Huyi Chen

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Algebra, Chapter 0
By Paolo Aluffi

Contents

Chapter I. Preliminaries: Set theory and categories

§3. Categories

3.1 Let C be a category. Consider a structure C^{op} with:

- $\mathrm{Obj}(\mathsf{C}^{op}) := \mathrm{Obj}(\mathsf{C})$;
- for A, B objects of C^{op} (hence, objects of C), $\mathrm{Hom}_{\mathsf{C}^{op}}(A,B) := \mathrm{Hom}_{\mathsf{C}}(B,A)$

Show how to make this into a category (that is, define composition of morphisms in C^{op} and verify the properties listed in §3.1). Intuitively, the 'opposite' category C^{op} is simply obtained by 'reversing all the arrows' in C. [5.1, §VIII.1.1, §IX.1.2, IX.1.10]

- For every object A of C, there exists one identity morphism $1_A \in \mathrm{Hom}_{\mathsf{C}}(A,A)$. Since $\mathrm{Obj}(\mathsf{C}^{op}) := \mathrm{Obj}(\mathsf{C})$ and $\mathrm{Hom}_{\mathsf{C}^{op}}(A,A) := \mathrm{Hom}_{\mathsf{C}}(A,A)$, for every object A of C^{op}, the identity on A coincides with $1_A \in \mathsf{C}$.
- For A, B, C objects of C^{op} and $f \in \mathrm{Hom}_{\mathsf{C}^{op}}(A,B) = \mathrm{Hom}_{\mathsf{C}}(B,A)$, $g \in \mathrm{Hom}_{\mathsf{C}^{op}}(B,C) = \mathrm{Hom}_{\mathsf{C}}(C,B)$, the composition laws in C determines a morphism $f * g$ in $\mathrm{Hom}_{\mathsf{C}}(C,A)$, which deduces the composition defined on C^{op}:

$$\begin{aligned}\mathrm{Hom}_{\mathsf{C}^{op}}(A,B) \times \mathrm{Hom}_{\mathsf{C}^{op}}(B,C) &\longrightarrow \mathrm{Hom}_{\mathsf{C}^{op}}(A,C)\\ (f,g) &\longmapsto g \circ f := f * g\end{aligned}$$

- Associativity. If $f \in \mathrm{Hom}_{\mathsf{C}^{op}}(A,B)$, $g \in \mathrm{Hom}_{\mathsf{C}^{op}}(B,C)$, $h \in \mathrm{Hom}_{\mathsf{C}^{op}}(C,D)$, then

$$f \circ (g \circ h) = f \circ (h * g) = (h * g) * f = h * (g * f) = (g * f) \circ h = (f \circ g) \circ h.$$

- Identity. For all $f \in \mathrm{Hom}_{\mathsf{C}^{op}}(A,B)$, we have

$$f \circ 1_A = 1_A * f = f, \quad 1_B \circ f = f * 1_B = f.$$

Thus we get the full construction of C^{op}. ∎

§4. Morphisms

4.2 In Example 3.3 we have seen how to construct a category from a set endowed with a relation, provided this latter is reflexive and transitive. For what types of relations is the corresponding category a groupoid (cf. Example 4.6)? [§4.1]

For a reflexive and transitive relation $\sim$ on a set S, define the category C as follows:

- Objects: $\mathrm{Obj}(\mathsf{C}) = S$;
- Morphisms: if a, b are objects (that is: if $a, b \in S$) then let

$$\mathrm{Hom}_{\mathsf{C}}(a,b) = \begin{cases} (a,b) \in S \times S & \text{if } a \sim b \\ \emptyset & \text{otherwise} \end{cases}$$

In Example 3.3 we have shown the category. If the relation $\sim$ is endowed with symmetry, we have

$$(a,b) \in \mathrm{Hom}_{\mathsf{C}}(a,b) \implies a \sim b \implies b \sim a \implies (b,a) \in \mathrm{Hom}_{\mathsf{C}}(b,a).$$

Since

$$(a,b)(b,a) = (a,a) = 1_a, \quad (b,a)(a,b) = (b,b) = 1_b,$$

in fact (a,b) is an isomorphism. From the arbitrariness of the choice of (a,b), we show that C is a groupoid. Conversely, if C is a groupoid, we can show the relation $\sim$ is symmetric. To sum up, the category C is a groupoid if and only if the corresponding relation $\sim$ is an equivalence relation. ∎

§5. Universal properties

5.1 Prove that a final object in a category C is initial in the opposite category C_{op} (cf. Exercise 3.1).

An object F of C is final in C if and only if

$$\forall A \in \mathrm{Obj}(\mathsf{C}) : \mathrm{Hom}_{\mathsf{C}}(A, F) \text{ is a singleton.}$$

That is equivalent to

$$\forall A \in \mathrm{Obj}(\mathsf{C}_{op}) : \mathrm{Hom}_{\mathsf{C}_{op}}(F, A) \text{ is a singleton,}$$

which means F is initial in the opposite category C_{op}. ∎

Chapter II. Groups, first encounter

§1. Definition of group

1.1 Write a careful proof that every group is the group of isomorphisms of a groupoid. In particular, every group is the group of automorphisms of some object in some category.

Assume G is a group. Define a category C as follows:

- Objects: $\mathrm{Obj}(\mathsf{C}) = \{*\}$;
- Morphisms: $\mathrm{Hom}_{\mathsf{C}}(*, *) = \mathrm{End}_{\mathsf{C}}(*) = G$.

The composition of homomorphism is corresponding to the multiplication between two elements in G. The identity morphism on $*$ is $1_* = e_G$, which satisfies for all $g \in \mathrm{Hom}_{\mathsf{C}}(*, *)$,

$$ge_G = e_G g = g,$$

and

$$gg^{-1} = e_G, \ g^{-1}g = e_G.$$

Thus any homomorphism $g \in \mathrm{Hom}_{\mathsf{C}}(*, *)$ is an isomorphism and accordingly C is a groupoid. ∎

1.4 Suppose that $g^2 = e$ for all elements g of a group G; prove that G is commutative.

For all $a, b \in G$,

$$abab = e \implies a(abab)b = ab \implies (aa)ba(bb) = ab \implies ba = ab.$$

∎

§2. Examples of groups

2.1 One can associate an $n \times n$ matrix M_σ with a permutation $\sigma \in S_n$, by letting the entry at $(i, \sigma(i))$ be 1, and letting all other entries be 0. For example, the matrix corresponding to the permutation

$$\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \in S_3$$

would be

$$M_\sigma = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

Prove that, with this notation,

$$M_{\sigma\tau} = M_\sigma M_\tau$$

for all $\sigma, \tau \in S_n$, where the product on the right is the ordinary product of matrices.

By introducing the Kronecker delta function

$$\delta_{i,j} = \begin{cases} 0 & \text{if } i \neq j, \\ 1 & \text{if } i = j, \end{cases}$$

the entry at (i, j) of the matrix $M_{\sigma\tau}$ can be written as

$$(M_{\sigma\tau})_{i,j} = \delta_{\tau(\sigma(i)),j}$$

and the entry at (i, j) of the matrix $M_\sigma M_\tau$ can be written as

$$(M_\sigma M_\tau)_{i,j} = \sum_{k=1}^{n} (M_\sigma)_{i,k} (M_\tau)_{k,j} = \sum_{k=1}^{n} \delta_{\sigma(i),k} \cdot \delta_{\tau(k),j} = \sum_{k=1}^{n} \delta_{\sigma(i),k} \cdot \delta_{k,\tau^{-1}(j)} = \delta_{\sigma(i),\tau^{-1}(j)},$$

where the last but one equality holds by the fact

$$\tau(k) = j \iff k = \tau^{-1}(j).$$

Noticing that

$$\tau(\sigma(i)) = j \iff \sigma(i) = \tau^{-1}(j),$$

we see $M_{\sigma\tau} = M_\sigma M_\tau$ for all $\sigma, \tau \in S_n$. ∎

2.2 Prove that if $d \leq n$, then S_n contains elements of order d.

The cyclic permutation

$$\sigma = (1\ 2\ 3 \cdots d)$$

is an element of order d in S_n. ∎

2.3 For every positive integer n find an element of order n in $S_{\mathbb{N}}$.

The cyclic permutation

$$\sigma = (1\ 2\ 3 \cdots n)$$

is an element of order d in S_n. ∎

2.4 Define a homomorphism $D_8 \to S_4$ by labeling vertices of a square, as we did for a triangle in §2.2. List the 8 permutations in the image of this homomorphism.

The image of n rotations under the homomorphism are

$$\sigma_1 = e_{D_8},\ \sigma_2 = (1\ 2\ 3\ 4),\ \sigma_3 = (1\ 3)(2\ 4),\ \sigma_4 = (1\ 4\ 3\ 2).$$

The image of n reflections under the homomorphism are

$$\sigma_5 = (1\ 3),\ \sigma_6 = (2\ 4),\ \sigma_7 = (1\ 2)(3\ 4),\ \sigma_8 = (1\ 4)(3\ 2).$$

∎

2.11 Prove that the square of every odd integer is congruent to 1 modulo 8.

Given an odd integer $2k+1$, we have

$$(2k+1)^2 = 4k(k+1) + 1,$$

where $k(k+1)$ is an even integer. So $(2k+1)^2 \equiv 1 \mod 8$. ∎

2.12 Prove that there are no integers a, b, c such that $a^2 + b^2 = 3c^2$. (Hint: studying the equation $[a]_4^2 + [b]_4^2 = 3[c]_4^2$ in $\mathbb{Z}/4\mathbb{Z}$, show that a, b, c would all have to be even. Letting $a = 2k, b = 2l, c = 2m$, you would have $k^2 + l^2 = 3m^2$. What's wrong with that?)

$$a^2 + b^2 = 3c^2 \implies [a]_4^2 + [b]_4^2 = 3[c]_4^2.$$

Noting that $[0]_4^2 = [0]_4, [1]_4^2 = [1]_4, [2]_4^2 = [0]_4, [3]_4^2 = [1]_4$, we see $[c]_4^2$ must be $[0]_4$ and so do $[a]_4^2$ and $[b]_4^2$. Hence $[a]_4, [b]_4, [b]_4$ can only be $[0]_4$ or $[2]_4$, which justifies letting $a = 2k_1, b = 2l_2, c = 2m_1$. After substitution we have $k^2 + l^2 = 3m^2$. Repeating this process n times yields $a = 2^n k_n, b = 2^n l_n, c = 2^n m_n$. For a sufficiently large number N, the absolute value of k_N, l_N, m_N must be less than 1. Thus we conclude that $a = b = c = 0$ is the unique solution to the equation $a^2 + b^2 = 3c^2$. ∎

2.13 Prove that if $\gcd(m, n) = 1$, then there exist integers a and b such that $am + bn = 1$. (Use Corollary 2.5.) Conversely, prove that if $am + bn = 1$ for some integers a and b, then $\gcd(m, n) = 1$. [2.15, §V.2.1, V.2.4]

Applying corollary 2.5, we have $\gcd(m, n) = 1$ if and only if $[m]_n$ generates $\mathbb{Z}/n\mathbb{Z}$. Hence

$$\gcd(m, n) = 1 \iff a[m]_n = [1]_n \iff [am]_n = [1]_n \iff am + bn = 1.$$

∎

2.15 Let $n > 0$ be an odd integer.

- Prove that if $\gcd(m, n) = 1$, then $\gcd(2m + n, 2n) = 1$. (Use Exercise 2.13.)
- Prove that if $\gcd(r, 2n) = 1$, then $\gcd(\frac{r+n}{2}, n) = 1$. (Ditto.)
- Conclude that the function $[m]_n \to [2m + n]_{2n}$ is a bijection between $(\mathbb{Z}/n\mathbb{Z})^*$ and $(\mathbb{Z}/2n\mathbb{Z})^*$.

The number $\phi(n)$ of elements of $(\mathbb{Z}/n\mathbb{Z})^*$ is Eulers $\phi(n)$-function. The reader has just proved that if n is odd, then $\phi(2n) = \phi(n)$. Much more general formulas will be given later on (cf. Exercise V.6.8). [VII.5.11]

- According to Exercise 2.13,

  $$\gcd(m, n) = 1 \implies am + bn = 1 \implies \frac{a}{2}(2m + n) + \left(b - \frac{a}{2}\right) n = 1.$$

  If a is even, we have shown $\gcd(2m + n, 2n) = 1$. Otherwise we can let $a' = a + n$ be an even integer and $b' = b - m$. Then it holds that

  $$\frac{a'}{2}(2m + n) + \left(b' - \frac{a'}{2}\right) n = 1,$$

which also indicates $\gcd(2m+n, 2n) = 1$.

- If $\gcd(r, 2n) = 1$, then r must be an odd integer and accordingly

$$\gcd(2r+2n, 4n) = 1 \implies a(2r+2n) + b(4n) = 1 \implies 4a\frac{r+n}{2} + 4bn = 1,$$

which is $\gcd(\frac{r+n}{2}, n) = 1$.

- It is easy to check that the function $f : (\mathbb{Z}/n\mathbb{Z})^* \to (\mathbb{Z}/2n\mathbb{Z})^*$, $[m]_n \mapsto [2m+n]_{2n}$ is well-defined. The fact

$$\begin{aligned} f([m_1]_n) = f([m_2]_n) &\implies f([2m_1+n]_{2n}) = f([2m_2+n]_{2n}) \\ &\implies (2m_1+n) - (2m_2+n) = 2kn \\ &\implies m_1 - m_2 = kn \\ &\implies [m_1]_n = [m_2]_n \end{aligned}$$

indicates that f is injective. For any $[r]_{2n} \in (\mathbb{Z}/2n\mathbb{Z})^*$, we have

$$\gcd(r, 2n) = 1 \implies \gcd\left(\frac{r+n}{2}, n\right) = 1 \implies \left[\frac{r+n}{2}\right]_n \in (\mathbb{Z}/n\mathbb{Z})^*,$$

and

$$f\left(\left[\frac{r+n}{2}\right]_n\right) = [r+2n]_{2n} = [r]_{2n},$$

which indicates that f is surjective. Thus we show f is a bijection.

■

2.16 Find the last digit of $1238237^{18238456}$. (Work in $\mathbb{Z}/10\mathbb{Z}$.)

$$1238237^{18238456} \equiv 7^{18238456} \equiv (7^4)^{4559614} \equiv 2401^{4559614} \equiv 1 \mod 10,$$

which indicates that the last digit of $1238237^{18238456}$ is 1. ■

2.17 Show that if $m \equiv m' \mod n$, then $\gcd(m, n) = 1$ if and only if $\gcd(m', n) = 1$. [§2.3]

Assume that $m - m' = kn$. If $\gcd(m, n) = 1$, for any common divisor d of m' and n

$$d|m',\ d|n \implies d|(m'+kn) \implies d|m \implies d = 1,$$

which means $\gcd(m', n) = 1$. Likewise, we can show $\gcd(m', n) = 1 \implies \gcd(m, n) = 1$ ■

§3. The category Grp

3.1 Let $\varphi : G \to H$ be a morphism in a category C with products. Explain why there is a unique morphism

$$(\varphi \times \varphi) : G \times G \longrightarrow H \times H.$$

(This morphism is defined explicitly for $\mathsf{C} = \mathsf{Set}$ in §3.1.)

By the universal property of product in C, there exist a unique morphism $(\varphi \times \varphi) : G \times G \longrightarrow H \times H$ such that the following diagram commutes.

$$\begin{tikzcd}
G \arrow[r, "\varphi"] & H \\
G \times G \arrow[u, "\pi_G"] \arrow[r, "\varphi\times\varphi"] \arrow[d, "\pi_G"'] & H \times H \arrow[u, "\pi_H"'] \arrow[d, "\pi_H"] \\
G \arrow[r, "\varphi"] & H
\end{tikzcd}$$

■

3.2 Let $\varphi : G \to H, \psi : H \to K$ be morphisms in a category with products, and consider morphisms between the products $G \times G, H \times H, K \times K$ as in Exercise 3.1. Prove that

$$(\psi\varphi) \times (\psi\varphi) = (\psi \times \psi)(\varphi \times \varphi).$$

(This is part of the commutativity of the diagram displayed in §3.2.)

By the universal property of product in C, there exist a unique morphism

$$(\psi\varphi) \times (\psi\varphi) : G \times G \to K \times K$$

such that the following diagram commutes.

$$\begin{tikzcd}
G \arrow[r, "\psi\varphi"] & H \\
G \times G \arrow[u, "\pi_G"] \arrow[r, "(\psi\varphi)\times(\psi\varphi)"] \arrow[d, "\pi_G"'] & H \times H \arrow[u, "\pi_H"'] \arrow[d, "\pi_H"] \\
G \arrow[r, "\psi\varphi"] & H
\end{tikzcd}$$

As the following commuting diagram tells us the composition

$$(\psi \times \psi)(\varphi \times \varphi) : G \times G \to K \times K$$

can make the above diagram commute,

$$\begin{tikzcd}
G \arrow[r, "\varphi"] \arrow[rr, bend left, "\psi\varphi"] & H \arrow[r, "\psi"] & K \\
G \times G \arrow[u, "\pi_G"] \arrow[d, "\pi_G"'] \arrow[r, "\varphi\times\varphi"] & H \times H \arrow[u, "\pi_H"] \arrow[d, "\pi_H"'] \arrow[r, "\psi\times\psi"] & K \times K \arrow[u, "\pi_K"] \arrow[d, "\pi_K"'] \\
G \arrow[r, "\varphi"] \arrow[rr, bend right, "\psi\varphi"'] & H \arrow[r, "\psi"] & K
\end{tikzcd}$$

there must be $(\psi\varphi) \times (\psi\varphi) = (\psi \times \psi)(\varphi \times \varphi)$.

■

3.3 Show that if G, H are abelian groups, then $G \times H$ satisfies the universal property for coproducts in Ab.

Define two monomorphisms:

$$i_G : G \longrightarrow G \times H,\ a \longmapsto (a, 0_H)$$

$$i_H : H \longrightarrow G \times H,\ b \longmapsto (0_G, b)$$

We are proving that for any two homomorphisms $g : G \to M$ and $h : H \to M$ in Ab, the map

$$\begin{aligned} \varphi : \quad & G \times H \longrightarrow M, \\ & (a, b) \longmapsto g(a) + h(b) \end{aligned}$$

is a homomorphism and makes the following diagram commute.

$$\begin{tikzcd}
G \arrow[d, "i_G"'] \arrow[dr, "g"] & \\
G \times H \arrow[r, "\varphi"] & M \\
H \arrow[u, "i_H"] \arrow[ur, "h"'] &
\end{tikzcd}$$

Exploiting the fact that g, h are homomorphisms and M is an abelian group, it is easy to check that φ preserves the addition operation

$$\begin{aligned}
\varphi((a_1, b_1) + (a_2, b_2)) &= \varphi((a_1 + a_2, b_1 + b_2)) \\
&= g(a_1 + a_2) + h(b_1 + b_2) \\
&= (g(a_1) + g(a_2)) + (h(b_1) + h(b_2)) \\
&= (g(a_1) + h(b_1)) + (g(a_2) + h(b_2)) \\
&= g(a_1 + b_1) + h(a_2 + b_2) \\
&= \varphi((a_1, b_1)) + \varphi((a_2, b_2))
\end{aligned}$$

and the diagram commutes

$$\varphi \circ i_G(a) = \varphi((a, 0_H)) = g(a) + h(0_H) = g(a) + 0_M = g(a),$$

$$\varphi \circ i_H(b) = \varphi((0_G, b)) = g(0_G) + h(b) = 0_M + h(b) = h(b).$$

To show the uniqueness of the homomorphism φ we have constructed, suppose a homomorphism φ' can make the diagram commute. Then we have

$$\varphi'((a, b)) = \varphi'((a, 0_H) + (0_G, b)) = \varphi'(i_G(a)) + \varphi'(i_H(b)) = g(a) + h(b) = \varphi((a, b)),$$

that is $\varphi' = \varphi$. Hence we show that there exist a unique homomorphism φ such that the diagram commutes, which amounts to the universal property for coproducts in Ab.

■

3.4 Let G, H be groups, and assume that $G \cong H \times G$. Can you conclude that H is trivial? (Hint: No. Can you construct a counterexample?)

Consider the function

$$\begin{aligned}
\varphi : \mathbb{Z} \times \mathbb{Z}[x] &\longrightarrow \mathbb{Z}[x] \\
(n, f(x)) &\longmapsto n + x f(x)
\end{aligned}$$

Firstly, we can show φ is a homomorphism as follows

$$\begin{aligned}
\varphi((n_1, f_1(x)) + (n_2, f_2(x))) &= \varphi((n_1 + n_2, f_1(x) + f_2(x))) \\
&= (n_1 + n_2) + x(f_1(x) + f_2(x)) \\
&= (n_1 + x f_1(x)) + (n_2 + x f_2(x)) \\
&= \varphi((n_1, f_1(x))) + \varphi((n_2, f_2(x))).
\end{aligned}$$

Secondly, we are to show φ is a monomorphism. It follows by

$$\varphi((n, f(x))) = n + x f(x) = 0 \implies n = 0,\ f(x) = 0 \implies \ker \varphi = \{(0, 0)\}.$$

Lastly, since the cardinal numbers of both $\mathbb{Z} \times \mathbb{Z}[x]$ and $\mathbb{Z}[x]$ are $\aleph_0$, φ is indeed an isomorphism. Therefore, as a counterexample we have $\mathbb{Z}[x] \cong \mathbb{Z} \times \mathbb{Z}[x]$. ∎

3.5 Prove that $\mathbb{Q}$ is not the direct product of two nontrivial groups.

Consider the additive group of rationals $(\mathbb{Q}, +)$. Assume that φ is a isomorphism between the product $G \times H = \{(a, b) | a \in G, b \in H\}$ and $(\mathbb{Q}, +)$. Note that $\{e_G\} \times H$ and $G \times \{e_H\}$ are subgroups in $G \times H$ and their intersection is the trivial group $\{(e_G, e_H)\}$. It is easy to check that bijection φ satisfies $\varphi(A \cap B) = \varphi(A) \cap \varphi(B)$. So applying the fact we have

$$\varphi(\{(e_G, e_H)\}) = \varphi(\{e_G\} \times H \cap G \times \{e_H\}) = \varphi(\{e_G\} \times H) \cap \varphi(G \times \{e_H\}) = \{0\}.$$

Suppose both $\varphi(\{e_G\} \times H)$ and $\varphi(G \times \{e_H\})$ are nontrivial groups. If $\dfrac{p}{q} \in \varphi(\{e_G\} \times H) - \{0\}$ and $\dfrac{r}{s} \in \varphi(G \times \{e_H\}) - \{0\}$, there must be

$$rp = rq \cdot \frac{p}{q} = ps \cdot \frac{r}{s} \in \varphi(\{e_G\} \times H) \cap \varphi(G \times \{e_H\}),$$

which implies $rp = 0$. Since both $\dfrac{p}{q}$ and $\dfrac{r}{s}$ are non-zero, it leads to a contradiction. Thus without loss of generality we can assume $\varphi(\{e_G\} \times H)$ is a trivial group $\{0\}$. Since φ is isomorphism, we see that for all $h \in H$,

$$\varphi(e_G, h) = \varphi(e_G, e_H) = 0 \iff h = e_H.$$

That is, H is a trivial group. Therefore, we have shown $(\mathbb{Q}, +)$ will never be isomorphic to the direct product of two nontrivial groups. ∎

3.6 Consider the product of the cyclic groups C_2, C_3 (cf. §2.3): $C_2 \times C_3$. By Exercise 3.3, this group is a coproduct of C_2 and C_3 in Ab. Show that it is not a coproduct of C_2 and C_3 in Grp, as follows:

- find injective homomorphisms $C_2 \to S_3$, $C_3 \to S_3$;
- arguing by contradiction, assume that $C_2 \times C_3$ is a coproduct of C_2, C_3, and deduce that there would be a group homomorphism $C_2 \times C_3 \to S_3$ with certain properties;
- show that there is no such homomorphism.

- Monomorphisms $g : C_2 \to S_3$, $h : C_3 \to S_3$ can be constructed as follows:

$$g([0]_2) = e, g([1]_2) = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}.$$

$$h([0]_3)=e, h([1]_3)=\begin{pmatrix}1&2&3\\3&1&2\end{pmatrix}, h([2]_3)=\begin{pmatrix}1&2&3\\2&3&1\end{pmatrix}.$$

- Supposing that $C_2 \times C_3$ is a coproduct of C_2, C_3, there would be a unique group homomorphism $\varphi : C_2 \times C_3 \to S_3$ such that the following diagram commutes

$$\begin{tikzcd} C_2 \arrow[d, "i_{C_2}"'] \arrow[dr, "g"] & \\ C_2 \times C_3 \arrow[r, "\varphi"] & S_3 \\ C_3 \arrow[u, "i_{C_3}"] \arrow[ur, "h"'] & \end{tikzcd}$$

In other words, for all $a \in C_2, b \in C_3$,

$$\begin{aligned}\varphi(a,b) &= \varphi(([0]_2,b)+(a,[0]_3)) = \varphi(([0]_2,b))\varphi((a,[0]_3)) = \varphi(i_{C_3}(b))\varphi(i_{C_2}(a)) = h(b)g(a)\\ &= \varphi((a,[0]_3)+([0]_2,b)) = \varphi((a,[0]_3))\varphi(([0]_2,b)) = \varphi(i_{C_2}(a))\varphi(i_{C_3}(b)) = g(a)h(b).\end{aligned}$$

- Since

$$g([1]_2)h([1]_3)=\begin{pmatrix}1&2&3\\1&3&2\end{pmatrix}\begin{pmatrix}1&2&3\\3&1&2\end{pmatrix}=\begin{pmatrix}1&2&3\\3&2&1\end{pmatrix},$$
$$h([1]_3)g([1]_2)=\begin{pmatrix}1&2&3\\3&1&2\end{pmatrix}\begin{pmatrix}1&2&3\\1&3&2\end{pmatrix}=\begin{pmatrix}1&2&3\\2&1&3\end{pmatrix},$$

we see $g(a)h(b) \neq h(b)g(a)$ not always holds. The derived contradiction shows that $C_2 \times C_3$ is not a coproduct of C_2, C_3 in Grp.

∎

3.7 Show that there is a surjective homomorphism $Z * Z \to C_2 * C_3$. ($*$ denotes coproduct in Grp.)

Consider the mapping

$$\begin{aligned}\varphi : \mathbb{Z} * \mathbb{Z} &\longrightarrow C_2 * C_3\\ x^{m_1}y^{n_1}\cdots x^{m_k}y^{n_k} &\longmapsto x^{[m_1]_2}y^{[n_1]_3}\cdots x^{[m_k]_2}y^{[n_k]_3}\end{aligned}$$

Since

$$\begin{aligned}&\varphi(x^{m_1}y^{n_1}\cdots x^{m_k}y^{n_k}x^{m'_1}y^{n'_1}\cdots x^{m'_{k'}}y^{n'_k})\\ =&x^{[m_1]_2}y^{[n_1]_3}\cdots x^{[m_k]_2}y^{[n_k]_3}x^{[m'_1]_2}y^{[n'_1]_3}\cdots x^{[m'_k]_2}y^{[n'_k]_3},\\ =&\varphi(x^{m_1}y^{n_1}\cdots x^{m_k}y^{n_k})\varphi(x^{m'_1}y^{n'_1}\cdots x^{m'_{k'}}y^{n'_k})\end{aligned}$$

φ is a homomorphism. It is clear that φ is surjective. Thus we show there exists a surjective homomorphism $Z * Z \to C_2 * C_3$. ∎

3.8 Define a group G with two generators x, y, subject (only) to the relations $x^2 = e_G, y^3 = e_G$. Prove that G is a coproduct of C_2 and C_3 in Grp. (The reader will obtain an even more concrete description for $C_2 * C_3$ in Exercise 9.14; it is called the modular group.) [§3.4, 9.14]

Given the maps $i_1 : C_2 \to G, [m]_2 \mapsto x^m$ and $i_2 : C_3 \to G, [n]_3 \mapsto y^n$, we can check that i_1, i_2 are homomorphisms. We are to show that for every group H endowed with two homomorphisms $f_1 : C_2 \to H$, $f_2 : C_3 \to H$, there would be a unique group homomorphism $\varphi : G \to H$ such that the following diagram commutes

$$\begin{tikzcd}
C_2 \arrow[d, "i_1"'] \arrow[dr, "f_1"] & \\
G \arrow[r, "\varphi"] & H \\
C_3 \arrow[u, "i_2"] \arrow[ur, "f_2"'] &
\end{tikzcd}$$

or

$$\varphi(i_1([m]_2)) = \varphi(x^m) = \varphi(x)^m = f_1([m]_2),$$
$$\varphi(i_2([n]_3)) = \varphi(y^n) = \varphi(y)^n = f_2([n]_3).$$

Define $\phi : G \to H$ as $\phi(x^m y^n) = f_1([m]_2)f_2([n]_3)$, $\phi(y^n x^m) = f_2([n]_3)f_1([m]_2)$. It is clear to see ϕ makes the diagram commute. Moreover, if φ makes the diagram commute, it follows that for all $x^m y^n, y^n x^m \in G$,

$$\varphi(x^m y^n) = \varphi(x^m)\varphi(y^n) = f_1([m]_2)f_2([n]_3),$$

$$\varphi(y^n x^m) = \varphi(y^n)\varphi(x^m) = f_2([n]_3)f_1([m]_2),$$

which implies $\varphi = \phi$. Thus we can conclude G is the coproduct of C_2 and C_3 in Grp.

■