PROGRAMMING LOGIC AND TECHNIQUES

Numeral systems

Introduction

- Recall the definitions of the terms “digit”, “numeral”, and “number” that we saw in the previous lesson on numbers:

- The term “**digit**” is used to designate a **symbol** out of the following set:
  - { “0”, “1”, “2”, “3”, “4”, “5”, “6”, “7”, “8”, “9” }

- The term “**numeral**” is used to designate a **symbol** composed of an arrangement of one or more digits. For example:
  - “2”, “34”, “3 ¾”, “0.567”, “2^4”, “–56”, “3,800,000”

Introduction

- The term "**number**" is used to designate a **mathematical quantity**, as opposed to a linguistic symbol.

- We can use linguistic symbols, such as digits, numerals, and other mathematical expressions, to:
  - Represent and refer to numbers
  - Make statements about numbers (which can be true or false)
  - Ask and answer questions about numbers
  - Think and reason about numbers
  - Perform calculations involving numbers

Introduction

- However, it's important to note that, in practice, the distinction between a) digits/numerals, and b) numbers, is not always made clear and explicit, and sometimes people's usage of this terminology is ambiguous or inconsistent.

  - "Number" is sometimes used to refer to a symbol (digit/numeral).

  - "Digit" is sometimes used to refer to a number (the number represented by a digit symbol).

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- Regardless, the distinction is an important one!

- Even if we sometimes speak casually, ambiguously, or loosely for certain practical purposes, we should make sure to never confuse symbols with what they represent.

- For example, just as we should never confuse the word "apple" with an apple, we should never confuse the numeral "12" with the number twelve.

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- As an example, consider the following linguistic symbols:
  - “four” in English
  - “quatre” in French
  - “Vier” in German
  - “चार” in Hindi
  - “أربعة” in Arabic
  - “넷” in Korean
  - “4” in the Hindu–Arabic numeral system
  - “IV” in the Roman numeral system
- All of these different symbols from different languages represent one and the same number – the number four!

Introduction

- A **numeral system** is a system of mathematical notation used for expressing numbers according to a set of formal rules. The numeral systems we will be studying in this course are systems of **positional notation.**

- A positional numeral system provides a systematic method of representing or encoding numbers by forming numerals out of a pre-defined set of digits.

- The most familiar positional numeral system is called the **decimal system**, and it is what we all use every day.

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- The **number of unique digits** that a positional numeral system uses (or the size of its “alphabet”) is called the **base** of the system. A system’s base determines how it works and how we classify it.

- If a positional numeral system uses **n** unique digits, then we call it a **base-n** system.

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- The decimal system uses ten digit symbols, as we have seen. So its base is ten, and it is also called the **base-10 system**, or simply **base-10**.

- But of course, the decimal system is only one out of many different base-**n** numeral systems. In addition to the decimal systems, we will be interested in the following systems:
  - **binary** (**base-2**)
  - **octal** (**base-8**)
  - **hexadecimal** (**base-16**)

  These systems are codes that are widely used in logic and in the world of computers.

Introduction

- **Decimal code** is a numeral system whose base is 10.
- In this system, we only use the following 10 digits:

{ “0”, “1”, “2”, “3”, “4”, “5”, “6”, “7”, “8”, “9” }

Introduction

- **Binary code** is a numeral system whose base is 2.
- In this system, we only use the following 2 digits:

{ "0", "1" }

Introduction

- **Hexadecimal code** is a numeral system whose base is 16.

- In this system, we only use the following 16 digits:

{ "0", "1", "2", "3", "4", "5", "6", "7", "8", "9",
"A", "B", "C", "D", "E", "F" }

"A" represents: ten
"B" represents: eleven
"C" represents: twelve
"D" represents: thirteen
"E" represents: fourteen
"F" represents: fifteen

Introduction

- Since we all have been using the decimal system since our youth, it is very likely that our usage and understanding of it has become automatic and intuitive, causing us to forget the fundamental principles of the decimal system.

- Let's do a little review of the decimal system. We can write a decimal number in **expanded form**, with each digit multiplied by a power of 10.

Introduction

Example:

$$4253_{(10)} = 4 \times 1000 + 2 \times 100 + 5 \times 10 + 3 \times 1$$

$$= 4 \times 10^3 + 2 \times 10^2 + 5 \times 10^1 + 3 \times 10^0$$

Introduction

- The other numeral systems are founded on the same principle, but involve replacing 10 with 2, 8, or 16, according to the case.

- Observing this will allow us to establish correspondences between these new bases and our good old base-10.

Conversions into decimal

a) Conversions from **binary into decimal:**

Example: $1101_{(2)} = \_\_\_\__{(10)}$

$$\begin{aligned} 1101_{(2)} &= 1 \times 2^3 + 1 \times 2^2 + 0 \times 2^1 + 1 \times 2^0 \\ &= 8 + 4 + 0 + 1 \\ &= 13_{(10)} \end{aligned}$$

Conversions into decimal

b) Conversions from **hexadecimal into decimal:**

Example : $1AC_{(16)} = \_\_\_\__{(10)}$

$$
\begin{aligned}
1AC_{(16)} &= 1 \times 16^2 + A \times 16^1 + C \times 16^0 \\
&= 1 \times 256 + 10 \times 16 + 12 \times 1 \\
&= 428_{(10)}
\end{aligned}
$$

Conversions from decimal

a) Conversions from **decimal into binary**:

To apply the transformation in the opposite direction, we must decompose the value into sum of factors involving powers of 2 (or powers of 8 or 16 for octal or hexadecimal).

For example, if we want to know the value of $43_{(10)}$ in binary form, we must establish that:

$$43 = 32 + 8 + 2 + 1$$

$$43 = 1 \times 2^5 + 0 \times 2^4 + 1 \times 2^3 + 0 \times 2^2 + 1 \times 2^1 + 1 \times 2^0$$

Therefore: $43_{(10)} = 101011_{(2)}$

Conversions from decimal

- As this decomposition is not easy to visualize, we propose using the following method:
  - 1. Using whole-number division, we divide the number by 2 (for binary), and write down the quotient and the remainder.
  - 2. We take the quotient just obtained and repeat the process starting at 1, using the quotient as the new number to divide. We continue repeating until we obtain a quotient equal to 0.
  - 3. Once finished, we take all of the remainders from the divisions, and read them backwards.

Conversions from decimal

Example:

43	$\div$ 2	=	21	remainder:	1	↑
21	$\div$ 2	=	10	remainder:	1	
10	$\div$ 2	=	5	remainder:	0	
5	$\div$ 2	=	2	remainder:	1	
2	$\div$ 2	=	1	remainder:	0	
1	$\div$ 2	=	**0**	remainder:	1	

Once finished, we read the remainders in reverse order (the opposite of the order in which they were obtained): 101011

$$43_{(10)} = 101011_{(2)}$$

Conversions from decimal

b) Conversions from **decimal into hexadecimal**:

Same method as previously, but dividing by16.

The division remainders that are between 10 and 15 are recoded to A, B, C, D, E and F.

Example :

$$43 \div 16 = 2 \qquad \text{remainder: } 11 \ \ (B)$$

$$2 \div 16 = \mathbf{0} \qquad \text{remainder: } 2 \ \ (2)$$

$$43_{(10)} = 2B_{(16)}$$

Conversions from binary into hexadecimal

- To convert from **binary into hexadecimal**:
  - 1. Separate the binary numeral into 4-digit segments, starting from the right.
  - 2. Convert each 4-digit segment into decimal.
  - 3. Convert these decimal numerals into hexadecimal.

Conversions from binary into hexadecimal

1. Separate numeral into 4-digit slices:	1	1011	0101	1100	1111
2. Convert each segment into decimal:	1	11	5	12	15
3. Convert each segment into hexadecimal:	1	B	5	C	F

- Final value in hexadecimal: $1B5CF_{(16)}$
- Note that the decimal values in row 2 will always be less than or equal to 15.

Conversions from hexadecimal into binary

- To obtain the inverse transformation, i.e. to convert from **hexadecimal into binary**, you must perform the same three steps as above, but in reverse order:
  - 1. Convert each hexadecimal digit into decimal.
  - 2. Convert each decimal numeral into binary, creating a segment of 4 binary digits.
  - 3. Combine all of the 4-digit segments together into a single binary numeral.

Conversions from hexadecimal into binary

- 1. Convert hexadecimal digits into decimal:
- 2. Convert decimal numerals into binary segments:

D	4	2	E	F	7
13	4	2	14	15	7
1101	0100	0010	1110	1111	0111

- 3. Combine the segments together into the final binary numeral:

$$1101\ 0100\ 0010\ 1110\ 1111\ 0111_{(2)}$$

Exercises

- Transform each of the following binary numerals into decimal numerals.
  - $1100_{(2)}$ =
  - $1000_{(2)}$ =
  - $10010_{(2)}$ =
  - $100100_{(2)}$ =
  - $10111_{(2)}$ =
  - $11111_{(2)}$ =
  - $1100011_{(2)}$ =
  - $1111111_{(2)}$ =

Exercises

- Transform each of the following binary numerals into decimal numerals.
  - $1100_{(2)} = 12$
  - $1000_{(2)} = 8$
  - $10010_{(2)} = 18$
  - $100100_{(2)} = 36$
  - $10111_{(2)} = 23$
  - $11111_{(2)} = 31$
  - $1100011_{(2)} = 99$
  - $1111111_{(2)} = 127$

Exercises

- Transform each of the following hexadecimal codes to decimal codes.
  - $6AF_{(16)}$ =
  - $1CD_{(16)}$ =
  - $C12_{(16)}$ =
  - $39E_{(16)}$ =
  - $B3D_{(16)}$ =
  - $5FA_{(16)}$ =
  - $48C_{(16)}$ =

Exercises

- Transform each of the following hexadecimal codes to decimal codes.
  - $6AF_{(16)}$ = 1711
  - $1CD_{(16)}$ = 461
  - $C12_{(16)}$ = 3090
  - $39E_{(16)}$ = 926
  - $B3D_{(16)}$ = 2877
  - $5FA_{(16)}$ = 1530
  - $48C_{(16)}$ = 1164

Exercises

- Transform each of the following decimal codes to binary codes.
  - $6_{(10)}$ =
  - $11_{(10)}$ =
  - $13_{(10)}$ =
  - $79_{(10)}$ =
  - $101_{(10)}$ =
  - $234_{(10)}$ =
  - $61_{(10)}$ =

Exercises

- Transform each of the following decimal codes to binary codes.
  - $6_{(10)} = 110$
  - $11_{(10)} = 1011$
  - $13_{(10)} = 1101$
  - $79_{(10)} = 1001111$
  - $101_{(10)} = 1100101$
  - $234_{(10)} = 11101010$
  - $61_{(10)} = 111101$

Exercises

- Transform each of the following decimal codes to hexadecimal codes.
  - $42_{(10)}$ =
  - $67_{(10)}$ =
  - $113_{(10)}$ =
  - $226_{(10)}$ =
  - $152_{(10)}$ =
  - $135_{(10)}$ =
  - $391_{(10)}$ =

Exercises

- Transform each of the following decimal codes to hexadecimal codes.
  - $42_{(10)}$ = 2A
  - $67_{(10)}$ = 43
  - $113_{(10)}$ = 71
  - $226_{(10)}$ = E2
  - $152_{(10)}$ = 98
  - $135_{(10)}$ = 87
  - $391_{(10)}$ = 187