Asymptotic Behavior in Rational Functions

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Abstract

The behavior of rational functions around asymptotes, especially horizontal and so-called "oblique" asymptotes, is poorly taught and poorly understood in the context of high school algebra. It is necessary to define a new approach to such topics, for the purpose of better understanding and simpler computation on the part of students. In this, we will outline an alternative method for finding asymptotes for students and a new way to think about rational expressions for teachers.

1 Background

1.1 Rational Functions

A rational function is any function that can be represented as a fraction where the numerator and denominator are polynomials.[1] This includes, but is not limited to, examples such as

$$\frac{1}{x^3} \quad \text{or} \quad \frac{x-2}{x+2} \quad \text{or} \quad \frac{4x^4+3x^3+2x^2+x}{x-1}$$

Despite the fact that polynomials themselves may have domains that encompass the set of real numbers $\mathbb{R}$ (and indeed the set of complex numbers $\mathbb{C}$ as well), many rational functions have restrictions in their domains. This can be demonstrated by inputting -2 into $\frac{x-2}{x+2}$. The result is undefined, and

[1] Wikipedia

the surrounding function behaves asymptotically.

The tendency of the funtion towards infinity or negative infinity around this x value is refered to as a vertical asymptote, because the function, as one moves up or down the output values, approaches the asymptotic input, but does not reach it. This accelerated growth of the input occurs because, as the denominator of the fraction approaches 0, the fraction itself becomes larger and larger.

Likewise, a graph of the same function reveals that, as we follow the graph towards $x = +\infty$, the output tends towards $f(x) = 1$. Logically, this is sensible; as the x in the numerator and the x in the denominator both increase, the effects of the ± 2 become negligible. Note, however, that, just as with the vertical asymptote, the output of 1 is never reached, because of the slight difference between the numerator and denominator.

1.2 Motivation

The method commonly taught for finding asymptotes is patchwork at best. Finding vertical asymptotes is accomplished through a method similar to the one described here, so it will be covered later. But to discover horizontal or oblique asymptotes, the process is more complicated.

First, students note the highest powers of the input variable in both the numerator and denominator of the function. If the power of the numerator is (exactly) one power higher, then it is a slant asymptote. If one wants to find the equation of that asymptote, one then must do long division (as described below). If the powers of the numerator and denominator are the same or the numerator has the lesser power, it is a horizontal asymptote. In the case of the former, the ratio of the coefficients gives the output value of the asymptote. In the case of the latter, the asymptote is at $f(x) = 0$.

Unfortunately, this method for determining the presence of non-vertical asymptotes relies entirely on the memory of the student. They must remember two different rules and when to use each of three methods to find the asymptote. It also depends on the end behavior of the graphs, which in turn relies on computation involving infinities, a concept with unnecesary complications for the high school level.

These shortcomings of the current method couple with a few benefits of a more reliable method. For one, a better way of finding asymptotes will assist in graphing (the bane of any algebra student's existence). In addition, this method can be better used to predict actual behavior around non-vertical asymptotes, as we will see later.

1.3 Holes

2 The Two-Form Method

In this section, we will introduce a new method for approaching rational functions in order to discover asymptotic behavior. As the name implies, this method relies on converting the given form of a rational funtion into two different, distinct forms, in order to obtain information about asymptotic behavior.

2.1 Factored Form

We start with the form closest to the conventional method, the factored version of the rational function. This will give us domain restrictions and zeroes of the function. Obtaining this form is relatively simply: we just find the factored forms of the numerator and denominator of the function. For instance,

$$\frac{x^2-3x+2}{x^2-4x+3} \quad \text{becomes} \quad \frac{(x-2)(x-1)}{(x-3)(x-1)}$$

Note that the original form is functionally identical to the factored form. Graphs of the two will demonstrate this. Importantly, this means that, as any one of the individual factors approaches zero, the numerator or denominator will also approach zero. As the numerator of the function approaches zero, the function itself will approach $\frac{0}{g(x)}$, meaning that, unless whatever $g(x)$ is evaluates to zero at that same input coordinate, the function will have a zero at that input.

As the denominator of the function approaches zero, however, the function will approach $\pm\infty$.[2] Thus, each zero of the polynomial in the denominator is a vertical asymptote.

2.2 Divided Form

This second form for the function is the key part of the new approach. Essentially, we will express the rational function as the sum of a polynomial and a remainder, an indivisible rational function. This will, most importantly, always identify horizontal or oblique asymptotes, without the need for the many conditions of the conventional method.

In order to obtain this form from the given, we simply use polynomial long division. This leaves us with a polynomial (the highest power of which will be the difference of the highest power of the numerator and the highest power of the denominator of the given), and a remainder, a rational function that cannot be further divided (that is, the highest power of its numerator is less than the highest power of its denominator). We may, for the time being, ignore the remainder and focus on the polynomial.

The polynomial part of the divided form is the non-vertical asymptote. If one were to isolate that portion of the function and graph it alongside the given, that would be clear. The power of the input variable in the polynomial function determines the type of asymptote created, just as it would an actual polynomial. If it is raised to the first power, the asymptote is oblique, while having no input variables present creates a horizontal asymptote. As should be clear, this eliminates the need for analyzing the given to discover the type of asymptote, as most algebra students can better handle polynomials than rational functions themselves.

To better understand this, let us consider an example. Using long division,

$$\frac{x^3 + 3x^2 - 10x - 24}{x^2 - 3x + 2} \quad \text{reduces to} \quad x + 6 + \frac{6x - 36}{x^2 - 3x + 2} \tag{1}$$

The asymptote can be described by $f(x) = x + 6$. Because this function

[2]Except, as we will see, in the case of "holes."

has slope, the asymptote is oblique. Had we used the current method, we would have first had to check the powers of the input variable, remembered the type of asymptote to which the positive difference corresponds, and then still do long division to find the actual equation of the asymptote. The method being proposed eliminates the need to remember the cases, because it relies on intuition of polynomials that students should already have.

One way to think about this method (for teachers or more invested students) is as a transformation of a polynomial. The asymptote may be thought of as the "original" function (note the distinction with the "given" functions described earlier), and the irreducible rational expression that is added — the remainder — as the transformation upon the original function. After one adds the two (original and transformation) together, the result may be transformed to a rational function in the form with which we are familiar (the "given" form) by finding a common denominator between the two and multiplying the original by an identity that will give it this denominator.

This transformation has a few consequences: the asymptotes themselves. As the input approaches one of the domain restrictions in the transformational expression (provided it is not a "hole"), the tranformation part of the expression will grow, creating deviation from the original in the transformed function. Thus, the rational function created has the same vertical asymptotes as the transformation itself did. But, as the input progresses further from the domain restrictions, the the transformation tends towards zero. This is because, since the tranformation is required to be irreducible by the method of long division used, the power of the denominator is smaller than that of the numerator, and thus grows more rapidly than the numerator. But, if this tendency towards zero is added to the original function, it becomes a tendency towards the original function, or an asymptote.

It may be noted, if the example above were to be graphed, that the function itself actually crosses the oblique asymptote. This is explained by the distinction made between vertical and non-vertical asymptotes: vertical asymptotes are created by domain restrictions such as the ones described in 2.1, but horizontal and slant asymptotes are created whenever a vertical asymptote is added, through the transformation described. Vertical asymptotes are the only type whose implementation requires that the points on the asymptote are never reached by the function. Because horizontal and oblique asymptotes are determined by the value of the remainder at a given input, the asymptote can be crossed at the zeroes of the transformation function. In the case of the above example, $x = 6$ makes the remainder evaluate to zero, and thus the asymptote is crossed.

3 Further Benefits

This new method, and the accompanying interpretation, carry with them some opportunities for generalization. For example, the old method only allows for a difference of powers (numerator minus denominator) of one or less. This method, however, equips us to deal with other varieties of rational functions. Consider, for instance,

$$\frac{x^6 + 2x^5 - 34x^4 - 52x^3 + 233x^2 + 50x - 200}{x^3 - 5x^2 - 2x + 24} \tag{2}$$

(not, unfortunately, a very appealing equation). Factoring reveals it to be equivalent to

$$\frac{(x+1)(x-1)(x-2)(x+4)(x-5)(x+5)}{(x+2)(x-3)(x-4)} \tag{3}$$

while long division yields

$$x^3 + 7x^2 + 3x - 47 + \frac{-164x^2 - 116x + 928}{x^3 - 5x^2 - 2x + 24} \tag{4}$$

This is difficult enough to deal with (or graph) without the fact that the commonly used method for dealing with this simply cannot account for this type of problem. But looking at the two forms, we can clearly see how to approach this problem. From our divided form, we find that the asymptote has the equation $x^3 + 7x^2 + 3x - 47$. It is neither oblique nor horizontal, but a higher-powered — a cubic, to be precise — function. This is, for lack of a proper term, a curved asymptote. Furthermore, by solving the numerator of the remainder at zero, we find that the given function crosses the asymptote at $x = \frac{-29 \pm 3\sqrt{4321}}{82}$.[3]

[3]Some graphing utilities may display another crossing at $x = 3$, but evaluating this directly will reveal a vertical asymptote.