Fall 2010

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Math 216: Foundations of algebraic geometry, I(2.2–4.6, pp1–73)

8/27/10 – 10/2/10

Fri - 8/27

2.2 <u>Categories and functors</u>

<u>2.2.A</u> Groupoid

(a) Gp elts are the set of mors (each an <u>isom</u>).

Mpln given by composition.

The identity elt is the identity mor.

Have inverses since are isoms.

Ass ✓

Comm, not necess.

✓ (b) - <u>The fundamental gpd</u> $\Pi(X)$ of a top space X

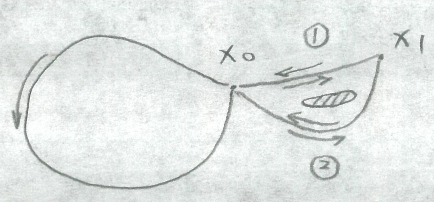
Obj: pts of X

Mor: hmtp equiv classes of paths from one pt to another

- There is a canonical gpd str on a G-set.
- There is a canonical gpd str on a set w/ an equiv relation on it.

<u>2.2.B</u> More about the fundamental gpd

In the case where X is conn, and $\pi_1 X$ is <u>not ab</u>, this illustrates the fact that for a conn gpd — a mor from one obj to another always exists — the autom gps of the objs are all isom, <u>but not canonically isom</u>:



<u>Ex 2.2.8</u> Poset

Think of diags as cats! The "obvious" mors are omitted; only the "generating" mors are depicted.

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d] & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd} \qquad \cdots \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet$$

Ex 2.2.10 / 2.2.15 In particular, a subcat $\mathcal{A}$ in which $\mathrm{Hom}_{\mathcal{A}}(A, A') = \mathrm{Hom}_{\mathcal{B}}(A, A')$ for every A, A' in $\mathcal{A}$ is called a full subcat. (Note that this is stronger than "closed under composition".) We often refer to it as "the full subcat on the objs $\mathrm{Obj}(\mathcal{A})$". More generally, we have full fcns, and the inclusion of a full subcat is full(y faithful). Equiv btwn cats = fully faithful + essentially surj... (cf 2.2.22 on p22)

Exs 2.2.13

- $\pi_1(-)$, not to be confused w/ 2.2.B.
  The source cat is the cat of pted top spaces w/ basept-preserving cts maps

✓ Ex 2.2.14 $A \in \mathcal{C}$ $\quad h^A : \mathcal{C} \longrightarrow Sets$

$\qquad\qquad\qquad\qquad B \longmapsto \mathrm{Mor}(A, B)$

2.2.C Define $g: V \longrightarrow V^{**}$ st $g(v)(w^*) = w^*(v)$. This is an isom of finite-dimensional k-vector spaces (the same dim proves surj, and inj is easy by checking def).
Cf "Why isn't there a cleaner pf that a finite-dimensional vector space is isom to its double dual?" by Timothy Gowers.

Sat - 7/28

2.3 Universal properties determine an object up to unique isomorphism

2.3.1 Localization

A a ring, S a mpl subset of A (containing 1).

Note $\frac{a_1}{s_1} = \frac{a_2}{s_2}$ in $S^{-1}A \iff \exists\, s \in S$ st $s(a_1 s_2 - a_2 s_1) = 0$

(This implies: $0 \in S \Rightarrow S^{-1}A = \{0\}$.)

2.3.A The universal property of $S^{-1}A$: $S^{-1}A$ is initial among A-algs B where every elt of S is sent to a unit in B.

$$\begin{tikzcd} S^{-1}A \arrow[rr, dashed, "\exists!"', "\psi"] & & B \\ & A \arrow[ul] \arrow[ur, "\phi"'] & \end{tikzcd}$$

ϕ sends S to units in B

Naturally, define $\psi : S^{-1}A \longrightarrow B$

$\frac{a}{s} \longmapsto \phi(a)\phi(s)^{-1}$

Well-defined? ✓

ring homom? ✓

unique? ✓

✓ 2.3.B NOT always do we have $A \hookrightarrow S^{-1}A$!

Next, localization of modules (How does it specialize to $S^{-1}A$?)

M an A-module. Define $S^{-1}M$ as being initial among A-modules N for which $s\times\cdot : N \to N$ is an isom for all $s \in S$, and for which there is a map $M \to N$ of A-modules. (cf p23 of later versions.)

2.3.C (a) Cf Sat-8/28-1.

(b) $\frac{a}{s} \cdot m := (s\times\cdot)^{-1}(a \cdot m)$

2.3.D ✓

2.3.E It suffices to show that $S^{-1}M_1 \times \cdots \times S^{-1}M_n$ satisfies the universal property of $S^{-1}(M_1 \times \cdots \times M_n)$, or vice versa!

Approach 1 Check that $S^{-1}M_1 \times \cdots \times S^{-1}M_n$ satisfies the two

Explicitly, the isom is given by $S^{-1}(M_1 \times \cdots \times M_n) \longrightarrow S^{-1}M_1 \times \cdots \times S^{-1}M_n$

$$\frac{(m_1, \cdots, m_n)}{s} \mapsto \left(\frac{m_1}{s}, \cdots, \frac{m_n}{s}\right)$$

Note that localization does not necess commute w/ infinite products (eg $(1, \frac{1}{2}, \frac{1}{3}, \cdots) \in \mathbb{Q} \times \mathbb{Q} \times \mathbb{Q} \times \cdots$). This is a manifestation of $S^{-1}(\cdot)$ being a left adjoint (cf 2.5.G and 2.6.10) and that finite products are coproducts in Mod_A (cf 2.6.2).

properties of N and

$$\exists! \quad S^{-1}M_1 \times \cdots \times S^{-1}M_n \longrightarrow N$$

$$\left(\frac{m_1}{s_1}, \cdots, \frac{m_n}{s_n}\right)$$

$$\parallel$$

$$\left(\frac{s_2 \cdots s_n \cdot m_1}{s_1 \cdots s_n}, \cdots, \frac{s_1 \cdots s_{n-1} \cdot m_n}{s_1 \cdots s_n}\right) \longmapsto (s_1 \cdots s_n \times \cdot)^{-1} \Big| \phi(s_2 \cdots s_n \cdot m_1, \cdots, s_1 \cdots s_{n-1} m_n)$$

Approach 2 This time we concern about the universal property of a product; For simplicity, assume $n = 2$

$$\begin{tikzcd}
P \arrow[dr, dashed, "\exists!\phi"] \arrow[drr, "\nu"] \arrow[ddr, "\mu"'] & & \\
& S^{-1}(M_1 \times M_2) \arrow[r] \arrow[d] & S^{-1}M_2 \\
& S^{-1}M_1 &
\end{tikzcd}$$

Suppose $\mu(x) = \frac{m_1}{s_1} = \frac{s_2 \cdot m_1}{s_1 s_2}$, $\nu(x) = \frac{m_2}{s_2} = \frac{s_1 \cdot m_2}{s_1 s_2}$

Define $\phi(x) = \frac{(s_2 \cdot m_1, s_1 \cdot m_2)}{s_1 s_2}$.

2.3.F Given an arbitrary $m \otimes n \in \mathbb{Z}/(10) \otimes_{\mathbb{Z}} \mathbb{Z}/(12)$,

$10(m \otimes n) = (10m) \otimes n = 0 \otimes n = (12 \cdot 0) \otimes n = 0 \otimes (12n) = 0 \otimes 0 = 0$

$12(m \otimes n) = \cdots = 0$

$\Rightarrow 2(m \otimes n) = 0$

Now, $m \otimes n = 1 \otimes mn$. Hence there are only

$0 = 1 \otimes 0$, $1 \otimes 1$, $1 \otimes 2$, $\cdots$, $1 \otimes 11$.

But $1 \otimes 1 = 1 \otimes 3 = \cdots = 1 \otimes 11$ since $1 \otimes 3 = 1 \otimes 1 + (1 \otimes 1 + 1 \otimes 1) = 1 \otimes 1 + 0$. $\boxed{\mathbb{Z}/2}$

Sun - 8/29

2.3.G

(1) Given $M' \xrightarrow{g} M \xrightarrow{f} M'' \to 0$,

show $M' \otimes_A N \xrightarrow{g \otimes id_N} M \otimes_A N \xrightarrow{f \otimes id_N} M'' \otimes_A N \to 0$.

Nontrivial part: $\ker(f \otimes id_N) \subset \mathrm{im}(g \otimes id_N)$

(2) Give a counterex to

$$0 \to M' \xrightarrow{g} M$$

$$\Rightarrow \quad 0 \to M' \otimes_A N \xrightarrow{g \otimes id_N} M \otimes_A N$$

(2).

$$0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z}$$

$$\not\Rightarrow \quad 0 \to \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/(2) \xrightarrow{2 \otimes 1} \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/(2)$$

$$\mathbb{Z}/(2) \xrightarrow{0} \mathbb{Z}/(2)$$

Show the right-exactness of $\cdot \otimes_A N$ by the universal property.

✓ (1) ("After the first time you come up w/ such an argument as a solution to an exe, your life will never again be the same.")

As $\mathrm{im}(g \otimes id_N) \subset \ker(f \otimes id_N)$, there is a nat map

$$\phi : M \otimes_A N / \mathrm{im}(g \otimes id_N) \longrightarrow M'' \otimes_A N$$
$$m \otimes n + \mathrm{im}(g \otimes id_N) \longmapsto f(m) \otimes n.$$

It suffices to show that this is an isom, as then in

$$\begin{tikzcd} M \otimes_A N \arrow[r, "\pi"] \arrow[dr, "f \otimes id_N"'] & M \otimes_A N / \mathrm{im}(g \otimes id_N) \arrow[d, "\phi"] \\ & M'' \otimes_A N \end{tikzcd}$$

$\ker(f \otimes id_N) = \ker(\phi \circ \pi) = \ker \pi = \mathrm{im}(g \otimes id_N)$ (actually ϕ being inj suffices). We construct an inverse $\psi : M'' \otimes_A N \longrightarrow M \otimes_A N / \mathrm{im}(g \otimes id_N)$ by the univ property of $M'' \otimes_A N$. Define an A-bilinear map $\psi_0 : M'' \times N \longrightarrow M \otimes_A N / \mathrm{im}(g \otimes id_N)$ as follows. For any $m'' \in M''$, $\exists\, m \in M$ st $f(m) = m''$. Then $\psi_0(m'', n) := m \otimes n + \mathrm{im}(g \otimes id_N)$. This is well defined, as we take quotient by $\mathrm{im}(g \otimes id_N)$. It is clearly A-bilinear, and clearly the induced map ψ satisfies $\psi \circ \phi = id$.

(Exploiting the univ property in the above argument is necess, as it would be impossible to check that the directly defined map $m''\otimes n \mapsto m\otimes n + \operatorname{im}(g\otimes \mathrm{id}_N)$ is well defined – too many representatives for $m''\otimes n$!)

<u>2.3.H</u> Here "unique up to unique isom" means: If $(T_1, t_1 : M\times N \to T_1)$ and $(T_2, t_2 : M\times N \to T_2)$ are two such pairs, then there exists a unique isom $f : T_1 \xrightarrow{\sim} T_2$ st

$$\begin{tikzcd} M\times N \arrow[r, "t_1"] \arrow[dr, "t_2"'] & T_1 \arrow[d, "f"] \\ & T_2 \end{tikzcd}$$

commutes.

(It follows that if $(T, t_1 : M\times N \to T)$ and $(T, t_2 : M\times N \to T)$ are two such pairs, then $t_1 = t_2$, as we can take $f = \mathrm{id}_T$.)

<u>Pf</u>

$$\begin{tikzcd} M\times N \arrow[r, "t_1"] \arrow[d, "t_1"'] \arrow[dr, "t_2"] & T_1 \arrow[d, dashed, "f_1"] \arrow[dl, bend left, "\mathrm{id}_{T_1}"] \\ T_1 & T_2 \arrow[l, dashed, "f_2"] \end{tikzcd}$$

<u>2.3.I</u> ✓

<u>2.3.J</u> (a) ✓

(b) ✓

<u>2.3.K</u> Consider

$$(S^{-1}A)\otimes_A M \longleftrightarrow S^{-1}M$$

$$\frac{a}{s}\otimes m \longmapsto \frac{a\cdot m}{s}$$

$$\frac{1}{s}\otimes m \longleftarrow \frac{m}{s}$$

Mon-8/30

<u>2.3.L</u> ✓

<u>2.3.M</u>

	Sets	Rings	Top	2.2.9
initial	ϕ	$\mathbb{Z}$	ϕ	ϕ
final	$\{*\}$	0	$\{*\}$	X

(There's a unique empty fcn: $\phi \to A$.)

2.3.O ✓

2.3.P intersection

2.3.Q ✓ Cf 2.3.M. A special case that we've seen before is that the usual product of two schs is the same as the fiber product of them over $\mathrm{Spec}(\mathbb{Z})$.

2.3.R

$$\begin{tikzcd}
T \arrow[drr, bend left] \arrow[dr, dashed, "2\ \exists!"] \arrow[ddr, dashed, "1\ \exists!"'] \arrow[dddr, bend right] & & \\
& U \arrow[r] \arrow[d] & V \arrow[d] \\
& W \arrow[r] \arrow[d] & X \arrow[d] \\
& Y \arrow[r] & Z
\end{tikzcd}$$

2.3.S ✓

2.3.T (One can think of the lower left Y as $Y \times_Y Y$.) The commutativity of the square follows from the universal property of $Y \times_Z Y$ as we have

$$
\begin{tikzcd}
X_1 \times_Y X_2 \arrow[r] \arrow[d] & X_2 \arrow[d] & & \\
X_1 \arrow[r] & Y \arrow[dr, dashed] \arrow[drr, bend left, "id"] \arrow[ddr, "id"'] & & \\
& & Y \times_Z Y \arrow[r] \arrow[d] & Y \arrow[d] \\
& & Y \arrow[r] & Z
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
X_1 \times_Y X_2 \arrow[dr, dashed] \arrow[drrr, bend left] \arrow[dddr] & & & \\
& X_1 \times_Z X_2 \arrow[rr] \arrow[dr, dashed] \arrow[dd] & & X_2 \arrow[d] \\
& & Y \times_Z Y \arrow[r] \arrow[d] & Y \arrow[d] \\
& X_1 \arrow[r] & Y \arrow[r] & Z
\end{tikzcd}
$$

For the universal property, it suffices to show that a W that makes

$$
\begin{tikzcd}
W \arrow[drr] \arrow[ddr] & & \\
& X_1 \times_Y X_2 \arrow[r] \arrow[d] & X_1 \times_Z X_2 \arrow[d] \\
& Y \arrow[r] & Y \times_Z Y
\end{tikzcd}
$$

commute also makes

$$
\begin{tikzcd}
W \arrow[r] \arrow[d] & X_2 \arrow[d] \\
X_1 \arrow[r] & Y
\end{tikzcd}
$$

(where $W \to X_1$ and $W \to X_2$ come from $W \to X_1 \times_Z X_2$)

commute. In fact, from $W \to X_1 \times_Z X_2$ we have a comm diag

$$
\begin{tikzcd}
W \arrow[r] \arrow[d] & X_2 \arrow[r] & Y \arrow[dd] \\
X_1 \arrow[d] & & \\
Y \arrow[rr] & & Z
\end{tikzcd}
$$

and thus

$$
\begin{tikzcd}
W \arrow[r] \arrow[d] \arrow[dr, dashed, "\exists!"] & X_2 \arrow[r] & Y \arrow[dd] \\
X_1 \arrow[d] & Y \times_Z Y \arrow[ur] \arrow[dl] & \\
Y \arrow[rr] & & Z
\end{tikzcd}
$$

By uniqueness of the dotted arrow we have the comm diag

$$\begin{tikzcd}
W \arrow[r] \arrow[d] \arrow[dr] & X_2 \arrow[r] & Y \arrow[ddd] \\
X_1 \arrow[dd] & Y \arrow[d] & \\
 & Y\times_Z Y \arrow[uur] \arrow[dl] & \\
Y \arrow[rr] & & Z
\end{tikzcd}$$

and we can further complete it as

$$\begin{tikzcd}
W \arrow[r] \arrow[d] \arrow[dr] & X_2 \arrow[r] & Y \arrow[ddd] \\
X_1 \arrow[dd] & Y \arrow[d] \arrow[ur, "id"] \arrow[ddl, "id"'] & \\
 & Y\times_Z Y \arrow[uur] \arrow[dl] & \\
Y \arrow[rr] & & Z
\end{tikzcd}$$

and get

$$\begin{tikzcd}
W \arrow[rr] \arrow[dd] \arrow[dr] & & X_2 \arrow[dd] \\
 & Y \arrow[dr, "id"] & \\
X_1 \arrow[rr] & & Y
\end{tikzcd}$$

2.3.U ✓

2.3.V "⇒":

$$\begin{tikzcd}
X \arrow[dr, "\phi"] \arrow[drr, bend left, "id_X"] \arrow[ddr, bend right, "id_X"'] & & \\
 & X\times_Y X \arrow[r, "\psi'"] \arrow[d, "\psi"] & X \arrow[d, "f"] \\
 & X \arrow[r, "f"] & Y
\end{tikzcd}$$

We have $\psi \circ \phi = id_X$

$\underline{\psi \circ \phi} \circ \psi \overset{?}{=} \underline{\psi' \circ \phi} \circ \psi$ Yes! $\Rightarrow$ $\phi \circ \psi = id_{X \times_Y X}$

"$\Leftarrow$": By universal property of $X \times_Y X$.

2.3.W Unlike the top right diag on π8, as here $Y \to Z$ is a monom, we have a genuine comm diag

$$\begin{tikzcd}
X_1 \times_Y X_2 \arrow[rr] \arrow[dd] \arrow[dr, "\phi"] & & X_2 \arrow[dd] & \\
& X_1 \times_Z X_2 \arrow[ur] \arrow[dl] & & \\
X_1 \arrow[rr] & & Y \arrow[dr] & \\
& & & Z
\end{tikzcd}$$

Now we need to construct an inverse to ϕ:

$$\begin{tikzcd}
X_1 \times_Y X_2 \arrow[rrr] \arrow[ddd] \arrow[dr, "\phi"] & & & X_2 \arrow[ddd] & \\
& X_1 \times_Z X_2 \arrow[urr] \arrow[ddl] \arrow[dr, "\psi"] & & & \\
& & X_1 \times_Y X_2 \arrow[uur] \arrow[dll] & & \\
X_1 \arrow[rrr] & & & Y \arrow[dr] & \\
& & & & Z
\end{tikzcd}$$

Note that this ψ is defined for the entire $X_1 \times_Z X_2$, and by uniqueness $\psi \circ \phi = id_{X_1 \times_Y X_2}$;

also by uniqueness $\phi \circ \psi = id_{X_1 \times_Z X_2}$.

2.3X A coproduct is a diag

$$\begin{tikzcd} & Y \arrow[d] \\ X \arrow[r] & X \sqcup Y \end{tikzcd}$$

st for any obj Z w/ $X \to Z$ and $Y \to Z$ there exists a unique mor $X \sqcup Y \to Z$ st

$$\begin{tikzcd} & Y \arrow[d] \arrow[ddr] & \\ X \arrow[r] \arrow[drr] & X \sqcup Y \arrow[dr] & \\ & & Z \end{tikzcd}$$

commutes.

2.3.Y

$$\begin{tikzcd} A \arrow[r] \arrow[d] & C \arrow[d] \arrow[ddr, "\nu"] & \\ B \arrow[r] \arrow[drr, "\mu"'] & B \otimes_A C \arrow[dr, dashed, "\exists ! \phi"] & \\ & & D \end{tikzcd}$$

$$\begin{aligned} & \phi(b \otimes c) \\ &= \phi(b \otimes 1) \cdot \phi(1 \otimes c) \\ &= \mu(b)\nu(c) \end{aligned}$$

Never confuse the univ property of the tensor product (coproduct) of two A-alg w/ that of the tensor product of two A-modules!

2.3.Z (a) Given $\eta : h^A \to h^B$, we have $f := \eta_A(\mathrm{id}_A) \in \mathrm{Mor}(B, A)$.

It turns out that f determines η completely:

$$\begin{tikzcd} \mathrm{Mor}(A,A) \arrow[r, "\eta_A"] \arrow[dd, "g_*"'] & \mathrm{Mor}(B,A) \arrow[dd, "g_*"] \\ \mathrm{id}_A \arrow[r, mapsto] \arrow[d, mapsto] & f \arrow[d, mapsto] \\ g \arrow[r, mapsto, "?"] & g \circ f \\ \mathrm{Mor}(A,C) \arrow[r, "\eta_C"'] & \mathrm{Mor}(B,C) \end{tikzcd}$$

Thus the corr is bijective.

Note $h^- : \mathcal{C}^{op} \longrightarrow Sets^{\mathcal{C}}$ gives the Yoneda embedding.

(b) There is a bijection between the natural transformations $h_A \to h_B$ of contravariant fcrs $\mathcal{C} \to Sets$ and the mors $A \to B$.

Proof of Yoneda's lemma.

(c) Pf: similar to (a).

Suppose F is a contravariant fcr $\mathcal{C} \to Sets$. There is a bijection between natural transformations $h_A \to F$ and $F(A)$.

Note A contravariant fcr F from $\mathcal{C}$ to Sets is said to be representable if there is a natural isom $\xi : F \xrightarrow{\sim} h_A = Mor(-, A)$.

A covariant ... corepresentab[le] ... $h^A = Mor(A, -)$.

"representable" ~ "limit" ~ "map in"
"corepresentable" ~ "colimit" ~ "map out"

Tue - 8/31

2.4 Limits and colimits

2.4.A Thus, in Sets (and Mod_A, Vec_k, Ab, ...), a limit is a subset of the product.
Cf 2.4.C below.

2.4.4 Interpret kernel / cokernel as limit / colimit:

kernel as equalizer of $A \overset{f}{\underset{0}{\rightrightarrows}} B$

cokernel as coequalizer of $A \overset{f}{\underset{0}{\rightrightarrows}} B$

(Here the index cat is not a poset.)

OR

$$\begin{array}{ccc} & & 0 \\ & & \downarrow \\ A & \xrightarrow{f} & B \end{array}$$

cf p36.

2.4.B ✓

We define the filtered index cat I w/ objs elts of S, and mors induced by mpln in S (in particular, $1 \in S$ is the initial obj). The "ambient" cat $\mathcal{C}$ has objs "sets of fractions in A" (eg $\frac{1}{s}A$), and mors "induced by mpln in S" (in par, $A = \frac{A}{1}$ is the initial obj).

2.4.6 <u>filtered</u> category, may be remembered as "coproducts and coequalizers exist (fogetting their universal properties)".

<u>2.4.C</u> Thus, in a cat where the objs are "set-like",
a limit is a <u>subset of the product</u>, so that the entries are compatible w/ the underlying index cat, and
a filtered colimit is a <u>quotient of the coproduct</u>, so that elts get identified as representatives and there is no redundancy (so as to satisfy the universal property).
(These seemingly oddly asymmetric interpretations result from the inherit asymmetry in the target and source of a map.)

<u>2.4.D</u> ✓

<u>2.4.E</u> Good viewpt. Note that a mpl set is viewed as a filtered set!

<u>2.4.F</u> In 2.4.C, any elt of a <u>filtered</u> colimit genuinely comes from some A_i; in other words, we can pick a representative for any elt in the colimit (cf the pf of lemma 2.6.14 on pp56-7 of [ha] for how we do this using the "upper bound" of index).
In contrast, here, <u>w/o</u> the filtered condition, an elt of a colimi is ~~less~~ (no longer) tangible/explicit. (Here a_i is the image of $a_i \in A_i$ in $\bigoplus_{i \in I} A_i$ under the canonical map.) In this case, we CANNOT find explicit representatives, only knowing killing off the vast elts in $\bigoplus_{i \in I} A_i$

by the equally vast relations.

2.4.G The same phenomenon occurs in the case of limit (ex 2.4.3)

2.5 Adjoints

2.5.A ✓

2.5.B Consider $\tau_{AF(A)} : \mathrm{Mor}_{\mathcal{B}}(F(A), F(A)) \longrightarrow \mathrm{Mor}_{\mathcal{A}}(A, GF(A))$

$$\mathrm{id}_{F(A)} \longmapsto \eta_A$$

By naturality,

$$\begin{tikzcd}
\mathrm{Mor}_{\mathcal{B}}(F(A), F(A)) \arrow[r, "g_*"] \arrow[d, "\tau_{AF(A)}"'] & \mathrm{Mor}_{\mathcal{B}}(F(A), B) \arrow[d, "\tau_{AB}"] \\
\mathrm{Mor}_{\mathcal{A}}(A, GF(A)) \arrow[r, "Gg_*"'] & \mathrm{Mor}_{\mathcal{A}}(A, G(B))
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{id}_{F(A)} \arrow[r, mapsto] \arrow[d, mapsto] & g \arrow[d, mapsto] \\
\eta_A \arrow[r, mapsto] & Gg \circ \eta_A
\end{tikzcd}$$

Similarly, consider $\tau_{G(B)B} : \mathrm{Mor}_{\mathcal{B}}(FG(B), B) \longrightarrow \mathrm{Mor}_{\mathcal{A}}(G(B), G(B))$

$$\varepsilon_B \longmapsto \mathrm{id}_{G(B)}$$

By naturality,

$$\begin{tikzcd}
\mathrm{Mor}_{\mathcal{B}}(FG(B), B) \arrow[r, "Ff^*"] \arrow[d, "\tau_{G(B)B}"'] & \mathrm{Mor}_{\mathcal{A}}(F(A), B) \arrow[d, "\tau_{AB}"] \\
\mathrm{Mor}_{\mathcal{A}}(G(B), G(B)) \arrow[r, "f^*"'] & \mathrm{Mor}_{\mathcal{A}}(A, G(B))
\end{tikzcd}$$

$$\begin{tikzcd}
\varepsilon_B \arrow[r, mapsto] \arrow[d, mapsto] & \varepsilon_B \circ Ff \arrow[d, mapsto] \\
\mathrm{id}_{G(B)} \arrow[r, mapsto] & f
\end{tikzcd}$$

Thus τ is determined by either $\eta : \mathrm{id}_{\mathcal{A}} \to GF$
or $\varepsilon : FG \to \mathrm{id}_{\mathcal{B}}$

| Cf 2.3.Z on p28
| and 2.5.1 below
| Cf Tue-8/31-2.

Wed - 9/1

2.5.C $\mathrm{Mor}_A(M \otimes_A N, P) \longleftrightarrow \mathrm{Mor}_A(M, \mathrm{Mor}_A(N, P))$

Given $f: M \longrightarrow \mathrm{Mor}_A(N, P)$, by the univ property of tensor produc

$\exists!\ g: M \otimes_A N \longrightarrow P$

$$m \otimes n \longmapsto (f(m))(n)$$

Conversely, given $h: M \otimes_A N \longrightarrow P$, define

$$t: M \longrightarrow \mathrm{Mor}_A(N, P)$$

$$m \longmapsto t(m): N \longrightarrow P$$

$$n \longmapsto h(m \otimes n).$$

One checks that the above are inverses.

2.5.D I honestly checked the naturality.

2.5.1

Claim The left adjoint determines the right adjoint up to unique nat isom.

Suppose we have $\tau_{AB}: \mathrm{Mor}_{\mathcal{B}}(F(A), B) \xrightarrow{\sim} \mathrm{Mor}_{\mathcal{A}}(A, G(B))$

and $\tau'_{AB}: \mathrm{Mor}_{\mathcal{B}}(F(A), B) \xrightarrow{\sim} \mathrm{Mor}_{\mathcal{A}}(A, G'(B))$.

Construct $\xi_B: G(B) \longrightarrow G'(B)$ by

$$\tau'_{G(B)B}: \mathrm{Mor}_{\mathcal{B}}(F(G(B)), B) \xrightarrow{\sim} \mathrm{Mor}_{\mathcal{A}}(G(B), G'(B))$$

$$\varepsilon_B \longmapsto \xi_B.$$

(Cf the second half of the pf of 2.5.B on π 14; the above def is really nat / canonical.)

Similarly, ε'_B gives rise to $\xi'_B : G'(B) \longrightarrow G(B)$.
Clearly both ξ_B and ξ'_B are nat in B.
Now we want to show that both $\xi'_B \circ \xi_B$ and $id_{G(B)}$ corr to the same mor in $\mathcal{B}$ via

$$\tau_{G(B)B} : Mor_{\mathcal{B}}(F(G(B)), B) \xrightarrow{\sim} Mor_{\mathcal{A}}(G(B), G(B)).$$

Clearly $id_{G(B)}$ corrs to $F(G(B)) \xrightarrow{F\,id_{G(B)}} F(G(B)) \xrightarrow{\varepsilon_B} B$

$$= F(G(B)) \xrightarrow{\varepsilon_B} B.$$

On the other hand, $\xi'_B \circ \xi_B$ corrs to

$$F(G(B)) \xrightarrow{F\xi_B} F(G'(B)) \xrightarrow{F\xi'_B} F(G(B)) \xrightarrow{\varepsilon_B} B$$

$$= F(G(B)) \xrightarrow{F\xi_B} F(G'(B)) \xrightarrow{\varepsilon'_B} B$$

$$= F(G(B)) \xrightarrow{\varepsilon_B} B.$$

Thus we've set up a nat isom between G and G'.

<u>We can phrase the above a little differently.</u>

Suppose we have nat bijections τ for F, G and τ' for F, G'. As in 2.5.B on p32, they are given by $\varepsilon : FG \to id$ and $\varepsilon' : FG' \to id$ resp. Then, for each $B \in \mathcal{B}$,

$\exists!\ \xi_B : G(B) \to G'(B)$ st $\varepsilon'_B \circ F\xi_B = \varepsilon_B$ via $\tau'_{G(B)B}$, and

$\exists!\ \xi'_B : G'(B) \to G(B)$ st $\varepsilon_B \circ F\xi'_B = \varepsilon'_B$ via $\tau_{G'(B)B}$.

But $\varepsilon_B \circ F(\xi'_B \circ \xi_B) = \varepsilon_B \circ F\xi'_B \circ F\xi_B = \varepsilon'_B \circ F\xi_B = \varepsilon_B$, and hence by "the univ property argument" $\xi'_B \circ \xi_B = id_{G(B)}$. Similarly, $\xi_B \circ \xi'_B = id_{G'(B)}$. Thus, we've set up a unique isom $\xi_B : G(B) \xrightarrow{\sim} G'(B)$ st $\varepsilon'_B \circ F\xi_B = \varepsilon_B$ and it is clearly nat.

For <u>uniqueness</u> of the nat isom: So far, the best way to fully prove the statement is to follow [ct]: the pf of cor 1 on p85, where $\eta_x : x \to GFx$ being a univ arrow is shown on p81 (by prop 1 on p59, when the condition giving rise to a univ arrow is precisely the adjunction isom).

Claim $F: \mathcal{A} \to \mathcal{B}$, $G: \mathcal{B} \to \mathcal{A}$, $\eta: id \to GF$, $\varepsilon: FG \to id$ st $\varepsilon F \circ F\eta = id_F$ and $G\varepsilon \circ \eta G = id_G$ (ie for all $B \in \mathcal{B}$, the composit

$G(B) \xrightarrow{\eta G} GFG(B) \xrightarrow{G\varepsilon} G(B)$ is the identity) completely determine an adjunction.

(This is thm 2(v) on p83 of [ct].)

The bijection is set up as in 2.5.B, while the conditions imposed on η and ε guarantee that composites are the identities. It is not.

Thu-9/2

Groupification and the Grothendieck construction

2.5.E

- Addition: $(a_1, b_1) + (a_2, b_2) := (a_1 + a_2, b_1 + b_2)$

  ass ✓

  zero: $(0,0)$

  inverse: $-(a,b) = (b,a)$ (Here we need that the semi-gp was ab.)

  comm: ✓

- The semigp map $i: S \to H(S)$: $a \mapsto (a, 0)$

  $0 \mapsto (0,0)$ ✓

  (Note that for a semigp map, zero mapping to zero is not automatic but sth imposed.)

- The universal property:

$$\begin{tikzcd} S \arrow[r] \arrow[dr, "f"'] & H(S) \arrow[d, "\exists ! g"] \\ & G' \end{tikzcd}$$

Define $g: H(S) \longrightarrow G'$

$(a,b) \longmapsto f(a) - f(b)$ well defined ✓

$(a,0) \mapsto f(a)$ determined by comm of the diag

$(0,b) \mapsto -f(b)$ determined b/c $(b,0)+(0,b)=(0,0) \mapsto$

- <u>How about defining $(a,b)\sim(c,d) \iff a+d=b+c$?</u>

No, we won't have the transitivity of an equiv relation then!

- <u>K-gps</u>

In fact, we've seen this before – the defs of $K(X)$ and $\tilde{K}(X)$ (cf pp39–40 of [vbkt])! Note that the direct sum opn on vector bdles is <u>ab.</u> $K(X)$ is defined via the equiv $(a,b)\sim(c,d)$ iff $a+d+\varepsilon^n = b+c+\varepsilon^n$ for some trivial vector bdle ε^n, and is thus the gpification of the semigp $\mathrm{Vect}(X)$ of isom classes of vector bdles over X under direct sum. On the other hand, $\tilde{K}(X)$ is defined via the equiv $a\sim b$ iff $a+\varepsilon^m = b+\varepsilon^n$. We have a map of (semi)gps $K(X)\to\tilde{K}(X)$, $(a,\varepsilon^n)\longmapsto a$, and a splitting $K(X)\cong\tilde{K}(X)\oplus\mathbb{Z}$. Conversely, on p200 of [ccat], $\tilde{K}(X)$ (of a based X) is defined as the ker of the dim fcn $d: K(X)\to\mathbb{Z}$ that sends a vector bdle to the dim of its restricti to the comp of the basept, and we have a map $\tilde{K}(X)\to K(X)$, $a\mapsto(a,\varepsilon^{\dim a})$. The pt is that this map doesn't fit into the triangle so as to imply $K(X)\cong\tilde{K}(X)$.

$$\begin{tikzcd} \mathrm{Vect}(X) \arrow[r] \arrow[dr] & K(X) \arrow[d, dashed, "\exists!"] \\ & \tilde{K}(X) \end{tikzcd}$$

And this explicitly shows that $\tilde{K}(X)$ is not the gpification:

$$\begin{tikzcd} \mathrm{Vect}(X) \arrow[r] \arrow[dr] & \tilde{K}(X) \arrow[d, dashed, "\nexists"] & a \arrow[d, mapsto] \\ & K(X) & (a,0) \end{tikzcd}$$

is not well def

- <u>"The Grothendieck construction"</u>

On p199 of [ccat], there is a more elegant way of phrasing the gpification: One takes the quotient of the free ab gp generated by the elts of S by the subgp generated by the set of elts of the form $m+n-m\oplus n$, where $\oplus$ is the sum in S.

- <u>"Groupoidification"</u>

Cf Fri-9/3-1.

- <u>The adjunction $\mathrm{Mor}_{Ab}(HS, G)\cong\mathrm{Mor}_{AbS}(S, FG)$</u>

Given $f: HS\to G$, we have $S\xrightarrow{i}HS\xrightarrow{f}G$

Given $g: S\to FG=G$, we have a unique $HS\to G$

Note that by the univ property, H has to be a <u>left</u> adjoint

Z.S.F ✓

2.5.G What we really need to do: the adjunction

$$\mathrm{Mor}_{S^{-1}A}(S^{-1}M, N) \cong \mathrm{Mor}_A(M, FN)$$

Again, by the universal property of $S^{-1}M$ (cf p23).

Fri - 9/3

2.6 Kernels, cokernels, and exact sequences: A brief introduction to abelian categories

im(f) ✓ 2.6.3 The *image* of a mor $f: A \to B$ is defined as $\mathrm{im}(f) = \ker(\mathrm{coker}\, f)$. It is the unique factorization

$$A \xrightarrow{\text{epi}} \mathrm{im}(f) \xrightarrow{\text{monic}} B.$$

It is the coker of the ker, and the ker of the coker. It is unique up to unique isom.

Pf The defining diag of $\mathrm{im}(f) := \ker(\mathrm{coker}\, f)$:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, dashed, "\exists!"'] & B \arrow[r, "p_1"] & C_1 \\ & K_1 =: \mathrm{im}(f) \arrow[u, "i_1", "\text{monic}"'] & \end{tikzcd}$$

w/ two byproducts:

(1) $\exists!\ A \to K_1$,

(2) i_1 is monic.

First we want to set up a unique isom between $\ker(\mathrm{coker}\, f)$ and $\mathrm{coker}(\ker f)$:

$$\begin{tikzcd} K_2 \arrow[r, "i_2"] & A \arrow[r, "f"] \arrow[d, "p_2", "\text{epi}"'] \arrow[dr, dashed, "\exists!"] & B \arrow[r, "p_1"] & C_1 \\ & C_2 \arrow[ur, dashed, "\exists!"] & K_1 \arrow[u, "i_1"', "\text{monic}"] & \end{tikzcd}$$

Since $C_2 \overset{\exists!}{\dashrightarrow} B \xrightarrow{p_1} C_1$ is zero (p_2 is epi), $\exists!\ C_2 \to K_1$. Dually, $\exists!\ C_2 \leftarrow K_1$, and by a univ property argument, C_2 and K_1 are uniquely isom. Thus we have a factorization

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "p_2", "\text{epi}"'] & & B \\ & \mathrm{im}(f) \arrow[ur, "i_1", "\text{monic}"'] & \end{tikzcd}$$

It remains to show that such a factorization is unique. We use the defining properties of an ab cat: By (2), we have

$$\begin{tikzcd} & & C_3 \arrow[dr, dashed, bend left, "\exists!"] & \\ A \arrow[r, "f"] \arrow[dr, "p", "\text{epi}"'] & B \arrow[ur, "p_3"] \arrow[rr, "p_1"] & & C_1 \arrow[ul, dashed, "\exists!"] \\ & I = \ker p_3 \arrow[u, "i", "\text{monic}"'] & & \end{tikzcd}$$

but by a universal property argument C_1 and C_3 are uniquely isom ($\exists!\ C_3 \to C_1$ b/c p is epi), and thus $I \cong \mathrm{im}(f)$. $\square$

Mon - 9/6

"The alternative definition of homology"

2.6.A Suppose we have a cpx:

$A^\bullet$: $\cdots \longrightarrow A^{i-1} \xrightarrow{f^{i-1}} A^i \xrightarrow{f^i} A^{i+1} \xrightarrow{f^{i+1}} \cdots$

Then we have ses's

$$0 \to \ker f^i \to A^i \to \mathrm{im}\, f^i \to 0 \quad ✓$$

$$0 \to \mathrm{im}\, f^{i-1} \to \ker f^i \to H^i(A^\bullet) \to 0 \quad ✓$$

and dually,

$$0 \to \mathrm{im}\, f^i \to A^{i+1} \to \mathrm{coker}\, f^i \to 0 \quad ✓$$

$$0 \to H^i(A^\bullet) \to \mathrm{coker}\, f^{i-1} \to \mathrm{im}\, f^i \to 0$$

This can be thought of as an "alternative def of hmlg".

ie $0 \to \ker f^i/\mathrm{im}\, f^{i-1} \to A^i/\mathrm{im}\, f^{i-1} \to A^i/\ker$

<u>2.6.B</u> We need to show $\sum(-1)^i \dim A^i = \sum(-1)^i \dim \ker(d^i)/\operatorname{im}(d^{i-1})$.

We have canonical isoms $A^i/\ker(d^i) \xrightarrow{\sim} \operatorname{im}(d^i)$,

which give $\dim A^i - \dim \ker(d^i) = \dim \operatorname{im}(d^i)$.

We want to show

$$\sum_{i=1}^{n} (-1)^i \left(\dim \ker(d^i) + \dim \operatorname{im}(d^i)\right) = \sum_{i=1}^{n} (-1)^i \left(\dim \ker(d^i) - \dim \operatorname{im}(d^{i-1})\right)$$

but this is

$$-(\dim \ker(d^1) + \dim \operatorname{im}(d^1)) + \cdots + (-1)^n\left(\dim \ker(d^n) + \cancel{\dim \operatorname{im}(d^n)}^{\,0}\right)$$

$$= -(\dim \ker(d^1) - \cancel{\dim \operatorname{im}(d^0)}^{\,0}) + \cdots + (-1)^n\left(\dim \ker(d^n) - \dim \operatorname{im}(d^{n-1})\right).$$

✓ In particular, if $A^\bullet$ is exact, then $\sum(-1)^i \dim A^i = 0$.

<u>2.6.C</u> What we're supposed to show:

Ad1: $\operatorname{Mor}(A^\bullet, B^\bullet)$ is an ab gp. ✓ $\left(\text{Have } \begin{array}{c} A^\bullet \\ \downarrow 0 \\ B^\bullet \end{array}\right)$

Ad2: The zero obj is $\cdots \to 0 \to 0 \to 0 \to \cdots$

Ad3: $A^\bullet \times B^\bullet$ exists.

$$A^\bullet \times B^\bullet : \quad \cdots \to A^{i-1} \times B^{i-1} \xrightarrow{f^{i-1} \times g^{i-1}} A^i \times B^i \xrightarrow{f^i \times g^i} A^{i+1} \times B^{i+1} \xrightarrow{f^{i+1} \times g^{i+1}} \cdots$$

Ab1: $A^\bullet \xrightarrow{\phi^\bullet} B^\bullet$ has a ker and a coker.

$$\ker \phi^\bullet : \quad \cdots \longrightarrow \ker \phi^{i-1} \xrightarrow{\tilde{f}^{i-1}} \ker \phi^i \xrightarrow{\tilde{f}^i} \ker \phi^{i+1} \xrightarrow{\tilde{f}^{i+1}} \cdots$$

$$\operatorname{coker} \phi^\bullet : \quad \cdots \longrightarrow \begin{array}{c} \operatorname{coker} \phi^{i-1} \\ \parallel \\ B^{i-1}/\operatorname{im}\phi^{i-1} \end{array} \xrightarrow{\tilde{g}^{i-1}} \begin{array}{c} \operatorname{coker} \phi^i \\ \parallel \\ B^i/\operatorname{im}\phi^i \end{array} \xrightarrow{\tilde{g}^i} \operatorname{coker} \phi^{i+1} \xrightarrow{\tilde{g}^{i+1}} \cdots$$

$$g^{i-1}(\operatorname{im}\phi^{i-1}) \subset \operatorname{im}\phi^i$$

Ab2: Every monic mor is the ker of its coker. $\Big\}$ Unwind defs.
Ab3: Every epi mor is the coker of its ker.

Thus, <u>$\mathrm{Com}_{\mathcal{C}}$ is an ab cat!</u>

<u>2.6.D</u> Suppose we have

$$\begin{tikzcd}
A^\bullet \arrow[d, "\phi^\bullet"'] & : & \cdots \arrow[r] & A^{i-1} \arrow[r, "f^{i-1}"] \arrow[d, "\phi_{i-1}"'] & A^i \arrow[r, "f^i"] \arrow[d, "\phi_i"'] & A^{i+1} \arrow[r, "f^{i+1}"] \arrow[d, "\phi_{i+1}"'] & \cdots \\
B^\bullet & & \cdots \arrow[r] & B^{i-1} \arrow[r, "g^{i-1}"] & B^i \arrow[r, "g^i"] & B^{i+1} \arrow[r, "g^{i+1}"] & \cdots
\end{tikzcd}$$

We want

$$\begin{tikzcd}
H(A^\bullet) \arrow[d, "\phi^\bullet_*"'] & : & \cdots \arrow[r] & \ker f^{i-1}/\operatorname{im} f^{i-2} \arrow[r] \arrow[d] & \ker f^i/\operatorname{im} f^{i-1} \arrow[r] \arrow[d] & \ker f^{i+1}/\operatorname{im} f^i \arrow[r] \arrow[d] & \cdots \\
H(B^\bullet) & & \cdots \arrow[r] & \ker g^{i-1}/\operatorname{im} g^{i-2} \arrow[r] & \ker g^i/\operatorname{im} g^{i-1} \arrow[r] & \ker g^{i+1}/\operatorname{im} g^i \arrow[r] & \cdots
\end{tikzcd}$$

(Note that $H(A^\bullet)$ is an infinite cpx.)

<u>2.6.E</u> Suppose $A' \xrightarrow{f} A \xrightarrow{g} A''$ is exact, ie $\operatorname{im} f = \ker g$.

Then $0 \to \operatorname{im} f \to A \xrightarrow{g} A''$ is exact.

Thus $0 \to F(\operatorname{im} f) \to F(A) \xrightarrow{Fg} F(A'')$ is exact.

We want to show $0 \to \operatorname{im} Ff \to F(A) \xrightarrow{Fg} F(A'')$ is exact.

But $F(\operatorname{im} f) \to F(A)$ is uniquely isom to $\operatorname{im} Ff \to F(A)$, as it fits into the triangle (cf p36):

need right-exactness of F

$$\begin{tikzcd}
F(A') \arrow[r, "Ff"] \arrow[d, "\text{epi}"'] & F(A) \\
F(\operatorname{im} f) \arrow[ur, "\text{monic}"'] &
\end{tikzcd}$$

epi b/c $A' \to \operatorname{im} f \to 0$

monic b/c $0 \to \operatorname{im} f \to A$.

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Note that the converse is not necess true unless F is faithful (cf $\pi 29$): Suppose we have $A' \xrightarrow{f} A \xrightarrow{g} A''$ unknown, but $F(A') \xrightarrow{Ff} F(A) \xrightarrow{Fg} F(A'')$ exact.

We have $0 \to \operatorname{im} Ff \to F(A) \xrightarrow{Fg} F(A'')$ exact

$\Rightarrow 0 \to F(\operatorname{im} f) \to F(A) \xrightarrow{Fg} F(A'')$ exact (by the above triangle)

but we also have $0 \to F(\operatorname{im} f) \to F(A) \to F(\operatorname{coker} f)$ exact, so $F(A'') \cong F(\operatorname{coker} f)$

If this implies $A'' \cong \operatorname{coker} f$ then we're done.

2.6.F On the other hand, to show that F is exact, clearly it suffices to show that $A \to B \to C \Rightarrow F(A) \to F(B) \to F(C)$

(a) By 2.6.E, it suffices to show that

$L \xrightarrow{f} M \xrightarrow{g} N$ is exact

$\Rightarrow S^{-1}L \to S^{-1}M \to S^{-1}N$ is exact.

Clearly $\operatorname{im} S^{-1}f \subset \ker S^{-1}g$.

$$S^{-1}g\left(\frac{m}{s}\right) = \frac{g(m)}{s} = 0 \Rightarrow g(m) = 0$$

$$\Rightarrow m = f(l)$$

$$\Rightarrow\!\!\!\!\not\Leftarrow \quad \frac{m}{s} = \frac{f(l)}{s} = S^{-1}f\left(\frac{l}{s}\right).$$

More exactness of $\otimes$ and Hom functors with counterexamples

(b) ✓, and we've shown that it is not left-exact (cf $\pi 5$).

(c) The less trivial spot is

$\ker(\operatorname{Hom}(M,N) \to \operatorname{Hom}(M,N'')) \subseteq \operatorname{im}(\operatorname{Hom}(M,N') \to \operatorname{Hom}(M,N))$,

where we do need the inj of $N' \to N$.

Not right exact: We want to figure out a counterex via the adjunction $\operatorname{Hom}(L \otimes M, N) \cong \operatorname{Hom}(L, \operatorname{Hom}(M,N))$.

We had in (b): $0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z}$ exact

$\not\Rightarrow 0 \to \mathbb{Z} \otimes \mathbb{Z}/2 \to \mathbb{Z} \otimes \mathbb{Z}/2$ exact

Now: $\mathbb{Z} \to \mathbb{Z}/2 \to 0$ exact

$\not\Rightarrow \operatorname{Hom}(\mathbb{Z}/2, \mathbb{Z}) \to \operatorname{Hom}(\mathbb{Z}/2, \mathbb{Z}/2) \to 0$ exact

($\mathbb{Z} \to \mathbb{Z}/2$ is not inj.)

(d) The quickest way may be by showing the fcr $\mathcal{C} \to \mathcal{C}^{op}$ is exact.

$$A \xrightarrow{f} B \xrightarrow{g} C$$

$$\operatorname{im} f = \ker g$$

$$\Updownarrow \qquad \text{cf 2.6.A on p38.}$$

$$\operatorname{coker} f = \operatorname{im} g = B/\ker g$$

$$\ker f = \qquad B/\operatorname{coker} g = \operatorname{im} g$$

$$A \xleftarrow{f} B \xleftarrow{g} C$$

im = ker (coker)

im = coker (ker)

An explicit counterex for (not) right exact:

$$0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \quad \text{exact}$$

$$\not\Rightarrow \operatorname{Hom}(\mathbb{Z}, \mathbb{Z}/2) \to \operatorname{Hom}(\mathbb{Z}, \mathbb{Z}/2) \to 0 \quad \text{exact}$$

Q: Can we prove (c) and (d) conceptually by the Yoneda embedding?

Exactness of the fcr $\operatorname{Hom}(M, \cdot)$ is to fix M and let $\cdot$ vary, whereas the Yoneda embedding concerns about the collection of fcrs $\operatorname{Hom}(M, \cdot)$ ie lets M vary.

2.6G We want to show $S^{-1}\operatorname{Hom}_A(M, N) \cong \operatorname{Hom}_{S^{-1}A}(S^{-1}M, S^{-1}N)$.
We have $A^{\oplus q} \to A^{\oplus p} \to M \to 0$ (note that we won't use any finiteness assumption on N). We know S^{-1} is exact (2.6.F(a)), and Hom is left exact (2.6.F(d)).

Thus,

$$0 \to \mathrm{Hom}_A(M,N) \to \mathrm{Hom}_A(A^{\oplus p}, N) \to \mathrm{Hom}_A(A^{\oplus q}, N)$$

$$0 \to \mathrm{Hom}_A(M,N) \to A^{\oplus p} \to A^{\oplus q}$$

$$0 \to S^{-1}\mathrm{Hom}_A(M,N) \to S^{-1}A^{\oplus p} \to S^{-1}A^{\oplus q} \quad (2.3.E)$$

On the other hand,

$$S^{-1}A^{\oplus q} \to S^{-1}A^{\oplus p} \to S^{-1}M \to 0$$

$$0 \to \mathrm{Hom}_{S^{-1}A}(S^{-1}M, S^{-1}N) \to \mathrm{Hom}_{S^{-1}A}(S^{-1}A^{\oplus p}, S^{-1}N) \to \mathrm{Hom}_{S^{-1}A}(S^{-1}A^{\oplus q}, S^{-1}N)$$

$$0 \to \mathrm{Hom}_{S^{-1}A}(S^{-1}M, S^{-1}N) \to S^{-1}A^{\oplus p} \to S^{-1}A^{\oplus q}$$

Thus $S^{-1}\mathrm{Hom}_A(M,N)$ and $\mathrm{Hom}_{S^{-1}A}(S^{-1}M, S^{-1}N)$ are the ker of the <u>same map</u> $S^{-1}A^{\oplus p} \to S^{-1}A^{\oplus q}$.

<u>Counterex for non-finitely presented</u> Cf 2.3.F'.

Tue-9/7 <u>Interaction of hmlg and (right/left-) exact fcts</u>

FHHF ✓ <u>2.6.H</u>

(a) We have $C^\bullet: \cdots \to C^{i-1} \xrightarrow{d^{i-1}} C^i \xrightarrow{d^i} C^{i+1} \xrightarrow{d^{i+1}} \cdots$ a cpx, Then

$$FH^\bullet: \cdots \to F\left(\frac{\ker d^{i-1}}{\mathrm{im}\, d^{i-2}}\right) \to F\left(\frac{\ker d^i}{\mathrm{im}\, d^{i-1}}\right) \to F\left(\frac{\ker d^{i+1}}{\mathrm{im}\, d^i}\right) \to \cdots$$

$$H^\bullet F: \cdots \to \frac{\ker Fd^{i-1}}{\mathrm{im}\, Fd^{i-2}} \to \frac{\ker Fd^i}{\mathrm{im}\, Fd^{i-1}} \to \frac{\ker Fd^{i+1}}{\mathrm{im}\, Fd^i} \to \cdots$$

We want $F\left(\frac{\ker d^i}{\mathrm{im}\, d^{i-1}}\right) \to \frac{\ker Fd^i}{\mathrm{im}\, Fd^{i-1}}$.

We have $C^i \xrightarrow{d^i} C^{i+1} \to \mathrm{coker}\, d^i \to 0$

$$\Rightarrow FC^i \xrightarrow{Fd^i} FC^{i+1} \to F(\mathrm{coker}\, d^i) \to 0$$

in particular, $F(\mathrm{coker}\, d^i) \cong \mathrm{coker}\, Fd^i$.

Now, $0 \to \operatorname{im} d^i \to C^{i+1} \to \operatorname{coker} d^i \to 0$ (2.6.4.4)

$$\Rightarrow F(\operatorname{im} d^i) \to FC^{i+1} \to F(\operatorname{coker} d^i) \to 0$$

$$\Rightarrow F(\operatorname{im} d^i) \twoheadrightarrow \operatorname{im}\left(F(\operatorname{im} d^i) \to FC^{i+1}\right)$$

$$= \ker\left(FC^{i+1} \to F(\operatorname{coker} d^i)\right)$$

$$= \operatorname{im}\left(FC^i \xrightarrow{Fd^i} FC^{i+1}\right)$$

$$= \operatorname{im} Fd^i.$$

Also, $0 \to \dfrac{\ker d^i}{\operatorname{im} d^{i-1}} \to \operatorname{coker} d^{i-1} \to \operatorname{im} d^i \to 0$ (2.6.4.4, "the alternative def of homlg")

$$\Rightarrow F\left(\frac{\ker d^i}{\operatorname{im} d^{i-1}}\right) \to F(\operatorname{coker} d^{i-1}) \to F(\operatorname{im} d^i) \to 0$$

Thus we have a map

$$F\left(\frac{\ker d^i}{\operatorname{im} d^{i-1}}\right) \twoheadrightarrow \operatorname{im}\left(F\left(\frac{\ker d^i}{\operatorname{im} d^{i-1}}\right) \to F(\operatorname{coker} d^{i-1})\right)$$

$$= \ker\left(\operatorname{coker} Fd^{i-1} \to F(\operatorname{im} d^i)\right)$$

$$\hookrightarrow \ker\left(\operatorname{coker} Fd^{i-1} \to \operatorname{im} Fd^i\right)$$

$$= \ker\left(\frac{FC^i}{\operatorname{im} Fd^{i-1}} \to \frac{FC^i}{\ker Fd^i}\right)$$

$$= \frac{\ker Fd^i}{\operatorname{im} Fd^{i-1}}.$$

(We've described a map $F(\operatorname{im} d^i) \to \operatorname{im} Fd^i$.
How about $F(\ker d^i) \to \ker Fd^i$? Easy!
Consider $0 \to \ker d^i \to C^i \to \operatorname{im} d^i \to 0$ (2.6.4.3)

$$\Rightarrow F(\ker d^i) \to FC^i \to F(\operatorname{im} d^i) \to 0$$

$$\Rightarrow F(\ker d^i) \twoheadrightarrow \operatorname{im}\left(F(\ker d^i) \to FC^i\right)$$

$$= \ker\left(FC^i \to F(\operatorname{im} d^i)\right)$$

$$= \ker Fd^i.$$
)

(b) This time we want $\dfrac{\ker Fd^i}{\operatorname{im} Fd^{i-1}} \longrightarrow F\left(\dfrac{\ker d^i}{\operatorname{im} d^{i-1}}\right)$, and we try to carry out a <u>dual</u> argument (w/ a view toward (c)).

We have $0 \to \ker d^i \to C^i \xrightarrow{d^i} C^{i+1}$

$$\Rightarrow 0 \to F(\ker d^i) \to FC^i \xrightarrow{Fd^i} FC^{i+1}$$

in particular, $F(\ker d^i) \cong \ker Fd^i$.

Now, $0 \to \ker d^i \to C^i \to \operatorname{im} d^i \to 0$ (2.6.4.3)

$$\Rightarrow 0 \to F(\ker d^i) \to FC^i \to F(\operatorname{im} d^i)$$

$$\Rightarrow \operatorname{im} Fd^i = \frac{FC^i}{\ker(FC^i \xrightarrow{Fd^i} FC^{i+1})}$$

$$= \frac{FC^i}{\operatorname{im}(F(\ker d^i) \to FC^i)}$$

$$= \frac{FC^i}{\ker(FC^i \to F(\operatorname{im} d^i))}$$

$$\hookrightarrow F(\operatorname{im} d^i).$$

Also, $0 \to \operatorname{im} d^{i-1} \to \ker d^i \to \dfrac{\ker d^i}{\operatorname{im} d^{i-1}} \to 0$ (2.6.4.3)

$$\Rightarrow 0 \to F(\operatorname{im} d^{i-1}) \to F(\ker d^i) \to F\left(\frac{\ker d^i}{\operatorname{im} d^{i-1}}\right).$$

Thus we have a map

$$\frac{\ker Fd^i}{\operatorname{im} Fd^{i-1}} \twoheadrightarrow \frac{\ker Fd^i}{F(\operatorname{im} d^{i-1})}$$

$$= \frac{F(\ker d^i)}{\ker\left(F(\ker d^i) \to F\left(\frac{\ker d^i}{\operatorname{im} d^{i-1}}\right)\right)}$$

$$\hookrightarrow F\left(\frac{\ker d^i}{\operatorname{im} d^{i-1}}\right).$$

This is a really nice exe to explore the duality among the quartet $\{\ker, \operatorname{coker}, \operatorname{im}, H\}$.

(c) " Show that the mors of (a) and (b) are inverses and thus isom

Let's not cheat and at least do examine carefully the composite

$$\boxed{F\left(\frac{\ker d^i}{\operatorname{im} d^{i-1}}\right) \twoheadrightarrow} \operatorname{im}\left(F\left(\frac{\ker d^i}{\operatorname{im} d^{i-1}}\right) \to F(\operatorname{coker} d^{i-1})\right)$$

$$= \ker\left(\operatorname{coker} Fd^{i-1} \to F(\operatorname{im} d^i)\right)$$

$$\hookrightarrow \ker\left(\operatorname{coker} Fd^{i-1} \to \operatorname{im}\left(F(\operatorname{im} d^i) \to FC^{i+1}\right)\right)$$

$$= \ker\left(\operatorname{coker} Fd^{i-1} \to \ker\left(FC^{i+1} \to F(\operatorname{coker} d^i)\right)\right)$$

$$= \ker\left(\operatorname{coker} Fd^{i-1} \to \operatorname{im}\left(FC^i \xrightarrow{Fd^i} FC^{i+1}\right)\right)$$

$$= \ker\left(\operatorname{coker} Fd^{i-1} \to \operatorname{im} Fd^i\right)$$

$$= \ker\left(\frac{FC^i}{\operatorname{im} Fd^{i-1}} \to \frac{FC^i}{\ker Fd^i}\right)$$

$$= \boxed{\frac{\ker Fd^i}{\operatorname{im} Fd^{i-1}}}$$

$$= \frac{\ker Fd^i}{FC^{i-1} / \ker\left(FC^{i-1} \xrightarrow{Fd^{i-1}} FC^i\right)}$$

$$= \frac{\ker Fd^i}{FC^{i-1} / \operatorname{im}\left(F(\ker d^{i-1}) \to FC^{i-1}\right)}$$

$$= \frac{\ker Fd^i}{FC^{i-1} / \ker\left(FC^{i-1} \to F(\operatorname{im} d^{i-1})\right)}$$

$$\twoheadrightarrow \frac{\ker Fd^i}{F(\operatorname{im} d^{i-1})}$$

$$= \frac{F(\ker d^i)}{\ker\left(F(\ker d^i) \to F\left(\frac{\ker d^i}{\operatorname{im} d^{i-1}}\right)\right)}$$

$$\boxed{\hookrightarrow F\left(\frac{\ker d^i}{\operatorname{im} d^{i-1}}\right)}.$$

Thus, for 2.6.E on p40: Consider $\cdots \to 0 \to \ker f \to \overset{i-1}{A'} \xrightarrow{f} \overset{i}{A} \xrightarrow{g} \overset{i+1}{A''} \to \operatorname{coker} \to 0 \to \cdots$

$A^{i-1} \to A^i \to A^{i+1}$ exact

$\Leftrightarrow H^i = 0$

$\Rightarrow FH^i = 0$

$\underset{F\text{ exact}}{\overset{(c)}{\Leftrightarrow}} H^iF = 0$

$\Leftrightarrow FA^{i-1} \to FA^i \to FA^{i+1}$ exact

And for 2.6.G: S^{-1} is exact and $\mathrm{Hom} \sim H^0$, so the isom is just $FH^0 \cong H^0F$ (cf Wed-9/8-1).

Wed-9/8

<u>Interaction of adjoints, (co)limits, and (left/right-)exactness</u>

<u>2.6.I</u> We want $\varprojlim \ker h_i \cong \ker(\varprojlim A_i \to \varprojlim B_i)$.

For $\rightarrow$:

$$\begin{tikzcd}
\varprojlim \ker h_i \arrow[dr, dashed, "\exists!"] \arrow[drr, bend right] \arrow[drrr] & & & \\
& \ker(\varprojlim A_i \to \varprojlim B_i) \arrow[d] \arrow[rr] & & 0 \arrow[d] \\
& \varprojlim A_i \arrow[rr] & & \varprojlim B_i
\end{tikzcd}$$

$$\begin{tikzcd}
\varprojlim \ker h_i \arrow[d] \arrow[dr] & \\
\ker h_j \arrow[r] \arrow[d] & \ker h_k \arrow[d] \\
A_j \arrow[r] & A_k
\end{tikzcd}
\qquad
\begin{tikzcd}
\varprojlim A_i \arrow[d] \arrow[dr] & \\
A_j \arrow[r] & A_k
\end{tikzcd}$$

$$\begin{tikzcd}
{} \arrow[d, "h_j"] & & {} \arrow[dr, "h_k"] & \\
B_j \arrow[rrr] & & & B_k
\end{tikzcd}$$

(We see that it's important that ker is a limit and h is a nat transformation.)

For $\Leftarrow$:

$$\begin{tikzcd}
\ker(\varprojlim A_i \to \varprojlim B_i) \arrow[d] \arrow[dr] & \\
\ker h_j \arrow[r] & \ker h_k
\end{tikzcd}$$

$$\begin{tikzcd}
\ker(\varprojlim A_i \to \varprojlim B_i) \arrow[dr, dashed, "\exists!"] \arrow[drr, bend left] \arrow[ddr] & & \\
& \ker h_j \arrow[r] \arrow[d] & 0 \arrow[d] \\
& A_j \arrow[r] & B_j
\end{tikzcd}$$

$$\begin{tikzcd}
\ker(\varprojlim A_i \to \varprojlim B_i) \arrow[d] & \\
\varprojlim A_i \arrow[d] \arrow[dr] & \\
A_j \arrow[r] & A_k
\end{tikzcd}$$

Thus, we can now show that limits are left exact:
Given $0 \to A_i \to B_i \to C_i \to 0$, we want $0 \to \varprojlim A_i \to \varprojlim B_i \to \varprojlim C_i$,
but this is equiv to $\varprojlim \ker(B_i \to C_i) = \ker(\varprojlim B_i \to \varprojlim C_i)$.

Calculus of fcns?

2.6.J Limits comm w/ limits, ie

$$\varprojlim_I \varprojlim_J A_{ij} = \varprojlim_J \varprojlim_I A_{ij}$$

where

(*) each $i \to i'$ in I gives rise to a nat transformation $\{g_j : A_{ij} \to A_{i'j}\}_{j\in J}$

(**) $j \to j'$ J $\{h_i : A_{ij} \to A_{ij'}\}_{i\in I}$

Note that in 2.6.I, $J = \left\{ \begin{array}{c} \cdot \\ \downarrow \\ \cdot \to \cdot \end{array} \right\}$, (*) is automatic, and (**) is required. The pf must still be a univ property argument.

2.6.K In an ab cat, colimits over filtered index cats are (left-)exact

First idea Think of filtered ~ $\exists$ common denominator. We had 2.4.E on p31 interpreting localizations as colimits (precisely, localization of a ring as a colimit over a filtered set). Conversely, let's try to interpret colimits over filtered index cats as localizations, so that we may show the exactness similarly as in 2.6.F(a) on p40 and π23.

Precisely, we want to show:

$$\{A_i \xrightarrow{f_i} B_i \xrightarrow{g_i} C_i\}_{i\in I}$$

$$\Rightarrow \varinjlim A_i \to \varinjlim B_i \to \varinjlim C_i, \text{ ie } \operatorname{im} \varinjlim f_i = \ker \varinjlim g_i,$$

where $\varinjlim f_i$ is gotten by $A_i \to B_i \to \varinjlim B_i$, all $i \in I$

$$\Rightarrow \exists!\ \varinjlim A_i \to \varinjlim B_i.$$

Problem Note that here the objs in the ses may no longer be sets, so it doesn't make sense to write for ex $\ker \varinjlim g_i \subset \operatorname{im} \varinjlim f_i$ (the nontrivial part on π23), and we can't argue by "elts", not even as in the general statement between 2.4.C and 2.4.D on p31

<u>Second idea</u> How about showing $\operatorname{im} \varinjlim f_i = \ker \varinjlim g_i$ by a universal property argument?

$\longrightarrow$ is easy:

$$\begin{tikzcd}
\operatorname{im} \varinjlim f_i \arrow[dr, dashed, "\exists !"] \arrow[drr, bend left] \arrow[ddr] & & \\
& \ker \varinjlim g_i \arrow[r] \arrow[d] & 0 \arrow[d] \\
& \varinjlim B_i \arrow[r, "\varinjlim g_i"'] & \varinjlim C_i
\end{tikzcd}$$

$$\begin{tikzcd}
\varinjlim A_i \arrow[r, "epi"] \arrow[rr, bend right, "\varinjlim f_i"'] & \operatorname{im} \varinjlim f_i \arrow[r, "monic"] \arrow[rr, bend left, "\therefore = 0"] & \varinjlim B_i \arrow[r, phantom] \arrow[r, bend right, "\varinjlim g_i"'] & \varinjlim C_i
\end{tikzcd}$$

$$\varinjlim g_i \circ \varinjlim f_i = \varinjlim (g_i \circ f_i) = 0$$

$\longleftarrow$:

$$\begin{tikzcd}
\ker \varinjlim g_i \arrow[dr, dashed, "\exists !"] \arrow[drr, bend left] \arrow[ddr, "?"'] & & \\
& \operatorname{im} \varinjlim f_i \arrow[r] \arrow[d] & 0 \arrow[d] \\
& \varinjlim B_i \arrow[r] & \operatorname{coker} \varinjlim f_i
\end{tikzcd}$$

$$\begin{tikzcd}
\ker \varinjlim g_i \arrow[r, "monic"] \arrow[rr, bend right, "0\,?"'] & \varinjlim B_i \arrow[r, "epi"] & \operatorname{coker} \varinjlim f_i
\end{tikzcd}$$

This is the pt! "$\ker \varinjlim g_i \subset \operatorname{im} \varinjlim f_i$"

Precisely,

$$\begin{tikzcd}
 & & \varinjlim A_i \arrow[d, "\varinjlim f_i"] & \\
\ker \varinjlim g_i \arrow[urr, dashed] \arrow[rr, "\text{monic}"] \arrow[drr, "0?"'] & & \varinjlim B_i \arrow[r, "\varinjlim g_i"] \arrow[d, "\text{epi}"] & \varinjlim C_i \\
 & & \operatorname{coker} \varinjlim f_i &
\end{tikzcd}$$

We hope that the dotted arrow exists by the assumption that I is filtered.

Conclusion

After looking at p57 of [ha], we know that such a lift may never be possible ....

Note Whenever see "filtered", think about whether or not "set-like".

2.6.L ✓

Thu - 9/9

3.1 Motivating example: The sheaf of differentiable functions

Germs as colimits If $p \in U$, we get a map $\mathcal{O}(U) \to \mathcal{O}_p$.

Recall that we have the cat of open sets in a top space X (2.2.9 on p19) as a poset w/ the order given by inclusion (in par, ϕ is initial and X is final). The index cat of the above colimit is in fact the opposite cat of this poset. To see it is a colimit or a limit, we judge by the arrow $\mathcal{O}(U) \to \mathcal{O}_p$ (which can be viewed as a quotient map).

3.1.A Note that here we need that the fcns in $\mathcal{O}$ be nice so that if $f(p) \neq 0$, then $f(x) \neq 0$ for all x in some open nbhd of p.

3.1.B $\mathfrak{m}/\mathfrak{m}^2$ is a module over $\mathcal{O}_p/\mathfrak{m} \cong \mathbb{R}$, ie it is a real vector space.

$$\mathcal{O}_p/\mathfrak{m} \times \mathfrak{m}/\mathfrak{m}^2 \to \mathfrak{m}/\mathfrak{m}^2$$

$$(r+\mathfrak{m}) \cdot (x+\mathfrak{m}) = rx + \mathfrak{m}^2$$

$$r - r' \in \mathfrak{m} \Rightarrow rx - r'x = (r-r')x \in \mathfrak{m}^2. \text{ well defined.}$$

For the pf of $\mathfrak{m}/\mathfrak{m}^2 \cong$ the cotangent space to the mfld at p,

cf thm 4 on p143 of the red book perhaps.

Mon - 9/13

3.2 Definition of sheaf and presheaf

3.2.A ✓. Note "precisely". The notion of a presheaf, though seemingly more informative, is nothing but a contravariant fcn out of a particular nice/useful cat. Cf π33, "germs as colimits".

Ex $U \cap V = \phi \Rightarrow \mathcal{F}(U \cup V) = \mathcal{F}(U) \times \mathcal{F}(V)$

For the universal arrow of the product $\mathcal{F}(U) \times \mathcal{F}(V)$,

identity axiom $\leftrightarrow$!

gluability axiom $\leftrightarrow$ $\exists$

$$\begin{tikzcd}
X \arrow[dr, "\exists!"] \arrow[drr, bend left] \arrow[ddr, bend right] & & & \\
& \mathcal{F}(U \cup V) \arrow[r] \arrow[d] & \mathcal{F}(U) \arrow[d] & \phi \arrow[dl, dash] \\
& \mathcal{F}(V) \arrow[r] & \mathcal{F}(U \cap V) &
\end{tikzcd}$$

recall that 0 is a branch pt for the fcn of square root.

3.2.B ✓ Good simple exs. Also cf π 14 exs 3-5 of the red book ①.
(For (b)), $\mathbb{C}-\{0\}$ is an open subset of $\mathbb{C}$, so failure of gluability of this open subset makes this fail to be a sh.)

3.2.C In view of the exact seq in 3.2.7, $\mathcal{F}(\bigcup_{i\in I} U_i)$ is the ker of $\prod \mathcal{F}(U_i) \rightrightarrows \prod \mathcal{F}(U_i \cap U_j)$ (or equalizer of), but we know ker is a limit. More directly, $\mathcal{F}(\bigcup_{i\in I} U_i) = \lim \mathcal{F}(U)$, where the filtered index cat is the opposite cat of the full subcat of the cat of open sets in $\bigcup_{i\in I} U_i$ w/ objs U_i and $U_i \cap U_j$. In fact we need the identity axiom as well so that the universal arrow is unique.

Compare the facts that "global scns" is a limit and that "stalks" is a colimit.

3.2.D ✓

skyscraper sheaf

3.2.9 (skyscraper sh)

This is a sh of sets on a top space X, ass to a given pt $p \in X$ and a given set S, denoted by S_p, st

$$S_p(U) = \begin{cases} S & \text{if } p \in U \\ \{*\} & \text{if } p \notin U \end{cases}$$

More generally,

- $\{*\}$ is replaced by the final obj in the cat where the sh takes values
- $\{p\}$ can be generalized to any closed subset Z and we have the process of extending the const sh on Z by zero (cf p104 of the red book and 1.19 (a) on p68 of [ag]). Relevant comments: Sehan on Sep 10 at 4:48 pm, and Sam Lichtenstein on Aug 30 at 10:40pm w/ ravivakil's reply.

3.2.E

constant sheaf

(a) presh ✓

the identity axiom ✓

need not form a sh: the gluability axiom fails

Consider $U_i \cap U_j = \phi$, where we must have $\mathcal{F}(\phi) = \{*\}$.

(b) A locally constant sh is the sh of continuous fcns into a discrete set - the discrete top forces the fcns to be locally const. Recall that there is a unique empty fcn: $\phi \to S$ (π 7).
Note that $\underline{S}$ is sheafification of $\underline{S}^{pre}$ by "adding in": $\mathcal{F}(U) = S$ can be think of as the set of globally constant fcns, each corr to an elt in S, and we add in locally const fcns.

3.2.F The issue is that the naturally glued fcn must be continuous which follows from the facts that open sets of open subspaces are open in the ambient space and that unions of open sets are open.

3.2.G

Sheaf of sections and the espace étalé.

(a) Cf p150 of the red book. This is in fact a subsh of the sh of cts maps to Y in the previous exe. Again, the issue is that the glued fcn must be the right inverse to f. But being the identity map is checked by pts.

(b) ✓

3.2.10 (The espace étalé of a (pre)sh)

(a) Compare the defs in 3.2.10 and 3.2.G(a): the nat identification comes from the fact that "each sec s of $\mathcal{F}$ over an open set U determines a sec of $Y \to X$ over U", and vice versa (using the fact of the "weakest" top st "these" secs are cts).

(b) First, we've checked that the sh of secs of $Y \to X$ is indeed a sh (3.2.G(a)), and we have a canonical mor of preshs. Then use the universal property diag on p69. In fact, this sheafification is nothing more than the original presh but w/ additional (minimal) continuity assumption. So we can view the sh axioms as certain assumptions for continuity. A concrete ex is indeed 3.2.E: we added in locally const fcns for continuity to happen.

(This process is analogous to how we endow the top for a product of spaces.)

3.2.H ✓. f^{-1} is nice in preserving openness and commuting w/ unions and inclusions.

In particular, as we noted on π35, the skyscraper sh can be interpreted as the pushforward of the const sh $\underline{S}$ on a one-pt space $\{p\}$, under the mor $f : \{p\} \to X$.

3.2.I Define $(f_* \mathcal{F})_y \longrightarrow \mathcal{F}_x$ in terms of

germs: $(a, V) \mapsto (a, f^{-1}(V))$, with $a \in f_*\mathcal{F}(V)$, $V \ni y$, $a \in \mathcal{F}(f^{-1}(V))$, $f^{-1}(V) \ni x$. well defined ✓

checking this we see that the asymmetry of the direction of this nat mor results from that of the the cts f.

$$\begin{tikzcd}
\mathcal{F}(f^{-1}(V_i)) \arrow[d, equal] \arrow[rrd, bend left] & \mathcal{F}(f^{-1}(V_j)) \arrow[d, equal] \arrow[rd, bend left] & \\
f_*\mathcal{F}(V_i) \arrow[r] \arrow[d] & f_*\mathcal{F}(V_j) \arrow[dl] & \\
\varinjlim_{V \ni y} f_*\mathcal{F}(V) \arrow[rr, dashed, "\exists !"] & & \varinjlim_{U \ni x} \mathcal{F}(U)
\end{tikzcd}$$

colimits: $\varinjlim_{V \ni y} f_*\mathcal{F}(V) \overset{\exists !}{\dashrightarrow} \varinjlim_{U \ni x} \mathcal{F}(U)$

That

$$\begin{tikzcd} \mathrm{Sets}_X \arrow[r, "f_*"] \arrow[d] & \mathrm{Sets}_Y \arrow[d] \\ \mathrm{Sets} \arrow[r] & \mathrm{Sets} \end{tikzcd}$$

commutes means we've defined a fcr (taking stalks).

✓ Warning Taking stalks is exact (cf 3.4.N and O on p70, and 1.2(C on p66 of [ag]), but NOT for the reason of being a filtered colimit (cf 2.6.10 on p41): the surj of $\phi: \mathcal{F} \to \mathcal{G}$ is NOT the same as the surj of $\phi(U): \mathcal{F}(U) \to \mathcal{G}(U)$ for all open $U \subset X$.

3.2.11 $\underline{\mathbb{Z}}$ - module = sh of ab gps.

action: $\underline{\mathbb{Z}}(U) \times \mathcal{F}(U) \longrightarrow \mathcal{F}(U)$

f a locally const $\mathbb{Z}$-valued fcn

G an ab gp

$\|$ espace étalé

$x \mapsto G_x$ $\qquad$ $x \mapsto f_x \cdot G_x$

(cf Mon-9/13-3)

Tue - 9/14

3.3 Morphisms of presheaves and sheaves

3.3.A The following commutes:

$$\begin{tikzcd} \mathcal{F}(V) \arrow[rr, "(f_*)_{\mathcal{F}}"] \arrow[dr, "\phi(V)"] \arrow[dd, "\mathrm{res}_{V,U}"'] & & f_*\mathcal{F}(V) = \mathcal{F}(f^{-1}(V)) \arrow[dr, "f_*\phi(V)"] \arrow[dd, dashed] & \\ & \mathcal{G}(V) \arrow[rr, "(f_*)_{\mathcal{G}}"] \arrow[dd] & & f_*\mathcal{G}(V) = \mathcal{G}(f^{-1}(V)) \arrow[dd] \\ \mathcal{F}(U) \arrow[rr, dashed, "\mathrm{res}_{V,U}"] \arrow[dr, "\phi(U)"'] & & f_*\mathcal{F}(U) = \mathcal{F}(f^{-1}(U)) \arrow[dr, dashed, "f_*\phi(U)"] & \\ & \mathcal{G}(U) \arrow[rr] & & f_*\mathcal{G}(U) = \mathcal{G}(f^{-1}(U)) \end{tikzcd}$$

3.3 B Note that it is NOT defined as $\mathcal{H}om(\mathcal{F}, \mathcal{G})(U) := \mathrm{Hom}(\mathcal{F}(U), \mathcal{G}(U))$ – we may not have nat restriction maps this way.

Check $\mathcal{H}om(\mathcal{F}, \mathcal{G})(U) := \mathrm{Hom}(\mathcal{F}|_U, \mathcal{G}|_U)$ is a sh (of sets on X).

It is naturally a presh.

Identity axiom:
Gluability axiom: $\Big\}$ Maps are identified / defined (glued) by the effect on their targets (here the sh $\mathcal{G}$), and the sh properties of $\mathcal{G}$ enable these. In par, it suffices if we only assume $\mathcal{F}$ as a presh

"Warning"

$$\varinjlim_{U \ni p} \mathrm{Hom}(\mathcal{F}|_U, \mathcal{G}|_U) \hookrightarrow \mathrm{Hom}\Big(\varinjlim_{V \ni p} \mathcal{F}(V), \varinjlim_{W \ni p} \mathcal{G}(W)\Big)$$

always compatible w/ restrictions

forget $\mathcal{F}_p$ and $\mathcal{G}_p$ as stalks, just view them as sets

Counterex

Sat – 9/18

3.3.C

(a) $\mathcal{H}om(\underline{\{p\}}, \mathcal{F})(U) := \mathrm{Mor}(\underline{\{p\}}|_U, \mathcal{F}|_U) \underset{\uparrow}{\cong} \mathcal{F}(U)$

$\underline{\{p\}}(V) = \{*$: the const map to $p\}$

Also the restriction maps agree.

If two shs have the same data as preshs, then they must be the same as shs, since the cat of shs on X is a full subcat of the cat of preshs on X.

(b) The ab gp str on $\underline{\mathbb{Z}}(U)$ is given by ptwise addition, which is generated (ptwisely) by the const fcn to 1. So

$\mathcal{H}om(\underline{\mathbb{Z}}|_U, \mathcal{F}|_U) \cong \mathcal{H}om(\underline{\{1\}}|_U, \mathcal{F}|_U) \cong \mathcal{F}(U)$.

(For a mor $\underline{\mathbb{Z}}|_U \to \mathcal{F}|_U$, if you know where the const fcn $\underline{\{1\}}$ goes, then you can write down the image of any locally const $\mathbb{Z}$-valued fcn.)

(c) Similar.

3.3.D ✓

3.3.E ✓

3.3.F This is an "evaluation" fcn. ✓

3.3.G "$\Rightarrow$": ✓

"$\Leftarrow$": ✓

3.3.H As a subobj of the sh $\mathcal{F}$, $\ker_{pre} f$ inherits the identity axiom and the gluability axiom (we should check that the the glued elt $x \in \ker f(U)$, which boils down to the identity axiom of $\mathcal{G}$ as we want to see whether $f(x) = 0$ in $\mathcal{G}(U)$ by looking at its restrictions).

3.3.I This must be a classic ex, though its verification is not complicated (cf 3.2.B (b) on p61 and π35). The intuition of this failure in general is somewhat explained on π14 of the red book, and that explanation fits well to this specific ex!

3.4 Properties determined at the level of stalks, and sheafification

3.4.A 2.4.C on p31 of [FOAG1] is important ("a very central exe to understand not just in one's head (but also in one's heart)"), and we will make essential use of it here. Recall that stalks are filtered colimits. We want to show that two s.cns having the same imag in every $\mathcal{F}_p$ must be the same to start w/. But if they agree in $\mathcal{F}_p$, they must already agree on a nbhd U_p of p. As $\{U_p\}$ form an open cover of U, the identity axiom gives the result.
Let's immediately work out 3.4.F (a) – we want to check on a presh not satisfying the identity axiom. The ex of locally const fcns modulo const ones on a non-conn space in the red book (π14) does the trick.

3.4.B Glue $s_p' \in \mathcal{F}(U_p)$, $p \in U$. The agreement on an overlap is checked using 3.4.A.

3.4.C Explicitly,

$$
\begin{tikzcd}
\mathcal{F}_p = \varinjlim_{U \ni p} \mathcal{F}(U) \arrow[d, dashed, "\exists!"] & \mathcal{F}(U) \arrow[l] \arrow[d, "\phi(U)"] & \mathcal{F}(U') \arrow[l] \arrow[d, "\phi(U')"] \arrow[ll, bend right] \\
\mathcal{G}_p = \varinjlim_{V \ni p} \mathcal{G}(V) & \mathcal{G}(U) \arrow[l] & \mathcal{G}(U') \arrow[l] \arrow[ll, bend left]
\end{tikzcd}
$$

In the language of fcrs, we have the fcr of taking stalks:

$$
\begin{array}{ccc}
\mathrm{Sets}_X & \longrightarrow & \mathrm{Sets} \\
\mathcal{F} & \longmapsto & \mathcal{F}_p
\end{array}
$$

, where p is a fixed pt in X.

Cf 3.2.I on p63 (taking $f = \mathrm{id}_X$).

3.4.D

$$
\begin{tikzcd}
\mathcal{F}(U) \arrow[r] \arrow[d] & \mathcal{G}(U) \arrow[d] \\
\prod_{p \in U} \mathcal{F}_p \arrow[r] & \prod_{p \in U} \mathcal{G}_p
\end{tikzcd}
\quad \text{coming from} \quad
\begin{tikzcd}
\mathcal{F}(U) \arrow[r] \arrow[d] & \mathcal{G}(U) \arrow[d] \\
\mathcal{F}_p \arrow[r] & \mathcal{G}_p
\end{tikzcd}
\quad \text{each } p \in U
$$

Really think of secns = elts consisting of compatible germs.

The "collective" mors of stalks are obviously compatible w/ restrictions.

The worst thing is that an arbitrary collective mor may not map any set of compatible germs to a set of compatible germs. But still we can say that mors of shs are determined by mors of stalks in the same sense as in 3.4.A — different mors of shs can't have the same mors of stalks.

3.4.E Note that we have isom $\Rightarrow$ monom + epim, but in general the converse is not true (cf Sun-9/19-3). For the cat of shs it is in fact true, and this is what we want to show in "$\Leftarrow$" below.

"$\Rightarrow$": Use the fact that taking stalks is a fcrl (cf 3.4.C).

"$\Leftarrow$": (This is prop1 on p104 of the red book.) Injectivity follows from 3.4.D. We need to show that $\mathcal{F}(U) \to \mathcal{G}(U)$ is surj for all U. Given any scn $t \in \mathcal{G}(U)$, it corrs to $\prod_{p \in U} t_p$ consisting of compatible germ

Since the <u>induced</u> maps on stalks are surj, each t_p has a preimage $s_p \in \mathcal{F}_p$. Are the s_p's compatible germs? They should be. In fact, let $(U_p, t_p' \in \mathcal{G}(U_p))$ be a representative for t_p st the germ of t_p' at all $y \in U_p$ is t_y. Let $(V_p \subset U_p, s_p' \in \mathcal{F}(V_p))$ be a representative for s_p, and let t_p'' be the image of s_p' in $\mathcal{G}(V_p)$. Then the images of t_p' and t_p'' in $\mathcal{G}_p$ are both t_p:

$$\begin{tikzcd} \mathcal{F}(V_p) \arrow[r] \arrow[d] & \mathcal{G}(V_p) \arrow[d] \\ \mathcal{F}_p \arrow[r] & \mathcal{G}_p \end{tikzcd} \qquad \begin{tikzcd} s_p' \arrow[r, mapsto] \arrow[d, mapsto] & t_p'' \arrow[d, mapsto] \\ s_p \arrow[r, mapsto] & t_p \end{tikzcd}$$

so they must agree on some $W_p \subset V_p$ (2.4.C on p31 is so useful!). Now, for all $y \in W_p$, both the germ of s_p' at y and s_y map to t_y:

$$\begin{tikzcd} \mathcal{F}(W_p) \arrow[r] \arrow[d] & \mathcal{G}(W_p) \arrow[d] \\ \mathcal{F}_y \arrow[r] & \mathcal{G}_y \end{tikzcd} \qquad \begin{tikzcd} s_p' \arrow[r, mapsto] \arrow[d, mapsto] & t_p''=t_p' \arrow[d, mapsto] \\ (s_p')_y \arrow[r, mapsto] & t_y \end{tikzcd}$$

and thus by the injectivity of the induced maps on stalks, the s_p's are indeed compatible. Hence $\prod_{p \in U} s_p$ corrs to a scn $s \in \mathcal{F}(U)$ that maps to t (by 3.4.B and 3.4.D).

(The above pf doesn't explicitly mention "gluability", as we quote 3.4.B at the last step. For a slightly different phrasing, including <u>idea</u>, cf ππ116-7 of the red book. We notice that once we've built up some "machinary" as we did here, the pf becomes more straightforwar[d] and formal, the goal being clearer and the idea more natural/systemati[c]

<u>3.4.F</u>

(a) Cf 3.4.A on π41.

(b)(c). Let $\mathcal{F}$ be the presh in (a) of locally const fcns modulo const over on a non-conn space. Consider $\mathcal{F} \underset{\cdot 0}{\overset{id}{\rightrightarrows}} \mathcal{F}$.

<u>3.4.G</u> Second nature?

The compatible germs construction of sheafification

<u>3.4.H</u> The construction of $\mathcal{F}^{sh}$ really makes scns "stalkwise", and the "tautological restriction maps" turn out to be purely set-theoretic and naive, and really trivialize the two axioms! A "sheaf" is just this! Cf 3.4.3 remark on p68: the subtlety is the word "appropriately".

Compare w/ Mumford's "description" of sheafification: identify, then add in.

<u>3.4.I</u> "Clearly any scn s of $\mathcal{F}$ over U gives a choice of compatible germs for U." (p67) Here the $\mathcal{F}$ can be just a presh! So the nat map of preshs is just

$$\begin{aligned} sh: \mathcal{F} &\longrightarrow \mathcal{F}^{sh} \\ \mathcal{F}(U) &\longrightarrow \mathcal{F}^{sh}(U) \\ s &\longmapsto (s_x \in \mathcal{F}_x)_{x \in U} \end{aligned}$$

Must be compatible!

<u>3.4.J</u>

$$\begin{tikzcd} \mathcal{F} \arrow[r, "sh"] \arrow[dr, "g"'] & \mathcal{F}^{sh} \arrow[d, dashed, "\exists! f"] \\ & \mathcal{G} \end{tikzcd} \qquad \begin{tikzcd} \mathcal{F}^{sh}(U) \arrow[d] \\ \mathcal{G}(U) \end{tikzcd} \qquad \begin{tikzcd} (s_x \in \mathcal{F}_x)_{x \in U} \arrow[d, mapsto, "(g_x)_{x \in U}"] \\ (g_x(s_x) \in \mathcal{G}_x)_{x \in U} \end{tikzcd}$$

Use 3.4.D. to show uniqueness

3.4.K

$$\mathrm{Mor}_{sh}(\mathcal{F}^{sh}, \mathcal{G}) \cong \mathrm{Mor}_{presh}(\mathcal{F}, F\mathcal{G})$$

$$f \xleftarrow{\exists!} g$$

$$f \twoheadleftarrow f \circ sh$$

"Explain": coker and $\otimes$ are colimits, which commute w/ left-adjoints, so that if $\mathcal{F}$ is a sh, then

$$\mathrm{coker}\, \mathcal{F} = \mathrm{coker}(F\mathcal{F})^{sh} = (\mathrm{coker} F\mathcal{F})^{sh}.$$

On the other hand, ker commutes w/ right adjoints, so that if $\mathcal{F}$ is a sh, then

$$F(\ker \mathcal{F}) = \ker F\mathcal{F}.$$

Also, here we see sheafification is right exact. Cf Sat-9/18-1.

3.4.L

$$\begin{tikzcd} \mathcal{F} \arrow[r, "sh"] \arrow[d, "\phi"'] & \mathcal{F}^{sh} \arrow[d, dashed, "\exists!\, \phi^{sh}"] \\ \mathcal{G} \arrow[r, "sh"'] & \mathcal{G}^{sh} \end{tikzcd}$$

3.4.M $\mathcal{F} \longrightarrow \mathcal{F}^{sh}$

$$\mathcal{F}(U) \longrightarrow \mathcal{F}^{sh}(U) \qquad \rightsquigarrow \qquad \mathcal{F}_p \longrightarrow \mathcal{F}^{sh}_p$$

$$s \longmapsto (s_x)_{x \in U} \qquad \qquad s_p \longmapsto s_p$$

In fact, "on the level of open sets" can be thought of as "taking global scns", and we can extend the exact seq farther to the right: cf 3.5.E on p72.

3.4.7 Remark Indeed. Compare the compatible germs construction on p69 and the espace étalé construction on p62: a family of compatible germs corrs to a continuous scn of $Y \to X$, each germ f_x being the value of the scn at x.

Injectivity / surjectivity on the levels of sheaves, of stalks, of open sets

3.4.N

(a) ⇒ (b): Recall that $\phi : \mathcal{F} \to \mathcal{G}$ being a monom means that for any sh $\mathcal{H}$, the natural map $\mathrm{Mor}(\mathcal{H},\mathcal{F}) \to \mathrm{Mor}(\mathcal{H},\mathcal{G})$ is inj (cf p27). Let $\mathcal{H} = \{*\}_x = (i_x)_*\{*\}$, the one-floor skyscraper sh of a one-pt set over x. Then one checks that $\mathrm{Mor}(\{*\}_x, \mathcal{F}) \cong \mathcal{F}_x$! (cf Sun-9/19-1) More generally, when showing (a) ⇒ (c), we use the "indicator sh" $\mathcal{H}$ w/ one scn over every open set contained in U and no scn over any other open set so that $\mathrm{Mor}(\mathcal{H},\mathcal{F}) \cong \mathcal{F}(U)$.

(b) ⇒ (c): 3.4.D.

(c) ⇒ (a): Check def.

3.4.O

(b) ⇒ (a): Note that when proving surj in 3.4.E, we use the inj on stalks, which we don't have here. This is why (c) is missing from the equiv conditions. (c) is stronger than (a) ⇔ (b).

Let's check by def. $\mathcal{F} \xrightarrow{\phi} \mathcal{G} \underset{\psi_2}{\overset{\psi_1}{\rightrightarrows}} H$ $\quad \psi_1 \circ \phi = \psi_2 \circ \phi$

$$\Rightarrow \psi_{1p} \circ \phi_p = \psi_{2p} \circ \phi_p$$

$$\Rightarrow \psi_{1p} = \psi_{2p}$$

$$\Rightarrow \psi_1 = \psi_2 \quad (3.4.D)$$

(a) ⇒ (b): Unfortunately, the method in 3.4.N (a) ⇒ (b) is not dualizable in the desired fashion.

Suppose on the contrary that $\exists x \in X$ st $\phi_x : \mathcal{F}_x \to \mathcal{G}_x$ is not surj. Then $\exists S$ and $\mathcal{G}_x \underset{\eta_x}{\overset{\xi_x}{\rightrightarrows}} S$ st $\xi_x \circ \phi_x = \eta_x \circ \phi_x$ but $\xi_x \neq \eta_x$. Consider the skyscraper sh S_x. We have $\mathcal{F} \xrightarrow{\phi} \mathcal{G} \underset{\eta}{\overset{\xi}{\rightrightarrows}} S_x$, where

$$\mathcal{G}(U) \xrightarrow{\xi(U)} S_x(U)$$

$$s \longmapsto \begin{cases} \xi_x(s_x) & \text{if } U \ni x \\ * & \text{if } U \not\ni x \end{cases}$$

so that $\xi \circ \phi = \eta \circ \phi$ (by 3.4.D) but $\xi \neq \eta$, contra.

nowhere vanishing holomorphic functions vs functions admitting a holomorphic logarithm

3.4.P In 3.3.I on p67 (cf π41 also), we have an exact seq of presh s on $X = \mathbb{C}$:

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O}_X \xrightarrow{f \mapsto \exp 2\pi i f} \mathcal{F} \longrightarrow 1$$

where $\mathcal{F}$ is the presh (not a sh) of fcns admitting a holom log.
Here we have an exact seq of shs on $X = \mathbb{C}$:

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\times 2\pi i} \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^* \longrightarrow 1$$

(when to apply $\times 2\pi i$ is not substantial) where $\mathcal{O}_X^*$ is the sh of invertible (nowhere zero) holom fcns.

To show that $\mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^*$ describes $\mathcal{O}_X^*$ as a quotient sh of $\mathcal{O}_X$, we look at the maps on stalks (values at a pt. cf warning abov 3.2.6 on p60), which are certainly surj.

To see that exp is not surj on the level of open sets, we just note once again that there is no fcn over $\mathbb{C} - \{0\}$ admitting a holom log: nowhere vanishing holom fcns have logs locally, but they need not globally.

This is indeed a "great ex".

Sun - 9/19

3.5 Sheaves of abelian groups, and $\mathcal{O}_X$-modules, form abelian categories

3.5.A Using the concrete description of stalks, consider the map

$$\ker(\mathcal{F} \to \mathcal{G})_x \xrightarrow{\mathrm{id}} \ker(\mathcal{F}_x \to \mathcal{G}_x)$$

$(s, U) \mapsto (s, U)$

$U \ni x,\ s \in \mathcal{F}(U),\ f(U)(s) = 0$

We need only check the surj. once again using 2.4.C.

3.5.B We dualize 3.5.A and use 3.4.M on p69.

3.5.C As we've shown that shs of ab gps on X form an ab cat, the image sh $\operatorname{im}\phi$ for a mor $\phi: \mathcal{F} \to \mathcal{G}$ is defined as $\operatorname{coker}(\ker\phi)$, which is the sheafification of the presh $\operatorname{coker}^{pre}\ker\phi$ $= \operatorname{coker}^{pre}(\ker^{pre}\phi)$, ie the image presh.

$$\begin{aligned}\operatorname{im}(\mathcal{F}\to\mathcal{G})_x &= \operatorname{coker}(\ker(\mathcal{F}\to\mathcal{G})\to\mathcal{F})_x \\ &= \operatorname{coker}(\ker(\mathcal{F}\to\mathcal{G})_x \to \mathcal{F}_x) \\ &= \operatorname{coker}(\ker(\mathcal{F}_x\to\mathcal{G}_x)\to\mathcal{F}_x) \\ &= \operatorname{im}(\mathcal{F}_x\to\mathcal{G}_x)\end{aligned}$$

As a consequence, exactness of a seq of shs may be checked at the level of stalks, ie a seq of shs is exact iff each seq of stalks is exact. In particular, taking the stalk of a sh of ab gps is an exact fcr.

3.5.D ✓

3.5.E $0 \to \mathcal{F} \xrightarrow{\phi} \mathcal{G} \xrightarrow{\psi} \mathcal{H}$

"By hand":

$0 \to \mathcal{F} \to \mathcal{G} \Rightarrow 0 \to \mathcal{F}(U) \to \mathcal{G}(U)$ by 3.4.N.

For the exactness at $\mathcal{G}(U)$:

- The global scn fcr is additive, so that
  $\psi(U)\circ\phi(U) = (\psi\circ\phi)(U) = 0(U) = 0$

  Or, suppose $x = \phi(U)(y) \in \mathcal{G}(U)$. Then, for each $p \in U$, $\psi_p(x_p) = 0$ (by 3.5.D). Thus $\psi(U)(x) = 0$ by 3.4.D.

- Suppose $x \in \mathcal{G}(U)$ st $\psi(U)(x) = 0$. Repeat the argument on $\overline{\text{IIII}}$

"Use the fact that sheafification is an adjoint":

Counterex for right-exactness: 3.4.P on p70.

3.5.F $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H}$ exact

$\overset{3.5.E}{\Rightarrow}$ $0 \to \mathcal{F}(f^{-1}(U)) \to \mathcal{G}(f^{-1}(U)) \to \mathcal{H}(f^{-1}(U))$ exact

$\Rightarrow$ $0 \to f_*\mathcal{F}(U) \to f_*\mathcal{G}(U) \to f_*\mathcal{H}(U)$ exact

filtered colim exact $\Rightarrow$ $0 \to f_*\mathcal{F}_p \to f_*\mathcal{G}_p \to f_*\mathcal{H}_p$ exact

$\Rightarrow$ $0 \to f_*\mathcal{F} \to f_*\mathcal{G} \to f_*\mathcal{H}$ exact.

3.5.G ✓

3.5.H

(a) Categorical def of tensor product of two $\mathcal{O}_X$-modules:

$$\begin{tikzcd}
\mathcal{F} \times \mathcal{G} \arrow[r, "\mathcal{O}_X\text{-bilinear}"] \arrow[dr, "\mathcal{O}_X\text{-bilinear}"'] & \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G} \arrow[d, dashed, "\exists!\ \mathcal{O}_X\text{-linear}"] \\
 & \mathcal{H}
\end{tikzcd}$$

Explicit construction: Take the presh tensor product defined as a quotient of preshs (cf 2.3.2 on p23), and sheafify. The univ property is checked exactly as we did for the sh coker on p71.

(b) ✓

Tue - 9/21

3.6 <u>The inverse image sheaf</u>

<u>3.6.A</u> ✓

The adjunction of f^{-1} and f_*. (also cf π53) <u>3.6.B</u> Let's first assume this adjunction, and look at a special case, where we can actually prove by hand.

<u>3.6.3</u> Consider $f: X = \{p\} \hookrightarrow Y$, and $\mathcal{F} = \underline{S}$. Then

$$\mathrm{Mor}_X(f^{-1}\mathcal{G}, \mathcal{F}) \longleftrightarrow \mathrm{Mor}_Y(\mathcal{G}, f_*\mathcal{F})$$

becomes

$$\mathrm{Mor}_{Sets}(\mathcal{G}_p, S) \longleftrightarrow \mathrm{Mor}_Y(\mathcal{G}, S_p)$$

ie maps of shs on Y from $\mathcal{G}$ to the skyscraper sh S_p w/ set supported at p are just maps of sets from the stalk $\mathcal{G}_p$ to S

<u>By hand</u> $\mathcal{G}(U) \longrightarrow S_p(U) = \begin{cases} S & U \ni p \\ \{*\} & U \not\ni p \end{cases}$

are determined by

$$\begin{tikzcd} \mathcal{G}(U) \arrow[d] \arrow[dr] & \\ \mathcal{G}(V) \arrow[r] & S \end{tikzcd}, \qquad U \supset V \ni p$$

which is equiv to

$$\mathcal{G}_p = \varinjlim_{U \ni p} \mathcal{G}(U) \longrightarrow S.$$

Now, return to the general case: $\mathrm{Mor}_X(f^{-1}\mathcal{G}, \mathcal{F}) \leftrightarrow \mathrm{Mor}_Y(\mathcal{G}, f_*\mathcal{F})$, where $f: X \to Y$ is cts. Given a mor $f^{-1}\mathcal{G} \to \mathcal{F}$, it uniquely (3.4.D) induces mors on stalks $f^{-1}\mathcal{G}_x \to \mathcal{F}_x$ for all $x \in X$.

From the special case, we guess $f^{-1}\mathcal{G}_x = \mathcal{G}_{f(x)}$,

as stalk is really a local notion. Since sheafification induce isom of stalks (3.4.M), we can check explicitly that (cf 3.2.I on π37, where we only have the arrow in one direction $f_* \mathcal{F}_{f(x)} \to \mathcal{F}_x$: a manifestation of the asymmetry of a map $f: X \to Y$)

by a univ property argument

$$f^{-1}\mathcal{G}_x = \varinjlim_{U \ni x} \varinjlim_{V \supset f(U)} \mathcal{G}(V) = \varinjlim_{V \ni f(x)} \mathcal{G}(V) = \mathcal{G}_{f(x)}.$$

Thus we have a map of sets $\mathcal{G}_{f(x)} \to \mathcal{F}_x$ for each $x \in X$. And these suffice to give a mor $\mathcal{G} \to f_* \mathcal{F}$, as $f_* \mathcal{F}_y = \{*\}$ if $y \notin f(X)$ ($\mathcal{F}(\phi) = \{*\}$). Clearly the above process can be traced backward.
(Cf Nick's solution for a pf on the level of open sets.)

3.6.C Proven above.

- $f^{-1}\mathcal{G}_x \cong \mathcal{G}_{f(x)}$ means that the inverse image sh $f^{-1}\mathcal{G}$ really gets its compatible germs from $\mathcal{G}$: for each $x \in X$ there is a unique $f(x) \in Y$.
- $f_* \mathcal{F}_{f(x)} \to \mathcal{F}_x$ means that the direct image sh $f_* \mathcal{F}$ doesn't get its compatible germs from $\mathcal{F}$ at merely one pt in the preimage: for each $y \in Y$, $f^{-1}(y)$ may consist of more than one pt. Do we then have $f_* \mathcal{F}_y \overset{\sim}{\underset{\leftarrow ---}{\longrightarrow}} \prod_{x \in f^{-1}(y)} \mathcal{F}_x$?

3.6.3 (cont)

- 3.4.M (use the "special case") Recall that in 3.4.I on π44, we see the mor $\mathcal{F} \to \mathcal{F}^{sh}$ takes a scn to the corr family of compatible germs, ie $s \mapsto (s_x \in \mathcal{F}_x)_{x \in U}$. Restricted to a nbhd U of p, it is equiv to the mor $\mathcal{F} \to (\mathcal{F}_p)_p$, which is the same as the map of sets $\mathcal{F}_p \to \mathcal{F}_p$. More precisely, the mor defined above corrs to the identity map of $\mathcal{F}_p$.
- 3.4.O (use the "special case" for the harder direction (a) ⇒ (b))

Consider the composite checked by def

$$\mathcal{F} \xrightarrow{epi} \mathcal{G} \xrightarrow{epi} (\mathcal{G}_x)_x$$

which is the same as

$$\mathcal{F}_x \longrightarrow \mathcal{G}_x$$

Now

$$\mathcal{F}_x \longrightarrow \mathcal{G}_x \rightrightarrows S$$

translates to

$$\mathcal{F} \xrightarrow{epi} (\mathcal{G}_x)_x \rightrightarrows S_x .$$

(All these good things happen b/c the skyscraper sh is so special that it can really be used to connect sheaves and stalks.)

<u>3.6.D</u>

"Explicit description": $\quad i^{-1}\mathcal{G}(V) = \varinjlim_{U \supset V} \mathcal{G}(U) = \mathcal{G}(V)$

"universal property": $\quad \mathrm{Mor}_U(\mathcal{G}|_U, \mathcal{F}) \overset{?}{\leftrightarrow} \mathrm{Mor}_Y(\mathcal{G}, \underline{i_* \mathcal{F}})$

extension by 0

More generally, we can ask: What is $f^{-1}\mathcal{G}$ if $f: X \to Y$ is an open map?

$$f^{-1}\mathcal{G}(V) = \varinjlim_{U \supset f(V)} \mathcal{G}(U) = \mathcal{G}(f(V))$$

This is sth more symmetric to

$$f_* \mathcal{F}(W) = \mathcal{F}(f^{-1}(W)).$$

The pt is that openness of the map f sort of excludes the asymmetry of f: If the map takes several pts to one, it is not likely to be open.

<u>3.6.E</u> Cf 3.6.3

Fri - 9/24

More thoughts on the adjunction of f^{-1} and f_*

Given $f: X \to Y$
$$x \mapsto y$$
we have $\mathrm{Mor}_X(f^{-1}\mathcal{G}, \mathcal{F}) \leftrightarrow \mathrm{Mor}_Y(\mathcal{G}, f_*\mathcal{F})$.

Moreover, in terms of stalks, we have

$f^{-1}\mathcal{G}_x \cong \mathcal{G}_y$ (3.6.C)

$f_*\mathcal{F}_y \to \mathcal{F}_x$ (3.2.I)

We ask: - When do we have an isom $f_*\mathcal{F}_y \cong \mathcal{F}_x$? Under certain assumptions on f (~~open~~, inj)?

(This is inspired by 3.6.D. Certainly there are cts ma[ps] open but not inj, eg covering spaces.)

- In general, what is $f_*\mathcal{F}_y$ isom to? $\prod_{x \in f^{-1}(y)} \mathcal{F}_x$?

3.6.F $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ exact in Ab_Y

$\Rightarrow 0 \to \mathcal{F}_y \to \mathcal{G}_y \to \mathcal{H}_y \to 0$ exact $\forall y \in Y$

$\overset{3.6.C}{\Rightarrow} 0 \to (f^{-1}\mathcal{F})_x \to (f^{-1}\mathcal{G})_x \to (f^{-1}\mathcal{H})_x \to 0$ exact $\forall x \in X$

$\Rightarrow 0 \to f^{-1}\mathcal{F} \to f^{-1}\mathcal{G} \to f^{-1}\mathcal{H} \to 0$ exact in Ab_X.

As is remarked, the right-exactness of f^{-1} follows from f^{-1} being a left adjoint, and the left-exactness, which is equiv to that on the level of open sets, follows from f^{-1} being defined as a filtered colimit.

By the way, by the stalkwise argument above, it is also easily seen that sheafification is exact.

3.6.G

(a) Z is closed in Y.

$$(i_* \mathcal{F})_y = \varinjlim_{V \ni y} \mathcal{F}(i^{-1}(V)) = \begin{cases} \mathcal{F}(\phi) = \{*\} & y \notin Z \quad \text{(need closedness)} \\ \mathcal{F}_y & y \in Z \end{cases}$$

(b) For $y \in Z$,

$$\mathcal{F}_y \xrightarrow{?} (i_* i^{-1} \mathcal{F})_y \xrightarrow{\sim} (i^{-1}\mathcal{F})_y \cong \mathcal{F}_y$$

cf π53.
This is an ex where we don't have an open map but still get an isom!

precisely the process in the adjunction (cf π50)

so $? = \sim$

For $y \notin Z$,

$$\begin{array}{ccc} \mathcal{F}_y & \longrightarrow & (i_* i^{-1}\mathcal{F})_y \\ \text{Supp}\,\mathcal{F} \subset Z \,\| & & \| \text{ part(a)} \\ \{*\} & & \{*\} \end{array}$$

cf 1.19 on p68 of [ag]. 3.6.H

(a) $$(i_! \mathcal{F})_y = \varinjlim_{V \ni y} i_! \mathcal{F}(V) = \begin{cases} \mathcal{F}_y & y \in U \quad \text{(need openness)} \\ 0 & y \notin U \end{cases}$$

Note Compare 3.6.G(a) and 3.6.H(a). It seems that the defs are different for "extending" a closed set and "extending" an open set but the effects are the same on stalks. Properties determined at the level of stalks are really intrinsic to shs

(b) Check stalks.

(c) Left exactness can be checked on open sets.

$$i_! i^{-1} \mathcal{F}(W) = \begin{cases} i^{-1}\mathcal{F}(W) = \mathcal{F}(i^{-1}(W)) = \mathcal{F}(W), & W \subset U \\ 0 & \text{o/w} \end{cases}$$

$$\hookrightarrow \mathcal{F}(W).$$

(d) ✓

3.7 Recovering sheaves from a "sheaf on a base"

3.7.A

"we can determine the stalks": $\mathcal{F}_x = \varinjlim_{U \ni x} \mathcal{F}(U) \underset{\substack{\uparrow \\ \text{upa}}}{=} \varinjlim_{B \ni x} \mathcal{F}(B)$

"we can determine when germs are compatible": Recall the def of compatible germs:

$$\{ (f_x \in \mathcal{F}_x)_{x \in U} \mid \forall x \in U,\ \exists x \in V \subset U \text{ and } s \in \mathcal{F}(V) \text{ st } f_y = s_y \ \forall y \in V$$

$$\Updownarrow$$

$$\{ (f_x \in \mathcal{F}_x)_{x \in U} \mid \underline{\forall x \in U},\ \exists x \in B \subset U \text{ and } s \in \mathcal{F}(B) \text{ st } f_y = s_y \ \forall y \in B$$

In fact, $\forall x \in U$ must be contained in some B'. This is why we can take "the sh of compatible germs $\underline{\text{of } F}$" in thm 3.7.1.

3.7.1 Define $\mathcal{F}(U) := \{ (f_x \in F_x)_{x \in U} \mid \forall x \in B' \subset U,\ \exists x \in B \subset U$ and $s \in F(B$ st $f_y = s_y \ \forall y \in B \}$ (cf 3.4.6 on p69). This is a sh (cf 3.4.H on p69) w/ isoms $\mathcal{F}(B_i) \cong F(B_i)$ agreeing w/ the restriction maps (cf 3.4.B on p67 or 3.4.G on p69). For the uniqueness of such $\mathcal{F}$ extending F, we would have defined $\mathcal{F}$ by a univ property

$$\begin{tikzcd} F \arrow[r, "\phi(B)"] \arrow[dr, "\psi(B)"'] & \mathcal{F} \arrow[d, dashed, "\exists!\, \eta(U)"] \\ & \mathcal{G} \end{tikzcd}$$

and check that the $\mathcal{F}$ construted above satisfies this univ property. Then it boils down to the facts that mors of shs are determined by stalks and that we can determine the stalks as well as when germs are compatible (so that mors on stalks glue) once we have a sh on a base (cf 3.7.A on p75).

3.7.B See above. (The pt is probably that we have an isom on stalks; see p287.)

3.7.C

(a). 3.4.D on p68.

(b) Such a (coherent) mor descends to give a mor on compatible germs, where we defined our induced shs. More precisely, the compatibility of the diag gives rise to a mor $F_p \to G_p$ for each $p \in X$. Thus by def in 3.7.1, we have $\mathcal{F}(U) \to \mathcal{G}(U)$ for all $U \subset X$, which are trivially compatible w/ restrictions.

3.7.D

We first construct a sh F on a base using the info of $\mathcal{F}_i$ on U_i, and then extend it to a sh $\mathcal{F}$ on X. Take the base on X to consist of those open sets contained in some U_i. By def on p75, the $\mathcal{F}_i$'s give rise to a sh F on the base, which extends uniquely to a sh $\mathcal{F}$ on X st $\mathcal{F}_i = \mathcal{F}|_{U_i}$.

(the compatibility on triple overlaps guarantees that F is well defined for $B \subset U_i \cap U_j$, there are two possibilities for F(B); it also implies that ϕ_{ij} and ϕ_{ji} are inverses)

3.7.E ✓ The usefulness of this criterion is that although it is still stronger than the surj on stalks (intrinsic, equiv to the mor of shs being an epim), it is weaker than the surj on all open sets.

4.1 Toward schemes

4.1.A Locally, given $\phi \in \mathcal{O}_Y(V)$, $f^\# \phi(\underline{x}) := \phi(f(\underline{x}))$, $\underline{x} \in U$.
Taking $\phi(\underline{y}) = \underline{y}$ in par, we see the differentiability of $f^\# \phi$ is the same as that of f.

4.1.B The above map $f^\#$ descends to stalks as it is obviously compatible w/ restrictions.

To see that $f^\#(\mathfrak{m}_{Y,q}) \subset \mathfrak{m}_{X,p}$, recall that the maximal ideal consists of (germs of) fcns vanishing at the pt. As $\mathcal{O}_{X,p}$ is a local ring w/ maximal ideal $\mathfrak{m}_{X,p}$, we see that $f^\#$ is a mor of local rings (cf "local homom" on p78 of the red book).

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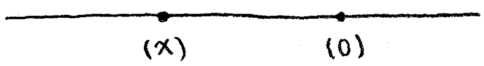
4.2 The underlying set of affine schemes

4.2.A

Spec$(k[\varepsilon]/(\varepsilon^2))$ (a) Let p be a prime ideal of $k[\varepsilon]/(\varepsilon^2)$. Then p must contain an elt of the form $a\varepsilon + b$ w/ $a \neq 0$. Thus $\varepsilon \in \mathfrak{p}$. Let $\alpha\varepsilon + \beta$ be any elt in $\mathfrak{p}$. Then $\beta = 0$; o/w $\mathfrak{p}$ would contain some nonzero elt in k. Thus $\mathfrak{p} = (\varepsilon)$. Since $(k[\varepsilon]/(\varepsilon^2))/(\varepsilon) \cong k$, $\mathfrak{p}$ is a maximal ideal of $k[\varepsilon]/(\varepsilon^2)$. Hence the set Spec $k[\varepsilon]/(\varepsilon^2)$ consists of a single pt $[(\varepsilon)]$. Note that ε is a nonzero fcn whose value at the pt $[(\varepsilon)]$ is zero. Cf ex I on p76 of the red book.

(b) $k[x]_{(x)} = \left\{ \frac{f}{g} \;\middle|\; f, g \in k[x],\ g(0) \neq 0 \right\}$. The prime ideals are (0) and (x). (This is a DVR, w/ $[(0)]$ the open pt and $[(x)]$ the closed pt. Cf p89)



4.2.B The canonical isom is defined as

$$\mathbb{R}[x]/(x^2+ax+b) \longrightarrow \mathbb{C}$$

$$\alpha x+\beta+(x^2+ax+b) \longmapsto \alpha t+\beta$$

where t is a cpx root of $x^2+ax+b=0$.

4.2.C $\mathbb{A}^1_{\mathbb{Q}} = \operatorname{Spec} \mathbb{Q}[x]$

still a field

There are irred polys of higher deg.
The quotient ring may not be a field.
Again a cpx plane, folded along the real axis w/ rational pts scattered on it, and the gluing being more complicated, including gluing irrational pts on the real axis.

4.2.D Follow the hint. If $\mathfrak{p}$ is a principal prime ideal, then it must be generated by an irred poly. Suppose $\mathfrak{p}$ is not principal. Then there must exist $f(x,y)$ and $g(x,y) \in \mathfrak{p}$ w/ no common factor. Since $k(x)[y]$ is a Euclidean domain, we can apply the division algorithm and obtain a nonzero $h(x) \in (f,g) \subset \mathfrak{p}$. Since $\mathfrak{p}$ is prime, one of the linear factors of h $x-a$ must be contained in $\mathfrak{p}$. Similarly, $y-b \in \mathfrak{p}$ for some b. Thus $(x-a, y-b) \subset \mathfrak{p}$. But we've seen that $(x-a, y-b)$ is maximal (essentially for dimension reason), so $\mathfrak{p} = (x-a, y-b)$.

4.2.E The maximal ideal corr $(\sqrt{2}, \sqrt{2})$ and $(-\sqrt{2}, -\sqrt{2})$ is $(x^2-2, x-y)$

Consider $\mathbb{Q}[x,y] \longrightarrow \mathbb{R}$

$f(x,y) \longmapsto f(\sqrt{2}, \sqrt{2})$

We have $\mathbb{Q}[x,y] / (x^2-2, x-y) \cong \mathbb{R}$.

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prime ideals under quotients and localizations

4.2.F For any prime $P \subset A/I$, $\phi^{-1}P$ can be easily checked as a prime ideal containing I. The corr is clearly inclusion-preserving and inj. For surj, suppose Q is a prime ideal containing I; then $Q = \phi^{-1}(Q/I)$. Clearly Q/I is an ideal. It remains to show that Q/I is prime, but $(A/I)/(Q/I) \cong A/Q$ is a domain.

4.2.G Consider the nat map $\psi: A \to S^{-1}A$ (cf p23). For any prime $P \subset S^{-1}A$, $\varphi^{-1}P$ can be easily checked as a prime ideal not meeting S. (The corr is clearly inclusion-preserving and inj.) For surj, suppose Q is a prime ideal not meeting S; Then $Q \cdot S^{-1}A$ can be easily checked as a prime ideal st $\varphi^{-1}(Q \cdot S^{-1}A) = Q$.

quotient vs localization

Note Compare Q/I and $Q \cdot S^{-1}A$. In general, the bijection in 4.2.F is true for ideals not necess prime, whereas it is not so for 4.2.G. Cf prop 6.1 on p320 and cor 11.18 on p909 of [alg], 4.4.H on Π65- 5.5.G on Π89 and 8.2.B on Π135-1

4.2.H

One feature of quotient which localization doesn't have is surjectivity

Approach 1 Explicit isom: $(k[x,y]/(xy))_x \longrightarrow k[x]_x$

$$x \longmapsto x$$
$$y \longmapsto 0$$
$$\frac{1}{x} \longmapsto \frac{1}{x}$$

Approach 2 Univ property of localization:

$$\begin{tikzcd} k[x] \arrow[r] \arrow[dr] & k[x]_x \arrow[d, dashed, "\exists!"] \\ & B \end{tikzcd}$$

every elt x^n is sent to a unit in B

Check that $(k[x,y]/(xy))_x$ satisfies this.

4.2.I ✓ Moreover, $\mathfrak{p}$ maximal in $A \not\Rightarrow \phi^{-1}(\mathfrak{p})$ maximal in B. cf Wed-9/29-1

4.2.J ✓

$[(y-\sqrt{a})], [(y+\sqrt{a})] \longmapsto [(x-a$

4.2.K map of cpx mflds: $\mathbb{C} \longrightarrow \mathbb{C}$, $c \longmapsto c^2$, $\pm\sqrt{a} \longmapsto a$

map of spec: $\operatorname{Spec}\mathbb{C}[y] \longrightarrow \operatorname{Spec}\mathbb{C}[x]$, $[(y-c)] \longmapsto [(x-c^2)]$,

map of rings: $\mathbb{C}[y] \leftarrow \mathbb{C}[x]$ $\quad y^2 \longleftarrow x$

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For the fiber $f^{-1}([(0)])$, it consists of prime ideals whose preimage under $\mathbb{Z} \hookrightarrow \mathbb{Z}[x_1, \cdots, x_n]$ is (0). Such prime ideals are precisely those not meeting the multiplicative set S of nonzero integers. Thus they are in bijection w/ pts of $\operatorname{Spec} S^{-1}\mathbb{Z}[x_1, \cdots, x_n] \cong \operatorname{Spec} \mathbb{A}^n_{\mathbb{Q}}$. (We have similar pictures for general rings other than $\mathbb{Z}$.)

4.2.L

(a) Any pt in $\operatorname{Spec} \mathbb{C}[x_1, \cdots, x_m]/I$ can be identified as a prime ideal $\mathfrak{p}$ st $I \subset \mathfrak{p} \subset \mathbb{C}[x_1, \cdots, x_m]$. Since $\phi(J) \subset I$, $\phi^{-1}(\mathfrak{p})$ is a prime ideal st $J \subset \phi^{-1}\phi(J) \subset \phi^{-1}(I) \subset \phi^{-1}(\mathfrak{p}) \subset \mathbb{C}[y_1, \cdots, y_n]$, ie a pt in $\operatorname{Spec} \mathbb{C}[y_1, \cdots, y_n]/J$.

(b) In par, the maximal ideal

$$(X_1 - x_1, \cdots, X_m - x_m)$$

maps to the maximal ideal

$$(Y_1 - f_1(x_1, \cdots, x_m), \cdots, Y_n - f_n(x_1, \cdots, x_m))$$

4.2.M A pt in the fiber $f^{-1}([(p)])$ corrs to a prime ideal of $\mathbb{Z}[x_1, \cdots, x_n]$ whose preimage under $\mathbb{Z} \hookrightarrow \mathbb{Z}[x_1, \cdots, x_n]$ is (p). We can check that such prime ideals are precisely those containing (p). Thus they are in bijection w/ pts of $\operatorname{Spec} \mathbb{Z}[x_1, \cdots, x_n]/(p) \cong \operatorname{Spec} \mathbb{A}^n_{\mathbb{F}_p}$. (In the case of $n=1$, the maximal ideals of $\mathbb{Z}[X]$ are (p, f), where f is a monic integral poly irred modulo p (cf (iii) in ex H on pp 74-5 of the red book); such an f corrs to a closed pt in $\mathbb{A}^1_{\mathbb{F}_p}$ (cf ex 6 on p 86), and moreover, in view of (p) in (p, f), the induced corr between the (p, f)'s and the closed pts in $\mathbb{A}^1_{\mathbb{F}_p}$ is a bijection.)

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4.2.N

(a) We've seen that $\operatorname{Spec} B/I \to \operatorname{Spec} B$ is a bijection between primes in B/I and primes in B containing I. It suffices to show that any primes in B contains I, which is clear.

(b) ✓

4.2.O We want to show that $\mathfrak{N}(A) \supset \cap \mathfrak{p}_i$. It suffices to show that if $x \notin \mathfrak{N}(A)$, ie $x^m \neq 0$ for all m, then $\exists\, \mathfrak{p}_i \not\ni x$. Consider the localization $\varphi: A \to A_x$. One easily checks that $1 \neq 0$ in A_x, ie A_x is not the 0-ring, and thus Spec A_x is not empty (by the axiom of choice, (0) is a proper ideal of A_x, so it must be contained in a maximal ideal), whose pts are prime ideals not containing x. Let $\mathfrak{p}$ be one of them. Then $\varphi^{-1}\mathfrak{p}$ is a prime ideal in A not containing x.

4.2.P
$$f(x+\varepsilon) = f(x) + f'(x)\,\varepsilon.$$
$$= f(x) + f'(x)\,\varepsilon + f''(x)\,\varepsilon^2 + \cdots$$

Summary 4.3 Visualizing schemes I: generic points

- Visualizing pts:

	ALG	GEOM	EX
traditional (or classical)	$\mathbb{A}^n_k$ w/ k alg closed Hilbert's weak Nullstellensatz		$\mathbb{A}^1_{\mathbb{C}}$
	$\mathbb{A}^n_k$ w/ k not alg closed Hilbert's Nullstellensatz	traditional pts are glued together into clumps by Galois conjugation	$\mathbb{A}^1_{\mathbb{R}}$, $\mathbb{A}^1_{\mathbb{F}_p}$
additional	irred closed alg subsets	irred curves	$\mathbb{A}^2_{\mathbb{C}}$
generic		almost everywhere. "near" every other pt	

- Visualizing subsets (eg affine vrts)

ALG	GEOM	EX
Spec A/I: prime ideals of A containing I	things cut out by equations involving a finite number of variables over an alg closed field	Spec $\mathbb{C}[x,y]/(x+y)$, a line contained in the xy-plane
Spec $S^{-1}A$: prime ideals of A not meeting S		
- Spec A_f: primes not containing f	pts on which the fcn f doesn't vanish (distinguished open set)	$(k[x,y]/(xy))_x \cong k[x]_x$
- Spec $A_{\mathfrak{p}}$: primes contained in $\mathfrak{p}$	a shred of the space Spec A near the subset corr to $\mathfrak{p}$	- Spec $k[x,y]_{(x,y)}$ - Spec $k[x]_{(x)}$, the "germ of a smooth curve", where one pt is the "classical pt", and the other is the "generic pt of the curve"

4.4 The Zariski topology: The underlying topological space of an affine scheme

4.4.A In terms of alg, the ideals $(y,z) \supset \{xy, yz\}$
$\updownarrow$
the x-axis

In terms of geom, both fcns xy and yz vanish over the $\underbrace{x\text{-axis } y=z=0}_{\text{cor at the pt } [(y,z)]}$

4.4.B Recall that $V(S) := \{[\mathfrak{p}] \in \operatorname{Spec} A \mid S \subset \mathfrak{p}\}$

"vanishing at a pt" = "contained in a prime"

We need to show that $S \subset \mathfrak{p} \Leftrightarrow (S) \subset \mathfrak{p}$. ✓

4.4.C

(a) $\operatorname{Spec} A = V(0)$, $\phi = V(1) = V(A)$ ($\Sigma_i I_i$ is the ideal generated by $\bigcup_i I$)

(b) Both sides describe the pts where all elts in all I_i vanish.

Formally, $I_i \subset \Sigma_i I_i \quad \forall i$

$\Rightarrow V(I_i) \supset V(\Sigma_i I_i) \quad \forall i$

$\Rightarrow \bigcap_i V(I_i) \supset V(\Sigma_i I_i)$

On the other hand, suppose $[\mathfrak{p}] \in \bigcap_i V(I_i)$, ie $I_i \subset \mathfrak{p}, \forall i$.

Then $\Sigma I_i = \{\sum_{i=1}^{n} a_i \mid a_i \in I_i, n \in \mathbb{N}\} \subset \mathfrak{p}$.

(c) Both sides describe the pts where either all elts in I_1 or all elts in I_2 vanish (less obvious for the RHS: if $f_1 \in I_1$ and $f_2 \in I_2$ don't vanish at $[\mathfrak{p}]$, then $f_1 f_2 \in I_1 I_2$ doesn't vanish at $[\mathfrak{p}]$). Thus the finiteness sort of comes from the defining property of any prime ideal.

Formally, $I_i \supset I_1 I_2 \quad i = 1, 2$

$\Rightarrow V(I_i) \subset V(I_1 I_2) \quad i = 1, 2$

$\Rightarrow V(I_1) \cup V(I_2) \subset V(I_1 I_2)$

The other inclusion can be checked by hand using the above description.

4.4.D We need to show $V(\sqrt{I}) \supset V(I)$. This is by primality: Suppose $r^n \in I$ is contained in $\mathfrak{p}$; then $r \in \mathfrak{p}$.

(Note that in the def of $\sqrt{I}$, $r^0 = 1$, and we can equivly just define using $n \in \mathbb{Z}^{>0}$.)

In par, prime ideals are radical.

<u>Cor</u> $(I\cap J)^2 \subset IJ \subset I\cap J$

$$\Rightarrow V((I\cap J)^2) \supset V(IJ) \supset V(I\cap J)$$

$$\begin{array}{ccc} V((I\cap J)^2) & & V(I\cap J) \\ \| & & \| \\ V(\sqrt{(I\cap J)^2}) & \subset & V(\sqrt{I\cap J}) \\ & \nearrow\!\!\!\nearrow & \\ \sqrt{(I\cap J)^2} & \supset & \sqrt{I\cap J} \end{array}$$

This clearly generalizes to more factors.

<u>4.4.E</u> ✓ Once one writes down the pf, one sees where the finiteness of the index set is used.

<u>Nonex</u> $I_i = (2i)$. Then $\sqrt{\bigcap_{i=1}^\infty I_i} = (0)$ whereas $\bigcap_{i=1}^\infty \sqrt{I_i} = (2)$.

Note that we always have $\sqrt{\bigcap_i I_i} \subset \bigcap_i \sqrt{I_i}$.

<u>4.4.F</u> Note that 4.2.10 can be viewed as a special case when $I=(0)$. It turns out that we can show the general case by reducing it to that. In fact, consider $\phi: A \longrightarrow A/I$. Then under ϕ^{-1}, $\mathfrak{p}_i \subset A/I$ are in bijection w/ $I \subset \mathfrak{q}_i \subset A$. Note that $\sqrt{I} = \phi^{-1}(\mathfrak{N}(A/I))$, and thus $\sqrt{I} = \phi^{-1}(\mathfrak{N}(A/I)) = \phi^{-1}(\bigcap_{\mathfrak{p}_i \subset A/I} \mathfrak{p}_i) = \bigcap_{I\subset \mathfrak{q}_i \subset A} \mathfrak{q}_i$.

<u>4.4.G</u> For any closed set $V(J) \subset \operatorname{Spec} B$, where J is an ideal in B and $V(J) = \{[\mathfrak{q}] \in \operatorname{Spec} B \mid \mathfrak{q} \supset J\}$, $\pi^{-1}V(J) = \{[\mathfrak{p}] \in \operatorname{Spec} A \mid \phi^{-1}\mathfrak{p} \supset J\} = \{[\mathfrak{p}] \in \operatorname{Spec} A \mid \mathfrak{p} \supset \phi(J)\} = V(\phi(J))$.

<u>4.4.H</u> $\operatorname{Spec} B/I \cong \{[\mathfrak{p}] \in \operatorname{Spec} B \mid \mathfrak{p} \supset I\} = V(I)$

To show that the Zariski top on $\operatorname{Spec} B/I$ (resp $\operatorname{Spec} S^{-1}B$) is the subspace topology induced by inclusion into $\operatorname{Spec} B$, we need to show that the closed subsets in Spec B/I (resp Spec $S^{-1}B$) are precisely the intersections of closed subsets in Spec B w/ Spec B/I (resp Spec $S^{-1}B$). In fact, any closed subset in Spec B/I is of the form

$$\begin{aligned} V(J) &= \{ [q] \in \operatorname{Spec} B/I \mid q \supset J \} \\ &\cong \{ [p] \in \operatorname{Spec} B \mid p \supset I,\ p \supset \phi^{-1} J \} \quad \text{need } \phi^{-1} J \supset I \\ &= \operatorname{Spec} B/I \cap V(\phi^{-1} J) \end{aligned}$$

where ϕ is the nat map $B \to B/I$, and any closed subset in Spec $S^{-1}B$ is of the form

$$\begin{aligned} V(J) &= \{ [q] \in \operatorname{Spec} S^{-1}B \mid q \supset J \} \\ &\cong \{ [p] \in \operatorname{Spec} B \mid p \cap S = \phi,\ p \supset \psi^{-1} J \} \quad \text{need } \psi^{-1} J \cap S = \phi \\ &= \operatorname{Spec} S^{-1}B \cap V(\psi^{-1} J) \end{aligned}$$

where ψ is the nat map $B \to S^{-1}B$.

In par, if $I \subset \mathfrak{N}$ is an ideal of nilps, then the bijection Spec $B/I \to$ Spec B is a homeom. Thus nilps don't affect the topological space. The difference will be in the str sh.
Note It isn't true that Spec $S^{-1}B$ is always an open set - it's true for ex if S is $\{1, f, f^2, \cdots\}$, but isn't true if $B = k[x]$ and $S = B - (x)$.

4.4.I f vanishes on $V(I)$

$$\begin{aligned} &\iff f \in p, \text{ for all } p \supset I \\ &\iff f \in \bigcap_{p \supset I} p = \sqrt{I}. \end{aligned}$$

4.4.J On π58, we've seen that as a set Spec $k[x]_{(x)}$ consists of two pts $[(0)]$ and $[(x)]$ (as a shred of the space $\mathbb{A}^1_k$ near the pt $[(x)]$, cf π63). Now $\{[(x)]\} = V(x)$, and thus $[(x)]$ is a closed pt. Hence $[(0)]$ is an open pt; it is not closed since there is no $S \subset k[x]_{(x)}$ st $\{[(0)]\} = V(S)$.

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4.5 A base of the Zariski topology on Spec A: Distinguished open sets

4.5.A It suffices to show that given $S \subset A$, $\operatorname{Spec} A - V(S) = \bigcup_{f \in S} D(f)$

Indeed, $\operatorname{Spec} A - V(S) = \{[\mathfrak{p}] \in \operatorname{Spec} A \mid \mathfrak{p} \not\supset S\}$

$$= \{[\mathfrak{p}] \in \operatorname{Spec} A \mid \exists f \in S \text{ st } f \notin \mathfrak{p}\}$$

$$= \bigcup_{f \in S} D(f).$$

4.5.B $\operatorname{Spec} A = \bigcup_{i \in J} D(f_i) = \operatorname{Spec} A - V(\{f_i\}_{i \in J})$

$\iff V(\{f_i\}_{i \in J}) = \phi$

$\iff A = (\{f_i\}_{i \in J}) = \sum_{i \in J} (f_i)$

$\iff \exists a_i \in A$ for all $i \in J$, all but finitely many 0, st

$\sum_{i \in J} a_i f_i = 1$ ("partition of unity". cf p357).

4.5.C Use the second equiv and very useful condition in 4.5.B.

✓ Note The quasicptness of Spec A is true for any A; that of any open subset is true when A is ntl (but not necess) (cf 4.6.K and L on p100).

4.5.D In 4.4.C(c) we've shown that $V(I_1) \cup V(I_2) = V(I_1 I_2)$, in par. $V(f) \cup V(g) = V(fg)$. The result follows by taking complement.

4.5.E $D(f) \subset D(g)$

$\iff V(f) \supset V(g)$

$\iff f$ vanishes on $V(g)$

$\overset{4.4.I}{\iff} f \in \sqrt{(g)}$

$\iff f^n \in (g)$ for some n

$\Leftrightarrow g$ is a unit in A_f.

<u>Note</u> It's helpful to use the geom interpretation on p97 and think over 4.2.10, 4.2.11, 4.4.I and 4.5.E.

<u>4.5.F</u> $D(f) \subset \operatorname{Spec} A$. $D(f) = \phi$ means there's nowhere f doesn't vanish, ie f vanishes everywhere, so $f \in \mathfrak{N}$.

4.6 <u>Topological definitions</u>

<u>4.6.A</u> Suppose $U \subset X$ is a nonempty open set. If $\bar{U} \neq X$, then $X = \bar{U} \cup \overline{(X-\bar{U})}$. But $U \not\subset \overline{(X-\bar{U})}$, contra.

<u>4.6.B</u> $\mathfrak{p}$ is maximal

$\Leftrightarrow \{[\mathfrak{p}]\} = V(\mathfrak{p}) = \overline{\{[\mathfrak{p}]\}}$

$\Leftrightarrow [\mathfrak{p}]$ is closed.

<u>4.6.C</u>

(a) ✓

(b) A is the prototypical ex of a non-nt ring!

$\operatorname{Spec} A - V(\mathfrak{m}) = \bigcup_{i=1}^{\infty} D(x_i)$

<u>4.6.D</u>

(a) ✓

(b) ✓ Think of this as $[0,1] - \{0\} = (0,1] = \bigcup_{n=1}^{\infty} (\frac{1}{n}, 1]$.

<u>4.6.2</u> $\overline{\{[(y-x^2)]\}} = V((y-x^2)) = \{[(y-x^2)], [(x-a, y-b)] \text{ w/ } b=a^2\}$

$[(y-x^2)]$ is the generic pt of the irred curve $V((y-x^2))$
$= \operatorname{Spec} \mathbb{C}[x,y]/(y-x^2)$

<u>4.6.E</u> $[\mathfrak{p}] \in \overline{\{[\mathfrak{q}]\}} = V(\mathfrak{q})$

$\Leftrightarrow \mathfrak{p} \supset \mathfrak{q}$ ie $\mathfrak{q}$ vanishes at $[\mathfrak{p}]$ (cf the ex in 4.6.2).

<u>4.6.F</u> ✓

<u>4.6.G</u>

(a) Let's first show that $V(I)$ irred $\overset{\Leftarrow}{\underset{\Rightarrow?}{}}$ I prime.

"$\Leftarrow$": Suppose $V(I) = V(I_1) \cup V(I_2) = V(I_1 I_2)$.

Claim that either $I_1 \subset I$ or $I_2 \subset I$ and hence $V(I)$ is irred.

O/w, $\exists\, i_1 \in I_1 - I,\ i_2 \in I_2 - I$.

Since $i_1 i_2 \in I_1 I_2$ and $[I] \in V(I) = V(I_1 I_2)$, $i_1 i_2 \in I$, contra.

"$\Rightarrow$?": Suppose $ab \in I$.

Then $V(I) \subset V(ab) = V(a) \cup V(b)$,

and thus $V(I) = (V(I) \cap V(a)) \cup (V(I) \cap V(b))$.

Assume $V(I) = V(I) \cap V(a)$.

Then $V(a) \supset V(I)$, and thus a vanishes on $V(I)$, ie $a \in \sqrt{I}$.

Thus actually we can only show that I is <u>primary</u>.

We need some reducedness condition to have a genuine prime idea

Back to Spec $k[w,x,y,z]/I$. It suffices to show that I is prime. Let's show that the quotient ring is a domain by showing that it is isom to the subring of $k[a,b]$ generated by monomials of deg divisible by 3. Consider the ring homom:

$$k[w,x,y,z] \longrightarrow k[a,b]$$

$$\begin{aligned} w &\longmapsto a^3 \\ x &\longmapsto a^2b \\ y &\longmapsto ab^2 \\ z &\longmapsto b^3. \end{aligned}$$

(cf p10 of the red book).

(b) $\begin{cases} wz - xy = 0 \\ wy - x^2 = 0 \\ xz - y^2 = 0 \end{cases} \iff \operatorname{rank}\begin{pmatrix} w & x & y \\ x & y & z \end{pmatrix} \leq 1.$

4.6.H ✓

4.6.4

- Being nt is preserved under quotients and localizations.
  The corr in 4.2.F (cf π59) holds for general ideals not necess prime, whereas that in 4.2.G (cf ππ59-60) doesn't. However, we can still lift ideals in $S^{-1}A$ injectively to ideals in A, and thus prove the nt property.

4.6.I

"⇒": Take the strictly ascending chain inside the non-fg ideal

"⇐": Take the ideal $\bigcup_{i=1}^{\infty} I_i$.

A is noetherian ⇒ A[[X]] is noetherian

4.6.J First of all, to see that $A[[X]] = \varprojlim A[X]/X^n$, consider the inverse system $\cdots \longrightarrow A[X]/X^{n+1} \longrightarrow A[X]/X^n \longrightarrow \cdots \longrightarrow A[X]/X^2 \longrightarrow A[X]/X \cong A$, where the maps are projections to lower terms. Note that we cannot write it as a colimit (union) of subsets of polys of deg $\leq n$, as these are not rings.

Suppose $I \subset A[\![x]\!]$ is an ideal. Let's show that I is fg. Let $I_n \subset$ be the ideal of the initial coeffs in deg n that appear in the elts of I. Since A is nt, the ascending chain $I_1 \subset I_2 \subset \cdots$ stabilizes, say at I_r so that $I_r = I_{r+1} = \cdots$. Moreover, each I_n, $n = 1, \cdots, r$, is fg, the gens corr to a finite set of power series in I. We claim that they gen the entire I. In fact, given an arbitrary power series in I starting in deg n, say $n \leq r$, we can first use the gens of deg n to kill the deg n term, and then inductively kill the higher terms up to deg r. For $n > r$, we use the gens of deg r times x^{n-r} to kill the deg n term, and inductively kill all the higher terms. In doing this, the coeffs of the gens would be power series rather than polys, as we may work thru infinitely many steps.

Note that the "dual" version of the above gives another pf of the Hilbert basis thm.

Also note that $A[\![x]\!]$ (and $A[x]$) is an ex of being nt but not at ($(x) \supset (x^2) \supset \cdots$); recall that at $\Rightarrow$ nt.

The nonex of a nt ring is $A[x_1, x_2, \cdots]$, not $A[\![x]\!]$.

4.6.K A strictly descending chain of infinite length of closed subsets of Spec A, $V(I_1) \supset V(I_2) \supset \cdots$, would give rise to a strictly ascending chain of infinite length of ideals of A, $I_1 \subset I_1 + I_2 \subset \cdots$.

$A = k[x_1, x_2, \cdots]$.

(For the "aside", cf Sat-10/2-1.)

4.6.L We know that Spec A must be nt as a topological space. Suppose $U = \bigcup U_i$ is an open subset which is not quasicpt. Then the descending chain of closed subsets

$$\operatorname{Spec} A - U_1 \supset \operatorname{Spec} A - U_1 \cup U_2 \supset \cdots$$

would never stabilize.

4.6.M We have a bijection between irred closed subsets of Spec A and prime ideals of A (cf. 4.7.E on p103). Thus

$V(\mathfrak{p})$ is maximal

$\Leftrightarrow$ $V(\mathfrak{q}) \supset V(\mathfrak{p}) \Rightarrow V(\mathfrak{q}) = V(\mathfrak{p})$

$\mathfrak{p}, \mathfrak{q}$ are prime

$\Leftrightarrow$ $\mathfrak{q} \subset \mathfrak{p} \Rightarrow \mathfrak{q} = \mathfrak{p}$

$\Leftrightarrow$ $\mathfrak{p}$ is minimal.

4.6.N ✓ An integral domain has only one minimal prime ideal (0). $k[\varepsilon]/(\varepsilon^2)$ is not an integral domain but has only one minimal prime ideal (ε) (cf 4.2.A1 on p57)!

4.6.O Let's use geom to solve an alg problem! Consider Spec $k[x,y]/(xy)$. Corr to the minimal primes, the irred comps are the x-axis and the y-axis. Thus the minima primes are $k[x]/(xy)$ and $k[y]/(xy)$. Check algebraically ($(x-a)/(xy)$ is no longer prime: $(x-a)y = -ay$)

4.6.P If it were not conn, then $X = U \cup V$, U, V nonempty open, $U \cap V = \phi$. Thus $X = U^c \cup V^c$, U^c, V^c closed proper.

4.6.Q Spec $k[x,y]/(xy)$. We've seen in 4.6.O that it is not irred (having two minimal prime ideals). For connes, we can rephrase the condition as the nonexistence of two nonempty closed subsets whose intersection is empty and whose union is the entire space, which can be explicitly checked using our knowledge of the list of closed subsets of Spec $k[x,y]/(xy)$.

4.6.R Let Y be a conn comp of Spec $A = Z_1 \cup \cdots \cup Z_n$ where Z_i are irred comps. Then $Y = \bigcup (Y \cap Z_i)$. Since Z_i is conn, if $Y \cap Z_i \neq \phi$, then $Y \cap Z_i = Z_i$ by maximality of Y. Thus Y is a union of the irred comps it meets.

Since Y is a finite union of closed subsets and so is its complement, Y is both closed and open.

4.6.S The nat maps of rings

$$\iota_i : A_i \hookrightarrow A_1 \times \cdots \times A_n$$

$$\pi_i : A_1 \times \cdots \times A_n \to A_i$$

induce maps Spec $\prod_{i=1}^{n} A_i \rightleftarrows \coprod_{i=1}^{n}$ Spec A_i, and we can check that they are inverses and cts (primes in $\prod_{i=1}^{n} A_i$ are componentwisely primes / whole rings (at least one proper)).

Spec $A_i = D(f_i)$, where $f_i = (0, \cdots, 0, 1, 0, \cdots, 0) \in \prod_{i=1}^{n} A_i = A$: 1 is not contained in any prime ideal whereas ${}^{i\text{th}}$ 0 is contained in any prime ideal.

"$\Leftarrow$": ✓

"$\Rightarrow$":

4.6.8 The ideal of "eventually zero" elts is proper (not prime: $(1,0,1,0,\cdots) \cdot (0,1,0,1,\cdots)$), contained in some "other" maximal ideal.

Sat - 10/2

✓ 1. p100. 4.6.K. ex for the aside?

Let $A = k[x_1, x_2, \cdots]$, $I = (x_1, x_2^2, x_3^3, \cdots)$ and $M = (x_1, x_2, x_3, \cdots)$ (in fact in the def of I we just need infinitely many exponents to be greater than 1). Then A/I is the ring we want. First of all, A/I is not nt, as we can produce an infinite ascending chain of ideals A_i/I by defining $A_i = (x_1, x_2, \cdots, x_i, x_{i+1}^{i+1}, x_{i+2}^{i+2}, \cdots)$. On the other hand, Spec A/I is nt as it consists of a single pt. To see this, note that M/I is a maximal ideal of A/I, as M is maximal in A. But every elt of M/I is nilp, so M/I is contained in the nilradical of A/I, which is the intersectio of all the prime ideals of A/I. Thus M/I is the only prim ideal of A/I.

2. p102. 4.6.S. "$\Rightarrow$"?

3. p68. 3.4.E

isom $\Rightarrow$ monom & epim

$\not\Leftarrow$

4. p72. 3.5.E use the fact that sheafification is an adjoint?

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3 Wed-9/29-1

1. $\phi: B \to A$, $\mathfrak{p}$ maximal in $A \not\Rightarrow \phi^{-1}(\mathfrak{p})$ maximal in B

If not, when is it true?

p90, 4.2.8. Do we have $[(a-a_0, b-b_0)] \mapsto [(x-a_0, y-b_0, z-b_0^2)]$?

Also p91. 4.2.L(b).

2. p91. Figure 4.5? Cf 10.3.3 on p208, over $\mathbb{Q}$. Over $\mathbb{C}$, we can take square root of + and −, so two branches.

Thu - 9/16

1. p65. Warning. Counterex?

Sun - 9/19

1. II.46. $\{*\}_X$ is not the const sh? It's just that in Sets, the final obj and the "1" are both $\{*\}$? Cf 3.3.C on p65.

2. 3.4.9 on p70 and 3.3.I on p67. Is $\mathcal{F} \longrightarrow \mathcal{O}_X^*$ the sheafification?

3. What precisely is the action of a $\mathbb{Z}$-module = sh of ab gps?

I hope that someone can explain to me (personally or on the next meeting) the idea of espace etale (3.2.10 on p62). (I was trying to see the module action of a sh of ab gps as a $\mathbb{Z}$-module (pp 63-4) using this viewpt.) In par, I am confused by the following comment: —— (What does it mean that "the 'espace etale' is constant"?).

p69. remark 3.4.7 [EH]?

Wed - 9/15

1. What is the exactness of the bifcr Hom(−,−) on p65?

Can the warning fit into the principle that colimits commute w/ left-adjoints (not right-adjoints)?

Do filtered colimits commute w/ right-adjoints and limits? (set-like)

exact

Are "cofiltered" limits useful?

Wed-9/8

1. p40. 2.6.G. interpretation? Could we just go as on π29? The finiteness assumption guarantees that "we do have a cohmlg thr".

② Mon-9/13

1. p61. Remark 3.7.2?

2. p63. 3.2.I Is the nat mor of stalks $(f_*\mathcal{F})_y \longrightarrow \underset{f^{-1}(y)}{\overset{\shortparallel}{\mathcal{F}_x}}$ a manifestati of the adjunction $f^{-1} : Shs_Y \rightleftarrows Shs_X : f_*$?

But this mor isn't an isom?

Cf π53, and the question there.

✓. p33. 2.5.E How about defining $(a,b) \sim (c,d) \Leftrightarrow a+d = b+c$?
My guess is that it still gives rise to a gp, but the univ property is lost (but it seems not) - under the previous equiv relation, more elts get identified.
But in the def given on p199 of [ccat], it seems to erase this (possible) differen

We won't have the transitivity of an equiv relation then!

✓. π18. How can we check explicitly that $\tilde{K}(X)$ is not the gpification
There should be an inclusion map $\tilde{K}(X) \to K(X)$ ($\tilde{K}(X)$ is defined as the ker of the dim map $d: K(X) \to \mathbb{Z}$ on p200 of [ccat]).
How can we define this here? ($a \mapsto (a, \varepsilon^0)$ is not well defined

Clarified on π18.

4. p35. 2.6.2 Why do additive fcrs preserve products?

Sun - 9/5

* 1 p36 So in an ab cat, monic $\Leftrightarrow$ monom? Cf comments. and new version!

2. p32. 2.5.13 Are there sth general underlying the resemblance between the pf of this and that of Yoneda's lemma? "Sth is completely determined by its value on the identity (mor)."

Another ex: Maps out of a free ab gp on one generator.

Thu-9/2

✓ pp32-3. 2.5.1 Uniqueness of the nat isom? (cf ππ 15-6)

Cf cor 1 on p85.

p81, prop 1 on pp59-60 of [ct]

Clarified on ππ 16-7.

Fri-9/3

1. p33 2.5.3 generalization? "gpdification" (used for other purpose

problem: for cats, compositions of mors are rarely comm.

(Gpdification is a form of categorification where we do linear alg w/ gpds instead of vector spaces.)

Further Q: Is it possible to define the gpification of a semigp not necess ab?

Sat - 8/28

Some discussion about localization of modules and the universal property (the two approaches for 2.3.E on p23)

✓1. p23 $S^{-1}M$ plus " and for which there is a map $M \to N$ of A-module

Yes, gets fixed in later version.

✓2. p23. 2.3.E $S^{-1}(-)$ preserves limits in general?

S^{-1} is a left adjoint (to the forgetful fcr), and hence preserves colimits. In Mod_A, finite products are coproducts as well.

(cf 2.6.10 on p41) (cf 2.6.2 on p35

Cf π4.

Sun - 8/29

✓1. π5. 2.3.G.(1)

cf π5.

Mon - 8/30

1. p27 "in some contexts". what contexts?

"injective" is more "set-like".

Tue - 8/31

1. p32. How to formulate adjointness for contravariant fcrs? Especiall the naturality squares?

- Longer meetings biweekly / Shorter meetings weekly?
- Take turns to tex up important / tough problem before/after meetings?

  Send emails / set up an underground website?

Robbie: (May come late) Fri after 2, the earlier the better

Nick: Mon / Tue / Wed after 1

Thu before 11 or 12:30–3 (prefer)

Yifei: Tue / Thu after 2:15

Wed / Fri

Denis:

Xiaoyi:

AL

Joel

Jason:

Weiwei

Seminars:

Potential math topics:

- Gpification and the Grothendieck construction (2.5.3 and 2.5.E on p33): Fri-9/3-1, ππ 17-8. [vbkt][ccat]
- The magic diag (2.3.T on pp26-7): ππ 7-9.
- Show the right-exactness of $\cdot \otimes_A N$ by the univ property (2.3.G on p24): ππ 5-6.