MA2033 - 1954 - Fri 10 - Lect 1

Saturday, February 5, 2022 3:00 PM

Pure maths
- Algebra - Linear Algebra
- Analysis (Real / complex)
- Topology

<u>Group</u> G with a binary operation $*$, $(G, *)$

1) $G \neq \emptyset$
2) $\forall a, b \in G$; $a * b \in G$: $*$ is closed
3) $\forall a, b, c \in G$; $a * (b * c) = (a * b) * c$: associative
4) $\exists e \in G\ \forall a \in G$; $e * a = a * e = a$: identity (e)
5) $\forall a \in G\ \exists \bar{a} \in G$; $a * \bar{a} = \bar{a} * a = e$: inverse.

<u>Abelian Group</u>

1) $(G, *)$ is a group
2) $\forall a, b \in G$; $a * b = b * a$: commutative.

Ex.: $(\mathbb{R}, +)$

$* = +$, $e = 0$, $\bar{a} = -a$

abelian group ✓

Ex. $(\mathbb{R} - \{0\}, \cdot) \leftarrow$ abelian group.

$* = \cdot$, $e = 1$, $\bar{a} = a^{-1}$

$\forall a \in \mathbb{R} - \{0\}\ \exists a^{-1} \in \mathbb{R} - \{0\}$, $a \cdot a^{-1} = a^{-1} \cdot a = 1$

Ex. $GL_2(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| a, b, c, d \in \mathbb{R};\ ad - bc \neq 0 \right\}$

$* = \cdot \leftarrow$ matrix multiplication

$I = e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in GL_2(\mathbb{R})$, $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$A \cdot I = I \cdot A = A$

$$A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -c \\ -b & a \end{pmatrix}^T = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \in GL_2(\mathbb{R})$$

$A A^{-1} = A^{-1} A = I$

$|A A^{-1}| = |A||A^{-1}| = |I| = 1$, $|A^{-1}| \neq 0$

$$AA^{-1} = A^{-1}A = I$$

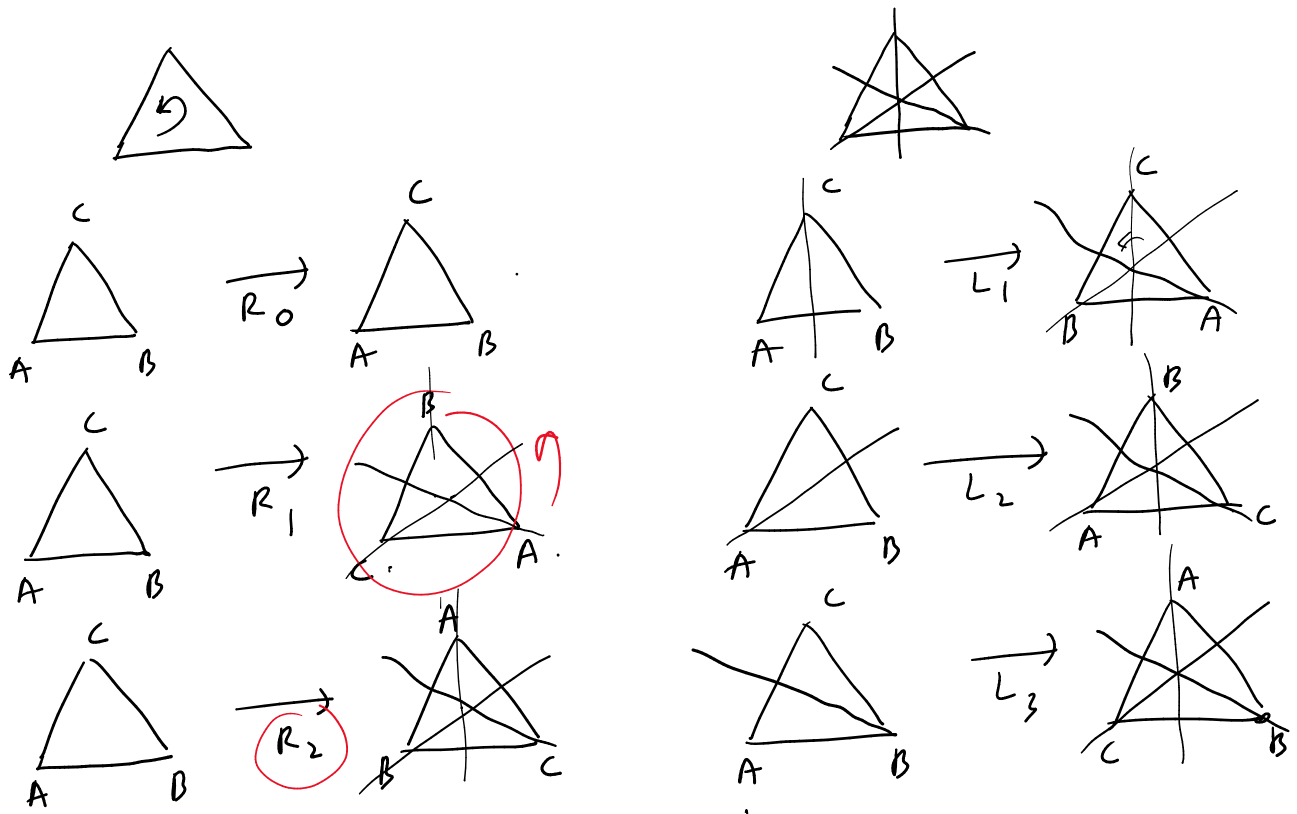
$$|AA^{-1}| = |A||A^{-1}| = |I| = 1, \quad |A^{-1}| \neq 0$$

group ✓

$A, B \in GL_2(\mathbb{R})$; $A \cdot B \neq B \cdot A$ in general.

not abelian.

<u>Ex.</u> D_3 = set of symmetries of a equilateral triangle. $= \{R_0, R_1, R_2, L_1, L_2, L_3\}$



$*$	R_0	R_1	R_2	L_1	L_2	L_3
R_0	R_0	R_1	R_2	L_1	L_2	L_3
R_1	R_1	R_2	R_0	L_3	L_1	L_2
R_2	R_2	R_0	R_1	L_2	L_3	L_1
L_1	L_1	L_2	L_3	R_0	R_1	R_2
L_2	L_2	L_3	L_1	R_2	R_0	R_1
L_3	L_3	L_1	L_2	R_1	R_2	R_0

$* = \circ$

$(f \circ g)(x) = f(g(x))$.

$e = R_0$

$R_0 * R_0 = R_0, \quad \overline{R_0} = R_0$

$\underline{R_1 * R_2} = \underline{R_2 * R_1} = R_0$

$, \quad \overline{R_1} = R_2, \quad \overline{R_2} = R_1$

$\underline{L_1 * L_1} = R_0, \quad \overline{L_1} = L_1$

$L_2 * L_2 = R_0, \quad \overline{L_2} = L_2$

$L_3 * L_3 = R_0, \quad \overline{L_3} = L_3$

L_3	L_3	L_1	L_2	R_1	R_2	R_0

$L_3 * L_3 = R_0, \quad L_3 - L_3$

Group.

$R_1 * L_2 = L_3 \quad , \quad L_2 * R_1 = L_1$

not abelian

Ex. $(B, +)$, $\quad B = \{0, 1\}, \quad + = \text{or} = *$

$e = 0 \quad , \quad 0 + 0 = 0 \quad , \quad \bar{0} = 0$

$1 + \cancel{X} = 0 \quad \bar{1}$ does not exist.

not a group.

Ex. $(B, \cdot)$, $\quad B = \{0, 1\}$, $\quad \cdot = \text{and} = *$

$e = 1 \quad , \quad 1 \cdot 1 = 1 \quad , \quad \bar{1} = 1$

$0 \cdot \underset{\uparrow \text{no}}{\cancel{X}} = 1 = e \quad \bar{0}$ D.N.E.

Ex. $(G, *)$ is a group $\Rightarrow \bar{\bar{a}} = a$

$(\mathbb{R}, +) \Rightarrow -(-a) = a$

$(\mathbb{R} - \{0\}, \cdot) \Rightarrow (a^{-1})^{-1} = a$.

Field F with two binary operations $+, \cdot$

$(F, +, \cdot)$

1) $(F, +)$ is an abelian group $\quad \Big| \quad 0 = e_+$
2) $(F - \{0\}, \cdot)$ " $\quad \Big| \quad 1 = e_\cdot$
3) $\forall a, b \in F$; $\quad a \cdot b \in F$
4) $\forall a, b, c \in F$; $\quad a \cdot (b + c) = (a \cdot b) + (a \cdot c)$

Ex. $(\mathbb{R}, +, \cdot)$ Field ✓

$+ \rightarrow \quad \bar{a} = -a$

$\cdot \rightarrow \quad \bar{a} = a^{-1}$

Ex. $(\mathbb{R}, \cdot, +)$

$a + (b \cdot c) = (a+b) \cdot (a+c)$ ✗

Ex. $(\mathbb{R}, \cdot, +)$

4) $a, b, c \in \mathbb{R}$; $\quad a + (b \cdot c) = (a+b)\cdot(a+c)$ ✗

Also $(\mathbb{R} - \{1\}, +)$ a group? no

$-1 \in \mathbb{R} - \{1\}$; $\quad -1 + x = 0$, $x = 1 \notin \mathbb{R} - \{1\}$

i.e. $-(-1) = 1 \in \mathbb{R} - \{1\}$.

Ex. $(\mathbb{Q}, +, \cdot)$

$\mathbb{Q} = \{ \frac{p}{q} \mid p, q \in \mathbb{Z};\ q \neq 0 \}$.

$$\frac{p_1}{q_1} + \frac{p_2}{q_2} = \frac{p_1 q_2 + p_2 q_1}{q_1 q_2 \neq 0} \in \mathbb{Q}.$$

$$\frac{p_1}{q_1} \cdot \frac{p_2}{q_2} = \frac{p_1 p_2}{q_1 q_2 \neq 0} \in \mathbb{Q}$$

$$\frac{p}{q} \neq 0 \ (p \neq 0); \quad \left(\frac{p}{q}\right)^{-1} = \frac{q}{p \neq 0} \in \mathbb{Q}.$$

$$0 = \frac{0}{1} \in \mathbb{Q}; \quad 1 = \frac{1}{1} \in \mathbb{Q}.$$

Ex. $(\{0, 1, 2\}, +_3, \cdot_3)$; $\quad 2 +_3 2 =_3 4 =_3 1$

+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

$0 +_3 0 =_3 0$, $\quad \bar{0} = 0$

$1 +_3 2 =_3 2 +_3 1 = 0$, $\bar{1} = 2$, $\bar{2} = 1$

$e = 0$

$\cdot$	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

$e = 1$, $\quad \{1, 2\}$

$1 \cdot_3 1 = 1$, $\quad \bar{1} = 1$

$2 \cdot_3 2 = 1$, $\quad \bar{2} = 2$

Field

Ex. $(\{0, 1, 2, 3\}, +_4, \cdot_4)$ a Field?

Thm: If p is prime $(\{0,1,2,\ldots,p-1\}, +_p, \cdot_p)$ is a field. $\mathbb{F}_p$

Thm: $(F, +, \cdot)$ is field.

1) $a \cdot 0 = 0$

$$a = a \cdot 1 = a \cdot (1+0) = (a \cdot 1) + (a \cdot 0)$$
$$a = a + (a \cdot 0)$$
$$-a + a = -a + (a + (a \cdot 0))$$
$$0 = (-a + a) + a \cdot 0$$
$$0 = 0 + (a \cdot 0)$$
$$0 = a \cdot 0$$

2) $1 \neq 0$. $(F, +, \cdot)$ is a field, $e_+ = 0$, $e_\cdot = 1$

$F - \{0\} \neq \emptyset$.

$\exists\, 1 \in F - \{0\}\ \forall a \in F - \{0\}$; $a \cdot 1 = 1 \cdot a = a$.

If $1 = 0 \Rightarrow 0 \in F - \{0\}$ #

So $1 \neq 0$. (at least 2 elements in a field)

vector space $(V, *, \circ)$ over a field $(F, +, \cdot)$

$+ : F \times F \to F$ ✓

$\cdot : F \times F \to F$ ✓

$* : V \times V \to V$ ✓ : vector addition

$\circ : F \times V \to V$: scalar multiplication.

Rules.

1) $(V, *)$ is an abelian group.

2) $(F, +, \cdot)$ is a field.

3) $\forall a \in F\ \forall x \in V$; $a \circ x \in V$

4) $\forall a \in F, \forall x, y \in V$; $a \circ (x * y) = (a \circ x) * (a \circ y)$

5) $\forall a, b \in F\ \forall x \in V$; $(a+b) \circ x = (a \circ x) * (b \circ x)$

4) $\forall a \in F, \forall x, y \in V$; $a \circ (x * y) = (a \circ x) * (a \circ y)$
5) $\forall a, b \in F, \forall x \in V$; $(a + b) \circ x = (a \circ x) * (b \circ x)$
6) $\forall a, b \in F \; \forall x \in V$; $(a \cdot b) \circ x = a \circ (b \circ x)$
7) $\forall x \in V$; $1 \circ x = x$.

Ex. $(\mathbb{R}^3, +, \cdot)$ with $(\mathbb{R}, +, \cdot)$ vector space ✓

$+$: vector addition : $*$

$\cdot$: scalar multiplication : $\circ$

$x = (a_1, a_2, a_3)$, $y = (b_1, b_2, b_3)$

$x * y = \underset{\uparrow \text{ vector } +}{x + y} = (a_1 + b_1, \underset{\uparrow \mathbb{R} +}{a_2 + b_2}, a_3 + b_3) \in \mathbb{R}^3$

$a \in \mathbb{R}$

$a \circ x = \underset{\uparrow \text{ scalar multiplication.}}{a \cdot (a_1, a_2, a_3)} = (a \cdot a_1, \underset{\uparrow \mathbb{R} \cdot}{a \cdot a_2}, a \cdot a_3) \in \mathbb{R}^3$

$e_* = \underline{0} = (0, 0, 0) \in \mathbb{R}^3$.

$\bar{x} = -x = -(a_1, a_2, a_3) = (\underset{\uparrow \mathbb{R}}{-a_1}, -a_2, -a_3) \in \mathbb{R}^3$

Ex $(\mathbb{R}^3, +, \cdot)$ over $(\mathbb{C}, +, \cdot)$

a vector space? no

$a \in \mathbb{C}$

$a \circ x = \underset{\uparrow \mathbb{C}}{a \cdot x} = (a \cdot a_1, a a_2, a a_3) \in \mathbb{C}^3$ $\mathbb{R}$

Ex. $(\mathbb{C}^3, +, \cdot)$ over $(\mathbb{R}, +, \cdot)$

a vector space? ✓

$a \in \mathbb{R}$

$a \circ x = a \cdot x = (a a_1, a a_2, a a_3) \in \mathbb{C}^3$

Ex. $(\underset{\parallel V}{\mathbb{R}^+}, *, \circ)$ over $\underline{(\mathbb{R}, +, \cdot)}$

Def. $\begin{cases} x, y \in \mathbb{R}^+ = V \\ x * y = x \underset{\uparrow \mathbb{R}}{\cdot} y \in \mathbb{R}^+ \end{cases}$

a + ...

Def. $\begin{cases} x * y = x \underset{\mathbb{R}}{\cdot} y \in \mathbb{R}^+ \\ a \in \mathbb{R} = F, \quad a \circ x = x^a \in \mathbb{R}^+, \quad (-1)^{\pi} ✗ \end{cases}$

Vector space.

1) $(\mathbb{R}^+, *)$ is an abelian group ✓

1.1) $1 \in \mathbb{R}^+$, $\mathbb{R}^+ \neq \emptyset$

1.2) ✓ $x, y, z \in \mathbb{R}^+$, $x * (y * z) = x \cdot (y \cdot z)$

$\underset{x, y, z \in \mathbb{R}}{\uparrow}$ $= (x \cdot y) \cdot z = (x * y) * z$

1.3) $\underbrace{1 \in \mathbb{R}^+, x \in \mathbb{R}^+}_{1, x \in \mathbb{R}}$; $1 * x = 1 \cdot x = x = x \cdot 1 = x * 1$

1.4) $x \in \mathbb{R}^+$, $x^{-1} \in \mathbb{R}^+$; $x * x^{-1} = x \cdot x^{-1} = 1 = x^{-1} \cdot x = x^{-1} * x$

1.5) ✓ $\underbrace{x, y \in \mathbb{R}^+}_{x, y \in \mathbb{R}}$, $x * y = x \cdot y = y \cdot x = y * x$

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2) $(\mathbb{R}, +, \cdot)$ is a field.

3) $a \in \mathbb{R}$, $x \in \mathbb{R}^+$; $a \circ x = x^a \in \mathbb{R}^+$

4) $a \in \mathbb{R}$; $x, y \in \mathbb{R}^+$; $a \circ (x * y) = a \circ (x \cdot y)$

$= (x \cdot y)^a = x^a \cdot y^a = x^a * y^a$

$= (a \circ x) * (a \circ y)$

5) $a, b \in \mathbb{R}$; $x \in \mathbb{R}^+$; $(a + b) \circ x = x^{a+b} = x^a \cdot x^b$

$= x^a * x^b = (a \circ x) * (b \circ x)$

6) $a, b \in \mathbb{R}$; $x \in \mathbb{R}^+$; $(a \cdot b) \circ x$

$= x^{a \cdot b} = (x^b)^a = (b \circ x)^a$

$= a \circ (b \circ x)$

7) $x \in \mathbb{R}^+$, $1 \circ x = x^1 = x$

so the above is a vector space.

Thm: $(V, *, \circ)$ over $(F, +, \cdot)$ is a vector space.

$0 \circ x = e$; $e = e_*$, $0 = e_+$, $x \in V$.

Thm.

$0 \circ x = e \; ; \quad e = e_*, \; 0 = e_+, \; x \in V.$

Proof:
$$\begin{aligned}
x &= 1 \circ x = (1+0) \circ x \\
&= (1 \circ x) * (0 \circ x) \\
x &= x * (0 \circ x) \\
\bar{x} * x &= \bar{x} * (x * (0 \circ x)) \\
e &= (\bar{x} * x) * (0 \circ x) \\
e &= e * (0 \circ x) \\
e &= 0 \circ x
\end{aligned}$$

Th: $a \in F, \; e = e_*$

$a \circ e = e$

Proof:
$$\begin{aligned}
a \circ x &= a \circ (x * e) \\
&= (a \circ x) * (a \circ e) \\
\overline{a \circ x} * (a \circ x) &= \overline{a \circ x} * ((a \circ x) * (a \circ e)) \\
e &= (\overline{a \circ x} * (a \circ x)) * (a \circ e) \\
e &= e * (a \circ e) \\
e &= a \circ e
\end{aligned}$$

Quiz 1: $a \circ x = e \Rightarrow a = 0$ or $x = e$.

$(V, *, \circ)$ over $(F, +, \cdot)$ is a vector space

$a \in F, \; x \in V, \quad e = e_*, \; 0 = e_+$

Note: $(\mathbb{R}^+, +, \cdot)$ over $(\mathbb{R}, +, \cdot)$

$0 \circ x = 0 \cdot (a_1, a_2, a_3) = (0, 0, 0) = e_+$ ← o vector, ← vector.

$a \circ e_+ = a(0, 0, 0) = (0, 0, 0) = e_+$ ✓

$a \circ e_+ = a(0,0,0) = (0,0,0) = e_+$

Note: $(\mathbb{R}^+, *, \circ)$ over $(\mathbb{R}, +, \cdot)$

$$x * y = x \cdot y \;; \quad a \circ x = x^a.$$

$$0 \circ x = \underset{\uparrow e_+}{0} \circ x = x^0 = 1 = e_\cdot = e_* \checkmark$$

$$a \circ e_* = a \circ 1 = 1^a = 1 = e_\cdot = e_* \checkmark$$
