PROGRAMMING LOGIC AND TECHNIQUES

Numbers

Definitions

- The term "**digit**" is used to designate a **symbol** out of the following set:
  - { "0", "1", "2", "3", "4", "5", "6", "7", "8", "9" }
- The term "**numeral**" is used to designate a **symbol** composed of an arrangement of one or more digits.
  For example:
  - "2", "34", "3 ¾", "0.567", "2^4", "−56", "3,800,000"
- The term "**number**" is used to designate a **mathematical quantity**, as opposed to a linguistic symbol.
- Digits and numerals are symbols that can represent numbers.

Natural numbers

- The set of all **non-negative whole numbers** is called the set of **natural numbers.**
  - { 0, 1, 2, 3, 4, 5, 6, 7, 8, 10, 11, 12, ... }
- These are the numbers we use for **counting**, among other things.

Natural numbers: addition

- To add two or more numbers by hand, we must place them one on other, taking care to form columns so that the corresponding digit positions of all the numbers are lined up – that is, a column with the ones, another with the tens, another with the hundreds, etc…

- The result of an addition is called the **sum**.

Natural numbers: addition

- *Example*: Find the sum of 23005 plus 9698.

```
 1  1 1
  23005
+  9698
  32703
```

Natural numbers: subtraction

- For subtraction, the same technique is used as for addition. The result is called the **difference.**

Natural numbers: subtraction

- *Example*: Find the difference of 367801 minus 47348.

$$\begin{array}{r} \scriptstyle 7\ 9\ \\ 367\cancel{8}\cancel{0}1 \\ -\underline{\quad 47348} \\ 320453 \end{array}$$

Natural numbers: multiplication

- Numbers that are multiplied together are **factors**.
- The **multiplicand** is the factor that is multiplied.
- The **multiplier** is the factor that multiplies.
- The **product** is the result of the multiplication.

- When multiplying by hand, we begin by placing the multiplier under the multiplicand with the units aligned.
- In general, the multiplier will be the smallest number or the one with the least number of nonzero digits.

Natural numbers: multiplication

- We then multiply the ones-digit of the multiplier by the full multiplicand, which gives us a first result.

- Then we multiply the tens-digit of the multiplier in the same way. However, when we place the second result under the first, we must make sure to align the ones-digit of the second result with the tens-digit of the first.

Natural numbers: multiplication

- We continue to multiply all of the multiplier digits, each time placing the new result under the previous one and shifting it leftward by one digit (by one column).

- When we have all the results of the multiplications, we add them together to obtain the final result (the product).

Natural numbers: multiplication

- *Example*: Find the product of 345 multiplied by 37.

$$
\begin{array}{rr}
 & 1\,1 \\
 & 3\,3 \\
 & 345 \\
\times & 37 \\
\hline
 & 2415 \\
+ & 1035 \\
\hline
 & 12765
\end{array}
$$

Natural numbers: multiplication

- Exercise: Find the product of 10200 multiplied by 269.

$$
\begin{array}{cccccccc}
 & & & & \cancel{1} & & & \\
 & & & & \cancel{1} & & & \\
 & & & 1 & 0 & 2 & 0 & 0 \\
 & \times & & & & 2 & 6 & 9 \\
\hline
 & & & 9 & 1 & 8 & 0 & 0 \\
 & & 6 & 1 & 2 & 0 & 0 & \\
+ & 2 & 0 & 4 & 0 & 0 & & \\
\hline
 & 2 & 7 & 4 & 3 & 8 & 0 & 0
\end{array}
$$

Natural numbers: division

- The **dividend** is the number that is divided.
- The **divisor** is the number that divides.
- The **quotient** is the result of the division.
- We divide the dividend by the divisor to calculate the quotient.

- There are several methods of doing division by hand. We suggest **long division**, a calculation method suitable for arbitrarily large numbers, which breaks up the problem into a series of simple steps.

Natural numbers: division

- *Example*: Find the quotient of 3978 divided by 26.

$$
\begin{array}{r}
153 \\
26\,\overline{)\,3978} \\
-\underline{26} \\
137 \\
-\underline{130} \\
78 \\
-\underline{78} \\
0
\end{array}
$$

Exercises

- 1.

  In an office, there is an 8 kg printer and four computers. One of the four computers is a 2 kg laptop, while the other three are composed of keyboards of 1 kg each, and central units weighing 8 kg each, with two screens weighing 5 kg each and the last screen weighing 6 kg.

  The office is moving and as you are in charge of the computer equipment, you are asked to take care of its transportation. What is the total weight of the items you are going to carry?

  Answer: $8 + 2 + 3(1+8) + 2\times 5 + 6 = 53$ kg

Exercices

- 2.

  A telecom cable is 100m long. You need 12m segments to reach the desks. How many pieces can you obtain from one cable and how much will be left?

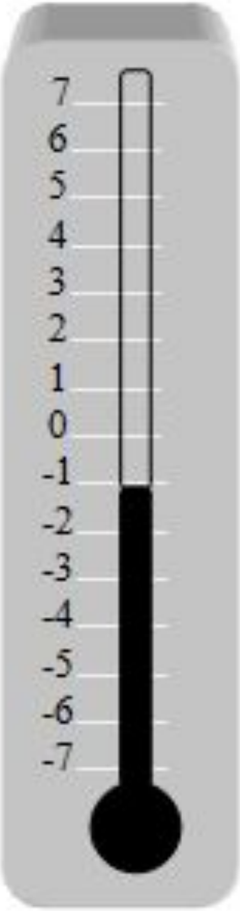
Answer: 100/12 = 8 pieces and 4m are left

Integers

- The set of all **whole numbers** is called the set of **integers.**
  - $\{\ldots, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, \ldots\}$
- This set can be constructed by taking the set of natural numbers and extending it by adding on to it the set of all negative whole numbers.

Integers: addition

- A good way to understand the addition of negative numbers is to give yourself a concrete representation of negative quantities.
- For example, let's use the thermometer.
- Observe the different temperatures listed on the thermometer. Positive temperatures are above zero and negative temperatures are below zero.

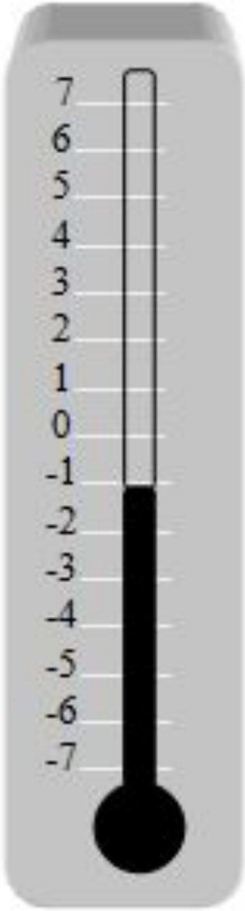


Integers: addition

- Given any integer **n** and any positive integer **x**:

- **n + x**: To add a positive integer **x** to **n**, go to the line on the thermometer corresponding to **n** and **move upward** by **x** lines.

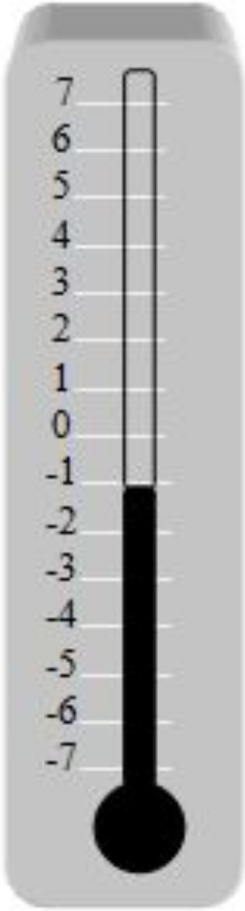
- **n + (−x)**: To add a negative integer (**−x**) to **n**, go to the line on the thermometer corresponding to **n** and **move downward** by **x** lines.



Integers: addition

- *Example*: Find the sum of −1 plus 5.

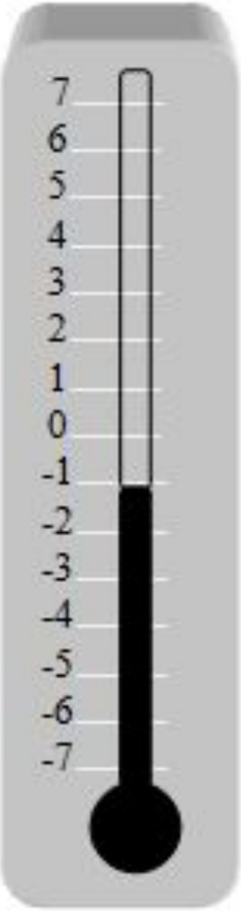
  Go to −1.
  Move upward by 5 lines.
  This gives us 4.
  Thus, $(-1) + (+5) = 4$.



Integers: addition

- *Example*: Find the sum of −1 plus −5.

Go to −1.
Move downward by 5 lines.
This gives us −6.
Thus, $(-1) + (-5) = -6$.

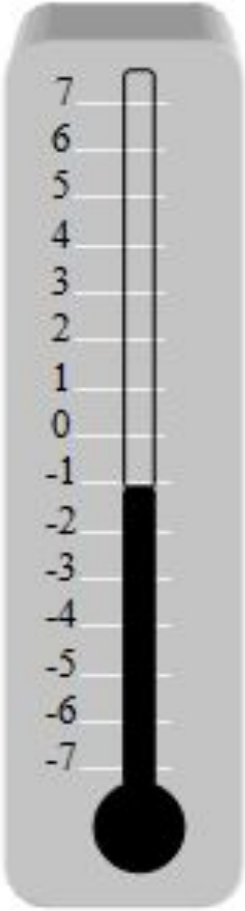


Integers: subtraction

- Given any number **n** and any positive number **x:**

- **n − x**: To subtract a positive number **x** from **n,** go to the line on the thermometer corresponding to **n** and **move downward** by **x** lines.

- **n − (−x)**: To subtract a negative number (**−x**) from **n,** go to the line on the thermometer corresponding to **n** and **move upward** by **x** lines.

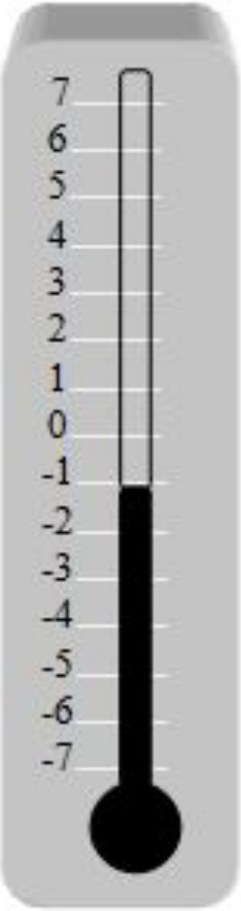


Integers: subtraction

- *Example*: Find the difference of −1 minus 5.

  Go to −1.
  Go downward by 5 lines.
  This gives us −6.
  Thus, $(-1) - (+5) = -6$.

  Compare to addition example:
  $$(-1) + (-5) = -6$$



Integers: subtraction

- *Example*: Find the difference of −1 minus −5.

  Go to −1.
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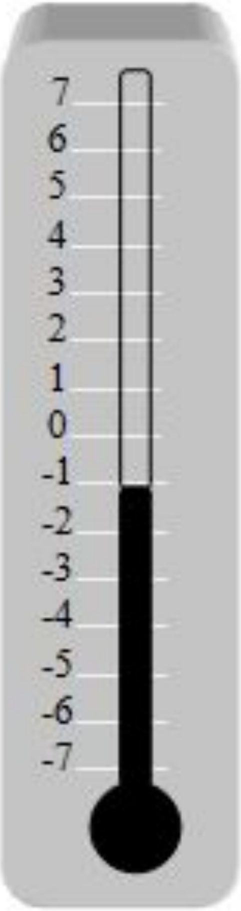
  This gives us 4.
  Thus, $(-1) - (-5) = 4$.

  Compare to addition example:
  $$(-1) + (+5) = 4$$



Integers: subtraction

- **Note:**

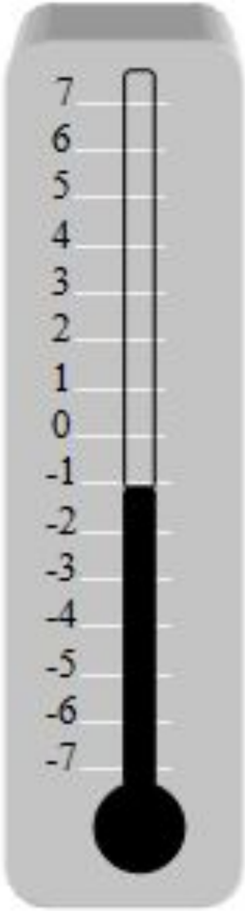
  When adding or subtracting, certain signs can be eliminated. When one sign follows another, the rule is: if the two signs are the same, then they can be replaced by a plus sign. But if the two signs are different, then they can be replaced by a minus sign:

  A + followed by a + becomes a +

  A – followed by a – becomes a +

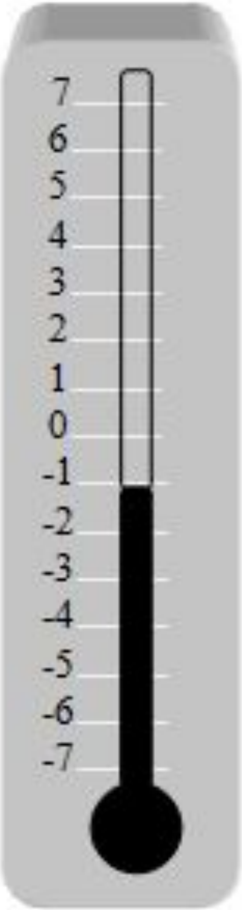
  A + followed by a – becomes a –

  A – followed by a + becomes a –



Integers: subtraction

- *Examples:*
  - $25 + (-10) = 25 - 10 = 15$
  - $(-6) - (-4) = (-6) + 4 = -2$



Integers: multiplication

- In the case of positive numbers, we know that multiplication is equivalent to the repeated addition of the same number.
  - *Example:* $5 \times 3 \;=\; 5 + 5 + 5 \;=\; 15$
- The same goes for negative numbers.
  - *Example:* $(-5) \times 3 \;=\; (-5) + (-5) + (-5) \;=\; -15$

Integers: multiplication

- It is, however, important to take into account the signs according to the following rules:

- When multiplying two numbers of different signs, the answer is always negative.

  Examples: $(-4) \times 20 = -80$ $7 \times (-3) = -21$

Integers: multiplication

- When multiplying several numbers, the following rules apply:

- When there is an even number of minus signs, the result is always positive.

- *Examples:*

$$-4 \times -2 \times -5 \times -3 = 120$$
$$4 \times -2 \times 5 \times -3 = 120$$
$$4 \times 2 \times 5 \times 3 = 120$$

Integers: multiplication

- When there is an odd number of minus signs, the result is negative.

- *Examples:*

$$-4 \times -2 \times 5 \times -3 = -120$$

$$4 \times 2 \times 5 \times -3 = -120$$

Integers: division

- When dividing negative numbers, the multiplication sign rules apply: the division of two different signs results in a negative and the division of two same signs results in a positive.

Examples: $4 \times 6 = 24$ so $24 \div 6 = 4$

$$-6 \div 3 = -2$$

$$-9 \div -3 = 3$$

$$12 \div -4 = -3$$

Exercises

- Calculate the result of these expressions without the help of a calculator:

a) $(-456) + (-280) =$ -736

b) $321 - 956 =$ -635

c) $31 \times (-3) =$ -93

d) $81 \div (-9) =$ -9

Exercises

- I had $100 in the bank and I performed 4 transactions:
  a $25 withdrawal, a $10 deposit, a $15 bill payment and I wrote a cheque for $20. In addition, I deposited my $50 paycheque.
  How much money do I have left in the bank?

$$100 - 25 + 10 - 15 - 20 + 50 = \$100$$

Real numbers

- The set of all **rational numbers** together with all **irrational numbers** is called the set of **real numbers.**

- An **irrational number** is a number that is written in infinite and non-periodic decimal form, such as the square root of 2, or π.

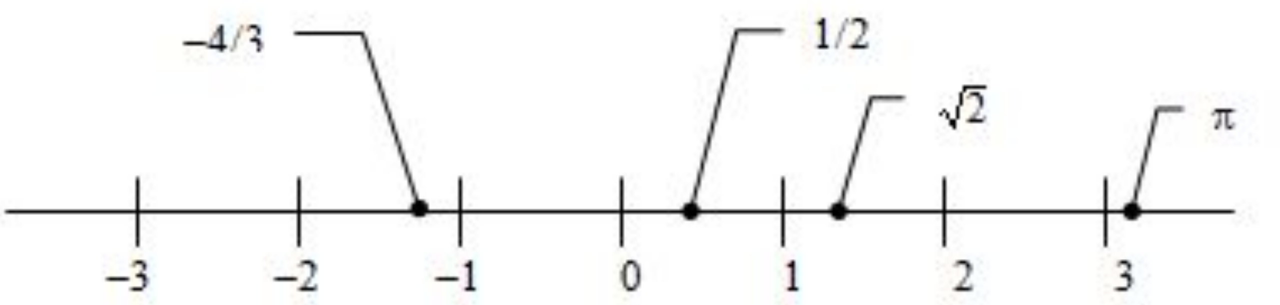
- Irrational numbers are not rational:
  A **rational number** is a number that can be written in the form **p/q**, where **p** and **q** are integers and **q** ≠ 0.

Real numbers

- *Example*: the square root of *5* =
  - 2.23606797749978969640917366873127623544...

Real numbers

- The geometric representation of real numbers, as points on a line (one-dimensional continuum), is called the **real axis.**
- Each real number corresponds to a single point on the real axis, and vice versa.



Classify the following numbers

Number	Natural	Integer	Rational	Real
0				
2.3				
$\sqrt{4}$				
$\sqrt{24}$				
-15				
7/3				
π				
22%				
3.1				

Modulo

- When is $13 + 5 = 0$?

- When is $12+7 = 7$?

- When is $1+7 = 1$?

Modulo

- The **modulo** operator looks for the remainder of a division, called the **modulus.**

- Therefore, the notation $a \pmod{b} = r$ means that

  r is the remainder of the division of the dividend a by the divisor b

- *Example*: $13 \pmod{2} = 1$
- *Example*: $13 \pmod{7} = 6$

Modulo

- The operation (mod 2) is oftentimes used to check if a number is even or odd.
- However, this is not the only use of modulo. Time-keeping is simultaneously calculated (mod 24) and (mod 7)

Modulo

- What is the full range of answers of m (mod n) ?

- *Example:* Try with m (mod 5), replacing m by different numbers

Modulo

- From this, a fundamental property of the modulo operator emerges: adding a multiple of the divisor to any value has no effect on the result of the modulus

- *Example:*

  4 mod 5 = 9 mod 5 = 24 mod 5 = -6 mod 5

Modulo

- We can use modulo to calculate any value in a repeated cycle.

- *Example*: What is the last digit of 2^{401} ?

Modulo

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Modulo

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- *Example*: What is the last digit of 2^{401} ?

Order of operations

- If you perform a calculation with multiple operations, the operations must always be evaluated in the order defined below, starting with the contents of the innermost parentheses:
  1. Parentheses
  2. Exponents
  3. Multiplication and division
  4. Addition and subtraction

Order of operations

- If there are parentheses nested within parentheses, the order of operations will look like this:
  - 1. Parentheses
    - 1.1 Parentheses
      - …
    - 1.2 Exponents
    - 1.3 Multiplication and division
    - 1.4 Addition and subtraction
  - 2. Exponents
  - 3. Multiplication and division
  - 4. Addition and subtraction

Order of operations

- *Examples:* $2 + 3 \times 5 =$

  1) $3 \times 5 = 15$

  2) $15 + 2 = 17$

  $12 - 8 \div 2 =$

  1) $8 \div 2 = 4$

  2) $12 - 4 = 8$

  $(12 - 8) \div 2 =$

  1) $12 - 8 = 4$

  2) $4 \div 2 = 2$

Zero

- **Zero** is a special number, which represents the absence of elements.
- Here are some rules for basic operations involving zero.
  - $\mathbf{x} + 0 \;=\; 0 + \mathbf{x} \;=\; \mathbf{x}$
  - $\mathbf{x} - 0 \;=\; \mathbf{x}$
  - $\mathbf{x} \times 0 \;=\; 0 \times \mathbf{x} \;=\; 0$
  - $0 \div \mathbf{x} \;=\; 0 \times (1 \div \mathbf{x}) \;=\; 0$
  - $\mathbf{x} \div 0 \;=\;$ undefined

One

- **One** is also a special number.

- Here are some rules for basic operations involving one.
  - $\mathbf{x} \times 1 = 1 \times \mathbf{x} = \mathbf{x}$
  - $\mathbf{x} \div 1 = \mathbf{x}$

Even numbers

- The set of all **integers divisible by 2** is called the set of **even numbers.**
  - $\{\ldots, -10, -8, -6, -4, -2, 0, 2, 4, 6, 8, 10, \ldots\}$

Odd numbers

- The set of **integers not divisible by 2** is called the set of **odd numbers.** The even numbers and the odd numbers are complementary sets of integers.
  - $\{\ldots, -11, -9, -7, -5, -3, -1, 1, 3, 5, 7, 9, 11, \ldots\}$

Prime numbers

- The set of **prime numbers** is the set of every natural number greater than 1 such that its only divisors are 1 and itself.
- Equivalently: a natural number greater than 1 is a prime number if and only if: it cannot be formed by multiplying two smaller natural numbers.

Example: The divisors of 8 are:

1, **2**, **4** and 8, because:

$1 \times 8 = 8$

$\mathbf{2 \times 4 = 8}$

So, 8 is not a prime number.

Prime numbers

Example: The divisors of 7 are:

1 and 7, because only:

$1 \times 7 = 7$

So, 7 is a prime number.

Example: The divisors of 12 are:

1, **2**, **3**, **4**, **6**, 12, because:

$1 \times 12 = 12$

$\mathbf{2 \times 6 = 12}$

$\mathbf{3 \times 4 = 12}$

So, 12 is not a prime number.

Prime numbers

- In ascending order, here are the first prime numbers:
  - 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, …

Exercises

- Calculate the results of the following expressions without using a calculator:

  - a) $4 + 8 \div 2 - 1 \times 3 \ =$
  - b) $(((4 + 8) \div 2) - \ 1) \times 3 \ =$
  - c) $(-2) + 0 \ =$
  - d) $0 \times 25.12 \ =$
  - e) $15 \div 0 \ =$
  - f) $0 \div 15 \ =$

Exponents

- Consider the following equation: $3^2 = 9$
- The 3 is the **base**, the 2 is the **exponent**, and the result, 9, is the **power.**
- The exponent tells us how many times the base has to be multiplied by itself.
- So, 3^2 is equivalent to the product of 3×3, which equals 9.

Exponents

- *Examples:* $5^3 = 5 \times 5 \times 5 = 125$

  $6^1 = 6$

  $1^4 = 1 \times 1 \times 1 \times 1 = 1$

  $0^2 = 0 \times 0 = 0$

- Any number with an exponent of 0 equals 1

  $8^0 = 1$

  $-3^0 = 1$

Exponents

$$7^{-1} = \frac{1}{7^1} = \frac{1}{7}$$

$$3^{-3} = \frac{1}{3^3} = \frac{1}{27}$$

Exponents

- **Scientific notation** enables us to write very large or very small numbers using fewer digits.

Example: $3{,}000{,}000 = 3 \times 1{,}000{,}000 = 3 \times 10^6$

Example: $5{,}200 = 5.2 \times 1{,}000 = 5.2 \times 10^3$

Example: $0.00064 = 6.4 \div 10{,}000 = 6.4 \times 10^{-4}$