Exercises

Exercises should be completed **on your own.**

1. **(1 pt.)** Suppose that $h : U \to \{0, \ldots, n-1\}$ is a uniformly random function. That is, $h(i)$ is distributed uniformly at random in the set $\{0, \ldots, n-1\}$ for all i, and $h(i)$ are independent for all i. Prove that for any $x \neq y \in U$,
$$\mathbb{P}_h\{h(x) = h(y)\} = \frac{1}{n}.$$
[We are expecting: A short but rigorous proof.]

2. **(2 pt.)** This exercise references the IPython notebook `HW4.ipynb` as well as the files `mysteryA.pickle` and `mysteryB.pickle`.

   In the IPython notebook, run the cells to load the hash families A and B. Both A and B are lists of functions $h : \{0, \ldots, 21\} \to \{0, 1, 2\}$. As shown in the notebook, at first glance both of these seem like reasonable hash families. **However,** one of them is a universal hash family and one of them is not. Which one is which? Play around with both hash families in Python until you figure it out.

   [We are expecting: Your answer, along with compelling numerical evidence (either numbers or a plot), and an explanation about why your numerical evidence is compelling and what it has to do with the definition of a universal hash family.]

Problems

You may talk with your fellow CS161-ers about the problems. However:

- Try the problems on your own *before* collaborating.
- Write up your answers yourself, in your own words. You should never share your typed-up solutions with your collaborators.
- If you collaborated, list the names of the students you collaborated with at the beginning of each problem.

---

1. **(3 pt.)** Your friend has a proposal for a new universal hash function $h : \mathcal{U} \to \{0, \ldots, n-1\}$, where $\mathcal{U} = \{0, \ldots, n^2 - 1\}$. Your friend thinks that you can skip this whole "choose uniformly from a universal hash family" stuff, and just go with

$$h(x) = x \bmod n.$$

More precisely, your friend says, just take the hash family $\mathcal{H} = \{h\}$ to be the set with just this one function in it.

   (a) **(1 pt.)** Your friend doesn't have a very good track record on these homework sets, so you are dubious even before you hear their argument. Prove to your friend that their choice does *not* satisfy the key property of a universal hash family.

   [We are expecting: A rigorous proof, using the definition of a universal hash family.]

   (b) **(1 pt.)** Even given your proof, your friend plows on. Their first point:

   > Let $h = x \bmod n$ be as above. If we choose $x \neq y$ uniformly at random[1] from $\mathcal{U}$, then $\mathbb{P}\{h(x) = h(y)\} \leq \frac{1}{n}$, where the probability is over the random choice of x and y.

   Do you agree?

   [We are expecting: Whether the statement is true or false, and a convincing argument either way.]

   (c) **(1 pt.)** Your friend continues:

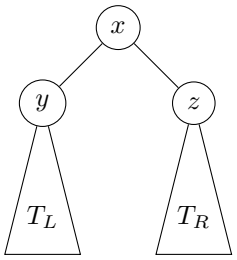
   > Given the computation above, we have
   >
   > $$\mathbb{P}\{h(x) = h(y)\} \leq \frac{1}{n}.$$
   >
   > This is the definition of a universal hash family, so $\{h\}$ must be a universal hash family.

   Do you agree?

   [We are expecting: Whether this conclusion correctly follows from the statement in part (b), and a convincing argument either way.]

[1] That is, choose x uniformly at random from $\mathcal{U}$ and then choose y uniformly at random from $\mathcal{U} \setminus \{x\}$

2. **(5 pt.)** Let T be a Red-Black tree with root x. Let T_L be the subtree rooted at x's left child, and let T_R be the subtree rooted at x's right child.



Decide if each of the following statements are true or false. If it is true, give a proof. If it is false, give a counter-example.

(a) **(2 pt.)** In the set-up above, we must have

$$|T_L| \geq \frac{|T|}{2} - 1 \qquad \text{and} \qquad |T_R| \geq \frac{|T|}{2} - 1, \tag{1}$$

where $|T|$ denotes the number of nodes in T (including the root, not including NIL nodes).

(b) **(3 pt.)** In the set-up above, we must have

$$|T_L| \geq \frac{\sqrt{|T|}}{2} - 1 \qquad \text{and} \qquad |T_R| \geq \frac{\sqrt{|T|}}{2} - 1, \tag{2}$$

where $|T|$ denotes the number of nodes in T (including the root, not including NIL nodes).

[We are expecting: For each, either a rigorous proof, or an explicit counter-example.]
[HINT: You may use a claim that we proved in class.]

3. **(6 pt.)** A large flock of T Colorful Geese will migrate south for the winter over the Gates building in the next few weeks. Colorful Geese are an interesting species. They can come in a huge number of colors—say, M colors—but each flock only has m colors represented, where $m < T$. You'd like to be able to answer queries about what colors of geese appeared in the flock. The birds will fly overhead one at a time, and after they have flown by they won't come back again.

   For example, if $T = 7$, $M = 100000$ and $m = 3$, then a flock of T colorful geese might look like:

`seabreeze, seabreeze, brick red, ultraviolet, brick red, ultraviolet, seabreeze`

   You'll see this sequence in order, and only once. After the birds have gone, you'll be asked questions like "How many `brick red` geese were there?" (Answer: 2), or "How many `neon orange` geese were there?" (Answer: 0).

   You have access to a universal hash family $\mathcal{H}$, so that each function $h \in \mathcal{H}$ maps the set of M colors into the set $\{0, \ldots, n-1\}$. For example, one function $h \in \mathcal{H}$ might have $h(\texttt{seabreeze}) = 3$.

   (a) **(3 pt.)** Suppose that $n = 10m$, and you only have space to store n numbers in the set $\{0, \ldots, T\}$, as well as one function h from $\mathcal{H}$. Use the universal hash family $\mathcal{H}$ to create a randomized data structure that fits in this space and that supports the following operations:

   - `Update(color)`: Update the data structure when you see a goose with color `color`.
   - `Query(color)`: Return the number of geese of color `color` that you have seen so far. For each query, your query should be correct with probability at least 9/10. That is, for all colors i,

$$\mathbb{P}\{\texttt{Query}(i) = \text{ the number of geese with color } i \} \geq \frac{9}{10}.$$

   You want each of these operations to be done in $O(1)$ time (in the worst case), assuming that you can evaluate a function $h \in \mathcal{H}$ in $O(1)$ time.

   [We are expecting: An explanation of how you will implement your operations, and a short but rigorous proof that your operations meet the requirements.]

   (b) **(3 pt.)** Suppose that you now have ten times the space you had in part (a). Adapt your data structure from part (a) so that the `Query` operation is correct with probability $1 - \frac{1}{10^{10}}$.

   [We are expecting: An explanation of how you will implement your operations, and a short but rigorous proof that your operations meet the requirements.]