

Algebra, Chapter 0

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Notation for Problems

\triangleright : those problems that are directly referenced from the text.

\neg : those problems that are referenced from other problems.

[§II.8.1]: related to the text in II.8.1 (Chapter II Section 8 Subsection 1).

[II.8.10]: related to the Definition/Example/Proposition/Lemma/Corollary 8.10 in Chapter II and (the 10th Definition/Example/Proposition/Lemma/Corollary in Chapter II Section 8).

Chapter I. Preliminaries: Set theory and categories

§1. Naive Set Theory

1.6 Define a relation \sim on the set \mathbb{R} of real numbers, by setting $a \sim b \iff b - a \in \mathbb{Z}$. Prove that this is an equivalence relation, and find a ‘compelling’ description for \mathbb{R}/\sim . Do the same for the relation \approx on the plane $\mathbb{R} \times \mathbb{R}$ defined by declaring $(a_1, a_2) \approx (b_1, b_2) \iff b_1 - a_1 \in \mathbb{Z}$ and $b_2 - a_2 \in \mathbb{Z}$. [§II.8.1, II.8.10]

Imaginatively, \mathbb{R}/\sim can be viewed as a ring of length 1 by bending the real line \mathbb{R} . Then we can rotate a ring around an axis of rotation to get $\mathbb{R} \times \mathbb{R}/\approx$, which makes a torus. ■

§2. Functions between sets

2.1 How many different bijections are there between a set S with n elements and itself? [§II.2.1]

There are $n!$ different bijections $S \rightarrow S$. ■

§3. Categories

3.1 Let \mathbf{C} be a category. Consider a structure \mathbf{C}^{op} with:

- $\text{Obj}(\mathbf{C}^{op}) := \text{Obj}(\mathbf{C})$;
- for A, B objects of \mathbf{C}^{op} (hence, objects of \mathbf{C}), $\text{Hom}_{\mathbf{C}^{op}}(A, B) := \text{Hom}_{\mathbf{C}}(B, A)$

Show how to make this into a category (that is, define composition of morphisms in \mathbf{C}^{op} and verify the properties listed in §3.1). Intuitively, the ‘opposite’ category \mathbf{C}^{op} is simply obtained by ‘reversing all the arrows’ in \mathbf{C} . [5.1, §VIII.1.1, §IX.1.2, IX.1.10]

- For every object A of \mathbf{C} , there exists one identity morphism $1_A \in \text{Hom}_{\mathbf{C}}(A, A)$. Since $\text{Obj}(\mathbf{C}^{op}) := \text{Obj}(\mathbf{C})$ and $\text{Hom}_{\mathbf{C}^{op}}(A, A) := \text{Hom}_{\mathbf{C}}(A, A)$, for every object A of \mathbf{C}^{op} , the identity on A coincides with $1_A \in \mathbf{C}$.
- For A, B, C objects of \mathbf{C}^{op} and $f \in \text{Hom}_{\mathbf{C}^{op}}(A, B) = \text{Hom}_{\mathbf{C}}(B, A)$, $g \in \text{Hom}_{\mathbf{C}^{op}}(B, C) = \text{Hom}_{\mathbf{C}}(C, B)$, the composition laws in \mathbf{C} determines a morphism $f * g$ in $\text{Hom}_{\mathbf{C}}(C, A)$, which deduces the composition defined on \mathbf{C}^{op} :

$$\begin{aligned} \text{Hom}_{\mathbf{C}^{op}}(A, B) \times \text{Hom}_{\mathbf{C}^{op}}(B, C) &\longrightarrow \text{Hom}_{\mathbf{C}^{op}}(A, C) \\ (f, g) &\longmapsto g \circ f := f * g \end{aligned}$$

- Associativity. If $f \in \text{Hom}_{\mathbf{C}^{op}}(A, B)$, $g \in \text{Hom}_{\mathbf{C}^{op}}(B, C)$, $h \in \text{Hom}_{\mathbf{C}^{op}}(C, D)$, then

$$f \circ (g \circ h) = f \circ (h * g) = (h * g) * f = h * (g * f) = (g * f) \circ h = (f \circ g) \circ h.$$

- Identity. For all $f \in \text{Hom}_{\mathcal{C}^{op}}(A, B)$, we have

$$f \circ 1_A = 1_A * f = f, \quad 1_B \circ f = f * 1_B = f.$$

Thus we get the full construction of \mathcal{C}^{op} . ■

3.3 ▷ Formulate precisely what it means to say that 1_a is an identity with respect to composition in Example 3.3, and prove this assertion. [§3.2]

Suppose S is a set, and \sim is a relation on S satisfying the reflexive and transitive property. Then we can encode this data into a category \mathcal{C} :

- Objects: the elements of S ;
- Morphisms: if a, b are objects (that is: if $a, b \in S$) then let $\text{Hom}(a, b)$ be the set consisting of the element $(a, b) \in S \times S$ if $a \sim b$, and $\text{Hom}(a, b) = \emptyset$. otherwise.

Given the composition of two morphisms

$$\begin{aligned} \text{Hom}_{\mathcal{C}}(A, B) \times \text{Hom}_{\mathcal{C}}(B, C) &\longrightarrow \text{Hom}_{\mathcal{C}}(A, C) \\ (a, b) \circ (b, c) &\longmapsto (a, c) \end{aligned}$$

we are asked to check $1_a = (a, a)$ is an identity with respect to this composition. ■

§4. Morphisms

4.2 In Example 3.3 we have seen how to construct a category from a set endowed with a relation, provided this latter is reflexive and transitive. For what types of relations is the corresponding category a groupoid (cf. Example 4.6)? [§4.1]

For a reflexive and transitive relation \sim on a set S , define the category \mathcal{C} as follows:

- Objects: $\text{Obj}(\mathcal{C}) = S$;
- Morphisms: if a, b are objects (that is: if $a, b \in S$) then let

$$\text{Hom}_{\mathcal{C}}(a, b) = \begin{cases} (a, b) \in S \times S & \text{if } a \sim b \\ \emptyset & \text{otherwise} \end{cases}$$

In Example 3.3 we have shown the category. If the relation \sim is endowed with symmetry, we have

$$(a, b) \in \text{Hom}_{\mathcal{C}}(a, b) \implies a \sim b \implies b \sim a \implies (b, a) \in \text{Hom}_{\mathcal{C}}(b, a).$$

Since

$$(a, b)(b, a) = (a, a) = 1_a, \quad (b, a)(a, b) = (b, b) = 1_b,$$

in fact (a, b) is an isomorphism. From the arbitrariness of the choice of (a, b) , we show that \mathbf{C} is a groupoid. Conversely, if \mathbf{C} is a groupoid, we can show the relation \sim is symmetric. To sum up, the category \mathbf{C} is a groupoid if and only if the corresponding relation \sim is an equivalence relation. ■

§5. Universal properties

5.1 Prove that a final object in a category \mathbf{C} is initial in the opposite category \mathbf{C}_{op} (cf. [Exercise I.3.1](#)).

An object F of \mathbf{C} is final in \mathbf{C} if and only if

$$\forall A \in \text{Obj}(\mathbf{C}) : \text{Hom}_{\mathbf{C}}(A, F) \text{ is a singleton.}$$

That is equivalent to

$$\forall A \in \text{Obj}(\mathbf{C}_{op}) : \text{Hom}_{\mathbf{C}_{op}}(F, A) \text{ is a singleton,}$$

which means F is initial in the opposite category \mathbf{C}_{op} . ■

Chapter II. Groups, first encounter

§1. Definition of group

1.1 Write a careful proof that every group is the group of isomorphisms of a groupoid. In particular, every group is the group of automorphisms of some object in some category.

Assume G is a group. Define a category \mathbf{C} as follows:

- Objects: $\text{Obj}(\mathbf{C}) = \{*\}$;
- Morphisms: $\text{Hom}_{\mathbf{C}}(*, *) = \text{End}_{\mathbf{C}}(*) = G$.

The composition of homomorphism is corresponding to the multiplication between two elements in G . The identity morphism on $*$ is $1_* = e_G$, which satisfies for all $g \in \text{Hom}_{\mathbf{C}}(*, *)$,

$$ge_G = e_Gg = g,$$

and

$$gg^{-1} = e_G, \quad g^{-1}g = e_G.$$

Thus any homomorphism $g \in \text{Hom}_{\mathbf{C}}(*, *)$ is an isomorphism and accordingly \mathbf{C} is a groupoid. Now we see $G = \text{End}_{\mathbf{C}}(*)$ is the group of isomorphisms of a groupoid. Moreover, supposing that $*$ is an object in some category \mathbf{D} , G would be the group of automorphisms of $*$, which is denoted as $\text{Aut}_{\mathbf{D}}(*)$. ■

1.4 Suppose that $g^2 = e$ for all elements g of a group G ; prove that G is commutative.

For all $a, b \in G$,

$$abab = e \implies a(abab)b = ab \implies (aa)ba(bb) = ab \implies ba = ab.$$

■

§2. Examples of groups

2.1 One can associate an $n \times n$ matrix M_σ with a permutation $\sigma \in S_n$, by letting the entry at $(i, \sigma(i))$ be 1, and letting all other entries be 0. For example, the matrix corresponding to the permutation

$$\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \in S_3$$

would be

$$M_\sigma = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

Prove that, with this notation,

$$M_{\sigma\tau} = M_\sigma M_\tau$$

for all $\sigma, \tau \in S_n$, where the product on the right is the ordinary product of matrices.

By introducing the Kronecker delta function

$$\delta_{i,j} = \begin{cases} 0 & \text{if } i \neq j, \\ 1 & \text{if } i = j, \end{cases}$$

the entry at (i, j) of the matrix $M_{\sigma\tau}$ can be written as

$$(M_{\sigma\tau})_{i,j} = \delta_{\tau(\sigma(i)),j}$$

and the entry at (i, j) of the matrix $M_\sigma M_\tau$ can be written as

$$(M_\sigma M_\tau)_{i,j} = \sum_{k=1}^n (M_\sigma)_{i,k} (M_\tau)_{k,j} = \sum_{k=1}^n \delta_{\sigma(i),k} \cdot \delta_{\tau(k),j} = \sum_{k=1}^n \delta_{\sigma(i),k} \cdot \delta_{k,\tau^{-1}(j)} = \delta_{\sigma(i),\tau^{-1}(j)},$$

where the last but one equality holds by the fact

$$\tau(k) = j \iff k = \tau^{-1}(j).$$

Noticing that

$$\tau(\sigma(i)) = j \iff \sigma(i) = \tau^{-1}(j),$$

we see $M_{\sigma\tau} = M_\sigma M_\tau$ for all $\sigma, \tau \in S_n$. ■

2.2 Prove that if $d \leq n$, then S_n contains elements of order d .

The cyclic permutation

$$\sigma = (1 \ 2 \ 3 \ \cdots \ d)$$

is an element of order d in S_n . ■

2.3 For every positive integer n find an element of order n in $S_{\mathbb{N}}$.

The cyclic permutation

$$\sigma = (1\ 2\ 3 \cdots n)$$

is an element of order d in S_n . ■

2.4 Define a homomorphism $D_8 \rightarrow S_4$ by labeling vertices of a square, as we did for a triangle in §2.2. List the 8 permutations in the image of this homomorphism.

The image of n rotations under the homomorphism are

$$\sigma_1 = e_{D_8}, \sigma_2 = (1\ 2\ 3\ 4), \sigma_3 = (1\ 3)(2\ 4), \sigma_4 = (1\ 4\ 3\ 2).$$

The image of n reflections under the homomorphism are

$$\sigma_5 = (1\ 3), \sigma_6 = (2\ 4), \sigma_7 = (1\ 2)(3\ 4), \sigma_8 = (1\ 4)(3\ 2).$$

■

2.11 Prove that the square of every odd integer is congruent to 1 modulo 8.

Given an odd integer $2k + 1$, we have

$$(2k + 1)^2 = 4k(k + 1) + 1,$$

where $k(k + 1)$ is an even integer. So $(2k + 1)^2 \equiv 1 \pmod{8}$. ■

2.12 Prove that there are no integers a, b, c such that $a^2 + b^2 = 3c^2$. (Hint: studying the equation $[a]_4^2 + [b]_4^2 = 3[c]_4^2$ in $\mathbb{Z}/4\mathbb{Z}$, show that a, b, c would all have to be even. Letting $a = 2k, b = 2l, c = 2m$, you would have $k^2 + l^2 = 3m^2$. What's wrong with that?)

$$a^2 + b^2 = 3c^2 \implies [a]_4^2 + [b]_4^2 = 3[c]_4^2.$$

Noting that $[0]_4^2 = [0]_4, [1]_4^2 = [1]_4, [2]_4^2 = [0]_4, [3]_4^2 = [1]_4$, we see $[c]_4^2$ must be $[0]_4$ and so do $[a]_4^2$ and $[b]_4^2$. Hence $[a]_4, [b]_4, [c]_4$ can only be $[0]_4$ or $[2]_4$, which justifies letting $a = 2k_1, b = 2l_2, c = 2m_1$. After substitution we have $k^2 + l^2 = 3m^2$. Repeating this process n times yields $a = 2^n k_n, b = 2^n l_n, c = 2^n m_n$. For a sufficiently large number N , the absolute value of k_N, l_N, m_N must be less than 1. Thus we conclude that $a = b = c = 0$ is the unique solution to the equation $a^2 + b^2 = 3c^2$. ■

2.13 Prove that if $\gcd(m, n) = 1$, then there exist integers a and b such that $am + bn = 1$. (Use Corollary 2.5.) Conversely, prove that if $am + bn = 1$ for some integers a and b , then $\gcd(m, n) = 1$. [2.15, §V.2.1, V.2.4]

Applying corollary 2.5, we have $\gcd(m, n) = 1$ if and only if $[m]_n$ generates $\mathbb{Z}/n\mathbb{Z}$. Hence

$$\gcd(m, n) = 1 \iff a[m]_n = [1]_n \iff [am]_n = [1]_n \iff am + bn = 1.$$

■

2.15 Let $n > 0$ be an odd integer.

- Prove that if $\gcd(m, n) = 1$, then $\gcd(2m + n, 2n) = 1$. (Use Exercise 2.13.)
- Prove that if $\gcd(r, 2n) = 1$, then $\gcd(\frac{r+n}{2}, n) = 1$. (Ditto.)
- Conclude that the function $[m]_n \rightarrow [2m + n]_{2n}$ is a bijection between $(\mathbb{Z}/n\mathbb{Z})^*$ and $(\mathbb{Z}/2n\mathbb{Z})^*$.

The number $\phi(n)$ of elements of $(\mathbb{Z}/n\mathbb{Z})^*$ is Euler's $\phi(n)$ -function. The reader has just proved that if n is odd, then $\phi(2n) = \phi(n)$. Much more general formulas will be given later on (cf. [Exercise V.6.8](#)). [VII.5.11]

- Since $2m + n$ is an odd integer, $\gcd(2m + n, 2n) = 1$ is actually equivalent to $\gcd(2m + n, n) = 1$. According to Exercise 2.13,

$$\gcd(m, n) = 1 \implies am + bn = 1 \implies \frac{a}{2}(2m + n) + \left(b - \frac{a}{2}\right)n = 1.$$

If a is even, we have shown $\gcd(2m + n, n) = 1$. Otherwise we can let $a' = a + n$ be an even integer and $b' = b - m$. Then it holds that

$$\frac{a'}{2}(2m + n) + \left(b' - \frac{a'}{2}\right)n = 1,$$

which also implies $\gcd(2m + n, n) = 1$.

- If $\gcd(r, 2n) = 1$, then r must be an odd integer and accordingly

$$\gcd(2r + 2n, 4n) = 1 \implies a(2r + 2n) + b(4n) = 1 \implies 4a\frac{r + n}{2} + 4bn = 1,$$

which is $\gcd(\frac{r+n}{2}, n) = 1$.

- It is easy to check that the function $f : (\mathbb{Z}/n\mathbb{Z})^* \rightarrow (\mathbb{Z}/2n\mathbb{Z})^*$, $[m]_n \mapsto [2m + n]_{2n}$ is well-defined. The fact

$$\begin{aligned} f([m_1]_n) = f([m_2]_n) &\implies f([2m_1 + n]_{2n}) = f([2m_2 + n]_{2n}) \\ &\implies (2m_1 + n) - (2m_2 + n) = 2kn \\ &\implies m_1 - m_2 = kn \\ &\implies [m_1]_n = [m_2]_n \end{aligned}$$

indicates that f is injective. For any $[r]_{2n} \in (\mathbb{Z}/2n\mathbb{Z})^*$, we have

$$\gcd(r, 2n) = 1 \implies \gcd\left(\frac{r+n}{2}, n\right) = 1 \implies \left[\frac{r+n}{2}\right]_n \in (\mathbb{Z}/n\mathbb{Z})^*,$$

and

$$f\left(\left[\frac{r+n}{2}\right]_n\right) = [r+2n]_{2n} = [r]_{2n},$$

which indicates that f is surjective. Thus we show f is a bijection. ■

2.16 Find the last digit of $1238237^{18238456}$. (Work in $\mathbb{Z}/10\mathbb{Z}$.)

$$1238237^{18238456} \equiv 7^{18238456} \equiv (7^4)^{4559614} \equiv 2401^{4559614} \equiv 1 \pmod{10},$$

which indicates that the last digit of $1238237^{18238456}$ is 1. ■

2.17 Show that if $m \equiv m' \pmod{n}$, then $\gcd(m, n) = 1$ if and only if $\gcd(m', n) = 1$. [§2.3]

Assume that $m - m' = kn$. If $\gcd(m, n) = 1$, for any common divisor d of m' and n

$$d|m', d|n \implies d|(m' + kn) \implies d|m \implies d = 1,$$

which means $\gcd(m', n) = 1$. Likewise, we can show $\gcd(m', n) = 1 \implies \gcd(m, n) = 1$ ■

§3. The category Grp

3.1 Let $\varphi : G \rightarrow H$ be a morphism in a category \mathbf{C} with products. Explain why there is a unique morphism

$$(\varphi \times \varphi) : G \times G \longrightarrow H \times H.$$

(This morphism is defined explicitly for $\mathbf{C} = \mathbf{Set}$ in §3.1.)

By the universal property of product in \mathbf{C} , there exist a unique morphism $(\varphi \times \varphi) : G \times G \longrightarrow H \times H$ such that the following diagram commutes.

$$\begin{array}{ccc} G & \xrightarrow{\varphi} & H \\ \pi_G \uparrow & & \uparrow \pi_H \\ G \times G & \xrightarrow{\varphi \times \varphi} & H \times H \\ \pi_G \downarrow & & \downarrow \pi_H \\ G & \xrightarrow{\varphi} & H \end{array}$$

■

3.2 Let $\varphi : G \rightarrow H, \psi : H \rightarrow K$ be morphisms in a category with products, and consider morphisms between the products $G \times G, H \times H, K \times K$ as in Exercise 3.1. Prove that

$$(\psi\varphi) \times (\psi\varphi) = (\psi \times \psi)(\varphi \times \varphi).$$

(This is part of the commutativity of the diagram displayed in §3.2.)

By the universal property of product in \mathbf{C} , there exists a unique morphism

$$(\psi\varphi) \times (\psi\varphi) : G \times G \rightarrow K \times K$$

such that the following diagram commutes.

$$\begin{array}{ccc} G & \xrightarrow{\psi\varphi} & H \\ \pi_G \uparrow & & \uparrow \pi_H \\ G \times G & \xrightarrow{(\psi\varphi) \times (\psi\varphi)} & H \times H \\ \pi_G \downarrow & & \downarrow \pi_H \\ G & \xrightarrow{\psi\varphi} & H \end{array}$$

As the following commutative diagram tells us the composition

$$(\psi \times \psi)(\varphi \times \varphi) : G \times G \rightarrow K \times K$$

can make the above diagram commute,

$$\begin{array}{ccccc} & & \psi\varphi & & \\ & \curvearrowright & & \curvearrowleft & \\ G & \xrightarrow{\varphi} & H & \xrightarrow{\psi} & K \\ \pi_G \uparrow & & \pi_H \uparrow & & \pi_K \uparrow \\ G \times G & \xrightarrow{\varphi \times \varphi} & H \times H & \xrightarrow{\psi \times \psi} & K \times K \\ \pi_G \downarrow & & \pi_H \downarrow & & \pi_K \downarrow \\ G & \xrightarrow{\varphi} & H & \xrightarrow{\psi} & K \\ & \curvearrowleft & & \curvearrowright & \\ & \psi\varphi & & & \end{array}$$

there must be $(\psi\varphi) \times (\psi\varphi) = (\psi \times \psi)(\varphi \times \varphi)$. ■

3.3 Show that if G, H are abelian groups, then $G \times H$ satisfies the universal property for coproducts in \mathbf{Ab} .

Define two monomorphisms:

$$i_G : G \longrightarrow G \times H, a \longmapsto (a, 0_H)$$

$$i_H : H \longrightarrow G \times H, b \longmapsto (0_G, b)$$

We are to show that for any two homomorphisms $g : G \rightarrow M$ and $h : H \rightarrow M$ in **Ab**, the mapping

$$\begin{aligned} \varphi : G \times H &\longrightarrow M, \\ (a, b) &\longmapsto g(a) + h(b) \end{aligned}$$

is a homomorphism and makes the following diagram commute.

$$\begin{array}{ccc} & G & \\ i_G \downarrow & \searrow g & \\ G \times H & \xrightarrow{\varphi} & M \\ i_H \uparrow & \nearrow h & \\ & H & \end{array}$$

Exploiting the fact that g, h are homomorphisms and M is an abelian group, it is easy to check that φ preserves the addition operation

$$\begin{aligned} \varphi((a_1, b_1) + (a_2, b_2)) &= \varphi((a_1 + a_2, b_1 + b_2)) \\ &= g(a_1 + a_2) + h(b_1 + b_2) \\ &= (g(a_1) + g(a_2)) + (h(b_1) + h(b_2)) \\ &= (g(a_1) + h(b_1)) + (g(a_2) + h(b_2)) \\ &= g(a_1 + b_1) + h(a_2 + b_2) \\ &= \varphi((a_1, b_1)) + \varphi((a_2, b_2)) \end{aligned}$$

and the diagram commutes

$$\varphi \circ i_G(a) = \varphi((a, 0_H)) = g(a) + h(0_H) = g(a) + 0_M = g(a),$$

$$\varphi \circ i_H(b) = \varphi((0_G, b)) = g(0_G) + h(b) = 0_M + h(b) = h(b).$$

To show the uniqueness of the homomorphism φ we have constructed, suppose a homomorphism φ' can make the diagram commute. Then we have

$$\varphi'((a, b)) = \varphi'((a, 0_H) + (0_G, b)) = \varphi'(i_G(a)) + \varphi'(i_H(b)) = g(a) + h(b) = \varphi((a, b)),$$

that is $\varphi' = \varphi$. Hence we show that there exist a unique homomorphism φ such that the diagram commutes, which amounts to the universal property for coproducts in **Ab**. ■

3.4 Let G, H be groups, and assume that $G \cong H \times G$. Can you conclude that H is trivial? (Hint: No. Can you construct a counterexample?)

Consider the function

$$\begin{aligned}\varphi : \mathbb{Z} \times \mathbb{Z}[x] &\longrightarrow \mathbb{Z}[x] \\ (n, f(x)) &\longmapsto n + xf(x)\end{aligned}$$

Firstly, we can show φ is a homomorphism as follows

$$\begin{aligned}\varphi((n_1, f_1(x)) + (n_2, f_2(x))) &= \varphi((n_1 + n_2, f_1(x) + f_2(x))) \\ &= (n_1 + n_2) + x(f_1(x) + f_2(x)) \\ &= (n_1 + xf_1(x)) + (n_2 + xf_2(x)) \\ &= \varphi(n_1, f_1(x)) + \varphi(n_2, f_2(x)).\end{aligned}$$

Secondly, we are to show φ is a monomorphism. It follows by

$$\varphi(n, f(x)) = n + xf(x) = 0 \implies n = 0, f(x) = 0 \implies \ker \varphi = \{(0, 0)\}.$$

Lastly, since given any $f(x) = \sum_{n \geq 0} a_n x^n \in \mathbb{Z}[x]$ we have

$$\varphi\left(a_0, \sum_{n \geq 1} a_n x^{n-1}\right) = a_0 + \sum_{n \geq 1} a_n x^n = f(x),$$

we claim φ is surjective and indeed an isomorphism. Therefore, as a counterexample we have $\mathbb{Z}[x] \cong \mathbb{Z} \times \mathbb{Z}[x]$ where \mathbb{Z} is non-trivial. \blacksquare

3.5 Prove that \mathbb{Q} is not the direct product of two nontrivial groups.

Consider the additive group of rationals $(\mathbb{Q}, +)$. Assume that φ is a isomorphism between the product $G \times H = \{(a, b) | a \in G, b \in H\}$ and $(\mathbb{Q}, +)$. Note that $\{e_G\} \times H$ and $G \times \{e_H\}$ are subgroups in $G \times H$ and their intersection is the trivial group $\{(e_G, e_H)\}$. It is easy to check that bijection φ satisfies $\varphi(A \cap B) = \varphi(A) \cap \varphi(B)$. So applying the fact we have

$$\varphi(\{(e_G, e_H)\}) = \varphi(\{e_G\} \times H \cap G \times \{e_H\}) = \varphi(\{e_G\} \times H) \cap \varphi(G \times \{e_H\}) = \{0\}.$$

Suppose both $\varphi(\{e_G\} \times H)$ and $\varphi(G \times \{e_H\})$ are nontrivial groups. If $\frac{p}{q} \in \varphi(\{e_G\} \times H) - \{0\}$ and $\frac{r}{s} \in \varphi(G \times \{e_H\}) - \{0\}$, there must be

$$rp = rq \cdot \frac{p}{q} = ps \cdot \frac{r}{s} \in \varphi(\{e_G\} \times H) \cap \varphi(G \times \{e_H\}),$$

which implies $rp = 0$. Since both $\frac{p}{q}$ and $\frac{r}{s}$ are non-zero, it leads to a contradiction. Thus without loss of generality we can assume $\varphi(\{e_G\} \times H)$ is a trivial group $\{0\}$. Since φ is isomorphism, we see that for all $h \in H$,

$$\varphi(e_G, h) = \varphi(e_G, e_H) = 0 \iff h = e_H.$$

That is, H is a trivial group. Therefore, we have shown $(\mathbb{Q}, +)$ will never be isomorphic to the direct product of two nontrivial groups. ■

3.6 Consider the product of the cyclic groups C_2, C_3 (cf. §2.3): $C_2 \times C_3$. By [Exercise 3.3](#), this group is a coproduct of C_2 and C_3 in **Ab**. Show that it is not a coproduct of C_2 and C_3 in **Grp**, as follows:

- find injective homomorphisms $C_2 \rightarrow S_3, C_3 \rightarrow S_3$;
- arguing by contradiction, assume that $C_2 \times C_3$ is a coproduct of C_2, C_3 , and deduce that there would be a group homomorphism $C_2 \times C_3 \rightarrow S_3$ with certain properties;
- show that there is no such homomorphism.

- Monomorphisms $g : C_2 \rightarrow S_3, h : C_3 \rightarrow S_3$ can be constructed as follows:

$$g([0]_2) = e, g([1]_2) = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}.$$

$$h([0]_3) = e, h([1]_3) = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}, h([2]_3) = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}.$$

- Supposing that $C_2 \times C_3$ is a coproduct of C_2, C_3 , there would be a unique group homomorphism $\varphi : C_2 \times C_3 \rightarrow S_3$ such that the following diagram commutes

$$\begin{array}{ccc} C_2 & & \\ i_{C_2} \downarrow & \searrow g & \\ C_2 \times C_3 & \xrightarrow{\varphi} & S_3 \\ i_{C_3} \uparrow & \nearrow h & \\ C_3 & & \end{array}$$

In other words, for all $a \in C_2, b \in C_3$,

$$\begin{aligned} \varphi(a, b) &= \varphi([0]_2, b) + (a, [0]_3) = \varphi([0]_2, b)\varphi(a, [0]_3) = \varphi(i_{C_3}(b))\varphi(i_{C_2}(a)) = h(b)g(a) \\ &= \varphi(a, [0]_3) + ([0]_2, b) = \varphi(a, [0]_3)\varphi([0]_2, b) = \varphi(i_{C_2}(a))\varphi(i_{C_3}(b)) = g(a)h(b). \end{aligned}$$

- Since

$$\begin{aligned} g([1]_2)h([1]_3) &= \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}, \\ h([1]_3)g([1]_2) &= \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}, \end{aligned}$$

we see $g(a)h(b) \neq h(b)g(a)$ not always holds. The derived contradiction shows that $C_2 \times C_3$ is not a coproduct of C_2, C_3 in **Grp**.

■

3.7 Show that there is a surjective homomorphism $Z * Z \rightarrow C_2 * C_3$. ($*$ denotes coproduct in Grp.)

Consider the mapping

$$\begin{aligned}\varphi : \mathbb{Z} * \mathbb{Z} &\longrightarrow C_2 * C_3 \\ x^{m_1} y^{n_1} \dots x^{m_k} y^{n_k} &\longmapsto x^{[m_1]_2} y^{[n_1]_3} \dots x^{[m_k]_2} y^{[n_k]_3}\end{aligned}$$

Since

$$\begin{aligned}&\varphi(x^{m_1} y^{n_1} \dots x^{m_k} y^{n_k} x^{m'_1} y^{n'_1} \dots x^{m'_{k'}} y^{n'_{k'}}) \\ &= x^{[m_1]_2} y^{[n_1]_3} \dots x^{[m_k]_2} y^{[n_k]_3} x^{[m'_1]_2} y^{[n'_1]_3} \dots x^{[m'_{k'}]_2} y^{[n'_{k'}]_3}, \\ &= \varphi(x^{m_1} y^{n_1} \dots x^{m_k} y^{n_k}) \varphi(x^{m'_1} y^{n'_1} \dots x^{m'_{k'}} y^{n'_{k'}})\end{aligned}$$

φ is a homomorphism. It is clear that φ is surjective. Thus we show there exists a surjective homomorphism $Z * Z \rightarrow C_2 * C_3$. ■

3.8 Define a group G with two generators x, y , subject (only) to the relations $x^2 = e_G, y^3 = e_G$. Prove that G is a coproduct of C_2 and C_3 in Grp. (The reader will obtain an even more concrete description for $C_2 * C_3$ in Exercise 9.14; it is called the modular group.) [§3.4, 9.14]

Given the maps $i_1 : C_2 \rightarrow G, [m]_2 \mapsto x^m$ and $i_2 : C_3 \rightarrow G, [n]_3 \mapsto y^n$, we can check that i_1, i_2 are homomorphisms. We are to show that for every group H endowed with two homomorphisms $f_1 : C_2 \rightarrow H, f_2 : C_3 \rightarrow H$, there would be a unique group homomorphism $\varphi : G \rightarrow H$ such that the following diagram commutes

$$\begin{array}{ccc} C_2 & & \\ i_1 \downarrow & \searrow f_1 & \\ G & \xrightarrow{\varphi} & H \\ i_2 \uparrow & \nearrow f_2 & \\ C_3 & & \end{array}$$

or

$$\begin{aligned}\varphi(i_1([m]_2)) &= \varphi(x^m) = \varphi(x)^m = f_1([m]_2), \\ \varphi(i_2([n]_3)) &= \varphi(y^n) = \varphi(y)^n = f_2([n]_3).\end{aligned}$$

Define $\phi : G \rightarrow H$ as $\phi(x^m y^n) = f_1([m]_2) f_2([n]_3)$, $\phi(y^n x^m) = f_2([n]_3) f_1([m]_2)$. It is clear to see ϕ makes the diagram commute. Moreover, if φ makes the diagram commute, it follows that for all $x^m y^n, y^n x^m \in G$,

$$\varphi(x^m y^n) = \varphi(x^m) \varphi(y^n) = f_1([m]_2) f_2([n]_3),$$

$$\varphi(y^n x^m) = \varphi(y^n) \varphi(x^m) = f_2([n]_3) f_1([m]_2),$$

which implies $\varphi = \phi$. Thus we can conclude G is the coproduct of C_2 and C_3 in \mathbf{Grp} . ■

§4. Group homomorphisms

4.1 Check that the function π_m^n defined in §4.1 is well-defined, and makes the diagram commute. Verify that it is a group homomorphism. Why is the hypothesis $m|n$ necessary? [§4.1]

In §4.1 the function π_m^n is defined as

$$\begin{aligned} \pi_m^n : \mathbb{Z}/n\mathbb{Z} &\longrightarrow \mathbb{Z}/m\mathbb{Z} \\ [a]_n &\longmapsto [a]_m \end{aligned}$$

with the condition $m|n$. We can check that π_m^n is well-defined as

$$[a_1]_n = [a_2]_n \iff a_1 - a_2 = kn = (kl)m \implies [a_1]_m = [a_2]_m \iff \pi_m^n([a_1]_n) = \pi_m^n([a_2]_n).$$

Note $\pi_m^n(\pi_n(a)) = \pi_m^n([a]_n) = [a]_m = \pi_m(a)$. The diagram in §4.1 must commute.

$$\begin{array}{ccc} \mathbb{Z} & & \\ \pi_n \downarrow & \searrow \pi_m & \\ \mathbb{Z}/n\mathbb{Z} & \xrightarrow{\pi_m^n} & \mathbb{Z}/m\mathbb{Z} \end{array}$$

Since

$$\pi_m^n([a]_n + [b]_n) = [a + b]_m = [a]_m + [b]_m = \pi_m^n([a]_n) + \pi_m^n([b]_n),$$

it follows that π_m^n is a group homomorphism. Actually we have shown that without the hypothesis $m|n$, π_m^n may not be well-defined. ■

4.2 Show that the homomorphism $\pi_2^4 \times \pi_2^4 : C_4 \rightarrow C_2 \times C_2$ is not an isomorphism. In fact, is there any nontrivial isomorphism $C_4 \rightarrow C_2 \times C_2$?

Let calculate the order of each non-zero element in both C_4 and $C_2 \times C_2$. For the group C_4 ,

$$|[2]_4| = 2, \quad |[1]_4| = |[3]_4| = 4.$$

For the group $C_2 \times C_2$,

$$|([1]_2, [0]_2)| = |([0]_2, [1]_2)| = |([1]_2, [1]_2)| = 2.$$

Since isomorphism must preserve the order, we can assert that there is no such isomorphism $C_4 \rightarrow C_2 \times C_2$. ■

4.3 Prove that a group of order n is isomorphic to $\mathbb{Z}/n\mathbb{Z}$ if and only if it contains an element of order n . [§4.3]

Assume some group G is isomorphic to $\mathbb{Z}/n\mathbb{Z}$. Since $|[1]_n| = n$ and isomorphism preserves the order, we can affirm that there is an element of order n in G .

Conversely, assume there is a group G of order n in which g is an element of order n . By definition we see $g^0, g^1, g^2 \dots g^{n-1}$ are distinct pairwise. Noticing group G has exactly n elements, G must consist of $g^0, g^1, g^2 \dots g^{n-1}$. We can easily check that the function

$$\begin{aligned} f : G &\longrightarrow \mathbb{Z}/n\mathbb{Z} \\ g^k &\longmapsto [k]_n \end{aligned}$$

is an isomorphism. ■

4.4 Prove that no two of the groups $(\mathbb{Z}, +)$, $(\mathbb{Q}, +)$, $(\mathbb{R}, +)$ are isomorphic to one another. Can you decide whether $(\mathbb{R}, +)$, $(\mathbb{C}, +)$ are isomorphic to one another? (Cf. Exercise VI.1.1.)

Suppose there exists an isomorphism $f : \mathbb{Z} \rightarrow \mathbb{Q}$. Let $f(1) = p/q$ ($p, q \in \mathbb{Z}$). If $p = 1$, for all $n \in \mathbb{Z}$, we have

$$f(n) = \frac{n}{q} \neq \frac{1}{2q}.$$

If $p \neq 1$, for all $n \in \mathbb{Z}$, we have

$$f(n) = \frac{np}{q} \neq \frac{p+1}{q}.$$

In both cases, it implies $f(\mathbb{Z}) \not\subseteq \mathbb{Q}$. Hence we see f is not a surjection, which contradicts the fact that $f : \mathbb{Z} \rightarrow \mathbb{Q}$ is an isomorphism. Compare the cardinality of \mathbb{Z} , \mathbb{Q} , \mathbb{R}

$$|\mathbb{Z}| = |\mathbb{Q}| < |\mathbb{R}|$$

and we show there exists no such isomorphisms like $f : \mathbb{Z} \rightarrow \mathbb{R}$ or $f : \mathbb{Q} \rightarrow \mathbb{R}$.

We can prove $(\mathbb{R}, +)$, $(\mathbb{C}, +)$ are isomorphic, if considering the both as vector spaces over \mathbb{Q} . ■

4.5 Prove that the groups $(\mathbb{R} \setminus \{0\}, \cdot)$ and $(\mathbb{C} \setminus \{0\}, \cdot)$ are not isomorphic.

Suppose $f : \mathbb{R} \rightarrow \mathbb{C}$ is an isomorphism. Then there exists a real number x such that $f(x) = i$.

$$f(x^4) = f(x)^4 = i^4 = 1.$$

Since isomorphism preserves the identity, we have

$$f(1) = 1 = f(x^4).$$

which indicates $x^4 = 1$. Noticing that $x \in \mathbb{R}$, there must be $x^2 = 1$. Now we see

$$f(1) = f(x^2) = f(x)^2 = i^2 = -1,$$

which derives a contradiction. Thus we can conclude that groups $(\mathbb{R} \setminus \{0\}, \cdot)$ and $(\mathbb{C} \setminus \{0\}, \cdot)$ are not isomorphic. ■

4.6 We have seen that $(\mathbb{R}, +)$ and $(\mathbb{R}_{>0}, \cdot)$ are isomorphic (Example 4.4). Are the groups $(\mathbb{Q}, +)$ and $(\mathbb{Q}_{>0}, \cdot)$ isomorphic?

Suppose $f : \mathbb{Q} \rightarrow \mathbb{Q}_{>0}$ is an isomorphism. Since isomorphism preserves the multiplication, we have

$$f(1) = f\left(n \cdot \frac{1}{n}\right) = f\left(\frac{1}{n}\right)^n \quad (n \in \mathbb{Z}_{>0}),$$

which implies

$$f\left(\frac{1}{n}\right) = f(1)^{\frac{1}{n}}.$$

Assume

$$f(1) = \frac{p}{q} = \frac{p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k}}{q_1^{s_1} q_2^{s_2} \cdots q_l^{s_l}}$$

where p_i, q_i are pairwise distinct positive prime numbers. Then let

$$M = \max\{p, q\} + 1 > \max\{r_1, \dots, r_k, s_1, \dots, s_l\}.$$

Thus we assert

$$f\left(\frac{1}{M}\right) = \left(\frac{p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k}}{q_1^{s_1} q_2^{s_2} \cdots q_l^{s_l}}\right)^{\frac{1}{M}} \notin \mathbb{Q},$$

which can be proved by contradiction. In fact, Suppose

$$\left(\frac{p}{q}\right)^{\frac{1}{M}} = \frac{a}{b} \in \mathbb{Q}$$

or say

$$pb^M = qa^M,$$

where a, b are coprime. Note that b^M, a^M are also coprime and that the prime factorization of a^M can be written as $a_1^{Mt_1} a_2^{Mt_2} \cdots a_j^{Mt_j}$ where a_i are pairwise distinct positive prime numbers.

That forces

$$p = p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k} = N \cdot a_1^{Mt_1} a_2^{Mt_2} \cdots a_j^{Mt_j}.$$

Noticing that a_i must coincide with one number in $\{p_1, p_2, \dots, p_k\}$, we can assume $a_1 = p_1$ without loss of generality. However, since $M > \max\{r_1, \dots, r_k\}$, we see the exponent of p_1 is distinct from that of a_1 , which violates the unique factorization property of \mathbb{Z} . Hence

we get a contradiction and verify $f\left(\frac{1}{M}\right) \notin \mathbb{Q}$. Moreover, it contradicts our assumption that $f : \mathbb{Q} \rightarrow \mathbb{Q}_{>0}$ is an isomorphism. Eventually we show that the groups $(\mathbb{Q}, +)$ and $(\mathbb{Q}_{>0}, \cdot)$ are not isomorphic. ■

4.7 Let G be a group. Prove that the function $G \rightarrow G$ defined by $g \mapsto g^{-1}$ is a homomorphism if and only if G is abelian. Prove that $g \mapsto g^2$ is a homomorphism if and only if G is abelian.

Given the function

$$\begin{aligned} f : G &\longrightarrow G \\ g &\longmapsto g^{-1} \end{aligned}$$

we have

$$f(g_1g_2) = (g_1g_2)^{-1} = g_2^{-1}g_1^{-1}, \quad f(g_1)f(g_2) = g_1^{-1}g_2^{-1}.$$

If G is abelian, it is clear to see $f(g_1g_2) = f(g_1)f(g_2)$. If f is a homomorphism, $\forall h_1, h_2 \in G$,

$$h_1h_2 = (h_2^{-1}h_1^{-1})^{-1} = f(h_2^{-1}h_1^{-1}) = f(h_2^{-1})f(h_1^{-1}) = h_2h_1.$$

Given the function

$$\begin{aligned} h : G &\longrightarrow G \\ g &\longmapsto g^2 \end{aligned}$$

we have

$$h(g_1g_2) = (g_1g_2)^2 = g_1g_2g_1g_2, \quad h(g_1)h(g_2) = g_1^2g_2^2 = g_1g_1g_2g_2.$$

If G is abelian, it is clear to see $h(g_1g_2) = h(g_1)h(g_2)$. If h is a homomorphism, by cancellation we have

$$h(g_1g_2) = h(g_1)h(g_2) \implies g_2g_1 = g_1g_2.$$

■

4.8 Let G be a group, and $g \in G$. Prove that the function $\gamma_g : G \rightarrow G$ defined by $(\forall a \in G) : \gamma_g(a) = gag^{-1}$ is an automorphism of G . (The automorphisms γ_g are called ‘inner’ automorphisms of G .) Prove that the function $G \rightarrow \text{Aut}(G)$ defined by $g \mapsto \gamma_g$ is a homomorphism. Prove that this homomorphism is trivial if and only if G is abelian.

Since

$$\gamma_g(ab) = gabg^{-1} = gag^{-1}gbg^{-1} = \gamma_g(a)\gamma_g(b),$$

γ_g is an automorphism of G . For all $a \in G$, we have

$$\gamma_{g_1g_2}(a) = g_1g_2ag_2^{-1}g_1^{-1} = \gamma_{g_1}(g_2ag_2^{-1}) = (\gamma_{g_1} \circ \gamma_{g_2})(a),$$

which implies $\gamma_{g_1g_2} = \gamma_{g_1} \circ \gamma_{g_2}$ and $g \mapsto \gamma_g$ is a homomorphism. If G is abelian, for all g the homomorphism

$$\gamma_g(a) = gag^{-1} = gg^{-1}a = a$$

is the identity in $\text{Aut}(G)$. That is, the homomorphism $g \mapsto \gamma_g$ is trivial. If the homomorphism $g \mapsto \gamma_g$ is trivial, we have for all $g, a \in G$,

$$gag^{-1} = a,$$

which implies for all $a, b \in G$,

$$ab = bab^{-1}b = ba.$$

Thus we show the homomorphism $g \mapsto \gamma_g$ is trivial if and only if G is abelian. \blacksquare

4.9 Prove that if m, n are positive integers such that $\gcd(m, n) = 1$, then $C_{mn} \cong C_m \times C_n$.

Define a function

$$\begin{aligned} \varphi : C_m \times C_n &\longrightarrow C_{mn} \\ ([a]_m, [b]_n) &\longmapsto [anp + bmq]_{mn} \end{aligned}$$

where $[pn]_m = [1]_m$ and $[qm]_n = [1]_n$, as $\gcd(m, n) = 1$ guarantees the existence of p, q (see textbook p56). First of all, we have to check whether φ is well-defined. Note that

$$[(anp_1 + bmq_1) - (anp_2 + bmq_2)]_m = [a(p_1n - p_2n) + b(q_1m - q_2m)]_m = [0]_m$$

$$[(anp_1 + bmq_1) - (anp_2 + bmq_2)]_n = [a(p_1n - p_2n) + b(q_1m - q_2m)]_n = [0]_n$$

and $\gcd(m, n) = 1$. Thus we have

$$[(anp_1 + bmq_1) - (anp_2 + bmq_2)]_{mn} = [0]_{mn},$$

or

$$[anp_1 + bmq_1]_{mn} = [anp_2 + bmq_2]_{mn}.$$

Then we show φ is a homomorphism.

$$\begin{aligned} \varphi([a_1]_m, [b_1]_n) + \varphi([a_2]_m, [b_2]_n) &= \varphi([a_1 + a_2]_m, [b_1 + b_2]_n) \\ &= [(a_1 + a_2)np + (b_1 + b_2)mq]_{mn} \\ &= [a_1np + b_1mq]_{mn} + [a_2np + b_2mq]_{mn} \\ &= \varphi([a_1]_m, [b_1]_n) + \varphi([a_2]_m, [b_2]_n). \end{aligned}$$

In order to show φ is a monomorphism, we can check

$$\begin{aligned} \varphi([a_1]_m, [b_1]_n) &= \varphi([a_2]_m, [b_2]_n) \\ \implies [a_1np + b_1mq]_{mn} &= [a_2np + b_2mq]_{mn} \\ \implies [(a_1 - a_2)np + (b_1 - b_2)mq]_{mn} &= [0]_{mn} \\ \implies [(a_1 - a_2)np + (b_1 - b_2)mq]_m &= [a_1 - a_2]_m = [0]_m, \\ [(a_1 - a_2)np + (b_1 - b_2)mq]_n &= [b_1 - b_2]_n = [0]_n \\ \implies [a_1]_m &= [a_2]_m, [b_1]_n = [b_2]_n. \end{aligned}$$

Since $|C_m \times C_n| = |C_{mn}| = mn$, we can conclude φ is an isomorphism. Thus we complete proving $C_{mn} \cong C_m \times C_n$. ■

§5. Free groups

5.1 Does the category \mathcal{F}^A defined in §5.2 have final objects? If so, what are they?

Yes, they are functions from A to any trivial group, for example $T = \{t\}$.

$$\begin{array}{ccc} G & \xrightarrow{\exists! \varphi} & \{t\} \\ j \uparrow & \nearrow e & \\ A & & \end{array}$$

For any object (j, G) in \mathcal{F}^A , the trivial homomorphism $\varphi : g \mapsto t$ is the unique homomorphism such that the diagram commutes. That is, $\text{Hom}((j, G), (e, T)) = \{\varphi\}$. ■

5.2 Since trivial groups T are initial in \mathbf{Grp} , one may be led to think that (e, T) should be initial in \mathcal{F}^A , for every A : e would be defined by sending every element of A to the (only) element in T ; and for any other group G , there is a unique homomorphism $T \rightarrow G$. Explain why (e, T) is not initial in \mathcal{F}^A (unless $A = \emptyset$).

Let $G = C_2 = \{[0]_2, [1]_2\}$. Note that $\varphi \circ e(A)$ must be the trivial subgroup $\{[0]_2\}$. If $x \in A$ and $j(x) = [1]_2$, we see $\varphi \circ e \neq j$ and the following diagram does not commute.

$$\begin{array}{ccc} T & \xrightarrow{\varphi} & G \\ e \uparrow & \nearrow j & \\ A & & \end{array}$$

That implies (e, T) is not initial in \mathcal{F}^A unless $A = \emptyset$. ■

5.3 Use the universal property of free groups to prove that the map $j : A \rightarrow F(A)$ is injective, for all sets A . (Hint: it suffices to show that for every two elements a, b of A there is a group G and a set-function $f : A \rightarrow G$ such that $f(a) \neq f(b)$. Why? and how do you construct f and G ?) [§III.6.3]

Let $G = S_A$ be the symmetric group over A . Define functions $g_a : A \rightarrow A$, $x \mapsto a$ sending every element of A to a . Since $g_a \in S_A$, we can define an injection

$$\begin{aligned} f : A &\longrightarrow S_A \\ a &\longmapsto g_a \end{aligned}$$

In light of the commutative diagram

$$\begin{array}{ccc} F(A) & \xrightarrow{\varphi} & S_A \\ j \uparrow & \nearrow f & \\ A & & \end{array}$$

we have $\forall a, b \in A$,

$$j(a) = j(b) \implies \varphi(j(a)) = \varphi(j(b)) \implies f(a) = f(b) \implies a = b.$$

■

5.4 In the ‘concrete’ construction of free groups, one can try to reduce words by performing cancellations in any order; the ‘elementary reductions’ used in the text (that is, from left to right) is only one possibility. Prove that the result of iterating cancellations on a word is independent of the order in which the cancellations are performed. Deduce the associativity of the product in $F(A)$ from this. [§5.3]

We use induction on the length of w . If w is reduced, there is nothing to show. If not, there must be some pair of symbols that can be cancelled, say the underlined pair

$$w = \cdots \underline{xx}^{-1} \cdots$$

(Let’s allow x to denote any element of A' , with the understanding that if $x = a^{-1}$ then $x^{-1} = a$.) If we show that we can obtain every reduced form of w by cancelling the pair xx^{-1} first, the proposition will follow by induction, because the word $w^* = \cdots \underline{xx}^{-1} \cdots$ is shorter.

Let w_0 be a reduced form of w . It is obtained from w by some sequence of cancellations. The first case is that our pair xx^{-1} is cancelled at some step in this sequence. If so, we may as well cancel xx^{-1} first. So this case is settled. On the other hand, since w_0 is reduced, the pair xx^{-1} can not remain in w_0 . At least one of the two symbols must be cancelled at some time. If the pair itself is not cancelled, the first cancellation involving the pair must look like

$$\cdots x^{-1} \underline{xx}^{-1} \cdots \quad \text{or} \quad \cdots \underline{xx}^{-1} x \cdots$$

Notice that the word obtained by this cancellation is the same as the one obtained by cancelling the pair xx^{-1} . So at this stage we may cancel the original pair instead. Then we are back in the first case, so the proposition is proved.

■

5.5 Verify explicitly that $H^{\oplus A}$ is a group.

Assume the A is a set and H is an abelian group. $H^{\oplus A}$ are defined as follows

$$H^{\oplus A} := \{\alpha : A \rightarrow H \mid \alpha(a) \neq e_H \text{ for only finitely many elements } a \in A\}.$$

Now that $H^{\oplus A} \subset H^A := \text{Hom}_{\text{Set}}(A, H)$, we can first show $(H^A, +)$ is a group, where for all $\phi, \psi \in H^A$, $\phi + \psi$ is defined by

$$(\forall a \in A) : (\phi + \psi)(a) := \phi(a) + \psi(a).$$

Here is the verification:

- Identity: Define a function $\varepsilon : A \rightarrow H, a \mapsto e_H$ sending all elements in A to e_H . Then for any $\alpha \in H^A$ we have

$$(\forall a \in A) : (\alpha + \varepsilon)(a) = \alpha(a) + \varepsilon(a) = \alpha(a),$$

which is $\alpha + \varepsilon = \alpha$. Because of the commutativity of the operation $+$ defined on H^A , ε is the identity indeed.

- Associativity: This follows by the associativity in H :

$$(\forall a \in A) : ((\alpha + \beta) + \gamma)(a) = (\alpha + \beta)(a) + \gamma(a) = \alpha(a) + (\beta + \gamma)(a) = (\alpha + (\beta + \gamma))(a).$$

- Inverse: Every function $\phi \in H^A$ has inverse $-\phi$ defined by

$$(\forall a \in A) : (-\phi)(a) = -\phi(a).$$

Thus H^A makes a group.

Then it is time to show $H^{\oplus A}$ is a subgroup of H^A . For all $\alpha, \beta \in H^{\oplus A}$, let $N_\alpha = \{a \in A \mid \alpha(a) \neq e_H\}$, $N_\beta = \{a \in A \mid \beta(a) \neq e_H\}$, $N_{\alpha-\beta} = \{a \in A \mid (\alpha - \beta)(a) \neq e_H\}$. Since

$$(\forall a \in A) : (\alpha - \beta)(a) = \alpha(a) - \beta(a),$$

we have

$$(\alpha - \beta)(a) \neq e_H \implies \alpha(a) \neq e_H \text{ or } \beta(a) \neq e_H,$$

which implies $N_{\alpha-\beta} \subset N_\alpha \cup N_\beta$. Note that N_α, N_β are both finite sets, which forces $N_{\alpha-\beta}$ to be finite. So there must be $\alpha - \beta \in H^{\oplus A}$. Now we see $H^{\oplus A}$ is closed under additions and inverses. And $e_{H^A} = \varepsilon \in H^{\oplus A}$ means that $H^{\oplus A}$ is nonempty. Finally we can conclude $H^{\oplus A}$ is a subgroup of H^A . ■

5.6 Prove that the group $F(\{x, y\})$ (visualized in Example 5.3) is a coproduct $\mathbb{Z} * \mathbb{Z}$ of \mathbb{Z} by itself in the category **Grp**. (Hint: with due care, the universal property for one turns into the universal property for the other.) [§3.4, 3.7, 5.7]

Define two homomorphisms

$$\begin{aligned} i_1 : \mathbb{Z} &\longrightarrow F(\{x, y\}), & n &\longmapsto x^n, \\ i_2 : \mathbb{Z} &\longrightarrow F(\{x, y\}), & n &\longmapsto y^n. \end{aligned}$$

We need to show that for any group G with two homomorphisms $f_1, f_2 : \mathbb{Z} \rightarrow G$, there exists a unique homomorphism φ such that the following diagram commutes.

$$\begin{array}{ccc}
 \mathbb{Z} & & \\
 i_1 \downarrow & \searrow f_1 & \\
 F(\{x, y\}) & \xrightarrow{\varphi} & G \\
 i_2 \uparrow & \nearrow f_2 & \\
 \mathbb{Z} & &
 \end{array}$$

Given the notation of indicator function

$$\mathbf{1}_A(x) := \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A, \end{cases}$$

we can define a function

$$\begin{aligned}
 \varphi : F(\{x, y\}) &\longrightarrow G, \\
 z_1^{n_1} \cdots z_k^{n_k} &\longmapsto f_1(n_1)^{\mathbf{1}_{\{x\}}(z_1)} f_2(n_1)^{\mathbf{1}_{\{y\}}(z_1)} \cdots f_1(n_k)^{\mathbf{1}_{\{x\}}(z_k)} f_2(n_k)^{\mathbf{1}_{\{y\}}(z_k)}, \quad z_i \in \{x, y\}
 \end{aligned}$$

and check that it is a homomorphism indeed. For all $n \in \mathbb{Z}$, we have

$$\begin{aligned}
 (\varphi \circ i_1)(n) &= \varphi(x^n) = f_1(n), \\
 (\varphi \circ i_2)(n) &= \varphi(y^n) = f_2(n),
 \end{aligned}$$

that is, the diagram commutes. Now we see φ exists. For the uniqueness of φ , let φ^* be another homomorphism that makes diagram commute. For all $z_1^{n_1} \cdots z_k^{n_k} \in F(\{x, y\})$, $z_i \in \{x, y\}$, we have

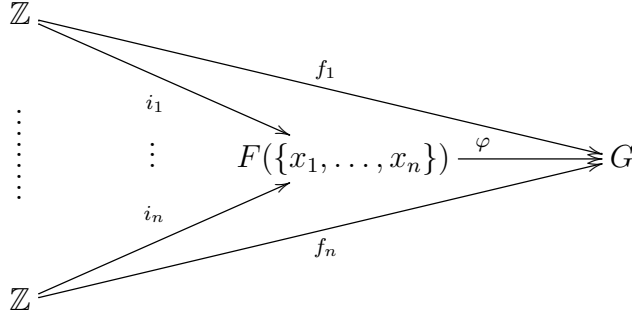
$$\begin{aligned}
 \varphi^*(z_1^{n_1} \cdots z_k^{n_k}) &= \varphi^*(z^{n_1}) \cdots \varphi^*(z^{n_k}) \\
 &= \varphi^*(i_1(n_1))^{\mathbf{1}_{\{x\}}(z_1)} \varphi^*(i_2(n_1))^{\mathbf{1}_{\{y\}}(z_1)} \cdots \varphi^*(i_1(n_k))^{\mathbf{1}_{\{x\}}(z_k)} \varphi^*(i_2(n_k))^{\mathbf{1}_{\{y\}}(z_k)} \\
 &= f_1(n_1)^{\mathbf{1}_{\{x\}}(z_1)} f_2(n_1)^{\mathbf{1}_{\{y\}}(z_1)} \cdots f_1(n_k)^{\mathbf{1}_{\{x\}}(z_k)} f_2(n_k)^{\mathbf{1}_{\{y\}}(z_k)} \\
 &= \varphi(z_1^{n_1} \cdots z_k^{n_k}).
 \end{aligned}$$

To sum up, we have shown that the group $F(\{x, y\})$ is a coproduct $\mathbb{Z} * \mathbb{Z}$ of \mathbb{Z} by itself in the category **Grp**. ■

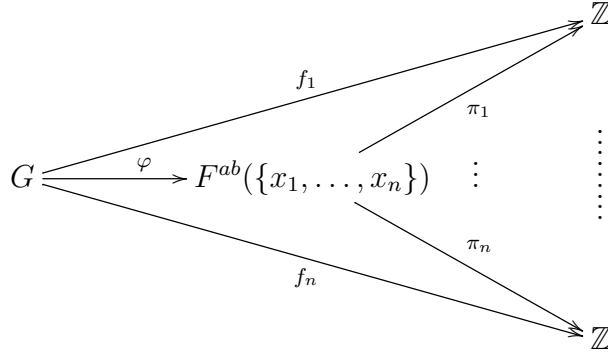
5.7 Extend the result of Exercise 5.6 to free groups $F(\{x_1, \dots, x_n\})$ and to free abelian groups $F^{ab}(\{x_1, \dots, x_n\})$. [§3.4, §5.4]

Let $*$ be coproduct. Then we have $\underbrace{\mathbb{Z} * \mathbb{Z} * \cdots * \mathbb{Z}}_{n \text{ times}} \cong F(\{x_1, \dots, x_n\})$, as the following dia-

gram demonstrates:



Dually, let \times be product. Then we have $\underbrace{\mathbb{Z} \times \mathbb{Z} \times \cdots \times \mathbb{Z}}_{n \text{ times}} \cong F^{ab}(\{x_1, \dots, x_n\})$, as the following diagram demonstrates:



■

5.8 Still more generally, prove that $F(A \amalg B) = F(A) * F(B)$ and that $F^{ab}(A \amalg B) = F^{ab}(A) \oplus F^{ab}(B)$ for all sets A, B . (That is, the constructions F, F^{ab} 'preserve coproducts'.)

In order to show $F(A) * F(B)$ is a free group generated by $A \amalg B$, we should first set an appropriate function $\psi : A \amalg B \rightarrow F(A) * F(B)$ and then prove that given any (θ, G) there exists a unique group homomorphism g such that the following diagram commutes.

$$\begin{array}{ccccc}
 A \amalg B & \xrightarrow{\psi} & F(A) * F(B) & \xrightarrow{\exists! g} & G \\
 & & \searrow & \nearrow \theta & \\
 & & & &
 \end{array}$$

The complete proof can be divided into three steps, by decomposing the following diagram

into parts.

$$\begin{array}{ccccc}
 A & \xrightarrow{j_1} & F(A) & & \\
 \downarrow i_1 & & \downarrow f_1 & \searrow \varphi_1 & \\
 A \amalg B & \xrightarrow{\psi} & F(A) * F(B) & \xrightarrow{g} & G \\
 \uparrow i_2 & \searrow \theta & \uparrow f_2 & \nearrow \varphi_2 & \\
 B & \xrightarrow{j_1} & F(B) & &
 \end{array}$$

Step 1. Construct $\psi : A \amalg B \longrightarrow F(A) * F(B)$.

Define injective functions

$$\begin{aligned}
 i_1 : A &\longrightarrow A \amalg B, & a &\longmapsto (a, 1), \\
 i_2 : B &\longrightarrow A \amalg B, & b &\longmapsto (b, 2), \\
 j_1 : A &\longrightarrow F(A), & a &\longmapsto a, \\
 j_2 : B &\longrightarrow F(B), & b &\longmapsto b.
 \end{aligned}$$

Let f_1, f_2 be the homomorphisms specified by the coproduct in **Grp**. Since $A \amalg B$ is a coproduct in **Set**, the universal property guarantees a unique mapping $\psi : A \amalg B \rightarrow F(A) * F(B)$ such that the following diagram commutes

$$\begin{array}{ccc}
 A & \xrightarrow{j_1} & F(A) \\
 \downarrow i_1 & & \downarrow f_1 \\
 A \amalg B & \xrightarrow{\exists! \psi} & F(A) * F(B) \\
 \uparrow i_2 & & \uparrow f_2 \\
 B & \xrightarrow{j_1} & F(B)
 \end{array}$$

That is,

$$\exists! \psi : A \amalg B \longrightarrow F(A) * F(B) \quad (\psi \circ i_1 = f_1 \circ j_1) \wedge (\psi \circ i_2 = f_2 \circ j_2).$$

Step 2. Prove the existence of g .

$$\begin{array}{ccc}
 A & \xrightarrow{j_1} & F(A) \\
 \downarrow i_1 & & \searrow \exists! \varphi_1 \\
 A \amalg B & \xrightarrow{\theta} & G \\
 \uparrow i_2 & & \nearrow \exists! \varphi_2 \\
 B & \xrightarrow{j_1} & F(B)
 \end{array}$$

Given some (θ, G) , according to the universal property of free groups $F(A)$, $F(B)$, we have

$$\begin{aligned}\exists! \varphi_1 : F(A) &\longrightarrow G & (\varphi_1 \circ j_1 = \theta \circ i_1), \\ \exists! \varphi_2 : F(B) &\longrightarrow G & (\varphi_2 \circ j_2 = \theta \circ i_2).\end{aligned}$$

$$\begin{array}{ccc} & F(A) & \\ & \downarrow f_1 & \searrow \varphi_1 \\ F(A) * F(B) & \xrightarrow{\exists! g} & G \\ & \uparrow f_2 & \swarrow \varphi_2 \\ & F(B) & \end{array}$$

Then according to the universal property of coproduct $F(A) * F(B)$ in **Grp**, we have

$$\exists! g : F(A) * F(B) \longrightarrow G \quad (g \circ f_1 = \varphi_1) \wedge (g \circ f_2 = \varphi_2).$$

The commutative diagram tells us

$$\begin{aligned}g \circ \psi \circ i_1 &= g \circ f_1 \circ j_1 = \varphi_1 \circ j_1 = \theta \circ i_1, \\ g \circ \psi \circ i_2 &= g \circ f_2 \circ j_2 = \varphi_2 \circ j_2 = \theta \circ i_2.\end{aligned}$$

Note that $A \amalg B = i_1(A) \cup i_2(B)$. For all $x \in A \amalg B$, x must be either $i_1(a)$ or $i_2(b)$. If $x = i_1(a)$, then

$$g \circ \psi(x) = g \circ \psi \circ i_1(a) = \theta \circ i_1(a) = \theta(x).$$

If $x = i_2(b)$, then

$$g \circ \psi(x) = g \circ \psi \circ i_2(b) = \theta \circ i_2(b) = \theta(x).$$

Hence we show that given some (θ, G) there exists $g : F(A) * F(B) \longrightarrow G$ such that $g \circ \psi = \theta$.

Step 3. Prove the uniqueness of g .

Assume there exists another homomorphism h such that $h \circ \psi = \theta$. We have

$$\begin{aligned}h \circ f_1 \circ j_1 &= h \circ \psi \circ i_1 = \theta \circ i_1, \\ h \circ f_2 \circ j_2 &= h \circ \psi \circ i_2 = \theta \circ i_2.\end{aligned}$$

Since

$$\begin{aligned}\exists! \varphi_1 : F(A) &\longrightarrow G & (\varphi_1 \circ j_1 = \theta \circ i_1), \\ \exists! \varphi_2 : F(B) &\longrightarrow G & (\varphi_2 \circ j_2 = \theta \circ i_2),\end{aligned}$$

there must be

$$\begin{aligned} h \circ f_1 &= \varphi_1, \\ h \circ f_2 &= \varphi_2. \end{aligned}$$

Again by universal property

$$\exists! g : F(A) * F(B) \longrightarrow G \quad (g \circ f_1 = \varphi_1) \wedge (g \circ f_2 = \varphi_2)$$

we get $h = g$, which implies g is unique.

Conclusion.

To sum up, we prove that there exists a unique group homomorphism g such that the first diagram in this proof commutes. As a result, we have $F(A \amalg B) = F(A) * F(B)$. Note that if **Grp** turns into **Ab**, the method of diagram chasing applied here also works. In the light of the following diagram, we can get $F^{ab}(A \amalg B) = F^{ab}(A) \oplus F^{ab}(B)$ step by step.

$$\begin{array}{ccccc} A & \xrightarrow{j_1} & F^{ab}(A) & & \\ \downarrow i_1 & & \downarrow f_1 & \searrow \varphi_1 & \\ A \amalg B & \xrightarrow{\psi} & F^{ab}(A) \oplus F^{ab}(B) & \xrightarrow{g} & G \\ \uparrow i_2 & \searrow \theta & \uparrow f_2 & \swarrow \varphi_2 & \\ B & \xrightarrow{j_1} & F^{ab}(B) & & \end{array}$$

■

5.9 Let $G = \mathbb{Z}^{\oplus \mathbb{N}}$. Prove that $G \times G \cong G$.

Define a function

$$\begin{aligned} \varphi : G \times G &\longrightarrow G \\ ((a_1, a_2, \dots), (b_1, b_2, \dots)) &\longmapsto (a_1, b_1, a_2, b_2, \dots) \end{aligned}$$

It is plain to check that φ is a homomorphism

$$\begin{aligned} &\varphi[((a_1, a_2, \dots), (b_1, b_2, \dots)) + ((a'_1, a'_2, \dots), (b'_1, b'_2, \dots))] \\ &= \varphi[((a_1 + a'_1, a_2 + a'_2, \dots), (b_1 + b'_1, b_2 + b'_2, \dots))] \\ &= (a_1 + a'_1, b_1 + b'_1, a_2 + a'_2, b_2 + b'_2, \dots) \\ &= (a_1, b_1, a_2, b_2, \dots) + (a'_1, b'_1, a'_2, b'_2, \dots) \\ &= \varphi[((a_1, a_2, \dots), (b_1, b_2, \dots))] + \varphi[((a'_1, a'_2, \dots), (b'_1, b'_2, \dots))]. \end{aligned}$$

Since $\ker \varphi = \{(0, 0, \dots)\}$ and $\varphi(G \times G) = G$, we can conclude that φ is an isomorphism and accordingly $G \times G \cong G$. ■

5.10 \neg Let $F = F^{ab}(A)$.

- Define an equivalence relation \sim on F by setting $f \sim f'$ if and only if $f - f' = 2g$ for some $g \in F$. Prove that F/\sim is a finite set if and only if A is finite, and in that case $|F/\sim| = 2^{|A|}$.
- Assume $F^{ab}(B) \cong F^{ab}(A)$. If A is finite, prove that so is B , and $A \cong B$ as sets. (This result holds for free groups as well, and without any finiteness hypothesis. See Exercises 7.13 and VI.1.20.)

[7.4, 7.13]

- If $|A| = \infty$, let $F = F^{ab}(A) = \mathbb{Z}^{\oplus A}$ and accordingly every element of $\mathbb{Z}^{\oplus A}$ can be written uniquely as a finite sum

$$\sum_{a \in A} m_a j(a), \quad m_a \neq 0 \text{ for only finitely many } a.$$

Apparently, the elements in $j(A) = \{j(a) \mid a \in A\}$ are not equivalent pairwise. Note that j is an injection. Hence we see

$$|F/\sim| \geq |j(A)| = A > \infty.$$

In other words, F/\sim is a finite set only if A is finite.

If $|A| = n < \infty$, we can set $F = F^{ab}(A) = \mathbb{Z}^{\oplus n}$. Assume $f = (a_1, a_2, \dots, a_n)$, $f' = (a'_1, a'_2, \dots, a'_n)$. Then $f \sim f'$ if and only if $a_i - a'_i \in 2\mathbb{Z}$ ($i = 1, 2, \dots, n$). Let $[f]$ denote the equivalence class including f . Thus we get

$$F/\sim = \{[(k_1, k_2, \dots, k_n)] \mid k_i = 0 \text{ or } 1, i = 1, 2, \dots, n\}$$

and accordingly $|F/\sim| = 2^{|A|}$.

- Assume $\varphi : F^{ab}(A) \rightarrow F^{ab}(B)$ is a group isomorphism. Since for all $f, f' \in F^{ab}(A)$,

$$\begin{aligned} f \sim f' &\iff \exists g \in F^{ab}(A), f - f' = 2g \\ &\iff \exists \varphi(g) \in F^{ab}(B), \varphi(f) - \varphi(f') = 2\varphi(g) \\ &\iff \varphi(f) \sim \varphi(f') \end{aligned}$$

in **Set** we have

$$F^{ab}(A)/\sim \simeq F^{ab}(B)/\sim.$$

If A is finite, then $F^{ab}(A)/\sim$ is finite. Furthermore it follows that

$$|F^{ab}(A)/\sim| = |F^{ab}(B)/\sim| \implies 2^{|A|} = 2^{|B|} \implies |A| = |B|.$$

Hence we see B is finite and $A \cong B$ in **Set**.

■

§6. Subgroups

6.1 \neg (If you know about matrices.) The group of invertible $n \times n$ matrices with entries in \mathbb{R} is denoted $\mathrm{GL}_n(\mathbb{R})$ (Example 1.5). Similarly, $\mathrm{GL}_n(\mathbb{C})$ denotes the group of $n \times n$ invertible matrices with complex entries. Consider the following sets of matrices:

- $\mathrm{SL}_n(\mathbb{R}) = \{M \in \mathrm{GL}_n(\mathbb{R}) \mid \det(M) = 1\}$;
- $\mathrm{SL}_n(\mathbb{C}) = \{M \in \mathrm{GL}_n(\mathbb{C}) \mid \det(M) = 1\}$;
- $\mathrm{O}_n(\mathbb{R}) = \{M \in \mathrm{GL}_n(\mathbb{R}) \mid MM^t = M^t M = I_n\}$;
- $\mathrm{SO}_n(\mathbb{R}) = \{M \in \mathrm{O}_n(\mathbb{R}) \mid \det(M) = 1\}$;
- $\mathrm{U}_n(\mathbb{C}) = \{M \in \mathrm{GL}_n(\mathbb{C}) \mid MM^\dagger = M^\dagger M = I_n\}$;
- $\mathrm{SU}_n(\mathbb{C}) = \{M \in \mathrm{U}_n(\mathbb{C}) \mid \det(M) = 1\}$.

Here I_n stands for the $n \times n$ identity matrix, M^t is the transpose of M , M^\dagger is the conjugate transpose of M , and $\det(M)$ denotes the determinant of M . Find all possible inclusions among these sets, and prove that in every case the smaller set is a subgroup of the larger one.

These sets of matrices have compelling geometric interpretations: for example, $\mathrm{SO}^3(\mathbb{R})$ is the group of ‘rotations’ in \mathbb{R}^3 . [8.8, 9.1, III.1.4, VI.6.16]

The following diagram commutes, where all arrows are inclusions.

$$\begin{array}{ccc}
 \mathrm{GL}_n(\mathbb{R}) & \longrightarrow & \mathrm{GL}_n(\mathbb{C}) \\
 \uparrow & & \uparrow \\
 \mathrm{SL}_n(\mathbb{R}) & \longrightarrow & \mathrm{SL}_n(\mathbb{C}) \\
 \uparrow & & \uparrow \\
 \mathrm{O}_n(\mathbb{R}) & \longrightarrow & \mathrm{U}_n(\mathbb{C}) \\
 \uparrow & & \uparrow \\
 \mathrm{SO}_n(\mathbb{R}) & \longrightarrow & \mathrm{SU}_n(\mathbb{C})
 \end{array}$$

■

6.2 \neg Prove that the set of 2×2 matrices

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$$

with a, b, d in \mathbb{C} and $ad \neq 0$ is a subgroup of $\mathrm{GL}_2(\mathbb{C})$. More generally, prove that the set of $n \times n$ complex matrices $(a_{ij})_{1 \leq i, j \leq n}$ with $a_{ij} = 0$ for $i > j$, and $a_{11} \cdots a_{nn} \neq 0$, is a subgroup of $\mathrm{GL}_n(\mathbb{C})$. (These matrices are called ‘upper triangular’, for evident reasons.) [IV.1.20]

Let A, B are $n \times n$ upper triangular matrices. If $i > j$,

$$(AB)_{ij} = \sum_{k=1}^n a_{ik}b_{kj} = \sum_{k=1}^{i-1} a_{ik}b_{kj} + \sum_{k=i}^n a_{ik}b_{kj} = \sum_{k=1}^{i-1} 0b_{kj} + \sum_{k=i}^n a_{ik}0 = 0,$$

which means the set of upper triangular matrices is closed with respect to the matrix multiplication. Thus it is a subgroup of $\text{GL}_n(\mathbb{C})$. \blacksquare

6.3 \neg Prove that every matrix in $\text{SU}_2(\mathbb{C})$ may be written in the form

$$\begin{pmatrix} a + bi & c + di \\ -c + di & a - bi \end{pmatrix}$$

where $a, b, c, d \in \mathbb{R}$ and $a^2 + b^2 + c^2 + d^2 = 1$. (Thus, $\text{SU}_2(\mathbb{C})$ may be realized as a three-dimensional sphere embedded in \mathbb{R}^4 ; in particular, it is simply connected.) [8.9, III.2.5]

Let

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in \text{SU}_2(\mathbb{C})$$

and we have

$$AA^\dagger = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} \overline{a_{11}} & \overline{a_{21}} \\ \overline{a_{12}} & \overline{a_{22}} \end{pmatrix} = \begin{pmatrix} |a_{11}|^2 + |a_{12}|^2 & a_{11}\overline{a_{21}} + a_{12}\overline{a_{22}} \\ a_{21}\overline{a_{11}} + a_{22}\overline{a_{12}} & |a_{21}|^2 + |a_{22}|^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21} = 1$$

Note

$$\begin{aligned} \overline{a_{11}a_{12}} &= \overline{a_{11}a_{12}} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = \begin{vmatrix} |a_{11}|^2 & |a_{12}|^2 \\ a_{21}\overline{a_{11}} & a_{22}\overline{a_{12}} \end{vmatrix} = \begin{vmatrix} |a_{11}|^2 & |a_{11}|^2 + |a_{12}|^2 \\ a_{21}\overline{a_{11}} & a_{21}\overline{a_{11}} + a_{22}\overline{a_{12}} \end{vmatrix} = \begin{vmatrix} |a_{11}|^2 & 1 \\ a_{21}\overline{a_{11}} & 0 \end{vmatrix} = -a_{21}\overline{a_{11}} \\ &\implies \overline{a_{11}}(\overline{a_{12}} + a_{21}) = 0, \end{aligned}$$

and

$$\begin{aligned} \overline{a_{21}a_{22}} &= \overline{a_{21}a_{22}} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = \begin{vmatrix} a_{11}\overline{a_{21}} & a_{12}\overline{a_{22}} \\ |a_{21}|^2 & |a_{22}|^2 \end{vmatrix} = \begin{vmatrix} a_{11}\overline{a_{21}} & a_{11}\overline{a_{21}} + a_{12}\overline{a_{22}} \\ |a_{21}|^2 & |a_{21}|^2 + |a_{22}|^2 \end{vmatrix} = \begin{vmatrix} a_{11}\overline{a_{21}} & 0 \\ |a_{21}|^2 & 1 \end{vmatrix} = a_{11}\overline{a_{21}} \\ &\implies \overline{a_{21}}(\overline{a_{11}} - a_{22}) = 0. \end{aligned}$$

If $\overline{a_{11}} \neq 0$, it must be $\overline{a_{12}} + a_{21} = 0$. If $\overline{a_{11}} = 0$, then $|a_{12}|^2 = 1$, $a_{12}\overline{a_{22}} = 0$ and accordingly $a_{22} = 0$. Since $-a_{12}a_{21} = 1 = a_{12}\overline{a_{12}}$, we also have $\overline{a_{12}} + a_{21} = 0$, that is $a_{12} = c + di$, $a_{21} = -c + di$. Likewise, we can show $\overline{a_{11}} - a_{22} = 0$ and $a_{11} = a + bi$, $a_{22} = a - bi$. And we have

$$|a_{11}|^2 + |a_{12}|^2 = a^2 + b^2 + c^2 + d^2 = 1.$$

■

6.4 Let G be a group, and $g \in G$. Verify that the image of the exponential map $\epsilon_g : \mathbb{Z} \rightarrow G$ is a cyclic group (in the sense of Definition 4.7).

If $|g| = \infty$, then $g^i \neq g^j (i \neq j)$. Define

$$\varphi : \mathbb{Z} \longrightarrow \epsilon_g(\mathbb{Z}), n \longmapsto g^n$$

and we can check it is an isomorphism.

If $|g| = k$, then $e_G, g, g^2, \dots, g^{k-1}$ are distinct. Define

$$\varphi : \mathbb{Z}/k\mathbb{Z} \longrightarrow \epsilon_g(\mathbb{Z}), [n]_k \longmapsto g^n$$

and we can check it is an isomorphism.

Since $\epsilon_g(\mathbb{Z})$ is isomorphic to \mathbb{Z} or $\mathbb{Z}/k\mathbb{Z}$, we show $\epsilon_g(\mathbb{Z})$ is a cyclic group. ■

6.6 Prove that the union of a family of subgroups of a group G is not necessarily a subgroup of G . In fact:

- Let H, H' be subgroups of a group G . Prove that $H \cup H'$ is a subgroup of G only if $H \subseteq H'$ or $H' \subseteq H$.
- On the other hand, let $H_0 \subseteq H_1 \subseteq H_2 \subseteq \dots$ be subgroups of a group G . Prove that $\cup_{i \geq 0} H_i$ is a subgroup of G .

- Let $H \cup H'$ be a subgroup of G . Suppose neither $H \subseteq H'$ nor $H' \subseteq H$ hold. Let $a \in H - H', b \in H' - H, h = ab^{-1} \in H \cup H'$. In the case of $h \in H$, we have $b = h^{-1}a \in H$, contradiction! In the case of $h \in H'$, we have $a = hb \in H'$, contradiction again! Therefore, there must be $H \subseteq H'$ or $H' \subseteq H$.
- For all $a, b \in \cup_{i \geq 0} H_i$, we can suppose $a \in H_j, b \in H_k$ and we have $a, b \in H_{\max\{j,k\}}$. Then $ab \in H_{\max\{j,k\}} \subseteq \cup_{i \geq 0} H_i$, implies that $\cup_{i \geq 0} H_i$ is closed and that $\cup_{i \geq 0} H_i$ is a subgroup of G .

■

6.7 \neg Show that inner automorphisms (cf. [Exercise II.4.8](#)) form a subgroup of $\text{Aut}(G)$; this subgroup is denoted $\text{Inn}(G)$. Prove that $\text{Inn}(G)$ is cyclic if and only if $\text{Inn}(G)$ is trivial if and only if G is abelian. (Hint: Assume that $\text{Inn}(G)$ is cyclic; with notation as in Exercise 4.8, this means that there exists an element $a \in G$ such that $\forall g \in G \exists n \in \mathbb{Z} \gamma_g = \gamma_a^n$. In particular, $gag^{-1} = a^n aa^{-n} = a$. Thus a commutes with every g in G . Therefore...) Deduce that if $\text{Aut}(G)$ is cyclic then G is abelian. [7.10, IV.1.5]

With notation as in Exercise 4.8, we assume $\gamma_g \in \text{Inn}(G)$ is defined by

$$\forall h \in G \quad (\gamma_g(h) = ghg^{-1}).$$

We have

$$\begin{aligned} & \text{Inn}(G) \text{ is cyclic} \\ \iff & \exists \gamma_a \in \text{Inn}(G), \text{Inn}(G) = \langle \gamma_a \rangle \\ \iff & \exists a \in G \forall g \in G \exists n \in \mathbb{Z} (\gamma_g = \gamma_a^n) \\ \implies & \exists a \in G \forall g \in G \exists n \in \mathbb{Z} (\gamma_g(a) = gag^{-1} = \gamma_a^n(a) = a^n aa^{-n} = a) \\ \implies & \exists a \in G \forall g \in G (ga = ag) \\ \implies & \forall h \in G, \gamma_a(h) = aha^{-1} = haa^{-1} = h \\ \implies & \text{Inn}(G) = \langle \text{id} \rangle \\ \implies & \text{Inn}(G) \text{ is trivial} \end{aligned}$$

$$\begin{aligned} & \text{Inn}(G) \text{ is trivial} \\ \implies & \forall g \in G \forall h \in G (\gamma_g(h) = ghg^{-1} = h) \\ \implies & \forall g \in G \forall h \in G (gh = hg) \\ \iff & G \text{ is abelian} \end{aligned}$$

$$\begin{aligned} & G \text{ is abelian} \\ \implies & \forall g \in G \forall h \in G (\gamma_g(h) = ghg^{-1} = h) \\ \implies & \text{Inn}(G) = \{\text{id}\} \\ \implies & \text{Inn}(G) \text{ is cyclic} \end{aligned}$$

If $\text{Aut}(G)$ is cyclic, its subgroup $\text{Inn}(G)$ is also cyclic. As we have shown, that means G is abelian. ■

6.8 Prove that an abelian group G is finitely generated if and only if there is a surjective homomorphism

$$\underbrace{\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}_{n \text{ times}} \twoheadrightarrow G$$

for some n .

Given any set $H \subseteq G$, there exists a unique homomorphism φ_H such that the following diagram commutes.

$$\begin{array}{ccc} F^{ab}(H) & \xrightarrow{\exists! \varphi} & G \\ j \uparrow & \nearrow i & \\ H & & \end{array}$$

The homomorphism image $\varphi_H(F^{ab}(H)) \leq G$ is called the subgroup generated by H in G , denoted by $\langle H \rangle$.

If G is finitely generated, there is a finite subset $G_n \subseteq G$ with n elements such that $\varphi_H(F^{ab}(G_n)) = \varphi_H(\mathbb{Z}^{\oplus n}) = G$. And φ_H is exactly the surjective homomorphism that we need.

If there is a surjective homomorphism $\psi : \mathbb{Z}^{\oplus n} \twoheadrightarrow G$ for some n . Suppose

$$\psi : \mathbf{1}_i = (0, \dots, 0, \underset{i\text{-th place}}{1}, 0, \dots, 0) \mapsto g_i$$

and $G_n = \{g_1, g_2, \dots, g_n\}$. Then define

$$j : G_n \longrightarrow \mathbb{Z}^{\oplus n}, \quad g_i \longmapsto \mathbf{1}_i.$$

We can check the following diagram commutes

$$\begin{array}{ccc} \mathbb{Z}^{\oplus n} & \xrightarrow{\psi} & G \\ j \uparrow & \nearrow i & \\ G_n & & \end{array}$$

which means $\langle G_n \rangle = \psi(\mathbb{Z}^{\oplus n})$. Since ψ is surjective, we have $\langle G_n \rangle = G$. Hence we show G is finitely generated. ■

6.9 Prove that every finitely generated subgroup of \mathbb{Q} is cyclic. Prove that \mathbb{Q} is not finitely generated.

Given any two rationals

$$\begin{aligned} a_1 &= \frac{p_1}{q_1} \in \mathbb{Q}, (p_1, q_1) = 1, \\ a_2 &= \frac{p_2}{q_2} \in \mathbb{Q}, (p_2, q_2) = 1, \end{aligned}$$

there exists $r = \frac{1}{q_1 q_2} \in \mathbb{Q}$ such that $\langle a_1, a_2 \rangle \leq \langle r_1 \rangle$. Then for some a_3 we have $\langle a_1, a_2, a_3 \rangle \leq \langle r_1, a_3 \rangle \leq \langle r_2 \rangle$. In general, let's set $B_n = \{a_1, a_2, \dots, a_n\}$. If $\langle B_n \rangle \leq \langle r_{n-1} \rangle$. we have $\langle B_{n+1} \rangle = \langle B_n, a_{n+1} \rangle \leq \langle r_{n-1}, a_{n+1} \rangle \leq \langle r_n \rangle$. By induction we can prove $\langle a_1, a_2, \dots, a_n \rangle \leq \langle r_{n-1} \rangle$ for $n \in \mathbb{N}_+$. Since the subgroups of a cyclic group are also cyclic, we see finitely generated subgroup $\langle a_1, a_2, \dots, a_n \rangle \leq \mathbb{Q}$ is cyclic.

Supposing \mathbb{Q} is finitely generated, \mathbb{Q} must be a cyclic group, which contradicts the fact. Thus we show \mathbb{Q} is not finitely generated. ■

6.10 \neg The set of 2×2 matrices with integer entries and determinant 1 is denoted $\text{SL}_2(\mathbb{Z})$:

$$\text{SL}_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ such that } a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\}.$$

Prove that $\text{SL}_2(\mathbb{Z})$ is generated by the matrices:

$$s = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad t = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Let H be the subgroup generated by s and t . We can check that both

$$P = \begin{pmatrix} 1 & -p \\ 0 & 1 \end{pmatrix} = t^{-p} \quad \text{and} \quad Q = \begin{pmatrix} 1 & 0 \\ -q & 1 \end{pmatrix} = s^{-1}t^qs$$

are in H . Given an arbitrary matrix

$$m = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}),$$

it suffices to show that we can obtain the identity I_2 by multiplying m by matrices in H . Note that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & -p \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b - pa \\ c & d - pc \end{pmatrix}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -q & 1 \end{pmatrix} = \begin{pmatrix} a - qb & b \\ c - qd & d \end{pmatrix},$$

and c, d cannot be nonzero simultaneously. Without loss of generality, we can assume that $0 < c < d$ and perform Euclidean algorithm. Let $p_1 = \lfloor \frac{d}{c} \rfloor, d_1 = d - p_1c < c$. Multiplying m by $P_1 = \begin{pmatrix} 1 & -p_1 \\ 0 & 1 \end{pmatrix}$ on the right yields

$$m_1 = mP_1 \begin{pmatrix} a & b - p_1a \\ c & d_1 \end{pmatrix}.$$

Then let $q_1 = \lfloor \frac{c}{d_1} \rfloor, c_1 = c - q_1d_1 < d_1$ and right multiplying m by $Q_1 = \begin{pmatrix} 1 & 0 \\ -q_1 & 1 \end{pmatrix}$ yields

$$m_2 = mP_1Q_1 \begin{pmatrix} a - q_1(b - p_1a) & b - p_1a \\ c_1 & d_1 \end{pmatrix}.$$

We can repeat this procedure until some d_i or c_i reduce to 0. The Euclidean algorithm generates a sequence

$$d > c > d_1 > c_1 > d_2 > c_2 > \cdots.$$

If c_i, d_i never reduce to 0, we will get an infinite decreasing positive sequence, which is

impossible. Suppose d_N is the first number reducing to 0. Then

$$m_{2N-1} = mP_1Q_1 \cdots P_N = \begin{pmatrix} a_N & b_N \\ c_{N-1} & 0 \end{pmatrix} \in \text{SL}_2(\mathbb{Z}),$$

which implies

$$m_{2N-1} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and $m_{2N-1}s^{-1} = I_2$. Suppose c_N is the first number reducing to 0. Then

$$m_{2N} = mP_1Q_1 \cdots P_NQ_N = \begin{pmatrix} a_N & b_N \\ 0 & d_N \end{pmatrix} \in \text{SL}_2(\mathbb{Z}),$$

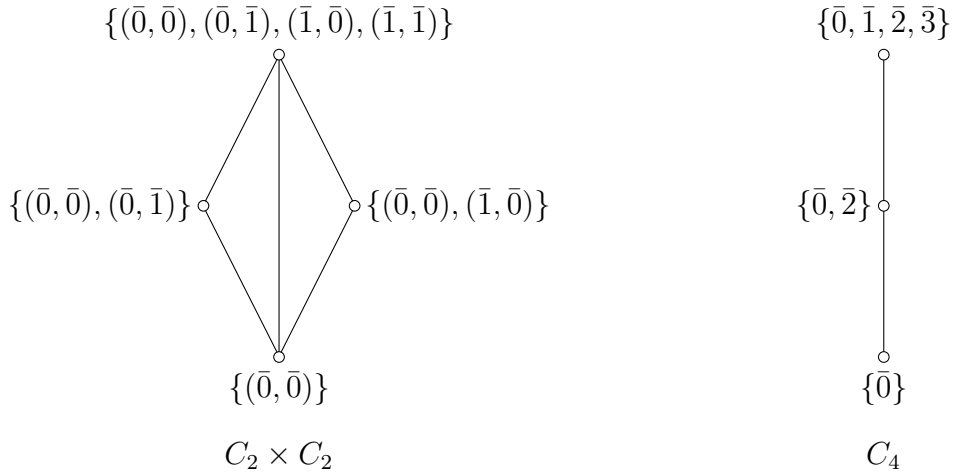
which implies

$$m_{2N} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2.$$

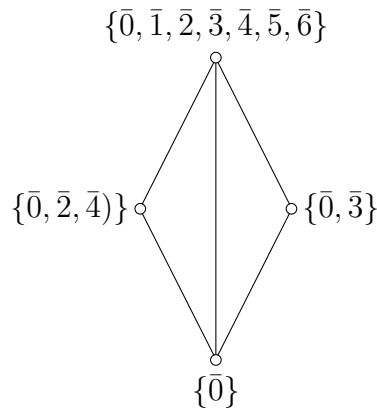
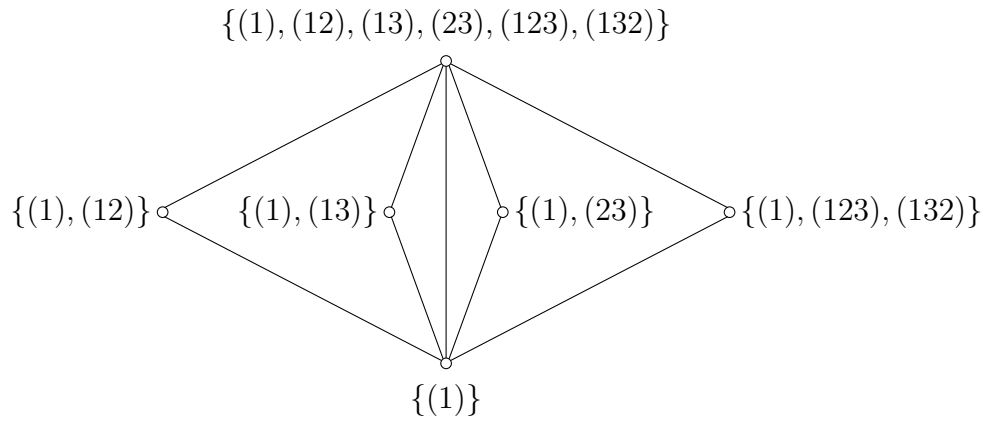
We have shown that we can obtain the identity I_2 by multiplying m by matrices in H , that is, m can be represented as a product of matrices in H . Thus we can conclude $\text{SL}_2(\mathbb{Z})$ is generated by s and t . ■

6.13 ▯ Draw and compare the lattices of subgroups of $C_2 \times C_2$ and C_4 . Draw the lattice of subgroups of S_3 , and compare it with the one for C_6 . [7.1]

Lattices of subgroups $C_2 \times C_2$ and C_4 are drawn as follows:



Lattices of subgroups S_3 and C_6 are drawn as follows:



■

§7. Quotient groups

7.1 ▷ List all subgroups of S_3 (cf. [Exercise II.6.13](#)) and determine which subgroups are normal and which are not normal. [§7.1]

The subgroups of S_3 are $\{(1)\}$, $\{(1), (12)\}$, $\{(1), (13)\}$, $\{(1), (23)\}$, $\{(1), (123), (132)\}$ and S_3 . We can check that $\{(1)\}$, $\{(1), (123), (132)\}$, S_3 are normal subgroups while others are not. ■

7.2 Is the image of a group homomorphism necessarily a normal subgroup of the target?

No. According to exercise 7.1 we have seen not all subgroups are normal. Suppose H is a subgroup of G but not normal. Then H itself is the image of the inclusion homomorphism $i : H \hookrightarrow G$, which makes a counterexample. ■

7.3 ▷ Verify that the equivalent conditions for normality given in §7.1 are indeed equivalent. [§7.1]

That a subgroup N of G is normal has four equivalent conditions:

- (i) $\forall g \in G, gNg^{-1} = N$;
- (ii) $\forall g \in G, gNg^{-1} \subseteq N$;
- (iii) $\forall g \in G, gN \subseteq Ng$;
- (iv) $\forall g \in G, gN = Ng$.

(i) \implies (ii) is straightforward.

(ii) \implies (iii). For any $g \in G$, the element $a \in gN$ can be written as $a = gn_1$ ($n_1 \in N$). Since $gn_1g^{-1} \in gNg^{-1} \subseteq N$, there exists an $n_2 \in N$ such that $gn_1g^{-1} = n_2$, which implies $gn_1 = n_2g \in Ng$. Thus we have $gN \subseteq Ng$.

(iii) \implies (iv). Given any $g \in G$, for all $n_1 \in N$, the element $g^{-1}n_1 \in g^{-1}N$ also belongs to Ng^{-1} , which implies that there exists $n_2 \in N$ such that $g^{-1}n_1 = n_2g^{-1}$, namely $n_1g = gn_2$. Thus we get $Ng \subseteq gN$ and accordingly $gN = Ng$.

(iv) \implies (i). For any $g \in G$, the element $b \in gNg^{-1}$ can be written as $a = gn_1g^{-1}$ ($n_1 \in N$). Since $gn_1 \in gN = Ng$, there exists an $n_2 \in N$ such that $gn_1 = n_2g$, which implies $gn_1g^{-1} = n_2 \in N$. Thus we have

$$\begin{aligned} & \forall g \in G, \quad gNg^{-1} \subseteq N \\ \implies & \forall g^{-1} \in G, \quad g^{-1}(gNg^{-1})g \subseteq gNg^{-1} \\ \implies & \forall g \in G, \quad N \subseteq gNg^{-1}. \end{aligned}$$

Hence we have $\forall g \in G, gNg^{-1} = N$. ■

7.4 Prove that the relation defined in [Exercise II.5.10](#) on a free abelian group $F = F^{ab}(A)$ is compatible with the group structure. Determine the quotient F/\sim as a better known group.

For all $f, f', h \in F$,

$$f \sim f' \iff f - f' = 2g, (g \in F) \implies (h + f) - (h + f') = 2g, (g \in F) \iff h + f \sim h + f'.$$

Since F is abelian, we see the relation \sim defined on a free abelian group $F = F^{ab}(A)$ is compatible with the group structure. By the notation of quotient group, we have

$$F/\sim = F/2F,$$

where $2F = \{2g \in F \mid g \in F\}$. ■

7.5 \neg Define an equivalence relation \sim on $\mathrm{SL}_2(\mathbb{Z})$ by letting $A \sim A' \iff A' = \pm A$. Prove that \sim is compatible with the group structure. The quotient $\mathrm{SL}_2(\mathbb{Z})/\sim$ is denoted $\mathrm{PSL}_2(\mathbb{Z})$, and is called the *modular group*; it would be a serious contender in a context for ‘the most important group in mathematics’, due to its role in algebraic geometry and number theory. Prove that $\mathrm{PSL}_2(\mathbb{Z})$ is generated by the (cosets of the) matrices

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}.$$

(You will not need to work very hard, if you use the result of [Exercise 6.10](#).) Note that the first has order 2 in $\mathrm{PSL}_2(\mathbb{Z})$, the second has order 3, and their product has infinite order. [9.14]

For all $A_1, A_2, B \in \mathrm{SL}_2(\mathbb{Z})$,

$$A_1 \sim A_2 \iff A_2 = \pm A_1 \iff BA_2 = \pm BA_1 \iff BA_1 \sim BA_2.$$

Hence \sim is compatible with the group structure and $\mathrm{PSL}_2(\mathbb{Z}) = \mathrm{SL}_2(\mathbb{Z})/\{I_2, -I_2\}$. In [Exercise 6.10](#) we have shown $\mathrm{SL}_2(\mathbb{Z})$ is generated by the matrices

$$s = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad t = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

It is clear that $\mathrm{SL}_2(\mathbb{Z})$ can also be generated by the matrices

$$s = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad ts = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix},$$

which implies $\mathrm{PSL}_2(\mathbb{Z})$ is generated by the cosets of the matrices s and ts . ■

7.6 Let G be a group, and let n be a positive integer. Consider the relation

$$a \sim b \iff (\exists g \in G) ab^{-1} = g^n.$$

- Show that in general \sim is not an equivalence relation.
- Prove that \sim is an equivalence relation if G is commutative, and determine the corresponding subgroup of G .

- Let G be the symmetric group S_4 and let $n = 2$. We can check that

$$\begin{aligned} (3\ 4)(2\ 3)^{-1} &= (2\ 4\ 3) = (2\ 3\ 4)^2 \implies (3\ 4) \sim (2\ 3) \\ (2\ 3)(1\ 2)^{-1} &= (1\ 3\ 2) = (1\ 2\ 3)^2 \implies (2\ 3) \sim (1\ 2) \end{aligned}$$

but $(3\ 4)(1\ 2)^{-1} = (1\ 2)(3\ 4)$ is not the square of any element in S_4 .

- Suppose that G is commutative. $aa^{-1} = e^n$ implies \sim is reflexive. Since

$$a \sim b \implies ab^{-1} = g^n \ (g \in G) \implies b^{-1}a = g^{-n} \ (g^{-1} \in G) \implies b \sim a,$$

\sim is symmetric. Since G is commutative, we have

$$\begin{aligned} a \sim b, b \sim c &\implies ab^{-1} = g_1^n, bc^{-1} = g_2^n \ (g_1, g_2 \in G) \\ &\implies ac^{-1} = ab^{-1}bc^{-1} = g_1^n g_2^n = (g_1 g_2)^n \ (g_1 g_2 \in G) \implies a \sim c, \end{aligned}$$

which means \sim is transitive. Thus we show that \sim is an equivalence relation. Since

$$a \sim b \implies ab^{-1} = g^n \implies ga(gb)^{-1} = (ag)(bg)^{-1} = g^n \implies ga \sim gb, ag \sim bg,$$

we see \sim is compatible with the group G and the equivalence class of the identity $H = \{g^n | g \in G\}$ is a subgroup of G . ■

7.7 Let G be a group, n a positive integer, and let $H \subseteq G$ be the subgroup generated by all elements of order n in G . Prove that H is normal.

For all $h \in H, g \in G$, we have

$$(ghg^{-1})^n = gh^n g^{-1} = gg^{-1} = e_G \implies ghg^{-1} \in H,$$

which means $gHg^{-1} \subseteq H$ for all $g \in G$. Thus we show that H is normal. ■

7.10 \neg Let G be a group, and $H \subseteq G$ a subgroup. With notation as in [Exercise II.6.7](#), show that H is normal in G if and only if $\forall \gamma \in \text{Inn}(G), \gamma(H) \subseteq H$. Conclude that if H is normal in G then there is an interesting homomorphism $\text{Inn}(G) \rightarrow \text{Aut}(H)$. [8.25]

Consistent with the notation as in [Exercise II.6.7](#), suppose

$$\gamma_g : G \longrightarrow G, \ h \longmapsto ghg^{-1}.$$

Then we have

$$\forall \gamma_g \in \text{Inn}(G), \gamma_g(H) \subseteq H \iff \forall g \in G, gHg^{-1} \subseteq H \iff H \text{ is normal in } G.$$

Thus we see that if H is normal in G , γ can be restricted to H so that $\gamma|_H : H \rightarrow H$ is an automorphism on H . Let

$$i : \text{Inn}(G) \longrightarrow \text{Aut}(H), \ \gamma \longmapsto \gamma|_H$$

and with the property of γ we have shown in [Exercise II.4.8](#), it is straightforward to check that

$$i(\gamma_{g_1} \gamma_{g_2}) = i(\gamma_{g_1 g_2}) = \gamma_{g_1 g_2}|_H = (\gamma_{g_1} \gamma_{g_2})|_H = \gamma_{g_1}|_H \gamma_{g_2}|_H = i(\gamma_{g_1}) i(\gamma_{g_2}).$$

That is, i is the interest homomorphism $\text{Inn}(G) \rightarrow \text{Aut}(H)$ that we expect. ■

7.11 ▷ Let G be a group, and let $[G, G]$ be the subgroup of G generated by all elements of the form $aba^{-1}b^{-1}$. (This is the commutator subgroup of G ; we will return to it in §IV.3.3.) Prove that $[G, G]$ is normal in G . (Hint: with notations in [Exercise II.4.8](#), $gaba^{-1}b^{-1}g^{-1} = \gamma_g(aba^{-1}b^{-1})$.) Prove that $[G, G]$ is normal in G . [7.12, §IV.3.3]

Since for all $g \in G$, $aba^{-1}b^{-1} \in [G, G]$, we have

$$gaba^{-1}b^{-1}g^{-1} = gag^{-1}gbg^{-1}ga^{-1}g^{-1}gb^{-1}g^{-1} = (gag^{-1})(gbg^{-1})(gag^{-1})^{-1}(gbg^{-1})^{-1} \in [G, G],$$

it follows that that $[G, G]$ is normal in G . Then we can show $[G, G]$ is normal in G by

$$[g_1][g_2] = [g_1g_2] = [g_1g_2(g_2^{-1}g_1^{-1}g_2g_1)] = [g_2g_1] = [g_2][g_1], \quad \forall [g_1], [g_2] \in [G, G].$$

■

7.12 ▷ Let $F = F(A)$ be a free group, and let $f : A \rightarrow G$ be a set-function from the set A to a commutative group G . Prove that f induces a unique homomorphism $F/[F, F] \rightarrow G$, where $[F, F]$ is the commutator subgroup of F defined in [Exercise II.7.11](#). (Use Theorem 7.12.) Conclude that $F/[F, F] \simeq F^{ab}(A)$. (Use Proposition I.5.4.) [§6.4, 7.13, VI.1.20]

By the universal property of free group, there exists a unique homomorphism $\varphi : F \rightarrow G$ such that $\forall a \in A$, $\varphi(j(a)) = f(a)$ where $j : A \rightarrow F(A)$ is a inclusion. Note that G is commutative, we have

$$\varphi(aba^{-1}b^{-1}) = \varphi(a)\varphi(b)\varphi(a)^{-1}\varphi(b)^{-1} = e_G,$$

which implies $[F, F] \subseteq \ker \varphi$. Theorem 7.12 indicates that there exists a unique group homomorphism $\tilde{\varphi} : F/[F, F] \rightarrow G$ so that $\tilde{\varphi} \circ \pi = \varphi$. Now we deduce that the diagram

$$\begin{array}{ccc} A & & \\ \downarrow j & \searrow f & \\ F & \xrightarrow{\exists! \varphi} & G \\ \downarrow \pi & \nearrow \exists! \tilde{\varphi} & \\ F/[F, F] & & \end{array}$$

commutes. For the diagram we see $\tilde{\varphi} \circ \pi \circ j = f$. Suppose there exists ψ such that $\psi \circ \pi \circ j = f$, which amounts to $(\psi \circ \pi) \circ j = \varphi \circ j$. By the uniqueness of φ we have $\psi \circ \pi = \varphi$. Then by the uniqueness of $\tilde{\varphi}$ we have $\psi = \tilde{\varphi}$. Thus we show that there exists unique $\tilde{\varphi}$ such that $\tilde{\varphi} \circ \pi \circ j = f$. According to the property of free abelian group, we can conclude that $F/[F, F] \simeq F^{ab}(A)$. ■

7.13 \neg Let A, B be sets, and $F(A), F(B)$ the corresponding free groups. Assume $F(A) \simeq F(B)$. If A is finite, prove that so is B , and $A \simeq B$. (Use [Exercise II.7.12](#) to upgrade [Exercise II.5.10](#).) [5.10, VI.1.20]

[Exercise II.7.12](#) tells us that the free abelian group generated by a set is merely determined by its free group, which means

$$F(A) \simeq F(B) \implies F(A)/[F(A), F(A)] \simeq F(B)/[F(B), F(B)] \implies F^{ab}(B) \cong F^{ab}(A).$$

Then under the auspices of the conclusion in [Exercise II.5.10](#) we complete the proof. ■

§8. Canonical decomposition and Lagrange's theorem

8.1 If a group H may be realized as a subgroup of two groups G_1 and G_2 , and

$$\frac{G_1}{H} \cong \frac{G_2}{H},$$

does it follow that $G_1 \cong G_2$. Give a proof or a counterexample.

A counterexample is given as follows. Take $H = C_3$, the cyclic group of order 3. Take $G_1 = D_6$ and $G_2 = C_6$, then one sees both G_1/H and G_2/H are C_2 . But obviously G_1 and G_2 are not isomorphic, one being abelian while the other is not. ■

8.2 \neg Extend Example 8.6 as follows. Suppose G is a group, and $H \subseteq G$ is a subgroup of index 2: that is, such that there are precisely two (say, left) cosets of H in G . Prove that H is normal in G . [9.11, IV.1.16]

Since $[G/H] = 2$, there must be $G/H = \{H, G - H\}$. For any $g \in G$:

- if $g \in H$, then $gH = Hg = H$;
- if $g \in G - H$, then $gH \neq H$ and $Hg \neq H$. Thus we have $gH = Hg = G - H$.

In either case $gH = Hg$ holds for all $g \in G$, which implies H is normal in G . ■

8.7 Let $(A|\mathcal{R})$, resp. $(A'|\mathcal{R}')$ be presentations for two groups G , resp. G' (cf. §8.2); we may assume that A, A' are disjoint. Prove that the group $G * G'$ presented by

$$(A \cup A' | \mathcal{R} \cup \mathcal{R}')$$

satisfies the universal property for the *coproduct* of G and G' in **Grp**. (Use the universal properties of both free groups and quotients to construct natural homomorphisms $G \rightarrow G * G'$, $G' \rightarrow G * G'$.) [§3.4, §8.2, 9.14].

Assume that $F(A)/R = (A|\mathcal{R})$, $F(A')/R' = (A'|\mathcal{R}')$, and $F(A \amalg A')/R'' = (A \cup A'|\mathcal{R} \cup \mathcal{R}')$.

$$\begin{array}{ccccc}
 & & G & & \\
 & \nearrow f & \uparrow \delta & \nwarrow f' & \\
 F(A)/R & \xrightarrow{\psi} & F(A \amalg A')/R'' & \xleftarrow{\psi'} & F(A')/R' \\
 \uparrow k & & \uparrow \pi & & \uparrow k' \\
 A & & F(A \amalg A') & & A' \\
 & \searrow i & \uparrow j & \swarrow i' & \\
 & & A \amalg A' & &
 \end{array}$$

According to [Lemma II.1](#), there exist unique ψ and ψ' such that

$$\psi \circ k = \pi \circ j \circ i, \quad \psi' \circ k' = \pi \circ j \circ i'.$$

Define

$$\begin{aligned}
 \delta : F(A \amalg A')/R'' &\longrightarrow G \\
 [\{a_1\} * \{a'_1\} * \cdots * \{a_n\} * \{a'_n\}] &\longmapsto f([\{a_1\}])f'([\{a'_1\}]) \cdots f([\{a_n\}])f'([\{a'_n\}]).
 \end{aligned}$$

where $*$ means the junction of words and $\{a_i\} = a_{i1} * a_{i2} * \cdots * a_{im_i}$, $a_{ij} \in A$ ($1 \leq i \leq n, 1 \leq j \leq m_i$) and $\{a'_i\} = a'_{i1} * a'_{i2} * \cdots * a'_{im'_i}$, $a'_{ij'} \in A$ ($1 \leq i \leq n, 1 \leq j' \leq m'_i$). It is routine to check that δ is a well-defined homomorphism such that

$$\delta \circ \psi = f, \quad \delta \circ \psi' = f'.$$

Then verify that if $\hat{\delta}$ is a homomorphism such that

$$\delta \circ \psi = f, \quad \delta \circ \psi' = f',$$

there must be $\hat{\delta} = \delta$. After these tasks are done, we can conclude that $F(A \amalg A')/R''$ satisfies the universal property of coproduct. ■

§9. Group actions

§10. Group objects in categories

Chapter III Rings and modules

§1. Definition of ring

1.1 \triangleright Prove that if $0 = 1$ in a ring R , then R is a zero-ring. [§1.2]

For any x in the ring R , we have

$$1 \cdot x = x, \quad 0 \cdot x = 0.$$

Since $0 = 1$ we see that $x = 0$, which implies R is a ring with only one element 0. \blacksquare

1.2 \neg Let S be a set, and define operations on the power set $\mathcal{P}(S)$ of S by setting $\forall A, B \in \mathcal{P}(S)$

$$A + B := (A \cup B) \setminus (A \cap B) \quad , \quad A \cdot B = A \cap B$$

Prove that $(\mathcal{P}(S), +, \cdot)$ is a commutative ring. [2.3, 3.15]

First, we need to check that $(\mathcal{P}(S), +)$ is an abelian group:

- associativity:

$$\begin{aligned} & (A + B) + C \\ &= ((A \cup B) \setminus (A \cap B)) + C \\ &= ((A \cup B) \cap (A^C \cup B^C)) + C \\ &= (A \cap (A^C \cup B^C)) \cup (B \cap (A^C \cup B^C)) + C \\ &= (A \cap B^C) \cup (A^C \cap B) + C \\ &= (((A \cap B^C) \cup (A^C \cap B)) \cap C^C) \cup (((A \cap B^C) \cup (A^C \cap B))^C \cap C) \\ &= ((A \cap B^C \cap C^C) \cup (A^C \cap B \cap C^C)) \cup ((A^C \cup B) \cap (A \cup B^C) \cap C) \\ &= ((A \cap B^C \cap C^C) \cup (A^C \cap B \cap C^C)) \cup ((A^C \cap B^C) \cup (A \cap B) \cap C) \\ &= (A \cap B^C \cap C^C) \cup (A^C \cap B \cap C^C) \cup (A^C \cap B^C \cap C) \cup (A \cap B \cap C) \\ &= (A \cap (B \cap C) \cup (B^C \cap C^C)) \cup ((A^C \cap B \cap C^C) \cup (A^C \cap B^C \cap C)) \\ &= (A \cap (B^C \cup C) \cap (B \cup C^C)) \cup ((A^C \cap B \cap C^C) \cup (A^C \cap B^C \cap C)) \\ &= (A \cap ((B \cap C^C) \cup (B^C \cap C))^C) \cup (A^C \cap ((B \cap C^C) \cup (B^C \cap C))) \\ &= A + ((B \cap C^C) \cup (B^C \cap C)) \\ &= A + (B + C); \end{aligned}$$

- commutativity:

$$A + B = (A \cup B) \setminus (A \cap B) = (B \cup A) \setminus (B \cap A) = B + A;$$

- additive identity: the additive identity is \emptyset since

$$A + \emptyset = (A \cup \emptyset) \setminus (A \cap \emptyset) = A; \emptyset \setminus \emptyset = \emptyset$$

- inverse: the inverse of some set A is just itself since

$$A + A = (A \cup A) \setminus (A \cap A) = A \setminus A = \emptyset.$$

Then we have to show that $(\mathcal{P}(S), \cdot)$ is a commutative monoid, which clearly holds with the multiplicative identity S . What is left to show is the distributive properties and the check is straightforward.

$$\begin{aligned} & (A + B) \cdot C \\ &= ((A \cap B^C) \cup (A^C \cap B)) \cap C \\ &= (A \cap B^C \cap C) \cup (A^C \cap B \cap C) \\ &= (A \cap C \cap (B^C \cup C^C)) \cup ((A^C \cup C^C) \cap (B \cap C)) \\ &= (A \cap C \cap (B \cap C)^C) \cup ((A \cap C)^C \cap (B \cap C)) \\ &= A \cdot C + B \cdot C. \end{aligned}$$

■

1.3 \neg Let R be a ring, and let S be any set. Explain how to endow the set R^S of set-functions $S \rightarrow R$ of two operations $+$, \cdot so as to make R^S into a ring, such that R^S is just a copy of R if S is a singleton. [2.3]

To make $(R^S, +, \cdot)$ a ring, for all $f, g \in R^S$ we define addition and multiplication as

$$\begin{aligned} f + g : S &\longrightarrow R, & x &\longmapsto f(x) + g(x) \\ f \cdot g : S &\longrightarrow R, & x &\longmapsto f(x) \cdot g(x). \end{aligned}$$

■

1.4 ▷ The set of $n \times n$ matrices with entries in a ring R is denoted $\mathcal{M}_n(R)$. Prove that componentwise addition and matrix multiplication makes $\mathcal{M}_n(R)$ into a ring, for any ring R . The notation $\mathfrak{gl}_n(R)$ is also commonly used, especially $R = \mathbb{R}$ or \mathbb{C} (although this indicates one is considering them as *Lie algebras*) in parallel with the analogous notation for the corresponding groups of units, cf. [Exercise II.6.1](#). In fact, the parallel continues with the definition of the following sets of matrices:

- $\mathfrak{sl}_n(\mathbb{R}) = \{M \in \mathfrak{gl}_n(\mathbb{R}) \mid \text{tr}(M) = 0\}$;
- $\mathfrak{sl}_n(\mathbb{C}) = \{M \in \mathfrak{gl}_n(\mathbb{C}) \mid \text{tr}(M) = 0\}$;
- $\mathfrak{so}_n(\mathbb{R}) = \{M \in \mathfrak{sl}_n(\mathbb{R}) \mid M + M^t = 0\}$;
- $\mathfrak{su}_n(\mathbb{C}) = \{M \in \mathfrak{sl}_n(\mathbb{C}) \mid M + M^\dagger = 0\}$.

Here $\text{tr}(M)$ is the trace of M , that is, the sum of its diagonal entries. The other notation matches the notation used in [Exercise II.6.1](#). Can we make rings of these sets, by endowing them of ordinary addition and multiplication of matrices? (These sets are all Lie algebras, cf. [Exercise VI.1.4](#).) [§1.2, 2.4, 5.9, VI.1.2, VI.1.4]

It is plain to show $\mathcal{M}_n(R)$ is a ring according to the definition. For multiplicative associativity, it follows that for all $A, B, C \in \mathcal{M}_n(R)$,

$$\begin{aligned}
& ((AB)C)_{\alpha,\delta} \\
&= \sum_{i=1}^n (AB)_{\alpha,i} c_{i,\delta} \\
&= \sum_{i=1}^n \left(\sum_{j=1}^n a_{\alpha,j} b_{j,i} \right) c_{i,\delta} \\
&= \sum_{i=1}^n \sum_{j=1}^n (a_{\alpha,j} b_{j,i}) c_{i,\delta} \\
&= \sum_{j=1}^n \sum_{i=1}^n a_{\alpha,j} (b_{j,i} c_{i,\delta}) \\
&= \sum_{j=1}^n a_{\alpha,j} \left(\sum_{i=1}^n b_{j,i} c_{i,\delta} \right) \\
&= \sum_{j=1}^n a_{\alpha,j} (BC)_{j,\delta} \\
&= (A(BC))_{\alpha,\delta}.
\end{aligned}$$

Under the ordinary addition and multiplication of matrices, $\mathfrak{sl}_n(\mathbb{R})$, $\mathfrak{sl}_n(\mathbb{C})$, $\mathfrak{so}_n(\mathbb{R})$, $\mathfrak{su}_n(\mathbb{C})$ are not rings. In fact, they are not closed under the multiplication. ■

1.5 Let R be a ring. If a, b are zero-divisors in R , is $a + b$ necessarily a zero-divisor?

That is not true. Let's take $\mathbb{Z}/6\mathbb{Z}$ as an counterexample. Though both $[2]_6$ and $[3]_6$ are zero-divisors, their sum $[5]_6$ is not a zero-divisor. ■

1.6 \neg An element a of a ring R is *nilpotent* if $a^n = 0$ for some n .

1. Prove that if a and b are nilpotent in R and $ab = ba$, then $a + b$ is also nilpotent.
2. Is the hypothesis $ab = ba$ in the previous statement necessary for its conclusion to hold?

[3.12]

1. Assume that $a^n = b^m = 0$ and let $k = 2 \max\{n, m\}$. If $ab = ba$, we can get

$$(a + b)^k = \sum_{p=0}^{\frac{k}{2}} \binom{k}{p} a^k b^{k-p} + \sum_{p=\frac{k}{2}+1}^k \binom{k}{p} a^k b^{k-p} = \sum_{p=0}^{\frac{k}{2}} \binom{k}{p} a^k \cdot 0 + \sum_{p=\frac{k}{2}+1}^k \binom{k}{p} 0 \cdot b^{k-p} = 0,$$

which means $a + b$ is also nilpotent.

2. The hypothesis $ab = ba$ is necessary. A counterexample can be found in the ring $\mathfrak{gl}_2(\mathbb{R})$. Let

$$a = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad b = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

and then we have $a^2 = b^2 = 0$. In other words, a and b are nilpotent. However, by diagonalization we see that

$$(a + b)^n = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^n = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}^n \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}^{-1} \neq \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Thus in such case, $a + b$ is no longer nilpotent. ■

1.8 Prove that $x = \pm 1$ are the only solutions to the equation $x^2 = 1$ in an integral domain. Find a ring in which the equation $x^2 = 1$ has more than 2 solutions.

It clearly holds that $1 \cdot 1 = 1$ and $(-1) \cdot (-1) = ((-1) \times (-1))1 \cdot 1 = 1$. That is to say, $x = \pm 1$ are the solutions to the equation $x^2 = 1$. Note that if there exists x in an integral domain such that $x^2 = 1$, then we have

$$(x - 1) \cdot (x + 1) = x^2 - 1 = 0,$$

which implies $x - 1 = 0$ or $x + 1 = 0$. Therefore, we can assert $x = \pm 1$ are the solutions. In the ring $\mathbb{Z}/8\mathbb{Z}$, $[3]_8$ and $[5]_8$ are also the solutions to the equation $x^2 = 1$. ■

1.10 Let R be a ring. Prove that if $a \in R$ is a right unit, and has two or more left-inverses, then a is not a left-zero-divisor, and is a right-zero-divisor.

Since $a \in R$ is a right unit, it cannot be a left-zero-divisor. Assume there exist two distinct elements $x, y \in R$ such that $xa = ya = 1$ and it deduces $(y - x)a = 0$. Thus we show that a is a right-zero-divisor. ■

1.11 Construct a field with 4 elements: as mentioned in the text, the underlying abelian group will have to be $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$; $(0, 0)$ will be the zero element, and $(1, 1)$ will be the multiplicative identity. The question is what $(0, 1) \cdot (0, 1)$, $(0, 1) \cdot (1, 0)$, $(1, 0) \cdot (1, 0)$ must be, in order to get a field. [§1.2, §V.5.1]

Define

$$(0, 1) \cdot (0, 1) = (0, 1), \quad (0, 1) \cdot (1, 0) = (0, 0), \quad (1, 0) \cdot (1, 0) = (1, 0),$$

and the the rest definition of multiplication will be determined uniquely according to field properties. For example, we have no alternatives but to define

$$(0, 1) \cdot (1, 1) = (0, 1) \cdot ((0, 1) + (1, 0)) = (0, 1) \cdot (0, 1) + (0, 1) \cdot (1, 0) = (0, 1) + (0, 0) = (0, 1).$$

Then we can check $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ forms a field by definition. ■

1.12 Just as complex numbers may be viewed as combinations $a + bi$, where $a, b \in \mathbb{R}$, and i satisfies the relation $i^2 = -1$ (and commutes with \mathbb{R}), we may construct a ring \mathbb{H} by considering linear combinations $a + bi + cj + dk$ where $a, b, c, d \in \mathbb{R}$, and i, j, k commute with \mathbb{R} and satisfy the following relations:

$$i^2 = j^2 = k^2 = -1 \quad , \quad ij = -ji = k \quad , \quad jk = -kj = i \quad , \quad ki = -ik = j.$$

Addition in \mathbb{H} is defined componentwise, while multiplication is defined by imposing distributivity and applying the relations. For example,

$$(1 + i + j) \cdot (2 + k) = 1 \cdot 2 + i \cdot 2 + j \cdot 2 + 1 \cdot k + i \cdot k + j \cdot k = 2 + 2i + 2j + k - j + i = 2 + 3i + j + k.$$

- (i) Verify that this prescription does indeed define a ring.
- (ii) Compute $(a + bi + cj + dk)(a - bi - cj - dk)$, where $a, b, c, d \in \mathbb{R}$.
- (iii) Prove that \mathbb{H} is a division ring. Elements of \mathbb{H} are called quaternions. Note that $Q_8 := \{\pm 1, \pm i, \pm j, \pm k\}$ forms a subgroup of the group of units of \mathbb{H} ; it is a noncommutative group of order 8, called the quaternionic group.
- (iv) List all subgroups of Q_8 , and prove that they are all normal.
- (v) Prove that Q_8, D_8 are not isomorphic.
- (vi) Prove that Q_8 admits the presentation $(x, y | x^2 y^{-2}, y^4, xyx^{-1}y)$.

[§II.7.1, 2.4, IV.1.12, IV.5.16, IV.5.17, V.6.19]

- (i) Verifying the $(\mathbb{H}, +)$ is an abelian group is immediate and we just omitted it. It is easy to see the multiplicative identity is 1 and the distributive properties are guaranteed by definition. The check of the associativity of multiplication looks straightforward but tedious.

$$\begin{aligned} & ((a_1 + b_1 i + c_1 j + d_1 k) \cdot (a_2 + b_2 i + c_2 j + d_2 k)) \cdot (a_3 + b_3 i + c_3 j + d_3 k) \\ &= [-c_3 (a_2 c_1 + a_1 c_2 + b_2 d_1 - b_1 d_2) - b_3 (a_2 b_1 + a_1 b_2 - c_2 d_1 + c_1 d_2) \\ &\quad + a_3 (a_1 a_2 - b_1 b_2 - c_1 c_2 - d_1 d_2) - d_3 (-b_2 c_1 + b_1 c_2 + a_2 d_1 + a_1 d_2)] \\ &\quad + [-c_3 (-b_2 c_1 + b_1 c_2 + a_2 d_1 + a_1 d_2) + a_3 (a_2 b_1 + a_1 b_2 - c_2 d_1 + c_1 d_2) \\ &\quad + b_3 (a_1 a_2 - b_1 b_2 - c_1 c_2 - d_1 d_2) + d_3 (a_2 c_1 + a_1 c_2 + b_2 d_1 - b_1 d_2)]i \\ &\quad + [b_3 (-b_2 c_1 + b_1 c_2 + a_2 d_1 + a_1 d_2) + a_3 (a_2 c_1 + a_1 c_2 + b_2 d_1 - b_1 d_2) \\ &\quad + c_3 (a_1 a_2 - b_1 b_2 - c_1 c_2 - d_1 d_2) - d_3 (a_2 b_1 + a_1 b_2 - c_2 d_1 + c_1 d_2)]j \\ &\quad + [a_3 (-b_2 c_1 + b_1 c_2 + a_2 d_1 + a_1 d_2) - b_3 (a_2 c_1 + a_1 c_2 + b_2 d_1 - b_1 d_2) \\ &\quad + c_3 (a_2 b_1 + a_1 b_2 - c_2 d_1 + c_1 d_2) + d_3 (a_1 a_2 - b_1 b_2 - c_1 c_2 - d_1 d_2)]k \end{aligned}$$

$$\begin{aligned}
& (a_1 + b_1i + c_1j + d_1k) \cdot ((a_2 + b_2i + c_2j + d_2k) \cdot (a_3 + b_3i + c_3j + d_3k)) \\
&= [-d_1(a_3d_2 + a_2d_3 - b_3c_2 + b_2c_3) - c_1(a_3c_2 + a_2c_3 + b_3d_2 - b_2d_3) \\
&\quad - b_1(a_3b_2 + a_2b_3 - c_3d_2 + c_2d_3) + a_1(a_2a_3 - b_2b_3 - c_2c_3 - d_2d_3)] \\
&\quad + [c_1(a_3d_2 + a_2d_3 - b_3c_2 + b_2c_3) - d_1(a_3c_2 + a_2c_3 + b_3d_2 - b_2d_3) \\
&\quad + a_1(a_3b_2 + a_2b_3 - c_3d_2 + c_2d_3) + b_1(a_2a_3 - b_2b_3 - c_2c_3 - d_2d_3)]i \\
&\quad + [-b_1(a_3d_2 + a_2d_3 - b_3c_2 + b_2c_3) + a_1(a_3c_2 + a_2c_3 + b_3d_2 - b_2d_3) \\
&\quad + d_1(a_3b_2 + a_2b_3 - c_3d_2 + c_2d_3) + c_1(a_2a_3 - b_2b_3 - c_2c_3 - d_2d_3)]j \\
&\quad + [a_1(a_3d_2 + a_2d_3 - b_3c_2 + b_2c_3) + b_1(a_3c_2 + a_2c_3 + b_3d_2 - b_2d_3) \\
&\quad - c_1(a_3b_2 + a_2b_3 - c_3d_2 + c_2d_3) + d_1(a_2a_3 - b_2b_3 - c_2c_3 - d_2d_3)]k
\end{aligned}$$

(ii) Expand it by distributive properties and we get

$$\begin{aligned}
& (a + bi + cj + dk)(a - bi - cj - dk) \\
&= a^2 - abi - acj - adk + abi + b^2 - bck + bdj + acj + bck + c^2 - cdi + adk - bdj + cdi + d^2 \\
&= a^2 + b^2 + c^2 + d^2.
\end{aligned}$$

(iii) Applying the results in (ii) we see that for any non-zero element $a + bi + cj + dk \in \mathbb{H}$,

$$(a + bi + cj + dk) \cdot \frac{a - bi - cj - dk}{a^2 + b^2 + c^2 + d^2} = \frac{a - bi - cj - dk}{a^2 + b^2 + c^2 + d^2} \cdot (a + bi + cj + dk) = 1,$$

which implies $a + bi + cj + dk$ is a two-sided unit. Thus we show that \mathbb{H} is a division ring.

- (iv) Q_8 has 6 subgroups: $\{1\}$, $\{1, -1\}$, $\{1, -1, i, -i\}$, $\{1, -1, j, -j\}$, $\{1, -1, k, -k\}$, Q_8 . We can just prove that they are all normal by the definition of normal subgroups.
- (v) Note that $D_8 = \{e, r, r^2, r^3, s_1, s_2, s_3, s_4\}$ has 7 subgroups: $\{e\}$, $\{e, r, r^2, r^3\}$, $\{e, s_1\}$, $\{e, s_2\}$, $\{e, s_3\}$, $\{e, s_4\}$, D_8 , while Q_8 has 6 subgroups. Thus Q_8, D_8 are not isomorphic.
- (vi) Let $P = (x, y|x^2y^{-2}, y^4, xyx^{-1}y)$. The relation $x^2y^{-2} = e$ implies $x^2 = y^2$ and the relation $xyx^{-1}y = e$ implies $yx = yx^{-1}x^2 = x^{-1}y^{-1}x^2 = x^3y^3x^2 = x^3y^5 = x^3y$. First, we can always replace yx by x^3y until we obtain a word of the form x^iy^j . Then applying $x^4 = y^4 = e$ and replace y^2 by x^2 , we can transform it into the form x^iy^j with $0 \leq i \leq 3$ and $0 \leq j \leq 1$. Thus we see P has at most 8 elements.

Next we will complete our proof by means of the [Lemma II.1](#) in the appendix. Define a mapping

$$\begin{aligned}
f : \{x, y\} &\longrightarrow Q_8, & x &\longmapsto i, \\
& & y &\longmapsto j.
\end{aligned}$$

Let $\varphi : F(\{x, y\}) \rightarrow Q_8$ be the unique homomorphism induced by the universal property of free group. Since

$$\begin{aligned}\varphi(x^2y^{-2}) &= i^2j^{-2} = 1, \\ \varphi(y^4) &= j^4 = 1, \\ \varphi(xy x^{-1}y) &= iji^{-1}j = 1,\end{aligned}$$

we see $\mathcal{R} = \{x^2y^{-2}, y^4, xyx^{-1}y\} \subset \ker \varphi$. And it is immediate to show that Q_8 can be generated by $\{i, j\}$. Thus according to the lemma, there exists a unique homomorphism $\psi : P \rightarrow Q_8$ such that $f = \psi \circ \pi \circ i$ and actually ψ is surjective.

$$\begin{array}{ccc} & P & \\ \pi \uparrow & \searrow \exists! \psi & \\ F(\{x, y\}) & \xrightarrow{\varphi} & Q_8 \\ i \uparrow & \nearrow f & \\ \{x, y\} & & \end{array}$$

Hence we get the inequality of cardinality $|P| \geq |Q_8|$. Since we have shown $|P| \leq 8 = |Q_8|$, there must be $|P| = |Q_8| = 8$, which implies ψ is indeed an isomorphism. Finally we conclude that $Q_8 \cong (x, y | x^2y^{-2}, y^4, xyx^{-1}y)$ and complete our proof. ■

1.14 ▷ Let R be a ring, and let $f(x), g(x) \in R[x]$ be nonzero polynomials. Prove that

$$\deg(f(x) + g(x)) \leq \max(\deg(f(x)), \deg(g(x))).$$

Assuming that R is an integral domain, prove that

$$\deg(f(x) \cdot g(x)) = \deg(f(x)) + \deg(g(x)).$$

[§1.3]

Assume

$$f(x) = \sum_{i \geq 0} a_i x^i, \quad g(x) = \sum_{i \geq 0} b_i x^i, \quad a_i, b_i \in R$$

and n, m are respectively the largest integers p, q for which a_p, b_q are non-zero. In others words, we have $a_n \neq 0, a_i = 0$ for $i > n$ and $b_m \neq 0, b_i = 0$ for $i > m$. Since

$$f(x) + g(x) = \sum_{i \geq 0} (a_i + b_i) x^i = \sum_{i=0}^{\max\{n, m\}} (a_i + b_i) x^i,$$

we see that

$$\deg(f(x) + g(x)) \leq \max\{n, m\} = \max(\deg(f(x)), \deg(g(x))).$$

Now Suppose that R is an integral domain. Noticing $a_n \neq 0$ and $b_m \neq 0$ implies $a_n b_m \neq 0$, we can see

$$f(x) \cdot g(x) = \sum_{k \geq 0} \sum_{i+j=k} a_i b_j x^{i+j} = \sum_{k=0}^{n+m} \sum_{i+j=k} a_i b_j x^{i+j}$$

has a degree of $n + m$. That is,

$$\deg(f(x) \cdot g(x)) = \deg(f(x)) + \deg(g(x)).$$

■

1.15 ▷ Prove that $R[x]$ is an integral domain if and only if R is an integral domain. [§1.3]

Assume R is an integral domain. [Exercise III.1.14](#) tells us if $f(x), g(x) \in R[x]$ are nonzero polynomials, we have

$$\deg(f(x) \cdot g(x)) = \deg(f(x)) + \deg(g(x)),$$

which implies $f(x) \cdot g(x)$ is also nonzero polynomial. Thus we show $R[x]$ is a integral domain.

Conversely, assume $R[x]$ is an integral domain. Note that given any $a, b \in R$, they also belong to $R[x]$. Hence we obtain

$$a \neq 0, b \neq 0 \implies ab \neq 0,$$

which means R is an integral domain.

■

1.16 Let R be a ring, and consider the ring of power series $R[[x]]$ (cf. §1.3).

1. Prove that a power series $a_0 + a_1x + a_2x^2 + \cdots$ is a unit in $R[[x]]$ if and only if a_0 is a unit in R . What is the inverse of $1 - x$ in $R[[x]]$?
2. Prove that $R[[x]]$ is an integral domain if and only if R is.

1. If a_0 is a unit in R then we can assume there exists $b_0 \in R$ such that $a_0 b_0 = 1$. Let

$$f(x) = \sum_{n \geq 0} a_n x^n, \quad g(x) = \sum_{n \geq 0} b_n x^n,$$

where

$$b_n = -b_0 \sum_{i=1}^n a_i b_{n-i}, \quad n \geq 1.$$

Noticing that

$$a_0 b_n = -a_0 b_0 \sum_{i=1}^n a_i b_{n-i} = - \sum_{i=1}^n a_i b_{n-i}, \quad n \geq 1,$$

we have

$$\begin{aligned} f(x)g(x) &= \sum_{n \geq 0} \sum_{i=0}^n a_{n-i} b_i x^n \\ &= 1 + \sum_{n \geq 1} \sum_{i=0}^n a_i b_{n-i} x^n \\ &= 1 + \sum_{n \geq 1} \left(a_0 b_n + \sum_{i=1}^n a_i b_{n-i} \right) x^n \\ &= 1 + \sum_{n \geq 1} (a_0 b_n - a_0 b_n) x^n \\ &= 1. \end{aligned}$$

Hence we show $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots$ is a unit.

For the other direction, supposing $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots$ is a unit, then there exists $g(x) = b_0 + b_1 x + b_2 x^2 + \cdots$ such that

$$f(x)g(x) = a_0 b_0 + \sum_{n \geq 1} \sum_{i=0}^n a_i b_{n-i} x^n = 1.$$

By comparing the both sides of the equality we can find $a_0 b_0 = 1$, which implies a_0 is a unit in R .

We can check that the inverse of $1 - x$ in $R[[x]]$ is $1 + x + x^2 + \cdots$ since

$$(1 - x) \sum_{i \geq 0} x^i = \sum_{i \geq 0} x^i - \sum_{i \geq 0} x^{i+1} = 1.$$

2. Suppose R is an integral domain. If $f(x), g(x) \in R[x]$ are nonzero polynomials, we can assume that

$$f(x) = \sum_{i \geq 0} a_i x^i, \quad g(x) = \sum_{i \geq 0} b_i x^i, \quad a_i, b_i \in R$$

and that n, m are respectively the smallest integers p, q for which a_p, b_q are non-zero. In others words, we have $a_n \neq 0$, $a_i = 0$ for $i < n$ and $b_m \neq 0$, $b_i = 0$ for $i < m$. Noticing $a_n \neq 0$ and $b_m \neq 0$ implies $a_n b_m \neq 0$, we can see

$$f(x) \cdot g(x) = \sum_{k \geq 0} \sum_{i+j=k} a_i b_j x^{i+j} = a_n b_m x^{n+m} + \sum_{k \geq n+m+1} \sum_{i+j=k} a_i b_j x^{i+j} \neq 0.$$

Thus we show $R[[x]]$ is an integral domain.

Conversely, assume that $R[[x]]$ is an integral domain. Note that given any $a, b \in R$, they also belong to $R[[x]]$. Hence we obtain

$$a \neq 0, b \neq 0 \implies ab \neq 0,$$

which means that R is also an integral domain. ■

§2. The category Ring

2.1 Prove that if there is a homomorphism from a zero-ring to a ring R , then R is a zero-ring [§2.1]

Suppose that φ is a homomorphism from a zero-ring O to a ring R . Since $\varphi(0_O) = 0_R$, $\varphi(1_O) = 1_R$, $0_O = 1_O$, we have $0_R = 1_R$, which implies that R is a zero-ring. ■

2.4 Define functions $\mathbb{H} \rightarrow \mathfrak{gl}_4(\mathbb{R})$ and $\mathbb{H} \rightarrow \mathfrak{gl}_2(\mathbb{C})$ (cf. [Exercise III.1.4](#) and [1.12](#)) by

$$\begin{aligned} a + bi + cj + dk &\longmapsto \begin{pmatrix} a & b & c & d \\ -b & a & -d & c \\ -c & d & a & -b \\ -d & -c & b & a \end{pmatrix} \\ a + bi + cj + dk &\longmapsto \begin{pmatrix} a + bi & c + di \\ -c + di & a - bi \end{pmatrix} \end{aligned}$$

for all $a, b, c, d \in \mathbb{R}$. Prove that both functions are injective ring homomorphisms. Thus, quaternions may be viewed as real or complex matrices.

Let f be the function $\mathbb{H} \rightarrow \mathfrak{gl}_4(\mathbb{R})$ described above. For simplicity, we omit trivial check and only verify f preserves multiplication

$$\begin{aligned} &f((a_1 + b_1i + c_1j + d_1k) \cdot (a_2 + b_2i + c_2j + d_2k)) \\ &= f((a_1a_2 - b_1b_2 - c_1c_2 - d_1d_2) + (a_2b_1 + a_1b_2 - c_2d_1 + c_1d_2)i \\ &\quad + (a_2c_1 + a_1c_2 + b_2d_1 - b_1d_2)j + (a_2d_1 + a_1d_2 - b_2c_1 + b_1c_2)k) \\ &= \begin{pmatrix} a_1 & b_1 & c_1 & d_1 \\ -b_1 & a_1 & -d_1 & c_1 \\ -c_1 & d_1 & a_1 & -b_1 \\ -d_1 & -c_1 & b_1 & a_1 \end{pmatrix} \begin{pmatrix} a_2 & b_2 & c_2 & d_2 \\ -b_2 & a_2 & -d_2 & c_2 \\ -c_2 & d_2 & a_2 & -b_2 \\ -d_2 & -c_2 & b_2 & a_2 \end{pmatrix} \\ &= f(a_1 + b_1i + c_1j + d_1k)f(a_2 + b_2i + c_2j + d_2k) \end{aligned}$$
■

2.5 The norm of a quaternion $w = a + bi + cj + dk$, with $a, b, c, d \in \mathbb{R}$, is the real number $N(w) = a^2 + b^2 + c^2 + d^2$. Prove that the function from the multiplicative group \mathbb{H}^* of nonzero quaternions to the multiplicative group \mathbb{R}^+ of positive real numbers, defined by assigning to each nonzero quaternion its norm, is a homomorphism. Prove that the kernel of this homomorphism is isomorphic to $\mathrm{SU}_2(\mathbb{C})$ (cf. [Exercise II.6.3](#)). [4.10, IV.5.17 V.6.19]

According to [Exercise III.2.4](#), $w \in \mathbb{H}^*$ can be viewed as a matrix $i(w) \in \mathfrak{gl}_2(\mathbb{C})$ where $i : \mathbb{H} \rightarrow \mathfrak{gl}_2(\mathbb{C})$ is a monomorphism in **Ring**. Then the function $N : \mathbb{H}^* \rightarrow \mathbb{R}^+$ can be just viewed as the determinant mapping $\det : i(\mathbb{H}^*) \subset \mathfrak{gl}_2(\mathbb{C}) \rightarrow \mathbb{R}^+$. More precisely, it means $N = \det \circ i$. We can check that

$$N(w_1 w_2) = \det(i(w_1 w_2)) = \det(i(w_1) i(w_2)) = \det(i(w_1)) \det(i(w_2)) = N(w_1) N(w_2)$$

and

$$w \in \ker N \iff N(w) = \det(i(w)) = 1 \iff i(w) \in \mathrm{SU}_2(\mathbb{C}).$$

Therefore, N is a homomorphism and $\ker N$ isomorphic to $\mathrm{SU}_2(\mathbb{C})$. ■

2.6 Verify the ‘extension property’ of polynomial rings, stated in Example 2.3. [§2.2]

Define the following ring homomorphisms

$$\begin{aligned} \alpha : R &\longrightarrow S, & r &\longmapsto \alpha(r) \\ \epsilon : R &\longrightarrow R[x], & r &\longmapsto r, \end{aligned}$$

and functions

$$\begin{aligned} j : \{s\} &\longrightarrow R[x], & s &\longmapsto x, \\ i : \{s\} &\longrightarrow S, & s &\longmapsto s. \end{aligned}$$

Assume that $s \in S$ is an element commuting with $\alpha(r)$ for all $r \in R$, we are to show that there exists a unique ring homomorphism $\bar{\alpha} : R[x] \rightarrow S$ such that the following diagram commutes.

$$\begin{array}{ccc} R & & \\ \epsilon \downarrow & \searrow \alpha & \\ R[x] & \xrightarrow{\exists! \bar{\alpha}} & S \\ j \uparrow & \nearrow i & \\ \{s\} & & \end{array}$$

Uniqueness. If $\bar{\alpha}$ exists, then the postulated commutativity of the diagram means that for all $f(x) = \sum_{n \geq 0} a_n x^n \in R[x]$, there must be

$$\bar{\alpha}(f(x)) = \bar{\alpha}\left(\sum_{n \geq 0} a_n x^n\right) = \sum_{n \geq 0} \bar{\alpha}(a_n) \bar{\alpha}(x)^n = \sum_{n \geq 0} \alpha(a_n) s^n.$$

That is, $\bar{\alpha}$ is unique.

Existence. The only choice is to define

$$\bar{\alpha} : R[x] \longrightarrow S, \quad \sum_{n \geq 0} a_n x^n \longmapsto \sum_{n \geq 0} \alpha(a_n) s^n$$

and to check whether it is a ring homomorphism.

1. Preserving addition:

$$\begin{aligned} \bar{\alpha} \left(\sum_{n \geq 0} a_n x^n + \sum_{n \geq 0} b_n x^n \right) &= \bar{\alpha} \left(\sum_{n \geq 0} (a_n + b_n) x^n \right) \\ &= \sum_{n \geq 0} \alpha(a_n + b_n) s^n \\ &= \sum_{n \geq 0} \alpha(a_n) s^n + \sum_{n \geq 0} \alpha(b_n) s^n \\ &= \bar{\alpha} \left(\sum_{n \geq 0} a_n x^n \right) + \bar{\alpha} \left(\sum_{n \geq 0} b_n x^n \right). \end{aligned}$$

2. Preserving multiplication:

$$\begin{aligned} \bar{\alpha} \left(\sum_{n \geq 0} a_n x^n \sum_{n \geq 0} b_n x^n \right) &= \bar{\alpha} \left(\sum_{n \geq 0} \sum_{i+j=n} a_i b_j x^n \right) \\ &= \sum_{n \geq 0} \alpha \left(\sum_{i+j=n} a_i b_j \right) s^n \\ &= \sum_{n \geq 0} \sum_{i+j=n} \alpha(a_i) s^i \alpha(b_j) s^j \\ &= \left(\sum_{n \geq 0} \alpha(a_n) s^n \right) \left(\sum_{n \geq 0} \alpha(b_n) s^n \right) \\ &= \bar{\alpha} \left(\sum_{n \geq 0} a_n x^n \right) \bar{\alpha} \left(\sum_{n \geq 0} b_n x^n \right). \end{aligned}$$

3. Preserving identity element:

$$\bar{\alpha}(1_R) = \alpha(1_R) = 1_S.$$

Integrating the two parts we finally conclude there exists a unique ring homomorphism $\bar{\alpha}$ such that the diagram commutes.

■

2.7 Let $R = \mathbb{Z}/2\mathbb{Z}$, and let $f(x) = x^2 - x$; note $f(x) \neq 0$. What is the polynomial function $R \rightarrow R$ determined by $f(x)$? [§2.2, §V.4.2, §V.5.1]

It determines a function $f : \mathbb{Z}/2\mathbb{Z} \rightarrow \mathbb{Z}/2\mathbb{Z}$ sends all elements to identity, that is, $f([0]_2) = [0]_2$, $f([1]_2) = [0]_2$. ■

2.8 Prove that every subring of a field is an integral domain.

Suppose $\varphi : R \hookrightarrow K$ is a inclusion homomorphism. If $a \neq 0$, we have

$$ab = ac \implies \varphi(a)\varphi(b) = \varphi(a)\varphi(c) \implies \varphi(b) = \varphi(c) \implies b = c.$$

Due to the commutativity of field it also holds that $ba = ca$. Thus we show R is an integral domain. ■

2.9 \neg The *center* of a ring R consists of the elements a such that $ar = ra$ for all $r \in R$. Prove that the center is a subring of R . Prove that the center of a division ring is a field. [2.11, IV.2.17, VII.5.14, VII.5.16]

Denote the center of R by $Z(R)$. We can check that

1. for all $x, y \in Z(R)$, for all $r \in R$,

$$(x - y)r = xr - yr = rx - ry = r(x - y) \implies x - y \in Z(R);$$

2. for all $r \in R$,

$$1r = r1 \implies 1 \in Z(R);$$

3. for all $x, y \in Z(R)$, for all $r \in R$,

$$(xy)r = xry = r(xy) \implies xy \in Z(R).$$

Thus we show that $Z(R)$ is a subring of R . If R is a division ring, then $Z(R)$ is also a division ring. Note that for all $x, y \in Z(R)$, $xy = yx$, we see that $Z(R)$ is a commutative division ring, namely field. ■

2.10 \neg The *centralizer* of an element a of a ring R consists of the elements $r \in R$ such that $ar = ra$. Prove that the centralizer of a is a subring of R , for every $a \in R$. Prove that the center of R is the intersection of all its centralizers. Prove that every centralizer in a division ring is a division ring. [2.11, IV.2.17, VII.5.16]

Denote the centralizer of an element a of R by $Z_a(R)$. That is,

$$Z_a(R) = \{r \in R \mid ar = ra\}.$$

We can check that

1. for all $x, y \in Z_a(R)$,

$$(x - y)a = xa - ya = ax - ay = a(x - y) \implies x - y \in Z_a(R);$$

2.

$$1a = a1 \implies 1 \in Z_a(R);$$

3. for all $x, y \in Z_a(R)$,

$$(xy)a = xay = a(xy) \implies xy \in Z_a(R).$$

Thus we show that $Z_a(R)$ is a subring of R .

By definition we have $Z(R) \subseteq Z_a(R)$ for all $a \in R$, which implies $Z(R) \subseteq \bigcap_{a \in R} Z_a(R)$. Assume $s \in \bigcap_{a \in R} Z_a(R)$, then we see $sa = as$ for all $a \in R$, which means $s \in Z(R)$ and accordingly $\bigcap_{a \in R} Z_a(R) \subseteq Z(R)$. Thus we deduce that $Z(R) = \bigcap_{a \in R} Z_a(R)$.

If R is a division ring and $r \in Z_a(R)$, we can assume that there exists $a \in R$ such as $ar = ra$, which means that

$$r^{-1}(ar)r^{-1} = r^{-1}(ra)r^{-1} \implies r^{-1}a = ar^{-1}.$$

According to the definition of $Z_a(R)$, we see $r^{-1} \in Z_a(R)$. Thus we show that $Z_a(R)$ is a division ring. ■

2.11 \neg Let R be a division ring consisting of p^2 elements, where p is a prime. Prove that R is commutative, as follows:

- If R is not commutative, then its center C ([Exercise III.2.9](#)) is a proper subring of R . Prove that $|C|$ would then consist of p elements.
- Let $r \in R, r \notin C$. Prove that the centralizer of r ([Exercise III.2.10](#)) contains both r and C .
- Deduce that the centralizer of r is the whole of R .
- Derive a contradiction, and conclude that R had to be commutative (hence, a field).

This is a particular case of Wedderburn's theorem: every finite division ring is a field. [IV.2.17, VII.5.16]

If R is not commutative, then its center $Z(R)$ is a proper subring of R , which means $|Z(R)| < p^2$. By considering $Z(R)$ as a subgroup of the underlying abelian group R , we can deduce that $|Z(R)|$ divides p^2 according to the Lagrange theorem. Thus we see that $Z(R)$ consists of p elements. Given any $r \in R - Z(R)$, in [Exercise III.2.10](#) we have shown that $Z_r(R)$ is a subring of R and $Z(R) \in Z_r(R)$. By the definition of $Z_r(R)$, it is clear that $r \in Z_r(R)$. Hence we have $Z(R) \cup \{r\} \subseteq Z_r(R)$ and $|Z_r(R)| > p$. Again by Lagrange theorem we have

$|Z_r(R)|$ divides p^2 , which forces $|Z_r(R)| = p^2$. Thus we show that $Z_r(R) = R$. Note that $Z_a(R) = R$ for all $a \in Z(R)$. We have $Z_a(R) = R$ for all $a \in R$. In [Exercise III.2.10](#), we have derived that $\bigcap_{a \in R} Z_a(R) \subseteq Z(R)$, which implies $R \subseteq Z(R)$. Thus we have $Z(R) = R$, which contradicts with the previous deduction that $Z(R)$ is a proper subring of R . Therefore, we can conclude that R is commutative. ■

2.15 For $m > 1$, the abelian groups $(\mathbb{Z}, +)$ and $(m\mathbb{Z}, +)$ are manifestly isomorphic: the function $\varphi : \mathbb{Z} \rightarrow m\mathbb{Z}, n \mapsto mn$ is a group isomorphism. Use this isomorphism to transfer the structure of ‘ring without identity’ $(m\mathbb{Z}, +, \cdot)$ back onto \mathbb{Z} : give an explicit formula for the ‘multiplication’ \bullet this defines on \mathbb{Z} (that is, such that $\varphi(a \bullet b) = \varphi(a) \cdot \varphi(b)$). Explain why structures induced by different positive integers m are non-isomorphic as ‘rings without 1’.

(This shows that there are many different ways to give a structure of ring without identity to the *group* $(\mathbb{Z}, +)$. Compare this observation with Exercise 2.16.) [§2.1] ■

§3. Ideals and quotient rings

3.1 Prove that the image of a ring homomorphism $\varphi : R \rightarrow S$ is a subring of S . What can you say about φ , if its image is an ideal of S ? What can you say about φ , if its kernel is a subring of R ?

We can see that $\text{im } \varphi$ is a subring of S from the canonical decomposition

$$R \twoheadrightarrow R/\ker \varphi \xrightarrow[\tilde{\varphi}]{\sim} \text{im } \varphi \hookrightarrow S$$

φ

If $\text{im } \varphi$ is an ideal, then $s \in S, 1 \in \text{im } \varphi \implies s \in \text{im } \varphi$. So $\text{im } \varphi = S$ and φ is an epimorphism. Since $\ker \varphi$ is a ideal, if it is also a subring, we have $\ker \varphi = R$. ■

3.2 Let $\varphi : R \rightarrow S$ be a ring homomorphism, and let J be an ideal of S . Prove that $I = \varphi^{-1}(J)$ is an ideal of R . [§3.1]

In **Ab** we see $\varphi^{-1}(J)$ is a subgroup of R . For all $r \in R, a \in \varphi^{-1}(J)$, we have

$$\varphi(ra) = \varphi(r)\varphi(a) \in J \implies ra \in \varphi^{-1}(J).$$

Similarly we can obtain $ar \in \varphi^{-1}(J)$. Therefore, we conclude that $I = \varphi^{-1}(J)$ is an ideal of R . ■

3.3 \neg Let $\varphi : R \rightarrow S$ be a ring homomorphism, and let J be an ideal of R .

- Show that $\varphi(J)$ need not be an ideal of S .
- Assume that φ is surjective; then prove that $\varphi(J)$ is an ideal of S .
- Assume that φ is surjective, and let $I = \ker \varphi$; thus we may identify S with R/I . Let $\bar{J} = \varphi(J)$, an ideal of R/I by the previous point. Prove that

$$\frac{R/I}{\bar{J}} \cong \frac{R}{I+J}$$

(Of course this is just a rehash of Proposition 3.11.) [4.11]

- Let $\varphi : \mathbb{Z} \hookrightarrow \mathbb{Q}$ and $J = \mathbb{Z}$. It is clear that $\varphi(J) = \mathbb{Z}$ is not an ideal of \mathbb{Q} .
- Assume that φ is surjective. In **Ab** we see $\varphi(J)$ is a subgroup of S . For all $a' = \varphi(a) \in \varphi(J)$, $r' = \varphi(r) \in S$,

$$ra \in J \implies r'a' = \varphi(r)\varphi(a) = \varphi(ra) \in \varphi(J).$$

Similarly we can obtain $a'r' \in \varphi(J)$. Therefore, we conclude that $\varphi(J)$ is an ideal of S .

- Assume that φ is surjective. The universal property yields a unique homomorphism

$$\begin{aligned} \psi : R/I &\longrightarrow R/(I+J), \\ r+I &\longmapsto r+I+J. \end{aligned}$$

Since

$$\begin{aligned} \ker \psi &= \{r+I \in R/I \mid r \in I+J\} \\ &= \{a+b+I \in R/I \mid a \in I, b \in J\} \\ &= \{b+I \in R/I \mid b \in J\} \\ &= \{\varphi(b) \in S \mid b \in J\} \\ &= \varphi(J) = \bar{J} \end{aligned}$$

and ψ is surjective,

$$\frac{R/I}{\bar{J}} = \frac{R/I}{\ker \psi} \cong \frac{R}{I+J}.$$

■

3.7 Let R be a ring, and let $a \in R$. Prove that Ra is a left-ideal of R , and aR is a right-ideal of R . Prove that a is a left-, resp. right-unit if and only if $R = aR$, resp. $R = Ra$.

For all $r \in R$, $r(Ra) \subseteq Ra$, $(aR)r \subseteq aR$. Therefore, Ra is a left-ideal of R , and aR is a right-ideal of R . Since $aR \subseteq R$, $R \subseteq aR$ actually amounts to $R = aR$.

$$a \text{ is a left-unit} \iff \exists b \in R, ab = 1 \implies \forall r \in R, r = abr \in aR \implies R \subseteq aR$$

$$R \subseteq aR \implies \forall r \in R, \exists r' \in R, r = ar' \implies \exists r' \in R, ar' = 1 \iff a \text{ is a left-unit}$$

Therefore, a is a left-unit if and only if $R = aR$. Similarly we can prove a is a right-unit if and only if $R = Ra$. ■

3.8 Prove that a ring R is a division ring if and only if its only left-ideals and right-ideals are $\{0\}$ and R .

In particular, a commutative ring R is a field if and only if the only ideals of R are $\{0\}$ and R . [3.9, §4.3]

Assume the only left-ideals and right-ideals that ring R have are $\{0\}$ and R . If $a \neq 0$, we have $Ra = aR = R$. As a result of [Exercise III.3.7](#), it implies that a is two-side unit and that accordingly R is a division ring.

Now assume that R is a division ring. Suppose I is a nonzero left-ideal of R and that $a \in I$ is not 0. Note that the condition of division ring guarantees there exists $b \in R$ such that $ba = 1$. Since for all $r \in R$, $r = (rb)a \in I$, there must be $I = R$. Supposing that I' is a nonzero right-ideal of R and that $a' \in I'$ is not 0, in a similar way we can deduce $I' = R$. Therefore, we conclude that the only left-ideals of R and right-ideals of R are $\{0\}$ and R . ■

3.11 Let R be a ring containing \mathbb{C} as a subring. Prove that there are no ring homomorphisms $R \rightarrow \mathbb{R}$.

Suppose $f : R \rightarrow \mathbb{R}$ is a homomorphism. On the one hand, we have

$$f(1) = f(1 * 1) = f(1)^2 \geq 0.$$

On the other hand, we can calculate $f(1)$ by

$$f(1) = f(-i * i) = -f(i)^2 \leq 0,$$

which forces $f(1)$ to be 0. Thus we see f sends some nonzero element in R to 0 in \mathbb{R} , which is a contradiction. ■

3.12 Let R be a commutative ring. Prove that the set of nilpotent elements of R is an ideal of R . (Cf. [Exercise III.1.6](#). This ideal is called the nilradical of R .)

Find a non-commutative ring in which the set of nilpotent elements is not an ideal. [3.13, 4.18, V.3.13, §VII.2.3]

Suppose N is the set of nilpotent elements of R . In [Exercise III.1.6](#) we have shown that if R is commutative, then $a + b \in N$ for all $a, b \in N$. Since for all $r \in R$, $a \in N$,

$$a^n = 0 \implies r^n a^n = a^n r^n = 0 \implies ra, ar \in N,$$

we prove that N is an ideal of R . A counterexample for non-commutative ring can be found in the ring $\mathfrak{gl}_2(\mathbb{R})$, as is shown in [Exercise III.1.6](#). ■

3.13 \neg Let R be a commutative ring, and let N be its nilradical (cf. [Exercise III.3.12](#)). Prove that R/N contains no nonzero nilpotent elements. (Such a ring is said to be reduced.) [4.6, VII.2.8]

Suppose there exists a nilpotent element $r + N \in R/N$ and $n > 0$ such that

$$r^n + N = N \iff r^n \in N.$$

Then we have $r^{nm} = 0$ for some $m > 0$, which implies $r \in N$. Therefore, the only nilpotent element in R/N is N . ■

3.14 \neg Prove that the characteristic of an integral domain is either 0 or a prime integer. Do you know any ring of characteristic 1?

Suppose the characteristic of the integral domain R is pq where p, q are positive prime integers. Then we have $p1_R \neq 0$ and $q1_R \neq 0$, since the order of 1_R is pq . However, we can deduce

$$(p1_R)(q1_R) = pq1_R = 0_R,$$

which contradicts the assumption that R is an integral domain.

If the characteristic of the integral domain R is 1, then the inclusion homomorphism $i : \mathbb{Z} \rightarrow R$ will send all integers to 0_R , which means $0_R = 1_R$ and R is actually a zero ring instead of an integral domain. Thus the characteristic of an integral domain is either be 0 or a prime integer. ■

3.17 Let I, J be ideals of a ring R . State and prove a precise result relating the ideals $(I + J)/I$ of R/I and $J/(I \cap J)$ of $R/(I \cap J)$. [§3.3]

As abelian groups, the second isomorphism theorem ensures $(I + J)/I \cong J/(I \cap J)$. ■

§4. Ideals and quotients: remarks and examples. Prime and maximal ideals

4.2 Prove that the homomorphic image of a Noetherian ring is Noetherian. That is, prove that if $\varphi : R \rightarrow S$ is a surjective ring homomorphism, and R is Noetherian, then S is Noetherian. [§6.4]

According to [Exercise III.3.2](#), given any ideal J of S , we see $\varphi^{-1}(J)$ is an ideal of R . Since R is a Noetherian ring, we have $\varphi^{-1}(J) = (a_1, a_2, \dots, a_n)$. Since φ is surjective, there must be

$$J = \varphi(\varphi^{-1}(J)) = (\varphi(a_1), \varphi(a_2), \dots, \varphi(a_n)),$$

which means J is finitely generated. Thus we conclude S is Noetherian. ■

4.3 Prove that the ideal $(2, x)$ of $\mathbb{Z}[x]$ is not principal.

Suppose $(f) = (2, x)$. Since it is easy to see $f \neq 0$ and $f \neq 1$, there must be

$$2 = gf \implies f = 2.$$

However, it is impossible to find some $h \in \mathbb{Z}[x]$ such that

$$2 + x = hf = 2h,$$

which leads to a contradiction. Thus we show that the ideal $(2, x)$ of $\mathbb{Z}[x]$ is not principal. ■

4.5 Let I, J be ideals in a ring R , such that $I + J = (1)$. Prove that $IJ = I \cap J$. [§4.1]

The notation (1) suggests R is commutative. For any $k \in IJ$, we can assume that $k = ab$, ($a \in I, b \in J$). Note that $k \in aJ = J$ and $k \in Ib = I$. It deduces that $k \in I \cap J$. Thus we show $IJ \subseteq I \cap J$.

Suppose $l \in I \cap J$. If $1 = a + b$ ($a \in I, b \in J$), Then we have $l = 1 * l = (a + b)l = al + lb \in IJ$, which implies that $I \cap J \subseteq IJ$. Therefore, we show $IJ = I \cap J$. ■

4.6 Let I, J be ideals in a ring R . Assume that $R/(IJ)$ is reduced (that is, it has no nonzero nilpotent elements; cf. [Exercise III.3.13](#)). Prove that $IJ = I \cap J$.

The notation (IJ) suggests R is commutative. As is shown in [Exercise III.4.5](#), it holds that $IJ \subseteq I \cap J$. Thus we are left to show $I \cap J \subseteq IJ$. Suppose $l \in I \cap J$. The condition that $R/(IJ)$ is reduced tells that $\forall r \in R$,

$$r^n \in IJ \implies r \in IJ.$$

Noticing $l \in I$ and $l \in J$, it is clear that $l^2 \in IJ$ which implies $l \in IJ$. There we show $I \cap J \subseteq IJ$ and complete the proof. ■

4.7 ▷ Let $R = k$ be a field. Prove that every nonzero (principal) ideal in $k[x]$ is generated by a unique *monic* polynomial. [§4.2, §VI.7.2]

Suppose I is a nonzero ideal in $k[x]$ and the least degree of nonzero polynomials in I is d . Since k is a field, we can find a monic polynomial $f(x) = k_0x^d + k_1x^{d+1} + \cdots + x^{d+n}$ in I . Given any $g(x) \in I$, there exist unique polynomials $q(x), r(x) \in k[x]$ such that $g(x) = f(x)q(x) + r(x)$ and $\deg r(x) < \deg f(x) = d$. Since $r(x) = g(x) - f(x)q(x) \in I$ and the least degree of nonzero polynomials in I is d , there must be $r(x) = 0$. Thus we show that I is generated by a monic polynomial $f(x)$. Suppose $I = (f(x))$ can be also generated by a monic polynomial $\bar{f}(x)$. Then we have $\bar{f}(x) = cf(x)$ for some $c \neq 0$. Since the two monic polynomials $\bar{f}(x), f(x)$ have the same degree, they are forced to be equal. Therefore, we conclude that every nonzero ideal in $k[x]$ is generated by a unique monic polynomial. ■

4.8 ▷ Let R be a ring, and $f(x) \in R[x]$ a monic polynomial. Prove that $f(x)$ is not a (left-, or right-) zero-divisor. [§4.2, 4.9]

Suppose $f(x) = x^d + a_{d-1}x^{d-1} + \cdots + a_1x + a_0$ is a monic polynomial in $R[x]$ and $f(x)g(x) = 0$ for some $g(x) = b_sx^s + b_{s-1}x^{s-1} + \cdots + b_1x + b_0 \in R[x]$. Since the term of the degree of $d + s$ of $f(x)g(x)$ is b_sx^{d+s} , there must be $b_s = 0$. Then the term of the degree of $d + s - 1$ of $f(x)g(x)$ is $b_{s-1}x^{d+s-1}$, which implies $b_{s-1} = 0$. Repeating this process we can show that $b_s = b_{s-1} = \cdots = b_0 = 0$, that is, $g(x) = 0$. Thus we see $f(x)$ is not a left-zero-divisor. In a similar way we can show that $f(x)$ is not a right-zero-divisor. ■

4.10 ▸ Let d be an integer that is not the square of an integer, and consider the subset of \mathbb{C} defined by

$$\mathbb{Q}(\sqrt{d}) := \{a + b\sqrt{d} \mid a, b \in \mathbb{Q}\}$$

- Prove that $\mathbb{Q}(\sqrt{d})$ is a subring of \mathbb{C} .
- Define a function $N : \mathbb{Q}(\sqrt{d}) \rightarrow \mathbb{Z}$ by $N(a + b\sqrt{d}) := a^2 - b^2d$. Prove that

$$N(zw) = N(z)N(w), \text{ and that } N(z) \neq 0 \text{ if } z \in \mathbb{Q}(\sqrt{d}), z \neq 0$$

The function N is a ‘norm’; it is very useful in the study of $\mathbb{Q}(\sqrt{d})$ and of its subrings. (Cf. also [Exercise III.2.5](#).)

- Prove that $\mathbb{Q}(\sqrt{d})$ is a field, and in fact the smallest subfield of \mathbb{C} containing both \mathbb{Q} and \sqrt{d} . (Use N .)
- Prove that $\mathbb{Q}(\sqrt{d}) \cong \mathbb{Q}[t]/(t^2 - d)$. (Cf. Example 4.8.)
[V.1.17, V.2.18, V.6.13, VII.1.12]

- We only show the check on multiplication

$$(a_1 + b_1\sqrt{d})(a_2 + b_2\sqrt{d}) = (a_1a_2 + b_1b_2d) + (a_1b_2 + a_2b_1)\sqrt{d} \in \mathbb{Q}(\sqrt{d}).$$

- It is immediate to check $N(zw) = N(z)N(w)$. Let $z \in \mathbb{Q}(\sqrt{d})$ and $z = a + b\sqrt{d} \neq 0$. Suppose $N(z) = a^2 - b^2d = 0$. If $b = 0$, we have $a = 0$, which contradicts with

$a + b\sqrt{d} \neq 0$. Otherwise we have $b \neq 0$ and $d = (a/b)^2$. Thus we get a contradiction again.

- We have known $\mathbb{Q}(\sqrt{d})$ is a commutative ring. For any $z = a + b\sqrt{d} \in \mathbb{Q}(\sqrt{d})$ such that $z \neq 0$,

$$N(z) = (a + b\sqrt{d})(a - b\sqrt{d}) = a^2 - b^2d \neq 0.$$

Therefore

$$(a + b\sqrt{d}) \left(\frac{a}{N(z)} - \frac{b}{N(z)}\sqrt{d} \right) = 1$$

and $\mathbb{Q}(\sqrt{d})$ is a field.

- The mapping

$$\begin{aligned} \bar{\varphi} : \mathbb{Q}[t]/(t^2 - d) &\longrightarrow \mathbb{Q}(\sqrt{d}), \\ a + bt + (t^2 - d) &\longmapsto a + b\sqrt{d}. \end{aligned}$$

is well-defined since if $(a_1 + b_1t) - (a_2 + b_2t) = g(t)(t^2 - d)$, then

$$\begin{aligned} \bar{\varphi}(a_1 + b_1t + (t^2 - d)) - \bar{\varphi}(a_2 + b_2t + (t^2 - d)) &= (a_1 + b_1\sqrt{d}) - (a_2 + b_2\sqrt{d}) \\ &= g(\sqrt{d}) \left((\sqrt{d})^2 - d \right) \\ &= 0. \end{aligned}$$

It is clear that $\bar{\varphi}$ preserves addition. Then we can check $\bar{\varphi}$ preserve multiplication:

$$\begin{aligned} &\bar{\varphi}((a_1 + b_1t + (t^2 - d))(a_2 + b_2t + (t^2 - d))) \\ &= \bar{\varphi}((a_1a_2 + (a_1b_2 + a_2b_1)t + b_1b_2t^2 + (t^2 - d))) \\ &= \bar{\varphi}(((a_1a_2 + b_1b_2d) + (a_1b_2 + a_2b_1)t + b_1b_2(t^2 - d) + (t^2 - d))) \\ &= (a_1a_2 + b_1b_2d) + (a_1b_2 + a_2b_1)\sqrt{d} \\ &= (a_1 + b_1\sqrt{d})(a_2 + b_2\sqrt{d}) \\ &= \bar{\varphi}(a_1 + b_1t + (t^2 - d)) \bar{\varphi}(a_2 + b_2t + (t^2 - d)). \end{aligned}$$

Thus we see $\bar{\varphi}$ is a ring homomorphism. Note

$$a + bt + (t^2 - d) \in \ker \bar{\varphi} \iff a + b\sqrt{d} = 0 \iff a = b = 0.$$

It implies that $\ker \bar{\varphi} = \{0 + (t^2 - d)\}$ and $\bar{\varphi}$ is injective. It is clear that $\bar{\varphi}$ is surjective. Therefore, $\bar{\varphi}$ is an isomorphism. ■

4.11 Let R be a commutative ring, $a \in R$, and $f_1(x), \dots, f_r(x) \in R[x]$.

- Prove the equality of ideals

$$(f_1(x), \dots, f_r(x), x - a) = (f_1(a), \dots, f_r(a), x - a)$$

- Prove the useful substitution trick

$$\frac{R[x]}{(f_1(x), \dots, f_r(x), x - a)} \cong \frac{R}{(f_1(a), \dots, f_r(a))}$$

(Hint: [Exercise III.3.3](#).)

- According to the polynomial remainder theorem, we have

$$f_i(x) = (x - a)q_i(x) + f_i(a),$$

which suffices to show that $(f_1(x), \dots, f_r(x), x - a) = (f_1(a), \dots, f_r(a), x - a)$.

- Define

$$\begin{aligned} \varphi : R[x] &\longrightarrow R, \\ f(x) &\longmapsto f(a). \end{aligned}$$

We can check that φ is a surjective ring homomorphism and $\ker \varphi = (x - a)$. According to [Exercise III.3.3](#), we have

$$\frac{R[x]}{(f_1(x), \dots, f_r(x), x - a)} \cong \frac{R[x]}{(f_1(a), \dots, f_r(a), x - a)} \cong \frac{R[x]/(x - a)}{(f_1(a), \dots, f_r(a))},$$

where

$$\overline{(f_1(a), \dots, f_r(a))} = (f_1(a) + (x - a), \dots, f_r(a) + (x - a)).$$

The ring isomorphism

$$\begin{aligned} \psi : R[x]/(x - a) &\longrightarrow R, \\ f(x) + (x - a) &\longmapsto f(a) \end{aligned}$$

gives the following isomorphism

$$\frac{R[x]/(x - a)}{(f_1(a), \dots, f_r(a))} \cong \frac{R}{(f_1(a), \dots, f_r(a))},$$

which completes the proof. ■

4.12 ▷ Let R be a commutative ring, and a_1, \dots, a_n elements of R . Prove that

$$\frac{R[x_1, \dots, x_n]}{(x_1 - a_1, \dots, x_n - a_n)} \cong R$$

[VII.2.2]

$$R \cong \frac{R[x_1]}{(x_1 - a_1)} \cong \frac{R[x_1, x_2]}{(x_1 - a_1, x_2 - a_2)}$$

The mapping

$$\begin{aligned} \bar{\varphi} : \frac{R[x_1, \dots, x_n]}{(x_1 - a_1, \dots, x_n - a_n)} &\longrightarrow \frac{R[x_1, \dots, x_{n-1}]}{(x_1 - a_1, \dots, x_{n-1} - a_{n-1})}, \\ f(x_1, \dots, x_n) + (x_1 - a_1, \dots, x_n - a_n) &\longmapsto f(x_1, \dots, x_{n-1}, a_n) + (x_1 - a_1, \dots, x_{n-1} - a_{n-1}) \end{aligned}$$

is well-defined since if $f_1(x_1, \dots, x_n) - f_2(x_1, \dots, x_n) = \sum_{i=1}^n g_i(x_1, \dots, x_n)(x_i - a_i)$, then

$$\begin{aligned} \bar{\varphi}\left(\overline{f_1(x_1, \dots, x_n)}\right) - \bar{\varphi}\left(\overline{f_2(x_1, \dots, x_n)}\right) &= f_1(x_1, \dots, x_{n-1}, a_n) - f_2(x_1, \dots, x_{n-1}, a_n) \\ &= \sum_{i=1}^{n-1} g_i(x_1, \dots, x_{n-1}, a_n)(x_i - a_i) + g_n(x_1, \dots, x_{n-1}, a_n)(a_n - a_n) \\ &= \sum_{i=1}^{n-1} g_i(x_1, \dots, x_{n-1}, a_n)(x_i - a_i). \end{aligned}$$

It is clear that $\bar{\varphi}$ preserves addition and multiplication. Thus we see $\bar{\varphi}$ is a ring homomorphism. Note

$$\begin{aligned} f(x_1, \dots, x_n) &\in \ker \bar{\varphi} \\ \iff f(x_1, \dots, x_{n-1}, a_n) &= \sum_{i=1}^{n-1} g_i(x_1, \dots, x_{n-1}, a_n)(x_i - a_i) \\ \iff f(x_1, \dots, x_{n-1}, x_n) &= \sum_{i=1}^{n-1} g_i(x_1, \dots, x_{n-1}, a_n)(x_i - a_i) + g_n(x_1, \dots, x_{n-1}, a_n)(x_n - a_n) \\ \iff f(x_1, \dots, x_n) &\in (x_1 - a_1, \dots, x_n - a_n), \end{aligned}$$

where the last but one line can be deduced by the polynomial remainder theorem if we fix x_1, \dots, x_{n-1} and regard x_n as a variable. It implies that $\ker \bar{\varphi} = \{0 + (x_1 - a_1, \dots, x_n - a_n)\}$ and $\bar{\varphi}$ is injective. It is clear that $\bar{\varphi}$ is surjective. Therefore, $\bar{\varphi}$ is an isomorphism and

$$R \cong \frac{R[x_1]}{(x_1 - a_1)} \cong \frac{R[x_1, x_2]}{(x_1 - a_1, x_2 - a_2)} \cong \dots \cong \frac{R[x_1, \dots, x_n]}{(x_1 - a_1, \dots, x_n - a_n)}.$$

■

4.17 \neg (If you know a little topology...) Let K be a compact topological space, and let R be the ring of continuous real-valued functions on K , with addition and multiplication defined pointwise.

- (i) For $p \in K$, let $M_p = \{f \in R \mid f(p) = 0\}$. Prove that M_p is a maximal ideal in R
- (ii) Prove that if $f_1, \dots, f_r \in R$ have no common zeros, then $(f_1, \dots, f_r) = (1)$ (Hint: consider $f_1^2 + \dots + f_r^2$)
- (iii) Prove that every maximal ideal M in R is of the form M_p for some $p \in K$. (Hint: you will use the compactness of K and (ii).)

Conclude that $p \mapsto M_p$ defines a bijection from K to the set of maximal ideals of R . (The set of maximal ideals of a commutative ring R is called the *maximal spectrum* of R ; it is contained in the (prime) spectrum $\text{Spec } R$ defined in §4.3. Relating commutative rings and "geometric" entities such as topological spaces is the business of algebraic geometry. The compactness hypothesis is necessary: cf. Exercise V.3.10. [V.3.10]

- (i) Suppose all functions in R that have same value in a neighborhood of p are identified. It is easy to check that M_p is an ideal and R/M_p is commutative. Given any $f \in R - M_p$, we have $f(p) \neq 0$ and

$$(f + M_p) \left(\frac{1}{f} + M_p \right) = 1 + M_p$$

Therefore, R/M_p is a field and M_p is a maximal ideal in R .

- (ii) If $f_1, \dots, f_r \in R$ have no common zeros, $(f_1, \dots, f_r) = (1)$ follows from

$$\sum_{i=1}^n \frac{f_i}{f_1^2 + \dots + f_n^2} f_i = 1.$$

- (iii) Let M be a maximal ideal in R . Then R/M is a field. For any $f \in R - M$, there exists $g \in R - M$ such that

$$(f + M)(g + M) = fg + M = 1 + M \implies fg - 1 \in M$$

■

§5. Modules over a ring

5.1 \triangleright Let R be a ring. The *opposite* ring R° is obtained from R by reversing the multiplication: that is, the product $a \bullet b$ in R° is defined to be $ba \in R$. Prove that the identity map $R \rightarrow R^\circ$ is an isomorphism if and only if R is commutative. Prove that $\mathcal{M}_n(\mathbb{R})$ is isomorphic to its opposite (not via the identity map!). Explain how to turn right- R -modules into left- R -modules and conversely, if $R \cong R^\circ$. [§5.1, VIII.5.19]

Let i denote the identity map $R \rightarrow R^\circ$. If R is commutative, we have

$$i(ab) = ab = b \bullet a = i(b) \bullet i(a) = i(a) \bullet i(b).$$

Given that $i(a + b) = a + b$ and identity map is a bijection, we see that i is an isomorphism.

If i is an isomorphism, we have

$$ab = b \bullet a = i^{-1}(b \bullet a) = i^{-1}(b)i^{-1}(a) = ba,$$

which implies that R is commutative.

Suppose $A, B \in \mathcal{M}_n(\mathbb{R})$. We can show that the transpose of matrix $\cdot^T : A \mapsto A^T$ is an isomorphism by checking

$$(AB)^T = B^T A^T = A^T \bullet B^T.$$

Let M be a right- R -module with right multiplication \odot . If $R \cong R^\circ$ and $f : R \rightarrow R^\circ$ is an isomorphism, then

$$f(ab) = f(a) \bullet f(b) = f(b)f(a)$$

Define left multiplication \odot_L as

$$r \odot_L m := m \odot f(r), \quad \forall r \in R, m \in M.$$

We can check that

$$1 \odot_L m = m \odot f(1) = 1,$$

$$\begin{aligned} (rs) \odot_L m &= m \odot f(rs) = m \odot (f(s)f(r)) = (m \odot f(s)) \odot f(r) \\ &= (s \odot_L m) \odot f(r) = r \odot_L (s \odot_L m), \end{aligned}$$

$$r \odot_L (m_1 + m_2) = (m_1 + m_2) \odot f(r) = m_1 \odot f(r) + m_2 \odot f(r) = r \odot_L m_1 + r \odot_L m_2.$$

Therefore, we show that M is a left- R -module with right multiplication \odot_L .

If M is a left- R -module with left multiplication $*$ and $f : R \rightarrow R^\circ$ is an isomorphism, then we can show that M is a right- R -module with right multiplication $*_R$ defined as

$$m *_R r := f^{-1}(r), \quad \forall r \in R, m \in M.$$

■

5.3 ▷ Let M be a module over a ring R . Prove that $0 \cdot m = 0$ and that $(-1) \cdot m = -m$, for all $m \in M$. [§5.2]

$$0 \cdot m = (0 + 0) \cdot m = 0 \cdot m + 0 \cdot m \implies 0 \cdot m = 0,$$

$$0 = (1 - 1) \cdot m = 1 \cdot m + (-1) \cdot m \implies (-1) \cdot m = -m.$$

■

5.4 \neg Let R be a ring. A nonzero R -module M is *simple* (or *irreducible*) if its only submodules are $\{0\}$ and M . Let M, N be simple modules, and let $\varphi : M \rightarrow N$ be a homomorphism of R -modules. Prove that either $\varphi = 0$, or φ is an isomorphism. (This rather innocent statement is known as Schur's lemma.) [5.10, 6.16, VI.1.16]

For convenience, we talk about the identity of modules up to isomorphism. Since the nonzero R -module M is simple, $\ker \varphi$ is either $\{0\}$ or M . Thus $\operatorname{im} \varphi = M / \ker \varphi$ is either $\{0\}$ or M . Note that $\operatorname{im} \varphi \subset N$ is either $\{0\}$ or N . If $\operatorname{im} \varphi = \{0\}$, then we have $\varphi = 0$. If $\operatorname{im} \varphi = M$, then we have $\operatorname{im} \varphi = M = N$. Therefore we show that either $\varphi = 0$, or φ is an isomorphism. ■

5.5 Let R be a ring, viewed as an R -module over itself, and let M be an R -module. Prove that $\operatorname{Hom}_{R\text{-Mod}}(R, M) \cong M$ as R -modules.

Define

$$\begin{aligned} \varphi : \operatorname{Hom}_{R\text{-Mod}}(R, M) &\longrightarrow M, \\ f &\longmapsto f(1) \end{aligned}$$

Since

$$\begin{aligned} \varphi(f + g) &= (f + g)(1) = f(1) + g(1) = \varphi(f) + \varphi(g), \\ \varphi(rf) &= (rf)(1) = rf(1) = r\varphi(f), \end{aligned}$$

we see φ is a homomorphism. If $\varphi(f_1) = \varphi(f_2)$, we have $f_1(1) = f_2(1)$. Multiply both sides by any $r \in R$ and we get

$$rf_1(1) = rf_2(1) \implies f_1(r) = f_2(r),$$

which means $f_1 = f_2$. Thus we show φ is injective. Given any $m \in M$, let

$$\begin{aligned} h_m : R &\longrightarrow M, \\ r &\longmapsto rm \end{aligned}$$

Since $\varphi(h_m) = h_m(1) = m$, we show that φ is surjective. Therefore, we conclude that φ is an isomorphism and $\operatorname{Hom}_{R\text{-Mod}}(R, M) \cong M$ as R -modules. ■

5.6 Let G be an abelian group. Prove that if G has a structure of \mathbb{Q} -vector space, then it has only one such structure. (Hint: first prove that every element of G has necessarily infinite order.)

Assume that G has two structures of \mathbb{Q} -vector space with scalar multiplication operations \cdot and $*$ respectively. Note that $1 \cdot g = 1 * g = g$ for all $g \in G$. With the conventional notation

$\sum_{i=1}^n g = ng$, we have for all $g \in G$,

$$\sum_{i=1}^n 1 \cdot g = \sum_{i=1}^n 1 * g = ng \implies \left(\sum_{i=1}^n 1 \right) \cdot g = \left(\sum_{i=1}^n 1 \right) * g = ng \implies n \cdot g = n * g = ng.$$

Since

$$\begin{aligned} \sum_{i=1}^m \frac{1}{m} \cdot h &= \left(\sum_{i=1}^m \frac{1}{m} \right) \cdot h = h, & \forall h \in G, \\ \sum_{i=1}^m \frac{1}{m} * h &= \left(\sum_{i=1}^m \frac{1}{m} \right) * h = h, & \forall h \in G, \end{aligned}$$

it holds that for all $h \in G$,

$$\sum_{i=1}^m \frac{1}{m} \cdot h = \sum_{i=1}^m \frac{1}{m} * h \implies \sum_{i=1}^m \left(\frac{1}{m} \cdot h - \frac{1}{m} * h \right) = 0 \implies m \cdot \left(\frac{1}{m} \cdot h - \frac{1}{m} * h \right) = 0.$$

According to the property of vector space, we have $m = 0$ or $\frac{1}{m} \cdot h - \frac{1}{m} * h = 0$. However, m is a positive integer, which forces $\frac{1}{m} \cdot h = \frac{1}{m} * h$. Thus we can deduce that for all $h \in G$, $n, m \in \mathbb{Z}_+$,

$$n \cdot \left(\frac{1}{m} \cdot h \right) = n * \left(\frac{1}{m} * h \right) \implies \frac{n}{m} \cdot h = \frac{n}{m} * h.$$

In other words, for all $h \in G$, $q \in \mathbb{Q}_+$, we have $q \cdot h = q * h$. Note that $(-q) \cdot h = (-q) * h$ and $0 \cdot h = 0 * h$, finally we obtain that for all $h \in G$, $q \in \mathbb{Q}$,

$$q \cdot h = q * h.$$

Therefore, the two scalar multiplication operations \cdot and $*$ coincide, which completes the proof. ■

5.7 Let K be a field, and let $k \subseteq K$ be a subfield of K . Show that K is a vector space over k (and in fact a k -algebra) in a natural way. In this situation, we say that K is an extension of k .

Define the scalar multiplication \cdot as

$$a \cdot x := ax, \quad \forall a \in k, x \in K.$$

Then we can check that for all $a, b \in k$, $x, y \in K$,

$$\begin{aligned} 1 \cdot x &= x, \\ (ab) \cdot x &= (ab)x = a(bx) = a \cdot (b \cdot x), \\ (a + b) \cdot x &= (a + b)x = ax + bx = a \cdot x + b \cdot x, \\ a \cdot (x + y) &= a(x + y) = ax + ay = a \cdot x + a \cdot y, \\ (a \cdot x)(b \cdot y) &= (ax)(by) = (ab)(xy) = (ab) \cdot (xy). \end{aligned}$$

Therefore, K is a k -vector space and a k -algebra as well. ■

5.8 What is the initial object of the category $R\text{-Alg}$?

The ring R can be seen as a R -algebra if it is endowed with a scalar multiplication \cdot in a natural way, that is

$$r \cdot x := rx, \quad \forall r \in R, x \in R.$$

Given any R -algebra A , define the following map

$$\begin{aligned} f : R &\longrightarrow A, \\ r &\longmapsto r \cdot 1_A. \end{aligned}$$

We can check that

$$\begin{aligned} f(r_1 + r_2) &= (r_1 + r_2) \cdot 1_A = r_1 \cdot 1_A + r_2 \cdot 1_A = f(r_1) + f(r_2), \\ f(r_1 r_2) &= (r_1 r_2) \cdot 1_A = r_1 \cdot (r_2 \cdot 1_A) = r_1 \cdot (1_A(r_2 \cdot 1_A)) = (r_1 \cdot 1_A)(r_2 \cdot 1_A) = f(r_1)f(r_2), \\ f(r_1 \cdot r_2) &= f(r_1 r_2) = r_1 \cdot (r_2 \cdot 1_A) = r_1 \cdot f(r_2). \end{aligned}$$

Hence f is a morphism in the category $R\text{-Alg}$.

Suppose $g : R \rightarrow A$ is a morphism in $R\text{-Alg}$. Then for all $r_1, r_2 \in R$,

$$g(r_1 r_2) = g(r_1 \cdot r_2) \implies g(r_1)g(r_2) = r_1 \cdot g(r_2) = r_1 \cdot (1_A g(r_2)) = (r_1 \cdot 1_A)g(r_2).$$

Take $r_2 = 1_R$ and then for all $r_1 \in R$,

$$g(r_1)g(1_R) = (r_1 \cdot 1_A)g(1_R) \implies g(r_1) = r_1 \cdot 1_A.$$

Thus we have $g = f$. Therefore, we show that for any R -algebra A , there exists a unique morphism $f : R \rightarrow A$ in $R\text{-Alg}$. In other words, R is the initial object of the category $R\text{-Alg}$. ■

§6. Products, coproducts, etc. in $R\text{-Mod}$

6.3 Let R be a ring, M an R -module, and $p : M \rightarrow M$ an R -module homomorphism such that $p^2 = p$. (Such a map is called a projection.) Prove that $M \cong \ker p \oplus \text{im } p$.

Since $x = p((p - id_M)x) \in M$, p must be an epimorphism and $M \cong \text{im } p \cong M/\ker p$. For all $x \in \ker p \cap \text{im } p$, we can assume $x = py$ and deduce that $0 = px = p^2y = py = x$. Thus we have $\ker p \cap \text{im } p = \{0\}$ and

$$\frac{\ker p \oplus \text{im } p}{\ker p} \cong \frac{\text{im } p}{\ker p \cap \text{im } p} \cong \text{im } p,$$

which implies

$$\frac{\ker p \oplus \text{im } p}{M} \cong \frac{\ker p \oplus \text{im } p / \ker p}{M / \ker p} \cong \frac{\text{im } p}{\text{im } p} \cong \{0\}.$$

Therefore we show that $M \cong \ker p \oplus \text{im } p$. ■

§7. Complexes and homology

7.1 Assume that the complex

$$\cdots \longrightarrow 0 \longrightarrow M \longrightarrow 0 \longrightarrow \cdots$$

is exact. Prove that $M \cong 0$. [§7.3]

Assume that $f : 0 \rightarrow M$ and $g : M \rightarrow 0$. Since the the complex is exact, we have

$$\{0\} = \text{im } f = \ker g = M.$$

■

Chapter V. Irreducibility and factorization in integral domains

Appendix

Lemma II.1 (von Dyck) Given a presentation $(A|\mathcal{R}) = F(A)/R$, where A is the set of generators, $\mathcal{R} \in F(A)$ is the set of relators and R is the smallest normal subgroup of $F(A)$ containing \mathcal{R} . Define inclusion mapping $i : A \rightarrow F(A)$ and projection $\pi : F(A) \rightarrow F(A)/R$. If f is a mapping from A to a group G , and every relations in \mathcal{R} holds in G via f , that is, $\mathcal{R} \subset \ker \varphi$ where φ is the unique homomorphism induced by the universal property of free group, then there exists a unique homomorphism $\psi : F(A)/R \rightarrow G$ such that $f = \psi \circ \pi \circ i$. If G is generated by $f(A)$, then ψ is surjective.

$$\begin{array}{ccc}
 & F(A)/R & \\
 \pi \uparrow & \searrow \exists! \psi & \\
 F(A) & \xrightarrow{\varphi} & G \\
 i \uparrow & \nearrow f & \\
 A & &
 \end{array}$$

Proof of the lemma. Since R is the smallest normal subgroup of $F(A)$ containing \mathcal{R} and the normal subgroup $\ker \varphi$ contains \mathcal{R} , we must have $R \subset \ker \varphi$. Then according to Theorem 7.12, there exists a unique homomorphism $\psi : F(A)/R \rightarrow G$ such that $\varphi = \psi \circ \pi$, which means the whole diagram commutes. If there exists a homomorphism $\zeta : F(A)/R \rightarrow G$ such that $f = \zeta \circ \pi \circ i$, then we have $\varphi \circ i = \zeta \circ \pi \circ i$, which implies $\varphi(t) = \zeta(\pi(t))$ for all $t \in A$. Note that a homomorphism defined on $F(A)$ can be specified only by its valuation on the set of generators A , we can assert that $\varphi = \zeta \circ \pi$. Since there exists a unique homomorphism $\psi : F(A)/R \rightarrow G$ such that $\varphi = \psi \circ \pi$, we have $\zeta = \psi$. Thus we show that there exists a unique homomorphism $\psi : F(A)/R \rightarrow G$ such that $f = \psi \circ \pi \circ i$.

Moreover, if G is generated by $f(A)$, then $\text{im} \psi = G$, since $f(A) = \psi(\pi(i(A))) \subset \text{im} \psi$ implies $G \subset \text{im} \psi$. \lrcorner

References