0+1 and 1+1 dimensional field theories

We have managed to write the single particle problem in terms of a path integral of $x(t)$. It turns out that this is a so-called $0+1$ dimensional field theory. Here 0 refers to the number of spatial dimensions and 1 to the number of time dimensions. This may seem strange at first sight, since we have thought of $x(t)$ as a spatial coordinate. But this is just a question of interpretation. We may just as well imagine that we have a scalar field ϕ (like the potential for an electric field) at one single position in space and dependent on time t. Then we would describe it by the same kind of path integral $\int D\phi(t)e^{iS[\phi]/\hbar}$ with some action for the scalar field.

Similarly, imagine that we start out with a path integral of N particles, described by coordinates $x_i(t)$ with $i = 1 \dots N$. Then in the limit $N \to \infty$ we can also interprete this path integral as a $1+1$ dimensional field theory. We just have to introduce a scalar field ϕ that is defined on a line with coordinates $x = ia$, where a is a lattice size. Then the field ϕ depends on two dimensions x and t: $\phi(x,t)$.

Connection of the path integral to statistical physics

One of the central themes of condensed matter physics are the equilibrium properties of many-body systems. The subject that deals with these questions is statistical physics. In statistical physics the central object is the partition function, denoted by Z. If we know the Z for a physical system, we know all of its equilibrium properties. In the statistical physics of quantum systems we start with a Hamiltonian, as for example our simple

$$H = \frac{P^2}{2m} + V(X)$$

and we find Z by computing a trace:

$$Z = \mathrm{Tr}[e^{-\beta H}] = \sum_n \langle n|e^{-\beta H}|n\rangle,$$

here the sum has to be extended over a complete set of states, say the energy eigenstates of H. Now we ask: how is this sum connected to path integrals? Instead of using energy eigenstates $|n\rangle$, let us switch to position eigenstates, by using $\mathbb{I} = \int dx|x\rangle\langle x|$:

$$Z = \int dx \sum_n \langle n|x\rangle\langle x|e^{-\beta H}|n\rangle = \int dx \sum_n \langle x|e^{-\beta H}|n\rangle\langle n|x\rangle = \int dx\langle x|e^{-\beta H}|x\rangle$$

This looks familiar, in fact the summand is reminiscent of a probability amplitude, except that instead of $-iH \cdot (t_f - t_i)/\hbar$ in the exponent, we have $-\beta H$ and also $x_f = x_i = x$. Let us take our result

$$\langle x_0|e^{-iHT/\hbar}|x_0\rangle = \int\limits_{\substack{x(0)=x_0 \\ x(T)=x_0}} Dx(t)\; e^{\frac{i}{\hbar}\int_0^T dt\left[\frac{m}{2}\dot{x}^2 - V(x)\right]},$$

where $t_i = 0, t_f = T$ and make the substitutions

$$T = -i\beta\hbar$$
$$t = -i\tau$$

then we get

$$\langle x_0|e^{-\beta H}|x_0\rangle = \int\limits_{\substack{x(0)=x_0\\x(T)=x_0}} Dx(t)\ e^{-\frac{1}{\hbar}\int\limits_0^{\beta\hbar} d\tau\left[\frac{m}{2}\dot{x}^2+V(x)\right]}.$$

Surprisingly the i in going from t to τ has changed the sign of the kinetic energy, such that in the exponent we have now the Hamiltonian and not the Lagrangian. Finally, in order to obtain the partition function we integrate over x_0. This changes the path integration such that in the boundary conditions x_0 can be anything, as long as $x(t)$ is periodic:

$$Z = \int\limits_{x(0)=x(T)} Dx(t)\ e^{-\frac{1}{\hbar}\int\limits_0^{\beta\hbar} d\tau\left[\frac{m}{2}\dot{x}^2+V(x)\right]}$$

The free particle

So far we have derived path integrals, but we haven't evaluated any of them. Let us evaluate the easiest, non-trivial path integral we can think of. We get this by putting $V(x) = 0$, i.e. we want to calculate the free particle propagator

$$K(x_f, T, x_i, 0) = \int\limits_{\substack{x(0)=x_i\\x(T)=x_f}} Dx(t) e^{\frac{i}{\hbar}\int\limits_0^T dt\frac{m}{2}\dot{x}^2}.$$

It is useful to go back to the discretized definition of the path integral

$$K(x_f, T, x_i, 0) = \lim_{N\to\infty}\left(\frac{m}{2\pi i\hbar\Delta t}\right)^{N/2}\int dx_1\cdots\int dx_N \exp\left(\frac{im}{2\hbar\Delta t}\sum_{i=0}^{N-1}(x_{i+1}-x_i)^2\right),$$

where just as before we have as endpoints $x_0 = x_i$, $x_N = x_f$ and $\Delta t = T/N$. This is a somewhat tricky integral, in that each integration involves two Gaussian factors, e.g. the integral over x_1 involves $\exp(\frac{im}{2\hbar\Delta t}(x_1 - x_0)^2) \times \exp(\frac{im}{2\hbar\Delta t}(x_2 - x_1)^2)$. A neat way of doing this can be found in the excellent book of Feynman and Hibbs. This way is based on the integral identity

$$\int\limits_{-\infty}^{\infty} dx e^{-a(x-x')^2-b(x''-x)^2} = \sqrt{\frac{\pi}{a+b}}\exp\left(-\frac{(x'-x'')^2}{\frac{1}{a}+\frac{1}{b}}\right).$$

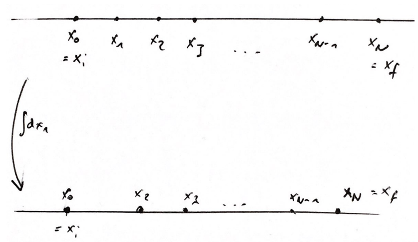

Figure 6: A variable x_i (except for x_0 ind x_N) is connected to its two neighbors x_{i-1} and x_{i+1}. After integrating over x_i, x_{i-1} and x_{i+1} become connected.

Let us begin by taking up first the x_1 integration:

$$\Big(\frac{m}{2\pi i\hbar\Delta t}\Big)^{2/2}\int dx_1\exp(\frac{im}{2\hbar\Delta t}(x_1-x_0)^2)\times\exp(\frac{im}{2\hbar\Delta t}(x_2-x_1)^2)$$

Here we used up two factors of $\big(\frac{m}{2\pi i\hbar\Delta t}\big)^{1/2}$ from the definition. The result of integrating over x_1 is

$$\Big(\frac{m}{2\pi i\hbar\ 2\Delta t}\Big)^{1/2}\exp(\frac{im}{2\hbar\ 2\Delta t}(x_2-x_0)^2).$$

This looks just like one of the factors we started with, except that everywhere Δt has now been replaced with $2\Delta t$. Also x_1 has been removed from the sequence shown in the figure and x_2 is now linked directly to x_0.

We continue with the integrations. Next up is the integral over the Gaussians involving x_2. We also take one factor of $\big(\frac{m}{2\pi i\hbar\Delta t}\big)^{1/2}$ from the definition:

$$\Big(\frac{m}{2\pi i\hbar\Delta t}\Big)^{1/2}\Big(\frac{m}{2\pi i\hbar\ 2\Delta t}\Big)^{1/2}\exp(\frac{im}{2\hbar\ 2\Delta t}(x_2-x_0)^2)\times\exp(\frac{im}{2\hbar\ \Delta t}(x_2-x_3)^2)$$

The result of this is:

$$\Big(\frac{m}{2\pi i\hbar\ 3\Delta t}\Big)^{1/2}\exp(\frac{im}{2\hbar\ 3\Delta t}(x_3-x_0)^2)$$

In the place of Δt we find $3\Delta t$ and the Gaussian connects x_3 to x_0.

Since we are already seeing the rule behind these integrations, we can now proceed recursively. After carrying out the last integration, the one over x_{N-1}, we are left with

$$K(x_f,T,x_i,0)=\Big(\frac{m}{2\pi i\hbar\ N\Delta t}\Big)^{1/2}\exp(\frac{im}{2\hbar\ N\Delta t}(x_N-x_0)^2)=\Big(\frac{m}{2\pi i\hbar\ T}\Big)^{1/2}\exp(\frac{im}{2\hbar\ T}(x_f-x_i)^2).$$

This is the final answer for the propagator. This is indeed the familiar result from doing ordinary quantum mechanics and is a good check that the path integral formalism does indeed work as expected and that furthermore the strange looking factors $\lim_{N\to\infty}\big(\frac{m}{2\pi i\hbar\Delta t}\big)^{N/2}$ are in fact needed.

Gaussian integrals and Wick's theorem

In working with path integrals the most important class of Lagrangians are quadratic ones, since the corresponding path integrals can be computed exactly. The underlying reason for this is that such path integrals are Gaussian. By starting with the simplest Gaussian integral and successively generalizing we end up with Gaussian path integrals.

Let us begin with the simplest Gaussian integral

$$\int_{-\infty}^{+\infty} dx\ e^{-\frac{1}{2}ax^2} = \sqrt{\frac{2\pi}{a}}$$

which is a result that holds whenever Re $a > 0$. A slightly more difficult one is, if there is also a linear term in the exponent:

$$Z[J] = \int_{-\infty}^{+\infty} dx\ e^{-\frac{1}{2}ax^2+Jx}$$

We can relate it to the previous version by completing the square:

$$Z[J] = \int_{-\infty}^{+\infty} dx\ e^{-\frac{1}{2}a(x-J/a)^2} e^{\frac{J^2}{2a}} = \sqrt{\frac{2\pi}{a}} e^{\frac{1}{2}Ja^{-1}J} \tag{2}$$

The reason why we write the exponent in the result in this strange way, will become clearer now. Let us define averages by:

$$\langle \dots \rangle = \left. \frac{\int_{-\infty}^{+\infty} dx\ \ (\dots)\, e^{-\frac{1}{2}ax^2+Jx}}{Z[J]} \right|_{J\to 0} \tag{3}$$

The denominator is just there to normalize and we can calculate it rightaway: $\left.\int_{-\infty}^{+\infty} dx\ e^{-\frac{1}{2}ax^2+Jx}\right|_{J\to 0} = \sqrt{\frac{2\pi}{a}}$.

The numerator is somewhat trickier. As an example, let us calculate $\langle x^2 \rangle$. We need to find

$$\int_{-\infty}^{+\infty} dx\ x^2 e^{-\frac{1}{2}ax^2+Jx}.$$

This can be done in many ways, but a particularly elegant way is to use derivatives with respect to J:

$$\left.\int_{-\infty}^{+\infty} dx\ x^2 e^{-\frac{1}{2}ax^2+Jx}\right|_{J\to 0} = \partial_J^2 \left.\int_{-\infty}^{+\infty} dx\ e^{-\frac{1}{2}ax^2+Jx}\right|_{J\to 0} = \left.\sqrt{\frac{2\pi}{a}} \partial_J^2 e^{\frac{1}{2}Ja^{-1}J}\right|_{J\to 0}$$

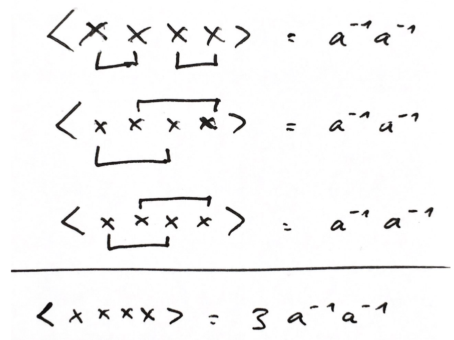

Figure 7: Three distinct ways to pair up each of the four x's

The square root factor is not important, since it will eventually be cancelled by the denominator in (3). In $\partial_J^2 e^{\frac{1}{2}Ja^{-1}J}\Big|_{J\to 0}$ it is useful to note that we set $J = 0$ after taking the derivative, thus terms with factors of J will vanish in the end. One differentiation brings down a J and the next differentiation has to annihilate it, or else it won't make a contribution. There are two ways to bring down a J for the first ∂_J, since there are two J's in $e^{\frac{1}{2}Ja^{-1}J}$. This cancels the factor of 1/2. In the end we are left with the simple result:

$$\langle x^2 \rangle = a^{-1}$$

Let us try to do the same for $\langle x^4 \rangle$. This gives

$$\langle x^4 \rangle = \partial_J^4 e^{\frac{1}{2}Ja^{-1}J}\Big|_{J\to 0}.$$

Again, each ∂_J acting on the exponential brings down a J and since we are taking the $J \to 0$ limit, this will only work if two ∂_J bring down J's and two ∂_J annihilate them. Which of the ∂_J annihilates which J does not matter, as long as there is one ∂_J that generates a J and there is another ∂_J to annihilate it. There is schematic way to count in how many ways this can be done. We write $\langle xxxx \rangle$ and pair up two x's as illustrated in the figure. This corresponds to saying which ∂_J generates a J and which ∂_J annihilates it. We see from the figure that there are 3 distinct ways. In general if one has $\langle x^{2m}\rangle$ there are $\frac{(2m)!}{2^m m!} = (2m-1)\cdot(2m-3)\cdot\dots\cdot 1$ ways. Thus the general result is

$$\langle x^{2m}\rangle = (2m-1)\cdot(2m-3)\cdot\dots\cdot 1\left(a^{-1}\right)^m .$$

Notice what we did: By introducing the term Jx into the exponential of the Gaussian, we found an elegant way to compute expectation values by taking derivatives of $Z[J]$. The variable J is called the source and by differentiating with respect to it, we can compute averages of powers of x.

Let us now generalize this integral to a slightly more difficult one, where we now integrate over $x_1, \dots, x_N$:

$$\int_{-\infty}^{+\infty} dx_1 \dots dx_N \; e^{-\frac{1}{2}\vec{x}^T A \vec{x}} = \int_{-\infty}^{+\infty} dx_1 \dots dx_N \; e^{-\frac{1}{2}x_i A_{ij} x_j} \tag{4}$$

Here A is a matrix and in the second equality we have written the exponent out using index notation. We use the summation convention, i.e. we sum over repeated indices. We can assume without loss of generality that A is symmetric. The reason is that we are summing over x_i and x_j, which would eliminate the antisymmetric part of A anyway.

But if A is symmetric, we can diagonalize it using an orthogonal transformation O:

$$A = O^T D O \tag{5}$$

Here D is a diagonal matrix. Let us insert this form into the exponent in (4):

$$-\frac{1}{2}\vec{x}^T A \vec{x} = -\frac{1}{2}\vec{x}^T O^T D O \vec{x} = -\frac{1}{2}\left(O\vec{x}\right)^T D \left(O\vec{x}\right)$$

We used that $\left(O\vec{x}\right)^T = \vec{x}^T O^T$. Now we can change to new variables $\vec{y} = O\vec{x}$ in the integration. This gives us the integral

$$I = \int_{-\infty}^{+\infty} dy_1 \dots dy_N \; e^{-\frac{1}{2}\vec{y}^T D \vec{y}} = \int_{-\infty}^{+\infty} dy_1 \dots dy_N \; e^{-\frac{1}{2}\sum_i D_{ii} y_i^2} .$$

In the last step we used the fact that D is diagonal. If all $D_{ii} > 0$ we can do this integral (else we get a diverging integral). The integrals are decoupled and we can do each of them seperately:

$$I = \prod_{i=1}^{N} \sqrt{\frac{2\pi}{D_i}} = \frac{(2\pi)^{N/2}}{\sqrt{D_{11}\dots D_{NN}}} = \frac{(2\pi)^{N/2}}{\sqrt{\det D}} = \frac{(2\pi)^{N/2}}{\sqrt{\det A}}$$

In the penultimate step we used the fact that the determinant of D is equal to product of its diagonal entries. In the last step we used the fact that a similarity transformation like (5) does not change the determinant. Finally, in going from variables x to y there would be a Jacobian, but this is 1 (here is another exercise).

In order to make this result useful for us we introduce a source term $\vec{J}\cdot\vec{x}$ into the exponent as before:

$$Z[J] = \int_{-\infty}^{+\infty} dx_1 \dots dx_N \; e^{-\frac{1}{2}\vec{x}^T A\vec{x}+\vec{J}\cdot\vec{x}}$$

We could derive this integral by going through the same steps as before, but it is just as easy to guess the result by looking at the 1D result (2):

$$Z[J] = \int_{-\infty}^{+\infty} dx_1 \dots dx_N \; e^{-\frac{1}{2}\vec{x}^T A\vec{x}+\vec{J}\cdot\vec{x}} = \frac{(2\pi)^{N/2}}{\sqrt{\det A}} e^{\frac{1}{2}\vec{J}^T A^{-1}\vec{J}} \tag{6}$$

This is our final result. We can use it to compute Gaussian averages that we define by

$$\langle \dots \rangle = \left. \frac{\int_{-\infty}^{+\infty} dx_1 \dots dx_N \; (\dots) e^{-\frac{1}{2}\vec{x}^T A\vec{x}+\vec{J}\cdot\vec{x}}}{Z[J]} \right|_{J\to 0}.$$

Notice once again that the denominator is just there to cancel the factor $\frac{(2\pi)^{N/2}}{\sqrt{\det A}}$ in (6). Let us begin with the average

$$\langle x_i x_j \rangle.$$

We obtain it from (6) by differentiating:

$$\langle x_i x_j \rangle = \partial_{J_i}\partial_{J_j} e^{\frac{1}{2}\vec{J}^T A^{-1}\vec{J}} \Big|_{J\to 0} = \left(A^{-1}\right)_{ij}$$

Again, since we are taking the $J \to 0$ limit, one operator ∂_j brings down a $\vec{J}$, the other annihilates it. Similarly we can look at

$$\langle x_i x_j x_k x_l \rangle$$

and apply four derivatives:

$$\partial_{J_i}\partial_{J_j}\partial_{J_k}\partial_{J_l} e^{\frac{1}{2}\vec{J}^T A^{-1}\vec{J}} \Big|_{J\to 0}$$

The end result is:

$$\langle x_i x_j x_k x_l \rangle = (A^{-1})_{ij}(A^{-1})_{kl} + (A^{-1})_{ik}(A^{-1})_{jl} + (A^{-1})_{il}(A^{-1})_{jk}$$

The general pattern is quite clear now. In order to compute

$$\langle x_{i_1} x_{i_2} \dots x_{i_{2m}} \rangle$$

$$\langle x_i x_j x_k x_l \rangle = (A^{-1})_{ij} \, (A^{-1})_{kl}$$

$$\langle x_i x_j x_k x_l \rangle = (A^{-1})_{ik} \, (A^{-1})_{jl}$$

$$\langle x_i x_j x_k x_l \rangle = (A^{-1})_{il} \, (A^{-1})_{jk}$$

$$\langle x_i x_j x_k x_l \rangle = (A^{-1})_{ij}(A^{-1})_{kl} + (A^{-1})_{ik}(A^{-1})_{jl} + (A^{-1})_{il}\,(A^{-1})_{jk}$$

Figure 8: Wick pairings in vector version

we just pair up (people say 'contract') the x's in all possible ways and for each pairup of index i_a with i_b we write the factor $(A^{-1})_{i_a i_b}$:

$$\langle x_{i_1} x_{i_2} \ldots x_{i_{2m}} \rangle = \sum_{\text{all pairings}} (A^{-1})_{ab} \ldots (A^{-1})_{cd}$$

This identity is called Wick's theorem and here it is shown in the special setting of Gaussian multiple integrals.

Now we can take up our final generalization to path integrals. A function $\phi(x)$ can be be discretized at certain points along the x-axis at points $x = ia$, where a is a lattice constant and $i = 0, \pm 1, \pm 2, \ldots$. Then at each one of these points in space the function takes on a value $\phi_i := \phi(x = ia)$. Thus a function can be viewed as a vector with infinitely many components ϕ_i. If we make take a progressively smaller, we obtain an ever more accurate description of the function $\phi(x)$. The continuum analogue of $\sum_i \phi_i \psi_i$ is then $\int dx \; \phi(x)\psi(x)$. Similarly, if one has a matrix A, then the continuum analogue of the expression $\sum_{ij} \phi_i A_{ij} \psi_j$ is given by $\int dx \int dy \; \phi(x) A(x,y) \psi(y)$.

Then how does one define the continuum inverse of $A(x,y)$? The inverse of a matrix is defined by

$$\sum_j A_{ij} (A^{-1})_{jk} = \delta_{ik}$$

and therefore the continuum inverse of $A(x,y)$ is defined by

$$\int dy \; A(x,y) A^{-1}(y,z) = \delta(x-z).$$

Now let us look at Gaussian integral with functions instead of vectors. By analogy with the identity in equation (6), we consider

$$Z[J] = \int Dx \; \exp\left(-\frac{1}{2} \int dt \int dt' \; x(t) A(t,t') x(t') + \int dt \; J(t) x(t) \right), \quad (7)$$

where we have decided to call the field x and to use as the 'component index' the symbols t and t'. This is intentional so we can recognize this integral as a path integral. The identity (6) also suggest the result for this integral. It is:

$$Z[J] \sim \frac{1}{\sqrt{\det A}} \int Dx \ \exp\left(\frac{1}{2}\int dt \int dt' \ J(t)A^{-1}(t,t')J(t')\right) \tag{8}$$

We have dropped the constant $(2\pi)^N$, which would diverge in our case. But since we will use this formula only to compute averages , the constant will not matter. Since A is an operator here, it is not immediately clear how the determinant is computed. We shall next demonstrate on a standard example how this is done.