Marked seminar – BE 3 – Random signal processing

In the second seminar, we studied theoretically a well-known signal processing technique dubbed “matched filter”. Herein, we propose to implement this filter in the context of radar. To that end, *before* the seminar prepare the preliminary questions of section 1.

Unlike the convention used in the course handouts, we use in this seminar notations commonly used in scientific papers. In particular, herein, we *do not use* uppercase letter to distinguish a random process from its realizations. Lowercase letters are systematically used instead to designate either the random process or one of its realization. Beware!

1 Preliminary questions: how generating an AWGN?

This preliminary questions aim at proposing a method to generate a complex-valued white Gaussian noise denoted by $\boldsymbol{n} \sim \mathcal{CN}\left(\mathbf{0}, \sigma^2 \boldsymbol{I}_K\right)$ where σ^2 is the noise power and $\boldsymbol{I}_K$ is the identity matrix of size $K \times K$.
For that purpose let us consider two real-valued Gaussian vectors $\boldsymbol{n}_r \sim \mathcal{N}\left(\mathbf{0}, \frac{\sigma^2}{2} \boldsymbol{I}_K\right)$ and $\boldsymbol{n}_i \sim \mathcal{N}\left(\mathbf{0}, \frac{\sigma^2}{2} \boldsymbol{I}_K\right)$.

Question 1

Express the PDF of each vector $\boldsymbol{n}_r$ and $\boldsymbol{n}_i$. (Use the Appendix of the course handouts where $|\boldsymbol{C}|$ denotes the determinant of the matrix $\boldsymbol{C}$.)

Question 2

Express the *joint* PDF of the random vectors $\boldsymbol{n}_r$ and $\boldsymbol{n}_i$ assuming that $\boldsymbol{n}_r$ and $\boldsymbol{n}_i$ are independent.

Question 3

(a) Develop and give a simple expression of the following squared norm $(\boldsymbol{n}_r + j\boldsymbol{n}_i)^H(\boldsymbol{n}_r + j\boldsymbol{n}_i)$.

(b) Give a simplified expression of the determinant $|\sigma^2 \boldsymbol{I}_K|$.

(c) Recognize the joint PDF of $\boldsymbol{n}_r$ and $\boldsymbol{n}_i$ as a complex Gaussian PDF. What are its mean and covariance matrix?

(d) Summarize the results by proposing a simple method to generate $\boldsymbol{n} \sim \mathcal{CN}\left(\mathbf{0}, \sigma^2 \boldsymbol{I}_K\right)$.

2 Summary of former theoretical results

2.1 Matched filtering in white noise

Consider a signal consisting of a SoI and an additive noise

$$x(t) = s(t) + n(t)$$

where $s(t)$ is the known[1] deterministic signature of the SoI with finite energy and $n(t)$ is the noise with known PSD. The filter that maximizes the SNR at its output is the matched filter. In case of white noise, the expression of the matched filter is

$$h(t) \propto s^*(t_0 - t)$$

where t_0 is the instant at which the SNR is maximized (cf. Fig 1).

[1] Up to a multiplicative factor.

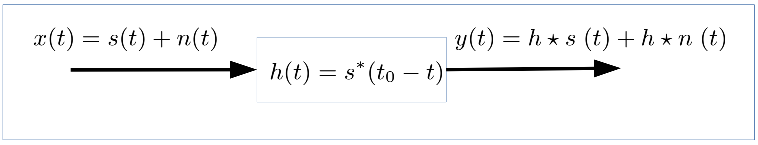


Figure 1: Matched filtering in white noise.

2.2 Range matched filtering

A direct application of matched filtering can be found in radar. It is used as a preprocessing stage in the receiver and is designed to maximize the SNR of the target's echo at an instant corresponding to its round-trip delay. In presence of just one nonmoving point target, the baseband received signal can be expressed as

$$\boxed{x(t) = \alpha\ e(t - \tau) + n(t)} \quad \text{with} \quad t \in [0, T_r] \tag{1}$$

where

$e(t)$ is the known transmitted waveform;

α, τ denotes the complex attenuation and the round-trip delay of the target due to propagation;

$n(t)$ is the internal noise of the receiver well represented by a white noise;

T_r is the so-called pulse repetition interval (PRI).

The range matched filter can thus, by design, be expressed as $\boxed{h(t) = e^*(-t)}$. It is worth noticing that the latter is known by the radar operator since it is a conjugated time-reversed version of the transmitted waveform. Input and output signals of the range matched filter are illustrated in figure 2 for a simple waveform $e(t) = \mathbb{I}_{[0,T]}(t)$ with T the pulse duration. For this specific waveform it was shown that the maximum output SNR in white noise is $\rho(\tau) = |\alpha|^2 T/(2N_0)$ with $2N_0$ the noise PSD.

3 Signal generation

3.1 From continuous to discrete-time signals

In practice, range matched filtering may be implemented in the digital domain. To that end, an analog-to-digital converter (ADC) is placed after downconversion so that one has to process samples gathered in the vector $\boldsymbol{x}$ (cf. figure 3)

$$\boldsymbol{x} = \begin{bmatrix} x_0 & \dots & x_k & \dots x_{K-1} \end{bmatrix}^T \quad \text{with} \quad x_k = x(kT_s) \quad \text{and} \quad K = \lfloor T_r/T_s \rfloor$$

where $\lfloor . \rfloor$ rounds to the nearest integer towards minus infinity and T_s is the sampling period. In what follows, we consider a simple waveform defined by $e(t) = \mathbb{I}_{[0,T]}(t)$[2] with $t \in [0, T_r]$. To simplify the implementation, we assume that

1. the target round-trip delay verifies $T < \tau < T_r - T$;
2. the target round-trip delay and the pulse duration T are both multiples of the sampling period, i.e.,

$$\tau = k_\tau T_s \quad \text{with} \quad k_\tau \in \mathbb{N}$$
$$T = k_T T_s \quad \text{with} \quad k_T \in \mathbb{N}$$

[2]Rigorously, this signal is not bandlimited. In practice, we take a sampling period $T_s \ll T$ to make sure that it can be, nonetheless, approximated this way.

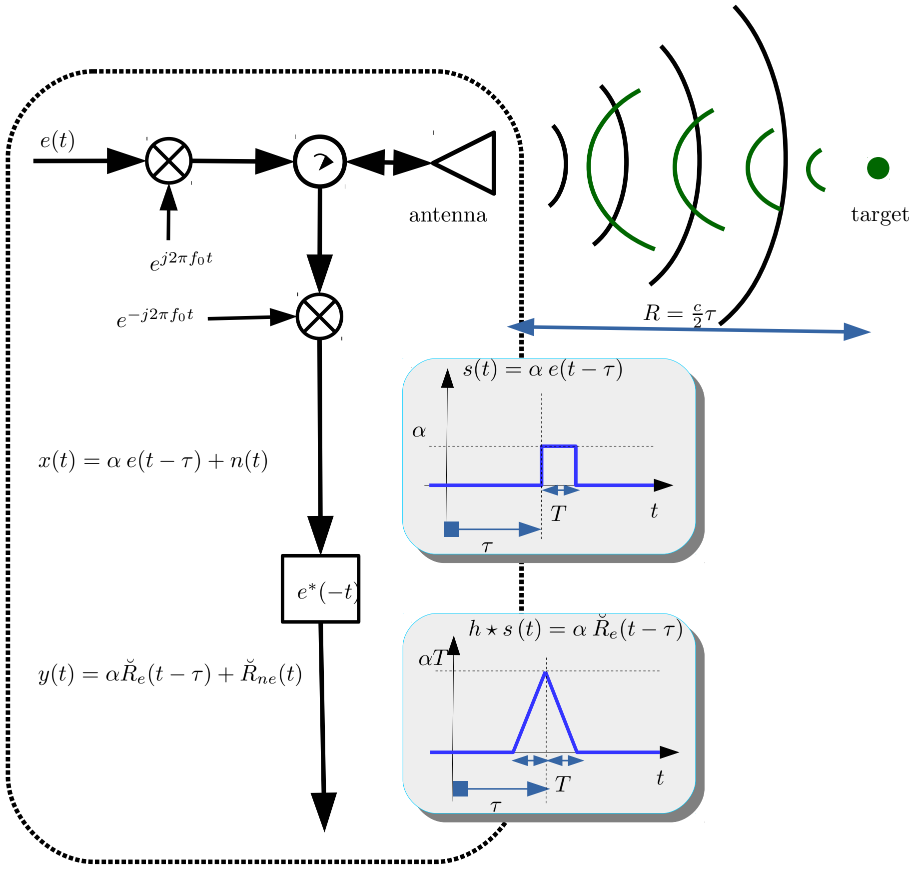


Figure 2: Radar scenario: transmitted and received waveforms; processing (downconversion + range matched filtering); a single nonmoving point target's echo amid internal noise. c is the speed of light. f_0 is the radar carrier frequency. $\breve{R}$ denotes the deterministic correlation function. Input/output of range matched filter illustrated for the non-realistic case where $\alpha \in \mathbb{R}$.

Given (1), the received signal vector can then be expressed as

$$\boxed{\boldsymbol{x} = \alpha\,\boldsymbol{a} + \boldsymbol{n}} \quad \text{with} \quad a_k = e_{k-k_\tau}. \tag{2}$$

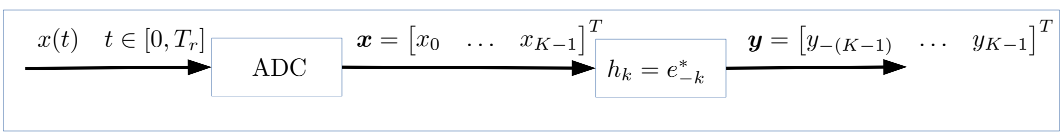


Figure 3: Range matched filtering with discrete-time signal.

3.2 SoI

Question 4

Generate and represent the transmitted waveform $\boldsymbol{e}$ as a function of time in seconds.

To generate a vector with linearly equally spaced points a simple syntax is `(start:step:end)`.

You may choose the following numerical values for signal generation.

Pulse repetition interval	$T_r = 1$ ms
Pulse duration	$T \ll T_r$ ms
Time sampling	$T_s \ll T$ ms
Thermal noise power	$\sigma^2 = 0$ dBW

Question 5

Generate and represent the SoI vector $\alpha\boldsymbol{a}$ as a function of time in seconds. Recall that the amplitude α is a priori complex valued.

When implementing a signal processing technique, one usually needs to define (at least) the following steps

- define parameters that describe the scenario and the processing;
- generate the signal;
- process the signal;
- display the results.

For that purpose you may create dedicated sections starting with `%%`. Also, if you are writing a script with Matlab, do not forget to clear variables from Matab memory.

3.3 Additive noise

Question 6

(a) Generate one realization of a real valued Gaussian white noise $\boldsymbol{n}_r \sim \mathcal{N}\left(\boldsymbol{0}, \frac{\sigma^2}{2}\boldsymbol{I}_K\right)$. For that purpose recall that for such random vector, one has

$$\mathcal{E}\left\{n_{rk}\right\} = 0 \quad \text{and} \quad \mathcal{E}\left\{n_{rk}n_{rk'}^*\right\} = \begin{cases} \frac{\sigma^2}{2} & \text{if } k = k' \\ 0 & \text{otherwise.} \end{cases}$$

This shows that the elements of $\boldsymbol{n}_r$ are uncorrelated. Since they are Gaussian distributed, it can be shown that they are actually independent.

To generate a real-valued Gaussian vector whose elements are independent and identically distributed (iid), you can either use `normrnd` or `randn`.

(b) Generate one realization of a real valued Gaussian white noise $\boldsymbol{n}_i \sim \mathcal{N}\left(\boldsymbol{0}, \frac{\sigma^2}{2}\boldsymbol{I}_K\right)$ considering that $\boldsymbol{n}_r$ and $\boldsymbol{n}_i$ are two independent random vectors.

(c) Using the results of the preliminary questions of section 1, generate the complex AWGN $\boldsymbol{n} \sim \mathcal{CN}\left(\boldsymbol{0}, \sigma^2 \boldsymbol{I}_K\right)$.

(d) Generate and represent the whole received signal $\boldsymbol{x}$.

Question 7

Within this section we invoke the principle of ergodicity to estimate the first order noise statistics and its spectrum.

(a) Verify that the mean, the variance and the PDF of n_r (n_i resp.) is conform to the theory.

> To perform a statistical analysis of a signal a lot of predefined functions can be used such as `mean`, `var`, `normpdf`, `histogram`.

(b) Estimate the noise PSD with the periodogram technique

$$S_n(f) = \frac{1}{K}\left|\sum_{k=0}^{K-1} n_k e^{-j2\pi k f}\right|^2$$

with f the normalized frequency. In practice, the sum is implemented via a Fast Fourier Transform (FFT).

(c) Represent then the PSD estimate in decibels wrt frequency axis expressed in Hertz and centered around 0 Hz.

> To perform and represent the spectral analysis of a signal (e.g., via a periodogram), the following functions are useful `fft`, `abs`, `fftshift`, `log10`.

4 Range matched filtering

Question 8

(a) Implement the matched filter as an operation of cross-correlation between the received signal and that transmitted, i.e., as demonstrated in the second seminar,

$$y_k = \breve{R}_{xe}(k) \stackrel{\text{def.}}{=} \sum_n x_n e^*_{n-k}$$

for the time-lag indices $k = -(K-1), \ldots, K-1$.

> To auto- or cross-correlate signals with Matlab you may use the function `xcorr`:
>
> - The second input argument is the signal used as a reference signal.
> - The second output argument is the vector of the time-lag indices.
>
> (Do not use any normalization option for this seminar.)

(b) Represent the matched filter output as a function of 1) the time-lag expressed in seconds (this time-scale is referred to as *fast*-time in radar); 2) the distance between the target and the radar in meters.

(c) It has been shown in the second seminar that the radar range resolution with such waveform is equal to T in seconds or, equivalently, to $\delta_R = (c/2)T$ in meters. Verify this result on you graph.

Question 9

(Facultative questions only for advanced students) Recalling that the input and output signals are given resp. by

$$
\begin{aligned}
x_k &= \alpha\, e_{k-k_\tau} + n_k \\
y_k &= \alpha\, \breve{R}_e(k-k_\tau) + \breve{R}_{ne}(k)
\end{aligned}
$$

(a) explain the value of the correlation peak observed on your graph;

(b) express the SNR before matched filtering at the time index k_τ as a function of α and σ^2;

(c) express the SNR after matched filtering at the time index k_τ as a function of α and σ^2 and k_T; you may either derive directly the SNR (for the denominator observe that the $\breve{R}_{ne}(k_\tau) = \boldsymbol{a}^H\boldsymbol{n}$) or use the result of the second seminar;

(d) give the processing gain so obtained.

5 Influence of the transmitted waveform

In this section, we consider another waveform, namely a so-called *chirp*

$$
e(t) = e^{j2\pi\frac{\beta}{2}t^2}\mathbb{I}_{[0,T]}(t) \quad \text{with } t \in [0, T_r]
$$

where β is a user-defined parameter.

> To understand the origin of the designation “chirp” (*gazoullis* in French) go to http://www.sigidwiki.com/wiki/Category:Radar and play one of the recorded radar data set with a frequency modulated (FM) waveform. You can also observe on this website other recorded radar waveform.

Question 10

Derive the instantaneous frequency of the transmitted waveform $e(t)$.

Question 11

How choosing the parameter β to span a given synthetic bandwidth B.

Question 12

As for the non-modulated waveform, generate and represent the transmitted and received signals as well as the range matched filter signal.

Question 13

It can be shown analytically that the range matched filter waveform $\breve{R}_e(t)$ has a functional form that is well approximated by the function $\mathrm{sinc}(\pi B t)$ (up to a multiplicative factor).

(a) What is the range resolution of the radar in seconds? in meters?

(b) Check this value on your graph.

6 Experimental radar data (facultative)

In this facultative section (for advanced student only), we consider experimental radar data that can be downloaded at `https://sourceforge.isae.fr/projects/ralf/wiki/Examples_(radar)`. We particularly consider the file `2017-07-27-17-07-46.mat`. A remote-control car is alternatively closing and receding from the radar in the corridor running along the RALF's room. A chirp waveform is used with the particular setting $T = T_r$; in that case the waveform is said to be "continuous" instead of "pulsed". With such waveform it is possible to implement the range matched filter via a simple inverse FFT on the so-called mixed signal defined as

$$s_{\text{mix}}(t) = x(t) \times e^*(t).$$

This operation is known as *destretching* or *deramping* [KB10, sec 2.2].

Question 14

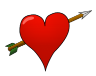
Download 1) the data file 2) the zip file `matlab_practical` from the LMS (Sujet du BE 3). Open the code `deramping.m`:

- M is the number of times the waveform is repeated (hence M range profiles can be obtained);
- L denotes the number of fast-time samples (i.e., $L = k_T$);
- the radar has two receiving channels so that 2 mixed signals are collected.

(a) Fill in the code: particularly implement the range transform via an IFFT.

(b) What is the range resolution?

(c) Observe the obtained range-time intensity plot: do you see the target trajectories?



Windows forever. If you are an aficionados of Windows, are at least, of its default shortcuts you may restore them even if you are working with Linux. To that end:

- go to the Matlab HOME;
- click on `Preferences`;
- in the left hand side, click on `Keyboard` then on `Shortcuts`;
- in the right hand side, choose `Windows Default Set` as `Active settings`

Acronyms

You are welcome to use the following acronyms without redefining them.

ACF Auto-Correlation Function

AWGN Additive white Gaussian Noise

CCF Cross-Correlation Function

CDF Cumulative Distribution Function

LTI Linear Time Invariant

PSD Power Spectral Density

PDF Probability Density Function

RP Random Process

RV Random Variable

SoI Signal of Interest

SNR Signal-to-Noise-Ratio

SSS Strict Sense Stationary

WSS Wide Sense Stationary

iid independent and identically distributed

wrt with respect to

resp. respectively

References

[KB10] Byron Murray Keel and J. Mike Baden. Advanced pulse compression waveform modulations and techniques. In Mark A. Richards, James A. Scheer, and William A. Holm, editors, *Principle of Modern Radar. Basic Principles*, chapter 2, pages 19–85. SciTech Publishing, 2010.