19053041 - Q.No: 12

$$\begin{bmatrix} 3 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 10 \\ 14 \\ 14 \end{bmatrix}$$

$$3x^{(k)} + 2y^{(k)} + z^{(k)} = 10 \longrightarrow (1)$$

$$2x^{(k+1)} + 3y^{(k+1)} + 2z^{(k)} = 14 \longrightarrow (2)$$

$$x^{(k+1)} + 2y^{(k+1)} + 3z^{(k+1)} = 14 \longrightarrow (3)$$

$$\begin{bmatrix} 3 & 0 & 0 \\ 2 & 3 & 0 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} x^{k+1} \\ y^{k+1} \\ z^{k+1} \end{bmatrix} + \begin{bmatrix} 0 & 2 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x^{k} \\ y^{k} \\ z^{k} \end{bmatrix} = \begin{bmatrix} 10 \\ 14 \\ 14 \end{bmatrix}$$

$$(D+L)X^{(k+1)} + UX^{(k)} = b$$

$$X^{(k+1)} = -(D+L)^{-1} \cdot U X^{(k)} + (D+L)^{-1} b$$

$$X^{(k+1)} = BX^{k} + C$$

$$B = (-1)\begin{bmatrix} 3 & 0 & 0 \\ 2 & 3 & 0 \\ 1 & 2 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 2 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -1/3 & 0 & 0 \\ 2/9 & -1/3 & 0 \\ -1/27 & 2/9 & -1/3 \end{bmatrix} \begin{bmatrix} 0 & 2 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -2/3 & -1/3 \\ 0 & 4/9 & -2/9 \\ 0 & -2/27 & 11/27 \end{bmatrix}$$

for eigen values,

$$|B - \lambda I| = 0.$$

$$\begin{vmatrix} -\lambda & -2/3 & -1/3 \\ 0 & \frac{4}{9} - \lambda & -4/9 \\ 0 & -2/27 & \frac{11}{27} - \lambda \end{vmatrix} = 0.$$

$$-\lambda\left[\left(\frac{4}{9} - \lambda\right)\left(\frac{11}{27} - \lambda\right) - \left(\frac{2}{27}\right)\left(\frac{4}{9}\right)\right] = 0.$$

$$\lambda = 0, \quad \lambda^2 - \frac{23}{27}\lambda + \frac{44}{243} - \frac{8}{243} = 0$$

$$\lambda^2 - \frac{23}{27}\lambda + \frac{4}{27} = 0.$$

$$27\lambda^2 - 23\lambda + 4 = 0.$$

$$\lambda = \frac{23 \pm \sqrt{23^2 - 4(27)4}}{2\times 27} = \frac{23 \pm \sqrt{17}}{2(27)} \simeq 0.6093,\ 0.243$$

$\rho(A) = \max|\lambda| = 0.6093 < 1$, $\therefore$ solution is converging.

$$-[D+L]^{-1}U = -\frac{1}{27}\begin{bmatrix} 0 & 18 & 9 \\ 0 & -12 & 12 \\ 0 & 2 & -11 \end{bmatrix} = B.$$

$$[D+L]^{-1}b = \begin{bmatrix} 1/3 & 0 & 0 \\ -2/9 & 1/3 & 0 \\ 1/27 & -2/9 & 1/3 \end{bmatrix}\begin{bmatrix} 10 \\ 14 \\ 14 \end{bmatrix} = \begin{bmatrix} 10/3 \\ 22/9 \\ 52/27 \end{bmatrix} = \frac{1}{27}\begin{bmatrix} 90 \\ 66 \\ 52 \end{bmatrix}$$

$$\begin{bmatrix} x^{k+1} \\ y^{k+1} \\ z^{k+1} \end{bmatrix} = -\frac{1}{27}\begin{bmatrix} 0 & 18 & 9 \\ 0 & -12 & 12 \\ 0 & 2 & -11 \end{bmatrix}\begin{bmatrix} x^k \\ y^k \\ z^k \end{bmatrix} + \frac{1}{27}\begin{bmatrix} 90 \\ 66 \\ 52 \end{bmatrix}$$

Assume,

$$\begin{bmatrix} x^{(0)} \\ y^{(0)} \\ z^{(0)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} x^{(1)} \\ y^{(1)} \\ z^{(1)} \end{bmatrix} = \frac{1}{27}\begin{bmatrix} 90 \\ 66 \\ 52 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x^{(2)} \\ y^{(2)} \\ z^{(2)} \end{bmatrix} = -\frac{1}{27^2}\begin{bmatrix} 0 & 18 & 9 \\ 0 & -12 & 12 \\ 0 & 2 & -11 \end{bmatrix}\begin{bmatrix} 90 \\ 66 \\ 52 \end{bmatrix} + \frac{1}{27}\begin{bmatrix} 90 \\ 66 \\ 52 \end{bmatrix} = \begin{bmatrix} 86/81 \\ 650/243 \\ 1946/729 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x^{(3)} \\ y^{(3)} \\ z^{(3)} \end{bmatrix} = \frac{-1}{27}\begin{bmatrix} 0 & 18 & 9 \\ 0 & -12 & 12 \\ 0 & 2 & -11 \end{bmatrix}\begin{bmatrix} 86/81 \\ 650/243 \\ 1946/729 \end{bmatrix} + \frac{1}{27}\begin{bmatrix} 90 \\ 66 \\ 52 \end{bmatrix} = \begin{bmatrix} 0.7069 \\ 0.81 \\ 2.759 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x^{(4)} \\ y^{(4)} \\ z^{(4)} \end{bmatrix} = \frac{-1}{27}\begin{bmatrix} 0 & 18 & 9 \\ 0 & -12 & 12 \\ 0 & 2 & -11 \end{bmatrix}\begin{bmatrix} 0.7069 \\ 0.81 \\ 2.759 \end{bmatrix} + \frac{1}{27}\begin{bmatrix} 90 \\ 66 \\ 52 \end{bmatrix} = \begin{bmatrix} 0.7411 \\ 2.3336 \\ 2.9628 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x^{(1)} \\ y^{(1)} \\ z^{(1)} \end{bmatrix} = -\frac{1}{27}\begin{bmatrix} 0 & 18 & 9 \\ 0 & -12 & 12 \\ 0 & 2 & -11 \end{bmatrix}\begin{bmatrix} 0.7481 \\ 2.3336 \\ 2.8638 \end{bmatrix} + \frac{1}{27}\begin{bmatrix} 90 \\ 66 \\ 52 \end{bmatrix} = \begin{bmatrix} 0.8229 \\ 2.2018 \\ 2.9178 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x^{(2)} \\ y^{(2)} \\ z^{(2)} \end{bmatrix} = -\frac{1}{27}\begin{bmatrix} 0 & 18 & 9 \\ 0 & -12 & 12 \\ 0 & 2 & -11 \end{bmatrix}\begin{bmatrix} 0.8229 \\ 2.2018 \\ 2.9178 \end{bmatrix} + \frac{1}{27}\begin{bmatrix} 90 \\ 66 \\ 52 \end{bmatrix} = \begin{bmatrix} 0.8875 \\ 2.1784 \\ 2.9918 \end{bmatrix}$$

$x^{(k)} \rightarrow 1$

$y^{(k)} \rightarrow 2$

$z^{(k)} \rightarrow 3$ //.