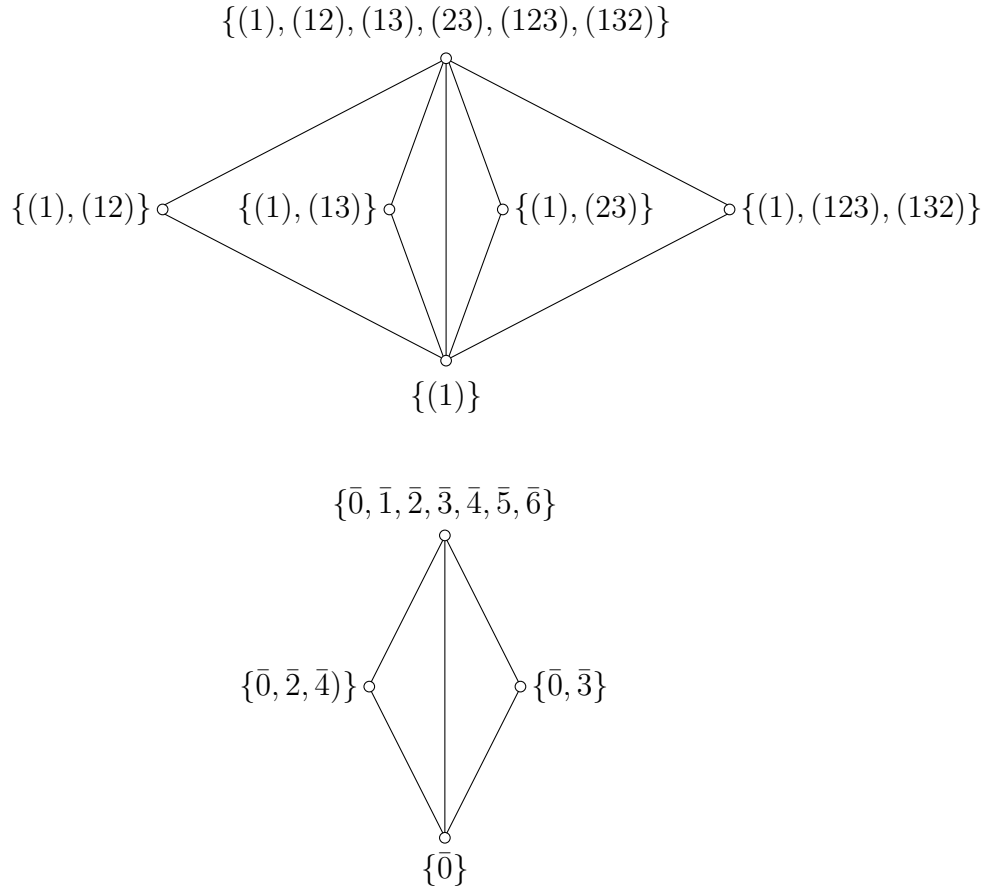


**6.13**  $\neg$  Draw and compare the lattices of subgroups of  $C_2 \times C_2$  and  $C_4$ . Draw the lattice of subgroups of  $S_3$ , and compare it with the one for  $C_6$ . [7.1]

Lattices of subgroups  $C_2 \times C_2$  and  $C_4$  are drawn as follows:



Lattices of subgroups  $S_3$  and  $C_6$  are drawn as follows:



■

## §7. Quotient groups

**7.1** ▷ List all subgroups of  $S_3$  (cf. Exercise 6.13) and determine which subgroups are normal and which are not normal. [§7.1]

The subgroups of  $S_3$  are  $\{(1)\}$ ,  $\{(1), (12)\}$ ,  $\{(1), (13)\}$ ,  $\{(1), (23)\}$ ,  $\{(1), (123), (132)\}$  and  $S_3$ . We can check that  $\{(1)\}$ ,  $\{(1), (123), (132)\}$ ,  $S_3$  are normal subgroups while others are not. ■

**7.2** Is the image of a group homomorphism necessarily a normal subgroup of the target?

No. According to exercise 7.1 we have seen not all subgroups are normal. Suppose  $H$  is a subgroup of  $G$  but not normal. Then  $H$  itself is the image of the inclusion homomorphism  $i : H \hookrightarrow G$ , which makes a counterexample. ■

**7.3** ▷ Verify that the equivalent conditions for normality given in 7.1 are indeed equivalent. [§7.1]

A subgroup  $N$  of  $G$  is normal has four equivalent conditions:

- (i)  $\forall g \in G, gNg^{-1} = N$ ;
- (ii)  $\forall g \in G, gNg^{-1} \subseteq N$ ;
- (iii)  $\forall g \in G, gN \subseteq Ng$ ;
- (iv)  $\forall g \in G, gN = Ng$ .

(i)  $\implies$  (ii) is straightforward.

(ii)  $\implies$  (iii). For any  $g \in G$ , the element  $a \in gN$  can be written as  $a = gn_1$  ( $n_1 \in N$ ). Since  $gn_1g^{-1} \in gNg^{-1} \subseteq N$ , there exists an  $n_2 \in N$  such that  $gn_1g^{-1} = n_2$ , which implies  $gn_1 = n_2g \in Ng$ . Thus we have  $gN \subseteq Ng$ .

(iii)  $\implies$  (iv). Given any  $g \in G$ , for all  $n_1 \in N$ , the element  $g^{-1}n_1 \in g^{-1}N$  also belongs to  $Ng^{-1}$ , which implies that there exists  $n_2 \in N$  such that  $g^{-1}n_1 = n_2g^{-1}$ , namely  $n_1g = gn_2$ . Thus we get  $Ng \subseteq gN$  and accordingly  $gN = Ng$ .

(iv)  $\implies$  (i). For any  $g \in G$ , the element  $b \in gNg^{-1}$  can be written as  $a = gn_1g^{-1}$  ( $n_1 \in N$ ). Since  $gn_1 \in gN = Ng$ , there exists an  $n_2 \in N$  such that  $gn_1 = n_2g$ , which implies  $gn_1g^{-1} = n_2 \in N$ . Thus we have

$$\begin{aligned} & \forall g \in G, \quad gNg^{-1} \subseteq N \\ \implies & \forall g^{-1} \in G, \quad g^{-1}(gNg^{-1})g \subseteq gNg^{-1} \\ \implies & \forall g \in G, \quad N \subseteq gNg^{-1}. \end{aligned}$$

Hence we have  $\forall g \in G, gNg^{-1} = N$ . ■

## §8. Canonical decomposition and Lagranges theorem

8.1 If a group  $H$  may be realized as a subgroup of two groups  $G_1$  and  $G_2$ , and

$$\frac{G_1}{H} \cong \frac{G_2}{H},$$

does it follow that  $G_1 \cong G_2$ . Give a proof or a counterexample.

■

## §9. Group actions

## §10. Group objects in categories

## References