190530H - Quiz 04

Show that V is a vector space over $\mathbb{R}$ where, V is the set of converging real sequence $\left((a_1, a_2, a_3, \ldots) \in \mathbb{R}^{\infty} \right\}$ $; \lim_{k \to \infty} a_k = a \in \mathbb{R})$

Let's consider V with $(+)$,

$(x_1, x_2, x_3, \ldots) + (y_1, y_2, y_3, \ldots) = (x_1 + y_1, x_2 + y_2, \ldots)$.

① (i) $V \neq \emptyset$ $[\exists (a, ar, ar^2, \ldots) \in \mathbb{R}$ where $|r| < 1$; $\lim_{k \to \infty} a_k = \frac{a}{1-r}]$

(ii) $\forall x_1, x_2 \in V$; $x_1 = (a_1, a_2, \ldots)$, $x_2 = (b_1, b_2, \ldots)$

$x_1 + x_2 = (a_1, a_2, \ldots) + (b_1, b_2, \ldots) = (a_1 + b_1, a_2 + b_2, \ldots)$

$\therefore x_1 + x_2 \in V \Rightarrow (+)$ is closed.

(iii) $\forall x_1, x_2, x_3 \in V$; $x_1 = (a_1, a_2, \ldots)$, $x_2 = (b_1, b_2, \ldots)$, $x_3 = (c_1, c_2, \ldots)$

$x_1 + (x_2 + x_3) = (a_1, a_2, \ldots) + [(b_1, b_2, \ldots) + (c_1, c_2, \ldots)]$.

$= (a_1, a_2, \ldots) + [b_1 + c_1, b_2 + c_2, \ldots]$.

$= [a_1 + b_1 + c_1, a_2 + b_2 + c_2, \ldots]$.

$= [(a_1 + b_1) + c_1, (a_2 + b_2) + c_2, \ldots]$.

$= (a_1 + b_1, a_2 + b_2, \ldots) + (c_1, c_2, \ldots)$

$= [(a_1, a_2, \ldots) + (b_1, b_2, \ldots)] + (c_1, c_2, \ldots)$

$= (x_1 + x_2) + x_3$.

$\therefore V$ is Associative //.

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(iv) let $e \in V$; $\forall x \in V$, $x+e = e+x = x$, $e = (e_1, e_2, e_3, \ldots)$.

$\Rightarrow (a_1, a_2, \ldots) + (e_1, e_2, \ldots) = (e_1, e_2, \ldots) + (a_1, a_2, \ldots) = (a_1, a_2, \ldots)$.

$\Rightarrow (a_1+e_1, a_2+e_2, \ldots) = (e_1+a_1, e_2+a_2, \ldots) = (a_1, a_2, \ldots)$.

$a_1+e_1 = e_1+a_1 = a_1$ $\quad \therefore e = (0, 0, \ldots)$.

$a_2+e_2 = e_2+a_2 = a_2$.

$\vdots \quad \vdots \quad \vdots$ Identity element exists.

(v) $\forall x \in V$, $x = (a_1, a_2, \ldots)$; $\exists y \in V$, $y = (-a_1, -a_2, \ldots)$

$(\forall a_k \in \mathbb{R}; \exists -a_k \in \mathbb{R})$.

$x+y = (a_1, a_2, \ldots) + (-a_1, -a_2, \ldots) = (0, 0, \ldots)$

$y+x = (-a_1, -a_2, \ldots) + (a_1, a_2, \ldots) = (0, 0, \ldots)$.

$y = x^{-1} \Rightarrow y = -x$. $\quad \therefore$ Inverse element exists.

(vi) $\forall x_1, x_2 \in V$; $\quad x_1 + x_2 = (a_1+b_1, a_2+b_2, \ldots)$

$x_2 + x_1 = (b_1+a_1, b_2+a_2, \ldots)$

$\therefore x_1 + x_2 = x_2 + x_1$

$\therefore$ commutative law satisfies.

$\therefore (V, +)$ is an abelian group.

(2) $(\mathbb{R}, +, \cdot)$ is a field.

(3) $\forall a \in \mathbb{R}$, $\forall x, y \in V$, $a(x+y) = a((a_1, a_2, \ldots) + (b_1, b_2, \ldots))$

$= a(a_1+b_1, a_2+b_2, \ldots)$.

$= (a(a_1+b_1), a(a_2+b_2), \ldots)$

$= [(a a_1 + ab_1), (aa_2 + ab_2, \ldots)]$.

$= [a(a_1, a_2, \ldots) + a(b_1, b_2, \ldots)]$.

$a(x+y) = ax + ay$

(4) $\forall a \in \mathbb{R}$, $\forall x \in V$, $ax = (aa_1, aa_2, \dots) \in V$

(5). $\forall a, b \in \mathbb{R}$, $\forall x \in V$; $(a+b)x = (a+b)(a_1, a_2, \dots)$.

$$= ((aa_1 + ba_1), (aa_2 + ba_2), \dots)$$

$$= a(a_1, a_2, \dots) + b(a_1, a_2, \dots)$$

$$= ax + bx$$

(6) $\forall a, b \in \mathbb{R}$, $\forall x \in V$; $(ab)x = (ab)(a_1, a_2, \dots)$.

$$= [(ab)a_1, (ab)a_2, \dots]$$

$$= [a(ba_1), a(ba_2), \dots]$$

$$= a[b(a_1, a_2, \dots)]$$

$$(ab)x = a(bx)$$

(7) $\forall x \in V$ $\quad 1.x = 1(a_1, a_2, \dots)$.

$$= (1.a_1, 1.a_2, \dots)$$

$$= (a_1, a_2, \dots)$$

$$1.x = x \in V.$$

(1) - (7) proves that $(V, +, \cdot)$ is a ~~field~~ vector space over $\mathbb{R}$.