Semiclassical approximation

We take the problem of a particle moving in one dimension in a potential as shown in the figure.The minimum of V is at $x = 0$ and $V(x = 0) = 0$. We want to compute the amplitude

$$\langle 0|\exp(-iHT/\hbar)|0\rangle = \int Dx(t)\ \exp(\frac{i}{\hbar}\int_0^T dt\left[\frac{m}{2}\dot{x}^2 - V(x)\right]).$$

If V is some general potential we will not be able to compute the result exactly. Thus we resort to an approximation that is know as the *semiclassical approximation.* In this approximation one assumes that the most dominant contribution to the path integral on the right hand side is the classical solution of the problem $x_{\rm cl}(t)$ plus a little bit of fluctuations $\eta(t)$:

$$x(t) = x_{\rm cl}(t) + \eta(t)$$

Inserting this into the action and expanding for small η, we obtain

$$S[x] = \int_0^T dt\left[\frac{m}{2}\dot{x}_{\rm cl}^2 - V(x_{\rm cl}) + \frac{m}{2}\dot{\eta}^2 - \frac{1}{2}\frac{\partial^2 V}{\partial x^2}\eta^2\right],$$

where $\frac{\partial^2 V}{\partial x^2}$ has to be evaluated at $x_{\rm cl}$. Notice that we have dropped all the linear parts in $x_{\rm cl}$. The reason is that this part is exactly zero, since the classical solution has the property that $\delta S[x_{\rm cl}\,] = 0$.

Inside the brackets we recognize the first two terms as the classical Lagrangian, thus we write

$$S[x] = S_{\rm cl} + \int_0^T dt\left[\frac{m}{2}\dot{\eta}^2 - \frac{1}{2}\frac{\partial^2 V}{\partial x^2}\eta^2\right].$$

So far everything is general and it is clear that we can always compute the resulting path integral, since the problem is quadratic in η.

We specialize now to our potential. We first have to find the classical solutions (there will in general be many) subject to the condition that the particle starts at $x = 0$ and returns to $x = 0$ at time T. There are many classical solutions that we can include, but the most important is $x_{\rm cl} = 0$. In this case the classical action is $S_{\rm cl} = 0$. Also, in order to make the action of the η part look like a harmonic oscillator, we set $\frac{\partial^2 V}{\partial x^2} = m\omega^2$. This results in the approximation for the amplitude:

$$\langle 0|\exp(-iHT/\hbar)|0\rangle = \int D\eta(t)\ \exp(\frac{i}{\hbar}\int_0^T dt\left[\frac{m}{2}\dot{\eta}^2 - \frac{m\omega^2}{2}\eta^2\right]).$$

We recognize this as the path integral (7) with $J = 0$. However, we first have

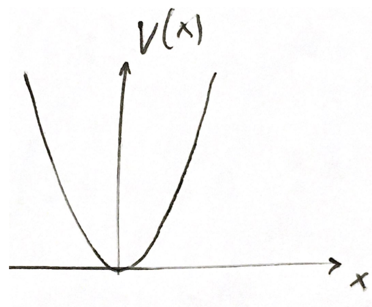

Figure 9: Potential of a 1D particle

to bring the quadratic part into the same form, which we can do by partially integrating the first term:

$$S[x] = \frac{m}{2} \int_0^T dt \left[\dot{\eta}^2 - \omega^2 \eta^2\right] = -\frac{m}{2} \int_0^T dt \int dt' \; \eta(t')\delta(t-t') \left[\partial_t^2 + \omega^2\right] \eta(t)$$

To bring it completely into the same form we have also introduced an integral over t' and multiplied the integrand with a δ(t-t'), but it turns out that this only clutters up the calculation and later on, when we are more familiar with functional determinants, we will not be so careful anymore. We now read off that

$$A(t,t') = m\delta(t-t') \left[\partial_t^2 + \omega^2\right]$$

and the result for the path integral is

$$Z[J=0] \sim \frac{1}{\sqrt{\det[A]}}.$$

How do we calculate this determinant? We make use of the fact about determinants that they are equal to the product of eigenvalues. Thus we have to study the eigenvalues of A, i.e. we look for solutions of

$$m \left[\partial_t^2 + \omega^2\right] \psi = \lambda \psi. \tag{9}$$

But we have to keep in mind that there are boundary conditions to the path integral. Since $x(T) = x(0)$ we have $\eta(T) = \eta(0)$. The solutions to (9) are given by

$$\psi(t) = \sin\left(\frac{n\pi}{T} t\right)$$

with $n = 1, 2 \ldots$. The eigenvalues are obtained by substituting back into the differential equation:

$$\lambda = m \left(\frac{n^2\pi^2}{T^2} - \omega^2\right).$$

The determinant is the product of these eigenvalues, thus

$$\det A = \prod_{n=1}^{\infty} \left[m \left(\frac{n^2\pi^2}{T^2} - \omega^2 \right) \right].$$

This looks like a horribly divergent result. But we remember that our result (8) was left unspecified up to a constant, which was infinitely large. We can guess that this is what will cancel the divergence here. In order to find the result, we proceed as follows. There is one result for the path integral that we have worked out carefully and that we know is right. It is the result for the free problem, which gave $\langle 0 | \exp(-iHT/\hbar) | 0 \rangle = \sqrt{\frac{m}{2\pi i \hbar T}}$. Our current problem reduces to the free problem, when we set $\omega = 0$ (this is the same as having $V = 0$ everywhere). Thus we are saying

$$\begin{aligned}
\langle 0 | \exp(-iHT/\hbar) | 0 \rangle &= \frac{\langle 0 | \exp(-iHT/\hbar) | 0 \rangle}{\langle 0 | \exp(-iHT/\hbar) | 0 \rangle_{\text{free}}} \sqrt{\frac{m}{2\pi i \hbar T}} \\
&= \prod_{n=1}^{\infty} \left[m \left(\frac{n^2\pi^2}{T^2} - \omega^2 \right) \right]^{-1/2} \prod_{n=1}^{\infty} \left[m \left(\frac{n^2\pi^2}{T^2} \right) \right]^{1/2} \sqrt{\frac{m}{2\pi i \hbar T}} \\
&= \prod_{n=1}^{\infty} \left[\left(1 - \frac{T^2\omega^2}{n^2\pi^2} \right) \right]^{-1/2} \sqrt{\frac{m}{2\pi i \hbar T}}
\end{aligned}$$

Now we can use a product formula that is due to L. Euler, who found it in working on the so-called Basel problem (1735):

$$\sin(x) = x \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{\pi^2 n^2} \right)$$

We thus obtain the final result:

$$\langle 0 | \exp(-iHT/\hbar) | 0 \rangle = \left[\frac{\sin \omega T}{\omega T} \right]^{-1/2} \sqrt{\frac{m}{2\pi i \hbar T}} = \sqrt{\frac{m\omega}{2\pi i \hbar \sin \omega T}}$$

In the limit $\omega \to 0$ we see that result reduces to the free problem. In the case, where the $V(x)$ is quadratic, this is an exact result. For other V's this is of course only an approximation.