Chapter 1

Differential Equations

1.1 Basic Concepts

Definition 1.1. *A differential equation is an equation involving an unknown function and its derivatives.*

If the unknown function depends on only one independent variable, the differential equation is called an ordinary differential equation (ODE). The differential equation is a partial differential equation (PDE) if the unknown function is dependent on two or more independent variables.

Remark 1.2. *Order of a differential equation is the order of the highest derivative that appears in the equation.*

Remark 1.3. *The highest power of the highest derivative is called as degree of the differential equation, after the differential equation has been made free from radicals and fractions as far as the derivatives are concerned.*

A general n^{th} order ODE on a given interval $I = [a, b]$ can be represent in the form,

$$f\left(x, y', y'', \ldots, y^{(n-1)}\right) = y^n, \quad x \in I, \tag{1.1}$$

where f is a real (complex) valued function and y is the unknown function of the independent variable x.

Example 1.4. *The following are differential equations involving the unknown function y.*

a) $\frac{dy}{dx} = -2x^3 + 3x^2 + 4,$ *(first order, first degree)*

b) $\frac{d^2y}{dx^2} = (\frac{dy}{dx})^2 + 3x$ *(second order, first degree)* *c)* $(\frac{dy}{dx})^2 + y$ *(first order, second degree)*

d) $(\frac{d^2y}{dx^2})^2 - \left(\frac{dy}{dx}\right)^3 - y = 0$ *(second order, second degree)*

A differential equation is said to be **linear** if the unknown function and its derivatives appears to the power one (products are not allowed) and non–linear otherwise. A general linear differential equation can be represented in the form,

$$a_0(x)y^{(n)}(x) + a_1(x)y^{(n-1)}(x) + \cdots + a_{n-1}(x)y'(x) + a_n(x)y(x) = r(x), \tag{1.2}$$

where $a_0(x), a_1(x), \ldots, a_n(x)$ and $r(x)$ are function of x and $a_0(x) \neq 0$. The functions $a_j(x), j = 1, \ldots, n$ are called coefficients of the equation. If $r(x) = 0$, then it is said to be **homogeneous**.

1.2 First Order Differential Equations

Definition 1.5. *Standard form for a first-order differential equation in the unknown function y(x) is,*

$$y' = f(x, y) \quad (explicit form) \tag{1.3a}$$

$$F(x, y, y') = 0 (implicit form) \tag{1.3b}$$

Definition 1.6. ***A solution*** *of a given first order differential equation on some open interval $a < x < b$ is a function $y = h(x)$ that has a derivative $y' = h'(x)$ and that satisfies the differential equation for all $x \in (a, b)$.*

Example 1.7. *Show that $y = x - 1$ is a solution of the differential equation $y'^2 - xy + y = 0$.*

1.2.1 Separable Differential Equations

If in an equation, it is possible to get all the functions of x and dx to one side and all the functions of y and dy to the other, the variables are said to be separable. Many first order differential equations can be reduced to the form,

$$g(y)y' = f(x) \tag{1.4}$$

and since $y' = \frac{dy}{dx}$, Equation 1.4 can be rewrite as

$$g(y)dy = f(x)dx. \tag{1.5}$$

By integrating both sides of 1.5 we have

$$\int g(y)dy = \int f(x)dx + c. \tag{1.6}$$

where c is an arbitrary constant.

Exercise 1.8. *Solve the following separable differential equations.*

a) $5yy' + 3x = 0$

b) $y' - y^2 - 1 = 0$

c) $y' = kxy, \quad k$ *is a constant*

d) $xy' + y = 1$

e) $x(x^2 - 1)y' = 2y(2x^2 - 1)$

1.2.2 Equations Reducible to Separable Equations Form

The equations of the form $\left(y' = g\left(\frac{y}{x}\right)\right)$ can be reduced to the separable form. Let use a change of variables by setting $u = \frac{y}{x}$. Thus by differentiation of $y = ux$, we have

$$y' = u'x + u$$

Substituting y and y' into given differential equation,

$$u'x + u = g(u)$$

By separating the variables

$$\frac{du}{g(u) - u} = \frac{dx}{x}$$

1.2.3 Equations Reducible to Homogeneous Form

The equations of the form

$$\frac{dy}{dx} = \frac{ax + by + c}{Ax + By + C}$$

can be reduced to the homogeneous form by substituting $x = X + h$ and $y = Y + k$, where h and k are the constants to be chosen so as to make the given equation homogeneous. Notice that,

$$\frac{dy}{dx} = \frac{dY}{dx}$$

Hence the equation becomes

$$\frac{dY}{dX} = \frac{a(X+h)+b(Y+k)+c}{A(X+h)+B(Y+k)+C} = \frac{(aX+bY)+(ah+bk+c)}{(AX+BY)+(Ah+Bk+C)}$$

Select h, k such that $ah+bk+c=0$ and $Ah+Bk+C=0$. Then the given equation becomes **homogeneous**.

$$\frac{dY}{dX} = \frac{(aX+bY)}{(AX+BY)}$$

Hence, we can apply variable separable method to solve the equation.

Exercise 1.9. *Reduce the following equations to the separable form and find the solutions.*

a) $2xyy' = y^2 - x^2$

b) $(x^3+3xy^2)dx + (y^3+3x^2y)dy = 0$

c) $\frac{dy}{dx} = \frac{2x+2y-2}{3+y-5}$

d) $(4x+3y)dy + (y-2x)dx = 0$

e) $y' = \frac{x+2y-3}{2x+y-3}$

f) $\frac{dy}{dx} = \frac{y-x+1}{y+x+5}$

g) $\frac{dy}{dx} = \frac{3y-7x+7}{7y-3x+3}$

h) $(x+2y)(dx-dy) = dx+dy$

1.2.4 Exact Equations

A differential equation of the form

$$P(x,y)dx + Q(x,y)dy = 0 \tag{1.7}$$

is said to be **exact** if and only if it satisfies the following condition.

$$\frac{\partial P(x,y)}{\partial y} = \frac{\partial Q(x,y)}{\partial x}$$

In other words, if the left hand side of the equation 1.7 is said to be exact of function $f(x,y)$ if

$$df = P(x,y)dx + Q(x,y)dy \tag{1.8a}$$

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy \tag{1.8b}$$

Thus, $P(x,y) = \frac{\partial f}{\partial x}$ and $P(x,y) = \frac{\partial f}{\partial y}$. Then the general solution can be written as

$$f(x,y) = c$$

for some arbitrary constant c.

Example 1.10. *Solve:* $(y^2e^{xy^2} + 4x^3)dx + (2xye^{xy^2} - 3y^2)dy = 0$.

1.2.5 Equations Reducible to the Exact Equations

Suppose $P(x,y)dx + Q(x,y)dy = 0$ is not an exact equation and there is a function $I(x,y) \neq 0$ such that $I(x,y)P(x,y)dx + I(x,y)Q(x,y)dy = 0$ is exact, then the function $I(x,y)$ is called an **integrating factor** (I) of the given equation.

a) If $\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)\frac{1}{Q}$ is a function of x alone $(g(x))$, then

$$I = e^{\int g(x)dx}$$

b) If $\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\frac{1}{P}$ is a function of y alone $(g(y))$, then

$$I = e^{\int g(y)dy}$$

c) If $Pdx + Qdy = 0$ is a homogeneous equation of x, y and $Px + Qy \neq 0$, then

$$I = \frac{1}{Px + Qy}$$

d) If $Pdx+Qdy = 0$ can be written in the form $yf_1(xy)dx+xf_2(xy)dy = 0$ and $f_1 \neq f_2$, then

$$I = \frac{1}{Px - Qy}$$

Exercise 1.11. *Are the following equations exact? If not, find an integrating factor and solve them.*

a) $(x + y + 5)dx + (x - y + 10)dy = 0$

b) $(ye^x + 2e^x + y^2)dx + (e^x + 2xy)dy = 0$

c) $(y - 2x^3)dx - x(1 - xy)dy = 0$

d) $\left(2x \sinh \frac{y}{x} + 3y \cosh \frac{y}{x}\right) dx - 3x \cosh \frac{y}{x} dy = 0$

e) $(x \sec^2 y - \cos y)dy + (\tan y - 3x^4)dx = 0$

f) $2(2x \ln x - xy)dy + 2ydx = 0$

g) $(x+y^2)dx + 2xydy = 0$

h) $y(xy+2x^2y^2)dx + x(xy - x^2y^2)dy = 0$

i) $(y^4+2y)dt + (4ty^3+2y^4+2t)dy = 0$

j) $(e^y+2)\sin t dt = e^y \cos t dy$

1.2.6 Linear First Order Equations

A differential equation of the form

$$\frac{dy}{dx} + P(x)y = Q(x) \tag{1.9}$$

is called a linear differential equation, where P and Q are independent of y. For this equation we can compute an integrating factor as follows. Let us multiply the equation 1.9 by $I(x)(\neq 0)$ and obtain

$$I(x)\frac{dy}{dx} + I(x)P(x)y = I(x)Q(x)$$

Choose $I(x)$ such that

$$I(x)P(x) = \frac{dI}{dx}$$

Then we have

$$\frac{d(I(x)y)}{dx} = I(x)Q(x)$$

Integrating both sides

$$I(x)y = \int I(x)Q(x)dx + C$$

where C is an arbitrary constant. To determine the integrating factor $I(x)$, we consider

$$\frac{dI}{dx} = P(x)I(x)$$

By separation of variables, we have

$$\int \frac{dI}{I(x)} = \int P(x)dx + c$$

$$\ln I = \int P(x)dx + c$$

$$|I| = e^{\int P(x)dx + c}$$

$$I = Ke^{\int P(x)dx}$$

where K is an arbitrary constant. Since we need one integrating factor, let us consider $K = 1$. Then

$$I = e^{\int P(x)dx}$$

Example 1.12. *Solve* $\frac{dy}{dx} + 2xy = x$ *for* $y(1) = 2$.

Example 1.13. *Solve* $x\frac{dy}{dx} - y = x^3$ *for* $y(1) = 1$.

Example 1.14. *Solve* $\frac{dy}{dx} + y\tan x = \sin x \cos x$ *for* $y(0) = 2$.

Example 1.15. *Solve* $(1 + \cos x)y' - (\sin x)y = 2x$.