Huyi Chen

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Algebra, Chapter 0
By Paolo Aluffi

Contents

Chapter I. Preliminaries: Set theory and categories

§1. Naive Set Theory

1.6 Define a relation $\sim$ on the set $\mathbb{R}$ of real numbers, by setting $a \sim b \iff b - a \in \mathbb{Z}$. Prove that this is an equivalence relation, and find a 'compelling' description for $\mathbb{R}/\sim$. Do the same for the relation $\approx$ on the plane $\mathbb{R} \times \mathbb{R}$ defined by declaring $(a_1, a_2) \approx (b_1, b_2) \iff b_1 - a_1 \in \mathbb{Z}$ and $b_2 - a_2 \in \mathbb{Z}$. [§II.8.1, II.8.10]

Imaginatively, $\mathbb{R}/\sim$ can be viewed as a ring of length 1 by bending the real line $\mathbb{R}$. Then we can rotate a ring around an axis of rotation to get $\mathbb{R} \times \mathbb{R}/\approx$, which makes a torus. ■

§2. Functions between sets

2.1 How many different bijections are there between a set S with n elements and itself? [§II.2.1]

There are $n!$ different bijections $S \to S$. ∎

§3. Categories

3.1 Let C be a category. Consider a structure C^{op} with:

- $\mathrm{Obj}(\mathsf{C}^{op}) := \mathrm{Obj}(\mathsf{C})$;
- for A, B objects of C^{op} (hence, objects of C), $\mathrm{Hom}_{\mathsf{C}^{op}}(A, B) := \mathrm{Hom}_{\mathsf{C}}(B, A)$

Show how to make this into a category (that is, define composition of morphisms in C^{op} and verify the properties listed in §3.1). Intuitively, the 'opposite' category C^{op} is simply obtained by 'reversing all the arrows' in C. [5.1, §VIII.1.1, §IX.1.2, IX.1.10]

- For every object A of C, there exists one identity morphism $1_A \in \mathrm{Hom}_{\mathsf{C}}(A, A)$. Since $\mathrm{Obj}(\mathsf{C}^{op}) := \mathrm{Obj}(\mathsf{C})$ and $\mathrm{Hom}_{\mathsf{C}^{op}}(A, A) := \mathrm{Hom}_{\mathsf{C}}(A, A)$, for every object A of C^{op}, the identity on A coincides with $1_A \in \mathsf{C}$.
- For A, B, C objects of C^{op} and $f \in \mathrm{Hom}_{\mathsf{C}^{op}}(A, B) = \mathrm{Hom}_{\mathsf{C}}(B, A)$, $g \in \mathrm{Hom}_{\mathsf{C}^{op}}(B, C) = \mathrm{Hom}_{\mathsf{C}}(C, B)$, the composition laws in C determines a morphism $f * g$ in $\mathrm{Hom}_{\mathsf{C}}(C, A)$, which deduces the composition defined on C^{op}:

$$\begin{aligned} \mathrm{Hom}_{\mathsf{C}^{op}}(A, B) \times \mathrm{Hom}_{\mathsf{C}^{op}}(B, C) &\longrightarrow \mathrm{Hom}_{\mathsf{C}^{op}}(A, C) \\ (f, g) &\longmapsto g \circ f := f * g \end{aligned}$$

- Associativity. If $f \in \mathrm{Hom}_{\mathsf{C}^{op}}(A, B)$, $g \in \mathrm{Hom}_{\mathsf{C}^{op}}(B, C)$, $h \in \mathrm{Hom}_{\mathsf{C}^{op}}(C, D)$, then

$$f \circ (g \circ h) = f \circ (h * g) = (h * g) * f = h * (g * f) = (g * f) \circ h = (f \circ g) \circ h.$$

- Identity. For all $f \in \mathrm{Hom}_{\mathsf{C}^{op}}(A, B)$, we have

$$f \circ 1_A = 1_A * f = f, \quad 1_B \circ f = f * 1_B = f.$$

Thus we get the full construction of C^{op}. ∎

§4. Morphisms

4.2 In Example 3.3 we have seen how to construct a category from a set endowed with a relation, provided this latter is reflexive and transitive. For what types of relations is the corresponding category a groupoid (cf. Example 4.6)? [§4.1]

For a reflexive and transitive relation $\sim$ on a set S, define the category C as follows:

- Objects: $\mathrm{Obj}(\mathsf{C}) = S$;
- Morphisms: if a, b are objects (that is: if $a, b \in S$) then let

$$\mathrm{Hom}_{\mathsf{C}}(a, b) = \begin{cases} (a, b) \in S \times S & \text{if } a \sim b \\ \emptyset & \text{otherwise} \end{cases}$$

In Example 3.3 we have shown the category. If the relation $\sim$ is endowed with symmetry, we have

$$(a, b) \in \mathrm{Hom}_{\mathsf{C}}(a, b) \implies a \sim b \implies b \sim a \implies (b, a) \in \mathrm{Hom}_{\mathsf{C}}(b, a).$$

Since

$$(a, b)(b, a) = (a, a) = 1_a, \quad (b, a)(a, b) = (b, b) = 1_b,$$

in fact (a, b) is an isomorphism. From the arbitrariness of the choice of (a, b), we show that C is a groupoid. Conversely, if C is a groupoid, we can show the relation $\sim$ is symmetric. To sum up, the category C is a groupoid if and only if the corresponding relation $\sim$ is an equivalence relation. ■

§5. Universal properties

5.1 Prove that a final object in a category C is initial in the opposite category C_{op} (cf. Exercise 3.1).

An object F of C is final in C if and only if

$$\forall A \in \mathrm{Obj}(\mathsf{C}) : \mathrm{Hom}_{\mathsf{C}}(A, F) \text{ is a singleton.}$$

That is equivalent to

$$\forall A \in \mathrm{Obj}(\mathsf{C}_{op}) : \mathrm{Hom}_{\mathsf{C}_{op}}(F, A) \text{ is a singleton,}$$

which means F is initial in the opposite category C_{op}. ■

Chapter II. Groups, first encounter

§1. Definition of group

1.1 Write a careful proof that every group is the group of isomorphisms of a groupoid. In particular, every group is the group of automorphisms of some object in some category.

Assume G is a group. Define a category C as follows:

- Objects: $\mathrm{Obj}(\mathsf{C}) = \{*\}$;
- Morphisms: $\mathrm{Hom}_{\mathsf{C}}(*, *) = \mathrm{End}_{\mathsf{C}}(*) = G$.

The composition of homomorphism is corresponding to the multiplication between two elements in G. The identity morphism on $*$ is $1_* = e_G$, which satisfies for all $g \in \mathrm{Hom}_{\mathsf{C}}(*, *)$,

$$ge_G = e_G g = g,$$

and

$$gg^{-1} = e_G, \; g^{-1}g = e_G.$$

Thus any homomorphism $g \in \mathrm{Hom}_{\mathsf{C}}(*, *)$ is an isomorphism and accordingly C is a groupoid. Now we see $G = \mathrm{End}_{\mathsf{C}}(*)$ is the group of isomorphisms of a groupoid. Moreover, supposing that $*$ is an object in some category D, G would be the group of automorphisms of $*$, which is denoted as $\mathrm{Aut}_{\mathsf{D}}(*)$. ∎

1.4 Suppose that $g^2 = e$ for all elements g of a group G; prove that G is commutative.

For all $a, b \in G$,

$$abab = e \implies a(abab)b = ab \implies (aa)ba(bb) = ab \implies ba = ab.$$

∎

§2. Examples of groups

2.1 One can associate an $n \times n$ matrix M_σ with a permutation $\sigma \in S_n$, by letting the entry at $(i, \sigma(i))$ be 1, and letting all other entries be 0. For example, the matrix corresponding to the permutation

$$\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \in S_3$$

would be

$$M_\sigma = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

Prove that, with this notation,

$$M_{\sigma\tau} = M_\sigma M_\tau$$

for all $\sigma, \tau \in S_n$, where the product on the right is the ordinary product of matrices.

By introducing the Kronecker delta function

$$\delta_{i,j} = \begin{cases} 0 & \text{if } i \neq j, \\ 1 & \text{if } i = j, \end{cases}$$

the entry at (i,j) of the matrix $M_{\sigma\tau}$ can be written as

$$(M_{\sigma\tau})_{i,j} = \delta_{\tau(\sigma(i)),j}$$

and the entry at (i,j) of the matrix $M_\sigma M_\tau$ can be written as

$$(M_\sigma M_\tau)_{i,j} = \sum_{k=1}^{n} (M_\sigma)_{i,k}(M_\tau)_{k,j} = \sum_{k=1}^{n} \delta_{\sigma(i),k} \cdot \delta_{\tau(k),j} = \sum_{k=1}^{n} \delta_{\sigma(i),k} \cdot \delta_{k,\tau^{-1}(j)} = \delta_{\sigma(i),\tau^{-1}(j)},$$

where the last but one equality holds by the fact

$$\tau(k) = j \iff k = \tau^{-1}(j).$$

Noticing that

$$\tau(\sigma(i)) = j \iff \sigma(i) = \tau^{-1}(j),$$

we see $M_{\sigma\tau} = M_\sigma M_\tau$ for all $\sigma, \tau \in S_n$. ■

2.2 Prove that if $d \leq n$, then S_n contains elements of order d.

The cyclic permutation

$$\sigma = (1\ 2\ 3 \cdots d)$$

is an element of order d in S_n. ■

2.3 For every positive integer n find an element of order n in $S_{\mathbb{N}}$.

The cyclic permutation

$$\sigma = (1\ 2\ 3 \cdots n)$$

is an element of order d in S_n. ■

2.4 Define a homomorphism $D_8 \to S_4$ by labeling vertices of a square, as we did for a triangle in §2.2. List the 8 permutations in the image of this homomorphism.

The image of n rotations under the homomorphism are

$$\sigma_1 = e_{D_8},\ \sigma_2 = (1\ 2\ 3\ 4),\ \sigma_3 = (1\ 3)(2\ 4),\ \sigma_4 = (1\ 4\ 3\ 2).$$

The image of n reflections under the homomorphism are

$$\sigma_5 = (1\ 3),\ \sigma_6 = (2\ 4),\ \sigma_7 = (1\ 2)(3\ 4),\ \sigma_8 = (1\ 4)(3\ 2).$$

■

2.11 Prove that the square of every odd integer is congruent to 1 modulo 8.

Given an odd integer $2k+1$, we have

$$(2k+1)^2 = 4k(k+1)+1,$$

where $k(k+1)$ is an even integer. So $(2k+1)^2 \equiv 1 \mod 8$. ■

2.12 Prove that there are no integers a, b, c such that $a^2+b^2 = 3c^2$. (Hint: studying the equation $[a]_4^2 + [b]_4^2 = 3[c]_4^2$ in $\mathbb{Z}/4\mathbb{Z}$, show that a, b, c would all have to be even. Letting $a = 2k, b = 2l, c = 2m$, you would have $k^2+l^2 = 3m^2$. What's wrong with that?)

$$a^2+b^2 = 3c^2 \implies [a]_4^2 + [b]_4^2 = 3[c]_4^2.$$

Noting that $[0]_4^2 = [0]_4, [1]_4^2 = [1]_4, [2]_4^2 = [0]_4, [3]_4^2 = [1]_4$, we see $[c]_4^2$ must be $[0]_4$ and so do $[a]_4^2$ and $[b]_4^2$. Hence $[a]_4, [b]_4, [b]_4$ can only be $[0]_4$ or $[2]_4$, which justifies letting $a = 2k_1, b = 2l_2, c = 2m_1$. After substitution we have $k^2+l^2 = 3m^2$. Repeating this process n times yields $a = 2^n k_n, b = 2^n l_n, c = 2^n m_n$. For a sufficiently large number N, the absolute value of k_N, l_N, m_N must be less than 1. Thus we conclude that $a = b = c = 0$ is the unique solution to the equation $a^2+b^2 = 3c^2$. ■

2.13 Prove that if $\gcd(m, n) = 1$, then there exist integers a and b such that $am+bn = 1$. (Use Corollary 2.5.) Conversely, prove that if $am+bn = 1$ for some integers a and b, then $\gcd(m, n) = 1$. [2.15, §V.2.1, V.2.4]

Applying corollary 2.5, we have $\gcd(m, n) = 1$ if and only if $[m]_n$ generates $\mathbb{Z}/n\mathbb{Z}$. Hence

$$\gcd(m, n) = 1 \iff a[m]_n = [1]_n \iff [am]_n = [1]_n \iff am+bn = 1.$$

■

2.15 Let $n > 0$ be an odd integer.

- Prove that if $\gcd(m, n) = 1$, then $\gcd(2m+n, 2n) = 1$. (Use Exercise 2.13.)
- Prove that if $\gcd(r, 2n) = 1$, then $\gcd(\frac{r+n}{2}, n) = 1$. (Ditto.)
- Conclude that the function $[m]_n \to [2m+n]_{2n}$ is a bijection between $(\mathbb{Z}/n\mathbb{Z})^*$ and $(\mathbb{Z}/2n\mathbb{Z})^*$.

The number $\phi(n)$ of elements of $(\mathbb{Z}/n\mathbb{Z})^*$ is Eulers $\phi(n)$-function. The reader has just proved that if n is odd, then $\phi(2n) = \phi(n)$. Much more general formulas will be given later on (cf. Exercise V.6.8). [VII.5.11]

- According to Exercise 2.13,

$$\gcd(m,n)=1 \implies am+bn=1 \implies \frac{a}{2}(2m+n)+\left(b-\frac{a}{2}\right)n=1.$$

If a is even, we have shown $\gcd(2m+n,2n)=1$. Otherwise we can let $a'=a+n$ be an even integer and $b'=b-m$. Then it holds that

$$\frac{a'}{2}(2m+n)+\left(b'-\frac{a'}{2}\right)n=1,$$

which also indicates $\gcd(2m+n,2n)=1$.

- If $\gcd(r,2n)=1$, then r must be an odd integer and accordingly

$$\gcd(2r+2n,4n)=1 \implies a(2r+2n)+b(4n)=1 \implies 4a\frac{r+n}{2}+4bn=1,$$

which is $\gcd(\frac{r+n}{2},n)=1$.

- It is easy to check that the function $f:(\mathbb{Z}/n\mathbb{Z})^*\to(\mathbb{Z}/2n\mathbb{Z})^*,\ [m]_n\mapsto[2m+n]_{2n}$ is well-defined. The fact

$$\begin{aligned} f([m_1]_n)=f([m_2]_n) &\implies f([2m_1+n]_{2n})=f([2m_2+n]_{2n}) \\ &\implies (2m_1+n)-(2m_2+n)=2kn \\ &\implies m_1-m_2=kn \\ &\implies [m_1]_n=[m_2]_n \end{aligned}$$

indicates that f is injective. For any $[r]_{2n}\in(\mathbb{Z}/2n\mathbb{Z})^*$, we have

$$\gcd(r,2n)=1 \implies \gcd\left(\frac{r+n}{2},n\right)=1 \implies \left[\frac{r+n}{2}\right]_n\in(\mathbb{Z}/n\mathbb{Z})^*,$$

and

$$f\left(\left[\frac{r+n}{2}\right]_n\right)=[r+2n]_{2n}=[r]_{2n},$$

which indicates that f is surjective. Thus we show f is a bijection.

■

2.16 Find the last digit of $1238237^{18238456}$. (Work in $\mathbb{Z}/10\mathbb{Z}$.)

$$1238237^{18238456}\equiv 7^{18238456}\equiv(7^4)^{4559614}\equiv 2401^{4559614}\equiv 1 \mod 10,$$

which indicates that the last digit of $1238237^{18238456}$ is 1. ■

2.17 Show that if $m\equiv m' \mod n$, then $\gcd(m,n)=1$ if and only if $\gcd(m',n)=1$. [§2.3]

Assume that $m - m' = kn$. If $\gcd(m, n) = 1$, for any common divisor d of m' and n

$$d|m',\ d|n \implies d|(m' + kn) \implies d|m \implies d = 1,$$

which means $\gcd(m', n) = 1$. Likewise, we can show $\gcd(m', n) = 1 \implies \gcd(m, n) = 1$ ■

§3. The category Grp

> **3.1** Let $\varphi : G \to H$ be a morphism in a category C with products. Explain why there is a unique morphism
>
> $$(\varphi \times \varphi) : G \times G \longrightarrow H \times H.$$
>
> (This morphism is defined explicitly for $\mathsf{C} = \mathsf{Set}$ in §3.1.)

By the universal property of product in C, there exist a unique morphism $(\varphi \times \varphi) : G \times G \longrightarrow H \times H$ such that the following diagram commutes.

$$\begin{tikzcd}
G \arrow[r, "\varphi"] & H \\
G \times G \arrow[u, "\pi_G"] \arrow[r, "\varphi\times\varphi"] \arrow[d, "\pi_G"'] & H \times H \arrow[u, "\pi_H"'] \arrow[d, "\pi_H"] \\
G \arrow[r, "\varphi"] & H
\end{tikzcd}$$

■

> **3.2** Let $\varphi : G \to H, \psi : H \to K$ be morphisms in a category with products, and consider morphisms between the products $G \times G, H \times H, K \times K$ as in Exercise 3.1. Prove that
>
> $$(\psi\varphi) \times (\psi\varphi) = (\psi \times \psi)(\varphi \times \varphi).$$
>
> (This is part of the commutativity of the diagram displayed in §3.2.)

By the universal property of product in C, there exists a unique morphism

$$(\psi\varphi) \times (\psi\varphi) : G \times G \to K \times K$$

such that the following diagram commutes.

$$\begin{tikzcd}
G \arrow[r, "\psi\varphi"] & H \\
G \times G \arrow[u, "\pi_G"] \arrow[r, "(\psi\varphi)\times(\psi\varphi)"] \arrow[d, "\pi_G"'] & H \times H \arrow[u, "\pi_H"'] \arrow[d, "\pi_H"] \\
G \arrow[r, "\psi\varphi"] & H
\end{tikzcd}$$

As the following commutative diagram tells us the composition

$$(\psi \times \psi)(\varphi \times \varphi) : G \times G \to K \times K$$

can make the above diagram commute,

$$\begin{tikzcd}
G \arrow[r, "\varphi"] \arrow[rr, bend left, "\psi\varphi"] & H \arrow[r, "\psi"] & K \\
G \times G \arrow[u, "\pi_G"] \arrow[r, "\varphi\times\varphi"] \arrow[d, "\pi_G"'] & H \times H \arrow[u, "\pi_H"] \arrow[r, "\psi\times\psi"] \arrow[d, "\pi_H"'] & K \times K \arrow[u, "\pi_K"] \arrow[d, "\pi_K"'] \\
G \arrow[r, "\varphi"] \arrow[rr, bend right, "\psi\varphi"'] & H \arrow[r, "\psi"] & K
\end{tikzcd}$$

there must be $(\psi\varphi) \times (\psi\varphi) = (\psi \times \psi)(\varphi \times \varphi)$.

■

3.3 Show that if G, H are abelian groups, then $G \times H$ satisfies the universal property for coproducts in Ab.

Define two monomorphisms:

$$i_G : G \longrightarrow G \times H,\ a \longmapsto (a, 0_H)$$

$$i_H : H \longrightarrow G \times H,\ b \longmapsto (0_G, b)$$

We are to show that for any two homomorphisms $g : G \to M$ and $h : H \to M$ in Ab, the mapping

$$\begin{aligned} \varphi : \quad & G \times H \longrightarrow M, \\ & (a, b) \longmapsto g(a) + h(b) \end{aligned}$$

is a homomorphism and makes the following diagram commute.

$$\begin{tikzcd}
G \arrow[d, "i_G"'] \arrow[dr, "g"] & \\
G \times H \arrow[r, "\varphi"] & M \\
H \arrow[u, "i_H"] \arrow[ur, "h"'] &
\end{tikzcd}$$

Exploiting the fact that g, h are homomorphisms and M is an abelian group, it is easy to check that φ preserves the addition operation

$$\begin{aligned}\varphi((a_1,b_1)+(a_2,b_2)) &= \varphi((a_1+a_2,b_1+b_2))\\ &= g(a_1+a_2)+h(b_1+b_2)\\ &= (g(a_1)+g(a_2))+(h(b_1)+h(b_2))\\ &= (g(a_1)+h(b_1))+(g(a_2)+h(b_2))\\ &= g(a_1+b_1)+h(a_2+b_2)\\ &= \varphi((a_1,b_1))+\varphi((a_2,b_2))\end{aligned}$$

and the diagram commutes

$$\varphi\circ i_G(a)=\varphi((a,0_H))=g(a)+h(0_H)=g(a)+0_M=g(a),$$

$$\varphi\circ i_H(b)=\varphi((0_G,b))=g(0_G)+h(b)=0_M+h(b)=h(b).$$

To show the uniqueness of the homomorphism φ we have constructed, suppose a homomorphism φ' can make the diagram commute. Then we have

$$\varphi'((a,b))=\varphi'((a,0_H)+(0_G,b))=\varphi'(i_G(a))+\varphi'(i_H(b))=g(a)+h(b)=\varphi((a,b)),$$

that is $\varphi'=\varphi$. Hence we show that there exist a unique homomorphism φ such that the diagram commutes, which amounts to the universal property for coproducts in Ab.

■

3.4 Let G,H be groups, and assume that $G\cong H\times G$. Can you conclude that H is trivial? (Hint: No. Can you construct a counterexample?)

Consider the function

$$\begin{aligned}\varphi:\mathbb{Z}\times\mathbb{Z}[x] &\longrightarrow \mathbb{Z}[x]\\ (n,f(x)) &\longmapsto n+xf(x)\end{aligned}$$

Firstly, we can show φ is a homomorphism as follows

$$\begin{aligned}\varphi((n_1,f_1(x))+(n_2,f_2(x))) &= \varphi((n_1+n_2,f_1(x)+f_2(x)))\\ &= (n_1+n_2)+x(f_1(x)+f_2(x))\\ &= (n_1+xf_1(x))+(n_2+xf_2(x))\\ &= \varphi((n_1,f_1(x)))+\varphi((n_2,f_2(x))).\end{aligned}$$

Secondly, we are to show φ is a monomorphism. It follows by

$$\varphi((n,f(x)))=n+xf(x)=0 \implies n=0,\ f(x)=0 \implies \ker\varphi=\{(0,0)\}.$$

Lastly, since the cardinal numbers of both $\mathbb{Z}\times\mathbb{Z}[x]$ and $\mathbb{Z}[x]$ are $\aleph_0$, φ is indeed an isomorphism. Therefore, as a counterexample we have $\mathbb{Z}[x]\cong\mathbb{Z}\times\mathbb{Z}[x]$. ■

3.5 Prove that $\mathbb{Q}$ is not the direct product of two nontrivial groups.

Consider the additive group of rationals $(\mathbb{Q}, +)$. Assume that φ is a isomorphism between the product $G \times H = \{(a,b) | a \in G, b \in H\}$ and $(\mathbb{Q}, +)$. Note that $\{e_G\} \times H$ and $G \times \{e_H\}$ are subgroups in $G \times H$ and their intersection is the trivial group $\{(e_G, e_H)\}$. It is easy to check that bijection φ satisfies $\varphi(A \cap B) = \varphi(A) \cap \varphi(B)$. So applying the fact we have

$$\varphi(\{(e_G, e_H)\}) = \varphi(\{e_G\} \times H \cap G \times \{e_H\}) = \varphi(\{e_G\} \times H) \cap \varphi(G \times \{e_H\}) = \{0\}.$$

Suppose both $\varphi(\{e_G\} \times H)$ and $\varphi(G \times \{e_H\})$ are nontrivial groups. If $\dfrac{p}{q} \in \varphi(\{e_G\} \times H) - \{0\}$ and $\dfrac{r}{s} \in \varphi(G \times \{e_H\}) - \{0\}$, there must be

$$rp = rq \cdot \frac{p}{q} = ps \cdot \frac{r}{s} \in \varphi(\{e_G\} \times H) \cap \varphi(G \times \{e_H\}),$$

which implies $rp = 0$. Since both $\dfrac{p}{q}$ and $\dfrac{r}{s}$ are non-zero, it leads to a contradiction. Thus without loss of generality we can assume $\varphi(\{e_G\} \times H)$ is a trivial group $\{0\}$. Since φ is isomorphism, we see that for all $h \in H$,

$$\varphi(e_G, h) = \varphi(e_G, e_H) = 0 \iff h = e_H.$$

That is, H is a trivial group. Therefore, we have shown $(\mathbb{Q}, +)$ will never be isomorphic to the direct product of two nontrivial groups. ■

3.6 Consider the product of the cyclic groups C_2, C_3 (cf. §2.3): $C_2 \times C_3$. By Exercise 3.3, this group is a coproduct of C_2 and C_3 in Ab. Show that it is not a coproduct of C_2 and C_3 in Grp, as follows:

- find injective homomorphisms $C_2 \to S_3$, $C_3 \to S_3$;
- arguing by contradiction, assume that $C_2 \times C_3$ is a coproduct of C_2, C_3, and deduce that there would be a group homomorphism $C_2 \times C_3 \to S_3$ with certain properties;
- show that there is no such homomorphism.

- Monomorphisms $g : C_2 \to S_3$, $h : C_3 \to S_3$ can be constructed as follows:

$$g([0]_2) = e, g([1]_2) = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}.$$

$$h([0]_3) = e, h([1]_3) = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}, h([2]_3) = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}.$$

- Supposing that $C_2 \times C_3$ is a coproduct of C_2, C_3, there would be a unique group homomorphism $\varphi : C_2 \times C_3 \to S_3$ such that the following diagram commutes

$$\begin{tikzcd} C_2 \arrow[d, "i_{C_2}"'] \arrow[dr, "g"] & \\ C_2 \times C_3 \arrow[r, "\varphi"] & S_3 \\ C_3 \arrow[u, "i_{C_3}"] \arrow[ur, "h"'] & \end{tikzcd}$$

In other words, for all $a \in C_2, b \in C_3$,

$$\begin{aligned}\varphi(a,b) &= \varphi(([0]_2,b)+(a,[0]_3)) = \varphi(([0]_2,b))\varphi((a,[0]_3)) = \varphi(i_{C_3}(b))\varphi(i_{C_2}(a)) = h(b)g(a) \\ &= \varphi((a,[0]_3)+([0]_2,b)) = \varphi((a,[0]_3))\varphi(([0]_2,b)) = \varphi(i_{C_2}(a))\varphi(i_{C_3}(b)) = g(a)h(b).\end{aligned}$$

- Since

$$g([1]_2)h([1]_3) = \begin{pmatrix}1 & 2 & 3\\ 1 & 3 & 2\end{pmatrix}\begin{pmatrix}1 & 2 & 3\\ 3 & 1 & 2\end{pmatrix} = \begin{pmatrix}1 & 2 & 3\\ 3 & 2 & 1\end{pmatrix},$$
$$h([1]_3)g([1]_2) = \begin{pmatrix}1 & 2 & 3\\ 3 & 1 & 2\end{pmatrix}\begin{pmatrix}1 & 2 & 3\\ 1 & 3 & 2\end{pmatrix} = \begin{pmatrix}1 & 2 & 3\\ 2 & 1 & 3\end{pmatrix},$$

we see $g(a)h(b) \neq h(b)g(a)$ not always holds. The derived contradiction shows that $C_2 \times C_3$ is not a coproduct of C_2, C_3 in Grp.

■

3.7 Show that there is a surjective homomorphism $Z * Z \to C_2 * C_3$. ($*$ denotes coproduct in Grp.)

Consider the mapping

$$\begin{aligned}\varphi : \mathbb{Z} * \mathbb{Z} &\longrightarrow C_2 * C_3 \\ x^{m_1}y^{n_1}\cdots x^{m_k}y^{n_k} &\longmapsto x^{[m_1]_2}y^{[n_1]_3}\cdots x^{[m_k]_2}y^{[n_k]_3}\end{aligned}$$

Since

$$\begin{aligned}&\varphi(x^{m_1}y^{n_1}\cdots x^{m_k}y^{n_k}x^{m'_1}y^{n'_1}\cdots x^{m'_{k'}}y^{n'_k}) \\ =&x^{[m_1]_2}y^{[n_1]_3}\cdots x^{[m_k]_2}y^{[n_k]_3}x^{[m'_1]_2}y^{[n'_1]_3}\cdots x^{[m'_k]_2}y^{[n'_k]_3}, \\ =&\varphi(x^{m_1}y^{n_1}\cdots x^{m_k}y^{n_k})\varphi(x^{m'_1}y^{n'_1}\cdots x^{m'_{k'}}y^{n'_k})\end{aligned}$$

φ is a homomorphism. It is clear that φ is surjective. Thus we show there exists a surjective homomorphism $Z * Z \to C_2 * C_3$. ■

3.8 Define a group G with two generators x, y, subject (only) to the relations $x^2 = e_G, y^3 = e_G$. Prove that G is a coproduct of C_2 and C_3 in Grp. (The reader will obtain an even more concrete description for $C_2 * C_3$ in Exercise 9.14; it is called the modular group.) [§3.4, 9.14]

Given the maps $i_1 : C_2 \to G, [m]_2 \mapsto x^m$ and $i_2 : C_3 \to G, [n]_3 \mapsto y^n$, we can check that i_1, i_2 are homomorphisms. We are to show that for every group H endowed with two homomorphisms $f_1 : C_2 \to H$, $f_2 : C_3 \to H$, there would be a unique group homomorphism $\varphi : G \to H$ such that the following diagram commutes

$$\begin{tikzcd} C_2 \arrow[d, "i_1"'] \arrow[dr, "f_1"] & \\ G \arrow[r, "\varphi"] & H \\ C_3 \arrow[u, "i_2"] \arrow[ur, "f_2"'] & \end{tikzcd}$$

or

$$\varphi(i_1([m]_2)) = \varphi(x^m) = \varphi(x)^m = f_1([m]_2),$$
$$\varphi(i_2([n]_3)) = \varphi(y^n) = \varphi(y)^n = f_2([n]_3).$$

Define $\phi : G \to H$ as $\phi(x^m y^n) = f_1([m]_2) f_2([n]_3)$, $\phi(y^n x^m) = f_2([n]_3) f_1([m]_2)$. It is clear to see ϕ makes the diagram commute. Moreover, if φ makes the diagram commute, it follows that for all $x^m y^n, y^n x^m \in G$,

$$\varphi(x^m y^n) = \varphi(x^m)\varphi(y^n) = f_1([m]_2) f_2([n]_3),$$

$$\varphi(y^n x^m) = \varphi(y^n)\varphi(x^m) = f_2([n]_3) f_1([m]_2),$$

which implies $\varphi = \phi$. Thus we can conclude G is the coproduct of C_2 and C_3 in Grp.

■

§4. Group homomorphisms

4.1 Check that the function π_m^n defined in §4.1 is well-defined, and makes the diagram commute. Verify that it is a group homomorphism. Why is the hypothesis $m|n$ necessary? [§4.1]

In §4.1 the function π_m^n is defined as

$$\begin{aligned} \pi_m^n \ : \mathbb{Z}/n\mathbb{Z} &\longrightarrow \mathbb{Z}/m\mathbb{Z} \\ [a]_n &\longmapsto [a]_m \end{aligned}$$

with the condition $m|n$. We can check that π_m^n is well-defined as

$$[a_1]_n = [a_2]_n \iff a_1 - a_2 = kn = (kl)m \implies [a_1]_m = [a_2]_m \iff \pi_m^n([a_1]_n) = \pi_m^n([a_2]_n).$$

Note $\pi_m^n(\pi_n(a)) = \pi_m^n([a]_n) = [a]_m = \pi_m(a)$. The diagram in §4.1 must commute.

$$\begin{tikzcd}
\mathbb{Z} \arrow[d, "\pi_n"'] \arrow[dr, "\pi_m"] & \\
\mathbb{Z}/n\mathbb{Z} \arrow[r, "\pi_m^n"'] & \mathbb{Z}/m\mathbb{Z}
\end{tikzcd}$$

Since

$$\pi_m^n([a]_n + [b]_n) = [a+b]_m = [a]_m + [b]_m = \pi_m^n([a]_n) + \pi_m^n([b]_n),$$

it follows that π_m^n is a group homomorphism. Actually we have shown that without the hypothesis $m|n$, π_m^n may not be well-defined. ∎

4.2 Show that the homomorphism $\pi_2^4 \times \pi_2^4 : C_4 \to C_2 \times C_2$ is not an isomorphism. In fact, is there any nontrivial isomorphism $C_4 \to C_2 \times C_2$?

Let calculate the order of each non-zero element in both C_4 and $C_2 \times C_2$. For the group C_4,

$$|[2]_4| = 2, \quad |[1]_4| = |[3]_4| = 4.$$

For the group $C_2 \times C_2$,

$$|([1]_2, [0]_2)| = |([0]_2, [1]_2)| = |([1]_2, [1]_2)| = 2.$$

Since isomorphism must preserve the order, we can assert that there is no such isomorphism $C_4 \to C_2 \times C_2$. ∎

4.3 Prove that a group of order n is isomorphic to $\mathbb{Z}/n\mathbb{Z}$ if and only if it contains an element of order n. [§4.3]

Assume some group G is isomorphic to $\mathbb{Z}/n\mathbb{Z}$. Since $|[1]_n| = n$ and isomorphism preserves the order, we can affirm that there is an element of order n in G.
Conversely, assume there is a group G of order n in which g is an element of order n. By definition we see $g^0, g^1, g^2 \cdots g^{n-1}$ are distinct pairwise. Noticing group G has exactly n elements, G must consist of $g^0, g^1, g^2 \cdots g^{n-1}$. We can easily check that the function

$$\begin{aligned} f \ : G &\longrightarrow \mathbb{Z}/n\mathbb{Z} \\ g^k &\longmapsto [k]_n \end{aligned}$$

is an isomorphism. ∎

4.4 Prove that no two of the groups $(\mathbb{Z}, +)$, $(\mathbb{Q}, +)$, $(\mathbb{R}, +)$ are isomorphic to one another. Can you decide whether $(\mathbb{R}, +)$, $(\mathbb{C}, +)$ are isomorphic to one another? (Cf. Exercise VI.1.1.)

Suppose there exists an isomorphism $f : \mathbb{Z} \to \mathbb{Q}$. Let $f(1) = p/q$ $(p, q \in \mathbb{Z})$. If $p = 1$, for all $n \in \mathbb{Z}$, we have

$$f(n) = \frac{n}{q} \neq \frac{1}{2q}.$$

If $p \neq 1$, for all $n \in \mathbb{Z}$, we have

$$f(n) = \frac{np}{q} \neq \frac{p+1}{q}.$$

In both cases, it implies $f(\mathbb{Z}) \nsubseteq \mathbb{Q}$. Hence we see f is not a surjection, which contradicts the fact that $f : \mathbb{Z} \to \mathbb{Q}$ is an isomorphism. Compare the cardinality of $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$

$$|\mathbb{Z}| = |\mathbb{Q}| < |\mathbb{R}|$$

and we show there exists no such isomorphisms like $f : \mathbb{Z} \to \mathbb{R}$ or $f : \mathbb{Q} \to \mathbb{R}$.
We can prove $(\mathbb{R}, +)$, $(\mathbb{C}, +)$ are isomorphic, if considering the both as vector spaces over $\mathbb{Q}$.

■

4.5 Prove that the groups $(\mathbb{R} \setminus \{0\}, \cdot)$ and $(\mathbb{C} \setminus \{0\}, \cdot)$ are not isomorphic.

Suppose $f : \mathbb{R} \to \mathbb{C}$ is an isomorphism. Then there exists a real number x such that $f(x) = i$.

$$f(x^4) = f(x)^4 = i^4 = 1.$$

Since isomorphism preserves the identity, we have

$$f(1) = 1 = f(x^4).$$

which indicates $x^4 = 1$. Noticing that $x \in \mathbb{R}$, there must be $x^2 = 1$. Now we see

$$f(1) = f(x^2) = f(x)^2 = i^2 = -1,$$

which derives a contradiction. Thus we can conclude that groups $(\mathbb{R} \setminus \{0\}, \cdot)$ and $(\mathbb{C} \setminus \{0\}, \cdot)$ are not isomorphic.

■

4.6 We have seen that $(\mathbb{R}, +)$ and $(\mathbb{R}_{>0}, \cdot)$ are isomorphic (Example 4.4). Are the groups $(\mathbb{Q}, +)$ and $(\mathbb{Q}_{>0}, \cdot)$ isomorphic?

Suppose $f : \mathbb{Q} \to \mathbb{Q}_{>0}$ is an isomorphism. Since isomorphism preserves the multiplication, we have

$$f(1) = f\left(n \cdot \frac{1}{n}\right) = f\left(\frac{1}{n}\right)^n \quad (n \in \mathbb{Z}_{>0}),$$

which implies

$$f\left(\frac{1}{n}\right) = f(1)^{\frac{1}{n}}.$$

Assume

$$f(1) = \frac{p}{q} = \frac{p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k}}{q_1^{s_1} q_2^{s_2} \cdots q_l^{s_l}}$$

where p_i, q_i are pairwise distinct positive prime numbers. Then let

$$M = \max\{p, q\} + 1 > \max\{r_1, \cdots, r_k, s_1, \cdots, s_l\}.$$

Thus we assert

$$f\left(\frac{1}{M}\right) = \left(\frac{p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k}}{q_1^{s_1} q_2^{s_2} \cdots q_l^{s_l}}\right)^{\frac{1}{M}} \notin \mathbb{Q},$$

which can be proved by contradiction. In fact, Suppose

$$\left(\frac{p}{q}\right)^{\frac{1}{M}} = \frac{a}{b} \in \mathbb{Q}$$

or say

$$pb^M = qa^M,$$

where a, b are coprime. Note that b^M, a^M are also coprime and that the prime factorization of a^M can be written as $a_1^{Mt_1} a_2^{Mt_2} \cdots a_j^{Mt_j}$ where a_i are pairwise distinct positive prime numbers. That forces

$$p = p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k} = N \cdot a_1^{Mt_1} a_2^{Mt_2} \cdots a_j^{Mt_j}.$$

Noticing that a_i must coincide with one number in $\{p_1, p_2, \cdots p_k\}$, we can assume $a_1 = p_1$ without loss of generality. However, since $M > \max\{r_1, \cdots, r_k\}$, we see the exponent of p_1 is distinct from that of a_1, which violates the unique factorization property of $\mathbb{Z}$. Hence we get a contradiction and verify $f\left(\frac{1}{M}\right) \notin \mathbb{Q}$. Moreover, it contradicts our assumption that $f : \mathbb{Q} \to \mathbb{Q}_{>0}$ is an isomorphism. Eventually we show that the groups $(\mathbb{Q}, +)$ and $(\mathbb{Q}_{>0}, \cdot)$ are not isomorphic.

$\blacksquare$

4.7 Let G be a group. Prove that the function $G \to G$ defined by $g \mapsto g^{-1}$ is a homomorphism if and only if G is abelian. Prove that $g \mapsto g^2$ is a homomorphism if and only if G is abelian.

Given the function

$$\begin{aligned} f\ : G &\longrightarrow G \\ g &\longmapsto g^{-1} \end{aligned}$$

we have

$$f(g_1 g_2) = (g_1 g_2)^{-1} = g_2^{-1} g_1^{-1}, \quad f(g_1) f(g_2) = g_1^{-1} g_2^{-1}.$$

If G is abelian, it is clear to see $f(g_1 g_2) = f(g_1) f(g_2)$. If f is a homomorphism, $\forall h_1, h_2 \in G$,

$$h_1 h_2 = (h_2^{-1} h_1^{-1})^{-1} = f(h_2^{-1} h_1^{-1}) = f(h_2^{-1}) f(h_1^{-1}) = h_2 h_1.$$

Given the function

$$\begin{aligned} h\ : G &\longrightarrow G \\ g &\longmapsto g^2 \end{aligned}$$

we have

$$h(g_1g_2) = (g_1g_2)^2 = g_1g_2g_1g_2, \quad h(g_1)h(g_2) = g_1^2g_2^2 = g_1g_1g_2g_2.$$

If G is abelian, it is clear to see $h(g_1g_2) = h(g_1)h(g_2)$. If h is a homomorphism, by cancellation we have

$$h(g_1g_2) = h(g_1)h(g_2) \implies g_2g_1 = g_1g_2.$$

■

> **4.8** Let G be a group, and $g \in G$. Prove that the function $\gamma_g : G \to G$ defined by $(\forall a \in G) : \gamma_g(a) = gag^{-1}$ is an automorphism of G. (The automorphisms γ_g are called 'inner' automorphisms of G.) Prove that the function $G \to \mathrm{Aut}(G)$ defined by $g \mapsto \gamma_g$ is a homomorphism. Prove that this homomorphism is trivial if and only if G is abelian.

Since

$$\gamma_g(ab) = gabg^{-1} = gag^{-1}gbg^{-1} = \gamma_g(a)\gamma_g(b),$$

γ_g is an automorphism of G. For all $a \in G$, we have

$$\gamma_{g_1g_2}(a) = g_1g_2ag_2^{-1}g_1^{-1} = \gamma_{g_1}(g_2ag_2^{-1}) = (\gamma_{g_1} \circ \gamma_{g_2})(a),$$

which implies $\gamma_{g_1g_2} = \gamma_{g_1} \circ \gamma_{g_2}$ and $g \mapsto \gamma_g$ is a homomorphism. If G is abelian, for all g the homomorphism

$$\gamma_g(a) = gag^{-1} = gg^{-1}a = a$$

is the identity in $\mathrm{Aut}(G)$. That is, the homomorphism $g \mapsto \gamma_g$ is trivial. If the homomorphism $g \mapsto \gamma_g$ is trivial, we have for all $g, a \in G$,

$$gag^{-1} = a,$$

which implies for all $a, b \in G$,

$$ab = bab^{-1}b = ba.$$

Thus we show the homomorphism $g \mapsto \gamma_g$ is trivial if and only if G is abelian. ■

> **4.9** Prove that if m, n are positive integers such that $\gcd(m, n) = 1$, then $C_{mn} \cong C_m \times C_n$.

Define a function

$$\begin{aligned} \varphi\ : C_m \times C_n &\longrightarrow C_{mn} \\ ([a]_m, [b]_n) &\longmapsto [anp + bmq]_{mn} \end{aligned}$$

where $[pn]_m = [1]_m$ and $[qm]_n = [1]_n$, as $\gcd(m, n) = 1$ guarantees the existence of p, q (see textbook p56). First of all, we have to check whether φ is well-defined. Note that

$$[(anp_1 + bmq_1) - (anp_2 + bmp_2)]_m = [a(p_1n - p_2n) + b(q_1m - q_2m)]_m = [0]_m$$

$$[(anp_1 + bmq_1) - (anp_2 + bmp_2)]_n = [a(p_1 n - p_2 n) + b(q_1 m - q_2 m)]_n = [0]_n$$

and $\gcd(m, n) = 1$. Thus we have

$$[(anp_1 + bmq_1) - (anp_2 + bmp_2)]_{mn} = [0]_{mn},$$

or

$$[anp_1 + bmq_1]_{mn} = [anp_2 + bmp_2]_{mn}.$$

Then we show φ is a homomorphism.

$$\begin{aligned}
\varphi(([a_1]_m, [b_1]_n) + ([a_2]_m, [b_2]_n)) &= \varphi([a_1 + a_2]_m, [b_1 + b_2]_n) \\
&= [(a_1 + a_2)np + (b_1 + b_2)mq]_{mn} \\
&= [a_1 np + b_1 mq]_{mn} + [a_2 np + b_2 mq]_{mn} \\
&= \varphi([a_1]_m, [b_1]_n) + \varphi([a_2]_m, [b_2]_n).
\end{aligned}$$

In order to show φ is a monomorphism, we can check

$$\begin{aligned}
&\varphi([a_1]_m, [b_1]_n) = \varphi([a_2]_m, [b_2]_n) \\
\Longrightarrow\ &[a_1 np + b_1 mq]_{mn} = [a_2 np + b_2 mq]_{mn} \\
\Longrightarrow\ &[(a_1 - a_2)np + (b_1 - b_2)mq]_{mn} = [0]_{mn} \\
\Longrightarrow\ &[(a_1 - a_2)np + (b_1 - b_2)mq]_m = [a_1 - a_2]_m = [0]_m, \\
&[(a_1 - a_2)np + (b_1 - b_2)mq]_n = [b_1 - b_2]_n = [0]_n \\
\Longrightarrow\ &[a_1]_m = [a_2]_m,\ [b_1]_m = [b_2]_m.
\end{aligned}$$

Since $|C_m \times C_n| = |C_{mn}| = mn$, we can conclude φ is an isomorphism. Thus we complete proving $C_{mn} \cong C_m \times C_n$.

■

§5. Free groups

5.1 Does the category $\mathscr{F}^A$ defined in §5.2 have final objects? If so, what are they?

Yes, they are functions from A to any trivial group, for example $T = \{t\}$.

$$\begin{tikzcd}
G \arrow[r, "\exists!\varphi"] & \{t\} \\
A \arrow[u, "j"] \arrow[ur, "e"'] &
\end{tikzcd}$$

For any object (j, G) in $\mathscr{F}^A$, the trivial homomorphism $\varphi : g \mapsto t$ is the unique homomorphism such that the diagram commutes. That is, $\operatorname{Hom}((j, G), (e, T)) = \{\varphi\}$. ■

5.2 Since trivial groups T are initial in Grp, one may be led to think that (e, T) should be initial in $\mathscr{F}^A$, for every A: e would be defined by sending every element of A to the (only) element in T ; and for any other group G, there is a unique homomorphism $T \to G$. Explain why (e, T) is not initial in $\mathscr{F}^A$ (unless $A = \emptyset$).

Let $G = C_2 = \{[0]_2, [1]_2\}$. Note that $\varphi \circ e(A)$ must be the trivial subgroup $\{[0]_2\}$. If $x \in A$ and $j(x) = [1]_2$, we see $\varphi \circ e \neq j$ and the following diagram does not commute.

$$\begin{tikzcd} T \arrow[r, "\varphi"] & G \\ A \arrow[u, "e"] \arrow[ur, "j"'] & \end{tikzcd}$$

That implies (e, T) is not initial in $\mathscr{F}^A$ unless $A = \emptyset$. ∎

5.3 Use the universal property of free groups to prove that the map $j : A \to F(A)$ is injective, for all sets A. (Hint: it suffices to show that for every two elements a, b of A there is a group G and a set-function $f : A \to G$ such that $f(a) = f(b)$. Why? and how do you construct f and G?) [§III.6.3]

Let $G = S_A$ be the symmetric group over A. Define functions $g_a : A \to A,\ x \mapsto a$ sending every element of A to a. Since $g_a \in S_A$, we can define an injection

$$\begin{aligned} f\ :\ A &\longrightarrow S_A \\ a &\longmapsto g_a \end{aligned}$$

In light of the commutative diagram

$$\begin{tikzcd} F(A) \arrow[r, "\varphi"] & S_A \\ A \arrow[u, "j"] \arrow[ur, "f"'] & \end{tikzcd}$$

we have $\forall a, b \in A$,

$$j(a) = j(b) \implies \varphi(j(a)) = \varphi(j(b)) \implies f(a) = f(b) \implies a = b.$$

∎

5.4 In the ‘concrete construction of free groups, one can try to reduce words by performing cancellations in any order; the ‘elementary reductions’ used in the text(that is, from left to right) is only one possibility. Prove that the result of iterating cancellations on a word is independent of the order in which the cancellations are performed. Deduce the associativity of the product in $F(A)$ from this. [§5.3]

We use induction on the length of w. If w is reduced, there is nothing to show. If not, there must be some pair of symbols that can be cancelled, say the underlined pair

$$w = \cdots \underline{xx^{-1}} \cdots .$$

(Let's allow x to denote any element of A', with the understanding that if $x = a^{-1}$ then $x^{-1} = a$.) If we show that we can obtain every reduced form of w by cancelling the pair xx^{-1} first, the proposition will follow by induction, because the word $w^* = \cdots \underline{\cancel{x}\cancel{x}^{-1}} \cdots$ is shorter.

Let w_0 be a reduced form of w. It is obtained from w by some sequence of cancellations. The first case is that our pair xx^{-1} is cancelled at some step in this sequence. If so, we may as well cancel xx^{-1} first. So this case is settled. On the other hand, since w_0 is reduced, the pair xx^{-1} can not remain in w_0. At least one of the two symbols must be cancelled at some time. If the pair itself is not cancelled, the first cancellation involving the pair must look like

$$\cdots \cancel{x}^{-1}\underline{\cancel{x}x^{-1}} \cdots \qquad \text{or} \qquad \cdots \underline{x\cancel{x}^{-1}}\cancel{x} \cdots$$

Notice that the word obtained by this cancellation is the same as the one obtained by cancelling the pair xx^{-1}. So at this stage we may cancel the original pair instead. Then we are back in the first case, so the proposition is proved.

■

5.5 Verify explicitly that $H^{\oplus A}$ is a group.

Assume the A is a set and H is an abelian group. $H^{\oplus A}$ are defined as follows

$$H^{\oplus A} := \{\alpha : A \to H | \alpha(a) \neq e_H \text{ for only finitely many elements } a \in A\}.$$

Now that $H^{\oplus A} \subset H^A := \mathrm{Hom}_{\mathsf{Set}}(A, H)$, we can first show $(H^A, +)$ is a group, where for all $\phi, \psi \in H^A$, $\phi + \psi$ is defined by

$$(\forall a \in A) : (\phi + \psi)(a) := \phi(a) + \psi(a).$$

Here is the verification:

- Identity: Define a function $\varepsilon : A \to H, a \mapsto e_H$ sending all elements in A to e_H. Then for any $\alpha \in H^A$ we have

  $$(\forall a \in A) : (\alpha + \varepsilon)(a) = \alpha(a) + \varepsilon(a) = \alpha(a),$$

  which is $\alpha + \varepsilon = \alpha$. Because of the commutativity of the operation $+$ defined on H^A, ε is the identity indeed.

- Associativity: This follows by the associativity in H:

  $$(\forall a \in A) : ((\alpha + \beta) + \gamma)(a) = (\alpha + \beta)(a) + \gamma(a) = \alpha(a) + (\beta + \gamma)(a) = (\alpha + (\beta + \gamma))(a).$$

- Inverse: Every function $\phi \in H^A$ has inverse $-\phi$ defined by

$$(\forall a \in A) : (-\phi)(a) = -\phi(a).$$

Thus H^A makes a group.

Then it is time to show $H^{\oplus A}$ is a subgroup of H^A. For all $\alpha, \beta \in H^{\oplus A}$, let $N_\alpha = \{a \in A | \alpha(a) \neq e_H\}$, $N_\beta = \{a \in A | \beta(a) \neq e_H\}$, $N_{\alpha-\beta} = \{a \in A | (\alpha - \beta)(a) \neq e_H\}$. Since

$$(\forall a \in A) : (\alpha - \beta)(a) = \alpha(a) - \beta(a),$$

we have

$$(\alpha - \beta)(a) \neq e_H \implies \alpha(a) \neq e_H \text{ or } \beta(a) \neq e_H,$$

which implies $N_{\alpha-\beta} \subset N_\alpha \cup N_\beta$. Note that N_α, N_β are both finite sets, which forces $N_{\alpha-\beta}$ to be finite. So there must be $\alpha - \beta \in H^{\oplus A}$. Now we see $H^{\oplus A}$ is closed under additions and inverses. And $e_{H^A} = \varepsilon \in H^{\oplus A}$ means that $H^{\oplus A}$ is nonempty. Finally we can conclude $H^{\oplus A}$ is a subgroup of H^A. ∎

5.6 Prove that the group $F(\{x, y\})$ (visualized in Example 5.3) is a coproduct $\mathbb{Z} * \mathbb{Z}$ of $\mathbb{Z}$ by itself in the category Grp. (Hint: with due care, the universal property for one turns into the universal property for the other.) [§3.4, 3.7, 5.7]

Define two homomorphisms

$$\begin{aligned} i_1 &: \mathbb{Z} \longrightarrow F(\{x, y\}), \quad n \longmapsto x^n, \\ i_2 &: \mathbb{Z} \longrightarrow F(\{x, y\}), \quad n \longmapsto y^n. \end{aligned}$$

We need to show that for any group G with two homomorphisms $f_1, f_2 : \mathbb{Z} \to G$, there exists a unique homomorphism φ such that the following diagram commutes.

$$\begin{tikzcd} \mathbb{Z} \arrow[d, "i_1"'] \arrow[dr, "f_1"] & \\ F(\{x,y\}) \arrow[r, "\varphi"] & G \\ \mathbb{Z} \arrow[u, "i_2"] \arrow[ur, "f_2"'] & \end{tikzcd}$$

Given the notation of indicator function

$$\mathbf{1}_A(x) := \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A, \end{cases}$$

we can define a function

$$
\begin{aligned}
\varphi : F(\{x, y\}) &\longrightarrow G, \\
z_1^{n_1} \cdots z_k^{n_k} &\longmapsto f_1(n_1)^{\mathbf{1}_{\{x\}}(z_1)} f_2(n_1)^{\mathbf{1}_{\{y\}}(z_1)} \cdots f_1(n_k)^{\mathbf{1}_{\{x\}}(z_n)} f_2(n_k)^{\mathbf{1}_{\{y\}}(z_n)},\ z_i \in \{x, y\}
\end{aligned}
$$

and check that it is a homomorphism indeed. For all $n \in \mathbb{Z}$, we have

$$
\begin{aligned}
(\varphi \circ i_1)(n) = \varphi(x^n) = f_1(n), \\
(\varphi \circ i_2)(n) = \varphi(y^n) = f_2(n),
\end{aligned}
$$

that is, the diagram commutes. Now we see φ exists. For the uniqueness of φ, let φ^* be another homomorphism that makes diagram commute. For all $z_1^{n_1} \cdots z_k^{n_k} \in F(\{x, y\})$, $z_i \in \{x, y\}$, we have

$$
\begin{aligned}
\varphi^*(z_1^{n_1} \cdots z_k^{n_k}) &= \varphi^*(z^{n_1}) \cdots \varphi^*(z^{n_k}) \\
&= \varphi^*(i_1(n_1))^{\mathbf{1}_{\{x\}}(z_1)} \varphi^*(i_2(n_1))^{\mathbf{1}_{\{y\}}(z_1)} \cdots \varphi^*(i_1(n_k))^{\mathbf{1}_{\{x\}}(z_1)} \varphi^*(i_2(n_k))^{\mathbf{1}_{\{y\}}(z_1)} \\
&= f_1(n_1)^{\mathbf{1}_{\{x\}}(z_1)} f_2(n_1)^{\mathbf{1}_{\{y\}}(z_1)} \cdots f_1(n_k)^{\mathbf{1}_{\{x\}}(z_n)} f_2(n_k)^{\mathbf{1}_{\{y\}}(z_n)} \\
&= \varphi(z_1^{n_1} \cdots z_k^{n_k}).
\end{aligned}
$$

To sum up, we have shown that the group $F(\{x, y\})$ is a coproduct $\mathbb{Z} * \mathbb{Z}$ of $\mathbb{Z}$ by itself in the category Grp. ∎

5.7 Extend the result of Exercise 5.6 to free groups $F(\{x_1, \dots, x_n\})$ and to free abelian groups $F^{ab}(\{x_1, \dots, x_n\})$. [3.4, 5.4]

Let $*$ be coproduct. Then we have $\underbrace{\mathbb{Z} * \mathbb{Z} * \cdots * \mathbb{Z}}_{n \text{ times}} \cong F(\{x_1, \dots, x_n\})$, as the following diagram demonstrates:

$$
\begin{tikzcd}
\mathbb{Z} \arrow[dr, "i_1"] \arrow[drr, "f_1"] & & \\
\vdots \quad \vdots & F(\{x_1, \dots, x_n\}) \arrow[r, "\varphi"] & G \\
\mathbb{Z} \arrow[ur, "i_n"'] \arrow[urr, "f_n"'] & &
\end{tikzcd}
$$

Dually, let $\times$ be product. Then we have $\underbrace{\mathbb{Z} \times \mathbb{Z} \times \cdots \times \mathbb{Z}}_{n \text{ times}} \cong F^{ab}(\{x_1, \cdots, x_n\})$, as the following diagram demonstrates:

$$\begin{tikzcd}
 & & & \mathbb{Z} \\
G \arrow[r, "\varphi"] \arrow[urrr, "f_1"] \arrow[drrr, "f_n"'] & F^{ab}(\{x_1, \dots, x_n\}) \arrow[urr, "\pi_1"] \arrow[drr, "\pi_n"'] & \vdots & \vdots \\
 & & & \mathbb{Z}
\end{tikzcd}$$

■

> **5.8** Still more generally, prove that $F(A \amalg B) = F(A) * F(B)$ and that $F^{ab}(A \amalg B) = F^{ab}(A) \oplus F^{ab}(B)$ for all sets A, B. (That is, the constructions F, F^{ab} 'preserve coproducts'.)

In order to show $F(A) * F(B)$ is a free group generated by $A \amalg B$, we should first set an appropriate function $\psi : A \amalg B \to F(A) * F(B)$ and then prove that given any (θ, G) there exists a unique group homomorphism g such that the following diagram commutes.

$$\begin{tikzcd}
A \amalg B \arrow[r, "\psi"] \arrow[rr, bend right, "\theta"'] & F(A) * F(B) \arrow[r, dashed, "\exists ! g"] & G
\end{tikzcd}$$

The complete proof can be divided into three steps, by decomposing the following diagram into parts.

$$\begin{tikzcd}
A \arrow[r, "j_1"] \arrow[d, "i_1"] & F(A) \arrow[d, "f_1"] \arrow[dr, dashed, "\varphi_1"] & \\
A \amalg B \arrow[r, dashed, "\psi"] \arrow[rr, bend right, "\theta"] & F(A) * F(B) \arrow[r, dashed, "g"] & G \\
B \arrow[r, "j_1"] \arrow[u, "i_2"] & F(B) \arrow[u, "f_2"] \arrow[ur, dashed, "\varphi_2"'] &
\end{tikzcd}$$

Step 1. Construct $\psi : A \amalg B \longrightarrow F(A) * F(B)$.
Define injective functions

$$\begin{aligned}
i_1 &: A \longrightarrow A \amalg B, \quad a \longmapsto (a, 1), \\
i_2 &: B \longrightarrow A \amalg B, \quad b \longmapsto (b, 2), \\
j_1 &: A \longrightarrow F(A), \quad a \longmapsto a, \\
j_2 &: B \longrightarrow F(B), \quad b \longmapsto b.
\end{aligned}$$

Let f_1, f_2 be the homomorphisms specified by the coproduct in Grp. Since $A \amalg B$ is a coproduct in Set, the universal property guarantees a unique mapping $\psi : A \amalg B \to F(A) *$

$F(B)$ such that the following diagram commutes

$$\begin{tikzcd}
A \arrow[r, "j_1"] \arrow[d, "i_1"] & F(A) \arrow[d, "f_1"] \\
A \amalg B \arrow[r, dashed, "\exists!\psi"] & F(A) * F(B) \\
B \arrow[r, "j_1"] \arrow[u, "i_2"] & F(B) \arrow[u, "f_2"]
\end{tikzcd}$$

That is,

$$\exists!\ \psi : A \amalg B \longrightarrow F(A) * F(B) \quad (\psi \circ i_1 = f_1 \circ j_1) \wedge (\psi \circ i_2 = f_2 \circ j_2).$$

Step 2. Prove the existence of g.

$$\begin{tikzcd}
A \arrow[r, "j_1"] \arrow[d, "i_1"] & F(A) \arrow[dr, dashed, "\exists!\varphi_1"] & \\
A \amalg B \arrow[rr, "\theta"] & & G \\
B \arrow[r, "j_1"] \arrow[u, "i_2"] & F(B) \arrow[ur, dashed, "\exists!\varphi_2"'] &
\end{tikzcd}$$

Given some (θ, G), according to the universal property of free groups $F(A)$, $F(B)$, we have

$$\exists!\ \varphi_1 : F(A) \longrightarrow G \quad (\varphi_1 \circ j_1 = \theta \circ i_1),$$
$$\exists!\ \varphi_2 : F(B) \longrightarrow G \quad (\varphi_2 \circ j_2 = \theta \circ i_2).$$

$$\begin{tikzcd}
F(A) \arrow[d, "f_1"] \arrow[dr, dashed, "\varphi_1"] & \\
F(A) * F(B) \arrow[r, dashed, "\exists!g"] & G \\
F(B) \arrow[u, "f_2"] \arrow[ur, dashed, "\varphi_2"'] &
\end{tikzcd}$$

Then according to the universal property of coproduct $F(A) * F(B)$ in Grp, we have

$$\exists!\ g : F(A) * F(B) \longrightarrow G \quad (g \circ f_1 = \varphi_1) \wedge (g \circ f_2 = \varphi_2).$$

The commutative diagram tells us

$$g \circ \psi \circ i_1 = g \circ f_1 \circ j_1 = \varphi_1 \circ j_1 = \theta \circ i_1,$$
$$g \circ \psi \circ i_2 = g \circ f_2 \circ j_2 = \varphi_2 \circ j_2 = \theta \circ i_2.$$

Note that $A \amalg B = i_1(A) \cup i_2(B)$. For all $x \in A \amalg B$, x must be either $i_1(a)$ or $i_2(b)$. If $x = i_1(a)$, then

$$g \circ \psi(x) = g \circ \psi \circ i_1(a) = \theta \circ i_1(a) = \theta(x).$$

If $x = i_2(b)$, then

$$g \circ \psi(x) = g \circ \psi \circ i_2(b) = \theta \circ i_2(b) = \theta(x).$$

Hence we show that given some (θ, G) there exists $g : F(A) * F(B) \longrightarrow G$ such that $g \circ \psi = \theta$.

Step 3. Prove the uniqueness of g.
Assume there exists another homomorphism h such that $h \circ \psi = \theta$. We have

$$\begin{aligned} h \circ f_1 \circ j_1 = h \circ \psi \circ i_1 = \theta \circ i_1, \\ h \circ f_2 \circ j_2 = h \circ \psi \circ i_2 = \theta \circ i_2. \end{aligned}$$

Since

$$\begin{aligned} \exists!\ \varphi_1 : F(A) \longrightarrow G \quad (\varphi_1 \circ j_1 = \theta \circ i_1), \\ \exists!\ \varphi_2 : F(B) \longrightarrow G \quad (\varphi_2 \circ j_2 = \theta \circ i_2), \end{aligned}$$

there must be

$$\begin{aligned} h \circ f_1 = \varphi_1, \\ h \circ f_2 = \varphi_2. \end{aligned}$$

Again by universal property

$$\exists!\ g : F(A) * F(B) \longrightarrow G \quad (g \circ f_1 = \varphi_1) \wedge (g \circ f_2 = \varphi_2)$$

we get $h = g$, which implies g is unique.

Conclusion.
To sum up, we prove that there exists a unique group homomorphism g such that the first diagram in this proof commutes. As a result, we have $F(A \amalg B) = F(A) * F(B)$. Note that if Grp turns into Ab, the method of diagram chasing applied here also works. In the light of the following diagram, we can get $F^{ab}(A \amalg B) = F^{ab}(A) \oplus F^{ab}(B)$ step by step.

$$\begin{tikzcd}
A \arrow[r, "j_1"] \arrow[d, "i_1"] & F^{ab}(A) \arrow[d, "f_1"] \arrow[dr, dashed, "\varphi_1"] & \\
A \amalg B \arrow[r, dashed, "\psi"] \arrow[rr, bend right, "\theta"] & F^{ab}(A) \oplus F^{ab}(B) \arrow[r, dashed, "g"] & G \\
B \arrow[u, "i_2"] \arrow[r, "j_1"] & F^{ab}(B) \arrow[u, "f_2"] \arrow[ur, dashed, "\varphi_2"'] &
\end{tikzcd}$$

$\blacksquare$

5.9 Let $G = \mathbb{Z}^{\oplus \mathbb{N}}$. Prove that $G \times G \cong G$.

Define a function

$$\begin{aligned} \varphi : G \times G &\longrightarrow G \\ ((a_1, a_2, \cdots), (b_1, b_2, \cdots)) &\longmapsto (a_1, b_1, a_2, b_2, \cdots) \end{aligned}$$

It is plain to check that φ is a homomorphism

$$\begin{aligned} &\varphi[((a_1, a_2, \cdots), (b_1, b_2, \cdots)) + ((a_1', a_2', \cdots), (b_1', b_2', \cdots))] \\ =&\varphi[((a_1 + a_1', a_2 + a_2', \cdots), (b_1 + b_1', b_2 + b_2', \cdots))] \\ =&(a_1 + a_1', b_1 + b_1', a_2 + a_2', b_2 + b_2', \cdots) \\ =&(a_1, b_1, a_2, b_2, \cdots) + (a_1', b_1', a_2', b_2', \cdots) \\ =&\varphi[((a_1, a_2, \cdots), (b_1, b_2, \cdots))] + \varphi[((a_1', a_2', \cdots), (b_1', b_2', \cdots))]. \end{aligned}$$

Since $\ker \varphi = \{(0, 0, \cdots)\}$ and $|G \times G| = |G| = \aleph_0$, we can conclude that φ is an isomorphism and accordingly $G \times G \cong G$. $\blacksquare$

5.10 $\neg$ Let $F = F^{ab}(A)$.

- Define an equivalence relation $\sim$ on F by setting $f \sim f'$ if and only if $f - f' = 2g$ for some $g \in F$. Prove that $F/\sim$ is a finite set if and only if A is finite, and in that case $|F/\sim| = 2^{|A|}$.

- Assume $F^{ab}(B) \cong F^{ab}(A)$. If A is finite, prove that so is B, and $A \cong B$ as sets. (This result holds for free groups as well, and without any finiteness hypothesis. See Exercises 7.13 and VI.1.20.)

[7.4, 7.13]

- If $|A| = \infty$, let $F = F^{ab}(A) = \mathbb{Z}^{\oplus A}$ and accordingly every element of $\mathbb{Z}^{\oplus A}$ can be written uniquely as a finite sum

  $$\sum_{a \in A} m_a j(a), \qquad m_a \neq 0 \text{ for only finitely many } a.$$

  Apparently, the elements in $j(A) = \{j(a) \mid a \in A\}$ are not equivalent pairwise. Note that j is an injection. Hence we see

  $$|F/\sim| \geq |j(A)| = A > \infty.$$

  In other words, $F/\sim$ is a finite set only if A is finite.
  If $|A| = n < \infty$, we can set $F = F^{ab}(A) = \mathbb{Z}^{\oplus n}$. Assume $f = (a_1, a_2, \cdots, a_n)$,

$f' = (a'_1, a'_2, \cdots, a'_n)$. Then $f \sim f'$ if and only if $a_i - a'_i \in 2\mathbb{Z}$ $(i = 1, 2, \cdots, n)$. Let $[f]$ denote the equivalence class including f. Thus we get

$$F/\sim = \{[(k_1, k_2, \cdots, k_n)] \mid k_i = 0 \text{ or } 1,\ i = 1, 2, \cdots, n\}$$

and accordingly $|F/\sim| = 2^{|A|}$.

- If A is finite, then $F^{ab}(A)$ is finite. $F^{ab}(B) \cong F^{ab}(A)$ guarantees that $F^{ab}(B)$ is finite. Hence we see B is finite. Furthermore it follows that

$$|F^{ab}(A)/\sim| = |F^{ab}(B)/\sim| \implies 2^{|A|} = 2^{|B|} \implies |A| = |B|.$$

That is, $A \cong B$ in Set.

∎

§6. Subgroups

6.1 ¬ (If you know about matrices.) The group of invertible $n \times n$ matrices with entries in R is denoted $\mathrm{GL}_n(\mathbb{R})$ (Example 1.5). Similarly, $\mathrm{GL}_n(\mathbb{C})$ denotes the group of $n \times n$ invertible matrices with complex entries. Consider the following sets of matrices:

- $\mathrm{SL}_n(\mathbb{R}) = \{M \in \mathrm{GL}_n(\mathbb{R}) \mid \det(M) = 1\}$;
- $\mathrm{SL}_n(\mathbb{C}) = \{M \in \mathrm{GL}_n(\mathbb{C}) \mid \det(M) = 1\}$;
- $\mathrm{O}_n(\mathbb{R}) = \{M \in \mathrm{GL}_n(\mathbb{R}) | MM^t = M^tM = I_n\}$;
- $\mathrm{SO}_n(\mathbb{R}) = \{M \in \mathrm{O}_n(\mathbb{R}) \mid \det(M) = 1\}$;
- $\mathrm{U}_n(\mathbb{C}) = \{M \in \mathrm{GL}_n(\mathbb{C}) | MM^\dagger = M^\dagger M = I_n\}$;
- $\mathrm{SU}_n(\mathbb{C}) = \{M \in \mathrm{U}_n(\mathbb{C}) \mid \det(M) = 1\}$.

Here In stands for the $n \times n$ identity matrix, M^t is the transpose of M, $M^\dagger$ is the conjugate transpose of M, and $\det(M)$ denotes the determinant of M. Find all possible inclusions among these sets, and prove that in every case the smaller set is a subgroup of the larger one.
These sets of matrices have compelling geometric interpretations: for example, $\mathrm{SO}^3(\mathbb{R})$ is the group of rotations in $\mathbb{R}^3$. [8.8, 9.1, III.1.4, VI.6.16]

The following diagram commutes, where all arrows are inclusions.

$$\begin{tikzcd}
\mathrm{GL}_n(\mathbb{R}) \arrow[r] & \mathrm{GL}_n(\mathbb{C}) \\
\mathrm{SL}_n(\mathbb{R}) \arrow[r] \arrow[u] & \mathrm{SL}_n(\mathbb{C}) \arrow[u] \\
\mathrm{O}_n(\mathbb{R}) \arrow[r] \arrow[u] & \mathrm{U}_n(\mathbb{C}) \arrow[u] \\
\mathrm{SO}_n(\mathbb{R}) \arrow[r] \arrow[u] & \mathrm{SU}_n(\mathbb{C}) \arrow[u]
\end{tikzcd}$$

■

6.2 ¬ Prove that the set of 2×2 matrices

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$$

with a, b, d in $\mathbb{C}$ and $ad \neq 0$ is a subgroup of $\mathrm{GL}_2(\mathbb{C})$. More generally, prove that the set of $n \times n$ complex matrices $(a_{ij})_{1\leq i,j\leq n}$ with $a_{ij} = 0$ for $i > j$, and $a_{11} \cdots a_{nn} \neq 0$, is a subgroup of $\mathrm{GL}_n(\mathbb{C})$. (These matrices are called 'upper triangular', for evident reasons.) [IV.1.20]

Let A, B are $n \times n$ upper triangular matrices. If $i > j$,

$$(AB)_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj} = \sum_{k=1}^{i-1} a_{ik}b_{kj} + \sum_{k=i}^{n} a_{ik}b_{kj} = \sum_{k=1}^{i-1} 0b_{kj} + \sum_{k=i}^{n} a_{ik}0 = 0,$$

which means the set of upper triangular matrices is closed with respect to the matrix multiplication. Thus it is a subgroup of $\mathrm{GL}_n(\mathbb{C})$. ■

6.3 ¬ Prove that every matrix in $\mathrm{SU}_2(\mathbb{C})$ may be written in the form

$$\begin{pmatrix} a+bi & c+di \\ -c+di & a-bi \end{pmatrix}$$

where $a, b, c, d \in \mathbb{R}$ and $a^2 + b^2 + c^2 + d^2 = 1$. (Thus, $\mathrm{SU}_2(\mathbb{C})$ may be realized as a three-dimensional sphere embedded in $\mathbb{R}^4$; in particular, it is simply connected.)[8.9, III.2.5]

Let

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in \mathrm{SU}_2(\mathbb{C})$$

and we have

$$AA^\dagger = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} \overline{a_{11}} & \overline{a_{21}} \\ \overline{a_{12}} & \overline{a_{22}} \end{pmatrix} = \begin{pmatrix} |a_{11}|^2 + |a_{12}|^2 & a_{11}\overline{a_{21}} + a_{12}\overline{a_{22}} \\ a_{21}\overline{a_{11}} + a_{22}\overline{a_{12}} & |a_{21}|^2 + |a_{22}|^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21} = 1$$

Note

$$\overline{a_{11}a_{12}} = \overline{a_{11}a_{12}} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = \begin{vmatrix} |a_{11}|^2 & |a_{12}|^2 \\ a_{21}\overline{a_{11}} & a_{22}\overline{a_{12}} \end{vmatrix} = \begin{vmatrix} |a_{11}|^2 & |a_{11}|^2 + |a_{12}|^2 \\ a_{21}\overline{a_{11}} & a_{21}\overline{a_{11}} + a_{22}\overline{a_{12}} \end{vmatrix} = \begin{vmatrix} |a_{11}|^2 & 1 \\ a_{21}\overline{a_{11}} & 0 \end{vmatrix} = -a_{21}\overline{a_{11}}$$

$$\implies \overline{a_{11}}(\overline{a_{12}} + a_{21}) = 0,$$

and

$$\overline{a_{21}a_{22}} = \overline{a_{21}a_{22}} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = \begin{vmatrix} a_{11}\overline{a_{21}} & a_{12}\overline{a_{22}} \\ |a_{21}|^2 & |a_{22}|^2 \end{vmatrix} = \begin{vmatrix} a_{11}\overline{a_{21}} & a_{11}\overline{a_{21}} + a_{12}\overline{a_{22}} \\ |a_{21}|^2 & |a_{21}|^2 + |a_{22}|^2 \end{vmatrix} = \begin{vmatrix} a_{11}\overline{a_{21}} & 0 \\ |a_{21}|^2 & 1 \end{vmatrix} = a_{11}\overline{a_{21}}$$

$$\implies \overline{a_{21}}(\overline{a_{11}} - a_{22}) = 0.$$

If $\overline{a_{11}} \neq 0$, it must be $\overline{a_{12}} + a_{21} = 0$. If $\overline{a_{11}} = 0$, then $|a_{12}|^2 = 1$, $a_{12}\overline{a_{22}} = 0$ and accordingly $a_{22} = 0$. Since $-a_{12}a_{21} = 1 = a_{12}\overline{a_{12}}$, we also have $\overline{a_{12}} + a_{21} = 0$, that is $a_{12} = c + di, a_{21} = -c + di$. Likewise, we can show $\overline{a_{11}} - a_{22} = 0$ and $a_{11} = a + bi, a_{22} = a - bi$. And we have

$$|a_{11}|^2 + |a_{12}|^2 = a^2 + b^2 + c^2 + d^2 = 1.$$

■

6.4 Let G be a group, and $g \in G$. Verify that the image of the exponential map $\epsilon_g : \mathbb{Z} \to G$ is a cyclic group (in the sense of Definition 4.7).

If $|g| = \infty$, then $g^i \neq g^j (i \neq j)$. Define

$$\varphi : \mathbb{Z} \longrightarrow \epsilon_g(\mathbb{Z}), n \longmapsto g^n$$

and we can check it is an isomorphism.
If $|g| = k$, then $e_G, g, g^2, \cdots, g^{k-1}$ are distinct. Define

$$\varphi : \mathbb{Z}/k\mathbb{Z} \longrightarrow \epsilon_g(\mathbb{Z}), [n]_k \longmapsto g^n$$

and we can check it is an isomorphism.
Since $\epsilon_g(\mathbb{Z})$ is isomorphic to $\mathbb{Z}$ or $\mathbb{Z}/k\mathbb{Z}$, we show $\epsilon_g(\mathbb{Z})$ is a cyclic group. ■

6.6 Prove that the union of a family of subgroups of a group G is not necessarily a subgroup of G. In fact:

- Let H, H' be subgroups of a group G. Prove that $H \cup H'$ is a subgroup of G only if $H \subseteq H'$ or $H' \subseteq H$.

- On the other hand, let $H_0 \subseteq H_1 \subseteq H_2 \subseteq \cdots$ be subgroups of a group G. Prove that $\cup_{i\geq 0} H_i$ is a subgroup of G.

- Let $H \cup H'$ be a subgroup of G. Suppose neither $H \subseteq H'$ nor $H' \subseteq H$ hold. Let $a \in H - H'$, $b \in H' - H$, $h = ab^{-1} \in H \cup H'$. In the case of $h \in H$, we have $b = h^{-1}a \in H$, contradiction! In the case of $h \in H'$, we have $a = hb \in H'$, contradiction again! Therefore, there must be $H \subseteq H'$ or $H' \subseteq H$.

- For all $a, b \in \cup_{i\geq 0} H_i$, we can suppose $a \in H_j, b \in H_k$ and we have $a, b \in H_{\max\{j,k\}}$. Then $ab \in H_{\max\{j,k\}} \subseteq \cup_{i\geq 0} H_i$, implies that $\cup_{i\geq 0} H_i$ is closed and that $\cup_{i\geq 0} H_i$ is a subgroup of G.

■

6.7 ¬ Show that inner automorphisms (cf. Exercise 4.8) form a subgroup of $\mathrm{Aut}(G)$; this subgroup is denoted $\mathrm{Inn}(G)$. Prove that $\mathrm{Inn}(G)$ is cyclic if and only if $\mathrm{Inn}(G)$ is trivial if and only if G is abelian. (Hint: Assume that $\mathrm{Inn}(G)$ is cyclic; with notation as in Exercise 4.8, this means that there exists an element $a \in G$ such that $\forall g \in G\ \exists n\ \in Z\ \gamma_g = \gamma_a^n$. In particular, $gag^{-1} = a^n a a^{-n} = a$. Thus a commutes with every g in G. Therefore...) Deduce that if $\mathrm{Aut}(G)$ is cyclic then G is abelian. [7.10, IV.1.5]

With notation as in Exercise 4.8, we assume $\gamma_g \in \mathrm{Inn}(G)$ is defined by

$$\forall h \in G\ \ (\gamma_g(h) = ghg^{-1}).$$

We have

$$\begin{aligned}
&\mathrm{Inn}(G) \text{ is cyclic} \\
\iff &\exists \gamma_a \in \mathrm{Inn}(G),\ \mathrm{Inn}(G) = \langle \gamma_a \rangle \\
\iff &\exists a \in G\ \forall g \in G\ \exists n \in \mathbb{Z}\ (\gamma_g = \gamma_a^n) \\
\implies &\exists a \in G\ \forall g \in G\ \exists n \in \mathbb{Z}\ (\gamma_g(a) = gag^{-1} = \gamma_a^n(a) = a^n a a^{-n} = a) \\
\implies &\exists a \in G\ \forall g \in G\ (ga = ag) \\
\implies &\forall h \in G, \gamma_a(h) = aha^{-1} = haa^{-1} = h \\
\implies &\mathrm{Inn}(G) = \langle \mathrm{id} \rangle \\
\implies &\mathrm{Inn}(G) \text{ is trivial}
\end{aligned}$$

$$
\begin{aligned}
&\mathrm{Inn}(G) \text{ is trivial} \\
\Longrightarrow\ & \forall g \in G\ \forall h \in G\ (\gamma_g(h) = ghg^{-1} = h) \\
\Longrightarrow\ & \forall g \in G\ \forall h \in G\ (gh = hg) \\
\Longleftrightarrow\ & G \text{ is abelian}
\end{aligned}
$$

$$
\begin{aligned}
&G \text{ is abelian} \\
\Longrightarrow\ & \forall g \in G\ \forall h \in G\ (\gamma_g(h) = ghg^{-1} = h) \\
\Longrightarrow\ & \mathrm{Inn}(G) = \{\mathrm{id}\} \\
\Longrightarrow\ & \mathrm{Inn}(G) \text{ is cyclic}
\end{aligned}
$$

If $\mathrm{Aut}(G)$ is cyclic, its subgroup $\mathrm{Inn}(G)$ is also cyclic. As we have shown, that means G is abelian. ∎

6.8 Prove that an abelian group G is finitely generated if and only if there is a surjective homomorphism

$$\underbrace{\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}_{n \text{ times}} \twoheadrightarrow G$$

for some n.

Given any set $H \subseteq G$, there exists a unique homomorphism φ_H such that the following diagram commutes.

$$
\begin{tikzcd}
F^{ab}(H) \arrow[r, "\exists!\varphi"] & G \\
H \arrow[u, "j"] \arrow[ur, hook, "i"'] &
\end{tikzcd}
$$

The homomorphism image $\varphi_H(F^{ab}(H)) \leq G$ is called the subgroup generated by H in G, denoted by $\langle H \rangle$.

If G is finitely generated, there is a finite subset $G_n \subseteq G$ with n elements such that $\varphi_H(F^{ab}(G_n)) = \varphi_H(\mathbb{Z}^{\oplus n}) = G$. And φ_H is exactly the surjective homomorphism that we need.

If there is a surjective homomorphism $\psi : \mathbb{Z}^{\oplus n} \twoheadrightarrow G$ for some n. Suppose

$$\psi : \mathbf{1}_i = (0, \cdots, 0, \underset{i\text{-th place}}{1}, 0, \cdots, 0) \longmapsto g_i$$

and $G_n = \{g_1, g_2, \cdots, g_n\}$. Then define

$$j : G_n \longrightarrow \mathbb{Z}^{\oplus n}, \quad g_i \longmapsto \mathbf{1}_i.$$

We can check the following diagram commutes

$$
\begin{tikzcd}
\mathbb{Z}^{\oplus n} \arrow[r, two heads, "\psi"] & G \\
G_n \arrow[u, "j"] \arrow[ur, hook, "i"'] &
\end{tikzcd}
$$

which means $\langle G_n \rangle = \psi(\mathbb{Z}^{\oplus n})$. Since ψ is surjective, we have $\langle G_n \rangle = G$. Hence we show G is finitely generated. ■

> **6.9** Prove that every finitely generated subgroup of $\mathbb{Q}$ is cyclic. Prove that $\mathbb{Q}$ is not finitely generated.

Given any two rationals

$$
\begin{aligned}
a_1 &= \frac{p_1}{q_1} \in \mathbb{Q}, (p_1, q_1) = 1, \\
a_2 &= \frac{p_2}{q_2} \in \mathbb{Q}, (p_2, q_2) = 1,
\end{aligned}
$$

there exists $r = \frac{1}{q_1 q_2} \in \mathbb{Q}$ such that $\langle a_1, a_2 \rangle \le \langle r_1 \rangle$. Then for some a_3 we have $\langle a_1, a_2, a_3 \rangle \le \langle r_1, a_3 \rangle \le \langle r_2 \rangle$. In general, let's set $B_n = \{a_1, a_2, \cdots, a_n\}$. If $\langle B_n \rangle \le \langle r_{n-1} \rangle$. we have $\langle B_{n+1} \rangle = \langle B_n, a_{n+1} \rangle \le \langle r_{n-1}, a_{n+1} \rangle \le \langle r_n \rangle$. By induction we can prove $\langle a_1, a_2, \cdots, a_n \rangle \le \langle r_{n-1} \rangle$ for $n \in \mathbb{N}_+$. Since the subgroups of a cyclic group are also cyclic, we see finitely generated subgroup $\langle a_1, a_2, \cdots, a_n \rangle$ is cyclic. ■

> **6.10** $\neg$ The set of 2×2 matrices with integer entries and determinant 1 is denoted $\mathrm{SL}_2(\mathbb{Z})$:
>
> $$\mathrm{SL}_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ such that } a, b, c, d \in \mathbb{Z},\ ad - bc = 1 \right\}.$$
>
> Prove that $\mathrm{SL}_2(\mathbb{Z})$ is generated by the matrices:
>
> $$s = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad t = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Let H be the subgroup generated by s and t. We can check that both

$$P = \begin{pmatrix} 1 & -p \\ 0 & 1 \end{pmatrix} = t^{-p} \quad \text{and} \quad Q = \begin{pmatrix} 1 & 0 \\ -q & 1 \end{pmatrix} = s^{-1} t^q s$$

are in H. Given an arbitrary matrix

$$m = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}),$$

it suffices to show that we can obtain the identity I_2 by multiplying m by matrices in H. Note that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & -p \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b-pa \\ c & d-pc \end{pmatrix}, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -q & 1 \end{pmatrix} = \begin{pmatrix} a-qb & b \\ c-qd & d \end{pmatrix},$$

and c, d cannot be nonzero simultaneously. Without loss of generality, we can assume that $0 < c < d$ and perform Euclidean algorithm. Let $p_1 = \lfloor \frac{d}{c} \rfloor, d_1 = d - p_1 c < c$. Multiplying m by $P_1 = \begin{pmatrix} 1 & -p_1 \\ 0 & 1 \end{pmatrix}$ on the right yields

$$m_1 = mP_1 \begin{pmatrix} a & b - p_1 a \\ c & d_1 \end{pmatrix}.$$

Then let $q_1 = \lfloor \frac{c}{d_1} \rfloor, c_1 = c - q_1 d_1 < d_1$ and right multiplying m by $Q_1 = \begin{pmatrix} 1 & 0 \\ -q_1 & 1 \end{pmatrix}$ yields

$$m_2 = mP_1Q_1 \begin{pmatrix} a - q_1(b-p_1a) & b - p_1 a \\ c_1 & d_1 \end{pmatrix}.$$

We can repeat this procedure until some d_i or c_i reduce to 0. The Euclidean algorithm generates a sequence

$$d > c > d_1 > c_1 > d_2 > c_2 > \cdots.$$

If c_i, d_i never reduce to 0, we will get an infinite decreasing positive sequence, which is impossible. Suppose d_N is the first number reducing to 0. Then

$$m_{2N-1} = mP_1Q_1 \cdots P_N = \begin{pmatrix} a_N & b_N \\ c_{N-1} & 0 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}),$$

which implies

$$m_{2N-1} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and $m_{2N-1}s^{-1} = I_2$. Suppose c_N is the first number reducing to 0. Then

$$m_{2N} = mP_1Q_1 \cdots P_N Q_N = \begin{pmatrix} a_N & b_N \\ 0 & d_N \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}),$$

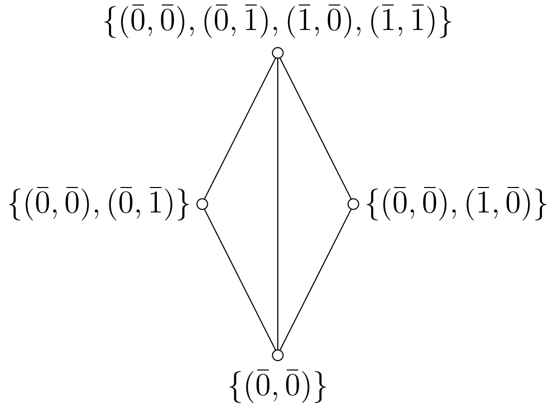
which implies

$$m_{2N} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2.$$

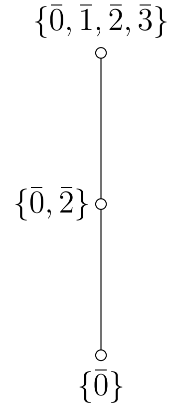
We have shown that we can obtain the identity I_2 by multiplying m by matrices in H, that is, m can be represented as a product of matrices in H. Thus we can conclude $\mathrm{SL}_2(\mathbb{Z})$ is generated by s and t. ∎

6.13 $\neg$ Draw and compare the lattices of subgroups of $C_2 \times C_2$ and C_4. Drawthe lattice of subgroups of S_3, and compare it with the one for C_6. [7.1]

Lattices of subgroups $C_2 \times C_2$ and C_4 are drawn as follows:

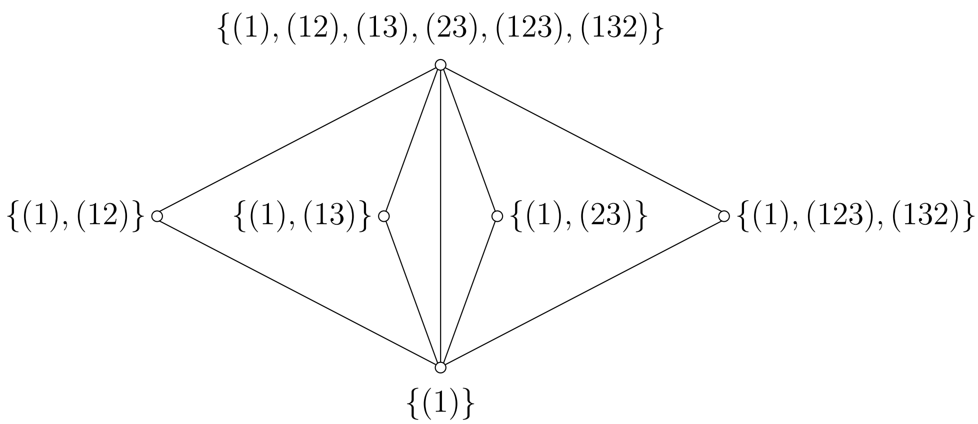


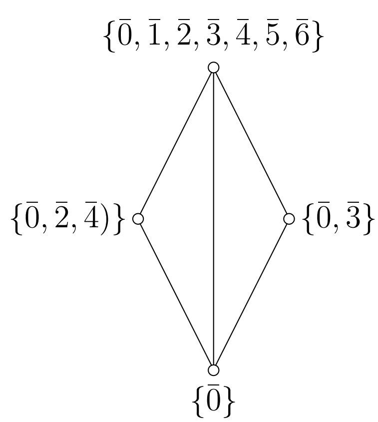
$C_2 \times C_2$



C_4

Lattices of subgroups S_3 and C_6 are drawn as follows:





■

§7. Quotient groups

7.1 ▷ List all subgroups of S_3 (cf. Exercise 6.13) and determine which subgroups are normal and which are not normal. [§7.1]

The subgroups of S_3 are $\{(1)\}, \{(1),(12)\}, \{(1),(13)\}, \{(1),(23)\}, \{(1),(123),(132)\}$ and S_3. We can check that $\{(1)\}, \{(1),(123),(132)\}, S_3$ are normal subgroups while others are not. ■

7.2 Is the image of a group homomorphism necessarily a normal subgroup of the target?

No. According to exercise 7.1 we have seen not all subgroups are normal. Suppose H is a subgroup of G but not normal. Then H itself is the image of the inclusion homomorphism $i: H \hookrightarrow G$, which makes a counterexample. ■

7.3 ▷ Verify that the equivalent conditions for normality given in 7.1 are indeed equivalent. [§7.1]

That a subgroup N of G is normal has four equivalent conditions:

(i) $\forall g \in G,\ gNg^{-1} = N$;

(ii) $\forall g \in G,\ gNg^{-1} \subseteq N$;

(iii) $\forall g \in G,\ gN \subseteq Ng$;

(iv) $\forall g \in G,\ gN = Ng$.

(i) $\Longrightarrow$ (ii) is straightforward.
(ii) $\Longrightarrow$ (iii). For any $g \in G$, the element $a \in gN$ can be written as $a = gn_1 (n_1 \in N)$. Since $gn_1g^{-1} \in gNg^{-1} \subseteq N$, there exists an $n_2 \in N$ such that $gn_1g^{-1} = n_2$, which implies $gn_1 = n_2g \in Ng$. Thus we have $gN \subseteq Ng$.
(iii) $\Longrightarrow$ (iv). Given any $g \in G$, for all $n_1 \in N$, the element $g^{-1}n_1 \in g^{-1}N_1$ also belongs to Ng^{-1}, which implies that there exists $n_2 \in N$ such that $g^{-1}n_1 = n_2g^{-1}$, namely $n_1g = gn_2$. Thus we get $Ng \subseteq gN$ and accordingly $gN = Ng$.
(iv) $\Longrightarrow$ (i). For any $g \in G$, the element $b \in gNg^{-1}$ can be written as $a = gn_1g^{-1}(n_1 \in N)$. Since $gn_1 \in gN = Ng$, there exists an $n_2 \in N$ such that $gn_1 = n_2g$, which implies $gn_1g^{-1} = n_2 \in N$. Thus we have

$$\begin{aligned} &\forall g \in G, \quad gNg^{-1} \subseteq N \\ \Longrightarrow &\forall g^{-1} \in G, \quad g^{-1}(gNg^{-1})g \subseteq gNg^{-1} \\ \Longrightarrow &\forall g \in G, \quad N \subseteq gNg^{-1}. \end{aligned}$$

Hence we have $\forall g \in G,\ gNg^{-1} = N$. ■

7.4 Prove that the relation defined in Exercise 5.10 on a free abelian group $F = F^{ab}(A)$ is compatible with the group structure. Determine the quotient $F/\sim$ as a better known group.

For all $f, f', h \in F$,

$$f \sim f' \iff f - f' = 2g,\ (g \in F) \implies (h+f)-(h+f') = 2g,\ (g \in F) \iff h+f \sim h+f'.$$

Since F is abelian, wee see the relation $\sim$ defined on a free abelian group $F = F^{ab}(A)$ is compatible with the group structure. By the notation of quotient group, we have

$$F/{\sim} = F/2F,$$

where $2F = \{2g \in F \mid g \in F\}$. ∎

7.5 ¬ Define an equivalence relation $\sim$ on $\mathrm{SL}_2(\mathbb{Z})$ by letting $A \sim A' \iff A' = \pm A$. Prove that $\sim$ is compatible with the group structure. The quotient $\mathrm{SL}_2(\mathbb{Z})/\sim$ is denoted $\mathrm{PSL}_2(\mathbb{Z})$, and is called the *modular group*; it would be a serious contender in a context for 'the most important group in mathematics', due to its role in algebraic geometry and number theory. Prove that $\mathrm{PSL}_2(\mathbb{Z})$ is generated by the (cosets of the) matrices

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}.$$

(You will not need to work very hard, if you use the result of Exercise 6.10.) Note that the first has order 2 in $\mathrm{PSL}_2(\mathbb{Z})$, the second has order 3, and their product has infinite order. [9.14]

For all $A_1, A_2, B \in \mathrm{SL}_2(\mathbb{Z})$,

$$A_1 \sim A_2 \iff A_2 = \pm A_1 \iff BA_2 = \pm BA_1 \iff BA_1 \sim BA_2.$$

Hence $\sim$ is compatible with the group structure and $\mathrm{PSL}_2(\mathbb{Z}) = \mathrm{SL}_2(\mathbb{Z})/\{I_2, -I_2\}$. In Exercise 6.10 we have shown $\mathrm{SL}_2(\mathbb{Z})$ is generated by the matrices

$$s = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad t = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

It is clear that $\mathrm{SL}_2(\mathbb{Z})$ can also be generated by the matrices

$$s = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad ts = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix},$$

which implies $\mathrm{PSL}_2(\mathbb{Z})$ is generated by the cosets of the matrices s and ts. ∎

§8. Canonical decomposition and Lagranges theorem

8.1 If a group H may be realized as a subgroup of two groups G_1 and G_2, and

$$\frac{G_1}{H} \cong \frac{G_2}{H},$$

does it follows that $G_1 \cong G_2$. Give a proof or a counterexample.

Take $H = C_3$, the cyclic group of order 3. Take $G_1 = D_6$ and $G_2 = C_6$, then one sees both G_1/H and G_2/H are C_2. But obviously G_1 and G_2 are not isomorphic, one being abelian while the other is not. ∎

8.2 ¬ Extend Example 8.6 as follows. Suppose G is a group, and $H \subseteq G$ is a subgroup of index 2: that is, such that there are precisely two (say, left) cosets of H in G. Prove that H is normal in G. [9.11, IV.1.16]

Since $[G/H] = 2$, there must be $G/H = \{H, G - H\}$. For any $g \in G$:

- if $g \in H$, then $gH = Hg = H$;
- if $g \in G - H$, then $gH \neq H$ and $Hg \neq H$. Thus we have $gH = Hg = G - H$.

To sum up, $gH = Hg$ follows for all $g \in G$, which implies H is normal in G. ∎

§9. Group actions

§10. Group objects in categories

References