Chapter 3

Partial Differential Equations

A partial differential equation (PDE) is a differential equation which involves partial derivatives of one or more dependent variable with respect to one or more independent variable. The solution of a partial differential is an explicit or implicit relation between the variables which does not contain derivatives. Moreover, the solution is identically satisfied the equation.

In general a partial differential equation of variables x, y can be written as

$$f(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}) = 0$$

where $u(x, y)$ is the solution.

Example 3.1. $\frac{\partial u}{\partial x} = x^2 + y^2$
Solution is

$$u = \int (x^2 + y^2)dx + g(y)$$

.

Remark 3.2. ***Order*** *of a partial differential equation is the same as that of the order of the highest differential co-efficient in it.*

Remark 3.3. *If the dependent variable and all its partial derivatives occurs linearly in any PDE then such an equation is called* ***linear PDE*** *otherwise a non-linear PDE.*

Remark 3.4. *A PDE is called as a* ***quasi–linear PDE*** *if all the terms with highest order derivatives of dependent variables occurs linearly, that is the coefficients of such terms are functions of only lower order derivatives of the dependent variables.*

Remark 3.5. *If all the terms of a PDE contain the dependent variable or its partial derivatives then such a PDE is called non–homogeneous PDE or homogeneous otherwise.*

Example 3.6. *The following are partial differential equations involving the independent variables x, y.*

a) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$ *(linear, homogeneous)*

b) $u_{xx} + u_{yy} = 0$ *(linear, homogeneous)*

c) $ux\frac{\partial^2 u}{\partial x^2} + u^2xy\frac{\partial^2 u}{\partial x \partial y} + uy\frac{\partial^2 u}{\partial y^2} + \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + u^2 = 0$ *(non-linear but quasi-linear, homogeneous)*

d) $u_{xx} + u_{yy} = \left[(u_x)^2 + (u_y)^2\right] u$ *(quasi-linear)*

e) $(x^2 + y^2)\frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x \partial y} - 3u = 0$ *(linear, 2nd order PDE)*

3.1 Classification of PDE

The classification of PDE's is an important concept because the general theory and methods of solution usually only to a given class of equations. The most general linear partial differential equation of second order with two independent variables is,

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial y \partial x} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu + G = 0 \tag{3.1}$$

where A, B, C, D, E, F and G are functions of x, y and constant terms. The equation 3.1 may be written in the form

$$Au_{xx} + Bu_{xy} + Cu_{yy} + f(x, y, u_x, u_y, u) = 0 \tag{3.2}$$

Assume that A, B, C are continuous function of x and y processing continuous partial derivatives of high orders as necessary

The classification of PDE is motivated by the classification of second order algebraic equation in two-variable. In other words, the nature of Equation 3.1 is determined by the principal part containing highest partial derivatives. i.e.

$$L_u = Au_{xx} + Bu_{xy} + Cu_{yy} \tag{3.3}$$

The classification of Equation 3.1 is done as follows.

1. If $B^2 - 4AC < 0$, the equation is said to be **elliptic**.
2. If $B^2 - 4AC > 0$, the equation is said to be **hyperbolic**.

3. If $B^2 - 4AC = 0$, the equation is said to be **parabolic**.

The above classification of 3.1 is still valid if the coefficients A, B, C, D, E and F depend on x, y. In this case,

1. If $B^2(x,y) - 4A(x,y)C(x,y) < 0$, the equation is said to be **elliptic** at (x,y).

2. If $B^2(x,y) - 4A(x,y)C(x,y) > 0$, the equation is said to be **hyperbolic** at (x,y).

3. If $B^2(x,y) - 4A(x,y)C(x,y) = 0$, the equation is said to be **parabolic** at (x,y).

Example 3.7. *Consider the following equations.*

1. $u_{xx} + 2u_{yy} = 1$ *is elliptic.*

2. $u_{xx} - u_{yy} = 1$ *is hyperbolic.*

3. $u_{xx} + 3u_{yy} - 2u_x + 24U_y + 5u = 0$ *is elliptic.*

4. $u_{xx} + u_{yy} = 0$ *(Laplace equation) is an elliptic.*

5. $u_t = u_{xx}$ *(Heat equation) is of parabolic type.*

6. $u_{tt} - u_{xx} = 0$ *(Wave equation) is of hyperbolic type.*

7. $u_{xx} + xu_{yy} = 0,\ x \neq 0$ *(Tricomi equation) is hyperbolic for* $x < 0$ *and elliptic for* $x > 0$. *This example shows that equations with variable coefficients can change form in the different regions of the domain.*

In general, a partial differential equation of order n has a solution which contains at most n arbitrary functions. Therefore the general solution can be written as the linear combination of n arbitrary functions. This general solution can be particularized to a unique solution if appropriate extra conditions are provided. These are classified as boundary conditions. The kind of boundary conditions we need to specify depend on the nature of the problem.

3.2 Techniques for Solving PDEs

Different types of equations usually require different solution techniques. However, there are some methods that work for most of the linearly partial differential equations with appropriate boundary conditions on a regular domain. These methods include separation of variables, series expansions, similarity solutions, hybrid methods, and integral transform methods.

3.2.1 Solution by direct integration

The simplest form of partial differential equation is such that a solution can be determined by direct partial integration.

Example 3.8. *Solve the differential equation* $u_{xx} = 12x^2(t+1)$ *given that* $x = 0, u = cos(2t)$ *and* $u_x = sint$.

Example 3.9. *Solve the differential equation* $u_{xy} = sin(x+y)$ *given that at* $y = 0, u_x = 1$ *and at* $x = 0, u = (y-1)^2$.

Initial conditions and boundary conditions: As with any differential equation, the arbitrary constants or arbitrary functions in any particular case are determines from the additional information given concerning the variables of the equation. These extra facts are called the initial conditions or, more generally the boundary conditions since they do not always refer to zero values of the independent variables.

3.2.2 Separation of Variables

This is a basic method which is very powerful for obtaining solutions of certain problems involving PDEs. Although the class of problems to which the method applied is relatively small, it nevertheless includes many problems of great physical interest.

The separation of variables attempts a solution of the form $u = X(x)Y(y)$ where $X(x)$ and $Y(y)$ are functions of x, y respectively. In order to determine these functions, they have to satisfy the partial differential equation and the required boundary conditions. As a result, the partial differential equation is usually transformed into two ordinary differential equations (ODEs). The final solution is then obtained by solving these ODEs. A solution that has this form is said to be separable in x and y and seeking solutions of this form is called the method of separation of variables. The following examples are to illustrate the method of solution by studying the wave equation, the heat equation and the Laplace equation.

Two Dimensional Heat Flow

Suppose we want to find the temperature distribution in a rectangular metal plate under certain conditions. The plate is covered on its top and bottom faces by layers of thermal insulating material so that heat is constrained to flow mainly in the X and Y directions as shown in the diagram below. Along the edges of the plate various conditions are applied. These are known as boundary conditions.

When formulating a simple mathematical model the following assumptions are made.

- The metal is uniform in the sense that its thermal conductivity is the same at all points of the plate.
- The plate is sufficiently thin so that we neglect any heat flow in the directions perpendicular to its face.
- The temperature distribution is in the steady state. i.e. temperature at any point in the plate does not depend on the time.

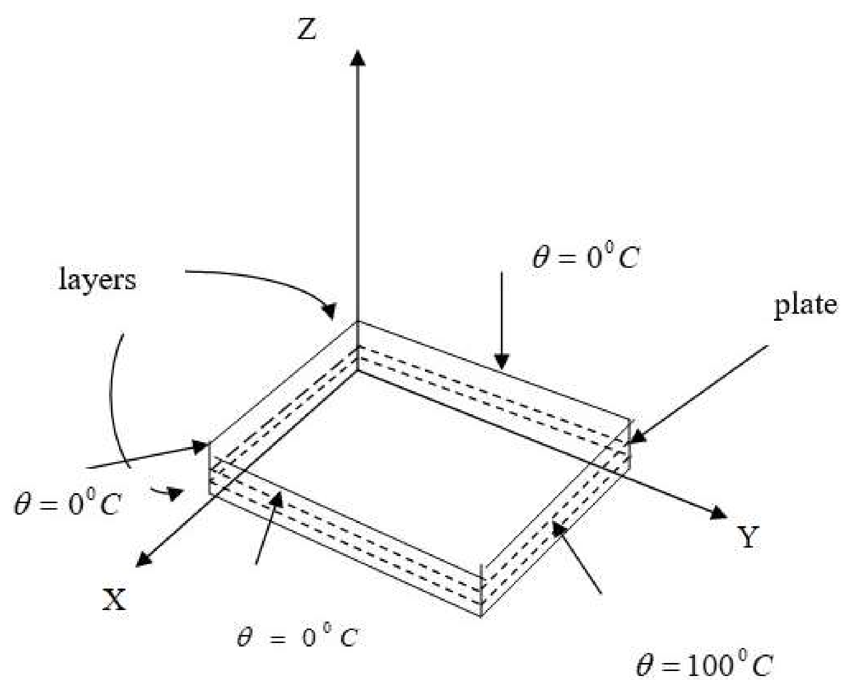

Figure 3.1: Example 3.1

Let us consider the temperature function $T(x, y)$ depends on x and y. It can be shown that $T(x, y)$ satisfies the Laplace equation given by

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0 \tag{3.4}$$

Laplace equation possesses infinitely many solutions. For a unique solution the following boundary conditions are used.

Boundary conditions:

$$T(0,y)=0, \quad \text{for all} y>0, \tag{3.5a}$$

$$T(L,y)=0, \quad \text{for all} y>0, \tag{3.5b}$$

$$T(x,y=\infty)=0, \quad \text{for all} 0<x<L, \tag{3.5c}$$

$$T(x,0)=100^\circ C, \quad \text{for all} 0<x<L. \tag{3.5d}$$

The method of separation of variables is to try to find solutions that are sums or products of functions of one variable. For the Laplace equation, we try to find solutions of the form,

$$T(x,y)=X(x)Y(y). \tag{3.6}$$

We now differentiate Equation 3.6 and substitute into the Equation 3.4.

$$Y(y)X''(x)+X(x)Y''(y)=0. \tag{3.7}$$

From this we obtain,

$$\frac{X''(x)}{X(x)}=-\frac{Y''(y)}{Y(y)}. \tag{3.8}$$

Since X is a function of x only, the left member of 3.8 is also a function of x only. Similarly, right member of 3.8 is a function of y only. Therefore, both of them must be equal to a constant $-k^2$. From this, we obtain the two ordinary differential equations.

$$X''(x)+k^2X(x)=0, \tag{3.9a}$$

$$Y''(y)-k^2Y(y)=0. \tag{3.9b}$$

Solving the two ordinary differential equation, we can obtain the general solution,

$$T(x,y)=(a\sin kx+b\cos kx)\left(ce^{ky}+de^{-ky}\right). \tag{3.10}$$

Applying the boundary conditions to the general solution, finally, we obtain

$$T(x,y)=100=\sum_{n=1}^{\infty}A_n\sin\frac{n\pi x}{L}. \tag{3.11}$$

This is the Fourier sine series for $T = 100$ with $0 < x < L$. Solving this, we obtain the solution,

$$T(x,y) = \sum_{n=1}^{\infty} \frac{400}{(2n-1)\pi} \sin \frac{(2n-1)\pi x}{L} e^{-\frac{(2n-1)\pi y}{L}}. \tag{3.12}$$

Exercise 3.10. *A rectangular plate with insulated surfaces is 8cm wide and so long compared to its width that it may be considered infinite in length. If the temperature along one short edge $y = 0$ is given by, $u(x,0) = 100 \sin \frac{x\pi}{8}$, $0 < x < 8$ while the two long edges $x = 0$ and $x = 8$, as well as the other edge is kept at $0°C$, find steady state temperature $u(x,y)$.*

Laplace Equation in Polar Coordinates

The Laplace equation in polar co-ordinates is

$$r^2 \frac{\partial^2 T}{\partial r^2} + r \frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial \theta^2} = 0. \tag{3.13}$$

where, r is the radius and θ is the angle. Let

$$T(r,\theta) = R(r)\Theta(\theta). \tag{3.14}$$

We now differentiate Equation 3.14 and substitute into the Equation 3.13.

$$r^2 \Theta(\theta) R''(r) + r\Theta(\theta) R'(r) + R(r)\Theta''(\theta) = 0. \tag{3.15}$$

From this we obtain,

$$\frac{r^2 R''(r) + r\Theta(\theta)R'(r)}{R} = \frac{-\Theta''(\theta)}{\Theta(\theta)} = h, \quad \text{h is a constant.} \tag{3.16}$$

Example 3.11. *The diameter of a semi circular plate of radius a, is kept at $0°C$ and the temperature at the semi circular boundary is $T_0^\circ C$. Find the steady state temperature in the plate.*

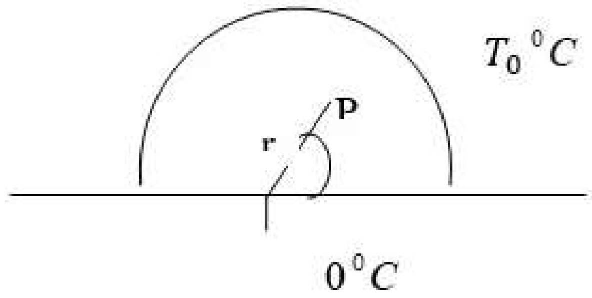

Figure 3.2: Example 3.2

Considering all three possibilities for h ($h = 0, h > 0, h < 0$) in Equation 3.16, It can be shown that possible solution is,

$$T(r,\theta) = (Ar^k + Br^{-k})(C\cos k\theta + D\sin k\theta) \tag{3.17}$$

The boundary conditions are,

- $T(r,0) = 0,\ 0 < r \leq a.$
- $T(r,\pi) = 0,\ 0 < r \leq a.$
- $T(a,\theta) = T_0,\ 0 < \theta < \pi.$

Then

$$T(r,\theta) = \sum_{n=1}^{\infty} A_n a^n \sin n\theta \tag{3.18}$$

This is the Fourier half range Sine series of T at all points. Therefore,

$$A_n a^n = \frac{2}{\pi}\int_0^{\pi} T_0 \sin n\theta d\theta. \tag{3.19}$$

Exercise 3.12. *A a long plane uniform plate is bounded by two parallel edges and an end at right angles to them. The breadth is π. This end is maintained at a temperature $100°C$ at all points and other ends are at a zero temperature. Determine the temperature at any point of the plate.*

Heat Flow in one dimension

Suppose that we have a long thin bar of length l which is aligned along the x axis. We wish to determine the temperature distribution $\theta(x,t)$ in the bar. Assume the bar is insulated along its sides, and that the heat flows in the x direction only.

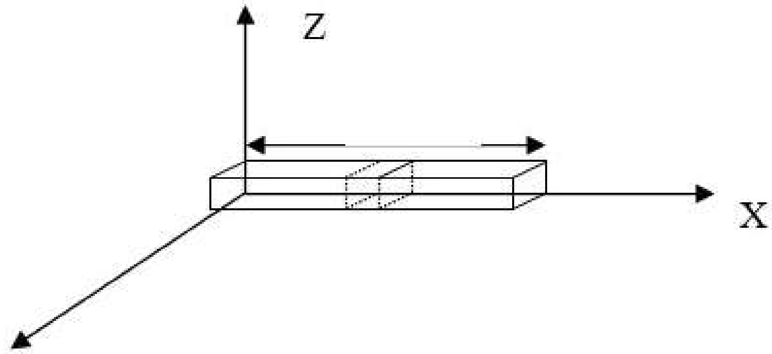

Figure 3.3: Example 3.3

The following laws of heat flow are used.

- The amount of heat in a body is proportional to its mass and to its temperature.
- The heat flows from a point at a higher temperature to a point at a lower temperature.
- The rate of flow of heat through a plane surface is proportional to the area of the surface and to the rate of change of temperature with respect to the distance in a direction perpendicular to the plane.

It can be shown that the temperature distribution $\theta(x,t)$ satisfies the following

$$\frac{\partial^2 \theta}{\partial x^2} = \frac{1}{k}\frac{\partial \theta}{\partial t}. \tag{3.20}$$

where k is a positive constant.

Example 3.13. *Obtain the solution of the equation 3.20, satisfying the following boundary conditions,*

- $\theta(0,t) = 0,\ t \geq 0$
- $\theta(l,t) = 0,\ t \geq 0$
- $\theta(x,0) = f(x),\ 0 < x < l$

$f(x)$ is a given function and l is a constant. Letting $\theta(x,t) = X(x)T(t)$ and following the steps of separation of variable, it is possible to obtain the general solution,

$$\theta(x,t) = \sum_{r=1}^{\infty} B_r \sin\frac{r\pi x}{l} e^{-\frac{r^2\pi^2 kt}{l^2}} \tag{3.21}$$

From Fourier half range sine series of $f(x)$

$$B_r = \frac{2}{l}\int_0^l f(x)\sin\frac{r\pi x}{l}dx. \tag{3.22}$$