Risk and PortFolio

Risk Taxonomy

- ***Market risk*** is defined as the risk to a financial portfolio from movements in market prices such as equity prices, foreign exchange rates, interest rates, and commodity prices.

- *Liquidity risk* is defined as the particular risk from conducting transactions in markets with low liquidity as evidenced in low trading volume and large bid-ask spreads.

- *Operational risk* is defined as the risk of loss due to physical catastrophe, technical failure, and human error in the operation of a firm, including fraud, failure of management, and process errors.

- ***Credit risk*** is defined as the risk that a counterparty may become less likely to fulfill its obligation in part or in full on the agreed upon date.

- *Business risk* is defined as the risk that changes in variables of a business plan will destroy that plan's viability, including quantifiable risks such as business cycle and demand equation risk, and nonquantifiable risks such as changes in competitive behavior or technology.

Risk Definition

- The definition of risk in investment area is *the standard deviation of return* (Markowitz).
- If R_p is a portfolio's total return, then the portfolio's standard deviation of return is denoted by

$$\sigma = Std[R] \; .$$

- We will typically quote this risk, or standard deviation of return, on a percent per year basis. We will also occasionally refer to this quantity as ***volatility***.

波动率计算

- 历史波动率：作为过往的日收益率的条件标准差计算。后面的模型是针对这种波动率定义。
- 隐含波动率：根据期权的理论公式如BS公式，从标的物价格和期权价格数据反解出模型中的波动率，这样的得到的波动率称为隐含波动率。隐含波动率倾向于比用日收益率建模得到的波动率数值要大。CBOE的VIX指数就是隐含波动率。
    - 期权价格影响因素：
        - 执行价格
        - 无风险利率
        - 当前股价
        - 股票的波动率
    - Black-Scholes模型：在假设股票价格服从几何布朗运动的条件下，给出了期权价格的解析解，其中包含不可观测的波动率。波动率（volatility）是股票价格的条件标准差。可以从股价以及BS模型求解出波动率，这样得到的波动率称为隐含波动率（implied volatility）。
- 实际波动率：利用一天内所有的收益率数据，如每5分钟的收益率，估计一天收益率的条件标准差

Logarithmic Returns

- 如果我们考察投资品在总共 T 期内的表现，那应该用对数收益率，而非算数收益率。算术平均值不能正确的反应一个投资品的收益率。比如一个投资品今年涨了 50%，明年跌了 50%，它的算数平均收益率为 0；但事实上，两年后该投资品亏损了最初资金的 25%。相反的，对数收益率由于具备可加性，它的均值可以正确反映出该投资品的真实收益率。比如这两年的对数收益率分别为 40.5% 和 -69.3%，平均值为 -28.77%，转换为百分比亏损就是 exp{-28.77%} - 1 = -25%。
- 假设初始资金 S_0（假设等于 1），ln(S_T) = ln(S_T/S_0) 就是整个 T 期的对数收益率。对数收益率的最大好处是它的可加性，把单期的对数收益率相加就得到整体的对数收益率。后面会使用对数收益率计算。

$$R_{t+1} = \ln(S_{t+1}) - \ln(S_t) = \ln(S_{t+1}/S_t) = \ln(1 + r_{t+1}) \approx r_{t+1}$$

$$R_{t+1:t+K} = \ln(S_{t+K}) - \ln(S_t) = \sum_{k=1}^{K} \ln(S_{t+k}) - \ln(S_{t+k-1}) = \sum_{k=1}^{K} R_{t+k}$$

- Arithmetic Return

$$R_{t+1:t+k} = ((S_{t+k} / S_t) - 1) = \prod_{K=1}^{K} (1 + R_{t+k}) - 1$$

Risk Measurement

- Return of the asset for T time periods

$$R_t = \ln\left(\frac{S_t}{S_{t-1}}\right) = \sum \ln(S_t) - \ln(S_{t-1}) \text{t = 1, 2, . . . ,T}$$

- Estimating the mean of returns

$$\hat{\mu} = \frac{1}{T}\sum_{t=0}^{T}(\ln(S_t) - \ln(S_{t-1})) = \frac{1}{T}(\ln(S_T) - \ln(S_0))$$

- Standard derivation sample estimate:

$$\hat{\sigma} = \sqrt{\frac{1}{T-1}\sum_{t=1}^{T}(R_t - \bar{R})^2}, \text{ with } \bar{R} = \frac{1}{T}\sum_1^T R_t$$

- Variance of Return

$$VAR_t = \sigma^2 = \frac{1}{T}\sum_1^T(\ln(S_t) - \ln(S_{t-1}) - \hat{\mu})^2$$

类似于利率， 度量波动率的时间区间一般也取为一年， 波动率一般是年化波动率。如果有了日收益率（条件标准差）， 可以将其乘以252转换成年化的波动率.

VaR (Value-at-Risk)

The value at risk (VaR) represents the potential loss in value of a position (expressed as a net present value) that could with a certain probability.

VaR is often defined in dollars, denoted by $VaR, the probability of getting an even larger loss as in

$$\Pr(\$Loss > \$VaR) = p$$

(1-*p*)*100% of the time, the $Loss will be smaller than the VaR. That is, we are (1 – p)100% confident that we will get a return better than VaR

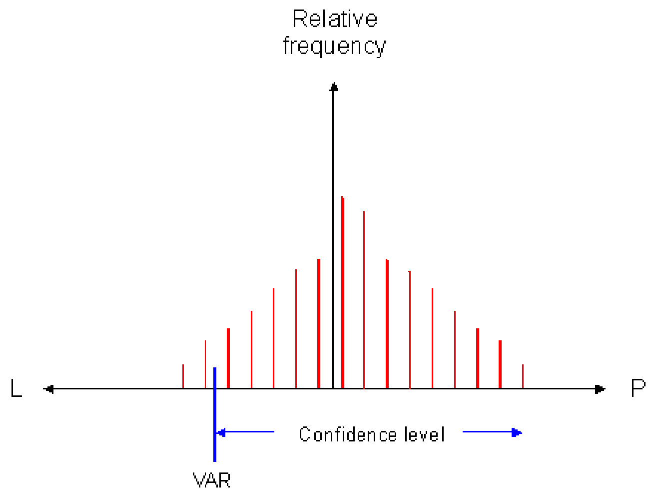
$$\Pr(-R_{PF} > VaR) = p \Leftrightarrow$$
$$\Pr(R_{PF} < -VaR) = p$$

$VaR^{P}_{t+1:t+k}$: *100% $*$p VaR for the next k days ahead return*

$$\$VaR = V_{PF}\,(1 - \exp(-VaR))$$

VaR Historical Simulation

- The purpose of historical simulation is to determine what profits or losses would be incurred if a market price development from the past were to occur today.
- 每次取一定长度的历史数据作为样本，将样本的分布看作是整体分布，在置信度p下，只需要找这些历史数据的前p-分位数，认为这些历史数据的p分位数就可以表示VaR.



$$VaR_{t+1}^{p} = -Percentile\left(\left\{R_{PF,t+1-\tau}\right\}_{\tau=1}^{m}, 100p\right)$$

Weighted Historical Simulation

- Weighted Historical Simulation (WHS): is designed to relieve the tension in the choice of m by assigning relatively more weight to the most recent observations and relatively less weight to the returns further in the past.
- HS方法认为过去每天数据包含的的信息量是一样的，WHS认为距离当天越近的数据，对于当天影响更大，应该赋予更高的权重.

Modeling Approach

$$\begin{aligned}
\Pr\left(R_{PF,t+1} < -VaR^p_{t+1}\right) = p \Leftrightarrow \\
\Pr\left(R_{PF,t+1}/\sigma_{PF,t+1} < -VaR^p_{t+1}/\sigma_{PF,t+1}\right) = p \Leftrightarrow \\
\Pr\left(z_{t+1} < -VaR^p_{t+1}/\sigma_{PF,t+1}\right) = p \Leftrightarrow \\
\Phi\left(-VaR^p_{t+1}/\sigma_{PF,t+1}\right) = p
\end{aligned}$$

where $\Phi(*)$ denotes the cumulative density function of the standard normal distribution.

$\Phi(z)$ calculates the probability of being below the number z, and $\Phi_p^{-1} = \Phi^{-1}(p)$ instead calculates the number such that $p \cdot 100\%$ of the probability mass is below Φ_p^{-1}. Taking $\Phi^{-1}(*)$ on both sides of the preceding equation yields the *VaR* as

$$\begin{aligned}
-VaR^p_{t+1}/\sigma_{PF,t+1} = \Phi^{-1}(p) \Leftrightarrow \\
VaR^p_{t+1} = -\sigma_{PF,t+1}\Phi_p^{-1}
\end{aligned}$$

上式右边第一项为资产收益率的波动率，第二项为正态分布累积函数的逆函数在p处的值.

正态分布累积逆函数(ICDF)

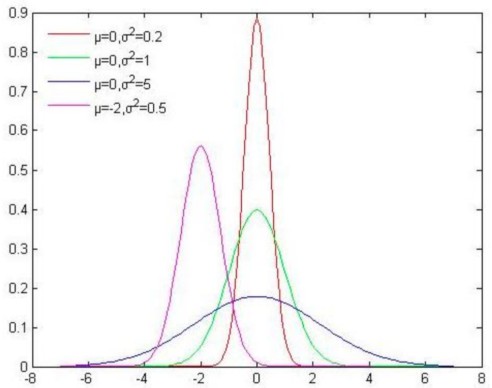
- 累积分布：分位点 -> 概率，
- 逆累积分布：概率 ->分位点

正态分布（*Normal distribution*）又名**高斯分布**（Gaussian distribution），是一个在数学、物理及工程等领域都非常重要的概率分布，在统计学的许多方面有着重大的影响力。

若随机变量X服从一个数学期望为μ、标准方差为σ^2的高斯分布，记为：$X \sim N(\mu, \sigma^2)$，

则其概率密度函数为

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$



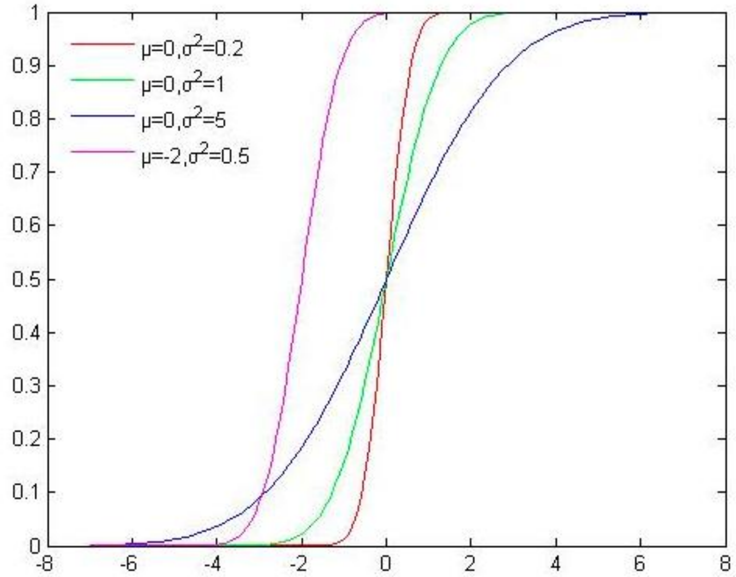
正态分布的期望值μ决定了其位置，其标准差σ决定了分布的幅度。因其曲线呈钟形，因此人们又经常称之为钟形曲线。我们通常所说的标准正态分布是$\mu = 0, \sigma = 1$的正态分布（见右图中绿色曲线）。

正态分布的概率密度函数均值为μ 方差为σ^2 (或标准差σ)是高斯函数的一个实例：

$$f(x;\mu,\sigma) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

累积分布函数

累积分布函数是指随机变量X小于或等于x的概率，用密度函数表示为$F(x;\mu,\sigma) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{x} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) dx.$



Modeling – AR Autoregressive Model 自回归

- Pearson's Correlation

$$Cov(X,Y) = E[(X-\mu_x)(Y-\mu_y)] \qquad \rho_{xy} = \frac{Cov(X,Y)}{\sqrt{Var(X)Var(Y)}}$$

- Autocorrelation Function, ACF

$$\rho_l = \frac{Cov(r_t, r_{t-l})}{\sqrt{Var(r_t)Var(r_{t-l})}} = \frac{Cov(r_t, r_{t-l})}{Var(r_t)}$$

- 前提：Weak stationary（弱平稳状态下 即该序列的均值，协方差不随时间而改变）

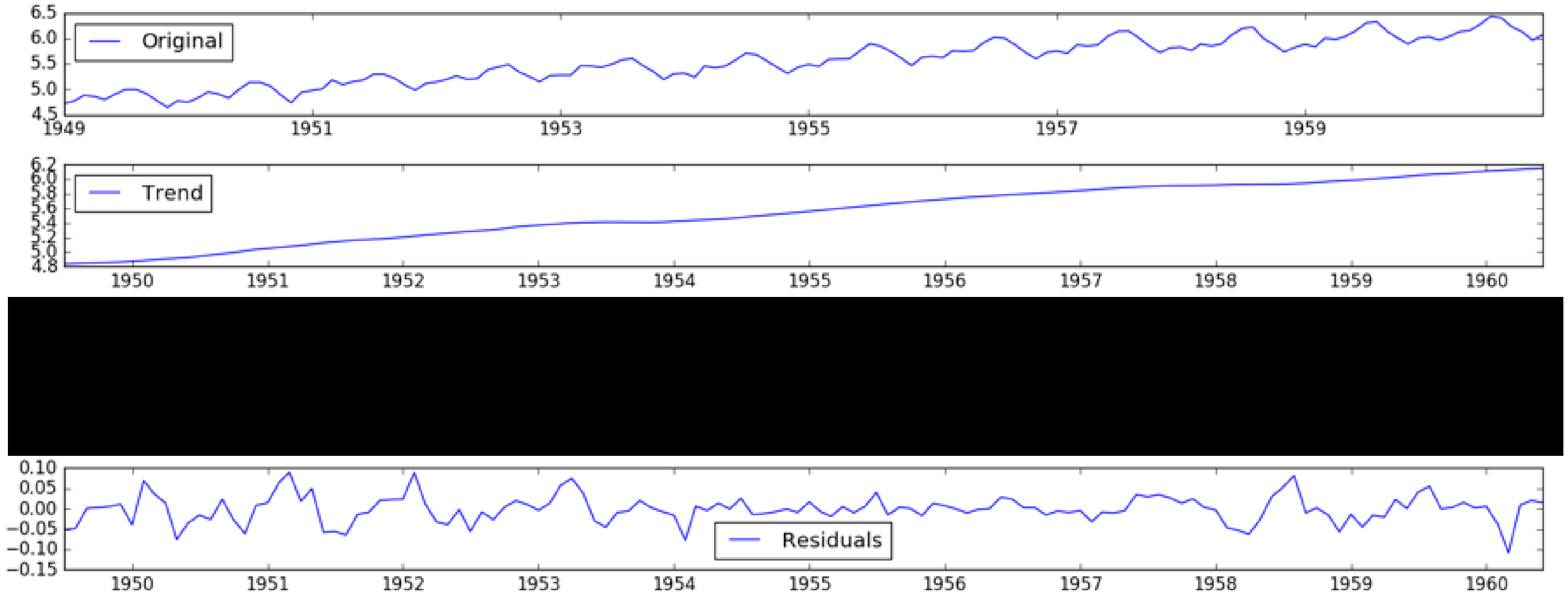
- **随机变量 X_t 和 X_s 的协方差为零，则称其为纯随机过程。**随机过程中，白噪声时间序列的定义是均值为零，方差恒定和相关性为零，白噪声无法进行预测。否则，可能可以改善这个模型

Modeling – AR

这说明在t−1时刻的数据，在预测t时刻t时可能是有用的。AR模型中, [Φ]是AR(p) 模型中的自回归系数， a_t是白噪声，也叫新息序列，也就是白噪声误差项。

$$r_t = \phi_0 + \phi_1 r_{t-1} + a_t$$

$$r_t = \phi_0 + \phi_1 r_{t-1} + \cdots + \phi_p r_{t-p} + a_t$$



Modeling – ARCH

- ARCH自回归条件异方差模型，波动率定义为条件标准差

计算均值方程残差：

$$a_t = r_t - \mu_t$$

- 存在波动率聚集(volatility clustering)
- 波动率随时间连续变化，一般不出现波动率的跳跃式变动
- 波动率一般在固定的范围内变化，意味着动态的波动率是平稳的
- 在资产价格大幅上扬和大幅下跌两种情形下， 波动率的反映不同， 大幅下跌时波动率一般也更大， 这种现象称为杠杆效应（leverage effect）

1.资产收益率序列的扰动$\{a_t\}$是序列不相关的，但是不独立。

2.$\{a_t\}$的不独立性可以用其延迟值的简单二次函数来描述。

具体而言，一个ARCH(m)模型为：

$$a_t = \sigma_t \varepsilon_t \quad \sigma_t^2 = \alpha_0 + \alpha_1 a_{t-1}^2 + \cdots + \alpha_m a_{t-m}^2 \quad \alpha_0 > 0; \forall i > 0, \alpha_i \geqslant 0$$

其中，ε_t为**均值为0，方差为1的独立同分布（iid）随机变量序列**，通常假定其服从标准正态分布。σ_t^2为条件异方差。

Modeling – ARCH

- 对资产收益率序列建立波动率模型需要如下四个步骤：

1. 通过检验序列的自相关性建立均值的方程， 必要时还可以引入适当的解释变量；
2. 对均值方程的残差作白噪声检验， 通过后，对残差检验ARCH效应；
3. 如果ARCH效应检验结果显著， 则指定一个波动率模型， 对均值方程和波动率方程进行联合估计；
4. 对得到的模型进行验证， 需要时做改进。

Modeling – GARCH

(Bollerslev 1986)提出了ARCH模型的一种重要推广模型， 称为GARCH模型。 对于一个对数收益率序列r_t，令$a_t = r_t - \mu_t = r_t - E(r_t|F_{t-1})$为其新息序列， 称$\{a_t\}$服从GARCH$(m, s)$模型， 如果$a_t$满足

$$a_t = \sigma_t \varepsilon_t, \quad \sigma_t^2 = \alpha_0 + \sum_{i=1}^{m} \alpha_i a_{t-i}^2 + \sum_{j=1}^{s} \beta_j \sigma_{t-j}^2 \tag{18.1}$$

其中$\{\varepsilon_t\}$为零均值单位方差的独立同分布白噪声列， $\alpha_0 > 0, \alpha_i \geq 0, \beta_j \geq 0$, $0 < \sum_{i=1}^{m} \alpha_i + \sum_{j=1}^{s} \beta_j < 1$， 这最后一个条件用来保证满足模型的$a_t$的无条件方差有限且不变， 而条件方差$\sigma_t^2$可以随时间$t$而变。

假定m，s都取1时候，GARCH(1,1)模型的新息序列为：

$$\sigma_t^2 = \omega + \alpha a_{t-1}^2 + \beta \sigma_{t-1}^2$$

Modeling – GARCH

- *波动率预测*

$\sigma_{t+1}^2 = 0.000004 + 0.05a_t^2 + 0.93\sigma_t^2$

JP Morgan's RiskMetrics model(SAP)

$$\sigma_{PF,t+1}^2 = 0.94\sigma_{PF,t}^2 + 0.06R_{PF,t}^2$$

收益率预测

$$r = 0.000636 - 0.038533a_t + 0.029812\, a_{t-1} + \ldots\ldots -0.007057\, a_{t-7}$$

- *VaR计算, 这里取使用NORM(0,1)的正态分布累积逆函数*

$$VAR_{t+1} = -\sigma_{t+1}\Phi^{-1}\,(0,1)$$

Cornish-Fisher Model

The VAR is calculated from the four moments of the distribution (expected value, variance, skewness and kurtosis) using Cornish/Fisher approximation

The Cornish-Fisher *VaR* with coverage rate p can then be calculated as

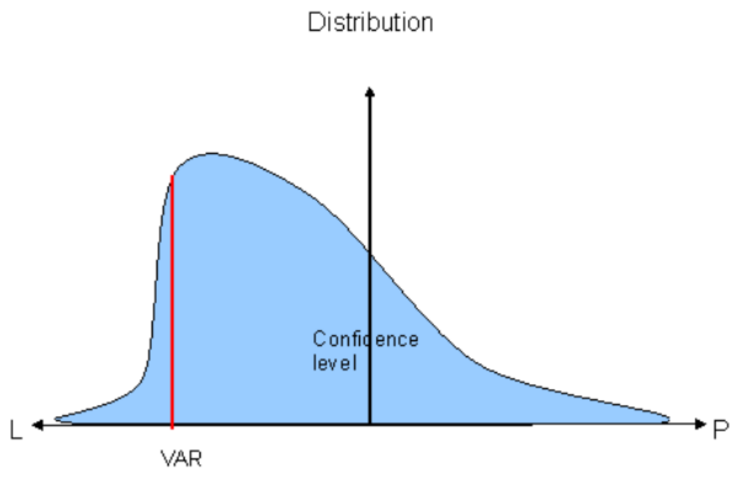
$$VaR_{t+1}^{p} = -\sigma_{PF,t+1} CF_p^{-1}$$

where

$$CF_p^{-1} = \Phi_p^{-1} + \frac{\zeta_1}{6}\left[(\Phi_p^{-1})^2 - 1\right] + \frac{\zeta_2}{24}\left[(\Phi_p^{-1})^3 - 3\Phi_p^{-1}\right] - \frac{\zeta_1^2}{36}\left[2(\Phi_p^{-1})^3 - 5\Phi_p^{-1}\right]$$

where ζ_1 is the skewness and ζ_2 is the excess kurtosis of the standardized returns, z_t. The Cornish-Fisher quantile can be viewed as a Taylor expansion around the normal distribution. Notice that if we have neither skewness nor excess kurtosis so that $\zeta_1 = \zeta_2 = 0$, then we simply get the quantile of the normal distribution

$$CF_p^{-1} = \Phi_p^{-1}, \quad \text{for } \zeta_1 = \zeta_2 = 0$$



Structured Monte Carlo

- In Monte Carlo simulation, it is assumed that there is a normal model of risk factor changes with an expected value of zero and a positive variance. The parameters required for the random process can be calculated from historical values.

$$VaR^{p}_{t+1:t+K} = -Percentile\left\{\left\{\check{R}_{i,t+1:t+K}\right\}_{i=1}^{MC}, 100p\right\}$$

Risk Calculation for Portfolio

- Covariance：

$$Cov(X,Y) = E[(X - \mu_x)(Y - \mu_y)]$$

$$\sigma_{xy} = Cov(X,Y)$$

- Pearson's Correlation

$$\rho_{xy} = \frac{Cov(X,Y)}{\sqrt{Var(X)Var(Y)}}$$

- If$1 \geq \rho \geq -1, if\ \rho = 0$ means that two variables don't have any linear relation.

Portfolio Volatility

- For a portfolio comprising of three assets A, B and C having weights in the portfolio of A, B and C respectively the portfolio volatility is given by the

$$
\begin{aligned}
\mathrm{var}(\mathbf{R}) &= \begin{pmatrix} \mathrm{var}(R_A) & \mathrm{cov}(R_A, R_B) & \mathrm{cov}(R_A, R_C) \\ \mathrm{cov}(R_B, R_A) & \mathrm{var}(R_B) & \mathrm{cov}(R_B, R_C) \\ \mathrm{cov}(R_C, R_A) & \mathrm{cov}(R_C, R_B) & \mathrm{var}(R_C) \end{pmatrix} \\
&= \begin{pmatrix} \sigma_A^2 & \sigma_{AB} & \sigma_{AC} \\ \sigma_{AB} & \sigma_B^2 & \sigma_{BC} \\ \sigma_{AC} & \sigma_{BC} & \sigma_C^2 \end{pmatrix} = \mathbf{\Sigma}.
\end{aligned}
$$

$$
\begin{aligned}
\sigma_{p,x}^2 &= \mathrm{var}(\mathbf{x}'\mathbf{R}) = \mathbf{x}'\Sigma\mathbf{x} = (x_A, x_B, x_C) \cdot \begin{pmatrix} \sigma_A^2 & \sigma_{AB} & \sigma_{AC} \\ \sigma_{AB} & \sigma_B^2 & \sigma_{BC} \\ \sigma_{AC} & \sigma_{BC} & \sigma_C^2 \end{pmatrix} \begin{pmatrix} x_A \\ x_B \\ x_C \end{pmatrix} \\
&= x_A^2\sigma_A^2 + x_B^2\sigma_B^2 + x_C^2\sigma_C^2 + 2x_Ax_B\sigma_{AB} + 2x_Ax_C\sigma_{AC} + 2x_Bx_C\sigma_{BC}.
\end{aligned}
$$

Markowitz Theory （最小方差投资组合）

- $$\min \delta^2(r_p) = \sum\sum w_i w_j cov(r_i, r_j)$$
- 式中：r_p——组合收益；
- r_i、r_j——第i种、第j种资产的收益；
- w_i、w_j——资产i和资产j在组合中的权重；
- $\delta^2(r_p)$——组合收益的方差即组合的总体风险；
- $cov(r_i, r_j)$——两种资产之间的协方差。

Sharpe ratio

The **Sharpe ratio** measures the return per unit of risk. The higher the Sharpe ratio, the better the combined performance of risk and return. To calculate the Sharpe ratio, the system uses the following formula:

$$SR = \frac{r_p - b}{\sigma_p}$$

Where:

r_p = annualized portfolio return
b = annualized benchmark return
σ_p = annualized portfolio risk (standard deviation of the portfolio return)

MVaR and VaRC

- The effect of the small changes in a part of the portfolio to the portfolio VaR is measured by marginal value at risk (MVaR). 边际VaR，即当组合中的某种资产变化一个单位或变化1%时投资组合VaR的变化值。
- The contribution of a position or a sub portfolio to the total VaR is measured by value at risk contribution (VaRC). 成分VaR，成分VaR的定义是投资组合的一个部分被剔除掉，投资组合的VaR的近似变化。

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