Q4)

Let V be the set of converging sequance $\Rightarrow \lim_{n\to\infty} a_n = b \in \mathbb{R}^{\infty}$ 190420V

We need to show V is a vector space over $\mathbb{R}$

Consider V with binary operation + ; $(x_1, x_2, x_3 \ldots) + (y_1, y_2, y_3 \ldots) = (x_1+y_1, x_2+y_2 \ldots)$

(1) (i) $V \neq \emptyset$ $\left[\because \exists (a, ar, ar^2 \ldots) \in \mathbb{R}^{\infty} \text{ with } |r|<1 ; \lim_{k\to\infty} a_k = \frac{a}{1-r}\right]$

(ii) $\forall x_1, x_2 \in V$; $x_1 = (a_1, a_2, a_3 \ldots)$ $x_2 = (b_1, b_2, b_3 \ldots)$

$x_1 + x_2 = (a_1, a_2, a_3 \ldots) + (b_1, b_2, b_3 \ldots)$

$= (a_1+b_1, a_2+b_2, a_3+b_3 \ldots) \in V$

$\therefore x_1 + x_2 \in V \Rightarrow$ + is closed for V

(iii) $\forall x_1, x_2, x_3 \in V$; $x_1 = (a_1, a_2, a_3 \ldots)$ $x_2 = (b_1, b_2, b_3 \ldots)$ $x_3 = (c_1, c_2, c_3 \ldots)$

$x_1 + (x_2 + x_3) = (a_1, a_2 \ldots) + ((b_1, b_2, b_3 \ldots) + (c_1 + c_2 \ldots))$

$= (a_1, a_2 \ldots) + (b_1+c_1, b_2+c_2, b_3+c_3 \ldots)$

$= (a_1+b_1+c_1, a_2+b_2+c_2, \ldots)$

$= ((a_1+b_1)+c_1, (a_2+b_2)+c_2 \ldots)$ ($\mathbb{R}$ is a field)

$= (a_1+b_1, a_2+b_2 \ldots) + (c_1+c_2+c_3 \ldots)$

$= ((a_1, a_2, a_3 \ldots) + (b_1, b_2, b_3 \ldots)) + (c_1+c_2+c_3 \ldots)$

$= (x_1 + x_2) + x_3$

$\therefore$ Associative law is satisfied.

(iv) Let $e \in V$; $\forall x \in V$ $x + e = e + x = x$ $e = (e_1, e_2, e_3 \ldots)$

$\Rightarrow (a_1, a_2 \ldots) + (e_1, e_2 \ldots) = (e_1, e_2 \ldots) + (a_1, a_2 \ldots) = (a_1, a_2, \ldots)$

$\Rightarrow (a_1+e_1, a_2+e_2 \ldots) = (e_1+a_1, e_2+a_2 \ldots) = (a_1, a_2 \ldots)$

$\Rightarrow (a_1+e_1) = (e_1+a_1) = a_1$
$a_2+e_2 = e_2+a_2 = a_2$
$\vdots$

$\therefore e = (0, 0, 0 \ldots)$

$\therefore$ identity element exists.

(v) $\forall x \in V$ $x = (a_1, a_2, a_3 \ldots)$; $\exists y \in V$ $y = (-a_1, -a_2, -a_3 \ldots)$

$x+y = (a_1, a_2 \ldots) + (-a_1, -a_2 \ldots) = (a_1 - a_1, a_2 - a_2 \ldots) = (0, 0, 0 \ldots)$ [$\because \mathbb{R}$ is a field for $\forall a_k \in \mathbb{R}$ $\exists -a_k \in \mathbb{R}$]

$y+x = (-a_1, -a_2 \ldots) + (a_1, a_2 \ldots) = (-a_1 + a_1, -a_2 + a_2 \ldots) = (0, 0, 0 \ldots)$

$\therefore y = x^{-1} \Rightarrow y = -x$

$\therefore$ Inverse element exists

(vi) $\forall x_1, x_2 \in V$ $x_1 + x_2 = (a_1 + b_1, a_2 + b_2 \ldots)$

$x_2 + x_1 = (b_1 + a_1, b_2 + a_2 \ldots)$

$= (a_1 + b_1, a_2 + b_2 \ldots)$ (convergin sequence and $\mathbb{R}$ satisfies the commutative)

$x_1 + x_2 = x_2 + x_1$

$\therefore$ commutative law satisfied.

$\therefore (V, +)$ is an abelian group

(2) $(\mathbb{R}, +, \cdot)$ is a field

(3) $\forall a \in \mathbb{R}$, $\forall x \in V$ $ax = (aa_1, aa_2, aa_3 \ldots) \in V$ ($\because x$ is converge)

(4) $\forall a \in \mathbb{R}$, $\forall x, y \in V$, $a(x+y) = a((a_1, a_2, a_3 \ldots) + (b_1, b_2, b_3 \ldots))$

$= a(a_1 + b_1, a_2 + b_2, a_3 + b_3 \ldots)$

$= (a(a_1 + b_1), a(a_2 + b_2), a(a_3 + b_3) \ldots)$

$= (aa_1 + ab_1 + aa_2 + ab_2 \ldots)$

$= a(a_1, a_2, a_3 \ldots) + a(b_1, b_2, b_3 \ldots)$

$a(x+y) = ax + ay$ //

(5) $\forall a, b \in \mathbb{R}$ $\forall x \in V$ $(a+b)x = (a+b)(a_1, a_2, a_3 \ldots)$

$= (aa_1 + ba_1, aa_2 + ba_2 \ldots)$

$= a(a_1, a_2 \ldots) + b(a_1, a_2 \ldots)$

$= ax + bx$ //

(6) $\forall a,b \in \mathbb{R}, \forall x \in V$ $(ab)x = (ab)(a_1, a_2, a_3 \cdots\cdots)$

$= ((ab)a_1, (ab)a_2 \cdots\cdots)$

$= (a(ba_1), a(ba_2) \cdots\cdots)$

$= a(b\underbrace{(a_1, a_2 \cdots\cdots)})$

$(ab)x = a(bx)$

(7) $\forall x \in V$ $1.x = 1(a_1, a_2, a_3 \cdots)$

$= (1a_1, 1a_2, 1a_3 \cdots)$

$= (a_1, a_2, a_3 \cdots)$

$1.x = x \in V$

from 1, 2, 3 ..... 6, 7 $(V, +, \cdot)$ over $(\mathbb{R}, +, \cdot)$ is a vector space

$\therefore$ V is a vector space over $\mathbb{R}$

Part 2 (Allow infinite sums)

Consider set $B \subset V$; $B = \{(1,0,0 \cdots), (0,1,0 \cdots), (0,0,1 \cdots) \cdots\}$

$= \{(a_1, a_2, a_3 \cdots) \mid \forall k \in \mathbb{Z}^+, a_k = 0 \text{ or } a_k = 1 \text{ and } \exists! k \in \mathbb{Z}^+, a_k = 1\}$

Consider any $x = (x_1, x_2, x_3 \cdots) \in V$ (all $x_k \in \mathbb{R}, k \in \mathbb{Z}^+$)

Let (all $b_k \in \mathbb{R}, k \in \mathbb{Z}^+$)

$(x_1, x_2, x_3 \cdots) = b_1(1,0,0 \cdots) + b_2(0,1,0 \cdots) + b_3(0,0,1 \cdots) \cdots$

$= (b_1, b_2, b_3 \cdots)$

$(x_1 - b_1, x_2 - b_2, x_3 - b_3 \cdots) = (0, 0, 0 \cdots)$

$\Rightarrow x_1 = b_1, x_2 = b_2, x_3 = b_3 \cdots$

$\Rightarrow$ for all k, $b_k = x_k \in \mathbb{R}$

$\therefore \exists b_k = x_k$ for all $k \in \mathbb{Z}^+$; $b_k \in \mathbb{R}$ $\therefore x \in \operatorname{span} B \Rightarrow V \subseteq \operatorname{Span} B$

$\Rightarrow \operatorname{span} B \subseteq V$

$\therefore V = \operatorname{Span} B \longrightarrow$ (A)

Now $\forall x_k \in B$, $(k \in \mathbb{Z}^+)$ and $b_k \in \mathbb{R}$ $x_1 = (1,0,0 \ldots)$ $x_2 = (0,1,0 \ldots)$

let $b_1x_1 + b_2x_2 + b_3x_3 \ldots = (0,0,0 \ldots)$

$\Rightarrow b_1(1,0,0\ldots) + b_2(0,1,0\ldots) + b_3(0,0,1\ldots) = (0,0,0\ldots)$

$\Rightarrow (b_1,0,0\ldots) + (0,b_2,0\ldots) + (0,0,b_3\ldots) = (0,0,0\ldots)$

$\Rightarrow b_1 = 0,\ b_2 = 0,\ b_3 = 0 \in$ ~~for~~ $\forall k \in \mathbb{Z}^+$ $b_k = 0$; B is linearly independent (B)

from (A), (B) B is a Basis ~~of~~ for V

~~Proofs~~ B is countable proofe

~~B~~ Let $x_1 = (1,0,0\ldots)$ $x_2 = (0,1,0\ldots)$ $x_3 = (0,0,1\ldots)$ ...

$x_k = (a_1, a_2, a_3 \ldots)$

Define function f ; $f(x_k) = k$ for all $k \in \mathbb{Z}^+$

$$\begin{bmatrix} f((1,0,0\ldots)) = 1 \\ f((0,1,0\ldots)) = 2 \\ f((0,0,1\ldots)) = 3 \end{bmatrix}$$

$f : B \to \mathbb{N}$

for any $x_p, x_q \in B$; $p, q \in \mathbb{Z}^+$

Let $f(x_p) = f(x_q)$

$\Rightarrow p = q$

$\Rightarrow x_p = x_q$

so $\forall x_p, x_q \in B$, $f(x_p) = f(x_q) \Rightarrow x_p = x_q$; ~~f~~ f is one-one

so $\exists f$; $f : B \to \mathbb{N}$; f is one-one

$\Rightarrow f(x_k) = k$ $\forall k \in \mathbb{Z}^+$, $x_k \in B$

Then B is countable

~~A~~ therefore if we allow infinite sums ~~and~~ we can come up with countable basis //