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# Algebra, Chapter 0
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**2.1** One can associate an $n \times n$ matrix $M_\sigma$ with a permutation $\sigma \in S_n$, by letting the entry at $(i, \sigma(i))$ be 1, and letting all other entries be 0. For example, the matrix corresponding to the permutation

$$\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \in S_3$$

would be

$$M_\sigma = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

Prove that, with this notation,

$$M_{\sigma\tau} = M_\sigma M_\tau$$

for all $\sigma, \tau \in S_n$, where the product on the right is the ordinary product of matrices.

With Kronecker delta function

$$\delta_{i,j} = \begin{cases} 0 & \text{if } i \neq j, \\ 1 & \text{if } i = j, \end{cases}$$

the entry at $(i, j)$ of the matrix $M_{\sigma\tau}$ can be written as

$$(M_{\sigma\tau})_{i,j} = \delta_{\tau(\sigma(i)),j}$$

and the entry at $(i, j)$ of the matrix $M_\sigma M_\tau$ can be written as

$$(M_\sigma M_\tau)_{i,j} = \sum_{k=1}^{n} (M_\sigma)_{i,k}(M_\tau)_{k,j} = \sum_{k=1}^{n} \delta_{\sigma(i),k} \cdot \delta_{\tau(k),j} = \sum_{k=1}^{n} \delta_{\sigma(i),k} \cdot \delta_{k,\tau^{-1}(j)} = \delta_{\sigma(i),\tau^{-1}(j)}.$$

Note that

$$\tau(\sigma(i)) = j \iff \sigma(i) = \tau^{-1}(j),$$

we see $M_{\sigma\tau} = M_\sigma M_\tau$ for all $\sigma, \tau \in S_n$.

∎

**2.2** Prove that if $d \leq n$, then $S_n$ contains elements of order $d$.

The cyclic permutation

$$\sigma = (1\ 2\ 3 \cdots d)$$

is an element of order $d$ in $S_n$. ■

**2.3** For every positive integer $n$ find an element of order $n$ in $S_{\mathbb{N}}$.

The cyclic permutation

$$\sigma = (1\ 2\ 3 \cdots n)$$

is an element of order $d$ in $S_n$. ■

**2.4** Define a homomorphism $D_8 \to S_4$ by labeling vertices of a square, as we did for a triangle in §2.2. List the 8 permutations in the image of this homomorphism.

The image of $n$ rotations under the homomorphism are

$$\sigma_1 = e_{D_8},\ \sigma_2 = (1\ 2\ 3\ 4),\ \sigma_3 = (1\ 3)(2\ 4),\ \sigma_4 = (1\ 4\ 3\ 2).$$

The image of $n$ reflections under the homomorphism are

$$\sigma_5 = (1\ 3),\ \sigma_6 = (2\ 4),\ \sigma_7 = (1\ 2)(3\ 4),\ \sigma_8 = (1\ 4)(3\ 2).$$

■

**3.1** Let $\varphi : G \to H$ be a morphism in a category $\mathsf{C}$ with products. Explain why there is a unique morphism

$$(\varphi \times \varphi) : G \times G \longrightarrow H \times H.$$

(This morphism is defined explicitly for $\mathsf{C} = \mathsf{Set}$ in §3.1.)

By the universal property of product in $\mathsf{C}$, there exist a unique morphism $(\varphi \times \varphi) : G \times G \longrightarrow H \times H$ such that the following diagram commutes.

$$\begin{tikzcd}
G \arrow[r, "\varphi"] & H \\
G \times G \arrow[u, "\pi_G"] \arrow[d, "\pi_G"'] \arrow[r, "\varphi\times\varphi"] & H \times H \arrow[u, "\pi_H"'] \arrow[d, "\pi_H"] \\
G \arrow[r, "\varphi"] & H
\end{tikzcd}$$

■

**3.2** Let $\varphi : G \to H, \psi : H \to K$ be morphisms in a category with products, and consider morphisms between the products $G \times G, H \times H, K \times K$ as in Exercise 3.1. Prove that

$$(\psi\varphi) \times (\psi\varphi) = (\psi \times \psi)(\varphi \times \varphi).$$

(This is part of the commutativity of the diagram displayed in §3.2.)

By the universal property of product in $\mathsf{C}$, there exist a unique morphism

$$(\psi\varphi) \times (\psi\varphi) : G \times G \to K \times K$$

such that the following diagram commutes.

$$\begin{tikzcd}
G \arrow[r, "\psi\varphi"] & H \\
G \times G \arrow[u, "\pi_G"] \arrow[d, "\pi_G"'] \arrow[r, "(\psi\varphi)\times(\psi\varphi)"] & H \times H \arrow[u, "\pi_H"'] \arrow[d, "\pi_H"] \\
G \arrow[r, "\psi\varphi"] & H
\end{tikzcd}$$

As the following commuting diagram tells us the composition

$$(\psi \times \psi)(\varphi \times \varphi) : G \times G \to K \times K$$

can make the above diagram commute,

$$\begin{tikzcd}
G \arrow[r, "\varphi"] \arrow[rr, bend left, "\psi\varphi"] & H \arrow[r, "\psi"] & K \\
G \times G \arrow[u, "\pi_G"] \arrow[d, "\pi_G"'] \arrow[r, "\varphi\times\varphi"] & H \times H \arrow[u, "\pi_H"] \arrow[d, "\pi_H"'] \arrow[r, "\psi\times\psi"] & K \times K \arrow[u, "\pi_K"] \arrow[d, "\pi_K"'] \\
G \arrow[r, "\varphi"] \arrow[rr, bend right, "\psi\varphi"'] & H \arrow[r, "\psi"] & K
\end{tikzcd}$$

there must be $(\psi\varphi) \times (\psi\varphi) = (\psi \times \psi)(\varphi \times \varphi)$.

■

**3.3** Show that if $G, H$ are abelian groups, then $G \times H$ satisfies the universal property for coproducts in $\mathsf{Ab}$.

Define two monomorphisms:

$$i_G : G \longrightarrow G \times H,\ a \longmapsto (a, 0_H)$$

$$i_H : H \longrightarrow G \times H,\ b \longmapsto (0_G, b)$$

We are proving that for any two homomorphisms $g : G \to M$ and $h : H \to M$ in $\mathsf{Ab}$, the map

$$\begin{aligned} \varphi : \quad & G \times H \longrightarrow M, \\ & (a, b) \longmapsto g(a) + h(b) \end{aligned}$$

is a homomorphism and makes the following diagram commute.

$$\begin{tikzcd} G \arrow[d, "i_G"'] \arrow[dr, "g"] & \\ G \times H \arrow[r, "\varphi"] & M \\ H \arrow[u, "i_H"] \arrow[ur, "h"'] & \end{tikzcd}$$

Exploiting the fact that $g, h$ are homomorphisms and $M$ is an abelian group, it is easy to check that $\varphi$ preserves the addition operation

$$\begin{aligned} \varphi((a_1, b_1) + (a_2, b_2)) &= \varphi((a_1 + a_2, b_1 + b_2)) \\ &= g(a_1 + a_2) + h(b_1 + b_2) \\ &= (g(a_1) + g(a_2)) + (h(b_1) + h(b_2)) \\ &= (g(a_1) + h(b_1)) + (g(a_2) + h(b_2)) \\ &= g(a_1 + b_1) + h(a_2 + b_2) \\ &= \varphi((a_1, b_1)) + \varphi((a_2, b_2)) \end{aligned}$$

and the diagram commutes

$$\varphi \circ i_G(a) = \varphi((a, 0_H)) = g(a) + h(0_H) = g(a) + 0_M = g(a),$$

$$\varphi \circ i_H(b) = \varphi((0_G, b)) = g(0_G) + h(b) = 0_M + h(b) = h(b).$$

To show the uniqueness of the homomorphism $\varphi$ we have constructed, suppose a homomorphism $\varphi'$ can make the diagram commute. Then we have

$$\varphi'((a, b)) = \varphi'((a, 0_H) + (0_G, b)) = \varphi'(i_G(a)) + \varphi'(i_H(b)) = g(a) + h(b) = \varphi((a, b)),$$

that is $\varphi' = \varphi$. Hence we show that there exist a unique homomorphism $\varphi$ such that the diagram commutes, which amounts to the universal property for coproducts in $\mathsf{Ab}$.

$\blacksquare$

**3.3** Prove that $\mathbb{Q}$ is not the direct product of two nontrivial groups.

$$\begin{tikzcd} G \arrow[d, "i_G"'] \arrow[dr, "g"] & \\ G \times H \arrow[r, "\varphi"] & \mathbb{Q} \\ H \arrow[u, "i_H"] \arrow[ur, "h"'] & \end{tikzcd}$$

Consider the additive group of rationals $(\mathbb{Q}, +)$. Assume the product $G \times H = \{(a,b)|a \in G, b \in H\}$ is isomorphic to $(\mathbb{Q}, +)$. Note that $\{e_G\} \times H$ and $G \times \{e_H\}$ are subgroups in $G \times H$ and there intersection is trivial group $\{e_G\} \times \{e_H\}$. The commutative diagram implies

$$\varphi(\{e_G\} \times H) = \varphi(i_H(H)) = h(H),$$

$$\varphi(G \times \{e_H\}) = \varphi(i_G(G)) = g(G).$$

It is easy to check bijection $\varphi$ satisfies $\varphi(A \cap B) = \varphi(A) \cap \varphi(B)$. Hence we have

$$\varphi(\{(e_G, e_H)\}) = \varphi(\{e_G\} \times H \cap G \times \{e_H\}) = \varphi(\{e_G\} \times H) \cap \varphi(G \times \{e_H\}) = h(H) \cap g(G) = \{0\}.$$

Suppose both $g(G)$ and $h(H)$ are nontrivial groups. If $\frac{p}{q} \in h(H) - \{0\}$ and $\frac{r}{s} \in g(G) - \{0\}$, there must be

$$rp = rq \cdot \frac{p}{q} = ps \cdot \frac{r}{s} \in h(H) \cap g(G).$$

Since $rp \neq 0$, it leads to a contradiction. Thus we can assume $g(G)$ is a trivial group. According to the dual commutative diagram,

$$\begin{tikzcd} G \arrow[d, "i_G"'] \arrow[dr, "g"] & \\ G \times H & \mathbb{Q} \arrow[l, "\varphi^{-1}"'] \\ H \arrow[u, "i_H"] \arrow[ur, "h"'] & \end{tikzcd}$$

we see that for all $a \in G$,

$$(a, e_H) = i_(a) = \varphi^{-1}(g(a)) = \varphi(0) = (e_G, e_H) \implies a = e_G.$$

that is, $G$ is a trivial group. Therefore, we have shown $(\mathbb{Q}, +)$ will never be isomorphic to the direct product of two nontrivial groups. ∎

Assume
$$\varphi(a_1, b_1) = g(a_1) + h(b_1) = 1,$$
By induction we can show for all $p \in \mathbb{N}$.
$$\varphi(a_1^p, b_1^p) = pg(a_1) + ph(b_1) = p,$$
For all $q \in \mathbb{N} - \{0\}$, there exist unique $(c_q, d_q) \in G \times H$ such that
$$\varphi(c_q, d_q) = g(c_q) + h(d_q) = \frac{1}{q},$$
namely
$$\varphi(c_q^q, d_q^q) = q\varphi(c_q, d_q) = 1 = \varphi(a_1, b_1) \implies (c_q^q, d_q^q) = (a_1, b_1).$$
Denote $c_q = a_1^{\frac{1}{q}},\ d_q = b_1^{\frac{1}{q}}$.
$$\varphi([(a_1^{\frac{1}{q}})^p]^q, [(b_1^{\frac{1}{q}})^p]^q) = \varphi((a_1^{\frac{1}{q}})^{pq}, (b_1^{\frac{1}{q}})^{pq}) = pq\varphi((a_1^{\frac{1}{q}}, b_1^{\frac{1}{q}})) = pq\frac{1}{q} = \varphi(a_1^p, b_1^p)$$
implies
$$[(a_1^{\frac{1}{q}})^p]^q = a_1^p,\ [(b_1^{\frac{1}{q}})^p]^q = b_1^p$$
Denote $(a_1^{\frac{1}{q}})^p = (a_1^p)^{\frac{1}{q}} = a_1^{\frac{p}{q}},\ (b_1^{\frac{1}{q}})^p = (b_1^p)^{\frac{1}{q}} = b_1^{\frac{p}{q}}$. Then
$$g(a_1^{\frac{p}{q}}) = pg(a_1^{\frac{1}{q}}) = \frac{p}{q}g(a_1)$$
$$h(b_1^{\frac{p}{q}}) = ph(b_1^{\frac{1}{q}}) = \frac{p}{q}h(b_1).$$
For all $p \in \mathbb{N}$, if $h(b_1) \neq 0$,
$$\begin{aligned} p &= \varphi\left(a_1^p, b_1^p\right) \\ &= (p+1)g(a_1) + \left(p - \frac{g(a_1)}{h(b_1)}\right)h(b_1) \\ &= g(a_1^{p+1}) + h\left(b_1^{p - \frac{g(a_1)}{h(b_1)}}\right) \\ &= \varphi\left(a_1^{p+1}, b_1^{p - \frac{g(a_1)}{h(b_1)}}\right) \end{aligned}$$
Therefore, $a_1^p = a_1^{p+1} \implies a_1 = e_G$

Hence for all $\frac{p}{q} \in \mathbb{Q}$, it holds that
$$\frac{p}{q} = p\varphi(a_1^{\frac{1}{q}}, b_1^{\frac{1}{q}}) = \varphi(a_1^{\frac{p}{q}}, b_1^{\frac{p}{q}}) = \frac{p}{q}g(a_1) + \frac{p}{q}h(b_1).$$

Suppose

$$\varphi(a_1, e_H) = \frac{r}{s} = \varphi(c_s^r, d_s^r),$$

which indicates

$$(a_1, e_H) = (c_s^r, d_s^r) \implies d_s^r = e_H.$$

Likewise, we suppose

$$\varphi(e_G, d_q^p) = \frac{m}{n} = \varphi(c_n^m, d_n^m),$$

and get $c_n^m = e_G$.

$$\varphi(c_q^p, d_q^p) = \varphi(c_q^p, e_H) + \varphi(e_G, d_q^p) = rg(c_s) + mh(d_n)$$